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On the Optimal Control of a System Governed by a Fractional Brownian Motion via Malliavin Calculus

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Dedication

I dedicate this work to:

the absolute dearest, most beloved and best appreciated couple in my whole life:

my parents: *Saliha and Hachemi,*

my dearest lovely brothers: *Mohamed and Brahim,*

my cutest little stepsisters:*Fatima Zahra and Nour,*

those who shared my journey with me, especially: *Ikram,*

and those who had helped me until I reached this glorious day:

my dearest friend *Brahim* and my colleagues: *Meriyam and Sabrina.*

Tayeb, November 2022 . . .

Abstract

In this thesis, we use the Malliavin calculus to derive the Pontryagin's stochastic maximum principle under the form of necessary and sufficient optimality conditions. In the introductory chapter 1, we state and build the framework that we use in the following chapters. We introduce the necessary tools from the Malliavin calculus, the Russo & Vallois integral, and apply the Doss-Sussmann transformation to our system, which is governed by backward doubly stochastic dynamics driven by standard Wiener and fractional Brownian motions. At the end of this chapter, we present important Girsanov theorems and uniqueness and existence result. In chapter 2, we derive the Pontryagin stochastic maximum principle for a system driven by standard and fractional Brownian motions, with Hurst parameter $H \in \left(\frac{1}{2}, 1\right)$. In chapter 3, we solve a stochastic optimization problem for backward stochastic differential equations driven by fractional Brownian motions, using the Malliavin calculus, where we minimize the cost functional, which is in the risk-sensitive type, with respect to the admissible control. In addition, we present the necessary and sufficient optimality conditions for this problem. Finally, we apply the pre-established results to an interesting linear-quadratic control problem.

Our work is considered an extension of the approaches of Buckdahn et al. in [12, 13] and Zähle in [62, 63] and the risk neutral stochastic maximum principle established by Yong in [61] to backward stochastic differential equations driven by fractional Brownian motions.

Keywords. Stochastic maximum principle, fractional Brownian motion, Malliavin derivative, risk-sensitive, variational equality, Doss-Sussmann transformation.

Résumé

Dans cette thèse, on utilise le calcul de Malliavin pour ètablir le principe de maximum stochastique de Pontryagin, sous la forme de conditions nécessaires et suffisantes d'optimalité. On la commence par un chapitre introductif 1, où on construit la machinerie qu'on utilise dans les chapitres suivants. On introduit les outils nécessaires du calcul de Malliavin, l'intégrale de Russo et Vallois, et on applique la transformation de Doss-Sussmann au notre système, qui est gouverné par une EDSDR dirigée par des mouvements Browniens standard et fractionnaire. Dans le chapitre 2, on dérive le principe de maximum stochastique pour un système dirigé par une EDSR gouvernée par un mouvement Brownien standard. On établit une égalité variationnelle et des conditions nécessaires d'optimalité. A la fin de ce chapitre, on présente quelques théorèmes de Girsanov pour les mouvements Browniens fractionnaires et un autre résultat sur l'existence et l'unicité des solutions. A la fin de cette thèse, en chapitre 3, En utilisant le calcul de Malliavin, on résout un problème d'optimisation stochastique pour une classe des équations différentielles stochastiques rétrogrades gouvernées par des mouvements Browniens fractionnaires. On minimise la fonction de coût, qui est à la forme risk-sensible, par rapport aux contrôles addmissibles. De plus, on introduit les conditions nécessaires et suffisantes d'optimalité et on applique la théorie qu'on a déjà construit à un problème de contrôle stochastique de type linéaire quadratique.

Notre travail est une extension des approches de Buckdahn et al. [12, 13] et Zähle [62, 63] et le principe de maximum avec probabilité risk-neutre établi par Yong en [61] aux équations stochastiques rétrogrades dirigées par des mouvements Browniens fractionnaires.

Mots clès. Principe de maximum stochastique, mouvement Brownien fractionnaire, la dérivée de Malliavin, risque sensible, égalité variationnelle, transformation de Doss-Sussmann.

Contents

Symbols and Abbreviations

The main abbreviations and symbols used in this thesis.

Abbreviations

FBM	:	fractional Brownian motion.
$BSDE$	:	Backward stochastic differential equation.
$fBSDE$	:	fractional backward stochastic differential equation.
$BDSDE$	:	Backward doubly stochastic differential equation.
$FBDSDE$	:	fractional backward doubly stochastic differential equation.
$i.e.$	:	Namely or that is.
$a.s.$	:	almost surely.
$a.e.$	:	almost everywhere.
càdlàg	:	continue à droite a limite à gauche.
SMP	:	Stochastic maximum principle.

Symbols

$\mathbb{R}$	:	The set of all real numbers.
$\mathcal{C}([0,T],\mathbb{R})$	:	The space of continuous functions assuming values in $\mathbb{R}$.
$\{\mathcal{F}_t^X\}_{0\leq t\leq T}$	:	The filtration generated by the process X over the time span $[0,T]$.
$(\Omega,\mathcal{F},\mathbb{P})$	:	Probability space.
$\mathbb{D}$	:	The Malliavin derivative.
$\inf g$	:	The infinimum of the functional g.
$L^2(\mathcal{F}_T,\mathbb{P})$	:	The space of $\mathcal{F}_T$-measurable and $\mathbb{P}$-square integrable functions.
$\left(\Omega,\mathcal{F},\mathbb{P},\{\mathcal{F}_t^X\}_{0\leq t\leq T}\right)$	:	Firltered probability space.
$\parallel x \parallel_\Omega$	:	The norm of the process x in the space Ω.
$<,>_\rho$	:	Endowed inner product or ρ-inner product.
$\mathbb{P}$	:	Probability measure.
$\mathbb{E}[.]$	:	Mathematical expectation with repect to the underlying probability measure.
$\widetilde{\mathbb{E}}[.]$	:	Quasi-conditional expectation.
W	:	Standard Wiener motion.
B^H	:	Fractional Brownian motion with *Hurst* parameter H
$\mathcal{N}$	:	The totality of null sets with respect to some probability measure.
$J(.)$	:	Risk-neutral cost functional.
δ_{ij}	:	Kronecker's symbol : $\begin{cases} 1 \text{ if } i=j, \\ 0 \text{ otherwise.} \end{cases}$
$\mathcal{U}$	:	The set of all admissible controls.
$\blacksquare$	:	End of proof.
$\mathbb{P}\ll\mathbb{Q}$	:	The probability measure $\mathbb{P}$ is absolutely continuous with respect to the measure $\mathbb{Q}$.
$\mathbb{P}\vert_{\mathcal{F}_T}$	:	The ristriction of the measure $\mathbb{P}$ to the Filrtation $\mathcal{F}_T$.
$\mathcal{H}$	:	The risk neutral Hamiltoniaion.
$\mathcal{H}^\theta$	:	The risk sensitive Hamiltonian.

Introduction

Historically, one may claim that the first rigorous appearance of optimization theory took place when Lagrange tried to solve the "*brachistochrone*[1] *problem*" proposed by Newton in 1699. Lagrange published two papers [35, 36]. The first one: "*Essai d'une nouvelle methode pour determiner les maxima et les minima des formules integrals indefinies*," and the second one: "*Mécanique Analytique*," in 1762 and 1788, respectively. At this time, great mathematicians as Johann Bernoulli, his brother Jacob, Newton, Leibniz, and Euler acclaimed the solution that Lagrange had produced. However, it was Euler, as officially documented, who collaborated with Lagrange. Ever since, the couple *Euler–Lagrange* have been formally considered godfathers[2] *of Variational Calculus.* This theory, ever since, alongside with optimization and optimal control theory have been used to model a variety of sciences which are related directly or indirectly to applied mathematics.

For instance, consider the rush hour in a megalopolis (New York, London or Paris.) It is evident that the daily behaviour (flow) of the passengers on a principal road is unpredictable, yet it is stochastic. However, without any previous study, one can tell that around the rush hour, since every worker is returning back to home, a potential heavy congestion is produced and almost every road is blocked. At this moment, governments should think of minimizing the flow of passengers and regulate it in order to avoid such blocks or overflows. Another interesting phenomena comes from finance. It is widely known that wages in public sectors are regulated and employees tend to draw them on the same day of the month (since wages

[1]Greek: *brachis*= short, *brachiston* = the shortest, *chrone* = time.

[2]Godfather: who started or developed something such as a style of music as the godfather of that thing. Oxford Learner's Dictionary.

are available at a specific time and public sector employees are supposed free at a certain time of the day.) In this case, the probability that a sudden crash would occur in the whole country is likely high. Consequently, if some employees do not get their wages, they are likely to go on a strike, and again the economy would be paralysed. Hence, governments should think of a solution to this cash overflow, bankruptcy or even inflation (e.g. bank accounts are empty or some people can not access or use their money).

In formal and rigorous mathematical framework, there exists mainly two major approaches to solve such problems of optimization (stochastic or deterministic): the programming dynamic principle and the Pontryagin's maximum principle.

The first approach consists of showing that the value functional associated to such problem satisfies a parabolic partial differential equation called: *the Hamilton-Jacobi-Bellman* (HJB PDE) equation of the form:

$$\begin{cases} \dfrac{\partial V}{\partial t} + \inf_{v \in \mathcal{U}} \{L^u V(t,y) + l(t,y,u)\} &= 0, \\ V(T,y) &= g(x), \end{cases}$$

where L^u refers to the infinitesimal generator associated to the diffusion solution to the HJB PDE. In our thesis, we are not concerned with this principle. Nevertheless, we can refer to an interesting pioneering works on this principle as: [2, 3, 16, 32, 41].

In this thesis, we use the second approach: the Pontryagin's[3] stochastic maximum principle to deal with an optimization problem called stochastic control problem for an Itô dynamics driven by both standard and fractional Brownian motions.

In [46, 47], Pardoux and Peng obtained their pivotal existence and uniqueness result for solutions to Backward SDEs and Backward Doubly SDEs. The first appearance of stochastic maximum principle theory was in the series of papers by: Bensoussan [4], Bismut in [7, 8],

[3]After the Soviet Mathematician: *Lev Semenovich Pontryagin* (03/09/1908–03/05/1988, Moscow, Soviet Union), his autobiography can be found in Russian at: `http://ega-math.narod.ru/LSP/book.htm`.

Haussmann [24], and Kushner [34]. In [48], Peng introduced a result on backward stochastic differential equations and applications to optimal control, then in [49], they obtained another important result on the optimal control for forward backward SDEs when the control domain is convex.

We use the Malliavin calculus to realise our objectives. In the first place, *Paul Malliavin* had introduced a proof of the existence and regularity of the density function of random vectors and the ellipticity of Hörmander operators in [38]. He used what became a cornerstone in the calculus named after him: *the integration by parts formula.* Since then, various extensions of this approach have taken place as: [9, 33, 58, 60].

Among these extensions, comes this thesis, where we use the Malliavin calculus to realize two major objectives. The first one [10] is the Pontryagin's stochastic maximum principle for risk neutral cost functional for a backward doubly stochastic differential equation driven by fractional Wiener and standard Brownian motions:

$$-dy_t = f\left(t, y_t, z_t, v_t\right) dt + g\left(y_t\right) dB_t^H - z_t dW_t. \tag{1}$$

The second one [11], is the risk sensitive Pontryagin's stochastic maximum principle for backward stochastic differential equations driven by fractional Wiener motion:

$$-dy_t = g\left(t, y_t, z_t, v_t\right) dt - z_t dB_t^H, \tag{2}$$

in equations (1) and (2), $v_t = v(t, \omega)$ is a stochastic process assuming values in a non-empty Borel set $U \subset \mathbb{R}$.

The main motivation for using the Malliavin calculus is the lack of the semi-martingale and Markov properties of the fractional Brownian motion when the Hurst parameter is different than one half. The problem that we treat in this thesis consists of making decision depending on what has happened to the state process (y_t, z_t) up to the moment t, in order to choose a suitable process $u_t = u(t, \omega)$, such that

$$J(u) = \inf_{v \in \mathcal{U}} J(v),$$

for a cost (performance) functional J, that we define according to our objective: risk neutral or risk sensitive performance. If the infinumum u_t is attained, i.e. exists, we call it *optimal control*. Since the stochastic term indicates the natural effect of the interference of a randomness on the state process, then such a problem is yet regarded as generalization of the deterministic case (i.e. where the state dynamics is an ordinary differential equation) and it broadens the range of its applications, mainly to finance, energy and physics, among others.

This is a thesis for the degree of *Doctorate* in *Applied Mathematics: Probability*. It is written upon two articles, and it is organized as follows:
In chapter 1, we introduce the main tools and build the general environment that we shall work in all along this thesis. We define the Malliavin calculus with respect to standard and fractional Brownian motions, in particular: the Malliavin derivative, the duality formula and the integration by parts formula. We introduce a brief introduction to stochastic calculus with respect to fractional Brownian motion, with *Hurst* parameter $H \in \left(\frac{1}{2}, 1\right)$, including the Itô-Russo-Vallois stochastic integral with respect to fractional Brownian motion. We finish this chapter by applying the Doss-Sussmann transformation to the underlying backward dynamics and obtaining new backward stochastic differential equation driven only by standard Brownian motion.
Chapter 2 presents the first main result of this thesis: *The Malliavin calculus used to derive Pontryagin's stochastic maximum principle for a system driven by fractional Bownian and standard Wiener motions.* We derive a variational equality in the first section, then in the second we derive necessary sufficient optimality conditions. At last, we present the Girsanov measure changing theorems to cover fractional calculus and we introduce a pioneering existence and uniqueness result obtained by Hu and Peng in [29].
Chapter 3 presents the second main result of this thesis: *Pontryagin's Risk-Sensitive Stochastic Maximum Principle for fractional Backward Stochastic Differential Equations Via Malliavin Calculus.* We optimize a risk-sensitive cost functional for a system driven by backward SDE

governed by fractional Brownian motion. We use an auxiliary result of Young [61], and derive the necessary and sufficient optimality conditions. At last, we apply the pre-introduced theory to an interesting linear quadratic example.

Chapter 1

Introduction to Malliavin Calculus

In this auxiliary chapter, upon two major sections, we build the framework that we work on all along this thesis. In the first section 1.1, we introduce the Malliavin calculus for standard Wiener motions and define the Malliavin derivative. In the second one 1.2, we introduce the fractional Brownian motion and extend the Malliavin theory from the first section to cover fractional calculus. Finally, we define an Itô stochastic integral called the Russo & Vallois integral, introduce the Doss-Sussmann transformation and give a uniqueness and existence result for a backward stochastic dynamics. Among others, the main references of this chapter are [6, 27, 42, 43, 44, 45].

Statement of the Problem

Let $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right) := \left(\Omega_1, \mathcal{F}^W, \left(\mathcal{F}_t^W\right)_{t\geq 0}, \mathbb{P}_1\right) \otimes \left(\Omega_2, \mathcal{F}^B, \left(\mathcal{F}_t^B\right)_{t\geq 0}, \mathbb{P}_2\right)$ be a filtered probability space, satisfying the usual conditions. $\Omega_1 := \mathcal{C}_0\left([0,T], \mathbb{R}\right)$ and $\Omega_2 := \mathcal{C}\left([0,T], \mathbb{R}\right)$ are two classical Wiener spaces endowed with the topology of uniform convergence, with time horizon $T > 0$. On Ω_1 we define one-dimensional standard Brownian motion $W := \{W_s(\omega), t \leq s \leq T\}$, and on Ω_2 we define a fractional Brownian motion $B^H := \left\{B_s^H(\omega), t \leq s \leq T\right\}$, with Hurst parameter $H \in \left(\frac{1}{2}, 1\right)$. We suppose that $\mathcal{G}_t^{(B,W)}$ is the $\mathbb{P}$-augmentation of the natural sub-filtrations of W and B^H defined for each $t \in [0,T]$ by

$$\begin{cases} \mathcal{G}_t^{(B,W)} &:= \mathcal{F}_{[t,T]}^W \vee \mathcal{F}_t^{B^H}, \\ \mathcal{F}_{[t,T]}^W &:= \sigma\left[W(T) - W(s); t \leq s \leq T\right] \vee \mathcal{N}, \\ \text{and} \quad \mathcal{F}_t^{B^H} &:= \sigma\left[B^H(s); 0 \leq s \leq t\right] \vee \mathcal{N}, \end{cases}$$

respectively, where $\mathcal{N}$ denotes the totality of $\mathbb{P}$-null sets and $\sigma_1 \vee \sigma_2$ denotes the σ-fields generated by $\sigma_1 \cup \sigma_2$. Note that the collection $\left\{\mathcal{G}_t^{(B,W)}\right\}$ is neither increasing nor decreasing and it does not constitute a classical filtration. Thus, we introduce the backward filtrations

$$\begin{cases} \mathbb{H} &:= \left(\mathcal{Q}_t^{(B,W)}\right)_{t\in[0,T]}, \\ \text{where} \quad \mathcal{Q}_t^{(B,W)} &:= \mathcal{F}_{[t,T]}^W \vee \mathcal{F}_T^{B^H}, \text{ for all } t \in [0,T], \\ \text{and} \quad \mathbb{F} &:= \left(\mathcal{F}_{[t,T]}^W\right)_{t\in[0,T]} \end{cases}$$

All along this thesis, we shall work in the following spaces.

1. $\mathcal{C}\left([0,T];\mathbb{R}\right) := \{\text{Continuous and } \mathbb{H}\text{-adapted processes } \xi := \{\xi_t(\omega)\}.\}$
2. $\mathcal{M}^2\left([0,T];\mathbb{R}\right) := \left\{\mathbb{F}\text{-adapted processes } \xi, \text{ such that } \mathbb{E}\left[\int_0^T |\xi_t|^2\, dt\right] < \infty.\right\}$
3. $\mathcal{H}_T^\infty(\mathbb{R}) := \{\ \xi : \mathbb{H}$-progressively measurable, such that there exists $\mathcal{F}_T^{B^H}$-measurable random variable $\zeta := \zeta(\omega)$ bounding ξ almost surely.$\}$
4. $\mathcal{H}_T^2(\mathbb{R}) := \{\ \xi$: real valued $\mathbb{H}$-progressively measurable processes, such that $\mathbb{E}\left[\int_0^T |\xi_t|^2\, dt \,\middle|\, \mathcal{F}_T^{B^H}\right] < \infty$. $\mathbb{P}$-a.s. $\}$

1.1 Malliavin Calculus with Respect to $W(.)$

For more details on the Malliavin Calculus with respect to standard Brownian motion W, may the reader consult [44, 45].

At the outset, we start by the Wiener-Itô chaos expansion. Such famous expansion is a pivotal tool in stochastic analysis, in particular, the Malliavin calculus. In 1938, Wiener proved its first version. Later in 1951, Itô showed that in the Wiener space setting the expansion could be expressed in terms of iterated Itô integrals.

Theorem 1.1.1 (The Wiener-Itô Expansion) *For all $\mathcal{F}_T$-measurable and square integrable random variable G: $G \in L^2\left(\mathcal{F}_T^{(W)}, \mathbb{P}_1\right)$, we have the following decomposition*

$$G = \sum_{n\geq 0} I_n(g_n), \tag{1.1.1}$$

for a unique sequence of symmetric deterministic functions $g_n \in L^2(\lambda^n)$, where λ is the Lebesgue measure on $[0, T]$ and

$$I_n(g_n) = n! \int_0^T \int_0^{t_n} \int_0^{t_{n-1}} \dots \int_0^{t_2} g_n(t_1, \dots, t_n)\, dW(t_1) \dots dW(t_n),$$

is viewed as the n-times iterated integral of g_n with respect to $W(.)$, for $n = 1, 2, \dots$ and $I_0(g_0) = g_0$ is a constant. In addition, we have the following isometry

$$\mathbb{E}\left(G^2\right) = \|G\|^2_{L^2(\mathbb{P}_1)} = \sum_{n\geq 0} n!\, \|g_n\|^2_{L^2(\lambda^n)}.$$

Proof. The proof of this theorem is based on the Itô representation theorem of an $\mathcal{F}_T$-measurable and square integrable random variable, for the details we refer to [45]. ■

Definition 1.1.1 (Malliavin Derivative) *We define the subspace of all $\mathcal{F}_T$-measurable and square integrable random variables satisfying*

$$\|G\|^2_{\mathcal{D}_{1,2}^{(W)}} := \sum_{n\geq 0} n.n!\, \|g_n\|^2_{L^2(\lambda^n)} < \infty, \tag{1.1.2}$$

and we denote it $\mathcal{D}_{1,2}^{(W)}$. *For all* $G \in \mathcal{D}_{1,2}^{(W)}$ *and* $t \in [0,T]$, *we define the Malliavin derivative* $D_t G$ *of* G *at point* t *with respect to* $W(.)$ *by*

$$D_t G = \sum_{n \geq 1} n I_{n-1}\left(g_n(.,t)\right),$$

where we keep the last variable $t_n = t$ *and the notation* $I_{n-1}\left(g_n(.,t)\right)$ *stands for applying n-1 times iterated integral to the first n-1 variables* $t_1, t_2, ..., t_{n-1}$ *of* $g_n(t_1, ..., t_n)$, *and we have the isometry*

$$\mathbb{E}\left[\int_0^T \left(D_t G\right)^2 dt\right] = \sum_{n \geq 1} n.n! \left\|g_n\right\|^2_{L^2(\lambda^n)} = \left\|G\right\|^2_{\mathcal{D}_{1,2}^{(W)}},$$

then the mapping $(t,w) \longmapsto D_t G(w)$ belongs to $L^2(\lambda \otimes \mathbb{P}_1)$.

The next two theorems concern the Malliavin derivative D_t. They provide us with smooth and fine properties that we use in the next chapters.

