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Introduction

Humans were naked worms; yet they had an internal model of the world. In the course of the time up to the present, this model has been updated many times, following the development of new experimental possibilities or the devlopment of their intellect. Sometimes the updating has been only quantitative, sometimes it has been qualitative. Inverse problem theory tries to describe the rules human beings should use for quantitative updatings. In fact, inverse problems are some of the most important mathematical problems in science and mathematics because they tell us about parameters (unknowns) that we cannot directly observe or measure. It is called an inverse problem because it is the process of calculating from a set of observations the causal factors that produced them in certain phenomenon. In other words, it starts with the effects and then calculates the causes. It is the inverse of a forward problem, which starts with the causes and then calculates the effects.

We find inverse problems in many scientific fields:

- Medical imaging (calculating an image in X-ray, scan, computed tomography)

- Petroleum engineering (prospecting by seismic methods, magnetic)

- Hydrogeology (identification of hydrolic permeabilities)

- Oceanography (underwater acoustics)

- Chemistry and physics (quantum mechicans)

- Geophysicss, radar, optics, astronomy, signal processing,...etc.

Inverse problems are challenging to solve because, according to Albert Tarantola[1] , the set of observations usually overdetermines some parameters while leaving others underdetermined. Schematically, there are two reasons for underdetermination: intrinsic lack of data and experimental uncertainties. The second reason is uncertainty of knowledge(observed values always have experimental uncertainties).

Underdetermination is handled easily by pure mathematicians like Jacques Hadamard, he defined **ill-posed problems** in 1923, in his book "Lectures on Cauchy's Problem in Linear Partial Differential Equations"[2] . In his opinion, ill-posed problems do not have physical sense, so he introduced the notion of the **well-posed** problem which, according to him, must satisfy three properties:

- The problem must have a solution;
- The solution must be unique, and;
- The solution must be stable under the small changes to the data.

The manuscriptis organized as follows:Chapter 1 talks about inverse problems in general with some examples. Chapter 2 is dedicated to basic functional analysis. While Chapter 3 reviews regularization methods of ill-posed PDEs problems. For the last chapter (4) , it presents a numerical illustration: inverse estimation of the initial condition for the heat equation.

Chapter 1

Generalities and some examples

In the late 1950's and early 1960's the theory of ill-posed problems attracted the attention of many mathematicians, due to the appearance of series of new approaches that became essential for this theory. Inverse and ill-posed problems gained popularity very rapidly, with the advent of computers. By the present day, the theory of inverse and ill-posed problems has developed into a powerful and field of science that has an impact on almost every area of mathematics. In most cases, inverse and ill-posed problems have one important property in common: instability.In these cases, inverse problems turn out to be ill-posed and, conversely, an ill-posed problem can usually be reduced to a problem that is inverse to some well-posed problem.

This chapter is an awarness of the consequences of ill-posed problems, it might be helpful to explain in more precise terms exactly what is meant by a nonwell posed problem and to provide an interesting examples of such a problem.

1.1 What is an inverse problem?

There are two types of inverse problems: linear and non-linear. Linear problems generally come down to solving integral equations of the first kind. However, non-linear problems are more difficult to solve and they are often found in habit. Let's explain on an example what is an inverse problem:[4]

Consider a rectangular iron bar that we heat at one of its ends. The diffusion of heat inside the bar is modalized by a boundary problem for a heat equation. The questions to be asked are: Can we determine the diffusion coefficient by measuring the temperature of the bar at the other end? How many measurements are needed to ensure that we determine an unique diffusion coefficient? In practice, we want to calculate this coefficient. We start by replacing the continuous model with a discrete model. So there is going to be a stability problem, that is to say: How can we control the disturbances on the diffusion coefficients by the errors that we make on the measurements?

Let Ω be a heated design domain fully occupied by conductive materials. We assume that Ω is an open and bounded domain of $\mathbb{R}^d$, $d = 2, 3$. This example can be modalized by the following boundary value problem:

$$\begin{cases} \frac{\partial u}{\partial t} - \operatorname{div}\left(a(z)\nabla u\right) = 0 & \text{in} \quad \Omega \times (0, T)\,, \\ u = 0 & \text{in} \quad \Omega \times \{0\}\,, \\ u = f & \text{on} \quad \Gamma_1 \times (0, T)\,, \\ \frac{\partial u}{\partial v} = 0 & \text{on} \quad \Gamma_2 \times (0, T)\,, \end{cases} \tag{1.1}$$

where f is a generated heat source on a smooth boundary Γ_1, Γ_2 is the rest of the boundary of Ω and $a(z)$ is the diffusion coefficient(assumed to be constant over time).

Can we then determine the coefficient $a(z)$ from the measurements $u = g$, $M \times (0, T)$, M being a part of Γ_2? For this example,we first examine the uniqueness (i.e.the injectivity which has a associates g).

Then, we are interested in stability. More precisely, we wish to establish an estimate of the form $d_1\left(a_1, a_2\right) \preceq \omega(d_2\left(g_1, g_2\right))$, for $d_1\left(a_1, a_2\right)$ neighbor of zero, ω is an increasing function, defined on $]0, +\infty[$ such as $\omega\left(s\right) \rightarrow 0$ when s converges to 0. d_1 and d_2 are distances defined respectively on the set of cofficients and the set of measures.

Note that, due the regularizing effect of elliptical and parabolic equations, the continuity module ω, here, is a logarithm or a power of this one. There are examples where it has been shown to be optimal.

Hence the notion of Hadamard's ill-posed problem[2] .For this reason, if we wish to calculate a from g, minimizing for example a functional of the form:$J(a) = \|u-g\|^2_{L^2(M\times(0,T))}$we have to use a regularisation method, for example Tikhonov type[3] .
Regarding the measures there are several possibilities. We can for example replace the above with:

$$u(.,t_i) = g_i \ \text{, on } M \ \text{ with } t_i \text{ , } 1 \preceq i \preceq N \text{ , points of } (0,T)\,.$$

We can also vary f . We give a finite or infinite set J .For each $j \in J$, we have a measure g_i .In this case the inverse problem consists in determining a by the application $A : f_j \to g_j$.
This problem is an inverse prolem which we try to determine a coefficient.

1.2 Well-posed and Ill-posed problem

We can formulate the problem as follows: $A : Z \to U$ such as (Z, ρ_z) and (U, ρ_u) are metric spaces, the problem: given $u \in U$, find $z \in Z$ such that

$$Az = u \tag{1.2}$$

is said to be well posed if [3] :
-for $u \in U$ there is $z_u \in Z$ such that $Az_u = u$ (the solution z_u existes for all u from U),
- z_u is unique (injectivity of A);
-A has a continuous inverse ($A^{-1}$is continuous: for all $\varepsilon \succ 0$ there is $\delta(\varepsilon, u) \succ 0$ such that $\rho_u(u, u_1) \prec \delta$ implies $\rho_z(z_u, z_{u_1}) \prec \varepsilon$).
If not the problem (1.2) is called ill-posed.

If a solution exists it is perfectly conceivable that different parameters lead to the same observations. The fact that the solution of an inverse problem may not exist is not a serious difficulty.It is usually possible to restore existence by relaxing the notion of solution. Non-uniquness is more serious problem.

If a problem has more several solutions, there must be a way to choose between them. For this, it is necessary to have additional informations(a priori information). The lack of continuity of A^{-1} is undoubtedly the most problematic, small disturbances on the data u can generate large differences on the solution z.

1.3 Examples of Ill-posed problems

Here are some examples of ill-posed problems:

Example 1.3.1 *(algebra, systems of linear algebraic equations).[5]*

Consider the system of linear algebraic equations

$$Aq = f \ , \tag{1.3}$$

where A is an $n \times m$ matrix, q and f are $n-$ and $m-$dimensional vectors, respectively. Let the rank of A be equal to $\min(n, m)$. For $m \prec n$ the system may have many solutions. For $m \succ n$ there may be no solutions. For $m = n$ the system has a unique solution for any right-hand side. In this case, there exists an inverse operator (matrix) A^{-1} . It is bounded, since it is a linear operator in a finite-dimensional space. Thus, all three conditions of well-posedness in the sense of Hadamard are satisfied.

We now analyze in detail the dependence of the solution on the perturbations of the right-hand side f in the case where the matrix A is nondegenerate. Subtracting the original equation (1.3) from the perturbed equation

$$A(q + \delta q) = f + \delta f \ ,$$

we obtain $A\delta q = \delta f$, which implies $\delta q = A^{-1}\delta f$ and $\|\delta q\| \leq \|A^{-1}\| \, \|\delta f\|$. We also have $\|A\| \, \|q\| \geq \|f\|$.

As a result, we have the best estimate for the relative error of the solution:

$$\frac{\|\delta q\|}{\|q\|} \le \|A\| \left\|A^{-1}\right\| \frac{\|\delta f\|}{\|f\|}$$

which shows that the error is determined by the constant $\mu(A) = \|A\| \|A^{-1}\|$ called the condition number of the system (matrix). Systems with relatively large condition number are said to be ill-conditioned. For normalized matrices ($\|A\| = 1$), it means that there are relatively large elements in the inverse matrix and, consequently, small variations in the right-hand side may lead to relatively large (although finite) variations in the solution. Therefore, systems with ill-conditioned matrices can be considered practically unstable, although formally the problem is well-posed and the stability condition $\|A^{-1}\| < \infty$ holds.

For example, In the system (1.3) ,where $A = \begin{pmatrix} 10 & 7 & 8 & 7 \\ 7 & 5 & 6 & 5 \\ 8 & 6 & 10 & 9 \\ 7 & 5 & 9 & 10 \end{pmatrix}$ and $f = \begin{pmatrix} 32 \\ 23 \\ 33 \\ 31 \end{pmatrix}$

We find

$$q = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix},$$

now we take a second member $\tilde{f}$ very slightly diffrent from f ,

let $\tilde{f} = \begin{pmatrix} 32.1 \\ 22.9 \\ 33.1 \\ 30.9 \end{pmatrix}$, then we verify that the solution of $Aq = \tilde{f}$ is $q = \begin{pmatrix} 9.2 \\ -12.6 \\ 4.5 \\ -1.1 \end{pmatrix}$.

In this case, we see that small perturbations on f have led to large variations on q . In this example $\mu(A) = 2984.0942$ (A is ill-conditioned matrix).If the determinant of A be zero. Then the system (1.3) may either have no solutions or more than one solution. It follows that the problem $Aq = f$ is ill-posed for degenerate matrices A ($\det A = 0$).

Example 1.3.2 *(minimization problems).[6]*

We consider the minimization problem:

$$\phi(u) = \inf \|Au - f\|$$

Assume that u_i is the infimum of $\phi(u) : \phi(u_i) \leq \phi(u)$ .If f is perturbed , thus:

$$\phi_\delta(u) = \inf \|Au - f_\delta\|, \quad \|f_\delta - f\| \leq \delta \ ,$$

the infimum of $\phi_\delta(u)$ cannot be reached at an element u_δ which is far from u_i, hence the graphe of $f \mapsto u_i$ can be non-continuous. In this case this problem is ill-posed.

Example 1.3.3 *(integral geometry on circles).[5]*

Consider the problem of determining a function of two variables $q(x, y)$ from the integral of this function over a collection of circles whose centers lie on a fixed line.

Assume that $q(x, y)$ is continuous for all $(x, y) \in \mathbb{R}^2$. Consider a collection of circles whose centers lie on a fixed line (for definiteness, let this line be the coordinate axis $y = 0$). Let $L(a, r)$ denote the circle $(x - a)^2 + y^2 = r^2$, which belongs to this collection. It is required to determine $q(x, y)$ from the function $f\ (x, y)$ such that

$$\int_{L(x,r)} q(\xi, \tau)\, dl = f(x, r), \tag{1.4}$$

and $f\ (x, r)$is defined for all $x \in (-\infty, +\infty)$ and $r > 0$.

The solution of this problem is not unique in the class of continuous functions, since for any continuous function $\tilde{q}(x, y)$ such that $\tilde{q}(x, y) = -\tilde{q}(x, -y)$ the integrals

$$\int_{L(x,r)} \tilde{q}(\xi, \tau)\, dl$$

vanish for all $x \in \mathbb{R}$ and $r > 0$. Indeed, using the change of variables $\xi = x + r\cos\varphi$, $\tau = r\sin\varphi$, we obtain

$$\int_{L(x,r)} \tilde{q}(\xi,\tau)\, dl = \int_0^{2\pi} \tilde{q}(x + r\cos\varphi, r\sin\varphi)\, r d\varphi \tag{1.5}$$
$$= \int_0^{\pi} \tilde{q}(x + r\cos\varphi, r\sin\varphi)\, r d\varphi + \int_{\pi}^{2\pi} \tilde{q}(x + r\cos\varphi, r\sin\varphi)\, r d\varphi$$

Putting $\bar{\varphi} = 2\pi - \varphi$ and using the condition $\tilde{q}(x,y) = -\tilde{q}(x,-y)$, we transform the last integral in the previous formula:

$$\int_{\pi}^{2\pi} \tilde{q}(x + r\cos\varphi, r\sin\varphi)\, r d\varphi = \int_{\pi}^{0} \tilde{q}(x + r\cos\bar{\varphi}, -r\sin\bar{\varphi})\, r d\bar{\varphi} = -\int_0^{\pi} \tilde{q}(x + r\cos\bar{\varphi}, r\sin\bar{\varphi})\, r d\bar{\varphi}$$

Substituting the result into (1.5) , we have

$$\int_{L(x,r)} \tilde{q}(\xi,\tau)\, dl = 0$$

for $x \in \mathbb{R}$, $r > 0$.

Thus, if $q(x,y)$ is a solution to the problem (1.4) , then $q(x,y) + \tilde{q}(x,y)$, where $\tilde{q}(x,y)$ is any continuous function such that $\tilde{q}(x,y) = -\tilde{q}(x,-y)$ is also a solution to(1.4) . For this reason, the problem can be reformulated as the problem of determining the even component of $q(x,y)$ with respect to y. The first well-posedness condition is not satisfied in the above problem: solutions may not exist for some $f(x,r)$.

