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Necessary and Sufficient Condition for Optimality of a Backward Non-Markovian System

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Abstract. *We consider a stochastic control problem in the case where the set of control domain is convex, the system is governed by a nonlinear backward stochastic differential equation with a non-Markovian system and constant terminal conditions. The paper reports on a derivation of a stochastic maximum principle for optimality with a minimized criterion in the general form, with initial costs.*

Key words : Backward Stochastic Differential Equations, Maximum Principle, Adjoint Equation, Variational Inequalities, Path Dependence.

AMS Subject Classifications : 60H10, 93E20

1. Introduction

The backward stochastic differential equation (BSDE) related to the stochastic maximum principle of Pontryagin was introduced in 1965 and 1972 by Kushner [4, 5] and by J. M Bismut [2] for the case when the generator f is linear with respect to the variables Y and Z. A stochastic maximum principle of BSDE systems was studied by El-Karoui et al [6], where the linear case is solved and some applications in finance are treated. Dokuchaev and Zhou [10] established necessary as well as sufficient optimality conditions for nonlinear controlled BSDE systems, where the control domain is not necessary convex. Bahlali et al [11], proved the existence of optimal strict control systems governed by linear BSDEs. The control domain and the cost functional are assumed convex and they established in this paper necessary as well as sufficient conditions of optimality, satisfied by an optimal control, in the form of Pontryagin stochastic maximum principle. The proof of this result is based on the convex optimization principle.

Peng et al developed in [8] and [9] a new type of PDE which are formulated through a classical BSDE in which the terminal values and the generators are allowed to be general functions of Brownian paths. For more information on this subject, the reader is directed to [8].

In this work, we study the maximum principle in a non-Markovian framework, using the approach developed by Bensoussan [1], to get the necessary conditions for optimality of control. Hence we assume that the control domain is convex. Apparently, an argument of convex perturbation can be used to derive the maximum principle.

Our objective here is to study a stochastic control problem where the system is governed by a nonlinear BSDE of non-Markovian type. We shall establish necessary and sufficient optimality conditions, in the form of a stochastic maximum principle, for this kind of systems.

The non-Markovians ystem under consideration is governed by a BSDE of the type

$$\begin{cases} dY^v_{\gamma_t}(t) = f(s, B^{\gamma_t}_s, Y^v_{\gamma_t}(s), Z^v_{\gamma_t}(s), v(s))dt - Z^v_{\gamma_t}(s)dB(s), \\ Y^v_{\gamma_t}(T) = \xi, \end{cases}$$

where f is given map, $B = (B_t)_{T \geq t \geq 0}$ is a standard Brownian motion, defined on a probability space $(\Omega, \mathcal{F}, \mathcal{P})$, and with values respectively in $\mathbb{R}$. The control variable $v = (v_t)$ is a $\mathcal{F}_t$-adapted process with values in a set U of $\mathbb{R}^n$. We denote by $\mathcal{U}$ the set of all admissible controls. The criterion to be minimized, over the set $\mathcal{U}$ of controls, has the form

$$J(v) = \mathbb{E}\left[g(B^{\gamma_t}(t), Y^v_{\gamma_t}(t)) + \int_t^T h(s, B^{\gamma_t}_s, Y^v_{\gamma_t}(s), Z^v_{\gamma_t}(s), v(s))ds \right],$$

where g and h are given functions, and $(\gamma, Y^v_{\gamma_t}, Z^v_{\gamma_t})$ is the trajectory of the system controlled by v.

The paper is organized as follows. In section 2 we introduce some preliminary background and definitions about our new kind of BSDEs of a non-Markovian type. In Section 3, we formulate the problem and give various assumptions used throughout the paper. Section 4 is devoted to some preliminary results, which will be used in the sequel. In Section 5, we derive necessary as well as sufficient optimality conditions in the form of a stochastic maximum principle.

2. Definitions and Assumptions

The following notations are mainly from Dupire [3]. Let $T > 0$ be fixed. For each $t \in [0, T]$, we denote by Λ_t the set of càdlàg $\mathbb{R}^d$ –valued functions on $[0, t]$.

For each $\gamma \in \Lambda_T$ the value of γ at time $s \in [0, T]$ is denoted by $\gamma(s)$. Thus $\gamma = \gamma(s)_{0 \leq s \leq T}$ is a càdlàg process on $[0, T]$ and its value at time s is $\gamma(s)$. The path of γ up to time t is denoted by γ_t, i.e., $\gamma_t = \gamma(s)_{0 \leq s \leq t} \in \Lambda_t$. We use the notation $\Lambda = \bigcup_{t \in [0,T]} \Lambda_t$, and sometimes also specifically write

$$\gamma_t = \gamma(s)_{0 \leq s \leq t} = (\gamma(s)_{0 \leq s < t}, \gamma(t)),$$

to indicate the terminal position $\gamma(t)$ of γ_t, which often plays a special role in this framework. For each $\gamma \in \Lambda$ and $x \in \mathbb{R}^d$ we denote $\gamma^x_t = (\gamma(s)_{0 \leq s < t}, \gamma(t) + x)$, which is also an element in Λ_t.

Our interest here is in a function f of path, i.e., $f : \Lambda \mapsto \mathbb{R}$. This function $f = f(\gamma_t),\ \gamma_t \in \Lambda$ can also be regarded as a family of real-valued function:

$$f(\gamma_t) = f(t, \gamma(s)_{0\le s\le t}) = f(t, \gamma(s)_{0\le s\le t}, \gamma_t) : \ \gamma_t \in \Lambda_t, \quad t \in [0,T].$$

We also use the notation $f(\gamma_t^x) := f(t, \gamma(s)_{0\le s\le t}, \gamma_t + x)$ for $\gamma_t \in \Lambda_t,\ x \in \mathbb{R}^d, \quad t \in [0,T]$.

Remark 2.1. It is very important to conceive $f(\gamma_t^x)$ as a function of t, $f(t, \gamma(s)_{0\le s\le t}, \gamma(t))$ and x. A typical case is $f(\gamma_t) = f(t, \gamma(s)_{0\le s\le t}) = f(t, \gamma(t_-), \gamma_t + x) : \gamma_t \in \Lambda_t, \quad t \in [0,T]$, where $\gamma(t_-) = \lim_{s\uparrow t} \gamma(s)$.