Theorem 1.1.2 (Chain Rule) *Consider a sequence from* $\mathcal{D}_{1,2}^{(W)}$: $\zeta_1, \zeta_2, ..., \zeta_m$ *and a real valued functional with continuous and bounded partial derivatives* $\Psi : \mathbb{R}^m \to \mathbb{R}$, $m \in \mathbb{N}^*$, *then*

$$\Psi(\zeta_1, ..., \zeta_m) \in \mathcal{D}_{1,2}^{(W)} \text{ and } D_t \Psi(\zeta_1, ..., \zeta_m) := \sum_{i=1}^{m} \frac{\partial \Psi}{\partial x_i}(\zeta_1, ..., \zeta_m) D_t \zeta_i. \tag{1.1.3}$$

Proof. See [44]. ■

Theorem 1.1.3 (Duality Formula) *Let* G *be in* $\mathcal{D}_{1,2}^{(W)}$ *and* $y(t)$ *be an* $\mathcal{F}_t^{(W)}$*-adapted process with*

$$\mathbb{E}\left[\int_0^T y^2(t)\, dt\right] < \infty.$$

Then

$$\mathbb{E}\left[G \int_0^T y(t)\, dW(t)\right] = \mathbb{E}\left[\int_0^T y(t)\, D_t G dt\right]. \tag{1.1.4}$$

Proof. See [44]. ■

1.2 Malliavin Calculus with Respect to $B^H(.)$

To the best of our knowledge, the most distinct references that the reader may use are the books of Biagini et al. [6] and Hu [27].

The existence of the fractional Brownian motion (FBM in short) follows from the general existence theorem of centered Gaussian processes with given covariance functions.
The FBM is divided into three different families corresponding to $0 < H < 1/2$, $H = 1/2$, and $1/2 < H < 1$. This last case is of concern to us in our work. It was B. Mandelbrot that named the parameter H of B^H after the British hydrologist *Harold Edwin Hurst*, who made a statistical study of yearly water run-offs of the Nile river. He considered the values $\delta_1, ..., \delta_n$ of n successive yearly run-offs and their corresponding cumulative value $\Delta_n = \sum_{i=1}^{n} \delta_i$ over the period from the year 662 until 1469. He discovered that the behavior of the normalized values of the amplitude of the deviation from the empirical mean was approximately cn^H, where $H = 0.7$. Moreover, the distribution of $\Delta_n = \sum_{i=1}^{n} \delta_i$ was approximately the same as $n^H\delta_1$, with $H > 1/2$. Hence, this phenomenon could not be modeled by using a process with independent increments, but rather the δ_i could be thought as the increments of a FBM. Because of this study, Mandelbrot introduced the name *Hurst index* (see [30, 50]).
Recently, there have been several approaches introducing integral representation for the fractional Brownian motion. We believe that the most suitable one for our work is the one introduced by Norros et al. in [42]. For more details on such approaches one can refer to [17, 39, 43, 55, 56] and the references therein.

1.2.1 Fractional Calculus

By the approach of Norros et al. in [42], for a canonical Wiener process B^0 defined on Ω_2, we have the integral representation

$$B_t^H = \int_0^t \kappa_H(t,s)\, dB_s^0,$$

where κ_H is the kernel of B^H defined as

$$\kappa_H(t,s) := \alpha_H s^{\left(\frac{1}{2}-H\right)} \int_s^t r^{\left(H-\frac{1}{2}\right)} (r-s)^{\left(H-\frac{3}{2}\right)}\, dr, \text{ with } \alpha_H := \sqrt{\frac{H(2H-1)}{\beta\left(2-2H, H-\frac{1}{2}\right)}},$$

and β is the beta function given by $\beta(a,b) := \int_0^1 r^{a-1}(1-r)^{b-1}\,dr, a, b \in [0,1]$.
By definition, B^H is a Gaussian process with zero mean and covariance function for all s,t$\in [0,T]$: $R_H(t,s) := \mathbb{E}\left(B_t^H B_s^H\right) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right)$.
We define the Hilbert space

$$\mathcal{L}_\rho^2(\mathbb{R}) := \left\{ f : \mathbb{R} \to \mathbb{R} / \ ||f||_\rho^2 := \int_0^T \int_0^T \ f(u)\, f(v)\, \rho(u,v)\, dudv < +\infty \right\},$$

endowed with the inner product

$$< f, g >_\rho := \int_0^T \int_0^T \ f(u)\, g(v)\, \rho(u,v)\, dudv,$$

for all $f,\ g \in \mathcal{L}_\rho^2(\mathbb{R})$, where for all $(u,v) \in [0,T]^2$, we have $\rho(u,v) = H(2H-1)|u-v|^{2H-2}$.

Definition 1.2.1 *Let $\mathcal{P}_T$ be the subspace of $\mathcal{L}_\rho^2(\mathbb{R})$, namely the set of all functions Θ of the form*

$$\Theta := f\left(\int_0^T \ \gamma_1(t) dB_t^H, \int_0^T \ \gamma_2(t) dB_t^H, ..., \int_0^T \ \gamma_m(t) dB_t^H\right), \tag{1.2.1}$$

where f is polynomial of m variables and $(\gamma_i)_{0 \le i \le m}$ is an orthogonal sequence of $\mathcal{L}_\rho^2(\mathbb{R})$, namely

$$< \gamma_i, \gamma_j >_\rho = \delta_{ij} ||\gamma_i||_\rho ||\gamma_j||_\rho, \textit{ for all } i, j = \overline{1,m}.$$

The Malliavin derivative of such a random variable $\Theta \in \mathcal{P}_T$ is defined for all $s \in [0,T]$

$$D_s^H \Theta := \sum_{i=1}^m \frac{\partial f}{\partial x_i}\left(\int_0^T \ \gamma_1(t) dB_t^H, \int_0^T \ \gamma_2(t) dB_t^H, ..., \int_0^T \ \gamma_n(t) dB_t^H\right)\gamma_i(s). \tag{1.2.2}$$

Definition 1.2.2 *Let $\mathbb{D}_{1,2}$ be the Banach space which is the completion of $\mathcal{P}_T$ with respect to the norm*

$$||\Theta||_{1,2}^2 := \mathbb{E}\left(||\Theta||_\rho^2 + ||D_t^H \Theta||_\rho^2\right),$$

and we define the derivative of all $\Theta \in \mathcal{P}_T$

$$\mathbb{D}_s^H \Theta := \int_0^T \ \rho(s,v)\, D_v^H \Theta dv, s \in [0,T]. \tag{1.2.3}$$

Definition 1.2.3 (Fractional Conditional Expectation)

1. *Let* $G = \sum_{n\geq 0} I_n^B(\bar{g}_n) \in \mathcal{D}_{1,2}^{(B)}$. *Then we define the fractional (or quasi-) conditional expectation of* G *with respect to* $\mathcal{F}_t^B$ *by*

$$\widetilde{\mathbb{E}}\left[G|\mathcal{F}_t^B\right] := \sum_{n=0}^{\infty} I_n^B\left(\bar{g}_n(s)\mathbb{1}_{0\leq s\leq t}\right) \tag{1.2.4}$$

for $t \in [0,T]$ *and a unique sequence of symmetric deterministic functions* $\bar{g}_n \in L^2(\lambda^n)$, *where* λ *is the Lebesgue measure on* $[0,T]$ *and*

$$I_n^B(\bar{g}_n) = n!\int_0^T \int_0^{t_n} \int_0^{t_{n-1}} ... \int_0^{t_2} \bar{g}_n(t_1, ..., t_n)\, dB^H(t_1)\, ... dB^H(t_n),$$

is viewed as the n-times iterated integral of $\bar{g}_n$ *with respect to* $B(.)$, *for* $n = 1, 2, ...$

2. *We say that* G *is* $\mathcal{F}_t^B$*-measurable if* $\widetilde{\mathbb{E}}\left[G|\mathcal{F}_t^B\right] = G,\ t \geq 0.$

Remark 1.1 *The fractional conditional expectation* $\widetilde{E}$ *is different from the ordinary expectation.*

Definition 1.2.4 (Quasi-Martingale) *The process* Y_t *is called an* $\mathbb{F}$*-quasi-martingale if there exists an* $\mathbb{F}$*-martingale process* $\{Y_1(t,\omega)\}$ *and a process* $\{Y_2(t,\omega)\}$ *with a.e. path function* $\{Y_2(.,\omega)\}$ *of bounded variations on* $[0,T]$*, such that* $\mathbb{P}(\mathcal{A}) = 1$*; where*

$$\mathcal{A} := \{Y(t) = Y_1(t) + Y_2(t); t \in [0,T]\} \subset \Omega,$$

given $\mathcal{A}$ *is a measurable subset.*

We introduce the following generalizations of the famous Itô and integration by parts formulas, respectively.

Theorem 1.2.1 *Let* $\pi \in \mathcal{C}^{1,2}([0,T]\times\mathbb{R})$ *and* $u, v : [0,T] \to \mathbb{R}$ *be two deterministic continuous functions such that* $u \in \mathcal{L}_\rho^2(\mathbb{R})$ *and* $\int_0^T |v(s)|ds < +\infty$. *We suppose that* $\|u(t)\|_\rho^2$ *is continuously differentiable as a function of* $t \in [0,T]$. *We denote*

$$\Xi_t = \Xi_0 + \int_0^t v(s)ds + \int_0^t u(s)dW_s^H,$$

where Ξ_0 *is a constant, then for all* $t \in [0,T]$ *we have*

$$\begin{aligned}\pi(t,\Xi_t) &= \pi(0,\Xi_0) + \int_0^t \frac{\partial \pi}{\partial s}(s,\Xi_s)ds \\ &+ \int_0^t \frac{\partial \pi}{\partial x}(s,\Xi_s)d\Xi_s + \frac{1}{2}\int_0^t \frac{\partial^2 \pi}{\partial x^2}(s,\Xi_s)\frac{d}{ds}||u(s)||_\rho^2 ds,\end{aligned} \tag{1.2.5}$$

where

$$\frac{d}{dt}||u(t)||_\rho^2 := u(t)\int_0^t u(s)\rho(t,s)ds \tag{1.2.6}$$

Proof. See [27]. ■

Theorem 1.2.2 *Let* $\Phi(t) = \int_0^t \varepsilon(u)dW_u^H$. $(\varepsilon(u), 0 \le u \le t)$ *is a stochastic process such that*

$$\mathbb{E}\left[||\varepsilon||_\rho^2 + \int_0^T \int_0^T |D_s^H \varepsilon(t)|^2 dsdt\right] < +\infty,$$

and there exists $\alpha > 1 - H$ *such that*

$$\mathbb{E}|\varepsilon(u) - \varepsilon(v)|^2 \le C|u-v|^{2\alpha},$$

where $|u - v| \le \delta$ *for* $\delta > 0$ *and*

$$\lim_{\substack{|u-v|\to 0\\ 0\le u,v\le t}} \mathbb{E}|\mathbb{D}_u^H\left[\varepsilon(u) - \varepsilon(v)\right]|^2 = 0,$$

D_t^H *is defined by (1.2.2) and* $\mathbb{D}_t^H$ *by (1.2.3). Let* h *be in* $C^{1,2}\left([0,T]\times\mathbb{R}\right)$ *with bounded derivatives. Moreover, we suppose that*

$$\begin{aligned}\mathbb{E}\Bigg[&\int_0^T |\varepsilon(s)\mathbb{D}_s^H\Phi(s)|ds + ||\frac{\partial h}{\partial x}(t,\Phi(t))\varepsilon(t)||_\rho^2 \\ &+ \int_0^T\int_0^T |\mathbb{D}_s^H\left[\frac{\partial h}{\partial x}(t,\Phi(t))\beta(t)\right]|^2 dsdt\Bigg] < +\infty,\end{aligned}$$

then for all $0 \le t \le T$ *we have*

$$h(t,\Phi(t)) = h(0,0) + \int_0^t \frac{\partial h}{\partial s}(s,\Phi(s))\,ds + \int_0^t \frac{\partial h}{\partial x}(s,\Phi(s))\,\varepsilon(s)dW_s^H$$
$$+\int_0^t \frac{\partial^2 h}{\partial x^2}(s,\Phi(s))\,\varepsilon(s)\mathbb{D}_s^H\Phi(s)ds \quad .a.s.$$

Proof. See [27]. ■

Theorem 1.2.3 *Let* χ_2, η_2 *be in* $\mathbb{D}_{1,2}$

$$||\chi_2||_{1,2}^2 = \mathbb{E}\left(||\chi_2||_\rho^2 + ||D_t^H\chi_2||_\rho^2\right) < +\infty \text{ and}$$
$$||\eta_2||_{1,2}^2 = \mathbb{E}\left(||\eta_2||_\rho^2 + ||D_t^H\eta_2||_\rho^2\right) < +\infty,$$

and assume that $\mathbb{D}_t^H\chi_2\,(s)$ *and* $\mathbb{D}_t^H\eta_2\,(s)$ *are continuously differentiable in* $(s,t)\in[0,T]^2$ *for* $\mathbb{P}$*-almost all* $\omega\in\Omega$*. Suppose that*

$$\mathbb{E}\left(\int_0^T\left[|\mathbb{D}_s^H\chi_2(s)|^2 + |\mathbb{D}_s^H\eta_2(s)|^2\right]ds\right) < +\infty,$$

in addition, for $i = 1, 2$,

$$\mathbb{E}\left(\int_0^T\left[|\chi_i(s)|^2 + |\eta_i(s)|^2\right]ds\right) < +\infty.$$

If we put

$$Z(t) = \int_0^t \chi_1(s)ds + \int_0^t \chi_2(s)dW_s^H \text{ and}$$
$$G(t) = \int_0^t \eta_1(s)ds + \int_0^t \eta_2(s)dW_s^H.$$

Then we have

$$(ZG)(t) = \int_0^t Z(s)\eta_1(s)ds + \int_0^t Z(s)\eta_2(s)dW_s^H + \int_0^t G(s)\chi_1(s)ds \tag{1.2.7}$$
$$+\int_0^t G(s)\chi_2(s)dW_s^H + \int_0^t\left[\chi_2(s)\mathbb{D}_s^H Z(s) + \chi_2(s)\mathbb{D}_s^H G(s)\right]ds.$$

Or under differential notation

$$dZ(t) = f_1(t)dt + f_2(s)dB_t^H \text{ and}$$
$$dG(t) = g_1(t)dt + g_2(t)dB_t^H.$$

Then

$$d(ZG)(t) = Z(t)dG(t) + G(t)dZ(t) + \left[\eta_2(t)\mathbb{D}_t^H Z(t) + \chi_2(t)\mathbb{D}_t^H G(t)\right] dt. \tag{1.2.8}$$

Proof. See [27]. ■

1.2.2 The Russo & Vallois Integral

Since the FBM fails the semimartingale property unless $H = \frac{1}{2}$, the classical stochastic Itô integration is no longer applicable to define a stochastic integral with respect to the FBM. Meanwhile, several approaches to give a sense to an integration with respect to the FBM have taken place in the literature, for instance [6, 27] and the references therein. Amongst these approaches, the one so-called: the Russo-Vallois backward stochastic integration introduced by Russo-Vallois in [51, 52], which we use in chapters 2 and 3.

Definition 1.2.5 *Let* $\nabla = \{\nabla(t), t \in [0,1]\}$ *and* $\Delta = \{\Delta(t), t \in [0,1]\}$ *be two continuous stochastic processes at* 0 *and* 1, *we set*

$$I^-(\varepsilon, \nabla, d\Delta) = \int_0^1 \nabla(t) \frac{\Delta[(t+\varepsilon) \wedge 1] - \Delta(t)}{\varepsilon} dt,$$
$$I^+(\varepsilon, \nabla, d\Delta) = \int_0^1 \nabla(t) \frac{\Delta(t) - \Delta[(t-\varepsilon) \vee 0]}{\varepsilon} dt,$$
$$\text{and } I^0 = \frac{I^- + I^+}{2}.$$

We define the limits, in probability

$$\lim_{\varepsilon\to 0} I^{-}\left(\varepsilon, \nabla, d\Delta\right) = \int_0^1 \nabla d^{-}\Delta,$$
$$\lim_{\varepsilon\to 0} I^{+}\left(\varepsilon, \nabla, d\Delta\right) = \int_0^1 \nabla d^{+}\Delta,$$
$$\textit{and } \lim_{\varepsilon\to 0} I^{0} = \frac{1}{2}\{\int_0^1 \nabla d^{-}\Delta + \int_0^1 \nabla d^{+}\Delta\}.$$

These limits are called forward, backward and symmetric, integral of ∇ *with respect to* Δ*, respectively.*

Remark 1.2 *In chapter 2, we shall derive a stochastic maximum principle for a class of backward doubly stochastic differential equations. However, due to the poor properties present by the fractional Brownian motion, we need to proceed by an intermediate step called: Doss-Sussmann transformation, which we explicit in the immediate next paragraph 1.2.3.*

1.2.3 Doss-Sussmann Transformation of Fractional BDSDE

In chapter 2, we study a system governed by backward doubly stochastic differential equation, driven by standard Wiener and fractional Brownian motions

$$\begin{cases} -dy_t &= f\left(t, y_t, z_t, v_t\right) dt + g\left(y_t\right) dB_t^H - z_t dW_t, \\ y_T &= \xi,\ t \in [0,T], \end{cases} \tag{1.2.3.1}$$

associated with the cost functional, with initial cost

$$J\left(v\right) = \mathbb{E}\left[\int_0^T \Pi\left(t, y_t^v, z_t^v, v_t\right) dt + \Psi\left(y^v\left(0\right)\right)\right], \tag{1.2.3.2}$$

which we want to minimize over $\mathcal{U}$, namely, to find the control u such that

$$J\left(u\right) = \inf_{v\in\mathcal{U}} J\left(v\right). \tag{1.2.3.3}$$

In (1.2.3.1), the integral with respect to the Brownian motion W is an Itô backward integral, while the integral with respect to the FBM B^H is in the Russo-Vallois sense, where

$$\begin{aligned} f : [0,T] \times \mathbb{R} \times \mathbb{R} \times U &\rightarrow \mathbb{R}, \\ \Pi : [0,T] \times \mathbb{R} \times \mathbb{R} \times U &\rightarrow \mathbb{R}, \\ \text{and } g, \Psi : \mathbb{R} &\rightarrow \mathbb{R}. \end{aligned}$$

For the well-posedness of the control problem {(1.2.3.1),(1.2.3.2),(1.2.3.3)}, we assume

Assumptions 1.1

1. *The function f is Lipshitz in (y,z).*

2. *For all $t \in [0,T] : |f(t,0,0)| \leq C$ uniformly.*

3. *The function $g : \mathbb{R} \rightarrow \mathbb{R}$ belongs to $\mathcal{C}_b^3([0,T];\mathbb{R})$.*

We define a solution to the system (1.2.3.1).

Definition 1.6 *We say that the couple of processes (y_s, z_s) is a solution of equation (1.2.3.1), if for all $s \in [0,T]$, we have*

1. *$(y_s, z_s) \in \mathcal{H}_T^\infty(\mathbb{R}) \times \mathcal{H}_T^2(\mathbb{R}^d)$.*

2. *The Russo-Vallois integral $\int_0^{\cdot} g(y_t)\, dB_t^H$ is well defined on $[0,T]$.*

3. *The equation (1.2.3.1) holds $\mathbb{P}$-a.s.*

Remark 1.3 *Unfortunately, as we have mentioned in remark 1.2, because of the properties of FBM B^H, the classical methods to solve such problem lead to an unevitable dead-end. However, Buckdahn and Ma introduced a new method to investigate this equation in [12, 13], using the Doss-Sussmann transformation, (may the reader see [62, 63]).*

[1]We define $\eta : [0,T] \times \mathbb{R} \rightarrow \mathbb{R}$ the unique solution of the stochastic flow as

[1]This paragraph is adapted from the paper of Zähle [63].

$$\eta(t,x) := x + \int_0^t g(\eta(s,x))\, dB_s^H,\ t \in [0,T], \tag{1.2.3.4}$$

in (1.2.3.4), the integral is defined in the Russo-Vallois sense.