Example 1.3.4 *(Fredholm integral equation of the first kind).[3]*

Consider the problem of solving a Fredholm integral equation of the first kind

$$\int_a^b K(x,s)\, q(s)\, ds = f(x), \qquad c \le x \le d \tag{1.6}$$

where the kernel $K(x,s)$ and the function $f(x)$ are given and it is required to find $q(s)$. It is assumed that $f(x) \in C[c,d]$, $q(s) \in C[a,b]$and $K(x,s)$, $K_x(x,s)$, and $K_s(x,s)$ are continuous in the rectangle $c \le x \le d$, $a \le s \le b$. The problem of solving equation (1.6) is ill-posed because solutions may not exist for some functions $f(x) \in C[c,d]$. For example, take a function $f(x)$ that is continuous but not differentiable on $[c,d]$. With such a right-hand side, the equation cannot have a continuous solution $q(s)$ since the conditions for the kernel $K(x,s)$ imply that the integral in the left-hand side of (1.6) is differentiable with respect to the parameter x for any continuous function $q(s)$. The condition of continuous dependence of solutions on the initial data is also not satisfied for equation (1.6) .

Example 1.3.5 *(Volterra integral equations of the first kind).[3]*

Consider the problem of solving a Volterra integral equation of the first kind

$$\int_0^x K(x,s)\, q(s)\, ds = f(x), \qquad 0 \leq x \leq 1 \tag{1.7}$$

For $K = 1$ the problem (1.7) is equivalent to differentiation $f'(x) = q(x)$. The sequence $f_n(x) = \cos(nx) \diagup \sqrt{n}$ demonstrates the instability of the problem.

Example 1.3.6 *(calculus, summing a Fourier series).[3]*

The problem of summing a Fourier series consists in finding a function $q(x)$ from its Fourier coefficients. We show that the problem of summation of a Fourier series is unstable with respect to small variations in the Fourier coefficients in the l_2 metric if the variations of the sum are estimated in the C metric. Let

$$q(x) = \sum_{k=1}^{\infty} c_k \cos(kx)$$

and let the Fourier coefficients c_k of the function $q(x)$ have small perturbations: $\tilde{c}_k = c_k + \frac{\varepsilon}{k}$. Set

$$\tilde{q}(x) = \sum_{k=1}^{\infty} \tilde{c}_k \cos(kx)$$

The coefficients of these series in the l_2 metric differ by

$$\left\{ \sum_{k=1}^{\infty} (c_k - \tilde{c}_k)^2 \right\}^{\frac{1}{2}} = \varepsilon \left\{ \sum_{k=1}^{\infty} \frac{1}{k^2} \right\}^{\frac{1}{2}} = \varepsilon \sqrt{\frac{\pi^2}{6}},$$

which vanishes as $\varepsilon \to 0$. However, the difference

$$q(x) - \tilde{q}(x) = \varepsilon \sum_{k=1}^{\infty} \frac{1}{k} \cos(kx)$$

can be as large as desired because the series diverges for $x = 0$.

Thus, if the C metric is used to estimate variations in the sum of the series, then summation of the Fourier series is not stable.

Example 1.3.7 *(differential equation of the second order).*

Suppose that a particle of unit mass is moving along a straight line. The motion is caused by a force $f(t)$ that depends on time. If the particle is at the origin $x = 0$ and has zero velocity at the initial instant $t = 0$, then, according to Newton's laws, the motion of the particle is described by a function $u(t)$ satisfying the Cauchy problem

$$\begin{cases} \frac{\partial^2 u}{\partial t^2} = f(t), \quad t \in [0, T], \\ u(0) = 0, \quad \frac{\partial u}{\partial t}(0) = 0, \end{cases} \tag{1.8}$$

where $u(t)$ is the coordinate of the particle at the instant t. Assume now that the force $f(t)$ is unknown, but the coordinate of the particle $u(t)$ can be measured at any instant of time (or at certain points of the interval $[0, T]$). It is required to reconstruct $f(t)$ from $u(t)$. Thus, we have the following inverse problem: determine the function $f(t)$ from the known solution $u(t)$ of the problem (1.8) .

We now prove that the inverse problem is unstable.

Let $u(t)$ be a solution to the direct problem for some $f(t)$. Consider the following perturbations of the solution to the direct problem:

$$u_n(t) = u(t) + \frac{1}{n} \cos(nt).$$

These perturbations correspond to the right-hand sides $f_n(t) = f(t) - n \cos(nt)$. Obviously, $\|u - u_n\|_{C[0,T]} \to 0$ as $n \to \infty$, and $\|f - f_n\|_{C[0,T]} \to \infty$ as $n \to \infty$.

Thus, the problem of determining the right-hand side of the linear differential equation (1.8) from its right-hand side is unstable.

Example 1.3.8 *(retrograde heat equation)*

Let the differntial equation

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} + \triangle u\left(x,t\right) = 0 \\ u\left(x,0\right) = v\left(x\right) \end{cases} \quad \text{for } x \in \mathbb{R}^d,\ t \in \mathbb{R}_+ \tag{1.9}$$

if $d = 1$ and $v\left(x\right) = n^{-1} \sin\left(nx\right)$, where $n \in \mathbb{N}^*$, then the solution is given by:

$u\left(x,t\right) = n^{-1} e^{n^2 t} \sin\left(nx\right)$, checked by substituting in the equation of (1.9) .

$\left\|v\right\|_\infty = n^{-1} \to 0$ as $n \to \infty$, $\left\|u\left(x,t\right)\right\|_\infty = n^{-1} e^{n^2 t} \to \infty$ as $n \to \infty$, it follows that the problem (1.9) is ill-posed.

In other words, finding the subsequent temperature propagation, knowing the initial temperature propagation, is a well-posed problem. However, finding the temperature propagation at final time is an ill-posed problem

Example 1.3.9 *(Cauchy problem for Laplace equation).[2]*

Consider Cauchy's problem relating to the Laplace equation in two-dimentional case(the example cited by Hadamard). Let $u = u\left(x,y\right)$ be a solution to the following problem:

$$\begin{cases} \triangle u = 0, \\ u \mid_{y=0} = 0, \\ \frac{\partial u}{\partial y} \mid_{y=0} = \varphi_n\left(x\right) = \frac{\sin(nx)}{n},\ x \in \mathbb{R}, \end{cases} \quad y > 0 \tag{1.10}$$

the solution of the problem (1.10) is given by

$$u\left(x,y\right) = \frac{\sin\left(nx\right)}{2n^2}\left(e^{ny} - e^{-ny}\right) \tag{1.11}$$

and it is unique (by the uniquness of the Cauchy problem solution for elliptic equations). J.Hadamard shows that for any fixed $y > 0$ and sufficiently large n, the value of the solution (1.11) can be as large as desired, while $\varphi_n(x)$ tends to zero as $n \to \infty$. Therefore, small variations in the data may lead to indefinitely large variation in the solution, which means that the problem (1.10) is ill-posed.

Example 1.3.10 *:*

Differentiation and integration are two inverse problems of each other. It is more usual to think of differentiation as a direct problem, and integration as an inverse problem.

In fact, integration has good mathematical properties which lead to considering it as a direct problem. And differentiation is the prototype of the ill-posed problem, as we will see.

Consider the Hilbert space $L^2(\Omega)$, and the integral operator A defined by:

$$Af(x) = \int_0^x f(t)\,dt.$$

It is clear that A is linear operator of $L^2(0,1)$. This operator is injective, hawever its image is the vector subspace

$$\mathrm{Im}(A) = \left\{f \in H^1(0,1) \ \ ,u(0) = 0\right\}$$

where $H^1(0,1)$ is Soblov space.Indeed, the equation: $Af = g \Leftrightarrow f(x) = g'(x)$ et $g(0) = 0$.

The image of A is not closed in $L^2(0,1)$ (of course, it is in $H^1(0,1)$). Consequently, the inverse of A is not continuous on $L^2(0,1)$, as shown in the following example:

Consider a function $g \in C^1([0,1])$, and $n \in \mathbb{N}$. Let $g_n(x) = g(x) + \frac{1}{n}\sin(n^2 x)$. Thus:

$$f_n(x) = g_n'(x) = g'(x) + n\cos\left(n^2 x\right).$$

$$\|g - g_n\|_2^2 = \int_0^1 |g_n(x) - g(x)|^2\,dx = \frac{1}{n^2}\int_0^1 \sin^2\left(n^2 x\right)dx = \frac{1}{2n^4}\left(n^2 + \sin n^2 + \cos n^2\right),$$

(since $\int \sin^2(x)\,dx = \frac{\sin(x)\cos(x)+x}{2} + cste$), we find

$$\|g - g_n\|_2 = \frac{1}{\sqrt{2}n^2}\sqrt{n^2 + \sin n^2 + \cos n^2}$$

thus

$$\|g - g_n\|_2 = \frac{1}{\sqrt{2}n^2}\sqrt{1 + \frac{\sin(2n^2)}{2n^2}}$$

$$\|f - f_n\|_2^2 = \int_0^1 n^2 \cos^2(n^2 x)\, dx = \frac{n^2 - \sin n^2 \cos n^2}{2}$$

so

$$\|f - f_n\|_2 = \sqrt{\frac{n^2}{2} - \frac{\sin(2n^2)}{4}}$$

thus,

$$\|f - f_n\|_2 = \frac{n}{2}\sqrt{1 - \frac{\sin(2n^2)}{2n^2}}$$

So the difference between f and f_n may be large, even though the difference between g and g_n is small. The derivation operator (the inverse of A) is not continuous, at least with the choice of norms. Instability of the inverse is typical of ill-posed problems. A small perturbations on the data (here is g) can have an arbitrarily large influence on the results (here f).

Example 1.3.11 *(perception).[7]*

Consider a mapping A from the distal stimulus X (e.g. a 3D object) to the proximal stimulus Y (e.g. its retinal image). If the object and its image are represented by homogeneous coordinates, the perspective mapping A is a linear transformation. Thus, one can write the following equation:

$$Y = AX. \tag{1.12}$$

Finding the proximal stimulus for a given distal stimulus is a direct (forward) problem and is expressed in the rules of physics. In contrast to the problem(1.12) , an observer is faced with an inverse problem. Namely, perception is about inferring the properties of the distal stimulus X given the proximal stimulus Y:

$$X = A^{-1}Y$$

This inverse problem is ill posed and/or ill-conditioned. This is related to the fact that finding a unique and stable A^{-1}, which is needed to determine X, is difficult.

For example, the image of a cube. This retinal image determines an infinite number of objects whose faces do not have to be planar, edges do not have to be straight-line segments, and the object does not have to be symmetric. Clearly, the problem of visual interpretation of the retinal image is ill posed.

Chapter 2

Some basic Functional Analysis

This chapter presents some basic mathematical tools, which we need to get into inverse problems.

2.1 Vector Analysis

2.1.1 Scalar field or "Scalar function"

Typical applications of scalar fields include:potential fields, temperature, humidity, pressure. Often these problems are governed by differential equation.

Let the scalar field be a function from two- or three-dimensional field ($\mathbb{R}^n, n = 2$ or 3)which values in $\mathbb{R}$, i.e: $f \ : \mathbb{R}^3 \to \mathbb{R}$ is a scalar field.

Definition 2.1.1 *(gradient)*

We call the vector :

$$\nabla f = \begin{pmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \\ \frac{\partial f}{\partial z} \end{pmatrix} \tag{2.1}$$

the **gradient** of the function $f\,(x, y, z)$ and we note it $\mathbf{grad}\,(f)$ or ∇f.

Definition 2.1.2 *(directional derivative)*

Let u a unit vector of $\mathbb{R}^3$. We call **directional derivative** of f in the direction u at point $M_0\left(x_0, y_0, z_0\right)$ the number:

$$f_u\left(M_0\right) = \nabla f\left(M_0\right).u$$

Remark 2.1.1 *:*

If the direction is given by a vector which is not unitary, it must be made unitary by dividing it by its norm: $f_u\left(M_0\right) = \nabla f\left(M_0\right).\frac{u}{\|u\|}$.

2.1.2 Vector field or "Vector function"

The force which associates the potential energy is a vector field, it can be obtined as a factor of the gradient. It includes for example:gravitational field, velocity, electric fiels.

Definition 2.1.3 *(vector field)*

A **vector field** on two-dimensional (or three-) space is a function V that assigns to each point (x, y) (or(x, y, z)) a two (or three) dimensional vector.

2.1.3 Differential operators

Definition 2.1.4 *(divergence)*

The **divergence** of a vector field $V = \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix}$ is noted by $\operatorname{div}(V)$or $\nabla.V$ and it is given by the expression:

$$\operatorname{div}(V) = \nabla.V = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + \frac{\partial V_z}{\partial z} \tag{2.2}$$

Formally, we write: $\left[\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right] \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix} = (\nabla.V)$.

Definition 2.1.5 *(rotational)*

The **rotational** is an differential operator that transforms a vector field to another vector field. We define the rotational of a vector fiald V by the relation:

$$rotV = \nabla \wedge V = \begin{bmatrix} \frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z} \\ \frac{\partial V_x}{\partial z} - \frac{\partial V_z}{\partial x} \\ \frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y} \end{bmatrix} \tag{2.3}$$

Formally, we write: $\begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{bmatrix} \wedge \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix} = \nabla \wedge V.$

2.1.4 Gradient field "Laplacian"

For a real-valued function $f(x, y, z)$, the **laplacian** of f ,denoted by Δf , is given by:

$$\Delta f = \nabla.\nabla f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^{2f}}{\partial z^2} \tag{2.4}$$

We say that a vector field V is a **gradient field** if there exists a function f such that at any point: $V = \text{grad}(f)$and we write $\Delta f = \text{div}(\text{grad}(f))$.

2.2 L^pSpaces and and Hölder Spaces

2.2.1 Elementary Definitions of L^p Spaces

Let Ω a set from $\mathbb{R}^n$. We denote by $L^1(\Omega)$ (or simply L^1), the space of integrable functions from Ω into $\mathbb{R}$ in the sense of Lebesgue.

For $f \in L^1(\Omega)$, we shall use the notation :

$$\|f\|_{L^1(\Omega)} = \|f\|_1 = \int_\Omega |f(x)|\,dx.$$

Definition 2.2.1 *(L^p space)*

Let $p \in \mathbb{R}$ with $1 \le p < \infty$; we set

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R}; f \text{ measurable and } |f|^p \in L^1(\Omega) \right\}$$

with

$$\|f\|_{L^p(\Omega)} = \|f\|_p = \left[\int_\Omega |f|^p \right]^{\frac{1}{p}}.$$

$\|f\|_p$ is a norm.