We now introduce a distance on Λ. Let $\langle . \rangle$ and $|.|$ denote respectively the inner product and norm in $\mathbb{R}^d$. Moreover, for each $0 \le t \le \bar{t} \le T$ and $\gamma_t, \bar{\gamma}_{\bar{t}} \in \Lambda$, let

$$\|\gamma_t\| := \sup_{r\in[0,t]} |\gamma(r)|, \text{ and}$$

$$d_\infty(\gamma_t, \bar{\gamma}_{\bar{t}}) := \max\left(\sup_{r\in[0,t]} \{|\gamma_t - \bar{\gamma}_{\bar{t}}|\}, \sup_{r\in[0,t]} \{|\gamma_t - \bar{\gamma}_{\bar{t}}|\} + |t - \bar{t}| \right).$$

It is obvious then that Λ_t is a Banach space with respect to $\|.\|$, and since Λ is not a linear space, d_∞ is not a norm.

Definition 2.1. (Continuity) A function $f : \Lambda \to \mathbb{R}$ is said Λ –continuous at $\gamma_t \in \Lambda$, if for any $\varepsilon > 0$ there exists $\delta > 0$, such that for each $\bar{\gamma}_{\bar{t}} \in \Lambda$ satisfying $d_\infty(\gamma_t, \bar{\gamma}_{\bar{t}}) < \delta$, we have $|f(\gamma_t) - f(\bar{\gamma}_{\bar{t}})| > \varepsilon$. f is said to be Λ –continuous if it is Λ –continuous at each $\gamma_t \in \Lambda$.

3. Formulation of the Problem

Let $\Omega = C([0,T], \mathbb{R}^d)$ and P the standard Wiener measure defined on $(\Omega, \mathcal{B}(\Omega))$, and consider the canonical process $B(t) = B(t,w) = w(t),\ t \in [0,T], w \in \Omega$. Then $B(t)_{0\le t\le T}$ is d –dimensional Brownian motion defined on the probability space $(\Omega, \mathcal{B}(\Omega), \mathbb{P})$. Let $\mathcal{N}$ be the collection of all $\mathbb{P}$ –null set in Ω. For any $0 \le t \le r \le T$, $\mathcal{F}_r^t$ denotes the completion of $\sigma\big(B(s) - B(t);\ t \le s \le r\big)$, i.e., $\mathcal{F}_r^t = \sigma\Big\{\big(B(s) - B(t);\ t \le s \le r\big) \vee \mathcal{N}\Big\}$. We also write $\mathcal{F}_r$ for $\mathcal{F}_r^0$ and $\mathcal{F}^t$ for $\mathcal{F}_T^t$.

For any $0 \le t \le T$, we denote by $L^2(\mathcal{F}_t)$ the set of all square integrable $\mathcal{F}_t$ –measurable random variables, $\mathcal{M}^2([t,T];\mathbb{R}^d)$ the space of all $\mathcal{F}_s^t$ – adapted, $\mathbb{R}^d$ –valued processes $(X(s))_{s\in[t,T]}$ with $\mathbb{E}\left[\int_0^T |(X(s))|^2 dt\right] < \infty$, and $\mathcal{S}^2([t,T];\mathbb{R}^d)$ the space of all $\mathcal{F}_s^t$ – adapted, $\mathbb{R}^d$ –valued continuous processes $(X(s))_{s\in[t,T]}$ with $\mathbb{E}\left[\sup_{s\in[t,T]} |(X(s))|^2\right] < \infty$.

Consider then a deterministic function $f : [0,T] \times \Lambda \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \to \mathbb{R}^m$, which will be, in the following, the generator of our BSDEs. For this f, we will make the following assumptions:

$\star$ $f(\gamma_t, y, z)$ is a given continuous function on $\Lambda \times \mathbb{R}^m \times \mathbb{R}^{m\times d}$.

$\star\star$ There exists constants $C \ge 0$ and $q \ge 0$ such that: for $\gamma_t, \bar{\gamma}_{\bar{t}} \in \Lambda, y, \bar{y} \in \mathbb{R}^m, z, \bar{z} \in \mathbb{R}^{m\times d}$,

$$|f(\gamma_t, y, z) - f(\bar{\gamma}_t, \bar{y}, \bar{z})| \le C(1 + \|\gamma_t\|^q + \|\bar{\gamma}_t\|^q)\|\gamma_t + \bar{\gamma}_t\|(|y - \bar{y}| + |z - \bar{z}|).$$

The following result on Backward stochastic differential equation (BSDEs) is now well known, and for its proof the reader is referred to Pardoux-Peng [7].

Lemma 3.1. *Let f satisfy the above conditions, then for each* $\xi \in L^2(\Omega,\mathcal{F}_T,\mathbb{P};\mathbb{R}^n)$, *the BSDE*

$$dY(t) = \xi + \int_t^T f(s,B_s,Y(s),Z(s))dt - \int_t^T Z(s)dB(s), \quad 0 \leq t \leq T,$$

has a unique adapted solution $(Y(t),Z(t))_{0\leq t\leq T} \in S^2([0,T];\mathbb{R}^d) \times M^2([0,T];\mathbb{R}^d)$.

Let T be a positive real number, U be a nonempty set of $\mathbb{R}^d$ and $B = (B_t)_{t\in[0,T]}$ is standard Brownian motion, defined on a complete probability space $(\Omega,\mathcal{F},\mathbb{P})$ and with values in $\mathbb{R}^d$. Let $L^2(\Omega,\mathcal{F}_T,\mathbb{P};\mathbb{R}^n)$ denote the space of all $\mathcal{F}_T$-measurable, one-valued, random variable satisfying $E|\xi|^2 < \infty$.

Definition 3.1. An admissible control v is $\mathcal{F}_t$ –adapted process with values in U such that $\mathbb{E}\left[\sup_{0\leq t\leq T}|v_t|^2\right] < \infty$, where $\mathcal{U}$ is the set of all admissible controls.

Given $\xi \in L^2(\Omega,\mathcal{F}_T,\mathbb{P};\mathbb{R}^n)$ and for any $v \in \mathcal{U}$, we consider the following controlled BSDE non-Markovian system

$$\begin{cases} dY^v_{\gamma_t}(t) = -f(s,B^{\gamma_t}_s,Y^v_{\gamma_t}(s),Z^v_{\gamma_t}(s),v(s))dt + Z^v_{\gamma_t}(s)dB(s), \\ Y^v_{\gamma_t}(T) = \xi, \end{cases} \tag{1}$$

where $f : [0,T] \times \Lambda \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \times U \to \mathbb{R}$, and $B^{\gamma_t}_s(u) := \gamma_t I_{[0,t]}(u) + (\gamma_t(t) + B(u) - B(t)) \times I_{(t,T]}(u)$.