Via the Doss-Sussmann transformation, the solution of this equation can be written as $\eta(t,x) = \zeta\left(x, B_t^H\right)$, where $(x,z) \longmapsto \zeta(x,z)$ is the solution of the partial differential equation

$$\begin{cases} \dfrac{\partial \zeta}{dz}(x,z) & = & g(\zeta(x,z)),\ \text{for } z \in \mathbb{R}, \\ \zeta(x,0) & = & x. \end{cases} \tag{1.2.3.5}$$

By the classic theory of partial differential equations, we know that for all $z \in \mathbb{R}$, the mapping $x \longmapsto \zeta(x,z)$ is a $\mathcal{C}_b^2$-diffeomorphism over $\mathbb{R}$. Hence, we can define its x-inverse and denote it by $\varphi(x,z)$, that is, $\zeta(\varphi(x,z)) = x$, for all $(x,z) \in \mathbb{R}^2$. By chain rule, It follows that

$$\begin{cases} \dfrac{\partial \zeta}{\partial x}(\varphi(x,z),z) . \dfrac{\partial \varphi}{\partial x}(x,z) & = 1 \\ \dfrac{\partial \zeta}{\partial z}(\varphi(x,z),z) + \dfrac{\partial \zeta}{\partial x}(\varphi(x,z),z)\dfrac{\partial \varphi}{\partial z}(x,z) & = 0. \end{cases} \tag{1.2.3.6}$$

Therefore

$$\begin{cases} \dfrac{\partial \varphi}{\partial z}(x,z) & = & -\dfrac{\partial \zeta}{\partial z}(\varphi(x,z),z)\left(\dfrac{\partial \zeta}{\partial x}(\varphi(x,z),z)\right)^{-1} = -g(x)\dfrac{\partial \varphi}{\partial x}(x,z), \\ \dfrac{\partial^2 \varphi}{\partial x^2}(x,z) & = & \left[\dfrac{\partial \zeta}{\partial x}(x,z)\right]^{-3}\dfrac{\partial^2 \zeta}{\partial x^2}(x,z). \end{cases} \tag{1.2.3.7}$$

Hence, the immediate consequence is that $\eta(t,.) = \zeta\left(., B_t^H\right) : \mathbb{R} \longmapsto \mathbb{R}$ is a $\mathcal{C}_b^2$-diffeomrphism, and we can define

$$\phi(t,x) := \zeta(x,.)^{-1}(x) = \varphi\left(x, B_t^H\right),\ \text{for all } (t,x) \in [0,T] \times \mathbb{R},$$

by the generalized Itô formula (1.2.2), it comes that

$$\phi(t,x) = x - \int_0^t g(x)\frac{\partial \phi}{\partial x}(s,x)\, dB_s^H,\ t \in [0,T].$$

We denote by $\bar{\Omega} := \left\{ w \in \Omega_2 : \sup_{0 \le s \le T} \left| B_s^H \right| < \infty \right\}$, then we have $\mathbb{P}_2\left(\bar{\Omega}\right) = 1$.
We define

$$y_t = \eta(t, Y_t) = \zeta\left(Y_t, B^H\right) \text{ then } Y_t = \phi(t, y_t) = \varphi\left(y_t, B^H\right). \tag{1.2.3.8}$$

Applying the Itô generalized formula (1.2.2) to $\phi(t, y_t)$ and using the identities (1.2.3.6) and (1.2.3.7) yield to

$$\begin{aligned} dY_t &= d\phi(t, y_t) = dy_t - g\left(y_t\right) \frac{\partial \phi}{\partial y}\left(t, y_t\right) dB_t^H \\ &= f\left(t, y_t, z_t\right) \frac{\partial \phi}{\partial y}\left(t, y_t\right) dt + g\left(y_t\right) \frac{\partial \phi}{\partial y}\left(t, y_t\right) dB_t^H - z_t \frac{\partial \phi}{\partial y}\left(t, y_t\right) dW_t \\ &- \frac{\left|z_t\right|^2}{2} \frac{\partial^2 \phi}{\partial y^2}\left(t, y_t\right) dt - g\left(y_t\right) \frac{\partial \phi}{\partial y}\left(t, y_t\right) dB_t^H \\ &= \left\{ f\left(t, y_t, z_t\right) \frac{\partial \phi}{\partial y}\left(t, y_t\right) - \frac{\left|z_t\right|^2}{2} \frac{\partial^2 \phi}{\partial y^2}\left(t, y_t\right) \right\} dt - z_t \frac{\partial \phi}{\partial y}\left(t, y_t\right) dW_t. \end{aligned}$$

If we put $z_t = Z_t \frac{\partial \eta}{\partial y}\left(t, Y_t\right)$, then the couple (Y, Z) satisfies the backward stochastic differential equation

$$\begin{cases} -dY_t &= F(t, Y_t, Z_t, v_t)dt - Z_t dW_t, \\ Y_T &= \xi, \ t \in [0, T], \end{cases} \tag{1.2.3.9}$$

where

$$F\left(t, Y_t, Z_t\right) = \frac{1}{\frac{\partial \eta}{\partial y}\left(t, Y_t\right)} \left\{ f\left(t, \eta\left(t, Y_t\right), \frac{\partial \eta}{\partial y}\left(t, Y_t\right) Z_t, v_t\right) + \frac{1}{2} ||Z_t||^2 \frac{\partial^2 \eta}{\partial y^2}\left(t, Y_t\right) \right\}. \tag{1.2.3.10}$$

Remark 1.4 *The BSDE (1.2.3.9) is driven only by the standard Brownian motion $W_t(\omega)$ for all $t \in [0, T]$. In [31], Kobylanski dealt with the class of functions as the generator $F(.,., Z)$, namely, functions with quadratic growth in the variable Z.*

Existence & Uniqueness Result

In order to assure the existence and uniqueness of the solution to equation (1.2.3.9), the following assumptions are presumed.

Assumptions 1.2.2

1. *For all* (Y,Z) $\mathbb{H}$*-progressively measurable, the function*

$$F:\Omega\times[0,T]\times\mathbb{R}\times\mathbb{R}\rightarrow\mathbb{R}$$

is measurable and continuous in (t,Y,Z).

2. *There exists a real valued and* $\mathcal{F}_T^{B^H}$*-measurable random variable*

$$K:\Omega_2\rightarrow\mathbb{R},\ \textit{such that}\ |F(t,Y,Z)|\leq K\left(1+|Z|^2\right).$$

3. *There exists some positive real valued* $\mathcal{F}_T^{B^H}$*-measurable random variables* C *and* ε*, and* $\mathcal{F}_T^{B^H}\otimes\mathcal{B}([0,T])$*-measurable functions* k, $l_\varepsilon:\Omega_2\times[0,T]\rightarrow\mathbb{R}$*, such that*

$$\left|\frac{\partial F}{\partial y}(t,Y,Z)\right|\leq k(t)+C|Z|\,,\ \textit{for all}\ (t,Y,Z)\,,\ \mathbb{P}-a.s.,$$
$$\left|\frac{\partial F}{\partial z}(t,Y,Z)\right|\leq l_\varepsilon(t)+\varepsilon|Z|^2\,,\ \textit{for all}\ (t,Y,Z)\,,\ \mathbb{P}-a.s.$$

Remarks 1.2.1

1. *The main reason of using a space of a.s. conditionally square integrable processes instead of the space of only square integrable processes is the fact that we may be provided with a.s. bounded conditional expectation of* $\int_0^T|Z_r|^2\,dr$ *by an a.s. finite process, as studied by Kobylanski in [31], since there does not exist a direct way to solve the BDSDE* (1.2.3.1).

2. *Assumptions 1.2.2 impose a quadratic growth on the generator* F *in the variable* Z*, as has been accentuated, such kind of BSDE was studied in [31], where an existence and uniqueness result of the solution to equation (1.2.3.9) was established (see Theorem 2.3*

and Theorem 2.6). We do not detail this result in this thesis. Nevertheless, we shall exaggerate the following assumptions.

Assumptions 1.2.3

1. *f, Π and Ψ are continuously differentiable in (y, z, v).*

2. *The derivatives of f, g and h are bounded by $C\left(1+|y|+|v|+||z||\right)$.*

3. *The function g belongs to $\mathcal{C}_b^3\left(\mathbb{R}\right)$.*

4. *The derivative of Ψ is bounded by $C\left(1+|y|\right)$.*

Given assumptions 1.2.3, for all $v \in \mathcal{U}$ the equation (1.2.3.1) admits a unique strong solution. The following theorem illustrates how we have transformed (1.2.3.1) into only BSDE (1.2.3.9) using the Doss-Sussmann transformation.

Theorem 1.2.4 *The process $(y^v, z^v) \in \mathcal{H}_T^\infty\left(\mathbb{R}\right) \times \mathcal{H}_T^2\left(\mathbb{R}^d\right)$ is the unique solution of (1.2.3.1), then if we put $y_s^v = \eta\left(s, Y_s^v\right)$ and $z_s^v = \dfrac{\partial \eta}{\partial y}\left(s, Y_s^v\right) Z_s^v$, where η is defined by 1.2.3.4, then $(Y^v, Z^v) \in \mathcal{H}_T^\infty\left(\mathbb{R}\right) \times \mathcal{H}_T^2\left(\mathbb{R}^d\right)$ is the unique solution of backward stochastic equation*

$$\begin{cases} -dY_t^v &= F\left(t, Y_t^v, Z_t^v, v_t\right) dt - Z_t^v dW_t, \\ Y_T^v &= \xi, \; t \in [0, T]. \end{cases} \tag{1.2.3.11}$$

Where F (given by (1.2.3.10)) is a continuous function with quadratic growth in Z.

1.3 Girsanov Theorems and Existence and Uniqueness Result for Systems Driven by Fractional Brownian Motions

We introduce Girsanov's theorems and change of probability measures formulas. In addition, we give an existence and uniqueness result for a class of backward stochastic differential equations driven by fractional Brownian motion that serves our goal in chapter 3.

1.3.1 Change of Probability Measures and Girsanov Transformations

Definition 1.3.1 *On $\left(\Omega_2, \mathcal{F}, (\mathcal{F}_t^H)_{0\le t\le T}, \mathbb{P}_2\right)$, we say that the probability measures $\mathbb{P}_2|_{\mathcal{F}_T}$ and $\mathbb{Q}$ are equivalent probability measures on Ω_2 if and only if*

$$\mathbb{P}_2|_{\mathcal{F}_T} \ll \mathbb{Q} \text{ and } \mathbb{Q} \ll \mathbb{P}_2|_{\mathcal{F}_T}, \text{ we write } \mathbb{P}_2|_{\mathcal{F}_T} \sim \mathbb{Q},$$

namely, if $\mathbb{P}_2|_{\mathcal{F}_T}$ and $\mathbb{Q}$ have the same zero sets on $\mathcal{F}_T$.

The following theorems are considered as extensions of the standard changing probability theorems of *I. W. Girsanov* [22].

Theorem 1.3.1 (Fractional Girsanov Formula I) *Let $\gamma \in \mathcal{L}_\rho^2(\mathbb{R}) \cap \mathcal{C}(\mathbb{R})$ and let $\psi \in \mathcal{L}_\rho^2(\mathbb{R})$. Define*

$$\begin{aligned}\tilde{\gamma}(t) &:= \int_{\mathbb{R}} \rho(t,s)\gamma(s)ds,\\ \varepsilon(\gamma) &:= \exp\{\int_{\mathbb{R}} \gamma(s)dB_s^H - \frac{1}{2}||\gamma||_\rho^2\}.\end{aligned} \tag{1.3.1}$$

Then the map $\omega \longmapsto \psi(\omega+\tilde{\gamma})$ belongs to $\mathcal{L}_\rho^2(\mathbb{P}_2)$ and

$$\int_{\Omega_2} \psi(\omega+\tilde{\gamma})d\mathbb{P}_2(\omega) = \int_{\Omega_2} \psi(\omega)\varepsilon(\gamma)d\mathbb{P}_2(\omega),$$

Proof. See [5] ■

Corollary 1.3.1 *Let $g : \mathbb{R} \longmapsto \mathbb{R}$ be bounded and ε as defined in (1.3.1), and $\gamma \in \mathcal{L}_\rho^2(\mathbb{R}) \cap \mathcal{C}(\mathbb{R})$. Then,*

$$\mathbb{E}\left[g(B_t^H + \int_0^t \tilde{\gamma}(s)ds)\right] = \mathbb{E}\left[g(B_t^H \varepsilon(\gamma))\right].$$

Proof. See [5] ■

Theorem 1.3.2 (Fractional Girsanov Formula II) *Let $T \geq 0$ and let γ be a continuous function with supp$\gamma \subset [0,T]$. Let K be a function with supp$K \subset [0,T]$ and such that*

$$< K, f >_{\rho} = < \gamma, f >_{L^2(\mathbb{R})}, \text{ for all } f \in L^2(R), \text{ supp} f \subset [0,T]$$

i.e.

$$\int_{\mathbb{R}} K(s)\rho(t,s)ds = \gamma(t); 0 \leq t \leq T.$$

On the σ-algebra $\mathcal{F}_T^H$ generated by $\left\{B_s^H : 0 \leq s \leq T\right\}$, define a probability measure $\mathbb{P}^{H,\gamma}$ by

$$\frac{\mathbb{P}^{H,\gamma}}{\mathbb{P}_2} := \exp\left\{-\int_0^t K(s)dB_s^H - \frac{1}{2}||K(t)||_{\rho}^2\right\}$$

Then $\hat{B}^H(t) := B_t^H + \int_0^t \gamma(s)ds, 0 \leq t \leq T$, is a fractional Brownian motion under $\mathbb{P}^{H,\gamma}$.

Proof. See [5] ■

1.4 Uniqueness and Existence Result

This section is adapted from the pioneering paper of Hu and Peng [29]. Throughout this section we let

$$\eta_t = \eta_0 + b_t + \int_0^r \sigma_s dB_s^H$$

unless otherwise stated, where η_0 is a given constant, b_t is a deterministic differentiable function of t, and σ_s is a deterministic continuous function such that $||\sigma||_t$ exists for all t and $\frac{d}{dt}(||\sigma||_t)$ exists and it is strictly positive. Set $\xi = g(\eta_T)$ where g is a continuous function. We consider the following backward stochastic differential equation:

$$\begin{cases} dy_t &= -f(t,\eta_t,y_t,z_t)dt - z_t dB_t^H \\ y_T &= \xi. \end{cases} \tag{1.4.1}$$

A solution (pair) to (1.4.1) is given by two adapted processes $((y_t, z_t), \eta_t, 0 \le t \le T)$

$$\begin{cases} \eta_t = \eta_0 + b_t + \int_0^r \sigma_s dB_s^H, \\ y_t = \xi + \int_t^T f(s, \eta_t, y_s, z_s)ds + \int_t^T z_s dB_s^H, 0 \le t \le T. \end{cases} \tag{1.4.2}$$

To find a pair of solutions we consider $u(t, \eta_t)$. Denote $\tilde{\sigma}_t = \dfrac{d}{dt}(||\sigma||_t^2)$. The Itô formula (see Theorem 2.3 in [19] and [27]) yields to

$$\begin{aligned} d\mu(t,\eta_t) &= \frac{\partial \mu}{\partial t}(t,\eta_t)dt + \frac{\partial \mu}{\partial x}(t,\eta_t)\left[b_t' dt + \sigma_t dB_t^H\right] + \frac{1}{2}\tilde{\sigma}_t \frac{\partial^2 \mu}{\partial x^2}(t,\eta_t)dt \\ &= \left[\frac{\partial \mu}{\partial t}(t,\eta_t) + b_t' \frac{\partial \mu}{\partial x}(t,\eta_t) + \frac{1}{2}\tilde{\sigma}_t \frac{\partial^2 \mu}{\partial x^2}(t,\eta_t)\right] dt + \sigma_t \frac{\partial \mu}{\partial x}(t,\eta_t) dB_t^H. \end{aligned}$$

If μ satisfies the quasi-linear partial differential equation (1.4.1) then

$$d\mu(t,\eta_t) = -f(t,\eta_t,\mu(t,\eta_t), -\sigma_t \frac{\partial \mu}{\partial x}(t,\eta_t))dt + \sigma_t \frac{\partial \mu}{\partial x}(t,\eta_t) dB_t^H. \tag{1.4.3}$$

Thus we have the following theorem.

Theorem 1.4.1 *Let $\mu(t,x)$ be the solution of the following PDE*

$$\begin{cases} \dfrac{\partial \mu}{\partial t} &= -\dfrac{1}{2}\dfrac{\tilde{\sigma}\partial^2 \mu}{\partial x^2} - b_t' \dfrac{\partial \mu}{\partial x} - f(t,x,\mu, -\dfrac{\partial \sigma}{\partial t}\dfrac{\partial \mu}{\partial x}) \\ \mu(T,x) &= g(x). \end{cases} \tag{1.4.4}$$

$\mu(t,x)$ is continuously differentiable with respect to t and twice continuously differentiable with respect to x. Then $(y_t, z_t) := (\mu(t,\eta_t), -\sigma_t \frac{\partial \mu}{\partial x}(t,\eta_t))$ satisfies the following backward stochastic differential equation

$$\begin{cases} dy_t &= -f(t,\eta_t,y_t,z_t)dt - z_t dB_t^H \\ y_T &= g(\eta_T). \end{cases} \tag{1.4.5}$$

The following lemma is important in dealing with the existence and uniqueness problem of the backward stochastic differential equation (1.4.1).

Lemma 1.4.1 *Let $b(s,x)$ and $a(s,x), 0 \leq s \leq T, x \in \mathbb{R}$, be continuous with respect to s and continuously differentiable with respect to x and let both of them be of polynomial growth. Let σ be continuous and let $||\sigma||_s$ be an increasing function of s. If*

$$\int_0^t b(s,\eta_s)ds + \int_0^t a(s,\eta_s)dB_s^H = 0, \forall t \in [0,T], \tag{1.4.6}$$

then

$$b(s,x) = a(s,x) = 0, \forall s \in [0,T], x \in \mathbb{R}. \tag{1.4.7}$$

Proof. See [29]. ∎

Proposition 1.4.1 *Let (1.4.1) have a solution of the forms $(y = \mu(t,\eta_t), z = \nu(t,\eta_t))$, where $\mu(t,x)$ is continuously differentiable with respect to t and twice continuously differentiable with respect with to x. Then $-\sigma_t \frac{\partial \mu}{\partial x}(t,x) = \nu(t,x)$.*

Proof. By the Itô formula, we have

$$\begin{aligned} d\mu(t,\eta_t) &= \frac{\partial \mu}{\partial t}(t,\eta_t)dt + \frac{\partial \mu}{\partial x}(t,\eta_t)\left[b_t' dt + \sigma_t dB_t^H\right] + \frac{1}{2}\tilde{\sigma}_t \frac{\partial^2 \mu}{\partial x^2}(t,\eta_t)dt \\ &= \left[\frac{\partial \mu}{\partial t}(t,\eta_t) + b_t' \frac{\partial \mu}{\partial x}(t,\eta_t) + \frac{1}{2}\tilde{\sigma}_t \frac{\partial^2 \mu}{\partial x^2}(t,\eta_t)\right]dt - \sigma_t \frac{\partial \mu}{\partial x}(t,\eta_t)dB_t^H. \end{aligned} \tag{1.4.8}$$

Or we can write

$$\mu(t,\eta_t) = \xi - \int_t^T \left[\frac{\partial \mu}{\partial s}(s,\eta_s) + b_s' \frac{\partial \mu}{\partial x}(s,\eta_s) + \frac{1}{2}\tilde{\sigma}_s \frac{\partial^2 \mu}{\partial x^2}(s,\eta_s)\right]ds - \int_t^T \sigma_s \frac{\partial \mu}{\partial x}(s,\eta_s)dB_s^H. \tag{1.4.9}$$

So

$$\begin{aligned} &- \int_t^T \left[\frac{\partial \mu}{\partial s}(s,\eta_s) + b_s' \frac{\partial \mu}{\partial x}(s,\eta_s) + \frac{1}{2}\tilde{\sigma}_s \frac{\partial^2 \mu}{\partial x^2}(s,\eta_s)\right]ds - \int_t^T \sigma_s \frac{\partial \mu}{\partial x}(s,\eta_s)dB_s^H. \\ &= \int_t^T f(s,\eta_s,\frac{\partial \mu}{\partial s}(s,\eta_s),\nu(s,\eta_s))ds + \int_t^T \nu(s,\eta_s)dB_s^H. \end{aligned}$$

This is also true for $t = 0$, namely,

$$
\begin{aligned}
&- \int_0^T \left[\frac{\partial \mu}{\partial s}(s,\eta_s) + b'_s \frac{\partial \mu}{\partial x}(s,\eta_s) + \frac{1}{2}\tilde{\sigma}_s \frac{\partial^2 \mu}{\partial x^2}(s,\eta_s)\right] ds - \int_0^T \sigma_s \frac{\partial \mu}{\partial x}(s,\eta_s) dB_s^H. \\
&= \int_0^T f(s,\eta_s,\frac{\partial \mu}{\partial s}(s,\eta_s),\nu(s,\eta_s))ds + \int_0^T \nu(s,\eta_s)dB_s^H.
\end{aligned}
$$

Substracting the two equations, we obtain

$$
\begin{aligned}
&- \int_0^t \left[\frac{\partial \mu}{\partial s}(s,\eta_s) + b'_s \frac{\partial \mu}{\partial x}(s,\eta_s) + \frac{1}{2}\tilde{\sigma}_s \frac{\partial^2 \mu}{\partial x^2}(s,\eta_s)\right] ds - \int_0^t \sigma_s \frac{\partial \mu}{\partial x}(s,\eta_s) dB_s^H. \\
&= \int_0^t f(s,\eta_s,\frac{\partial \mu}{\partial s}(s,\eta_s),\nu(s,\eta_s))ds + \int_0^t \nu(s,\eta_s)dB_s^H,
\end{aligned}
$$

for all $0 \leq t \leq T$. From Lemma 1.4.1, we have

$$
\nu(t,x) = -\sigma_t \frac{\partial \mu}{\partial x}(t,x); \forall t \in (0,T), x \in \mathbb{R}. \tag{1.4.10}
$$

This proves the proposition. ■

Remark 1.4.1 *From the above proof, we also see that if the nonlinear differential equation*

$$
\frac{\partial \mu}{\partial t}(t,x) + b'_t \frac{\partial \mu}{\partial x}(t,x) + \frac{1}{2}\tilde{\sigma}\frac{\partial^2 \mu}{\partial x^2}(t,x) + f(t,x,\mu(t,x),-\sigma_t \frac{\partial \mu}{\partial x}(t,x)) = 0 \tag{1.4.11}
$$

has a unique solution, then the backward stochastic differential equation (1.4.1) also has a unique solution.

In chapter 3, we use the machinery that we have established in chapter 1 to derive Pontryagin's risk-sensitive stochastic maximum principle for a fractional backward stochastic differential equation.