Definition 2.2.2 *(L^∞ space)*

We set

$$L^\infty(\Omega) = \left\{ f : \Omega \to \mathbb{R} \;\middle|\; \begin{array}{l} f \text{ is measurable and there is a constant } C \\ \text{such that } |f(x)| \le C \ a.e \text{ on } \Omega \end{array} \right\}$$

with

$$\|f\|_{L^\infty(\Omega)} = \|f\|_\infty = \inf\left\{ C; |f| \le C \ a.e \text{ on } \Omega \right\}.$$

$\|f\|_\infty$ is a norm.

Remark 2.2.1 *:*

For all $1 \le p \le \infty$,$L^p(\Omega)$ is a Banach space for the norm $\|.\|_p$(it is well known).

If $f \in L^\infty(\Omega)$ then we have $|f(x)| \le \|f\|_\infty$ a.e on Ω, this implies that $\|f\|_\infty$ is a norm.

Notation 2.2.1 *:*

Let $1 \leq p \leq \infty$, we denote by p' the **conjugate** exponent of p,

$$\frac{1}{p} + \frac{1}{p'} = 1.$$

Theorem 2.2.1 *(Hölder's inequality)*[8]

Assume that $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$ with $1 \leq p \leq \infty$. Then $fg \in L^1(\Omega)$ and

$$\|fg\|_{L^1(\Omega)} \leq \|f\|_p \|g\|_{p'}. \tag{2.5}$$

2.2.2 $L^p(a, b; X)$ Spaces

Let X a Banach space and $-\infty < a < b < +\infty$.

Definition 2.2.3 *(simple function)*

A function $f : [a, b] \to X$ is said a **simple function** if there exist measurable sets $E_1, ..., E_m$ from $[a, b]$ and $x_1, ..., x_m \in [a, b]$ such that:

$$f(t) = \sum_{i=1}^{m} \chi_{E_I}(t)\, x_i$$

with $\chi_{E_I} = \begin{cases} 1 \text{ if } t \in E_i \\ 0 \text{ else} \end{cases}$ $;\forall i \in \{1, 2, ..., m\}$and E_i disjoint two by two.

Definition 2.2.4 *(measurable function)*

We say that a function $f : [a, b] \to X$ is **measurable** if there exists a sequence of simple functions $(f_k)_{k\in\mathbb{N}}$,$f_k : [a, b] \to X$, such that $f_k \to f$ $a.e$ on $[a, b]$.

Definition 2.2.5 *(Bochner integrable function)*

A measurable function $f : [a, b] \to X$ is a **Bochner integrable** if there exists a sequence of integrable simple functions (f_k) such that:

$$\lim_k \int_a^b \|f - f_k\|_X \, dt = 0.$$

In this case, the **Bochner integral** is defined by:

$$\int f(t) \, dt = \lim_k \int_a^b f_k(t) \, dt.$$

Theorem 2.2.2 [4]

A measurable function $f : [a, b] \to X$ is integrable if and only if $\|f\|_X \in L^1(a, b)$.

For $1 \leq p \leq \infty$,let $L^p(a, b; X) = \{f : [a, b] \to X$ integrable such that $\|f\|_X \in L^p(a, b)\}$, provided with the norm

$$\|f\|_{L^p(a,b;X)} = \begin{cases} \left(\int \|f\|_X^p\right)^{\frac{1}{p}}, & \text{if } p < \infty \\ \inf\{C, \|f(t)\|_X \leq C a.e \text{ on } [a, b]\}, & \text{if } p = \infty \end{cases}$$

$L^p(a, b; X)$ is a Banach space.

2.2.3 Hölder Spaces

Let $\Omega \subset \mathbb{R}^n$ open, bounded , we say that $f \in C^0(\bar{\Omega})$ is γ-Hölderian if for $\gamma \in]0, 1[$:

$$[f]_\gamma = \sup_{\substack{x,y \in \bar{\Omega} \\ x \neq y}} \frac{|f(x) - f(y)|}{|x - y|^\gamma}.$$

Definition 2.2.6 *(Hölder space)*

The function space

$$C^{k,\gamma}\left(\bar{\Omega}\right)=\left\{f\in C^{0}\left(\bar{\Omega}\right)\mid\|f\|_{C^{k,\gamma}\left(\bar{\Omega}\right)}<\infty\right\}$$

is called the **Hölder space** with exponent γ such that for $f\in C^{k}\left(\bar{\Omega}\right)$ the Hölder norm

$$\|f\|_{C^{k,\gamma}\left(\bar{\Omega}\right)}=\sum_{|\alpha|\leq k}\|\partial^{\alpha}f\|_{C^{0}\left(\bar{\Omega}\right)}+\sum_{|\alpha|=k}\left[\partial^{\alpha}f\right]_{\gamma},\ \alpha\in\mathbb{N}^{n}$$

Theorem 2.2.3 *:*

The Hölder space with the Hölder norm is a Banach space, (i.e.$C^{k,\gamma}\left(\bar{\Omega}\right)$ is a vector space,$\|.\|_{C^{k,\gamma}\left(\bar{\Omega}\right)}$ is a norm and any Cauchy sequence in the Hölder space converges).

Proposition 2.2.1 *:*

If $T>0$ a given real number and $Q=\Omega\times(0,T)$, we denote $C^{\gamma,\frac{\gamma}{2}}\left(\bar{Q}\right)$ the space of functions $f\in C^{0}\left(\bar{Q}\right)$ such that:

$$[f]_{\gamma,\frac{\gamma}{2}}=\sup\left\{\frac{f(x,t)-f(y,s)}{\left[|x-y|^{2}+|t-s|\right]^{\frac{\gamma}{2}}};\ (x,t),(y,s)\in Q,(x,t)\neq(y,s)\right\}<\infty,$$

and for $k\geq 0$ an integer we denote:

$$C^{2k+\gamma,k+\frac{\gamma}{2}}\left(\bar{Q}\right)=\left\{f\in C^{2k,k}\left(\bar{Q}\right);\ \partial^{\alpha}\partial_{t}^{\beta}f\in C^{\gamma,\frac{\gamma}{2}}\left(\bar{Q}\right),\ \ (\alpha,\beta)\in\mathbb{N}^{n}\times\mathbb{N},\ \ |\alpha|+2\beta=2k\right\}$$

$C^{2k+\gamma,k+\frac{\gamma}{2}}\left(\bar{Q}\right)$ is a Banach space if we provide it with the norm

$$\|f\|_{C^{\gamma,\frac{\gamma}{2}}\left(\bar{Q}\right)}=\sum_{|\alpha|+2\beta\leq 2k}\left\|\partial^{\alpha}\partial_{t}^{\beta}f\right\|_{C^{0}\left(\bar{Q}\right)}+\sum_{|\alpha|+2\beta=2k}\left[\partial^{\alpha}\partial_{t}^{\beta}f\right]_{\gamma,\frac{\gamma}{2}},\ \ (\alpha,\beta)\in\mathbb{N}^{n}\times\mathbb{N}.$$

2.2.4 Regular Open

(H.1)Let Ω an open boundary from $\mathbb{R}^n$ and its border Γ.

(H.2)The boundary Γ is an infinitely differentiable variety of dimension $n-1$, Ω being locally on one side of Γ (i.e. we say that $\bar{\Omega}$ is a variety with boundary of class C^∞).

Under the hypotheses $(H.1)$ and $(H.2)$ there exist a finite family of couples$(\mathcal{O}_i, \varphi_i)_{i=\overline{1,N}}$such that:

a. $(\mathcal{O}_i)_{i=\overline{1,N}}$ is a family of bounded open sets covering Γ($\Gamma \subset \bigcup\limits_{i=1}^{N} \mathcal{O}_i$).

b. $(\varphi_i)_{i=\overline{1,N}}$ is a family of C^∞-diffeomorphisms of $(\mathcal{O}_i)$ into the open set Q of $\mathbb{R}^n$ defined by

$$Q = \left\{ y = (y', y_n) \in \mathbb{R}^{n-1} \times \mathbb{R},\ |y'| < 1 \text{ and } -1 < y_n < 1 \right\}.$$

with

$$\varphi_i\left(\mathcal{O}_i \cap \Omega\right) = Q_+ \overset{def}{=} \{y = (y', y_n) \in Q,\ y_n > 0\}$$

and so

$$\varphi_i\left(\mathcal{O}_i \cap \Gamma\right) = Q_0 \overset{def}{=} \{y = (y', y_n) \in Q,\ y_n = 0\}$$

and with the following compatibility conditions: if $\mathcal{O}_i \cap \mathcal{O}_j \neq \varnothing$, there exists a diffeomorphism $\boldsymbol{J}_{ij} : \varphi_i\left(\mathcal{O}_i \cap \mathcal{O}_j\right) \to \varphi_j\left(\mathcal{O}_i \cap \mathcal{O}_j\right)$, of class C^∞ with positive Jacobian such that:

$$\varphi_j\left(x\right) = \boldsymbol{J}_{ij}\left(\varphi_i\left(x\right)\right), \qquad \forall x \in \mathcal{O}_i \cap \mathcal{O}_j.$$

We say that Ω is open lipschitzian if we replace φ_j a C^∞-diffeomorphism from $\mathcal{O}_j$ onto Q with φ_j a bijection from $\mathcal{O}_j$ onto Q such that φ_j and φ_j^{-1} are lipschitzians for all j.

By modifying the regularity of functions φ_j ,we can easily guess how to define other regularity types of the open Ω : $C^k, C^{k,\gamma}; k$ integer and $1 < \gamma < 0$,etc.

2.3 Distributions

Let Ω an open subspace from $\mathbb{R}^n$ and $\varphi : \Omega \to \mathbb{C}$.

Definition 2.3.1 *(space of functions with compact support)*

We define the space of continuous functions on Ω **with compact support** by:

$$C_c(\Omega) = \{\varphi \in C(\Omega)\,;\ \varphi(x) = 0\ \forall x \in \Omega \setminus K,\ K \subset \Omega \textit{ is a compact}\},$$

we denote the support of function φ on Ω by $\mathrm{supp}(\varphi) = \overline{\{x \in \Omega;\ \varphi(x) \neq 0\}}$.

Notation 2.3.1 *:*

For a compact $K \subset \Omega$,we pose: $\mathcal{D}_K(\Omega) = \{\varphi \in C^\infty(\Omega)\,;\ \mathrm{supp}(\varphi) \subset \Omega\}$.

$\mathcal{D}_K(\Omega)$ is Fréchet space when it is provided with the topology which is defined by a family of seminorms: $p_{K,m}(\varphi) = \sup\limits_{|\alpha| \leq m, x \in K} |\partial^\alpha \varphi(x)|$.

Let $\mathcal{D}(\Omega) = \bigcup \mathcal{D}_K(\Omega)$,the union of all compacts K on Ω. We note that $\mathcal{D}(\Omega)$ is space of C^∞ functions with compact support.

Theorem 2.3.1 *:*

$\mathcal{D}(\Omega)$ is dense in $L^p(\Omega)$.That is to say: there exists a sequence $\{f_k\} \subset \mathcal{D}(\Omega)$ such that $f_k \to f$ in $L^p(\Omega)$, or:

$$\forall f \in L^p(\Omega),\ \forall \varepsilon > 0,\ \exists g \in \mathcal{D}(\Omega) \text{ such that } \|f - g\|_p < \varepsilon.$$

We consider $\mathcal{D}'(\Omega)$ the topological dual of $\mathcal{D}(\Omega)$,that is to say the space of linear continuous forms on $\mathcal{D}(\Omega)$.

Here is a simple criteria which verifies if a linear form on $\mathcal{D}(\Omega)$ is continuous:

Proposition 2.3.1 [9]

A linear form u on $\mathcal{D}(\Omega)$ is in $\mathcal{D}'(\Omega)$ if and only if, for any compact $K \subset \Omega$, there exists a positive constant C and a positive integer k such that:$|u(\varphi)| \leq C \sup\limits_{|\alpha| \leq k, x \in K} |\partial^\alpha \varphi(x)|$, $\varphi \in \mathcal{D}_K(\Omega)$,where $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$ is a multi-index, its length (or module): $|\alpha| = \alpha_1 + ... + \alpha_2$.
If $u \in \mathcal{D}'(\Omega)$, we denote: $\partial^\alpha u(\varphi) = (-1)^{|\alpha|} u(\partial^\alpha \varphi)$, $\varphi \in \mathcal{D}(\Omega)$.
It is clear that $\partial^\alpha u \in \mathcal{D}'(\Omega)$ according to the previous proposition.
Usually, we denote $C^\infty$by $\mathcal{E}(\Omega)$; we shall recall that $\mathcal{E}(\Omega)$ is Fréchet space if we provide it with a topology which is defined by a family of seminorms: $\|\varphi\|_{m,K} = \sup\limits_{|\alpha| \leq m, x \in K} |\partial^\alpha \varphi(x)|$ where m takes $\mathbb{N}$ and K takes a countable family of increasing compacts whose thier union aquals Ω.
We can prove that $\mathcal{E}'(\Omega)$ the topological dual of $\mathcal{E}(\Omega)$, is subspace from distributions of $\mathcal{E}'(\mathbb{R}^n)$ which are with compact support on Ω.;for more details on distributions theory see[9] .

2.3.1 Convolution Product

Let $C_c^k(\mathbb{R}^n)$ the space of functions of $C^k(\mathbb{R}^n)$ with compact support, for $k \geq 0$ an integer and $g \in L^1_{loc}(\mathbb{R}^n)$.