The functional cost to be minimized, is defined from $\mathcal{U}$ into $\mathbb{R}$ by

$$J(v) = \mathbb{E}\left[g(B^{\gamma_t}(t),Y^v_{\gamma_t}(t)) + \int_t^T h(s,B^{\gamma_t}_s,Y^v_{\gamma_t}(s),Z^v_{\gamma_t}(s),v(s))ds\right], \tag{2}$$

where $g : \Lambda \times \mathbb{R}^n \to \mathbb{R}$, $h : [0,T] \times \Lambda \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \times U \to \mathbb{R}$, $t \in [0,T], x \in \mathbb{R}, \gamma \in \Lambda, v \in U$. The control problem is to minimize the functional J over $\mathcal{U}$. If $u \in \mathcal{U}$ is an optimal solution, that is

$$J(u) = \inf_{v\in\mathcal{U}} J(v). \tag{3}$$

Now let us assume, as **A1**, that

- f is $\mathcal{F}_t$-progressively measurable and satisfies $b(w,t,0,0,0) \in \mathcal{M}^2(0,T;\mathbb{R}^n)$;
- f, and h are continuous and continuously differentiable with respect to (y,z,v);
- All the derivatives of f and h are bounded by $c > 0$.

Under the above assumptions, for every $v \in \mathcal{U}$, equation (1) has an unique strong solution $(Y^v_{\gamma_t},Z^v_{\gamma_t}) \in \mathcal{S}^2([0,T];\mathbb{R}^d) \times \mathcal{M}^2([0,T];\mathbb{R}^d)$ (see Pardoux-Peng [7]) and the functional cost J is well defined from $\mathcal{U}$ into $\mathbb{R}$.

We remark that assumptions (**A1**) imply in particular that there exist constants $c > 0$, such that for any $(y_1,z_1,v_1),(y_2,z_2,v_2) \in \mathbb{R}^n \times \mathbb{R}^{n\times d} \times U$, we have

$$|f(\gamma_t,y_1,z_1,v_1)-f(\gamma_t,y_2,z_2,v_2)|^2 \le c\left(|y_1-y_2|^2+\|z_1-z_2\|^2+|v_1-v_2|^2\right),$$
$$|h(\gamma_t,y_1,z_1,v_1)-h(\gamma_t,y_2,z_2,v_2)|^2 \le c\left(|y_1-y_2|^2+\|z_1-z_2\|^2+|v_1-v_2|^2\right),$$
$$|g(\gamma_t,y_1)-g(\gamma_t,y_2)|^2 \le c|y_1-y_2|^2.$$

4. Preliminary Results

Since the set $\mathcal{U}$ is convex, the classical way to derive necessary optimality conditions is to use the convex perturbation method. More precisely, let u be an optimal control and let $(Y^u_{\gamma_t}, Z^u_{\gamma_t})$ be the corresponding trajectory. Then, for each $s \in [t,T]$, we can define a perturbed control by

$$u^\varepsilon = u_t + \varepsilon(v(s)-u(s)),$$

where $\varepsilon > 0$ is sufficiently small and v is an arbitrary element of $\mathcal{U}$. Denote then by $(Y^\varepsilon_{\gamma_t}, Z^\varepsilon_{\gamma_t})$ the solution of (1) controlled by u^ε.

Since u is optimal, the variational inequality will be derived from the fact that

$$0 \le J(u^\varepsilon) - J(u).$$

Towards this end, we need the following lemmata.

Lemma 4.1.[9] *Let* $\delta(s) \in \mathcal{M}^2([0,T];\mathbb{R}^n), \beta(s) \in \mathcal{M}^2(t,T;\mathbb{R}^n)$, *be such that*

$$Y_{\gamma_t}(t) = \alpha_0 + \int_t^T \delta(s)ds - \int_t^T \beta(s)dB(s), t \in [0,T].$$

Then

$$|Y_{\gamma_t}(t)|^2 + \int_t^T |\beta(s)|^2 = |\alpha_0|^2 + 2\int_t^T \langle Y_{\gamma_t}(s), \delta(s)\rangle ds$$
$$- 2\int_t^T \langle Y_{\gamma_t}(s), \beta(s)\rangle dB(s),$$
$$\mathbb{E}|Y_{\gamma_t}(t)|^2 + \mathbb{E}\int_t^T |\beta(s)|^2 \le \mathbb{E}|\alpha_0|^2 + 2\mathbb{E}\int_t^T \langle Y_{\gamma_t}(t), \delta(s)\rangle ds.$$

Lemma 4.2. *Under the assumptions* (**A1**), *there holds*

$$\lim_{\varepsilon\to 0} \mathbb{E}\left[\sup_{t\le s\le T} |Y^\varepsilon_{\gamma_t}(s) - Y^u_{\gamma_t}(s)|^2\right] = 0, \quad (4)$$
$$\lim_{\varepsilon\to 0} \mathbb{E}\left[\int_t^T \|Z^\varepsilon_{\gamma_t}(s) - Z^u_{\gamma_t}(s)\|^2 ds\right] = 0. \quad (5)$$

Proof. Applying the generalized Itô formula (Lemma 4.1) to $|Y^\varepsilon_{\gamma_t}(s) - Y^u_{\gamma_t}(s)|^2$, we get

$$\mathbb{E}|Y^\varepsilon_{\gamma_t}(s) - Y^u_{\gamma_t}(s)|^2 + \mathbb{E}\int_t^T \|Z^\varepsilon_{\gamma_t}(s) - Z^u_{\gamma_t}(s)\|^2 ds$$
$$\le 2\mathbb{E}\int_t^T \langle (Y^\varepsilon_{\gamma_t}(s) - Y^u_{\gamma_t}(s)),$$
$$f(s, B^{\gamma_t}_s, Y^\varepsilon_{\gamma_t}(s), Z^\varepsilon_{\gamma_t}(s), u^\varepsilon(s)) - f(s, B^{\gamma_t}_s, Y^u_{\gamma_t}(s), Z^u_{\gamma_t}(s), u(s))\rangle\, ds. \quad (6)$$