Chapter 2

Pontryagin's SMP for a System Driven by Fractional Brownian and Standard Wiener Motions via Malliavin Calculus

This chapter presents the first main result of this thesis: The Malliavin calculus used to derive an SMP for a system driven by fractional and standard Brownian motions. This result is published in [10]. We are at the brink of solving some stochastic optimization problem for a class of backward doubly stochastic differential equations. The following triplet forms our problem: non linear backward doubly stochastic differential equation

$$\begin{cases} -dy_t &= f\left(t, y_t, z_t, v_t\right) dt + g\left(y_t\right) dB_t^H - z_t dW_t, \\ y_T &= \xi, t \in [0, T], \end{cases} \tag{2.1.1}$$

the cost functional, with initial cost

$$J\left(v\right) = \mathbb{E}\left[\int_0^T \Pi\left(t, y_t^v, z_t^v, v_t\right) dt + \Psi\left(y^v\left(0\right)\right)\right], \tag{2.1.2}$$

and we want to find an admissible control $u \in \mathcal{U}$ called *optimal*, that solves

$$J(u) = \inf_{v \in \mathcal{U}} J(v). \tag{2.1.3}$$

In solving the stochastic control problem {(2.1.1),(2.1.2),(2.1.3)}, we establish necessary and sufficient optimality conditions satisfied by some optimal control, in the form of Pontryagin's stochastic maximum principle. However, as has been accentuated in the previous chapter, there does not exist an explicit method to study the BDSDE (2.1.1) using classical methods, what makes us invite the Doss-Sussman transformation introduced in chapter 1, section 1.2.3 and apply it to the triplet {(2.1.1),(2.1.2),(2.1.3)}, what gives us the following new control problem

$$\begin{cases} -dY_t^v &= F(t, Y_t^v, Z_t^v, v_t)\, dt - Z_t^v dW_t, \\ Y_T^v &= \xi, t \in [0,T]. \end{cases} \tag{2.1.4}$$

where F is given by (1.2.3.10), and the cost functional becomes

$$\mathcal{J}(v) = \mathbb{E}\left[\int_0^T \Pi(t, Y_t^v, Z_t^v, v_t)\, dt + \Psi(Y^v(0))\right], \tag{2.1.5}$$

which we aim to minimize over $\mathcal{U}$, *i.e.* find the optimal control $u \in \mathcal{U}$ such that

$$\mathcal{J}(u) = \inf_{v \in \mathcal{U}} \mathcal{J}(v). \tag{2.1.6}$$

Remarks 2.1.1

1. *The transformed cost functional $\mathcal{J}$ defined in (2.1.5) results from injecting the transformation (1.2.3.8) into the cost functional (1.2.3.2) as follows:*

$$\Pi(t, Y_t^v, Z_t^v, v_t) \coloneqq \Pi(t, \phi(t, y_t^v), z_t \frac{\partial \phi}{\partial y}(t, y_t^v), v_t) \text{ and } \Psi(Y^v(0)) \coloneqq \Psi(\phi(0, y_0^v))$$

2. *By a simple verification of the new diver F, we have :*

$$|F(t,Y_t^v,Z_t^v) = \frac{\partial\phi}{\partial y}(t,y_t)\left\{f(t,\eta(t,y_t),Z_t\frac{\partial\eta}{\partial y}(t,y_t),v_t)+\frac{1}{2}Z^2\frac{\partial^2\eta}{\partial y^2}(t,y_t),v_t\right\}|$$
$$\leq \left|C\frac{\partial\phi}{\partial y}(t,y_t)\left\{1+|\eta(t,y_t)|+\left|Z_t\frac{\partial\eta}{\partial y}(t,y_t)\right|+|v_t|+\left|\frac{1}{2}Z^2\frac{\partial^2\eta}{\partial y^2}(t,y_t)\right|\right\}\right|.$$

Then F satisfies assumptions 1.2.2.

2.2 Variational Equality

For more details on the Malliavin derivative of backward stochastic differential equations, may the reader see the paper of El Karoui et al. [21]. In this section we shall derive stochastic maximum principle for the control problem {(2.1.4),(2.1.5),(2.1.6)}. We intoduce the following assumptions and definitions.

2.2.1 Assumptions and Definitions

Definition 2.2.1 *Let U be a non-empty subset of $\mathbb{R}$. An admissible control v is a $\mathcal{G}_t$-adapted process assuming values in U such that $\mathbb{E}\left[\sup_{0\leq t\leq T}|v_t|^2\right]<\infty$. We denote by $\mathcal{U}$ the set of all admissible controls.*

Assumption 2.2.1 *For all $t,r\in[0,T]$, $t\leq r$, and all bounded random variables $\theta=\theta(\omega)$, the control*

$$\gamma_\theta(s)=\theta(w)\mathbb{1}_{[t,r]}(s), \text{for all } s\in[0,T]$$

belongs to $\mathcal{U}$.

Assumption 2.2.2 *For all u, $\gamma\in\mathcal{U}$, with γ bounded, there exists $\sigma>0$, for all $\varepsilon\in(-\sigma,\sigma)$, the convex combination $u+\varepsilon\gamma$ belongs to $\in\mathcal{U}$.*

Under assumptions 1.2.2 and 1.2.3, we have the following lemma.

Lemma 2.2.1 *For all u, $\gamma \in \mathcal{U}$, with γ bounded, the process*

$$\widetilde{Y}(t) := D_\gamma Y(t) := \frac{d}{d\varepsilon} Y^{u+\varepsilon\gamma}(t)\big|_{\varepsilon=0}$$

exists and satisfies the equation

$$\begin{cases} -d\widetilde{Y}_t = & \left[\widetilde{Y}_t \dfrac{\partial F}{\partial y}(t, Y_t^u, Z_t^u, u_t) + \widetilde{Z}_t \dfrac{\partial F}{\partial z}(t, Y_t^u, Z_t^u, u_t) \right. \\ & \left. +\gamma(t) \dfrac{\partial F}{\partial u}(t, Y_t^u, Z_t^u, u_t)\right] dt - \widetilde{Z}_t^u dW_t, \\ \widetilde{Y}(T) = & 0, \ t \in [0,T]. \end{cases} \tag{2.2.1}$$

Proof. We know by a lemma (lemma 1.3.4) in [44] that an Itô integral is differentiable in the Malliavin sense if and only if the integrand is so. Supposing assumptions 1.2.2 and 1.2.3 hold, by Lebesgue's dominated convergence theorem and the chain rule 1.1.3, we get

$$\frac{d}{d\varepsilon}\int_t^T F\left(s, Y_s^{u+\varepsilon\gamma}, Z_s^{u+\varepsilon\gamma}, u_s + \varepsilon\gamma\right)\big|_{\varepsilon=0} ds = \int_t^T \left\{\widetilde{Y}_s \frac{\partial F}{\partial y}(s) + \widetilde{Z}_s \frac{\partial F}{\partial z}(s) + \gamma(s) \frac{\partial F}{\partial u}(s)\right\} ds,$$

and

$$\int_t^T D_\gamma Z_s dW_s = \int_t^T \widetilde{Z}_s dW_s.$$

Therefore, it follows that the process (Y_t, Z_t) is differentiable in the Malliavin sense, where $D_\gamma Z_s = \widetilde{Z}_s$, and its Malliavin derivative $\left(\widetilde{Y}_t, \widetilde{Z}_t\right)$ satisfies (2.2.1). ■

Having applied the Doss-Sussmann transformation to the control problem {(2.1.1),(2.1.2),(2.1.3)} and obtained the new control problem{(2.1.4),(2.1.5),(2.1.6)}, given assumptions 2.2.1 and 2.2.2 and proved lemma 2.2.1, we present the first main result of this chapter.

Theorem 2.2.1 *[Variational Equality] We suppose that assumptions 2.2.1 and 2.2.2 and lemma 2.2.1 are verified, and u is the optimal control of (2.1.6), then the following variational equation holds*

$$
\begin{aligned}
0 = \mathbb{E}\int_0^T L(t)\Big\{\widetilde{Y}_t\frac{\partial F}{\partial y}(t,Y_t^u,Z_t^u,u_t) + \widetilde{Z}_t\frac{\partial F}{\partial z}(t,Y_t^u,Z_t^u,u_t)\\
+\gamma(t)\frac{\partial F}{\partial u}(t,Y_t^u,Z_t^u,u_t)\Big\}\,dt - \mathbb{E}\int_0^T \widetilde{Z}_t D_t L(t)\,dt\\
+\mathbb{E}\int_0^T \widetilde{Z}_t\frac{\partial \Pi}{\partial z}(t,Y_t^u,Z_t^u,u_t)\,dt + \mathbb{E}\int_0^T \gamma(t)\frac{\partial \Pi}{\partial u}(t,Y_t^u,Z_t^u,u_t)\,dt,
\end{aligned}
\tag{2.2.2}
$$

where

$$
L(t) = \frac{\partial\Psi(Y_0^u)}{\partial y} + \int_0^t \frac{\partial \Pi}{\partial y}(s,Y_s^u,Z_s^u,u_s)\,ds. \tag{2.2.3}
$$

Notation 1 *Before making the proof of theorem 2.2.1, for the sake of simplicity we introduce the shorthand notations*

$$
A(t) = A(t,Y_t^u,Z_t^u,u_t),\ \textit{where } A = F,\ \frac{\partial F}{\partial y},\ \frac{\partial F}{\partial z},\ \frac{\partial F}{\partial u},\ \Pi,\ \frac{\partial \Pi}{\partial y},\ \frac{\partial \Pi}{\partial z}\ \textit{or}\ \frac{\partial \Pi}{\partial u}.
$$

Proof. We suppose that assumptions 2.2.1 and 2.2.2 hold, lemma 2.2.1, and let u be an optimal control. Then u is a critical point of the cost functional, in the sense

$$
0 := \frac{d}{d\varepsilon}\,\mathcal{J}(u+\varepsilon\gamma)\big|_{\varepsilon=0}\,.
$$

By the definition of $\mathcal{J}$ from (2.1.5), we get

$$
\begin{aligned}
0 &= \frac{d}{d\varepsilon}\,\mathbb{E}\left[\int_0^T \Pi\left(t,Y_t^{u+\varepsilon\gamma},Z_t^{u+\varepsilon\gamma},u+\varepsilon\gamma\right)dt + \Psi\left(Y_0^{u+\varepsilon\gamma}\right)\right]\bigg|_{\varepsilon=0}\\
&= \frac{d}{d\varepsilon}\,\mathbb{E}\left[\int_0^T \Pi\left(t,Y_t^{u+\varepsilon\gamma},Z_t^{u+\varepsilon\gamma},u+\varepsilon\gamma\right)dt\right]\bigg|_{\varepsilon=0} + \frac{d}{d\varepsilon}\,\mathbb{E}\left[\Psi\left(Y_0^{u+\varepsilon\gamma}\right)\right]\big|_{\varepsilon=0}\,.
\end{aligned}
\tag{2.2.4}
$$

We shall detail the proof upon the following two main arguments

$$
\frac{d}{d\varepsilon}\,\mathbb{E}\left[\int_0^T \Pi\left(t,Y_t^{u+\varepsilon\gamma},Z_t^{u+\varepsilon\gamma},u+\varepsilon\gamma\right)dt\right]\bigg|_{\varepsilon=0} \text{ and} \tag{2.2.5}
$$

$$
\frac{d}{d\varepsilon}\,\mathbb{E}\left[\Psi\left(Y_0^{u+\varepsilon\gamma}\right)\right]\big|_{\varepsilon=0}\,. \tag{2.2.6}
$$

In the first step we detail the first term (2.2.5).

Step 1: We start by permuting the derivative sign with the mathematical expectation and the Lebesgue integral by Lebesgue bounded convergence theorem as

$$\begin{aligned}&\frac{d}{d\varepsilon}\mathbb{E}\left[\int_0^T \Pi\left(t, Y_t^{u+\varepsilon\gamma}, Z_t^{u+\varepsilon\gamma}, u+\varepsilon\gamma\right)dt\right]\Big|_{\varepsilon=0}\\&=\mathbb{E}\left[\int_0^T \frac{d}{d\varepsilon}\Pi\left(t, Y_t^{u+\varepsilon\gamma}, Z_t^{u+\varepsilon\gamma}, u+\varepsilon\gamma\right)\big|_{\varepsilon=0}dt\right],\end{aligned}\tag{2.2.7}$$

then, by the chain rule (1.1.3), we obtain

$$\begin{aligned}&\mathbb{E}\left[\int_0^T \frac{d}{d\varepsilon}\Pi\left(t, Y_t^{u+\varepsilon\gamma}, Z_t^{u+\varepsilon\gamma}, u+\varepsilon\gamma\right)\big|_{\varepsilon=0}dt\right]\\=\;&\mathbb{E}\left(\int_0^T\left[\widetilde{Y}_t\frac{\partial\Pi}{\partial y}(t)+\widetilde{Z}_t\frac{\partial\Pi}{\partial z}(t)+\gamma(t)\frac{\partial\Pi}{\partial u}(t)\right]dt\right)\\=\;&\mathbb{E}\left(\int_0^T\left[\widetilde{Y}_t\frac{\partial\Pi}{\partial y}(t)\right]dt\right)+\mathbb{E}\left(\int_0^T\left[\widetilde{Z}_t\frac{\partial\Pi}{\partial z}(t)+\gamma(t)\frac{\partial\Pi}{\partial u}(t)\right]dt\right).\end{aligned}\tag{2.2.8}$$

In the first term of this last equation, we inject $\widetilde{Y}(t)$ by its value from (2.2.1), we get

$$\begin{aligned}&\mathbb{E}\left[\int_0^T\widetilde{Y}_t\frac{\partial\Pi}{\partial y}(t)\,dt\right]=\\&\mathbb{E}\int_0^T\frac{\partial\Pi}{\partial y}(t)\left\{\int_t^T\left[\widetilde{Y}_s\frac{\partial F}{\partial y}(s)+\widetilde{Z}_s\frac{\partial F}{\partial z}(s)+\gamma(s)\frac{\partial F}{\partial u}(s)\right]ds\right\}dt\\&-\mathbb{E}\int_0^T\frac{\partial\Pi}{\partial y}(t)\left(\int_t^T\widetilde{Z}_s dW_s\right)dt.\end{aligned}\tag{2.2.9}$$

In the last term of (2.2.9), by Fubini theorem, we permute the mathematical expectation with the Lebesgue integral

$$\mathbb{E}\int_0^T\frac{\partial\Pi}{\partial y}(t)\left(\int_t^T\widetilde{Z}_s dW_s\right)dt=\int_0^T\mathbb{E}\left[\frac{\partial\Pi}{\partial y}(t)\int_t^T\widetilde{Z}_s dW_s\right]dt.\tag{2.2.10}$$

Applying the duality formula (1.1.4) to (2.2.10), we get

$$\int_0^T\mathbb{E}\left[\frac{\partial\Pi}{\partial y}(t)\int_t^T\widetilde{Z}_s dW_s\right]dt=\int_0^T\left(\mathbb{E}\int_t^T\widetilde{Z}_s D_s\left[\frac{\partial\Pi}{\partial y}(t)\right]ds\right)dt.$$

We make a change of variable s to t : $0\le s\le t\le T$, and by Fubini theorem we permute the integrals with respect to dt and ds, we get

$$\begin{aligned}\mathbb{E}\int_0^T\int_t^T \widetilde{Z}_s D_s\left[\frac{\partial \Pi}{\partial y}(t)\right]dsdt &= \mathbb{E}\int_0^T\int_0^s \widetilde{Z}_s D_s\left[\frac{\partial \Pi}{\partial y}(t)\right]dtds\\ &= \mathbb{E}\int_0^T \widetilde{Z}_s\int_0^s D_s\left[\frac{\partial \Pi}{\partial y}(t)\right]dtds.\end{aligned}$$

After applying Lebesgue's theorem and the duality formula, (2.2.10) becomes:

$$\begin{aligned}&\mathbb{E}\left[\int_0^T \widetilde{Y}_t\frac{\partial \Pi}{\partial y}(t)\,dt\right]\\ =\ &\mathbb{E}\int_0^T\frac{\partial \Pi}{\partial y}(t)\left\{\int_t^T\left[\widetilde{Y}_s\frac{\partial F}{\partial y}(s)+\widetilde{Z}_s\frac{\partial F}{\partial z}(s)+\gamma(s)\frac{\partial F}{\partial u}(s)\right]ds\right\}dt\\ -\ &\mathbb{E}\int_0^T\widetilde{Z}_s\int_0^s D_s\left[\frac{\partial \Pi}{\partial y}(t)\right]dtds.\end{aligned}\tag{2.2.11}$$

Hence, we inject the new expression (2.2.11) in the first term of (2.2.8), we get the detailed form of (2.2.5) as

$$\begin{aligned}&\mathbb{E}\left[\int_0^T\frac{d}{d\varepsilon}\,\Pi\left(t,Y_t^{u+\varepsilon\gamma},Z_t^{u+\varepsilon\gamma},u+\varepsilon\gamma\right)\big|_{\varepsilon=0}\,dt\right]\\ =\ &\mathbb{E}\left(\int_0^T\left[\widetilde{Y}_t\frac{\partial \Pi}{\partial y}(t)+\widetilde{Z}_t\frac{\partial \Pi}{\partial z}(t)+\gamma(t)\frac{\partial \Pi}{\partial u}(t)\right]dt\right)\\ =\ &\mathbb{E}\int_0^T\frac{\partial \Pi}{\partial y}(t)\left\{\int_t^T\left[\widetilde{Y}_s\frac{\partial F}{\partial y}(s)+\widetilde{Z}_s\frac{\partial F}{\partial z}(s)+\gamma(s)\frac{\partial F}{\partial u}(s)\right]ds\right\}dt\\ -\ &\mathbb{E}\int_0^T\widetilde{Z}_s\int_0^s D_s\left[\frac{\partial \Pi}{\partial y}(t)\right]dtds+\mathbb{E}\left(\int_0^T\left[\widetilde{Z}_t\frac{\partial \Pi}{\partial z}(t)+\gamma(t)\frac{\partial \Pi}{\partial u}(t)\right]dt\right).\end{aligned}\tag{2.2.12}$$

In the second step we detail the second term (2.2.6).

Step 2: To explicit (2.2.6), we use similar tools and steps. At first, by Lebesgue's bounded convergence theorem, we permute the derivative with the mathematical expectation

$$\frac{d}{d\varepsilon}\,\mathbb{E}\left[\Psi\left(Y_0^{u+\varepsilon\gamma}\right)\right]\big|_{\varepsilon=0}=\mathbb{E}\left(\frac{d}{d\varepsilon}\,\left[\Psi\left(Y_0^{u+\varepsilon\gamma}\right)\right]\big|_{\varepsilon=0}\right).\tag{2.2.13}$$

Applying the chain rule (1.1.3) to (2.2.13), we have

$$\mathbb{E}\left(\frac{d}{d\varepsilon}\,\left[\Psi\left(Y_0^{u+\varepsilon\gamma}\right)\right]\big|_{\varepsilon=0}\right)=\mathbb{E}\left[\widetilde{Y}_0^u\frac{\Psi\left(Y_0^u\right)}{\partial y}\right].\tag{2.2.14}$$

In (2.2.14), we replace $\widetilde{Y}_0^u$ by its value from (2.2.1), we get

$$\begin{aligned}\mathbb{E}\left[\widetilde{Y}_0^u \frac{\Psi\left(Y_0^u\right)}{\partial y}\right] = \mathbb{E}\left[\frac{\Psi\left(Y_0^u\right)}{\partial y}\int_0^T \left\{\widetilde{Y}_t \frac{\partial F}{\partial y}(t) + \widetilde{Z}_t \frac{\partial F}{\partial z}(t) + \gamma(t)\frac{\partial F}{\partial u}(t)\right\} dt\right. \\ -\mathbb{E}\left[\frac{\Psi\left(Y_0^u\right)}{\partial y}\int_0^T \widetilde{Z}_t dW_t\right].\end{aligned} \tag{2.2.15}$$

We apply the duality formula (1.1.4) to the last term in the previous equation (2.2.15), we obtain

$$\mathbb{E}\left[\frac{\Psi\left(Y_0^u\right)}{\partial y}\int_0^T \widetilde{Z}_t dW_t\right] = \mathbb{E}\left[\int_0^T \widetilde{Z}_t D_t\left[\frac{\Psi\left(Y_0^u\right)}{\partial y}\right] dt\right]. \tag{2.2.16}$$

Inserting this simplified expression (2.2.16) in (2.2.15), (2.2.6) becomes

$$\begin{aligned}\mathbb{E}\left[\widetilde{Y}_0^u \frac{\Psi\left(Y_0^u\right)}{\partial y}\right] = \mathbb{E}\left[\frac{\Psi\left(Y_0^u\right)}{\partial y}\int_0^T \left\{\widetilde{Y}_t \frac{\partial F}{\partial y}(t) + \widetilde{Z}_t \frac{\partial F}{\partial z}(t) + \gamma(t)\frac{\partial F}{\partial u}(t)\right\} dt\right. \\ -\mathbb{E}\left[\int_0^T \widetilde{Z}_t D_t\left[\frac{\Psi\left(Y_0^u\right)}{\partial y}\right] dt\right].\end{aligned} \tag{2.2.17}$$

Now, by virtue of the results of Steps 1 and 2, we gather the simplified expressions of the terms (2.2.5) and (2.2.6) obtained in (2.2.12) and (2.2.17), respectively, in one equation

$$\begin{aligned} 0 &= \mathbb{E}\int_0^T \left(\int_0^t \frac{\partial \Pi}{\partial y}(s)\, ds\right)\left\{\widetilde{Y}_t \frac{\partial F}{\partial y}(t) + \widetilde{Z}_t \frac{\partial F}{\partial z}(t) + \gamma(t)\frac{\partial F}{\partial u}(t)\right\} dt \\ &- \mathbb{E}\int_0^T \widetilde{Z}_t\left(\int_0^t D_t\left[\frac{\partial \Pi}{\partial y}(s)\right] ds\right) dt \\ &+ \mathbb{E}\left(\int_0^T \left[\widetilde{Z}_t \frac{\partial \Pi}{\partial z}(t) + \gamma(t)\frac{\partial \Pi}{\partial u}(t)\right] dt\right) \\ &+ \mathbb{E}\left[\frac{\partial \Psi\left(Y_0^u\right)}{\partial y}\int_0^T \left\{\widetilde{Y}_t \frac{\partial F}{\partial y}(t) + \widetilde{Z}_t \frac{\partial F}{\partial z}(t) + \gamma(t)\frac{\partial F}{\partial u}(t)\right\}\right] dt \\ &- \mathbb{E}\left[\int_0^T \widetilde{Z}_t D_t\left[\frac{\partial \Psi}{\partial y}\left(Y_0^u\right)\right] dt\right], \end{aligned} \tag{2.2.18}$$

that is

$$
\begin{aligned}
0 &= \mathbb{E}\left[\left(\frac{\partial \Psi\left(Y_0^u\right)}{\partial y}+\int_0^t \frac{\partial \Pi}{\partial y}(s)\, ds\right) \int_0^T\left\{\widetilde{Y}_t \frac{\partial F}{\partial y}(t)+\widetilde{Z}_t \frac{\partial F}{\partial z}(t)+\gamma(t) \frac{\partial F}{\partial u}(t)\right\}\right] dt \\
&+ \mathbb{E}\left(\int_0^T\left[\widetilde{Z}_t \frac{\partial \Pi}{\partial z}(t)+\gamma(t) \frac{\partial \Pi}{\partial u}(t)\right] dt\right) \\
&- \mathbb{E} \int_0^T \widetilde{Z}_t\left(D_t\left[\frac{\partial \Psi}{\partial y}\left(Y_0^u\right)\right]+\int_0^t D_t\left[\frac{\partial \Pi}{\partial y}(s)\right] ds\right) dt.
\end{aligned} \tag{2.2.19}
$$

Taking into consideration $L(t)$ is given by (2.2.3), then (2.2.19) becomes

$$
\begin{aligned}
0 &= \mathbb{E}\left[\frac{\Psi\left(Y_0^u\right)}{\partial y} \int_0^T\left\{\widetilde{Y}_t \frac{\partial F}{\partial y}(t)+\widetilde{Z}_t \frac{\partial F}{\partial z}(t)+\gamma(t) \frac{\partial F}{\partial u}(t)\right\} dt\right] \\
&- \mathbb{E}\left[\int_0^T \widetilde{Z}_t D_t\left[\Psi\left(Y_0^u\right)\right] dt\right]+\mathbb{E}\left(\int_0^T\left[\widetilde{Y}_t \frac{\partial \Pi}{\partial y}(t)+\widetilde{Z}_t \frac{\partial \Pi}{\partial z}(t)+\gamma(t) \frac{\partial \Pi}{\partial u}(t)\right] dt\right) \\
&= \mathbb{E}\left[\int_0^T L_t\left\{\widetilde{Y}_t \frac{\partial F}{\partial y}(t)+\widetilde{Z}_t \frac{\partial F}{\partial z}(t)+\gamma(t) \frac{\partial F}{\partial u}(t)\right\} dt\right. \\
&+ \mathbb{E}\left(\int_0^T\left[\widetilde{Z}_t\left[\frac{\partial \Pi}{\partial z}(t)-D_t L_t\right]+\gamma(t) \frac{\partial \Pi}{\partial u}(t)\right] dt\right),
\end{aligned}
$$

which ends the proof. ■

2.3 Necessary Optimality Conditions

In this section 2.3, we shall use the results and tools from section 2.2 to build an adjoint process for our system and express necessary optimality condition in terms of this adjoint process, as the second main result of chapter 2.