Definition 2.3.2 *(convolution product)*

Let $f \in C_c^0(\mathbb{R}^n)$, the **convolution product** of f and g is defined by:

$$(f * g)(x) = \int\limits_{\mathbb{R}^n} f(x-y)\, g(y)\, dy. \tag{2.6}$$

Theorem 2.3.2 [9]

If $f \in C_c^k(\mathbb{R}^n)$ and $g \in L^1_{loc}(\mathbb{R}^n)$ then we have $f * g \in C^k(\mathbb{R}^n)$ and:$\partial^\alpha (f * g) = \partial^\alpha f * g$,$\alpha \in \mathbb{N}^n$, $|\alpha| \leq k$.
And if $g \in C^l(\mathbb{R}^n)$ then $f * g \in C^{k+l}(\mathbb{R}^n)$ and:

$$\partial^{\alpha+\beta}(f * g) = \partial^\alpha f * \partial^\beta g,\ \ \alpha, \beta \in \mathbb{N}^n,\ \ |\alpha| \leq k \text{ and } |\beta| \leq l.$$

Theorem 2.3.3 [9]

Let $u \in \mathcal{D}'(\mathbb{R}^n)$ and $v \in \mathcal{E}'(\mathbb{R}^n)$. Then there exists a unique element from $\mathcal{D}'(\mathbb{R}^n)$,noted by $u * v$ such that:

$$(u * v) * \varphi = u * (v * \varphi), \; \varphi \in \mathcal{D}(\mathbb{R}^n).$$

2.3.2 Fourier Transformation

Definition 2.3.3 *(Fourier transformation)*

Let $f \in L^1(\mathbb{R}^n)$, the **Fourier transformation** of f is given by:

$$\mathcal{F}f(\xi) = \int_{\mathbb{R}^n} e^{-ix\xi} f(x)\, dx, \; \xi \in \mathbb{R}^n. \tag{2.7}$$

Definition 2.3.4 *(Schwartz space)*

A function f is part of the space $\mathcal{S}(\mathbb{R}^n)$ when it is **indefinitely differentiable**, and if f and all its derivatives are **rapidly decreasing**, that is to say that

$$\mathcal{S}(\mathbb{R}^n) = \left\{ f \in C^\infty(\mathbb{R}^n) \mid \lim_{|x| \to +\infty} x^\alpha \partial^\beta f(x) = 0, \; \alpha, \beta \in \mathbb{N}^n \right\}.$$

Up next, we will use the derivation operator: $D_j = -i\partial_j$.

Thus, we intuitively visualize why $\mathcal{S}(\mathbb{R}^n)$ is invariant by Fourier transformation. Indeed, we have the following theorem:

Theorem 2.3.4 [9]

The operator $f \to \mathcal{F}f$ is an isomorphism from $\mathcal{S}(\mathbb{R}^n)$ onto $\mathcal{S}(\mathbb{R}^n)$ which verifies:

$$\mathcal{F}(D_j f) = \xi_j \mathcal{F}f \;\; \text{and} \;\; \mathcal{F}(x_j f) = -D_j \mathcal{F}f$$

and we have inversion formula: $\mathcal{F}^{-1}f = (2\pi)^n \tilde{f}$,where $\tilde{f}(\cdot) = f(-\cdot)$.

Proposition 2.3.2 [9]

Let $f, g \in \mathcal{S}(\mathbb{R}^n)$. Then :

$$\int_{\mathbb{R}^n} \mathcal{F}fg = \int_{\mathbb{R}^n} f\mathcal{F}g,$$

$$\int_{\mathbb{R}^n} f\overline{g} = (2\pi)^{-n} \int_{\mathbb{R}^n} \mathcal{F}f\overline{\mathcal{F}g},$$

$$\mathcal{F}(f * g) = \mathcal{F}f\mathcal{F}g,$$

$$\mathcal{F}(fg) = (2\pi)^{-n} \mathcal{F}f * \mathcal{F}g.$$

We recall that $\mathcal{S}'(\mathbb{R}^n)$ is the set of continuous linear formes on $\mathcal{S}(\mathbb{R}^n)$.

If $u \in \mathcal{S}'(\mathbb{R}^n)$, we define $\mathcal{F}u$ the unique element from $\mathcal{S}'(\mathbb{R}^n)$ such that:$\mathcal{F}u(f) = u(\mathcal{F}f)$, $f \in \mathcal{S}(\mathbb{R}^n)$.

We define $\tilde{u} \in \mathcal{S}'(\mathbb{R}^n)$ the unique element from $\mathcal{S}'(\mathbb{R}^n)$ such that: $\tilde{u}(f) = u\left(\tilde{f}\right), f \in \mathcal{S}(\mathbb{R}^n)$.

Theorem 2.3.5 [9]

$f \in \mathcal{S}(\mathbb{R}^n) \rightarrow \mathcal{F}f \in \mathcal{S}(\mathbb{R}^n)$ extends into an isomorphism over $L^2(\mathbb{R}^n)$. Plus, we have the Parceval formula: $\int_{\mathbb{R}^n} f\overline{g} = (2\pi)^{-n} \int_{\mathbb{R}^n} \mathcal{F}f\overline{\mathcal{F}g}$, $f, g \in L^2(\mathbb{R}^n)$.

2.4 Sobolev Spaces

2.4.1 $H^s(\mathbb{R}^n)$Spaces

Definition 2.4.1 *(Sobolev space)*

Let $s \in \mathbb{R}$. By $H^s(\mathbb{R}^n)$ we denote the space of all functions $u \in \mathcal{S}'(\mathbb{R}^n)$ with the property:

$$\left(1 + |\xi|^2\right)^{\frac{s}{2}} \mathcal{F}u \in L^2(\mathbb{R}^n),$$

for the Fourier transformation $\mathcal{F}u$ of u,with $|\xi|^2 = \xi_1^2 + \xi_2^2 + ... + \xi_n^2$. $H^s(\mathbb{R}^n)$ is called a **Sobolev space**. Frequently we will abbreviate $H^s = H^s(\mathbb{R}^n)$.

Theorem 2.4.1 [10]

The Sobolev space H^s is a Hilbert space with the scalar product defined by

$$\langle u, v\rangle_s = \int\limits_{\mathbb{R}^n} \left(1+|\xi|^2\right)^s \mathcal{F}u\left(\xi\right) \overline{\mathcal{F}v\left(\xi\right)} d\xi,$$

for $u, v \in H^s$ with Fourier tranformations $\mathcal{F}u$ and $\mathcal{F}v$,respectively. Note that the norm on H^s is given by:

$$\|u\|_s = \left(\int\limits_{\mathbb{R}^n} \left(1+|\xi|^2\right)^s \left|\mathcal{F}u\left(\xi\right)\right|^2 d\xi\right)^{\frac{1}{2}}.$$

Proof. It is clear that the space H^s is linear space and that $\langle \cdot, \cdot\rangle_s$ is a scalar product. That $\langle \cdot, \cdot\rangle_s$ is well defined can be concluded from the Cauchy-Schwartz inequality

$$\left|\int\limits_{\mathbb{R}^n} \left(1+|\xi|^2\right)^s \mathcal{F}u\left(\xi\right) \overline{\mathcal{F}v\left(\xi\right)} d\xi\right|^2 \leq \int\limits_{\mathbb{R}^n} \left(1+|\xi|^2\right)^s \left|\mathcal{F}u\left(\xi\right)\right|^2 d\xi \int\limits_{\mathbb{R}^n} \left(1+|\xi|^2\right)^s \left|\mathcal{F}v\left(\xi\right)\right|^2 d\xi.$$

To prove that H^s is complete, let $\left(u_j\right)_j$ be a Cauchy sequence from H^s, i.e., given $\varepsilon > 0$, there exists $\eta\left(\varepsilon\right) \in \mathbb{N}$ such that $\|u_m - u_k\|_s \leq \varepsilon$ for all $m, k \geq \eta\left(\varepsilon\right)$.

$$\|u_m - u_k\|_s \leq \varepsilon \Leftrightarrow \int\limits_{\mathbb{R}^n} \left(1+|\xi|^2\right)^s \left|\mathcal{F}u_m\left(\xi\right) - \mathcal{F}u_k\left(\xi\right)\right|^2 d\xi \leq \varepsilon^2$$

$$\Rightarrow \left(\left(1+|\xi|^2\right)^{\frac{s}{2}} \mathcal{F}u_j\left(\xi\right)\right)_j \text{ is a Cauchy sequence in } L^2$$

and because L^2 is complete then there exists $\mathcal{F}u_0 \in L^2$ such that

$$\left(1+|\xi|^2\right)^{\frac{s}{2}} \mathcal{F}u_j\left(\xi\right) \rightarrow \mathcal{F}u_0 in L^2,$$

that is to say that: $\forall\varepsilon > 0, \exists\eta\left(\varepsilon\right) \in \mathbb{N}, \forall m \geq \eta\left(\varepsilon\right), \int\limits_{\mathbb{R}^n} \left(1+|\xi|^2\right)^s \left|\mathcal{F}u_m\left(\xi\right) - \mathcal{F}u_0\left(\xi\right)\right|^2 d\xi \leq \varepsilon^2 . \left(1+|\xi|^2\right)^{-s} \mathcal{F}u_0\left(\xi\right) \in \mathcal{S}'$, let $v \in \mathcal{S}'$ such that $\mathcal{F}v\left(\xi\right) = \left(1+|\xi|^2\right)^{-s} \mathcal{F}u_0\left(\xi\right)$, then: $\forall\varepsilon > 0, \exists\eta\left(\varepsilon\right) \in \mathbb{N}, \forall j \geq \eta\left(\varepsilon\right), \int\limits_{\mathbb{R}^n} \left(1+|\xi|^2\right)^s \left|\mathcal{F}u_j\left(\xi\right) - \mathcal{F}v\left(\xi\right)\right|^2 d\xi \leq \varepsilon^2$ (u_j converges to v in H^s). Hence, H^s is complete. ∎

Theorem 2.4.2 [4]

If $s \geq t$,$H^s(\mathbb{R}^n)$ injects continuously into $H^t(\mathbb{R}^n)$.

For all $s, \mathcal{D}(\mathbb{R}^n)$ is dense in $H^s(\mathbb{R}^n)$.

$H^0(\mathbb{R}^n) = L^2(\mathbb{R}^n)$ which is identified by his dual, for all $s > 0$, we have $(H^s(\mathbb{R}^n))'$ algebrically and topologically coinsides with $H^{-s}(\mathbb{R}^n)$.

2.4.2 $W^{m,p}$ Spaces

In this section we will give some basic results on the generalized Lebesgue-Sobolev space $W^{m,p}(\Omega)$, where Ω is a bounded open of $\mathbb{R}^n$with boundary Γ.

$W^{m,p}(\Omega)$ is defined as $W^{m,p}(\Omega) = \{f \in L^p(\Omega)\,; \partial^\alpha f \in L^p(\Omega)\,, \alpha \in \mathbb{N}^n, |\alpha| \leq m\}$.

Remark 2.4.1

If $m = 0$, $W^{0,p}(\Omega) = L^p(\Omega)$.

If $p = 2$, $W^{m,2}(\Omega) = H^m(\Omega)$which is an Hilbert space.

$W^{m,p}(\Omega)$ can be equipped with the norm $\|f\|_{W^{m,p}(\Omega)}$as Banach space, where

$$\|f\|_{W^{m,p}(\Omega)} = \sum_{|\alpha| \leq m} \|\partial^\alpha f\|_{L^p(\Omega)}.$$

Theorem 2.4.3 [4]

Let $\Omega \subset \mathbb{R}^n$ a boundary open, if $W_0^{1,p}(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in $W^{1,p}(\Omega)$ then:

i/ $W_0^{1,p}(\Omega)$ can be embedded into $L^{\frac{np}{n-p}}(\Omega)$continuously for $p < n$, and into $C^0\left(\bar{\Omega}\right)$ for $p > n$.

ii/ There exists a constant $c = c(n,p)$ such that $\forall u \in W_0^{1,p}(\Omega)$:

$$\|u\|_{L^{\frac{np}{n-p}}(\Omega)} \leq c\,\|\nabla u\|_{L^p(\Omega)^n} \quad \text{,if } p < n,$$

$$\sup_{\Omega} |u| \leq c\,|\Omega|^{\frac{1}{n}-\frac{1}{p}}\,\|\nabla u\|_{L^p(\Omega)^n} \quad \text{,if } p > n.$$

Theorem 2.4.4 [4]

Let $\Omega \subset \mathbb{R}^n$ a boundary of class $C^{0,1}$.

i/ Assume that $mp < n$, $q < \frac{np}{n-mp}$ then there is a continuous and compact imbedding $W^{m,p}(\Omega) \to L^q(\Omega)$, and if $p^* = \frac{np}{n-mp}$ then we have the continuous embedding $W^{m,p}(\Omega) \to L^{p^*}(\Omega)$.

ii/ Assume k is an integer, if $0 \leq k < m - \frac{n}{p} < q + 1$ then $W^{m,p}(\Omega)$ can be embedded continuously into $C^{k,\alpha}\left(\bar{\Omega}\right)$, with $\alpha = m - \frac{n}{p} - k$, and the embedding $W^{m,p}(\Omega) \to C^{k,\beta}\left(\bar{\Omega}\right)$ is compact for all $\beta < \alpha$.

Theorem 2.4.5 *(trace embedding and extention theorem)*

Assuming that $\Omega \subset \mathbb{R}^n$ is a bounded domain of class $C^k, k \geq 1$. Let the application

$$\begin{aligned} \gamma_j u \quad : \quad & \mathcal{D}\left(\bar{\Omega}\right) \to \left(\mathcal{D}(\Gamma)\right)^k \\ & u \to \left(u, \partial_v u, ..., \partial_v^{k-1} u\right) \end{aligned}$$

(where $\partial_v^j u$,$j = 0, ..., k-1$ are „normal“ derivatives of u). Then the map $\gamma_j u$ can be extended (uniquely) to linear continuous map noted by

$$\begin{aligned} \gamma_j \quad : \quad & H^k(\Omega) \to \prod_{j=0}^{k-1} H^{k-j-\frac{1}{2}}(\Gamma) \\ & u \to \left(u, \partial_v u, ..., \partial_v^{k-1} u\right) \end{aligned}$$

and there exists a linear continuous operator

$$P : \prod_{j=0}^{k-1} H^{k-j-\frac{1}{2}}(\Gamma) \to H^k(\Omega)$$

such that, if $\varphi = (\varphi_0, ..., \varphi_{k-1})$, $\varphi_j \in \prod_{j=0}^{k-1} H^{k-j-\frac{1}{2}}(\Gamma)$, and $u = P\varphi$, then $\gamma_j u = \varphi_j$, $j = 0, ..., k-1$.

Theorem 2.4.6 [4]

Let Ω be an open set. If $u \in H^1(\Omega)$ then

$$u^+ = \sup(u, 0), \; u^- = \sup(-u, 0), \; |u| = u^+ + u^- \in H^1(\Omega),$$

and $\nabla u^+ = \chi_{[u>0]} \nabla u \; , \nabla u^- = \chi_{[u<0]} \nabla u$.