Then apply Young's formula, $2ab \leq \frac{2c}{(1-\alpha)}a^2 + \frac{(1-\alpha)}{2c}b^2$, to the term in the right hand side to obtain

$$\mathbb{E}|Y^{\varepsilon}_{\gamma_t}(s) - Y^{u}_{\gamma_t}(s)|^2 + \mathbb{E}\int_t^T \|Z^{\varepsilon}_{\gamma_t}(s) - Z^{u}_{\gamma_t}(s)\|^2 ds$$
$$\leq \frac{2c}{(1-\alpha)}\mathbb{E}\int_t^T |Y^{\varepsilon}_{\gamma_t}(s) - Y^{u}_{\gamma_t}(s)|^2 ds$$
$$+\frac{(1-\alpha)}{2c}\mathbb{E}\int_t^T |f(s, B^{\gamma_t}_s, Y^{\varepsilon}_{\gamma_t}(s), Z^{\varepsilon}_{\gamma_t}(s), u^{\varepsilon}(s)) - f(s, B^{\gamma_t}_s, Y^{u}_{\gamma_t}(s), Z^{u}_{\gamma_t}(s), u(s))|^2 ds.$$

By the Lipschitz conditions for f, we have

$$\mathbb{E}\int_t^T |f(s, B^{\gamma_t}_s, Y^{\varepsilon}_{\gamma_t}(s), Z^{\varepsilon}_{\gamma_t}(s), u^{\varepsilon}(s)) - f(s, B^{\gamma_t}_s, Y^{u}_{\gamma_t}(s), Z^{u}_{\gamma_t}(s), u(s))|^2 ds$$
$$\leq 3\mathbb{E}\int_t^T |f(s, B^{\gamma_t}_s, Y^{\varepsilon}_{\gamma_t}(s), Z^{\varepsilon}_{\gamma_t}(s), u^{\varepsilon}(s)) - f(s, B^{\gamma_t}_s, Y^{u}_{\gamma_t}(s), Z^{\varepsilon}_{\gamma_t}(s), u^{\varepsilon}(s))|^2 ds$$
$$+3\mathbb{E}\int_t^T |f(s, B^{\gamma_t}_s, Y^{u}_{\gamma_t}(s), Z^{\varepsilon}_{\gamma_t}(s), u^{\varepsilon}(s)) - f(s, B^{\gamma_t}_s, Y^{u}_{\gamma_t}(s), Z^{u}_{\gamma_t}(s), u^{\varepsilon}(s))|^2 ds$$
$$+3\mathbb{E}\int_t^T |f(s, B^{\gamma_t}_s, Y^{u}_{\gamma_t}(s), Z^{u}_{\gamma_t}(s), u^{\varepsilon}(s)) - f(s, B^{\gamma_t}_s, Y^{u}_{\gamma_t}(s), Z^{u}_{\gamma_t}(s), u(s))|^2 ds.$$

The use of the definition of u^{ε} leads to

$$\mathbb{E}\int_t^T |f(s, B^{\gamma_t}_s, Y^{\varepsilon}_{\gamma_t}(s), Z^{\varepsilon}_{\gamma_t}(s), u^{\varepsilon}(s)) - f(s, B^{\gamma_t}_s, Y^{u}_{\gamma_t}(s), Z^{u}_{\gamma_t}(s), u(s))|^2 ds$$
$$\leq 3M\mathbb{E}\int_t^T |Y^{\varepsilon}_{\gamma_t}(s) - Y^{u}_{\gamma_t}(s)|^2 ds + 3M\mathbb{E}\int_t^T |Z^{\varepsilon}_{\gamma_t}(s) - Z^{u}_{\gamma_t}(s)|^2 ds$$
$$+3\varepsilon^2 M\mathbb{E}\int_t^T |v(s) - u(s)|^2 ds.$$

Then we can rewrite (6) as follows

$$\mathbb{E}|Y^{\varepsilon}_{\gamma_t}(t) - Y^{u}_{\gamma_t}(t)|^2 + \mathbb{E}\int_t^T \|Z^{\varepsilon}_{\gamma_t}(s) - Z^{u}_{\gamma_t}(s)\|^2 ds$$
$$\leq \left(\frac{2c}{(1-\alpha)} + 3M\frac{(1-\alpha)}{2c}\right)\mathbb{E}\int_t^T |Y^{\varepsilon}_{\gamma_t}(s) - Y^{u}_{\gamma_t}(s)|^2 ds$$
$$+ 3M\frac{(1-\alpha)}{2c}\mathbb{E}\int_t^T |Z^{\varepsilon}_{\gamma_t}(s) - Z^{u}_{\gamma_t}(s)|^2 ds$$
$$+ \frac{3(1-\alpha)}{2c}\varepsilon^2 M\,\mathbb{E}\int_t^T |v(s) - u(s)|^2 ds,$$

where

$$M = \left(\frac{3(1-\alpha)}{c}\right)^{-1}. \tag{7}$$

From the above inequality, we derive the following two inequalities

$$\mathbb{E}|Y^{\varepsilon}_{\gamma_t}(t) - Y^{u}_{\gamma_t}(t)|^2 \leq c_2\mathbb{E}\int_t^T |Y^{\varepsilon}_{\gamma_t}(s) - Y^{u}_{\gamma_t}(s)|^2 ds + c_3\varepsilon^2, \tag{8}$$

$$\mathbb{E}\int_t^T \|Z^{\varepsilon}_{\gamma_t}(s) - Z^{u}_{\gamma_t}(s)\|^2 ds \leq 2c_2\mathbb{E}\int_t^T |Y^{\varepsilon}_{\gamma_t}(s) - Y^{u}_{\gamma_t}(s)|^2 ds + 2c_3\varepsilon^2. \tag{9}$$

Consideration of (8), Gronwall's lemma and the Bukholder–Davis–Gundy inequality yields (4). Finally, (5) is deduced from (9) and (4). ■

Lemma 4.3. *Let $(\tilde{y},\tilde{z})$ be the solution of the following linear equation (called variational equation):*

$$\begin{aligned} d\,\tilde{y}(s) &= -[f_y(s,B_s^{\gamma_t},Y_{\gamma_t}^u(s),Z_{\gamma_t}^u(s),u(s))\tilde{y}(s) \\ &\quad + f_z(s,B_s^{\gamma_t},Y_{\gamma_t}^u(s),Z_{\gamma_t}^u(s),u(s))\tilde{z}(s)\,]ds \\ &\quad - f_v(s,B_s^{\gamma_t},Y_{\gamma_t}^u(s),Z_{\gamma_t}^u(s),u(s))(v(s)-u(s))ds + \tilde{z}(s)dB(s), \\ \tilde{y}(T) &= 0, \end{aligned} \tag{10}$$

then

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[\sup_{t\le s\le T}\left|\frac{Y_{\gamma_t}^{\varepsilon}(s)-Y_{\gamma_t}^u(s)}{\varepsilon}-\tilde{y}(s)\right|^2\right]=0, \tag{11}$$