Theorem 2.3.1 (Necessary Optimality Condition) *Let u be an optimal control minimizing the functional $\mathcal{J}$ over $\mathcal{U}$ and (Y_t^u, Z_t^u) denote the corresponding optimal trajectory given by* (2.1.4) *. Then, in the sense of* (2.2.2) *, the following necessary optimality condition*

$$
\mathbb{E}\left[\frac{\partial}{\partial u}\left[\Pi\left(t, Y_t^u, Z_t^u, u_t\right)+p(t) F\left(t, Y_t^u, Z_t^u, u_t\right)\right] \Big| \mathcal{G}_t\right]=0. \tag{2.3.1}
$$

holds, where the process $L(t)$ is defined in (2.2.3), $p(t)$ and $M(t)$ are given

$$\begin{aligned}
p(t) &= L(t) + \int_0^t M(t,s)\,ds, \\
M(t,s) &= \frac{\partial N(s)}{\partial y} G(t,s), \\
N(s,Y,Z,u) &= L(s)F(s), \quad \text{and} \\
G(t,s) &= \exp\left(\int_s^t \left(\frac{\partial F}{\partial y}(r) - \frac{1}{2}\left(\frac{\partial F}{\partial z}(r)\right)^2 \right) dr + \int_s^t \frac{\partial F}{\partial z}(r)\,dW_r \right).
\end{aligned}$$

Proof. We consider the variational equation from (2.2.2)

$$\begin{aligned}
0 = \frac{d}{d\varepsilon}\mathcal{J}(u+\varepsilon\gamma)\big|_{\varepsilon=0} &= \mathbb{E}\int_0^T L(t)\left[\widetilde{Y}_t \frac{\partial F}{\partial y}(t) + \widetilde{Z}_t \frac{\partial F}{\partial z}(t)\right] dt \\
&+ \mathbb{E}\int_0^T \gamma(t)\left[L(t)\frac{\partial F}{\partial u}(t) + \frac{\partial \Pi}{\partial u}(t)\right] dt \\
&+ \mathbb{E}\int_0^T \widetilde{Z}_t \left[\frac{\partial \Pi}{\partial z}(t) - D_t L(t)\right] dt.
\end{aligned} \tag{2.3.2}$$

Let t, r be in $[0,T]$, such that $0 \le t-r \le t \le T$, we pick a bounded and $\mathcal{G}_t$-measurable random variable γ:

$$\gamma(s) := \gamma_\theta(s) := \theta(\omega)\,\mathbb{1}_{[t-r,t]}(s),$$

and inject it into equality (2.3.2), we obtain

$$\begin{aligned}
0 &= H_1 + H_2 + H_3 \\
&= \mathbb{E}\int_0^t L(s)\left[\widetilde{Y}_s \frac{\partial F}{\partial y}(t) + \widetilde{Z}_s \frac{\partial F}{\partial z}(s)\right] ds \\
&+ \mathbb{E}\int_{t-r}^t \theta\left[L(s)\frac{\partial F}{\partial u}(s) + \frac{\partial \Pi}{\partial u}(s)\right] ds \\
&+ \mathbb{E}\int_0^t \widetilde{Z}_s \left[\frac{\partial \Pi}{\partial z}(t)(s) - D_s L(s)\right] ds,
\end{aligned}$$

where

$$H_1 = \mathbb{E}\int_0^t L(s)\left[\widetilde{Y}_s \frac{\partial F}{\partial y}(s) + \widetilde{Z}_s \frac{\partial F}{\partial z}(s)\right] ds, \tag{2.3.3}$$

$$H_2 = \mathbb{E}\int_{t-r}^t \theta\left[L(s)\frac{\partial F}{\partial u}(s) + \frac{\partial \Pi}{\partial u}(s)\right] ds, \text{ and} \tag{2.3.4}$$

$$H_3 = \mathbb{E}\int_0^t \widetilde{Z}_s\left[\frac{\partial \Pi}{\partial z}(s) - D_s L(s)\right] ds. \tag{2.3.5}$$

From (2.2.1), for $t \leq s \leq T$,

$$\gamma_\theta(s) = 0 \text{ , then } \widetilde{Y}_s = 0, \text{ hence } \widetilde{Z}_s = 0.$$

Then, for $t - r \leq s \leq t$, equation (2.2.1) becomes

$$d\widetilde{Y}_s = \left[\widetilde{Y}_{s^+}\frac{\partial F}{\partial y}(s) + \widetilde{Z}_{s^+}\frac{\partial F}{\partial z}(s) + \theta\frac{\partial F}{\partial u}(s)\right] ds - \widetilde{Z}_{s^+} dW_s, \tag{2.3.6}$$

with terminal condition at time t given by $\widetilde{Y}_t$.

We shall make the proof upon three steps, as follows:

Step 1: Let be the geometric Brownian motion

$$G(t,s) = \exp\left(\int_s^t \left(\frac{\partial F}{\partial y}(r) - \frac{1}{2}\left(\frac{\partial F}{\partial z}(r)\right)^2\right) dr + \int_s^t \frac{\partial F}{\partial z}(r)\, dW_r\right),$$

by the Itô formula, we can simply show that

$$dG(t,s) = G(t,s)\left[\frac{\partial F}{\partial y}(t)\, dt + \frac{\partial F}{\partial z}(t)\, dW_t\right]; G(s,s) = 1, \tag{2.3.7}$$

hence, using (2.3.6) and (2.3.7) to apply the stochastic integration by parts formula to the product $\widetilde{Y}G$, we obtain

$$\begin{aligned} d\widetilde{Y}_s G(t,s) &= \widetilde{Y}_s dG(t,s) + G(t,s)\, d\widetilde{Y}_s + d < \widetilde{Y}_s, G(t,s) >_t \\ &= \widetilde{Y}_s G(t,s)\left[\frac{\partial F}{\partial y}(t) + \frac{\partial F}{\partial z}(t)\, dWt\right] \\ &- G(t,s)\left[\widetilde{Y}_{s^+}\frac{\partial F}{\partial y}(s) + \widetilde{Z}_{s^+}\frac{\partial F}{\partial z}(s) + \theta\frac{\partial F}{\partial u}(s)\right] ds - G(t,s)\,\widetilde{Z}_{s^+} dW_s \\ &+ G(t,s)\,\widetilde{Z}_{s^+}\frac{\partial F}{\partial z}(t)\, dt. \end{aligned}$$

Then

$$d\left(\widetilde{Y}_s G(t,s)\right) = -\theta G(t,s)\frac{\partial F}{\partial u}(s)\,ds + G(t,s)\left[\widetilde{Z}_{s^+} - \widetilde{Y}_{s^+}\frac{\partial F}{\partial y}(s)\right]dW_s. \tag{2.3.8}$$

We integrate (2.3.8) from s to t, we get the explicit solution of the dynamics (2.3.6):

$$\widetilde{Y}(s) = \widetilde{Y}(t)\,G(t,s) + \int_s^t \theta G(t,v)\frac{\partial F}{\partial u}(v)\,dv - \int_s^t G(t,v)\left(\widetilde{Z}_v - \widetilde{Y}_v\frac{\partial F}{\partial z}(v)\right)dW_v, \tag{2.3.9}$$

for $t - r \leq s$.

But at time $t = t - r$, the term including θ vanishes, and (2.3.9) becomes

$$\widetilde{Y}(s) = \widetilde{Y}(t-r)\,G(t-r,s) - \int_s^{t-r} G(t,v)\left(\widetilde{Z}_v - \widetilde{Y}_v\frac{\partial F}{\partial z}(v)\right)dW_v. \tag{2.3.10}$$

If we define

$$N(s,Y,Z,u) = L(s)\,F(s), \tag{2.3.11}$$

then replacing in (2.3.3), we obtain

$$H_1 = \mathbb{E}\int_0^t \left(\frac{\partial N(s)}{\partial y}\widetilde{Y}(s) + \frac{\partial N(s)}{\partial z}\widetilde{Z}(s)\right)ds.$$

At first, we differentiate (2.3.3) with respect to r at $r = 0$. We permute the expectation with Lebesgue's integral by Lebesgue's bounded convergence theorem, and using the fact that $\frac{\partial N}{\partial y}$ and $\frac{\partial N}{\partial z}$ are càglàd, we omit the following part from the derivative of (2.3.3)

$$\frac{d}{dr}\,\mathbb{E}\int_{t-r}^t \left(\frac{\partial N(s)}{\partial y}\widetilde{Y}(s) + \frac{\partial N(s)}{\partial z}\widetilde{Z}(s)\right)ds\bigg|_{r=0} =$$
$$\mathbb{E}\left[\frac{d}{dr}\int_{t-r}^t \left(\frac{\partial N(s)}{\partial y}\widetilde{Y}(s) + \frac{\partial N(s)}{\partial z}\widetilde{Z}(s)\right)ds\bigg|_{r=0}\right] = 0.$$

Without the part from $t - r$ to t, the derivative of (2.3.3) with respect to r at $r = 0$ becomes

$$\left.\frac{dH_1}{dr}\right|_{r=0} = \frac{d}{dr}\,\mathbb{E}\int_0^t \left(\frac{\partial N\left(s\right)}{\partial y}\widetilde{Y}\left(s\right) + \frac{\partial N\left(s\right)}{\partial z}\widetilde{Z}\left(s\right)\right) ds\bigg|_{r=0}$$

$$= \frac{d}{dr}\,\mathbb{E}\int_0^{t-r} \left(\frac{\partial N\left(s\right)}{\partial y}\widetilde{Y}\left(s\right) + \frac{\partial N\left(s\right)}{\partial z}\widetilde{Z}\left(s\right)\right) ds\bigg|_{r=0}. \tag{2.3.12}$$

Hence, we inject the explicit solution (2.3.10) in (2.3) and using the fact that G is independent from r, we obtain

$$\left.\frac{dH_1}{dr}\right|_{r=0} = \frac{d}{dr}\left\{\mathbb{E}\left[\int_0^t \frac{\partial N\left(s\right)}{\partial y}\left(\widetilde{Y}\left(t-r\right)G\left(t-r,s\right)\right.\right.\right.$$
$$\left.\left.\left.-\int_s^{t-r} G\left(t,v\right)\left(\widetilde{Z}_v - \widetilde{Y}_v\frac{\partial F}{\partial z}\left(v\right)\right)dW_v\right)ds\right]\right\}\bigg|_{r=0}$$
$$+\frac{d}{dr}\,\mathbb{E}\int_0^t \left(\frac{\partial N\left(s\right)}{\partial z}\widetilde{Z}\left(s\right)\right)ds\bigg|_{r=0}.$$

Removing again the part from $t-r$ to t, that is

$$\left.\frac{dH_1}{dr}\right|_{r=0} = \frac{d}{dr}\,\mathbb{E}\int_0^{t-r}\left(\frac{\partial N\left(s\right)}{\partial y}G\left(t-r,s\right)\widetilde{Y}\left(t-r\right)\right)ds\bigg|_{r=0}$$
$$+\frac{d}{dr}\,\mathbb{E}\int_0^{t-r}\widetilde{Z}\left(s\right)\frac{\partial N\left(s\right)}{\partial z}ds\bigg|_{r=0}$$
$$-\frac{d}{dr}\,\mathbb{E}\int_0^{t-r}\frac{\partial N\left(s\right)}{\partial y}\left(\int_s^{t-r}G\left(v,t\right)\left(\widetilde{Z}_s - \widetilde{Y}_s\frac{\partial F}{\partial z}\left(v\right)\right)dW_v\right)ds\bigg|_{r=0}$$
$$= \Lambda_1 + \Lambda_2 - \Lambda_3.$$

Step 2: By Lebesgue's bounded convergence theorem, we perumute the derivative with the integrals (the expectation and the Lebesgue's integral) and by Fubini's theorem, we permute the expectation and the Lebesgue integral, as follows

$$\Lambda_1 = \frac{d}{dr}\,\mathbb{E}\int_0^{t-r}\left[\frac{\partial N(s)}{\partial y}G(t-r,s)\,\widetilde{Y}(t-r)\right]ds\bigg|_{r=0} \tag{2.3.13}$$
$$= \int_0^t \frac{d}{dr}\mathbb{E}\left[\frac{\partial N(s)}{\partial y}G(t-r,s)\,\widetilde{Y}(t-r)\right]\bigg|_{\hat{=}0} ds$$
$$= \int_0^t \frac{d}{dr}\mathbb{E}\left[\frac{\partial N(s)}{\partial y}G(t,s)\,\widetilde{Y}(t-r)\right]\bigg|_{r=0} ds,$$
$$\Lambda_2 = \frac{d}{dr}\,\mathbb{E}\int_0^{t-r}\frac{\partial N(s)}{\partial y}\widetilde{Z}(s)\,ds\bigg|_{r=0}, \text{ and} \tag{2.3.14}$$
$$\Lambda_3 = \frac{d}{dr}\,\mathbb{E}\int_0^{t-r}\frac{\partial N(s)}{\partial y}\left[\int_s^{t-r}G(v,t)\left(\widetilde{Z}_s - \widetilde{Y}_s\frac{\partial F}{\partial z}(v)\right)dW_v\right]ds\bigg|_{r=0}. \tag{2.3.15}$$

By the definition of $\widetilde{Y}(s)$ in (2.2.1), we have

$$\widetilde{Y}(t-r) = \int_{t-r}^t\left[\frac{\partial F}{\partial y}(v)\,\widetilde{Y}_{v^+} + \frac{\partial F}{\partial z}(v)\,\widetilde{Z}_{v^+} + \theta\frac{\partial F}{\partial u}(v)\right]dv - \int_{t-r}^t \widetilde{Z}_{v^+}dW_v. \tag{2.3.16}$$

Therefore, replacing (2.3.16) in (2.3.13) and using Fubini's and Lebesgue's dominated convergence theorems, we have

$$\begin{aligned}
\Lambda_1 &= \int_0^t \frac{d}{dr}\mathbb{E}\left(\frac{\partial N(s)}{\partial y}G(t,s)\,\widetilde{Y}(t-r)\right)\bigg|_{r=0} ds \\
&= \int_0^t \frac{d}{dr}\mathbb{E}\,\frac{\partial N(s)}{\partial y}G(t,s)\int_{t-r}^t\frac{\partial F}{\partial y}(v)\,\widetilde{Y}_v dv\bigg|_{r=0} ds \\
&+ \int_0^t \frac{d}{dr}\mathbb{E}\,\frac{\partial N(s)}{\partial y}G(t,s)\int_{t-r}^t\frac{\partial F}{\partial z}(v)\,\widetilde{Z}_v dv\bigg|_{r=0} ds \\
&+ \int_0^t \frac{d}{dr}\mathbb{E}\,\frac{\partial N(s)}{\partial y}G(t,s)\int_{t-r}^t\theta\frac{\partial F}{\partial u}(v)\,dv\bigg|_{r=0} ds \\
&- \int_0^t \frac{d}{dr}\mathbb{E}\,\frac{\partial N(s)}{\partial y}G(t,s)\int_{t-r}^t\widetilde{Z}_v dW_v dv\bigg|_{r=0} ds.
\end{aligned} \tag{2.3.17}$$

By applying the duality formula (1.1.4) to the term (2.3.17), we obtain

$$\begin{aligned}\Lambda_1 = & \int_0^t \frac{d}{dr}\left[\mathbb{E}\frac{\partial N(s)}{\partial y}G(t,s)\int_{t-r}^t \frac{\partial F}{\partial y}(v)\widetilde{Y}_v dv\right]\Big|_{r=0} ds \\ & + \int_0^t \frac{d}{dr}\mathbb{E}\left[\frac{\partial N(s)}{\partial y}G(t,s)\int_{t-r}^t \frac{\partial F}{\partial z}(v)\widetilde{Z}_v dv\right]\Big|_{r=0} ds \\ & + \int_0^t \frac{d}{dr}\mathbb{E}\left[\frac{\partial N(s)}{\partial y}G(t,s)\int_{t-r}^t \theta\frac{\partial F}{\partial u}(v)dv\right]|_{r=0} ds \\ & - \int_0^t \frac{d}{dr}\mathbb{E}\left[\int_{t-r}^t \widetilde{Z}_v D_v\left(\frac{\partial N(s)}{\partial y}G(t,s)\right)dv\right]\Big|_{r=0} ds.\end{aligned}$$

By the fact $\widetilde{Y}_v = 0$ and $\widetilde{Z}_v = 0$, Lebesgue's bounded convergence theorem and that $\frac{\partial N}{\partial y}$ and $\frac{\partial N}{\partial z}$ are càglàd, we get

$$\begin{aligned}\Lambda_1 &= \int_0^t \frac{d}{dr}\mathbb{E}\left[\frac{\partial N(s)}{\partial y}G(t,s)\int_{t-r}^t \theta\frac{\partial F}{\partial u}(v)dv\right]\Big|_{r=0} ds \\ &= \int_0^t \mathbb{E}\left[\theta\frac{\partial N(s)}{\partial y}G(t,s)\frac{\partial F}{\partial u}(s)\right]ds \\ &= \mathbb{E}\int_0^t \theta M(t,s)\frac{\partial F}{\partial u}(s)\,ds,\end{aligned}$$

where

$$M(t,s) = \frac{\partial N(s)}{\partial y}G(t,s). \tag{2.3.18}$$

Similarly, we have for $s \le t-r$, $\widetilde{Y}_s = 0$ then $\widetilde{Z}_s = 0$, and using the fact that $\frac{\partial N}{\partial y}$ and $\frac{\partial F}{\partial z}$ are càglàd, this yields to

$$\begin{aligned}\Lambda_2 &= \frac{d}{dr}\mathbb{E}\int_0^{t-r}\frac{\partial H_0(s)}{\partial y}\widetilde{Z}(s)\,ds\Big|_{r=0} = \\ \Lambda_3 &= \frac{d}{dr}\mathbb{E}\int_0^{t-r}\frac{\partial H_0(s)}{\partial y}\left(\int_s^{t-r}G(v,t)\left(\widetilde{Z}_s + \widetilde{Y}_s\frac{\partial F}{\partial z}(v)\right)dW_v\right)ds\Big|_{r=0} \\ &= 0.\end{aligned}$$

Step 3: We have again

$$\frac{dH_3}{dr}\Big|_{r=0} = \frac{d}{dr}\mathbb{E}\int_0^t \widetilde{Z}_s\left[\frac{\partial \Pi}{\partial z}(s) - D_s L(s)\right]ds\Big|_{r=0} = 0.$$

Using the same arguments (Fubini and Lebesgue theorems), we have directly

$$\begin{aligned}\left.\frac{d\Lambda_2}{dr}\right|_{r=0} &= \frac{d}{dr}\mathbb{E}\int_{t-r}^{t}\theta\left[L(s)\frac{\partial F}{\partial u}(s)+\frac{\partial \Pi}{\partial u}(s)\right]ds\Big|_{r=0}\\ &= \mathbb{E}\left[\frac{d}{dr}\int_{t-r}^{t}\theta\left[L(s)\frac{\partial F}{\partial u}(s)+\frac{\partial \Pi}{\partial u}(s)\right]ds\Big|_{r=0}\right]\\ &= \mathbb{E}\left(\theta\left[L(t)\frac{\partial F}{\partial u}(t)+\frac{\partial \Pi}{\partial u}(t)\right]\right).\end{aligned}$$

We sum up all of the above calculus, we get

$$\begin{aligned}0 &= \left.\frac{dH_1}{dr}\right|_{r=0}+\left.\frac{dH_2}{dr}\right|_{r=0}\\ &= \Lambda_1+\mathbb{E}\theta\left(L(s)\frac{\partial F(t)}{\partial u}+\frac{\partial \Pi(t)}{\partial u}\right)\\ &= \mathbb{E}\int_0^t \theta M(t,s)\frac{\partial F}{\partial u}(s)\,ds+\mathbb{E}\theta\left(L(s)\frac{\partial F(t)}{\partial u}+\frac{\partial \Pi(t)}{\partial u}\right)\\ &= \mathbb{E}\left[\theta\left(\left(L(s)+\int_0^t M(t,s)\,ds\right)\frac{\partial F(t)}{\partial u}+\frac{\partial \Pi(t)}{\partial u}\right)\right].\end{aligned}$$

If we define

$$p(t)=L(t)+\int_0^t M(t,s)\,ds. \tag{2.3.19}$$

Since the last equation holds for all bounded random variable $\mathcal{G}_t$-measurable θ, Then we have

$$\mathbb{E}\left[\frac{\partial}{\partial u}\left[\Pi\left(t,Y_t^u,Z_t^u,u_t\right)+p(t)\,F\left(t,Y_t^u,Z_t^u,u_t\right)\right]\Big|\,\mathcal{G}_t\right]=0, \tag{2.3.20}$$

the proof is finished. ■

Remark 2.3.1 *Due to major difficulties, it is not easy to transform the BSDE (2.1.4) back to the BDSDE (2.1.1). For more details, we refer to [12, 13, 62, 63].*

Chapter 3

Malliavin Calculus Used to Derive Pontryagin's Risk-Sensitive SMP for BSDEs Driven by Fractional Brownian Motion

Chapter 3 presents the second main result of this thesis: Pontryagin's risk-sensitive stochastic maximum principle for fractional backward stochastic differential equations via the Malliavin calculus. This result is published in [11]. We start by establishing the problem formulation and introducing the risk-sensitive cost functional, then we study stochastic backward dynamics driven by fractional Brownian motion and derive risk neutral SMP. Next, we transform the adjoint equation in order to illustrate the main result: necessary and sufficient optimality conditions for risk-sensitive control problem under additional hypothesis. We finish this chapter by an application to linear quadratic control problem.