2.4.3 $H^k(a, b; X)$ Spaces

Let X Banach space and $-\infty \leq a < b \leq +\infty$. We call vector distribution on (a, b), any continuous linear map on $\mathcal{D}(a, b)$ in X ,that is to say: $\mathcal{D}'(a, b; X) = L(\mathcal{D}(a, b), X)$.

Assume $k \geq 0$ an integer, $u \in \mathcal{D}'(a, b; X)$. The map $\varphi \to (-1)^k u\left(\varphi^{(k)}\right)$ such that $\varphi \in \mathcal{D}(a, b)$ defines a distribution which we note as $u^{(k)}$.

For $k \geq 1$ an integer, the space $H^k(a, b; X)$ is defined as:

$$H^k(a, b; X) = \left\{u \in L^2(a, b; X); \; u^j \in L^2(a, b; X), \; j = 1, ..., k\right\}.$$

$H^k(a, b; X)$ is a Hilbert space with the norm $\|u\|_{H^k(a,b;X)} = \left(\sum_{j=0}^k \left\|u^{(j)}\right\|^2_{L^2(a,b;X)}\right)^{\frac{1}{2}}$.

2.4.4 Some formulas of integration by parts:

Let Ω bounded open set of class C^1 and Γ its boundary. The first classic formula of intrgration by parts is:$\int\limits_\Omega \partial_i uv = -\int\limits_\Omega u\partial_i v + \int\limits_\Gamma uv\nu_i \; , \; u, v \in H^1(\Omega)$. From this formula, we can easily deduce the following:

$$\int\limits_\Omega \Delta uv = -\int\limits_\Omega \nabla u.\nabla v + \int\limits_\Gamma \partial_\nu uv \; , \; u \in H^2(\Omega) \text{ and } v \in H^1(\Omega), \tag{2.8}$$

$$\int\limits_\Omega (\Delta uv - u\Delta v) = \int\limits_\Gamma (\partial_\nu uv - u\partial_\nu v) \; , \; u, v \in H^2(\Omega). \tag{2.9}$$

And if $Q = \Omega \times (0,T)$, $\Phi = \Gamma \times (0,T)$ then

$$\int\limits_{Q} (\Delta - \partial_t) uv - \int\limits_{Q} u (\Delta + \partial_t) v = \int\limits_{\Phi} (\partial_\nu uv \ - u\partial_\nu v \ ) - \int\limits_{\Omega} [u (.,T) v (.,T) - u (.;0) v (.,T)] , \tag{2.10}$$

$$\text{for } u, v \in L^2 \left(0, T; H^2 (\Omega)\right) \cap H^1 \left(0, T; L^2 (\Omega)\right) .$$

2.4.5 H_Δ Spaces

Let Ω bounded open set and Γ its boundary. We define the H_Δ space as:

$$H_\Delta (\Omega) = \left\{u \in H^1 (\Omega) ;\ \Delta u \in L^2 (\Omega)\right\} ,$$

provided with the norm $\|u\|_{H_\Delta(\Omega)} = \left(\|u\|_{H^1(\Omega)} + \|\Delta u\|_{L^2(\Omega)}\right)^{\frac{1}{2}}$.

$H_\Delta (\Omega)$ is an Hilbert space. The intrest of this space lies in the following theorem:

Theorem 2.4.7 [4]

Assuming that Ω is of class C^1 .

i/ The map

$$\partial_\nu :\ C^1 \left(\bar{\Omega}\right) \to C (\Gamma) : u \to\ \partial_\nu u_{|\Gamma}$$

extends into continuous map, defined on $H_\Delta (\Omega)$ into $H^{-\frac{1}{2}} (\Gamma)$, noted ∂_ν as well.

ii/ For all $u \in H_\Delta (\Omega)$ and $v \in H^1 (\Omega)$, we have the formula:

$$\int\limits_{\Omega} \Delta uv = -\int\limits_{\Omega} \nabla u.\nabla v + \langle \partial_\nu u, v\rangle_{H^{-\frac{1}{2}}(\Gamma), H^{\frac{1}{2}}(\Gamma)} .$$

2.4.6 Poincaré Inequalities

We consider a fixed, open and bounded subset Ω of $\mathbb{R}^n$. Let us recall some notations which are needed below.

Let ξ be a vector in $\mathbb{R}^n$ such that $|\xi| = 1$, $a, b \in \mathbb{R}$ and $d = b - a$. The set defined by $\pi_d(\xi) = \{x \in \mathbb{R}^n; \;\; a < x.\xi < b\}$ is called the strip of thickness d in thedirection ξ.

Proposition 2.4.1 *(Poincaré inequality)*[4]

Assume that Ω open subset of $\mathbb{R}^n$ such that there exits a strip $\pi_d(\xi)$ with $\Omega \in \pi_d(\xi)$. Then

$$\|u\|^2_{L^2(\Omega)} \leq \frac{d^2}{2} \||\nabla u|\|^2_{L^2(\Omega)}, \quad \forall u \in H^1_0(\Omega). \tag{2.11}$$

In the case of Ω is a bounded domain, we have the following Poincaré inequality:

Proposition 2.4.2 [4]

Let Ω open bounded set of $\mathbb{R}^n$ and $\lambda_1(\Omega)$ the first eigenvalue of Laplacian-Dirichlet. Then

$$\|u\|^2_{L^2(\Omega)} \leq \frac{1}{\lambda_1(\Omega)} \||\nabla u|\|^2_{L^2(\Omega)}, \quad \forall u \in H^1_0(\Omega). \tag{2.12}$$

2.5 Generalities on Partial Derivative Equations (PDEs)

A partial derivative equation (PDE) is an equation for some quantity u (dependent variable) which depends on the independent variables$x_1, ..., x_n, n \geq 2$, and involves derivatives of u with respect to at least some of the independent variables.

$$F\left(x_1, ..., x_n, u, \partial_{x_1} u, ..., \partial_{x_n} u, \partial^2_{x_1} u, \partial^2_{x_1 x_2} u, ..., \partial^n_{x_1 ... x_n} u\right) = 0. \tag{E}$$

Note:

1. In applications x_i are often space variables (e.g. x, y, z) and a solution may be required in some region $\Omega \subset \mathbb{R}^d$ of space. In this case there will be some conditions to be satisfied on the boundary $\partial\Omega$ such that $\partial\Omega = \bar{\Omega}/\Omega$; these are called **boundary conditions (BCs).**

2. Also in applications, one of the independent variables can be time(t say), then there will be some **initial conditions (ICs)** to be satisfied (i.e.,u is given at $t = 0$ everywhere inΩ).

3. The order $n \in \mathbb{N}$ of the PDE is the order of the highest (partial) direvative coefficient in the equation.

4. A linear equation is one in which F is a linear function of u and its derivatives, and it is called quasilinear of order n if f is linear on all partial derivatives of highest order.

5. If u satisfies the equation (E) then it is a solution of the PDE in $\Omega \subset \mathbb{R}^d$.

Here are some propreties that we will need:

- **Principle of superposition:** A linear equation has the useful property that if u_1 and u_2 both satisfy the equation(E) then so does $\alpha u_1 + \beta u_2$ for any $\alpha, \beta \in \mathbb{R}$. This is often used in constructing solutions to linear equations (for example, so as to satisfy boundary or initial conditions; c.f. Fourier series methods). This is not true for nonlinear equations.

- If u_h is solution of the linear homogeneous equation and u_p is solution of linear non-homogeneous equation, then $u_h + u_p$ is the solution of the complet equation.

- The general solution of a PDE is the one which makes it possible to find all the solutions of the equation by giving particular values to the arbitrary functions.

- To find particular solutions of PDE, starting from the general solution, we will impose restrictive conditions on the set of solutions.

Moving on now to see the most frequent constraints:

1/ **Initial conditions (ICs):**If u is a function of $(x,t) \in \mathbb{R}^d \times \mathbb{R}$ giving: $u(x, t_0) = \phi_0(x)$ where $\frac{\partial^p u(x,t_0)}{\partial x^p} = \phi_p(x)$.

2/ **Boundary conditions (BCs):**There are three types of boundary conditions for well-posed boundary value problems (BVPs), if u is a function of $x \in \Omega \subset \mathbb{R}^d$:

- Dirichlet condition: u takes prescribed values on the boundary $\partial\Omega$, $(u/_{\partial\Omega} = g)$;
- Neumann conditions: the normal derivative is prescribed on the boundary $\partial\Omega$,$(\frac{du}{dn}/_{\partial\Omega} = g)$;
- Robin conditions: a combination of u and its normal derivative such as $c(x)\,u + \bar{c}(x)\,\frac{du}{dn}$ is prescribed on the boundary $\partial\Omega$,$(c(x)\,u + \bar{c}(x)\,\frac{du}{dn} = g$ on $\partial\Omega)$.

If $g = 0$ we have boundary homogeneous conditions.

3/ Conditions at infinity: Ω is unbounded, we must impose conditions with the forme $u(x) \sim \phi(x)$ when $|x| \to \infty$, such that $\|u\|_2 < \infty$.

4/ Conditions on interfaces: if $\Omega = \Omega_1 \cup \Omega_2$ with $\bar{\Omega}_1 \cap \bar{\Omega}_2 = \partial\Omega_1 \cap \partial\Omega_2$, if we have determined u on Ω_1 and Ω_2 then detrmining u on Ω needs conditions for u, (resp $\frac{du}{dn}$on $\partial\Omega_1 \cap \partial\Omega_2$).

Second order PDEs:

The general forme of a linear, scalar of second order PDE is:

$$au + c.\nabla u + \operatorname{div}(A\nabla u) = f, \tag{2.13}$$

where $a : \Omega \to \mathbb{R}$, $c : \Omega \to \mathbb{R}^d$, $A : \Omega \to \mathbb{R}^{d\times d}$ and $f : \Omega \to \mathbb{R}$ are the coefficients of the PDE. In case $d = 1$, u is a scalar and the coefficients are constants, PDE becomes:

$$\alpha\frac{\partial^2 u}{\partial x^2} + \beta\frac{\partial^2 u}{\partial x\partial y} + \gamma\frac{\partial^2 u}{\partial y^2} + \delta\frac{\partial u}{\partial x} + \epsilon\frac{\partial u}{\partial y} + \xi u = f, \tag{2.14}$$

where $\alpha, \beta, \gamma, \delta, \epsilon$ and ξ are scalars.

We shall summarize the type of equation (2.14) in the following table:

Classification	Type of (2.14)	Example
$\beta^2 - 4\alpha\gamma < 0$	Elliptic	Laplace equation on Ω: $-\Delta u = f$
$\beta^2 - 4\alpha\gamma = 0$	Parabolic	Heat conduction(diffusion equation)on $Q = \mathbb{R}_+ \times \Omega$: $\partial_t u - \Delta u = f$
$\beta^2 - 4\alpha\gamma > 0$	Hyperbolic	Wave equation on $Q = \mathbb{R}_+ \times \Omega$: $\partial_{tt} u - \Delta u = f$

Chapter 3

On the Regularization of Ill-Posed PDEs Problems

For a long time it was assumed that mathematical problems which do not satisfy the correctness conditions due to Hadamard generally make no physical sense. This point of view changed and it can be stated (Tikhonov with Arsenin in1976): all problems which are related to real phenomena have stable solutions if proper **regularization methods** are applied.

In this chapter we will concentrate on this technique, however, there are other regularization methods:

1. Discrete methods: Least squares method; Singular Value Decomposition method (SVD).
2. Iterative methods: Landweber method ,see[11] ; Conjugate gradient (CG) type methods, see [11] ;Truncated Singular Value Decomposition (TSVD).

Iterative methods are used for large values problems. These methods construct a sequence of iterates approximating-for exact data-the solution; regularization is introduced by stopping the iteration based on a suitable discrepancy principle.

3.1 Least Squares Method (LS)

Consider the following system of linear equations:

$$Ax = b \tag{3.1}$$

where the matrix $A \in \mathcal{M}_{m\times n}(\mathbb{R})$ and the vector $b \in \mathbb{R}^m$, our aim is to find solution $x \in \mathbb{R}^n$ for the system (3.1) .

As we mentioned in example (1.3) in the first chapter, for $m > n$ there may be no solutions, even if there is a solution it will not be unique. But in practical cases we privilegier a solution and we choose x in order to approuch Ax to b.

The LS method leads to minimizing the residual$\|Ax - b\|_2$ where $\|\cdot\|_2$ is the Euclidean norm of $\mathbb{R}^n$, which is the sum of the squares (hence the name least squares).

Let $A \in \mathcal{M}_{m\times n}(\mathbb{R})$ and $b \in \mathbb{R}^m$ given. The LS problem is given by:

$$\min_{x\in\mathbb{R}^n} \|Ax - b\|_2^2 \tag{3.2}$$

We note by $\tilde{x}$ the solution of problem (3.2) .

Assume that:

$$E(x) = \|Ax - b\|_2^2$$
$$E(x) = (Ax - b)^t (Ax - b) = x^t A^t A x - b^t A x - x^t A^t b + \|b\|_2^2 ,$$

differentiating $E(x)$ yields the necessary condition $A^t A x = A^t b$, necessary the solution of LS problem $\tilde{x}$ verifies this condition, that is to say:

$$A^t A \tilde{x} = A^t b \tag{3.3}$$

(3.3) is called normal equations.

Proof. We can reformulate the LS problem (3.2) by:

$$\min_{x\in\mathbb{R}^n} J(x) = x^t G x - 2h^t x,$$

where $G = A^t A$ is symetric and $h = A^t b$ given vector.

Recalling that the quadratic function is a function defined by

$$J : \mathbb{R}^n \to \mathbb{R}$$
$$J(x) = x^t G x - 2h^t x, \tag{3.4}$$

where $G \in \mathcal{M}_{n\times n}(\mathbb{R})$ symetric and $h \in \mathbb{R}^n$a given vector.