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[\int_t^T\left|\frac{Z_{\gamma_t}^{\varepsilon}(s)-Z_{\gamma_t}^u(s)}{\varepsilon}-\tilde{z}(s)\right|^2 ds\right]=0. \tag{12}$$

Proof. For the sake of simplicity, let us use the notation

$$\widetilde{Y}^{\varepsilon}(s) = \frac{Y_{\gamma_t}^{\varepsilon}(s)-Y_{\gamma_t}^u(s)}{\varepsilon}-\tilde{y}(s),$$

$$\widetilde{Z}^{\varepsilon}(s) = \frac{Z_{\gamma_t}^{\varepsilon}(s)-Z_{\gamma_t}^u(s)}{\varepsilon}-\tilde{z}(s),$$

$$\begin{aligned}\Gamma^{\varepsilon}(s) &= \Big(s,Y_{\gamma_t}^u(s)+\lambda\varepsilon\big(\widetilde{Y}^{\varepsilon}(s)+\tilde{y}(s)\big),Z_{\gamma_t}^u(s)+\lambda\varepsilon\big(\widetilde{Z}^{\varepsilon}(s)+\tilde{z}(s)\big), \\ &\quad u(s)+\lambda\varepsilon(v(s)-u(s))\Big),\end{aligned}$$

and write

$$\begin{cases} d\widetilde{Y}^{\varepsilon}(s) = -\left[\int_0^1 f_y(\Gamma^{\varepsilon}(s))\widetilde{Y}^{\varepsilon}(s)d\lambda+\int_0^1 f_z(\Gamma^{\varepsilon}(s))\widetilde{Z}^{\varepsilon}(s)d\lambda+\alpha^{\varepsilon}(s)\right]ds \\ \qquad\qquad +\widetilde{Z}^{\varepsilon}(s)dB(s), \\ \tilde{y}(T) = 0, \end{cases}$$

where

$$\begin{aligned}\alpha_t^{\varepsilon} &= \left[f_y(s,B_s^{\gamma_t},Y_{\gamma_t}^u(s),Z_{\gamma_t}^u(s),u(s))-\int_0^1 f_y(\Gamma^{\varepsilon}(s))d\lambda\right]\tilde{y}(s) \\ &\quad+\left[f_z(s,B_s^{\gamma_t},Y_{\gamma_t}^u(s),Z_{\gamma_t}^u(s),u(s))-\int_0^1 f_z(\Gamma^{\varepsilon}(s))d\lambda\right]\tilde{z}(s) \\ &\quad+\left[f_v(s,B_s^{\gamma_t},Y_{\gamma_t}^u(s),Z_{\gamma_t}^u(s),u(s))-\int_0^1 f_v(\Gamma^{\varepsilon}(s))d\lambda\right] \\ &\quad\times(v(s)-u(s)).\end{aligned} \tag{13}$$

By applying the generalized Itô formula to $\left|\widetilde{Y}_t^{\varepsilon}\right|^2$, we get

$$\mathbb{E}\left|\widetilde{Y}^{\varepsilon}(s)\right|^2+\mathbb{E}\int_t^T\left|\widetilde{Z}^{\varepsilon}(s)\right|^2ds$$
$$\leq 2\mathbb{E}\int_t^T\left|\widetilde{Y}^{\varepsilon}(s)\cdot\left[\int_0^1 f_y(\Gamma^{\varepsilon}(s))\widetilde{Y}^{\varepsilon}(s)d\lambda+\int_0^1 f_z(\Gamma^{\varepsilon}(s))\widetilde{Z}^{\varepsilon}(s)d\lambda+\alpha^{\varepsilon}(s)\right]\right|ds,$$

and by using Young's formula in the term in the right hand side, for every $\delta_1>0$, we have

$$\mathbb{E}\left|\widetilde{Y}_t^{\varepsilon}\right|^2+\mathbb{E}\int_t^T\left|\widetilde{Z}_s^{\varepsilon}\right|^2ds$$
$$\leq \frac{1}{\delta_1}\mathbb{E}\int_t^T\left|\widetilde{Y}^{\varepsilon}(s)\right|^2ds$$
$$+\delta_1\mathbb{E}\int_t^T\left|\int_0^1 f_y(\Gamma^{\varepsilon}(s))\widetilde{Y}^{\varepsilon}(s)d\lambda+\int_0^1 f_z(\Gamma^{\varepsilon}(s))\widetilde{Z}^{\varepsilon}(s)d\lambda+\alpha^{\varepsilon}(s)\right|^2ds.$$

According to (**A1**), $f_y,\ f_z$ are bounded by $c>0$, and this allows for

$$\mathbb{E}\left|\widetilde{Y}^{\varepsilon}(s)\right|^2+\mathbb{E}\int_t^T\left|\widetilde{Z}^{\varepsilon}(s)\right|^2ds$$
$$\leq\left(\frac{1}{\delta_1}+3c\delta_1\right)\mathbb{E}\int_t^T\left|\widetilde{Y}^{\varepsilon}(s)\right|^2ds+3c\delta_1\mathbb{E}\int_t^T\left(\left|\widetilde{Z}^{\varepsilon}(s)\right|^2\right)ds$$
$$+3c\delta_1\mathbb{E}\int_t^T(|\alpha_s^{\varepsilon}|^2)ds.$$

A further choice of $\delta_1=\dfrac{1}{6c}$ in the previous inequality reduces it to

$$\mathbb{E}\left|\widetilde{Y}^{\varepsilon}(s)\right|^2+\tfrac{1}{2}\mathbb{E}\int_t^T\left|\widetilde{Z}^{\varepsilon}(s)\right|^2ds\leq C_1\mathbb{E}\int_t^T\left|\widetilde{Y}_s^{\theta}\right|^2ds+C_2\mathbb{E}\int_t^T(|\alpha_s^{\varepsilon}|^2)ds.$$

From this result we deduce the following two inequalities

$$\mathbb{E}\left|\widetilde{Y}^{\varepsilon}(s)\right|^2\leq C_1\mathbb{E}\int_t^T\left|\widetilde{Y}_s^{\theta}\right|^2ds+C_2\mathbb{E}\int_t^T(|\alpha_s^{\varepsilon}|^2)ds, \tag{14}$$

$$\mathbb{E}\int_t^T\left|\widetilde{Z}^{\varepsilon}(s)\right|^2ds\leq 2C_1\mathbb{E}\int_t^T\left|\widetilde{Y}_s^{\theta}\right|^2ds+2C_2\mathbb{E}\int_t^T(|\alpha_s^{\varepsilon}|^2)ds. \tag{15}$$

Now, since $f_y,\ f_z$ and b_v are continuous and bounded, then by using the Cauchy-Schwartz inequality, we may show that

$$\lim_{\varepsilon\to 0}\mathbb{E}\int_t^T(|\alpha_s^{\varepsilon}|^2)ds=0. \tag{16}$$

This relation is obtained by applying Gronwall's lemma and letting ε go to 0 in (13). Finally, (11) and (12) are deduced from (14), (15) and (16). ■

Since u is an optimal control, then

$$\tfrac{1}{\varepsilon}(J(u(s)+\varepsilon v(s))-J(u(s)))\geq 0. \tag{17}$$

And equipped with (17) and Lemma 4.3, we can state the lemma that follows.