3.1 Risk-Sensitive Stochastic Maximum Problem

Let $\left(\Omega_2, \mathcal{F}^B, \left(\mathcal{F}_t^B\right)_{t\geq 0}, \mathbb{P}_2\right)$ be a filtered probability space, satisfying the usual conditions, namely, $\Omega_2 \coloneqq \mathcal{C}\left([0,T], \mathbb{R}\right)$ is a Wiener space endowed with the topology of uniform convergence, with time horizon $T > 0$. Having established the main tools of our framework in chapter 1 , we consider the following controlled fractional BSDE (FBSDE)

$$\begin{cases} dy_t^v &= -g(t, \eta(t), y_t^v, z_t^v, v_t)dt + z_t^v dW_t^H, \\ y_T^v &= \xi;\ t \in [0,T], \end{cases} \tag{3.1}$$

where $g : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times U \to \mathbb{R}$. ξ is the terminal condition, an $\mathcal{F}_T^B$-measurable and square-integrable random variable. v is an admissible control. We keep the notation $\mathcal{U}$ of the set of all admissible controls, which we suppose convex. We define the criterion to be minimized, with initial risk-sensitive cost, as follows

$$J^\theta\left(v\right) = \mathbb{E}\left[exp\theta\left(\psi\left[y^v\left(0\right)\right] + \int_0^T f\left(t, \eta(t), y_t^v, z_t^v, v_t\right)dt\right)\right], \tag{3.2}$$

where $\mathbb{E}\left[.\right]$ denotes the mathematical expectation under the probability measure $\mathbb{P} \coloneqq \mathbb{P}_2$, θ is the risk-sensitive index,

$$\begin{aligned} \psi :&\quad \mathbb{R} \to \mathbb{R}, \\ f :&\quad [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times U \to \mathbb{R}, \end{aligned}$$

and η is to be defined in the very following assumptions (see *H1* from assumptions 3.1), that assure the well-posedness of the problem.

We intend to minimize the functional J^θ over the set $\mathcal{U}$, that is, to find an admissible control $u \in \mathcal{U}$ such that

$$J^\theta\left(u\right) = \inf_{v\in\mathcal{U}} J^\theta\left(v\right). \tag{3.3}$$

The following assumptions on the driver g of the system (3.1) and the cost functional (3.2) are essential for the well-posedness of the problem.

Assumptions 3.1

H1 : We assume that η is an Itô process of the form $\eta(t) = \eta(0) + \int_0^t F(s)dW_s^H$, where $\eta(0)$ is a given constant, and $F : \mathbb{R} \to \mathbb{R}$ is a deterministic continuous function such that $F(t) \neq 0$ for all $t \in [0,T]$.

H2 : $g, f : [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times U \to \mathbb{R}$ *are continuous with respect to* t *and continuously differentiable with respect to* (η, y, z, u)*, and having (with all the derivatives) a polynomial growth in all variables. Moreover, there exists* $C > 0$*, such that*

$$\forall y_1, z_1, y_2, z_2, x \in \mathbb{R} : |g(t, x, y_1, z_1) - g(t, x, y_2, z_2)| \leq C(|y_1 - y_2| + |z_1 - z_2|).$$

H3 : $\xi = h(\eta_T)$*, such that* $h : \mathbb{R} \to \mathbb{R}$ *is differentiable function with polynomial growth.*

H4 : ψ *is continuously differentiable and there exists* $C > 0$*, such that*

$$\forall y \in \mathbb{R} : |\psi(y)| \leq C(1 + |y|).$$

Given assumptions 3.1 (see *H1* to *H4*), the cost functional is well defined from $\mathcal{U}$ to $\mathbb{R}$. From chapter 1.3, we have the existence and uniqueness result: for all admissible control v, the system (3.1) admits a unique solution $(y, z) = (\phi_1(., \eta(.)), \phi_2(., \eta(.)))$ where ϕ_1 and ϕ_2 are continuously differentiable and having polynomial growth with respect to t and three times continuously differentiable with polynomial growth with respect to η.
An optimal control u is an admissible control solving the problem $\{(3.1), (3.2), (3.3)\}$. In the sequel, we establish necessary and sufficient optimality conditions, satisfied by such a control, under the form of stochastic maximum principle with risk-sensitive performance.

3.2 Risk-Sensitive Necessary Optimality Conditions

In order to solve our problem, we need to pass by an intermediate step, that is, we introduce an auxiliary state process Υ_t^v that satisfies the following stochastic differential equation

$$\begin{cases} d\Upsilon_t^v &= f(t, \eta(t), y_t^v, z_t^v, v_t)\, dt, \\ \Upsilon^v(0) &= 0. \end{cases}$$

Then, the control problem $\{(3.1), (3.2), (3.3)\}$ is equivalent to

$$\begin{cases} \inf_{v\in\mathcal{U}} \mathbb{E}\left[exp\theta\left(\psi\left[y^v\left(0\right)\right]+\Upsilon_T\right)\right]=\inf_{v\in\mathcal{U}}\mathbb{E}[\Lambda\left(y^v\left(0\right),\Upsilon_T\right)], \\ \text{subject to} \\ d\Upsilon_t^v = f\left(t,\eta(t),y_t^v,z_t^v,v_t\right)dt, \\ -dy_t^v = g\left(t,\eta(t),y_t^v,z_t^v,v_t\right)dt - z_t^v dW_t^H, \\ \Upsilon^v\left(0\right) = 0,\ y_T^v=\xi. \end{cases} \tag{3.2.1}$$

Define

$$\varpi_T^\theta := exp\theta\left(\psi\left[y^u\left(0\right)\right]+\int_0^T f\left(t,\eta(t),y_t^u,z_t^u,u_t\right)dt\right)$$

and

$$\Theta_T := \psi\left[y^u\left(0\right)\right]+\int_0^T f\left(t,\eta(t),y_t^u,z_t^u,u_t\right)dt,$$

the risk-sensitive loss functional is given by (see Chala et al. [14] or Tembine et al. [59])

$$\Theta_\theta := \frac{1}{\theta}\log\mathbb{E}\left[\exp\left(\psi\left[y^u\left(0\right)\right]+\int_0^T f\left(t,\eta(t),y_t^u,z_t^u,u_t\right)dt\right)\right] := \frac{1}{\theta}\log\mathbb{E}\left(\exp\theta\Theta_T\right). \tag{3.2.2}$$

Put θ in the neighbourhood of 0, the risk-neutral loss functional $\mathbb{E}\left(\Theta_T\right)$ may be considered as the limit of the risk-sensitive functional Θ_θ. Hence, by a Taylor expansion, the loss functional Θ_θ is expandable to

$$\mathbb{E}\left(\Theta_T\right)+\frac{\theta}{2}Var\left(\Theta_T\right)+O\left(\theta^2\right),$$

where $Var\left(\Theta_T\right)$ denotes the variance of Θ_T.

Notation 3.1 *We proceed with following notations. For* $v_t, u_t \in \mathcal{U}, \phi =: \widetilde{\mathcal{H}}^\theta, g, f, \psi,$ *or* η *(defined in H1 in assumptions* 3.1*), we put*

$$\begin{cases} \phi^u\left(t\right)=\phi\left(t,\eta(t),y_t^u,z_t^u,u_t\right), \\ \partial\phi\left(t\right)=\phi\left(t,\eta(t),y_t^v,z_t^v,v_t\right)-\phi\left(t,\eta(t),y_t^u,z_t^u,u_t\right), \\ \phi_\zeta\left(t\right)=\dfrac{\delta\phi}{\delta\zeta}\left(t,\eta(t),y_t^u,z_t^u,u_t\right),\ \zeta\in\{y,z,v\}\ \ \textit{and} \\ \phi^\theta\left(t\right)=\phi\left(t,\eta(t),y_t^{u^\theta},z_t^{u^\theta},u_t+\theta\right). \end{cases}$$

Suppose that assumptions *H1–H4* in assumptions 3.1 hold, then applying stochastic maximum principle for risk-neutral performance of forward-backward type control from [61] to the augmented state dynamics (Υ, y, z), one can find the adjoint equation satisfied by a unique $\mathcal{F}^B$-adapted pair of processes $\left(\left(p_1,q_1\right),\left(p_2,q_2\right)\right)$, that solves the following system of forward-backward stochastic differential equations

$$\begin{cases} d\overrightarrow{p}^{u}(t) = \begin{pmatrix} dp_1^u(t) \\ dp_2^u(t) \end{pmatrix} = -\begin{pmatrix} 0 & 0 \\ f_y(t) & g_y(t) \end{pmatrix}\begin{pmatrix} p_1^u(t) \\ p_2^u(t) \end{pmatrix} dt \\ \qquad + \begin{pmatrix} q_1^u(t) \\ -H_z(t) \end{pmatrix} dW_t^H, \\ p_1^u(T) = -\theta\varpi_T^\theta, \\ p_2^u(0) = -\theta\varpi_T^\theta\psi_y\left[y^u(0)\right], \end{cases} \tag{3.2.3}$$

with
$$\sum_{i=1}^{2}\|p_i^u(t)\|_\beta^2 + \|q_1^u(t)\|_\beta^2 < \infty.$$

We suppose here that $\mathcal{H}$ is the Hamiltonian associated with the optimal state dynamics (Υ^u, y^u, z^u) and the pair of adjoint process $(\overrightarrow{p}^u(t), \overrightarrow{q}^u(t))$ given by

$$\mathcal{H}(t, \eta(t), y_t^u, z_t^u, u_t, \overrightarrow{p}^u(t), \overrightarrow{q}^u(t)) := p_1^u(t)f(t) + p_2^u(t)g(t).$$

By some result of Yong [61], the stochastic maximum principle for risk-neutral performance of forward-backward type control in stated in the next theorem.

Theorem 3.1 *Suppose that assumptions H1 to H4 in assumptions 3.1 hold. If* (Υ, y^u, z^u) *is an optimal solution to the risk-neutral control problem* (3.2.1)*, then there exist pairs of* $\mathcal{F}^B$*-adapted processes* (p_1^u, q_1^u) *and* (p_2^u, q_2^u) *that satisfy* (3.2.3)*, such that*

$$(v_t - u_t)\mathcal{H}_v(t) \geq 0, \tag{3.2.4}$$

for all $v \in \mathcal{U}$*, almost every* $t \in [0,T]$ *and* $\mathbb{P}_2$*-a.s.*

Proof. At first, by the definition of the Hamiltonian, we have

$$\mathcal{H}(t, \eta(t), y_t^u, z_t^u, u_t, \overrightarrow{p}(t), \overrightarrow{q}(t)) := p_1^u(t)f(t) + p_2^u(t)g(t),$$

then
$$dp_1^u(t) = \mathcal{H}_x(t)dt + q_1^u(t)dW_t^H = q_1^u(t)dW_t^H,$$

and
$$\begin{aligned} dp_2^u(t) &= \mathcal{H}_y(t)dt + \mathcal{H}_z(t)dW_t^H \\ &= \left(p_1^u(t)f_y(t) + p_2^u(t)g_y(t)\right)dt + \left(p_1^u(t)f_z(t) + p_2^u(t)g_z(t)\right)dW_t^H, \end{aligned}$$

where
$$p_1^u(T) = \Lambda_x(\Upsilon_T, y_0) \quad \text{and } p_2^u(0) = \Lambda_y(\Upsilon_T, y_0).$$

Secondly, we shall derive the risk neutral stochastic maximum principle. We put

$$X = \lim_{\theta\searrow 0}\frac{\Upsilon(t+\theta) - \Upsilon(t)}{\theta} = \left(f_y(t)Y_t + f_z(t)Z_t\right)dt + \left(f^\theta(t) - f^u(t)\right)dt,$$

and
$$Y = \lim_{\theta\searrow 0}\frac{y(t+\theta) - y(t)}{\theta} = \left(g_y(t)Y_t + g_z(t)Z_t\right)dt + \left(g^\theta(t) - g^u(t)\right)dt.$$

Then, we obtain

$$J(u) = \mathbb{E}\left[\Upsilon_T^u, y^u(0)\right] \quad \text{and } J\left(u^\theta\right) = \mathbb{E}\left[\Lambda\left(\Upsilon_T^\theta, y^\theta(0)\right)\right].$$

Now, given $\mathcal{U}$ is convex, let $\hat{u} \in \mathcal{U}$ be optimal control. Then

$$\frac{1}{\theta}\left[j\left(u^\theta\right) - j(\hat{u})\right] \geq 0.$$

Hence

$$\frac{1}{\theta}\left\{\mathbb{E}\left[\Lambda\left(\Upsilon_T^\theta, y^\theta(0)\right)\right] - \mathbb{E}\left[\Lambda\left(\widehat{\Upsilon}_T, \widehat{y}(0)\right)\right]\right\} \geq 0.$$

By Taylor expansion of Λ at $\left(\Upsilon_T^\theta, y^\theta(0)\right)$, we get

$$\begin{aligned}\Lambda\left(\Upsilon_T^\theta, y^\theta(0)\right) &= \Lambda\left(\widehat{\Upsilon}_T, \widehat{y}(0)\right) + \Lambda_x\left(\widehat{\Upsilon}_T, \widehat{y}(0)\right)\left(\Upsilon_T^\theta - \widehat{\Upsilon}_T\right) \\ &+ \Lambda_y\left(\widehat{\Upsilon}_T, \widehat{y}(0)\right)\left(y^\theta(0) - \widehat{y}(0)\right).\end{aligned}$$

We put

$$\widetilde{x} = \frac{\Upsilon(t+\theta) - \Upsilon(t)}{\theta} - X,$$

then

$$\frac{\Upsilon(t+\theta) - \Upsilon(t)}{\theta} = X - \widetilde{x},$$

and

$$\widetilde{y} = \frac{y_0^\theta - \widehat{y}_0}{\theta} - Y,$$

hence

$$\frac{y_0^\theta - \widehat{y}_0}{\theta} = Y - \widetilde{y}.$$

Then

$$\begin{aligned}0 &\leq \frac{1}{\theta}\left[J\left(u^\theta\right) - J(\widehat{u})\right] \\ &= \mathbb{E}\left[\Lambda_x\left(\widehat{\Upsilon}_T, \widehat{y}(0)\right)\left(\Upsilon_T^\theta - \widehat{\Upsilon}_T\right)\right] + \mathbb{E}\left[\Lambda_y\left(\widehat{\Upsilon}_T, \widehat{y}(0)\right)\left(y^\theta(0) - \widehat{y}(0)\right)\right] \\ &= \mathbb{E}\left[\Lambda_x\left(\widehat{\Upsilon}_T, \widehat{y}(0)\right)(X - \widetilde{x})\right] + \mathbb{E}\left[\Lambda_y\left(\widehat{\Upsilon}_T, \widehat{y}(0)\right)(Y - \widetilde{y})\right].\end{aligned}$$

By Making $\theta \searrow 0$, it comes

$$\begin{aligned}\mathbb{E}\left[\Lambda_x\left(\widehat{\Upsilon}_T, \widehat{y}(0)\right)(\widetilde{x} - X)\right] &\longrightarrow \mathbb{E}\left[\Lambda_x\left(\widehat{\Upsilon}_T, \widehat{y}(0)\right)X_T\right] \\ \mathbb{E}\left[\Lambda_y\left(\widehat{\Upsilon}_T, \widehat{y}(0)\right)(\widetilde{y} - Y)\right] &\longrightarrow \mathbb{E}\left[\Lambda_y\left(\widehat{\Upsilon}_T, \widehat{y}(0)\right)Y_0\right].\end{aligned}$$

Hence, the following variational equality results

$$\mathbb{E}\left[\Lambda_x\left(\widehat{\Upsilon}_T, \widehat{y}(0)\right)X_T + \Lambda_y\left(\widehat{\Upsilon}_T, \widehat{y}(0)\right)Y_0\right] \geq 0.$$

Thus

$$\mathbb{E}\left[\Lambda_x\left(\widehat{\Upsilon}_T, \widehat{y}(0)\right)X_T\right] = \mathbb{E}\left[p_1^u(T)X_T\right]$$

and

$$\mathbb{E}\left[\Lambda_y\left(\widehat{\Upsilon}_T, \widehat{y}(0)\right)Y_0\right] = \mathbb{E}\left[p_2^u(0)Y_0\right].$$

Applying the generalized integration by parts formula 1.2.3 to $p_1^u(t)X_t$, we get

$$\begin{aligned}d\left(p_1^u(t)X_t\right) &= X_t dp_1^u(t) + p_1^u(t)dX_t + d < p_1^u, X >_t \\ &= X_t q_1^u(t)\,dW_t^H + p_1^u(t)\left(f_y(t)Y_t + f_z(t)Z_t\right)dt + p_1^u(t)\left(f^\theta(t) - \widehat{f}(t)\right).\end{aligned}$$

Then

$$\mathbb{E}\left[p_1^u(T)X_T\right] = \mathbb{E}\int_0^T p_1^u(t)\left(f_y^u(t)Y_t + f_z^u(t)Z_t\right)dt + \mathbb{E}\int_0^T p_1^u(t)\left(f^\theta(t) - f^u(t)\right)dt.$$

On the other hand, by the generalized integration by parts formula 1.2.3 applied to $p_2^u(0)Y_0$, we get

$$\begin{aligned} d\left(p_2^u(t)Y_t\right) &= Y_t dp_2^u(t) + p_2^u(t)dY_t + d < p_2^u, Y >_t \\ &= Y_t\left(p_1^u(t)f_y^u(t) + p_2^u(t)g_y^u(t)\right)dt + Y_t\left(p_1^u(t)f_y^u(t) + p_2^u(t)g_y^u(t)\right)dW_t^H \\ &+ p_2^u(t)\left(Y_t g_y^u(t) + Z_t g_z^u(t)\right)dt + p_2^u(t)\left(g^\theta(t) - g^u(t)\right)dt \\ &+ p_2^u(t)Z_t dW_t^H + Z_t\left(p_1^u(t)f_z^u(t) + p_2^u(t)g_z^u(t)\right)dt. \end{aligned}$$

Then

$$\begin{aligned} \mathbb{E}\left[p_2^u(0)Y_0\right] &= -\mathbb{E}\left[\int_0^T Y_t\left(p_1^u f_y(t) + p_2^u g_y(t)\right)dt\right] \\ &- \mathbb{E}\left[\int_0^T p_2^u(t)\left(Y_t g_y^u(t) + Z_t g_z(t)\right)dt\right] \\ &- \mathbb{E}\left[\int_0^T p_2^u(t)\left(g^\theta(t) - g(t)\right)dt\right] \\ &- \mathbb{E}\left[\int_0^T Z_t\left(p_1^u f_z(t) + p_2^u f_z(t)\right)dt\right]. \end{aligned}$$

Then

$$\begin{aligned} 0 &\leq \mathbb{E}\left[p_1^u(T)X_T\right] + \mathbb{E}\left[p_2^u(0)X_0\right] \\ &= \mathbb{E}\left[\int_0^T p_1^u(t)\left(f^\theta(t) - f(t)\right)dt\right] + \mathbb{E}\left[\int_0^{Tu} p_2^u(t)\left(g^\theta(t) - g(t)\right)dt\right]. \end{aligned}$$

Consequently

$$\mathbb{E}\left[\int_0^T \mathcal{H}_u\left(u_t^\theta - \widehat{u}_t\right)dt\right] \geq 0,$$

which ends the proof. ■

3.3 Transformation of the Adjoint Equation

In order to derive our result, we use the approach of Djehiche et al. in [18] that allows us to replace the first component of $\left(\left(p_1^u, q_1^u\right), \left(p_2^u, q_2^u\right)\right)$ and make it transformed into another one $\left(\widetilde{p}_2^u, \widetilde{q}_2^u\right)$ as the only appearing term when expressing the stochastic maximum principle with risk sensitive performance. First of all, we remark that

$$\begin{cases} dp_1^u(t) &= q_1^u(t)dW_t^H \\ p_1^u(T) &= -\theta\varpi_T^\theta. \end{cases}$$

By remark 3.4 in [29], the explicit solution of this fractional BSDE is

$$p_1^u(t) = -\theta\widetilde{\mathbb{E}}\left[\varpi_T^\theta \mid \mathcal{F}_t^B\right] = -\theta V^\theta(t), \tag{3.3.1}$$

where $\widetilde{\mathbb{E}}[.]$ denotes the quasi-conditional expectation introduced by Hu et al. in [26] and

$$V^\theta(t) := \widetilde{\mathbb{E}}\left[\varpi_T^\theta \mid \mathcal{F}_t^B\right],\ 0 \le t \le T. \tag{3.3.2}$$

Hence, we consider the transformation of $(\overrightarrow{p}^u, \overrightarrow{q}^u)$ to $(\widetilde{p}^u, \widetilde{q}^u)$, where
$\widetilde{p}_1^u(t) = \dfrac{1}{\theta V^\theta(t)} p_1^u(t) = -1$, namely,

$$\widetilde{p}^u(t) := \begin{pmatrix} \widetilde{p}_1^u(t) \\ \widetilde{p}_2^u(t) \end{pmatrix} := \frac{1}{\theta V^\theta(t)} \overrightarrow{p}^u(t),\ 0 \le t \le T. \tag{3.3.3}$$

By using (3.2.3) and (3.3.3), we have

$$\widetilde{p^u}_1(T) = -1 \text{ and } \widetilde{p}_2^u(0) = -\psi_y[y^u(0)].$$

To explicit the properties of the process $(\widetilde{p}^u(t), \widetilde{q}^u(t))$, we shall point out the following aspects of the quasi-martingale V^θ. Providing *H2 & H4* in assumptions 3.1 (i.e. the bounds of f and ψ by constant $C > 0$), we have

$$0 < \exp\{-(1+T)C\theta\} \le \varpi_T^\theta \le \exp\{(1+T)C\theta\}. \tag{3.3.4}$$

Then

$$0 < \exp\{-(1+T)C\theta\} \le V^\theta(t) \le \exp\{(1+T)C\theta\}. \tag{3.3.5}$$

The following lemma 3.3.1 is an auxiliary result of this chapter, and it serves us in our main objective in the next paragraphs.