Now,we know that if $J : \mathbb{R} \to \mathbb{R}$ is continuously derivable, has a minimum $\tilde{x} \in \mathbb{R}$ then $J'(\tilde{x}) = 0$. Same thing for $J : \mathbb{R}^n \to \mathbb{R}$ continuously derivable, then:

$$J(\tilde{x}) \le J(x),\ \forall x \in \mathbb{R}^n \Longrightarrow \nabla J(\tilde{x}) = 0,\ \text{where } \nabla \text{ is gradient operator.}$$

Calculating the gradient of function J represented in (3.4) :$\nabla J(x) = \left(\frac{\partial J}{\partial x_1}, \frac{\partial J}{\partial x_2}, ..., \frac{\partial J}{\partial x_n}\right)^t$. Developping J :

$$\begin{aligned} J(x) &= \sum\nolimits_{i=1}^n x_i (Gx)_i - 2\sum\nolimits_{i=1}^n h_i x_i, \\ \Longrightarrow \frac{\partial J}{\partial x_k} &= (Gx)_k + \sum\nolimits_{i=1}^n x_i \frac{\partial}{\partial x_k}(Gx)_i - 2h_k, \\ \text{and } \frac{\partial J}{\partial x_k}(Gx)_i &= \frac{\partial}{\partial x_k}\left(\sum\nolimits_{j=1}^n g_{ij}x_j\right) = g_{ik} = g_{ki}. \end{aligned}$$

Then $\frac{\partial J}{\partial x_k} = (Gx)_k + \sum_{i=1}^n x_i g_{ki} - 2h_k = (Gx)_k + (Gx)_k - 2h_k = 2(Gx)_k - 2h_k,$
It results: $\nabla J(x) = 2(Gx - h)$, and if $\tilde{x}$ is a solution of LS problem then $G\tilde{x} = h$, it follows: $A^t A\tilde{x} = A^t b$. ■

Back to our problem(3.2) , If the columns of A are linearly independent, then $A^t A$ is positive definite, i.e.E is strictly convex and the LS problem has unique solution $\tilde{x}$ given by: $A^t A\tilde{x} = A^t b$.

Indeed, if A has full rank n then $G = A^tA$ is positive definite,i.e.$x^tGx = \|Ax\|_2^2 \geq 0, \forall x \in \mathbb{R}^n$, in other side, if G is positive definite then it is inversible, it follows that thers exists an unique solution $\tilde{x}$ which verifies: $A^tA\tilde{x} = A^tb$.

Geometrically, $\tilde{x}$ is a solution of (3.2) if and only if the residual $b - Ax$ at $\tilde{x}$ is orthogonal to the range of A ($r = b - A\tilde{x} \perp \mathcal{R}(A)$), this is illustrated in Figure3.1 .

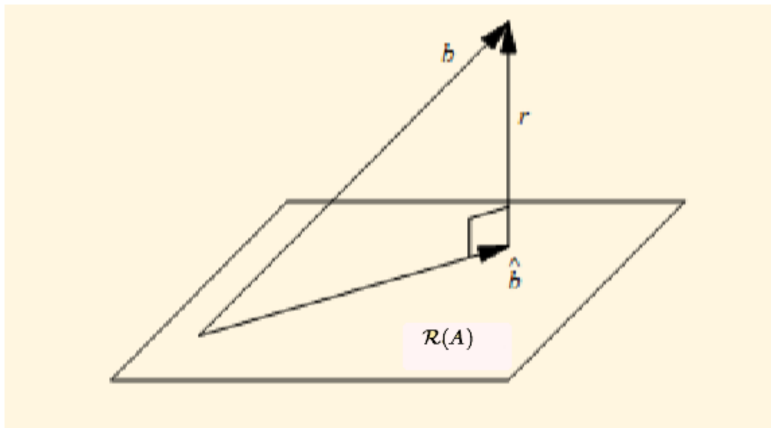

Figure 3.1: Projection of r onto $\mathcal{R}(A)$

If the columns of A are linearly independent, the solution $\tilde{x}$ can be obtained solving the normal equation(3.3) by the Cholesky factorization of $A^tA > 0$. However, A^tA may be badly conditioned, and then the solution obtained this way can be useless. In finite arithmetic the QR-decomposition of A is a more stable approach, for more details see[12] . A powerful tool for the analysis of the least squares problem is the singular value decomposition (SVD) of A, see[13] .

Theorem 3.1.1 *(Perturbation Theorem)*

Let $A \in \mathcal{M}_{m\times n}(\mathbb{R})$, $m \geq n$ have full rank n, let x be the unique solution of the LSP(3.2) , and let $\tilde{x}$ be the solution of a perturbed LS problem:

$$\min_{x\in\mathbb{R}^n} \|(A + \delta A)\, x - (b + \delta b)\|_2 \tag{3.5}$$

where the perturbation is not too large in the sense $\epsilon : = \max\left(\frac{\|\delta A\|}{\|A\|}, \frac{\|\delta b\|}{\|b\|}\right) < \frac{1}{k_2(A)}$, where

$k_2(A) : = \sigma_1/\sigma_2$ denotes the condition number of A. Then it holds that

$$\frac{\|x-\tilde{x}\|}{\|x\|} \leq \epsilon \left(\frac{2k_2(A)}{\cos(\theta)} + \tan(\theta) \cdot k_2^2(A) \right) + \mathcal{O}\left(\epsilon^2\right) \tag{3.6}$$

where θ is the angle between b and its projection onto$\mathcal{R}(A)$.

For a proof see the book of J. Demmel, Applied Linear Algebra.

Proposition 3.1.1 (3)

Let $x \in \mathbb{R}^n$ a solution of a LS problem. $\forall y \in \mathbb{R}^n$ we have: $\|Ax-b\|_2^2 \leq \|Ay-b\|_2^2$.

Example 3.1.1 *(Linear Regression)*

Let the line $(D) : y(t) = \alpha + \beta t$, the problem is finding a line (or curve) that best fits a set of data-in the standard formulation-a set of observations $(t_i, y_i)_{i=1,\dots,m}$.

We write forme matricielle: $\begin{pmatrix} 1 & t_1 \\ 1 & t_2 \\ \vdots & \vdots \\ 1 & t_m \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}$,the normal equation is:

$$\begin{pmatrix} 1 & 1 & \cdots & 1 \\ t_1 & t_2 & \cdots & t_m \end{pmatrix} \begin{pmatrix} 1 & t_1 \\ 1 & t_2 \\ \vdots & \vdots \\ 1 & t_m \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ t_1 & t_2 & \cdots & t_m \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} m & \sum_{i=1}^m t_i \\ \sum_{i=1}^m t_i & \sum_{i=1}^m t_i^2 \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \sum_{i=1}^m y_i \\ \sum_{i=1}^m y_i t_i \end{pmatrix}$$

Solving these equations gives the LS estimates of α and β as: $\alpha = \bar{y} - \beta\bar{t}$, $\beta = \frac{\sum_{i=1}^m y_i t_i - m\bar{y}\bar{t}}{\sum_{i=1}^m t_i^2 - m\bar{t}^2}$, where $\bar{y} = \frac{\sum_{i=1}^m y_i}{m}$ and $\bar{t} = \frac{\sum_{i=1}^m t_i}{m}$.

3.2 Tikhonov Method

Tikhonov regularization is the most used method for solving inverse problems that are ill-posed.

We assume throughout this section that K is a compact operator, let the inverse problem $Kx = y$, $K : X \to Y$ where X and Y are Hilbert spaces. We make the assumption that there exists a solution $x \in X$ of the unperturbed equation $Kx = y$.

In other words, we assume that $y \in \mathcal{R}(K)$. The injectivity of K implies that this solution is unique.

In practice, the right-hand side $y \in Y$ is never known exactly but only up to an error of,say, $\delta > 0$. Therefore, we assume that we know $\delta > 0$ and $y^\delta \in Y$ with $\left\|y - y^\delta\right\|_Y \leq \delta$.

It is our aim to solve the perturbed equation

$$Kx^\delta = y^\delta. \tag{3.7}$$

in general, (3.7) is not solvable because we cannot assume that the measured data y^δ are in $\mathcal{R}(K)$. Therefore, the best we can hope is to detrmine an approximation $x^\delta \in X$ to the exact solution x and x^δ should depend continuously on the data y^δ.

In other words, it is our aim to construct a suitable bounded approximation $R : Y \to X$ of the (unbounded) inverse operator $K^{-1} : \mathcal{R}(K) \to X$.

Definition 3.2.1 *(regularization strategy)*

A regularization strategy is a family of linear and bounded operators

$$R_\alpha : Y \to X \ , \alpha > 0$$

such that

$$\lim_{\alpha \to 0} R_\alpha K x = x \ \text{ for all } x \in X,$$

that is, that operator $R_\alpha K$ converge pointwise to the identity.

From this definition and the compactness of K, we conclude the following:

Theorem 3.2.1 [11]

Let R_α be a regularization strategy for a compact operator $K : X \to Y$ where $\dim X = \infty$. Then we have:

(1) The operators R_α are not uniformly bounded; that is, there exists a sequence $(\alpha_j) \subset \mathbb{R}^+$ with $\lim\limits_{j\to\infty} \left\|R_{\alpha_j}\right\|_{L(Y,X)} = \infty$.

(2) The sequence $(R_\alpha K x)$ does not converge uniformly on bounded subsets of X; that is, there is on convergence $R_\alpha K$ to the identity I in the operator norm.

Now, we define $x^{\alpha,\delta} : = R_\alpha y^\delta$ as an approximation of the solution x of $Kx = y$. The error splits into two parts, by the triangle inequality:

$$\left\|x^{\alpha,\delta} - x\right\|_X \leq \left\|R_\alpha y^\delta - R_\alpha y\right\| + \left\|R_\alpha y - x\right\| \leq \left\|R_\alpha\right\| \left\|y^\delta - y\right\| + \left\|R_\alpha K x - x\right\|$$

and thus $\left\|x^{\alpha,\delta} - x\right\|_X \leq \delta \left\|R_\alpha\right\|_{L(Y,X)} + \left\|R_\alpha K x - x\right\|_X$. The first term on the right-hand side describes the error in the data multiplied by the"condition number"$\|R_\alpha\|$ of the regularized problem, which tends to ∞ as α tends to 0, by Theorem3.2.1 The second term denotes the approximation error $\|(R_\alpha - K^{-1})y\|$ at the exact right-hand side $y = Kx$, which is by the definition of regularization strategy tends to 0 with α. Figure3.2 illusrates the situation:

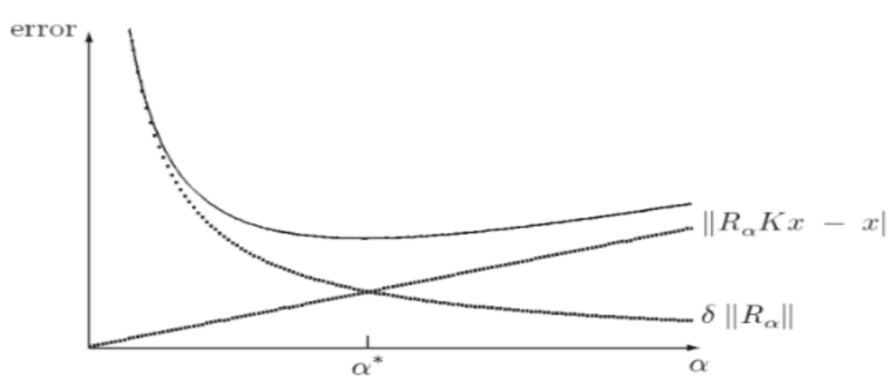

Figure 3.2: Behavior of the total error.

We need a regularization strategy in order to keep the total error as small as possible, this means that we would like to minimize $\delta \|R_\alpha\|_{L(Y,X)} + \|R_\alpha Kx - x\|_X$. A regularization strategy $\delta \longmapsto \alpha(\delta)$ is called admissible if $\alpha(\delta) \to 0$ and for every $x \in X$:

$$\sup_{y^\delta \in Y} \left\{ \left\|R_{\alpha(\delta)} y^\delta - x\right\|_X : y^\delta \in Y, \left\|Kx - y^\delta\right\|_Y \leq \delta \right\} \to 0,\ \delta \to 0 \tag{3.8}$$

In **Tikhonov regularization,** the approximate solution $x_\alpha \in X$ is defined as minimizer of the quadratic functional:

$$\|Kx - y\|_Y^2 + \alpha \|x\|_X^2 , \tag{3.9}$$

the basic idea of **Tikhonov regularization** is minimizing the functional in(3.9) , means to search for some x_α, providing at the same time a small residual $\|Kx - y\|_Y^2$ and a moderate value of the penalty function $x \mapsto \|x\|_X^2$. The existance and uniquness of the minimum is assured by the convexity of $x \mapsto \|x\|_X^2$.

If the **regularization parameter** α is chosen too small, (3.9) is too close to the original problem and instabilities have to be expected. If α is chosen too large, the problem we solve has only little connection with the original problem. Finding the optimal parameter is a tough problem.

Theorem 3.2.2 [10]

Let $K : X \to Y$ be a bounded linear operator and let $\alpha > 0$. Then for each $y \in Y$ there exists a unique $x_\alpha \in X$ such that

$$\|Kx_\alpha - y\|^2 + \alpha \|x_\alpha\|^2 = \inf_{x \in X} \left\{ \|Kx - y\|^2 + \alpha \|x\|^2 \right\} , \tag{3.10}$$

the minimizer x_α is given by the unique solution of the equation

$$\alpha x_\alpha + K^* K x_\alpha = K^* y, \tag{3.11}$$

and depends continuously on y .

Proof. From the equation

$$\|Kx-y\|^2+\alpha\|x\|^2=\|Kx_\alpha-y\|^2+\alpha\|x_\alpha\|^2+2\operatorname{Re}\left(x-x_\alpha,\alpha x_\alpha+K^*\left(Kx_\alpha-y\right)\right)$$
$$+\|K\left(x-x_\alpha\right)\|^2+\alpha\|x-x_\alpha\|^2,$$

which is valid for all $x\in X$, we observe that the condition(3.11) is necessary and sufficient for x_α to minimize the **Tikhonov functional** defined by(3.10) .