Lemma 4.4. *Let u be an optimal control minimizing the cost J over $\mathcal{U}$, and assume the validity of A1, then the following variational inequality holds.*

$$0 \leq \mathbb{E}[g_y(B^{\gamma_t}(t), Y^u_{\gamma_t}(t))\tilde{y}(t)] +$$
$$+ \mathbb{E}\int_t^T h_y(s, B^{\gamma_t}_s, Y^u_{\gamma_t}(s), Z^u_{\gamma_t}(s), u(s))\tilde{y}(s)ds$$
$$+ \mathbb{E}\int_t^T h_z(s, B^{\gamma_t}_s, Y^u_{\gamma_t}(s), Z^u_{\gamma_t}(s), u(s))\tilde{z}(s)ds$$
$$+ \mathbb{E}\int_t^T h_v(s, B^{\gamma_t}_s, Y^u_{\gamma_t}(s), Z^u_{\gamma_t}(s), u(s))(v(s) - u(s))ds. \tag{18}$$

Proof. Starting from (17), since u is optimal, we have

$$0 \leq \frac{1}{\varepsilon}\mathbb{E}[g(B^{\gamma_t}(t), Y^\varepsilon_{\gamma_t}(t)) - g(B^{\gamma_t}(t), Y^u_{\gamma_t}(t))]$$
$$+\frac{1}{\varepsilon}\mathbb{E}\int_t^T[h(s, B^{\gamma_t}_s, Y^\varepsilon_{\gamma_t}(s), Z^\varepsilon_{\gamma_t}(s), u^\varepsilon(s)) - h(s, B^{\gamma_t}_s, Y^u_{\gamma_t}(s), Z^u_{\gamma_t}(s), u(s))]ds$$
$$\leq \mathbb{E}\int_0^1 g_y\Big(Y^u_{\gamma_t}(t) + \lambda\varepsilon\Big(\widetilde{Y}^\varepsilon(t) + \tilde{y}(t)\Big)\Big)\Big(\widetilde{Y}^\varepsilon(t) + \tilde{y}(t)\Big)d\lambda$$
$$+ \mathbb{E}\int_t^T\int_0^1 f_y(\Gamma^\varepsilon(s))\Big(\widetilde{Y}^\varepsilon(s) + \tilde{y}(s)\Big)d\lambda ds$$
$$+ \mathbb{E}\int_t^T\int_0^1 f_z(\Gamma^\varepsilon(s))\Big(\widetilde{Y}^\varepsilon(s) + \tilde{y}(s)\Big)d\lambda ds$$
$$+ \mathbb{E}\int_t^T\int_0^1 f_v(\Gamma^\varepsilon(s))(v(s) - u(s))d\lambda ds + \eta^\varepsilon(t).$$

then

$$0 \leq \frac{1}{\varepsilon}\mathbb{E}[g(B^{\gamma_t}(t), Y^\varepsilon_{\gamma_t}(t)) - g(B^{\gamma_t}(t), Y^u_{\gamma_t}(t))]$$
$$+ \frac{1}{\varepsilon}\mathbb{E}\int_t^T[h(s, B^{\gamma_t}_s, Y^\varepsilon_{\gamma_t}(s), Z^\varepsilon_{\gamma_t}(s), u^\varepsilon(s))$$
$$-h(s, B^{\gamma_t}_s, Y^u_{\gamma_t}(s), Z^u_{\gamma_t}(s), u(s))]$$
$$\leq \mathbb{E}\int_0^1 g_y\Big(Y^u_{\gamma_t}(t) + \lambda\varepsilon\Big(\widetilde{Y}^\varepsilon(t) + \tilde{y}(t)\Big)\Big)(\tilde{y}(t))d\lambda$$
$$+ \mathbb{E}\int_t^T\int_0^1 f_y(\Gamma^\varepsilon(s))(\tilde{y}(s))d\lambda ds + \mathbb{E}\int_t^T\int_0^1 f_z(\Gamma^\varepsilon(s))(\tilde{y}(s))d\lambda ds$$
$$+ \mathbb{E}\int_t^T\int_0^1 f_v(\Gamma^\varepsilon(s))(v(s) - u(s))d\lambda ds + \eta^\varepsilon(t), \tag{19}$$

where $\eta^\varepsilon(t)$ is given by

$$\eta(t) = \mathbb{E}\int_0^1 g_y\Big(Y^u_{\gamma_t}(t) + \lambda\varepsilon\Big(\widetilde{Y}^\varepsilon(t) + \tilde{y}(t)\Big)\Big)\Big(\widetilde{Y}^\varepsilon(t)\Big)d\lambda$$
$$+ \mathbb{E}\int_t^T\int_0^1 f_y(\Gamma^\varepsilon(s))\Big(\widetilde{Y}^\varepsilon(s)\Big)d\lambda ds$$
$$+ \mathbb{E}\int_t^T\int_0^1 f_z(\Gamma^\varepsilon(s))\Big(\widetilde{Y}^\varepsilon(s)\Big)d\lambda ds.$$

Apply then the Cauchy Schwartz inequality, the fact that g_y, f_y and f_z are bounded, and use (12) with (13) to show that $\lim_{\varepsilon\to 0}\eta^\varepsilon(t) = 0$. Finally, by letting ε go to 0 in (19), the proof is completed. ■

5. Adjoint Equation and Maximum Principle

In this section, we derive the variational inequality from (18). For this end, we introduce the following adjoint equation

$$\begin{cases} -dp(s) = [f_y(s, B_s^{\gamma_t}, Y_{\gamma_t}^u(s), Z_{\gamma_t}^u(s), u(s))p(s) \\ \qquad + h_y(s, B_s^{\gamma_t}, Y_{\gamma_t}^u(s), Z_{\gamma_t}^u(s), u(s))]ds \\ \qquad + [f_z(s, B_s^{\gamma_t}, Y_{\gamma_t}^u(s), Z_{\gamma_t}^u(s), u(s))p(s) \\ \qquad + h_z(s, B_s^{\gamma_t}, Y_{\gamma_t}^u(s), Z_{\gamma_t}^u(s), u(s))]dB(s), \\ p(t) = g_y(B^{\gamma_t}(t), Y_{\gamma_t}^u(t)), \quad t \leq s \leq T, \end{cases} \tag{20}$$

with

$p \in L^2_{\mathcal{F}}([0,T]; \mathbb{R}^n)$.