Lemma 3.3.1 *If we define for all* $t \in [0, T]$

$$V^\theta(t) = \exp\theta\left[\Phi_t + \int_0^T f(s)ds\right], \tag{3.3.6}$$

then the process (Φ, l) *satisfies the following quadratic fractional BSDE*

$$\begin{cases} d\Phi_t &= -\theta\left[f(t) + \dfrac{\theta}{2}\dfrac{d}{dt}\|l_t\|_\rho^2\right]dt + \theta l_t dW_t^H, \\ \Phi_T &= \psi[y^v(0)], \end{cases}$$

where $\dfrac{d}{dt}\|.\|_\rho^2$ *is defined in* (1.2.6) *and* V^θ *is a uniformly bounded quasi-martingale.*

Proof. We proceed using a similar method to the proof of Lemma 3.2 in [15]. It is evident (by definition) that $\Phi_T = \psi[y^v(0)]$. Moreover, provided (3.3.5), we can consider the Logarithmic transformation (generalized version) established by El-Karoui and Hamadéne in [20], proposition 3.1, as follows

$$V^{\theta}(t) = \exp\theta\left[\Phi_t + \int_0^T f(s)ds\right] = \widetilde{\mathbb{E}}\left[\varpi_T^{\theta} \mid \mathcal{F}_t^H\right]. \tag{3.3.7}$$

Since V^{θ} is bounded by two square integrable processes in (3.3.5), then it is also $\mathbb{P}_2$-square integrable. Moreover, it is $\mathcal{F}^B$-adapted as a quasi conditional expectation), then we have by the fractional Clark-Ocone formula (See the extension of this formulae made by Hu in [27], page 106), there exists a unique $\mathcal{F}_T^B$-adapted square integrable process M such that

$$\exp\theta\left[\Phi_t + \int_0^T f(s)ds\right] = \mathbb{E}\left(\varpi_T^{\theta}\right) + \int_0^T M_s dW_s^H. \tag{3.3.8}$$

Suppose that Φ has the dynamics $d\Phi_t = -k_t dt + l_t dW_t^H$, where k and l are two processes to be identified. Then applying the fractional (generalized) Itô's theorem 1.2.1 to both sides of (3.3.8), we obtain

$$\begin{aligned} d\left(\exp\theta\left[\Phi_t + \int_0^T f(s)ds\right]\right) &= M_s dW_s^H \\ &= \exp\theta\left[\Phi_t + \int_0^T f(s)ds\right]\left[\theta f(t)dt + \theta d\Phi_t + \frac{\theta^2}{2}\frac{d}{dt}\|l_t\|_\rho^2 dt\right]. \end{aligned}$$

Then

$$\begin{aligned} M_s dW_s^H = \exp\theta\left[\Phi_t + \int_0^T f(s)ds\right]\Bigg(&\left[\theta f(t) + \frac{\theta^2}{2}\frac{d}{dt}\|l_t\|_\rho^2 - k_t\right]dt \\ &+\theta l_t dW_t^H\Bigg). \end{aligned}$$

Hence the second side is an $\mathcal{F}^H$-quasimartingale, which yields to

$$\begin{aligned} \exp\left(-\theta\left[\Phi_t + \int_0^T f(s)ds\right]\right) M_s dW_s^H = &\left[\theta f(t) + \frac{\theta^2}{2}\frac{d}{dt}\|l_t\|_\rho^2 - k_t\right]dt \\ &+\theta l_t dW_t^H. \end{aligned}$$

Identifying terms, we take

$$\begin{aligned} k_t &= \theta\left[f(t) + \frac{\theta}{2}\frac{d}{dt}\|l_t\|_\rho^2\right] \text{ and} \\ \theta l_t &= M_s \exp\left(-\theta\left[\Phi_t + \int_0^T f(s)ds\right]\right), \end{aligned}$$

which leads to the result. ■

Taking into consideration the previous lemma 3.3.1, we have in particular the following result.

Lemma 3.3.2 *V^{θ} satisfies the following fractional BSDE*

$$dV^{\theta}(t) = \theta l_t V^{\theta}(t)\, dW_t^H,\ V^{\theta}(T) = \varpi_T^{\theta}, \tag{3.3.9}$$

and the process M defined on $\left(\Omega, \mathcal{F}, \mathcal{F}_t^B, \mathbb{P}_2\right)$ by

$$M_t := \frac{\Phi_t}{\Phi_0} := \exp\left(\int_0^T \theta l_t dW_t^H - \frac{\theta^2}{2}\|l_t\|_\rho^2\right)$$

is an $\mathcal{F}_T^B$*-quasimartingale.*

Proof. A simple application of fractional Itô's formula 1.2.1 to (3.3.7), gives

$$
\begin{aligned}
& d\left[\exp\left(\theta\left[\Phi_t+\int_0^T f(s)ds\right]\right)\right] \\
&= \left[\theta f(t)dt+\theta d\Phi_t+\frac{\theta^2}{2}\frac{d}{dt}\|l_t\|_\rho^2 dt\right]\exp\theta\left[\Phi_t+\int_0^T f(s)ds\right] \\
&= \theta V^\theta(t)\left(f(t)dt-\left[f(t)+\frac{\theta}{2}\frac{d}{dt}\|l_t\|_\rho^2\right]dt+l_t dW_t^H+\frac{\theta}{2}\frac{d}{dt}\|l_t\|_\rho^2 dt\right) \\
&= \theta V^\theta(t) l_t dW_t^H,
\end{aligned}
$$

and the result follows immediately. ■

In the next, we state and prove the necessary and sufficient optimality conditions for a dynamics driven by fractional backward SDE for a risk sensitive performance functional.

Lemma 3.3.3 *The risk-sensitive dynamics for the adjoint equation satisfied by* $(\widetilde{p}_2^u, \widetilde{q}_2^u)$ *and* $\left(V^\theta, l\right)$ *becomes*

$$
\begin{cases}
d\widetilde{p}_2^u(t) &= -\widetilde{\mathcal{H}}_y^\theta(t)\,dt-\widetilde{\mathcal{H}}_z^\theta(t)\,dW_t^{\mathcal{H},\theta}, \\
dV^\theta(t) &= \theta l_t V^\theta(t)\,dW_t^H, \\
V^\theta(T) &= \varpi_T^\theta, \text{ and} \\
\widetilde{p}_2^u(0) &= -\psi_y\left[y^u(0)\right].
\end{cases}
\tag{3.3.10}
$$

The solution $\left(\widetilde{p}^u, \widetilde{q}^u, V^\theta, l\right)$ *of the system* (3.3.10) *is unique, such that*

$$
\|\widetilde{p}^u(t)\|_\beta^2+\|V^\theta(t)\|_\beta^2+\mathbb{E}\int_0^T\left(|\widetilde{q}^u(t)|^2+|l_t|^2\right)dt<\infty,
\tag{3.3.11}
$$

where

$$
\widetilde{\mathcal{H}}^\theta\left(t,\gamma_t,y_t,z_t,\begin{pmatrix}\widetilde{p}_2^u(t)\\ \widetilde{q}_2^u(t)\end{pmatrix},V^\theta(t),l_t\right)=\left(g(t)+\theta z_t l_t\right)\widetilde{p}_2^u(t)-f(t),
\tag{3.3.12}
$$

and

$$
dW_t^{H,\theta}=-2\theta l_t dt+dW_t^H.
$$

Proof. We want to identify the processes $\widetilde{\alpha}$ and $\widetilde{q}^u$ such that

$$
d\widetilde{p}^u(t)=-\widetilde{\alpha}(t)\,dt+\widetilde{q}^u(t)\,dW_t^H.
$$

By applying the fractional integration by parts formula 1.2.3 to the process $\overrightarrow{p}^u(t)=\theta V^\theta(t)\,\widetilde{p}^u(t)$, and using the expression of V^θ in (3.3.9), we obtain

$$
\begin{aligned}
d\overrightarrow{p}^u(t) &= \theta d(V^\theta(t)\,\widetilde{p}^u(t)) \\
&= \theta\left[V^\theta(t)\,d\widetilde{p}^u(t)+V^\theta(t)\,l_t\widetilde{p}^u(t)dW_t^H+2\widetilde{q}^u(t)V^\theta(t)\,l_t dt\right].
\end{aligned}
$$

Then

$$d\widetilde{p}(t)=\frac{1}{\theta V^{\theta}(t)}d\overrightarrow{p}(t)-2\widetilde{q}(t)\theta l_t dt-\theta l_t\widetilde{p}(t)dW_t^H.$$

Using (3.2.3), we get

$$\begin{aligned} d\widetilde{p}^u(t) &= -\frac{1}{\theta V^{\theta}(t)}\begin{pmatrix} 0 & 0 \\ f_y(t) & g_y(t)\end{pmatrix}\begin{pmatrix} p_1^u(t) \\ p_2^u(t)\end{pmatrix}dt-2\theta l_t\widetilde{q}^u(t)\,dt \\ &+ \frac{1}{\theta V^{\theta}(t)}\begin{pmatrix} q_1^u(t) \\ -\mathcal{H}_z(t)\end{pmatrix}dW_t^H-\theta l_t\widetilde{p}^u(t)\,dW_t^H. \end{aligned}$$

By identifying coefficients, we get the diffusion term

$$\widetilde{q}^u(t)=\frac{1}{\theta V^{\theta}(t)}\begin{pmatrix} q_1^u(t) \\ -\mathcal{H}_z(t)\end{pmatrix}-\theta l_t\widetilde{p}^u(t),$$

and the drift term of the process $\widetilde{p}^u(t)$

$$\widetilde{\alpha}(t)=\begin{pmatrix} 0 & 0 \\ f_y(t) & g_y(t)\end{pmatrix}\begin{pmatrix} \widetilde{p}_1^u(t) \\ \widetilde{p}_2^u(t)\end{pmatrix}+2\theta l_t\widetilde{q}^u(t).$$

Using the relation $\widetilde{p}^u(t)=\dfrac{1}{\theta V^{\theta}(t)}\overrightarrow{p}^u(t)$, the coefficient $\widetilde{q}^u(t)$ will be as

$$\widetilde{q}^u(t)=\begin{pmatrix} \widetilde{q}_1^u(t) \\ \dfrac{-\mathcal{H}_z(t)}{\theta V^{\theta}(t)}\end{pmatrix}-\theta l_t\widetilde{p}^u(t).$$

We finally obtain

$$d\widetilde{p}^u(t)=-\begin{pmatrix} 0 & 0 \\ f_y(t) & g_y(t)\end{pmatrix}\begin{pmatrix} \widetilde{p}_1^u(t) \\ \widetilde{p}_2^u(t)\end{pmatrix}dt-2\theta l_t\widetilde{q}^u(t)\,dt+\widetilde{q}^u(t)\,dW_t^H.$$

We can simply verify that

$$d\widetilde{p}_1^u(t)=\widetilde{q}_1^u(t)\left[-2\theta l_t dt+dW_t^H\right],\ \widetilde{p}_1^u(T)=-1.$$

Considering lemma 3.3.1, the fractional Girsanov's theorem 1.3.2 allows us to write

$$d\widetilde{p}_1^u(t)=\widetilde{q}_1^u(t)\,dW_t^{H,\theta},\ \widetilde{p}_1^u(T)=-1\ \mathbb{P}^{\theta}-a.s.,$$

where

$$dW_t^{H,\theta}=-2\theta l_t dt+dW_t^H$$

is a $\mathbb{P}^{\theta}$-fractional Brownian motion, and

$$\left.\frac{d\mathbb{P}^{\theta}}{d\mathbb{P}_2}\right|_{\mathcal{F}_T}:=\exp\left(\int_0^T 2\theta l_s dW_s^H-2\theta^2\frac{d}{dt}\|l_t\|_{\rho}^2 dt\right),\ 0\le t\le T.$$

In view of (3.3.4) and (3.3.7), the probability measures $\mathbb{P}^{\theta}$ and $\mathbb{P}_2$ are equivalent. Hence, noting that $\widetilde{p}_1^u(t) := \dfrac{1}{\theta V^\theta(t)} p_1^u(t)$ is square-integrable, we get that $\widetilde{p}_1^u(t) = \widetilde{\mathbb{E}}^{\mathbb{P}^\theta}\left[\widetilde{p}_1^u(T) \mid \mathcal{F}_t\right] = -1$. We can simply show that the process $\widetilde{q}_1^u(t)$ is of finite quadratic variation, such that $\mathbb{E}\displaystyle\int_0^T |\widetilde{q}_1^u(t)|^2\, dt = 0$. This implies that, for almost every $0 \leq t \leq T$, $\widetilde{q}_1^u(t) = 0$. $\mathbb{P}^\theta$ and $\mathbb{P}_2$-a.s.

$$d\widetilde{p}^u(t) = -\begin{pmatrix} 0 & 0 \\ f_y(t) & g_y(t) \end{pmatrix}\begin{pmatrix} \widetilde{p}_1^u(t) \\ \widetilde{p}_2^u(t) \end{pmatrix} dt + \widetilde{q}^u(t)\, dW_t^{H,\theta}.$$

By using the relation $\widetilde{q}^u(t) = \begin{pmatrix} \widetilde{q}_1^u(t) \\ \dfrac{-\mathcal{H}_z(t)}{\theta V^\theta(t)} \end{pmatrix} - \theta l_t \widetilde{p}^u(t)$, in the above equation, we obtain

$$\begin{aligned} d\widetilde{p}^u(t) &= -\begin{pmatrix} 0 & 0 \\ f_y(t) & g_y(t) \end{pmatrix}\begin{pmatrix} \widetilde{p}_1^u(t) \\ \widetilde{p}_2^u(t) \end{pmatrix} dt \\ &+ \left[\begin{pmatrix} \widetilde{q}_1^u(t) \\ -f_z(t)\widetilde{p}_1^u(t) - g_z(t)\widetilde{p}_2^u(t) \end{pmatrix} - \theta l_t \widetilde{p}^u(t)\right] dW_t^{H,\theta}. \end{aligned} \tag{3.3.13}$$

The second component of $\widetilde{p}_2^u$ is given by $(3.3.13)$, is

$$\begin{aligned} d\widetilde{p}_2^u(t) &= -\left[f_y(t)\widetilde{p}_1^u(t) + g_y(t)\widetilde{p}_2^u(t)\right] dt \\ &- \left[f_z(t)\widetilde{p}_1^u(t) + g_z(t)\widetilde{p}_2^u(t) + \theta l_t \widetilde{p}_2^u(t)\right] dW_t^{H,\theta}. \end{aligned}$$

The main risk-sensitive for the second adjoint equation satisfied by $(\widetilde{p}_2^u, \widetilde{q}_2^u)$ and $\left(V^\theta, l\right)$ becomes

$$\begin{cases} d\widetilde{p}_2^u(t) &= -\widetilde{\mathcal{H}}_y^\theta(t)\, dt - \widetilde{\mathcal{H}}_z^\theta(t)\, dW_t^{H,\theta}, \\ dV^\theta(t) &= \theta l_t V^\theta(t)\, dW_t^H, \\ V^\theta(T) &= A_T^\theta, \quad \text{and} \\ \widetilde{p}_2^u(0) &= -\psi_y\left[y^u(0)\right]. \end{cases}$$

The solution $\left(\widetilde{p}^u, \widetilde{q}^u, V^\theta, l\right)$ of the previous system (3.3.10) is unique, such that

$$\|\widetilde{p}^u(t)\|_\beta^2 + \|V^\theta(t)\|_\beta^2 + \mathbb{E}\int_0^T \left(|\widetilde{q}^u(t)|^2 + |l_t|^2\right) dt < \infty, \tag{3.3.14}$$

where

$$\widetilde{\mathcal{H}}^\theta\left(t, \gamma_t, y_t, z_t, \begin{pmatrix} \widetilde{p}_2^u(t) \\ \widetilde{q}_2^u(t) \end{pmatrix}, V^\theta(t), l_t\right) = -f(t) + \left(g(t) + \theta z_t l_t\right)\widetilde{p}_2^u(t).$$

■

The next theorem 3.1 summarizes the risk-sensitive fractional SMP.

Theorem 3.1 *(Risk-Sensitive Fractional SMP) Suppose that H1 in assumptions 3.1 holds, if (y, z, u) is an optimal solution to the risk-sensitive control problem* $\{(3.1), (3.2), (3.3)\}$, *then there exist pairs of $\mathcal{F}_T^B$-adapted processes* $\left(V^\theta, l\right)$ *and* (p^u, q^u) *that satisfy* (3.3.10) *and* (3.3.14) *such that*

$$(v_t - u_t)\widetilde{\mathcal{H}}_v^\theta(t) \geq 0, \tag{3.3.15}$$

for all $v \in \mathcal{U}$*, almost every* $0 \leq t \leq T$ *and* $\mathbb{P}_2$*-a.s., where*

$$\widetilde{\mathcal{H}}^\theta\left(t, y_t^u, z_t^u, \overrightarrow{p}^u, \overrightarrow{q}^u, u_t\right) = \theta V^\theta(t)\mathcal{H}\left(t, \Upsilon_t^u, y_t^u, z_t^u, \begin{pmatrix} \widetilde{p}_2^u(t) \\ \widetilde{q}_2^u(t) \end{pmatrix}, V^\theta(t), l_t, u_t\right).$$

3.4 Risk-Sensitive Sufficient Optimality Conditions

The following theorem presents the risk-sensitive sufficient optimality conditions.