Consider the operator $K_\alpha:X\to X$, given by $K_\alpha:=\alpha I+K^*K$. Since

$$\alpha\|x\|^2\leq\alpha\|x\|^2+\|Kx\|^2=\operatorname{Re}\left(K_\alpha x,x\right),x\in X,$$

the operator K_α is strictly coercive and has a bounded inverse $K_\alpha^{-1}:X\to X$,we can prove that $K_\alpha$has a bounded inverse by the next theorem. ■

The equation (3.11) , of course, coincides with the Tikhonov regularization introduced in the following theorem:

Theorem 3.2.3 [10]

Let $K:X\to Y$ be a compact linear operator. Then for each $\alpha>0$ the operator $K_\alpha:X\to X$, given by $K_\alpha:=\alpha I+K^*K$ has a bounded inverse. Furthermore,if K is injective then

$$R_\alpha:=\left(\alpha I+K^*K\right)^{-1}K^*$$

describes a **regularization scheme** with $\|R_\alpha\|_{L(Y,X)}\leq\frac{1}{2\sqrt{\alpha}}$.

3.3 Regularization Method of Fourier

3.3.1 Homogenous Retrograde Heat Problem

We presente this method passing by an ill-posed problem which is homogenous retrograde heat problem, we mentioned it before in example (1.3.8) .

In general, the solution of this problem exists but with restrictive conditions on the final situation. We find the exacte solution and we search for an approuch one using the regularization of Fourier. Let the following problem:

$$\begin{cases} u_t = u_{xx} \\ u(x,T) = \varphi_T(x) \end{cases} - \infty < x < +\infty,\ 0 \le t < T. \tag{3.12}$$

We search the solution u of this problem by the Fourier transformation, we can rewrite the problem (3.12) as:

$$\begin{aligned} &\begin{cases} \frac{\partial u}{\partial t}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) = 0 \\ u(x,T) = \varphi_T(x) \end{cases}, \\ \Rightarrow &\begin{cases} \mathcal{F}\left(\frac{\partial u}{\partial t}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t)\right)_{(\xi)} = \mathcal{F}(0)_{(\xi)} = 0 \\ \mathcal{F}(u(x,T))_{(\xi)} = \mathcal{F}(\varphi_T(x))_{(\xi)} \end{cases}, \\ \Rightarrow &\begin{cases} \mathcal{F}\left(\frac{\partial u}{\partial t}(x,t)\right)_{(\xi)} - \mathcal{F}\left(\frac{\partial^2 u}{\partial x^2}(x,t)\right)_{(\xi)} = 0 \\ \mathcal{F}(u(x,T))_{(\xi)} = \mathcal{F}(\varphi_T(x))_{(\xi)} \end{cases}. \end{aligned} \tag{3.13}$$

We know that: $\mathcal{F}(u(x,t))_{(\xi)} = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi x} u(x,t)\,dx$

$\Rightarrow \mathcal{F}\left(\frac{\partial u}{\partial t}(x,t)\right)_{(\xi)} = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi x} \frac{\partial u}{\partial t}(x,t)\,dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \frac{\partial}{\partial t}\left(e^{-i\xi x} u(x,t)\right) dx$, thus

$$\mathcal{F}\left(\frac{\partial u}{\partial t}(x,t)\right)_{(\xi)} = \frac{\partial}{\partial t} \mathcal{F}(u(x,t))_{(\xi)}. \tag{3.14}$$

We choose $\text{supp}(u) \subset \mathbb{R} \Rightarrow \exists R > 0$ such that $\text{supp}(u) \subset [-R, R]$, so:

$\mathcal{F}\left(\frac{\partial^2 u}{\partial x^2}(x,t)\right)_{(\xi)} = \frac{1}{\sqrt{2\pi}} \int_{-R}^{R} e^{-i\xi x} \frac{\partial^2 u}{\partial x^2}(x,t)\,dx$, using integration by parts on $\frac{1}{\sqrt{2\pi}} \int_{-R}^{R} e^{-i\xi x} \frac{\partial u}{\partial x}(x,t)\,dx$ we find:

$$\frac{1}{\sqrt{2\pi}} \int_{-R}^{R} e^{-i\xi x} \frac{\partial u}{\partial x}(x,t)\,dx = \frac{1}{\sqrt{2\pi}} \left[\left[e^{-i\xi x} u(x,t) \right]_{-R}^{R} + i\xi \int_{-R}^{R} e^{-i\xi x} u(x,t)\,dx \right]$$

and because $u(-R,t) = u(R,t)$, we have:

$$\begin{aligned}
\mathcal{F}\left(\frac{\partial u}{\partial x}(x,t)\right)_{(\xi)} &= \frac{1}{\sqrt{2\pi}}\int_{-R}^{R} e^{-i\xi x}\frac{\partial u}{\partial x}(x,t)\,dx = \frac{1}{\sqrt{2\pi}} i\xi \int_{-R}^{R} e^{-i\xi x} u(x,t)\,dx \\
&= i\xi\left(\frac{1}{\sqrt{2\pi}}\int_{-R}^{R} e^{-i\xi x}u(x,t)\,dx\right) = i\xi\mathcal{F}(u(x,t))_{(\xi)} \\
&\Rightarrow \mathcal{F}\left(\frac{\partial^2 u}{\partial x^2}(x,t)\right)_{(\xi)} = -\xi^2\mathcal{F}(u(x,t))_{(\xi)}\,. \qquad (3.15)
\end{aligned}$$

Replacing (3.14) and (3.15) in (3.13) , we obtain:

$$\left\{\begin{array}{c} \frac{\partial}{\partial t}\mathcal{F}(u(x,t))_{(\xi)} + \xi^2\mathcal{F}(u(x,t))_{(\xi)} = 0 \\ \mathcal{F}(u(x,T))_{(\xi)} = \mathcal{F}(\varphi_T(x))_{(\xi)} \end{array}\right. \Rightarrow \left\{\begin{array}{c} \partial_t\hat{u}(\xi,t) + \xi^2\hat{u}(\xi,t) = 0 \\ \hat{u}(\xi,T) = \hat{\varphi}_T(\xi) \end{array}\right. ,$$

from the equation $\partial_t\hat{u}(\xi,t) + \xi^2\hat{u}(\xi,t) = 0$, we have: $\frac{\partial_t\hat{u}(\xi,t)}{\hat{u}(\xi,t)} = -\xi^2$
$\Rightarrow \int \frac{\partial_t\hat{u}(\xi,t)}{\hat{u}(\xi,t)}dt = \int -\xi^2 dt \Rightarrow \hat{u}(\xi,t) = e^{-\xi^2 t}c(\xi)$, so

$$\left\{\begin{array}{c} \hat{u}(\xi,T) = e^{-\xi^2 T}c(\xi) \\ \hat{u}(\xi,T) = \hat{\varphi}_T(\xi) \end{array}\right. \Rightarrow c(\xi) = e^{\xi^2 T}\hat{\varphi}_T(\xi)\,,$$

thus $\hat{u}(\xi,t) = e^{\xi^2(T-t)}\hat{\varphi}_T(\xi)$, where $\hat{u}(\xi,t)$ is the Fourier transformation of $u(x,t)$ such that $u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{i\xi x}\hat{u}(\xi,t)\,d\xi$ and $\hat{u}(\xi,0) = c(\xi) = e^{\xi^2 T}\hat{\varphi}_T(\xi)$, thus the solution of the problem (3.12) is:

$$u(x,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} e^{i\xi x}e^{\xi^2(T-t)}\hat{\varphi}_T(\xi)\,d\xi. \qquad (3.16)$$

3.3.2 Regularization of Fourier and Error Estimation

We assume for $t = T$ the exacte solution $\varphi_T(x)$ and the perturbed solution is $\varphi_T^\delta(x)$, then there exists a contante $\delta > 0$ such that: $\left\|\varphi_T - \varphi_T^\delta\right\| \leq 0$. We note $\varphi_0(x) = u(x,0)$ and C a constante such that: $\|\varphi_0\|_{H^s} = \left(\int_{-\infty}^{+\infty} |\hat{\varphi}_0(\xi)|^2 (1+\xi^2)^s\,d\xi\right)^{\frac{1}{2}} \leq C$, $\forall s \geq 0$.

We have $\|u\|_{L^2(\mathbb{R})} = \|\hat{u}\|_{L^2(\mathbb{R})}$, where $u(x,t)$ is the exacte solution given by (3.16) .

Let:

$$u_{\delta,\xi_{\max}}(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{i\xi x} e^{\xi^2 (T-t)} \hat{\varphi}_T^{\delta}(\xi)\, \chi_{\max} d\xi \tag{3.17}$$

be the approuch solution to the exacte one u, such that $\xi_{\max}$ is a positive constante and $\chi_{\max}$ is the characteristic function of the compact $[-\xi_{\max}, \xi_{\max}]$, and $u_{\delta,\xi_{\max}}(x,t)$ exists, unique and stable. We have:

$$\|u(x,t) - u_{\delta,\xi_{\max}}(x,t)\| \leq C^{\left(1-\frac{t}{T}\right)} \delta^{\frac{t}{T}} \left(\ln \frac{C}{\delta}\right)^{-\frac{(T-t)s}{2T}} \left[1 + \left(\frac{\ln \frac{C}{\delta}}{\frac{1}{T}\ln \frac{C}{\delta} + \ln\left(\ln \frac{C}{\delta}\right)^{-\frac{s}{2T}}}\right)^{\frac{s}{2}}\right] \tag{3.18}$$

where $\xi_{\max} = \left(\ln\left(\left(\frac{C}{\delta}\right)^{\frac{1}{T}} \left(\ln \frac{C}{\delta}\right)^{-\frac{s}{2T}}\right)\right)^{\frac{1}{2}}$ and $\ln \frac{C}{\delta} > 1$, $\forall s > 0$, this chiose of $\xi_{\max}$ is to find a stability estimation of Hölder for the best approuch. For the proof, see[14].

Chapter 4

Application: Inverse Estimation of the Initial Condition for the Heat Equation

The direct (forward) problem consists of a transient heat conduction problem in a slab with adiabatic boundary condition and initially at a temperature denoted by $f(x)$. The mathematical formulation of this problem is given by the following heat equation:

$$\begin{cases} \frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} \quad 0 < x < L, t > 0, \\ u(0,t) = 0 \quad t > 0, \\ u(L,t) = 0 \quad t > 0, \\ u(x,0) = f(x) \quad 0 \le x \le L, \end{cases} \tag{4.1}$$

where $u(x,t)$: temperature, $f(x)$: initial condition, x: spatial variable, t: time variable and D denotes the dispersion coefficient.

For the direct problem where the initial condition $f(x)$ is specified, the problem given by equation (4.1) is concerned with the determination of the temperature distribution $u(x,t)$ in the interior region of the solid as a function of time and position.

Now, for the **inverse problem**, the initial condition $f(x)$ is regarded as being unknown. In addition, an overspecified condition is also considered available. To estimate the unknown coefficient $f(x)$, the additional information

$$u(x,T) = g(x) \tag{4.2}$$

is given at time T, over a specified space interval $0 \leq x \leq L$. We note that the measured over-specified condition $u(x,T) = g(x)$ should contain measurement errors. Therefore the inverse problem can be stated as follows: by utilizing the above mentioned measured data, estimate the unknown function $f(x)$.

In this chapter, we are going to demonstrate some numerical results for determining $f(x)$ in the inverse problem $(4.1) - (4.2)$. But first let us know the algorithm used for solving the problem.

The solution of the direct problem for a given initial condition $f(x)$ is explicitly obtained using separation of variables, for $0 < x < L$, $t \geq 0$ is:

$$u(x,t) = \int_0^L K(x,y,t) f(y)\, dy,$$

where

$$K(x,y,t) = \frac{2}{L} \sum_{n=1}^{\infty} e^{-\frac{(n\pi)^2 Dt}{L^2}} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi y}{L}\right)$$

is an infinite series. Numerically, we can't handle infinite sums. Limit the sum to a finite number of expansion terms 100 which guarantees the convergence of the series. So

$$K(x,y,t) = \frac{2}{L} \sum_{n=1}^{100} e^{-\frac{(n\pi)^2 Dt}{L^2}} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{n\pi y}{L}\right).$$

Thus initial inverse problem is reduced to solving integral equation of the first kind:

$$u(x,T) = g(x) = \int_0^L K(x,y,T) f(y)\, dy. \tag{4.3}$$

The first step in the numerical treatment used in this research consists in discretiztion of equation (4.3) by the quadrature formula. The interval $[0,L]$ can be subdivided into equal intervals of width $h = \Delta y = \frac{L}{N}$. Since the variable is either y or x, let $x_0 = y_0 = 0$, $x_N = y_N = L$ and $x_i = i\Delta y$ $(i.e.x_i = y_i)$, $y_j = j\Delta y$. Also denote $f(x_i)$ as f_i, $g(x_i)$ as g_i and $K(x_i, y_j, T)$ as K_{ij}.

Now if the trapezoid rule is used to approximate the given equation, then:

$$\begin{aligned}
g(x) &= \int_0^L K(x, y, T) f(y)\, dy \\
&\approx \Delta y \left[\frac{1}{2} K(x, y_0, T) f(y_0) + K(x, y_1, T) f(y_1) + \cdots \right. \\
&\left. + K(x, y_{N-1}, T) f(y_{N-1}) + \frac{1}{2} K(x, y_N, T) f(y_N) \right]
\end{aligned}$$

or moretersely:

$$\begin{aligned}
g(x) = \Delta y &\left[\frac{1}{2} K(x, y_0, T) f_0 + K(x, y_1, T) f_1 + \cdots \right. \\
&\left. + K(x, y_{N-1}, T) f_{N-1} + \frac{1}{2} K(x, y_N, T) f_N \right].
\end{aligned}$$

There are $N+1$ values of f_i,as $i = 0, 1, 2, ..., N$, therefore the equation becomes a set of $N+1$ equations in f_i

$$g_i = \Delta y \left[\frac{1}{2} K_{i0} f_0 + K_{i1} f_1 + \cdots + K_{i(N-1)} f_{N-1} + \frac{1}{2} K_{iN} f_N \right]$$

that give the approximate solution to $f(x_i)$ at $x = x_i$. This may also be written in matrix form

$$KF = G \tag{4.4}$$

where K is the matrix of coefficients

$$K = \Delta y \begin{bmatrix} \frac{1}{2} K(x_0, y_0, T) & K(x_0, y_1, T) & \cdots & \frac{1}{2} K(x_0, y_N, T) \\ \frac{1}{2} K(x_1, y_0, T) & K(x_1, y_1, T) & \cdots & \frac{1}{2} K(x_1, y_N, T) \\ \cdots & \cdots & \cdots & \cdots \\ \frac{1}{2} K(x_N, y_0, T) & K(x_N, y_1, T) & \cdots & \frac{1}{2} K(x_N, y_N, T) \end{bmatrix},$$

F is the matrix of solutions $F = [f(y_0), \cdots, f(y_i), \cdots, f(y_N)]^T$ and G is the matrix of the nonhomogeneous part $G = [g(x_0), \cdots, g(x_i), \cdots, g(x_N)]^T$.