By applying Itô's formula to $(p(s)\widetilde{y}(s))$ and invoking the expectation, we have

$$\begin{aligned} \mathbb{E}(p(t)\widetilde{y}(t)) &= \mathbb{E}(p(T)\widetilde{y}(T)) - \mathbb{E}\int_t^T h_y(s, B_s^{\gamma_t}, Y_{\gamma_t}^u(s), Z_{\gamma_t}^u(s), u(s))\widetilde{y}(s)ds \\ &\quad - \mathbb{E}\int_t^T h_z(s, B_s^{\gamma_t}, Y_{\gamma_t}^u(s), Z_{\gamma_t}^u(s), u(s))\widetilde{z}(s)ds \\ &\quad + \mathbb{E}\int_t^T [p(s) f_v(s, B_s^{\gamma_t}, Y_{\gamma_t}^u(s), Z_{\gamma_t}^u(s), u(s))](v(s) - u(s))ds. \end{aligned} \tag{21}$$

We remark that $Y(T) = 0$, and $p(t) = g_y(B^{\gamma_t}(t), Y_{\gamma_t}^u(t))$. Then (21) becomes

$$\begin{aligned} &\mathbb{E}(g_y(B^{\gamma_t}(t), Y_{\gamma_t}^u(t))\widetilde{y}(t)) \\ &= -\mathbb{E}\int_t^T h_y(s, B_s^{\gamma_t}, Y_{\gamma_t}^u(s), Z_{\gamma_t}^u(s), u(s))\widetilde{y}(s)ds \\ &\quad - \mathbb{E}\int_t^T h_z(s, B_s^{\gamma_t}, Y_{\gamma_t}^u(s), Z_{\gamma_t}^u(s), u(s))\widetilde{z}(s)ds \\ &\quad + \mathbb{E}\int_t^T [p(s) f_v(s, B_s^{\gamma_t}, Y_{\gamma_t}^u(s), Z_{\gamma_t}^u(s), u(s))](v(s) - u(s))ds. \end{aligned}$$

Finally, we can rewrite (18) as

$$0 \leq \mathbb{E}\int_t^T [H_v(s, B_s^{\gamma_t}, Y_{\gamma_t}^u(s), Z_{\gamma_t}^u(s), p(s), u(s))(v(s) - u(s))]ds, \tag{22}$$

where the Hamiltonian H is defined from $[0,T] \times \Lambda \times \mathbb{R}^n \times \mathcal{M}_{m\times d}(\mathbb{R}) \times \mathbb{R}^n \times U$ into $\mathbb{R}$ by

$$\begin{aligned} &H(s, B_s^{\gamma_t}, Y_{\gamma_t}^u(s), Z_{\gamma_t}^u(s), p(s), u(s)) \\ &= \quad f(s, B_s^{\gamma_t}, Y_{\gamma_t}^u(s), Z_{\gamma_t}^u(s), u(s))p(s) \\ &\quad + h(s, B_s^{\gamma_t}, Y_{\gamma_t}^u(s), Z_{\gamma_t}^u(s), u(s)). \end{aligned} \tag{23}$$

From the above variational inequality, we can straightforwardly derive the necessary conditions for optimality.

Theorem 5.1. (The necessary condition of optimality) *Let u be an optimal control minimizing the functional J over U and* $(Y^u_{\gamma_t}(s), Z^u_{\gamma_t}(s))$ *denotes corresponding optimal trajectory. Then there are two unique adapted processes* $p \in L^2_{\mathcal{F}}([0,T];\mathbb{R}^n)$, *which are respectively solutions of the stochastic differential equation* (20) *such that a.e., a.s., we have*

$$0 \leq \mathbb{E}\int_t^T H_v(s, B^{\gamma_t}_s, Y^u_{\gamma_t}(s), Z^u_{\gamma_t}(s), p(s), u(s))(v(s) - u(s))ds.$$

Proof. The prove flows directly from (22). ■

5.1. **Sufficient optimality conditions**

In this subsection, we study when the necessary optimality conditions (22) become sufficient. For any $v \in \mathcal{U}$, we denote by $(Y^v_{\gamma_t}(s), Z^v_{\gamma_t}(s))$ the solution of equation (1) controlled by v, to state the following result.

Theorem 5.2. (Sufficient optimality conditions) *Assume that the functions* $g(\gamma, y)$, *and* $(Y^v_{\gamma_t}(s), Z^v_{\gamma_t}(s)) \mapsto H(s, \gamma, Y^v_{\gamma_t}(s), Z^v_{\gamma_t}(s), p(s), v(s))$ *are convex, and for any* $v \in U$, $y^v(T) = 0$ *is an m –dimensional* F_t *–measurable random variable such that* $E|\xi|^2 < \infty$. *Then, u is an optimal solution of the control problem* $\{(1),(2),(3)\}$, *if it satisfies* (22).

Proof. Let u_2 be an arbitrary element of $\mathcal{U}$ (candidate to be optimal). For any $u_1 \in \mathcal{U}$, we have

$$\begin{aligned} &J(u_1) - J(u_2) \\ &= \mathbb{E}[g(B^{\gamma_t}(t), Y^{u_1}_{\gamma_t}(t)) - g(B^{\gamma_t}(t), Y^{u_2}_{\gamma_t}(t))] \\ &+ \mathbb{E}\int_t^T [h(s, B^{\gamma_t}_s, Y^{u_1}_{\gamma_t}(s), Z^{u_1}_{\gamma_t}(s), u_1(s)) - h(s, B^{\gamma_t}_s, Y^{u_2}_{\gamma_t}(s), Z^{u_2}_{\gamma_t}(s), u_2(s))]ds. \end{aligned}$$

Since $g(\gamma, y)$ is convex with respect to y, then

$$g(B^{\gamma_t}(t), Y^{u_1}_{\gamma_t}(t)) - g(B^{\gamma_t}(t), Y^{u_2}_{\gamma_t}(t)) \geq g(B^{\gamma_t}(t), Y^{u_2}_{\gamma_t}(t))(Y^{u_1}_{\gamma_t}(t) - Y^{u_2}_{\gamma_t}(t)).$$