Theorem 3.1 *Suppose that* ψ *is convex in* y *and the Hamiltonian* $\mathcal{H}$ *is convexe in* (y, u)*. For all* $v \in \mathcal{U}$ *and* $\mathcal{F}_T^B$*-measurable random variable* $y_T^v = \xi$*, such that* $\|\xi\|_\beta^2 < +\infty$*. Then* (y, z, u) *is an optimal solution to the problem* $\{(3.1), (3.2), (3.3)\}$ *if it satisfies* (3.3.15)*, for all* $v \in \mathcal{U}$*, almost every* $0 \leq t \leq T$ *and* $\mathbb{P}_2$*-a.s., where the Hamiltonian* $\mathcal{H}$ *associated with* (3.2.1)*, given by*

$$\mathcal{H}\left(t, y_t^u, z_t^u, \overrightarrow{p}^u(t), \overrightarrow{q}^u(t), u_t\right) = p_1^u(t)f(t) + p_2^u(t)g(t).$$

Proof. Let $v, u \in \mathcal{U}$ (u candidate to be optimal), then

$$J^\theta(v) - J^\theta(u) = \mathbb{E}\left[\exp\theta\left(\psi\left[y^v(0)\right] + \Upsilon_T^v\right) - \exp\theta\left(\psi\left[y^u(0)\right] + \Upsilon_T^u\right)\right].$$

By a Taylor's expansion of exp, we obtain

$$J^\theta(v) - J^\theta(u) = \mathbb{E}\left(\theta\left[\partial\psi(y_0) + \partial\Upsilon_T\right]\exp\theta\left(\psi\left[y^u(0)\right] + \Upsilon_T^u\right)\right),$$

but the function ψ is convex in y, hence

$$\partial\psi\left[y(0)\right] \geq \left[y^v(0) - y^u(0)\right]\psi_y\left[y^u(0)\right]. \tag{3.4.1}$$

Recall $\varpi_T^\theta := exp\theta\left(\psi\left[y^u(0)\right] + \Upsilon_T^u\right)$ and by inequality (3.4.1) and from the dynamics (3.2.3) we get

$$\begin{aligned}
J^\theta(v)-J^\theta(u) &\geq \mathbb{E}\left[\theta\varpi_T^\theta\left(\left[y^v(0)-y^u(0)\right]\psi_y\left[y^u(0)\right]+\Upsilon_T^v-\Upsilon_T^u\right)\right]\\
&= \mathbb{E}\left[\varpi_T^\theta\left(\Upsilon_T^v-\Upsilon_T^u\right)\right]+\mathbb{E}\left(\theta\varpi_T^\theta\left[y^v(0)-y^u(0)\right]\psi_y\left[y^u(0)\right]\right)\\
&= \mathbb{E}\left[p_1^u(T)\partial\Upsilon_T\right]-\mathbb{E}\left(p^u\left[y^u(0)\right]\left[y^v(0)-y^u(0)\right]\right).
\end{aligned}$$

On the one hand, from the dynamics (3.2.3), we have the following dynamics

$$p_1^u(t)=\theta\varpi_T^\theta-\int_t^T q_1^u(s)dW_s^H,$$

then the Malliavin derivative of p_1^u is $D_0^H p_1^u(\omega)=q_1^u(s,\omega)\mathbb{1}_{[0,T]}(s)$. Moreover,

$$\Upsilon_T^v-\Upsilon_T^u=\int_0^T\left[f^v(t)-f^u(t)\right]dt,$$

and

$$\Upsilon^v(0)-\Upsilon^u(0)=0.$$

Hence

$$D_0^H\left(\Upsilon_T^v-\Upsilon_T^u\right)(\omega)=0.\quad \mathbb{P}.a.s.$$

By the fractional integration by part formula (1.2.3), we get

$$\begin{aligned}
p_1^u(T)\left(\Upsilon_T^v-\Upsilon_T^u\right) &= p_1^u(0)\left(\Upsilon^v(0)-\Upsilon^u(0)\right)+\int_0^T p_1^u(t)\left[f^v(t)-f^u(t)\right]dt\\
&- \int_0^T q_1^u(t)\left(\Upsilon_t^v-\Upsilon_t^u\right)dW_t^H+\int_0^T\left[0\times q_1^u(t)+q_1^u(t)\times 0\right]dt.
\end{aligned}$$

Taking the mathematical expectation, we obtain

$$\mathbb{E}\left[p_1^u(T)\left(\Upsilon_T^v-\Upsilon_T^u\right)\right]=\mathbb{E}\int_0^T p_1^u(t)\left[f^v(t)-f^u(t)\right]dt. \tag{3.4.2}$$

On the other hand, we have

$$y^v(0)-y^u(0)=\int_0^T\left[g^v(t)-g^u(t)\right]dt-\int_0^T\left(z_t^v-z_t^u\right)dW_t^H,$$

then

$$D_0^H\left(y_T^v-y_T^u\right)(\omega)=\left[z_t^v(\omega)-z_t^u(\omega)\right]\mathbb{1}_{[0,T]}(t).$$

And again from the dynamics (3.2.3)

$$p_2^u(T)=-\theta\varpi_T^\theta\psi_y\left[y^u(0)\right]-\int_0^T\mathcal{H}_y(t)dt-\textstyle\int_0^T\mathcal{H}_z(t)dW_t^{\mathcal{H}},$$

we have

$$D_T^H p_2^u(t,\omega)=-\mathcal{H}_z(t,\omega)\mathbb{1}_{[0,T]}(t).$$

Using the fractional integration by parts formula 1.2.3, we obtain

$$
\begin{aligned}
p_2^u(T)(y_T^v - y_T^u) &= p_2^u(0)\left[y^v(0) - y^u(0)\right] + \int_0^T p_2^u(t)\left[g^v(t) - g^u(t)\right] dt \\
&- \int_0^T p_2^u(t)(z_t^v - z_t^u) dW_t^H - \int_0^T \mathcal{H}_y(t)(y_t^v - y_t^u) dt \\
&- \int_0^T (y_t^v - y_t^u)\mathcal{H}_z(t) dW_t^H \\
&+ \int_0^T \left(-\mathcal{H}_z(t)\left[-(z_t^v - z_t^u)\right] - (z_t^v - z_t^u)\mathcal{H}_z(t)\right) dt = 0.
\end{aligned}
$$

By taking the expectation, we get

$$
-\mathbb{E}\left(p_2^u(0)\left[y^v(0) - y^u(0)\right]\right) = \mathbb{E}\int_0^T \left(p_2^u(t)\left[g^v(t) - g^u(t)\right] - (y_t^v - y_t^u)\mathcal{H}_y(t)\right) dt. \tag{3.4.3}
$$

By combining equations (3.4.2) and (3.4.3), it comes

$$
\begin{aligned}
J^\theta(v) - J^\theta(u) \;\; \geq \;\; & \mathbb{E}\int_0^T \left(p_1^u(t)\left[f^v(t) - f^u(t)\right]\right. \\
& \left. + p_2^u(t)\left[g^v(t) - g^u(t)\right] - (y_t^v - y_t^u)\mathcal{H}_y(t)\right) dt,
\end{aligned}
$$

where $\phi^u(t) = \phi\left(t, y_t^u, z_t^u, u_t\right)$, for $\phi = f, g$ or $\mathcal{H}$.

By the convexity of the Hamiltonian $\mathcal{H}$ in (y, u), we have the following inequality

$$
\mathcal{H}^v(t) - \mathcal{H}^u(t) \geq (y_t^v - y_t^u)\mathcal{H}_y(t) + (v_t - u_t)\mathcal{H}_v(t).
$$

Then

$$
\mathcal{H}^v(t) - \mathcal{H}^u(t) - (y_t^v - y_t^u)\mathcal{H}_y(t) \geq (v_t - u_t)\mathcal{H}_v(t).
$$

Finally we get

$$
J^\theta(v) - J^\theta(u) \geq \mathbb{E}\int_0^T (v_t - u_t)\mathcal{H}_v(t) dt, \tag{3.4.4}
$$

since we have from (3.2.4) : $(v_t - u_t)\mathcal{H}_v(t) \geq 0$, the result follows immediately from the inequality (3.4.4), which proves the optimality of u. ■

3.5 Application

3.5.1 Linear Quadratic Risk-Sensitive Control Problem

Due to the advantages present by the FBM when modeling finance (i.e. long range dependence property), we would think of it as a realistic tool to model the unpredicted behaviour of some elements in a financial market. What provides us with a financial market called: fractional Black-Scholes Market with a wealth process V (V is independent on the quasi-martingale defined in (3.3.6)) defined on the filtred probability space $(\Omega, \mathcal{F}, \mathcal{F}^H, \mathbb{P})$, such that $\mathcal{F}_t^H = \sigma\left(W_s^H, 0 \leq s \leq t\right)$ and W^H is an FBM defined on $(\Omega, \mathcal{F}, \mathbb{P})$. We associate the wealth process with the investing strategy process (α, β) (It is obvious that α and β do not depend on any other notations in this thesis), which is $\mathcal{F}^H$-adapted such that

$$V_t = \alpha S_t + \beta B_t. \tag{3.5.1}$$

α : Signifies the amount of shares invested in stocks (risky assets) (S_t) with the dynamics

$$dS_t = S_t\left(\mu dt - \sigma dW_t^H\right), S_0 > 0.$$

β : Signifies the amount of shares invested in bonds (riskless assets) (B_t) with the dynamics

$$dB_t = rB_t dt, B_0 = 1.$$

The investing strategy is assumed self-financed, what provides us the following equality

$$\begin{aligned} dV_t &= \alpha dS_t + \beta dB_t \\ &= \alpha S_t\left(\mu dt - \sigma dW_t^H\right) + \beta r B_t dt. \end{aligned}$$

From (3.5.1), we get

$$dV_t = \left[rV_t + \left(\frac{\mu - r}{\sigma}\right)\alpha\sigma S_t\right]dt - \alpha\sigma S_t dW_t^H. \tag{3.5.2}$$

We denote

$$z_t = \alpha\sigma S_t \text{ and } \pi = \frac{\mu - r}{\sigma}.$$

π : Signifies the risk premium, μ and σ are the drift and the volatility of the process S.

The dynamics (3.5.2) becomes

$$dV_t = \left(rV_t + \pi z_t\right)dt - z_t dW_t^H,\ V_T = \bar{\xi}. \tag{3.5.3}$$

$\bar{\xi}$: is supposed to be an adapted contingent claim, with strike date T. We associate the FBSDE (3.5.3) with the following risk-sensitive cost functional

$$J^{\theta}(\pi) = \mathbb{E}\left(\exp\theta\left[\tfrac{-1}{2}\int_0^T\left(\pi^2+V_t^2\right)dt+\frac{V_0^2}{2}\right]\right), \tag{3.5.4}$$

which we aim to minimize over the set of all premium risks Π, that is, we seek an adapted process $\widehat{\pi}\in\Pi$, which satisfies the following equality

$$J^{\theta}(\widehat{\pi}) = \inf_{\pi\in\Pi} J^{\theta}(\pi). \tag{3.5.5}$$

On the whole, our fractional risk-sensitive linear quadratic control problem consists of the triplet $\{(3.5.3),(3.5.4),(3.5.5)\}$, which we solve in the sequel. Using the predefined formulas, we have the Hamiltonian

$$\widetilde{\mathcal{H}}^{\theta}(t,V_t,z_t,\pi,\widetilde{p}_2(t),l_t) = \frac{1}{2}\left(\pi^2+V_t^2\right)+\widetilde{p}_2(t)\left[rV_t+z_t\left(\theta l_t+\pi\right)\right].$$

Then, minimizing the Hamiltonian over Π leads to

$$\widetilde{\mathcal{H}}_{\pi}^{\theta}(t)=0\Leftrightarrow\widehat{\pi}=-\widehat{z}_t\widetilde{p}_2(t). \tag{3.5.6}$$

Hence the optimal wealth process $\widehat{V}$ has the dynamics

$$d\widehat{V}_t=\left[r\widehat{V}_t-\widehat{z}_t^2\widetilde{p}_2(t)\right]dt-\widehat{z}_t dW_t^H, \tag{3.5.7}$$

where there exists an $\mathcal{F}^B$-adapted process $\widetilde{p}_2$ satisfying the following fractional SDE

$$\begin{cases} d\widetilde{p}_2(t) &= -\mathcal{H}_V^{\theta}(t)dt-\mathcal{H}_z^{\theta}(t)dW_t^{\theta,H}\\ \widetilde{p}_2(0) &= \widehat{V}_0, \end{cases} \tag{3.5.8}$$

where

$$\mathcal{H}_V^{\theta}(t)=\widehat{V}_t+r\widetilde{p}_2(t),\quad \mathcal{H}_z^{\theta}(t)=\left(\theta\widehat{l_t}+\widehat{\pi}\right)\widetilde{p}_2(t),\ \text{and}\quad W_t^{\theta,H}=W_t^H-\int_0^t 2\theta\widehat{l}_s ds. \tag{3.5.9}$$

It remains to prove that $\widehat{\pi}$ is optimal.

Theorem 3.1 *We suppose that $\widehat{\pi}$ and $\widetilde{p}_2$, respectively, satisfy equations* (3.5.6) *and* (3.5.8), *respectively. Then $\widehat{\pi}$ is the unique solution to the fractional risk-sensitive linear quadratic control problem* $\{(3.5.3),(3.5.4),(3.5.5)\}$.

Proof. First of all, we know that the mapping $t\mapsto t^2$ is convexe, hence the Hamiltonian and the function $\psi(V_0) = \frac{V_0^2}{2}$, respectively, are convexe in (V, π) and π, respectively. Moreover, let $\pi \in \Pi$ be another control than $\widehat{\pi}$, then

$$(\pi - \widehat{\pi})\, \mathcal{H}_\pi(t) = (\pi - \widehat{\pi}) \left[\pi + z_t \widetilde{p}_2(t)\right].$$

From equation (3.5.6), we have

$$\begin{aligned} (\pi - \widehat{\pi})\, \mathcal{H}_\pi(t) &= \left[\pi + z_t \widetilde{p}_2(t)\right]\left[\pi + z_t \widetilde{p}_2(t)\right] \\ &= \left[\pi + z_t \widetilde{p}_2(t)\right]^2 \geq 0. \end{aligned}$$

It results by theorem 3.1 that $\widehat{\pi}$ is indeed optimal. In fact, the system (3.5.7) and (3.5.8) is fully coupled forward backawrd system, whose solution $((V, z), (\widetilde{p}_2, l)$ is difficult to express explicitely. Therefore, we consider the following linear combination of V and $\widetilde{p}_2$ as follows

$$\widetilde{p}_2 = \chi \widehat{V} + \varphi, \tag{3.5.10}$$

where χ and φ are two diterministic differentiable functions. Then applying the generalized integration by parts formula 1.2.3 to (3.5.10), we get

$$\begin{aligned} d\widetilde{p}_2(t) &= d\left(\chi_t \widehat{V}_t + \varphi_t\right) \\ &= \dot{\chi}_t \widehat{V}_t + \chi_t d\widehat{V}_t + \dot{\varphi}_t \\ &= \left[\dot{\chi}_t \widehat{V}_t + \chi_t \left(r\widehat{V}_t - \widehat{z}_t^2 \widetilde{p}_2(t)\right) + \dot{\varphi}_t\right] dt - \chi_t \widehat{z}_t dW_t^H \\ &= \left[\dot{\chi}_t \widehat{V}_t + \chi_t \left(r\widehat{V}_t - \widehat{z}_t^2 \left(\chi_t \widehat{V}_t + \varphi_t\right)\right) + \dot{\varphi}_t\right] dt - \chi_t \widehat{z}_t dW_t^H \\ &= \left(\dot{\chi}_t \widehat{V}_t + r\chi_t \widehat{V}_t - \widehat{V} \widehat{z}^2 \chi_t^2 - \chi_t \varphi \widehat{z}_t^2 + \chi_t \dot{\varphi} \widehat{z}_t\right) dt - \chi_t \widehat{z}_t dW_t^H, \end{aligned}$$

by taking into consideration (3.5.10) and (3.5.9) , we get

$$\begin{aligned}
d\widetilde{p}_2(t) &= -\mathcal{H}_V^{\theta}(t)dt - \mathcal{H}_z^{\theta}(t)dW_t^{\theta,H} \\
&= -\left(\widehat{V}_t + r\widetilde{p}_2(t)\right)dt - \widetilde{p}_2(t)(\theta\widehat{l}_t + \widehat{\pi})\left(dW_t^H - 2\theta\widehat{l}_t dt\right) \\
&= -\left[\widehat{V}_t + r\left(\chi_t\widehat{V}_t + \varphi_t\right)\right]dt - \left(\chi_t\widehat{V}_t + \varphi_t\right)(\theta\widehat{l}_t + \widehat{\pi})\left(dW_t^H - 2\theta\widehat{l}_t dt\right) \\
&= -\left[\widehat{V}_t + \left(r + 2\theta\widehat{l}_t\left(\theta\widehat{l}_t + \widehat{\pi}\right)\right)\left(\chi_t\widehat{V}_t + \varphi_t\right)\right]dt - \left(\chi_t\widehat{V}_t + \varphi_t\right)\left(\theta\widehat{l}_t + \widehat{\pi}\right)dW_t^H \\
&= -\left[\widehat{V}_t + \chi_t\widehat{V}_t\left(r + 2\theta^2\widehat{l}_t^2 + 2\theta\widehat{l}_t\widehat{\pi}\right) + \varphi_t\left(r + 2\theta^2\widehat{l}_t^2 + 2\theta\widehat{l}_t\widehat{\pi}\right)\right]dt \\
&- \left(\chi_t\widehat{V}_t + \varphi_t\right)\left(\theta\widehat{l}_t + \widehat{\pi}\right)dW_t^H.
\end{aligned}$$

Then

$$\begin{aligned}
\left(\dot{\chi}_t\widehat{V}_t + r\chi_t\widehat{V}_t - \widehat{V}\widehat{z}^2\chi_t^2 - \chi_t\varphi\widehat{z}_t^2 + \chi_t\dot{\varphi}\widehat{z}_t\right)dt - \chi_t\widehat{z}_t dW_t^H = \\
-\left[\widehat{V}_t + \chi_t\widehat{V}_t\left(r + 2\theta^2\widehat{l}_t^2 + 2\theta\widehat{l}_t\widehat{\pi}\right) + \varphi_t\left(r + 2\theta^2\widehat{l}_t^2 + 2\theta\widehat{l}_t\widehat{\pi}\right)\right]dt \\
-\left(\chi_t\widehat{V}_t + \varphi_t\right)\left(\theta\widehat{l}_t + \widehat{\pi}\right)dW_t^H,
\end{aligned}$$

Identifying diffusion terms yields to

$$\chi_t\widehat{z}_t = \left(\theta\widehat{l}_t + \widehat{\pi}\right)\left(\chi_t\widehat{V}_t + \varphi_t\right),$$

then

$$\widehat{l}_t = \frac{\chi_t\widehat{z} - \widehat{\pi}\left(\chi\widehat{V} + \varphi\right)}{\theta\left(\chi_t\widehat{V}_t + \varphi_t\right)}. \tag{3.5.11}$$

Similarly, identifying drift terms yields to the following Riccati and ordinary differential equations,respectively,

$$\begin{cases} \dot{\chi}_t + \widehat{z}_t^2\chi_t^2 + 2\chi_t\left(r + \theta^2\widehat{l}_t^2 + \theta\widehat{l}_t\widehat{\pi}\right) + 1 = 0 \\ \chi_0 = 1, \end{cases} \tag{3.5.12}$$

and

$$\begin{cases} \chi_t\widehat{z}_t\dot{\varphi}_t + \varphi_t\left(r + 2\theta^2\widehat{l}_t^2 + 2\theta\widehat{l}_t\widehat{\pi} - \chi_t\widehat{z}_t^2\right) = 0, \\ \varphi_0 = 0. \end{cases} \tag{3.5.13}$$

■

Theorem 3.2 *We suppose that χ and φ, respectively, are the unique solutions of* (3.5.12) *and* (3.5.13), *respectively, then the optimal control has the following feed-back state*

$$\widehat{\pi} = \chi_t\widehat{z}_t\widehat{V}_t + \widehat{z}_t\varphi_t.$$

3.5.2 Explicit Solution of the Riccati Equation

This paragraph is dedicated to finding the explicit solution of the Riccati differential equation (3.5.12), for simplicity we denote $h(t) = 2\left(r + \theta^2 \widehat{l}_t^2 + \theta \widehat{l}_t \widehat{\pi}\right)$, then (3.5.12) becomes

$$\dot{\chi}_t + \widehat{z}_t^2 \chi_t^2 + \chi_t h(t) + 1 = 0.$$

By simple algebra, we can show that

$$-dt = \frac{d\chi_t}{\left(\widehat{z}\chi_t + \dfrac{h(t)}{2\widehat{z}_t}\right)^2 - \nabla^2(t)}, \tag{3.5.1}$$

where $\nabla^2(t) = \dfrac{h^2(t)}{4\widehat{z}_t^2} - 1$. Now, we put $u_t = \widehat{z}_t \chi_t + \dfrac{h(t)}{2\widehat{z}_t}$, then (3.5.1) becomes

$$-\widehat{z}_t dt = \frac{du_t}{u_t^2 - \nabla^2(t)},$$

that is,

$$-2\nabla(t)\widehat{z}_t dt = \frac{du_t}{u_t - \nabla(t)} - \frac{du_t}{u_t + \nabla(t)}.$$

Integrating from 0 to t, we get

$$2\nabla(t)\widehat{z}_t t = \log\left(\frac{u_s + \nabla(s)}{u_s - \nabla(s)}\right)\Bigg|_0^t,$$

then

$$2\nabla(t)\widehat{z}_t t = \log\left(\frac{\widehat{z}_s \chi_s + \dfrac{h(s)}{2\widehat{z}_s} + \nabla(s)}{\widehat{z}_s \chi_s + \dfrac{h(s)}{2\widehat{z}_s} - \nabla(s)}\right)\Bigg|_0^t$$

$$= \log\left(\frac{\widehat{z}_t \chi_t + \dfrac{h(t)}{2\widehat{z}_t} + \nabla(t)}{\widehat{z}_t \chi_t + \dfrac{h(t)}{2\widehat{z}_t} - \nabla(t)}\right) - \log\left(\frac{\widehat{z}_0 + \dfrac{h(0)}{2\widehat{z}_0} + \nabla(0)}{\widehat{z}_0 + \dfrac{h(0)}{2\widehat{z}_0} - \nabla(0)}\right).$$

Put

$$C = \frac{\widehat{z}_0 + \dfrac{h(0)}{2\widehat{z}_0} + \nabla(0)}{\widehat{z}_0 + \dfrac{h(0)}{2\widehat{z}_0} - \nabla(0)},$$

we get

$$Ce^{2t\nabla(t)\widehat{z}_t} = \frac{\widehat{z}_t\chi_t + \dfrac{h(t)}{2\widehat{z}_t} + \nabla(t)}{\widehat{z}_t\chi_t + \dfrac{h(t)}{2\widehat{z}_t} - \nabla(t)},$$

which leads in the end to

$$\chi_t = \frac{\dfrac{h(t)}{2\widehat{z}_t}\left(1 - Ce^{2t\nabla(t)\widehat{z}_t}\right) + \nabla(t)\left(Ce^{2t\nabla(t)\widehat{z}_t} + 1\right)}{\widehat{z}_t\left(Ce^{2t\nabla(t)\widehat{z}_t} - 1\right)}. \tag{3.5.2}$$

Conclusion

In this thesis, we use the Malliavin calculus to establish two main results. Attaining Pontryagin's stochastic maximum principle for a backward doubly stochastic differential equation driven by Wiener process and fractional Brownian motion, with Hurst parameter $H \in \left(\frac{1}{2}, 1\right)$, is the first one. This result was obtained after applying the Doss-Sussmann transformation to the backward doubly dynamics (2.1.1). Lemma 2.2.1 and theorem 2.3.1 state our results. The second one is the risk sensitive necessary and sufficient optimality conditions for a fractional backward SDE. A variety of advanced mathematical tools are used such as: logarithmic transformation (see Lemma 3.3.1) as in [20], transforming the adjoint equation 3.3.3, and the convexity of the Hamiltonian (3.3.12) and the initial cost functional. The results are given in theorems 3.1 and 3.1, respectively.

• In our first result [10]: (chapter 2), we extend the Doss-Sussmann transformation introduced in [12, 13] and apply it to fractional doubly BSDE.

• In our second result [11]: (chapter 3), we use similar approach of Djehich et al. [18].

• Our logarithmic transformation in Lemma 3.3.1 may be considered as generalization of the one introduced by El Karoui an Hamadène in [20], from risk neutral to risk sensitive logarithmic quasi-martingale.

• We proceed as Yong in [61], and our result may be considered as generalization from risk neutral to risk sensitive stochastic maximum principle.

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