The problem (4.4) is ill-posed in the sense that the inverse operator A^{-1} of A exists but it is not continuous. Hence, although the problem (4.4) has a unique solution, solving it directly will not give a right solution. Indeed, the linear operator A is so badly conditioned that any numerical attempt to directly solve (4.4) may fail. So we go to regularization methods, see section 2 in chapter 3 (3.2) .

4.1 Numerical Examples:

We use the L_∞ error norm and the relative error to measure the difference between the numerical and analytical solutions. The L_∞ error norm is defined by: $L_\infty = \max\limits_{0 \le j \le N} \left| f(x_j) - \tilde{f}(x_j) \right|$ and the relative error RE is defined by: $RE = \sqrt{\dfrac{\sum_{j=0}^{N} \left| f(x_j) - \tilde{f}(x_j) \right|^2}{\sum_{j=0}^{N} |f(x_j)|^2}}$ where x_j are test points and N is the total number of uniformly distributed points on $[0, 1]$. $f(x)$ is the exact solution and $\tilde{f}(x)$ is the numerical solution. In our computations, we always take $N = 40$. The noisy data $\left\{ g^\delta(x_j) \right\} \big|_{j=0}^{N}$ were assumed to contain some random errors. However, in practical applications, the reduplicated measurements are fairly difficult and even are impossible. Hence, in the next, we consider the deterministic errors.

Assume the observed data has the following noised form: $g^\delta(x_j) = g(x_j) + \delta \sin(10\pi x_j)$, $j = 0, 1, 2, \cdots, N$.

Example 4.1.1 *In this example let us consider the following inverse problem:*

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} \quad 0 < x < 1, t > 0, \\ u(0, t) = 0 \quad t \ge 0, \\ u(1, t) = 0 \quad t \ge 0, \\ u(x, 0) = f(x) \quad 0 \le x \le 1, \end{cases}$$

the over specified condition: $u(x,1) = g(x) = e^{-\pi^2}\sin(\pi x)$.

The analytical solution of this example is

$$u(x,t) = e^{-\pi^2 t}\sin(\pi x) \text{ and } f(x) = \sin(\pi x)$$

The regularization parameter α is chosen using Newton's method (see [11]), the L_∞ error norm and relative error RE are presented in Table 4.1 . Also, the corresponding errors between the analytical and the estimated functions $f(x)$ in $x_j = 0.1j$ when $\delta = 0.1$ are listed in Table 4.2 .

δ	α	L_∞	RE
0.1	5.73185×10^{-14}	2.14202×10^{-5}	2.14247×10^{-5}
0.05	8.72765×10^{-15}	3.26344×10^{-6}	3.26231×10^{-6}
0.01	2.54233×10^{-15}	9.53403×10^{-7}	9.50310×10^{-7}
0.001	9.81035×10^{-16}	3.69100×10^{-7}	3.66705×10^{-7}

Table 4.1: The error norm and relative error RE for f(x)

j	Exact $f(x_j)$	Numerical $\tilde{f}(x_j)$	$\left\vert f(x_j) - \tilde{f}(x_j) \right\vert$
1	0.309016994374947	0.309016994374947	$6.621267118533 \times 10^{-6}$
2	0.587785252292473	0.587772658514349	$1.259377812457 \times 10^{-5}$
3	0.809016994374947	0.808999662566453	$1.733180849472 \times 10^{-5}$
4	0.951056516295154	0.951036138131767	$2.037816338640 \times 10^{-5}$
5	1	0.999978579729737	$2.142027026342 \times 10^{-5}$
6	0.951056516295154	0.951036139325882	$2.037696927137 \times 10^{-5}$
7	0.809016994374947	0.80899966217245	$1.733220249233 \times 10^{-5}$
8	0.587785252292473	0.587772660016492	$1.259227598149 \times 10^{-5}$
9	0.309016994374948	0.309010373872585	$6.620502362109 \times 10^{-6}$
10	$1.22464679914735 \times 10^{-16}$	$6.12310280754787 \times 10^{-17}$	$6.123365183926 \times 10^{-17}$

Table 4.2: The analytical and numerical results for f(x)

Example 4.1.2 *Let us consider the following inverse problem:*

$$\begin{cases} \frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2} \quad 0 < x < 1, t > 0, \\ u(0,t) = 0 \quad t > 0, \\ u(1,t) = 0 \quad t > 0, \\ u(x,0) = f(x) \quad 0 \leq x \leq 1, \end{cases}$$

the initial condition

$$f(x) = \begin{cases} 2x & 0 \leq x \leq 0.5 \\ 2(1-x) & 0.5 \leq x \leq 1 \end{cases}$$

where $D = 0.01$, the exact solution is given by using the separation of variables

$$u(x,t) = \sum_{n=0}^{\infty} \frac{8}{\pi^2 (2n+1)^2} \cos\left(\frac{(2n+1)\pi(2x-1)}{2}\right) e^{\left[-D\pi^2(2n+1)^2 t\right]}.$$

The experimental data $u(x,T) = g(x)$ (measured temperatures at $T = 1$) is obtained from the exact solution by taking the sum of the first one hundred terms. The L_∞ error norm and relative error RE are presented in Table 4.3 . Also, the corresponding errors between the analytical and the estimated functions $f(x)$ in $x_j = 0.1j$ when $\delta = 0.1$ are listed in Table 4.4 .

δ	α	L_∞	RE
0.1	0.00036282	0.0634335	0.0312757
0.05	0.00021836	0.0611471	0.0287479
0.01	0.00005996	0.0547497	0.0237954
0.001	0.00001008	0.0478144	0.0203349

Table 4.3: The error norm and relative error RE for f(x)

j	Exact $f(x_j)$	Numerical $\tilde{f}(x_j)$	$\lvert f(x_j) - \tilde{f}(x_j) \rvert$
1	0.2	0.203285479287456	0.0032854792874559
2	0.4	0.399171118110754	0.0008288818892464
3	0.6	0.590641061060895	0.0093589389391055
4	0.8	0.818696676546333	0.0186966765463328
5	1	0.945250250199793	0.0547497498002067
6	0.8	0.818696676542829	0.0186966765428294
7	0.6	0.590641061065385	0.0093589389346149
8	0.4	0.399171118106812	0.0008288818931877
9	0.2	0.203285479288568	0.0032854792885684
10	0	$3.39804782379576 \times 10^{-17}$	$3.39804782379576 \times 10^{-17}$

Table 4.4: The analytical and numerical results for f(x)

The graph of the analytical and the estimated functions for $f(x)$ with $\delta = 0.1$ for Example 1 is given in Figure4.1 , for Example 2, the graph is given in Figure4.2 . This chapter is based on the article [14] .

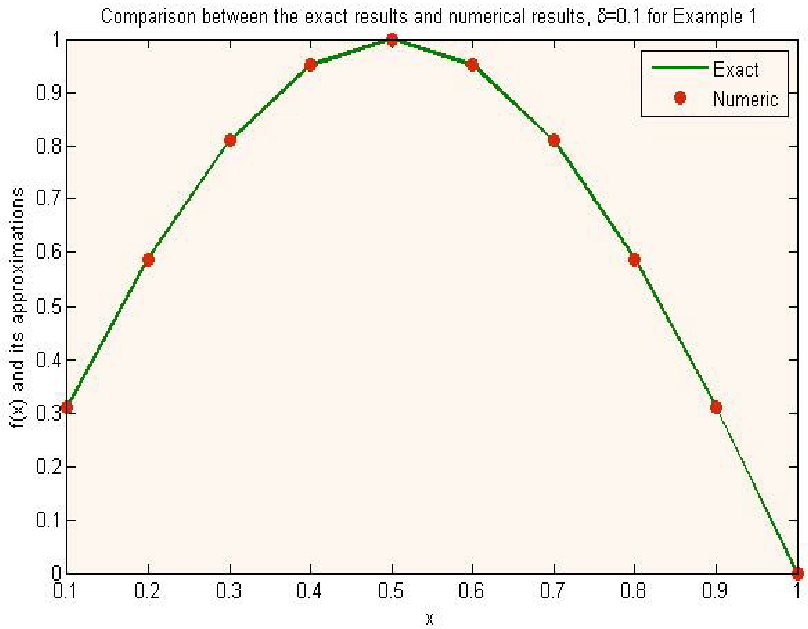

Figure 4.1: The comparison between exact and numerical solutions of $f(x)$ with noise $\delta = 0.1$, for Example 1.

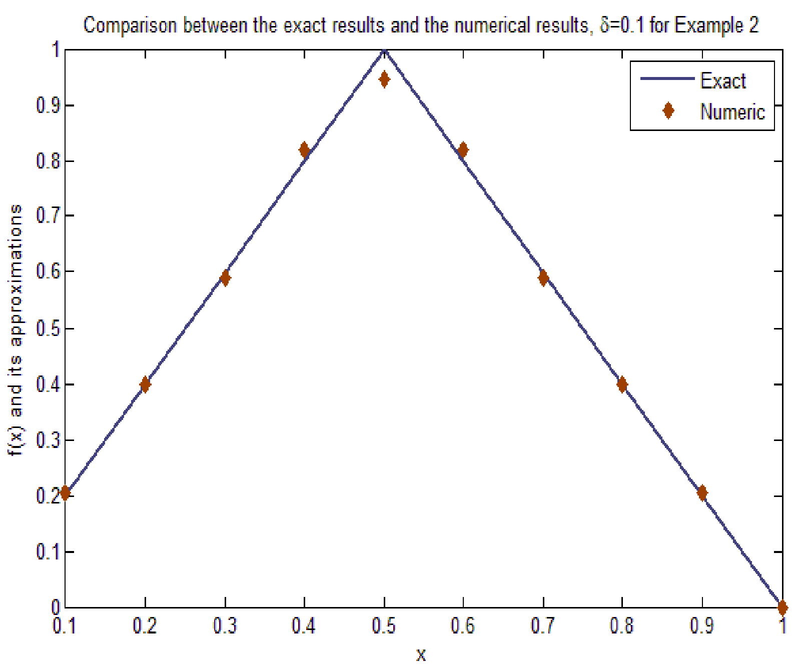

Figure 4.2: The comparison between exact and numerical solutions of $f(x)$ with noise $\delta = 0.1$, for Example 2.

Conclusion

Usually, an inverse problem is a situation in which we don't know the system (informations about matrials, initial conditions...), most of these problems are modeled (which is a difficult step, made with specialist in the studied field) in systems of partial defferential equations, hence the interest in deepening of PDEs notions (existence, regularity,...etc) something we provided in this memory.

Also, we presented in this memory some examples to explain ill-posed inverse problems and some regularization methods (LS method, Tikhonov method and method of Fourier). In fact, this work is indexed on regularization methods (insurance of existence and uniqueness of solution) and an application on estimation of the initial condition for heat equation.

Why this estimation?

In engineering, we have to solve heat transfer problems involving diffrent conditions such as cylindrical nuclear fuel element which involes internal heat source. Knowledge of temperture field is very important in thermal conduction, this estimation is to investigate the inverse problem in the heat equation involving the recovery of the initial temperture from measurements of the final temperture.

Eventually, inverse problems have a very wide domain, they constitute a branch of mathematical research whose importance continues to grow.

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Appendix: Index of Symbols

The various symbols used throught this thesis are explained below:

Symbol	Explication
$\mathbb{R}$	Real numbers.
$\mathbb{R}^2$	Tow-dimensional space.
$\mathbb{R}^3$	Three-dimensional space.
$\|\cdot\|$	Norm.
div	Divergence.
$\wedge$	Vector product.
∇	Nabla.
$a.e$	almost everywhere.
χ_E	characteristic function.
C^k	The set of k-differentiable functions.
C^∞	The set of infinitly differentiable functions(smooth functions).
$L^1_{loc}(\mathbb{R}^n)$	The set of locally 1-integrable on $\mathbb{R}^n$ functions.
$\mathcal{R}(A)$	*range* of A.

Abstract

Inverse problems for partial differential equations (PDEs in short) arise naturally in many areas, in geophysics, oil prospecting, in the design of optical devices and many other areas. Many of these problems may be regarded as the study of attempting an inversion of a mapping from the coefficients of PDEs to trace of same PDEs solutions on the boundary. This leads to the study of the existence, uniqueness and stability.

***Key words:* Inverse problems, PDEs, existence and uniqueness, stability, ill-posed problems, regularization methods, Heat equation.**

الملخص

تنشأ المشاكل العكسية للمعادلات التفاضلية الجزئية بشكل طبيعي في العديد من المجالات، في الجيوفيزياء، التنقيب عن النفط، في تصميم الأجهزة البصرية و العديد من المجالات الأخرى. و يمكن اعتبار العديد من هذه المشاكل بمثابة دراسة لمحاولة عكس رسم الخرائط من معاملات المعادلات التفاضلية الجزئية إلى تتبع بعض حلولها على الحدود. و هذا يؤدي إلى دراسة الوجود، الوحدانية و الاستقرار.

***الكلمات المفتاحية:* المشاكل العكسية، المعادلات التفاضلية الجزئية، الوجود و الوحدانية، الاستقرار، مشاكل أسيئت صياغتها، طرق التسوية، معادلة الحرارة.**

Résumé

Les problèmes inverses pour les équations aux dérivés partielles (EDP en abrégé) se posent naturellement dans nombreux domaines, en géophysique, en prospection pétrolière, dans la conception de dispositifs optiques et dans de nombreux autres domaines. Nombre de ces problèmes peuvent être considérés comme l'étude de la tentative d'inversion d'un mappage des coefficients des EDP à la trace de solutions d'EDP sur la frontière. Cela conduit à l'étude de l'existence, de l'unicité et de la stabilité.

***Mots clés:* Problèmes inverses, EDP, existence et unicité, stabilité, problèmes mal-posés, méthodes de régularisation, équation de Chaleur.**