Thus

$$\begin{aligned} &J(u_1) - J(u_2) \\ &= \mathbb{E}[g(B^{\gamma_t}(t), Y^{u_2}_{\gamma_t}(t))(Y^{u_1}_{\gamma_t}(t) - Y^{u_2}_{\gamma_t}(t))] \\ &+ \mathbb{E}\int_t^T [h(s, B^{\gamma_t}_s, Y^{u_1}_{\gamma_t}(s), Z^{u_1}_{\gamma_t}(s), u_1(s)) - h(s, B^{\gamma_t}_s, Y^{u_2}_{\gamma_t}(s), Z^{u_2}_{\gamma_t}(s), u_2(s))]ds. \end{aligned}$$

It follows from (20) that $p(t) = g_y(B^{\gamma_t}(t), Y^u_{\gamma_t}(t))$. Then we have

$$\begin{aligned} &J(u_1) - J(u_2) \\ &\geq \mathbb{E}[p(t)(Y^{u_1}_{\gamma_t}(t) - Y^{u_2}_{\gamma_t}(t))] \\ &+ \mathbb{E}\int_t^T [h(s, B^{\gamma_t}_s, Y^{u_1}_{\gamma_t}(s), Z^{u_1}_{\gamma_t}(s), u_1(s)) - h(s, B^{\gamma_t}_s, Y^{u_2}_{\gamma_t}(s), Z^{u_2}_{\gamma_t}(s), u_2(s))]ds. \end{aligned}$$

Applying Itô’s formula to $p(s)(Y^{u_1}_{\gamma_t}(s) - Y^{u_2}_{\gamma_t}(s))$ leads to

$$
\begin{aligned}
&\mathbb{E}[p(t)(Y^{u_1}_{\gamma t}(t)-Y^{u_2}_{\gamma t}(t))]\\
&=\mathbb{E}[p(T)(Y^{u_1}_{\gamma t}(T)-Y^{u_2}_{\gamma t}(T))]\\
&-\mathbb{E}\int_t^T H_y(s,B^{\gamma_t}_s,Y^u_{\gamma_t}(s),Z^u_{\gamma_t}(s),p(s),u_2(s))(Y^{u_1}_{\gamma t}(t)-Y^{u_2}_{\gamma t}(t))dt\\
&-\mathbb{E}\int_t^T H_z(s,B^{\gamma_t}_s,Y^u_{\gamma_t}(s),Z^u_{\gamma_t}(s),p(s),u_2(s))(Z^{u_1}_{\gamma t}(t)-Z^{u_2}_{\gamma t}(t))dt\\
&+\mathbb{E}\int_t^T p(s)[f(s,B^{\gamma_t}_s,Y^{u_1}_{\gamma t}(s),Z^{u_1}_{\gamma t}(s),u_1(s))-f(s,B^{\gamma_t}_s,Y^{u_2}_{\gamma t}(s),Z^{u_2}_{\gamma t}(s),u_2(s))]ds.
\end{aligned}
$$

Then

$$
\begin{aligned}
&J(u_1)-J(u_2)\geq\mathbb{E}\int_0^T[H(s,B^{\gamma_t}_s,Y^u_{\gamma_t}(s),Z^u_{\gamma_t}(s),p(s),u_1(s))\\
&-H(s,B^{\gamma_t}_s,Y^u_{\gamma_t}(s),Z^u_{\gamma_t}(s),p(s),u_2(s))]ds\\
&-\mathbb{E}\int_t^T H_y(s,B^{\gamma_t}_s,Y^u_{\gamma_t}(s),Z^u_{\gamma_t}(s),p(s),u_2(s))(Y^{u_1}_{\gamma t}(s)-Y^{u_2}_{\gamma t}(s))ds\\
&-\mathbb{E}\int_t^T H_z(s,B^{\gamma_t}_s,Y^u_{\gamma_t}(s),Z^u_{\gamma_t}(s),p(s),u_2(s))(Z^{u_1}_{\gamma t}(s)-Z^{u_2}_{\gamma t}(s))ds. \qquad (24)
\end{aligned}
$$

Since H is convex in $(Y_{\gamma_t}(s),Z_{\gamma_t}(s))$, then by using the Clarke generalized gradient of H evaluated at $(Y^{u_1}_{\gamma t}(s),Z^{u_1}_{\gamma t}(s),u)$ and the necessary optimality conditions we arrive at

$$
\begin{aligned}
&[H(s,B^{\gamma_t}_s,Y^u_{\gamma_t}(s),Z^u_{\gamma_t}(s),p(s),u_1(s))\\
&-H(s,B^{\gamma_t}_s,Y^u_{\gamma_t}(s),Z^u_{\gamma_t}(s),p(s),u_2(s))]ds\\
&\geq H_y(s,B^{\gamma_t}_s,Y^u_{\gamma_t}(s),Z^u_{\gamma_t}(s),p(s),u_2(s))(Y^{u_1}_{\gamma t}(s)-Y^{u_2}_{\gamma t}(s))ds\\
&+H_z(s,B^{\gamma_t}_s,Y^u_{\gamma_t}(s),Z^u_{\gamma_t}(s),p(s),u_2(s))(Z^{u_1}_{\gamma t}(s)-Z^{u_2}_{\gamma t}(s))ds,
\end{aligned}
$$

or equivalently

$$
\begin{aligned}
&0\leq\mathbb{E}\int_0^T[H(s,B^{\gamma_t}_s,Y^u_{\gamma_t}(s),Z^u_{\gamma_t}(s),p(s),u_1(s))\\
&\quad-H(s,B^{\gamma_t}_s,Y^u_{\gamma_t}(s),Z^u_{\gamma_t}(s),p(s),u_2(s))\Big]ds\\
&\quad-\mathbb{E}\int_t^T H_y(s,B^{\gamma_t}_s,Y^u_{\gamma_t}(s),Z^u_{\gamma_t}(s),p(s),u_2(s))(Y^{u_1}_{\gamma t}(s)-Y^{u_2}_{\gamma t}(s))ds\\
&\quad-\mathbb{E}\int_t^T H_z(s,B^{\gamma_t}_s,Y^u_{\gamma_t}(s),Z^u_{\gamma_t}(s),p(s),u_2(s))(Z^{u_1}_{\gamma t}(s)-Z^{u_2}_{\gamma t}(s))ds.
\end{aligned}
$$

Then from (24), we get $J(u_1)-J(u_2)\geq 0$. Here the proof completes. ■

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