Democratic and Popular Republic of Algeria
Ministry of Higher Education and Scientific Research

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Faculty of Exact Sciences, Natural and Life Sciences
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Doctoral Thesis Under the Title

The Numerical Solution of Nonlinear Weakly Singular Volterra Integral Equations

Presented by

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14/07/2021

Dedication

This thesis is dedicated to my parents who

support me and advise me in my academic career.

Indeed, there do not exist any words which can express

my gratitude to them. On the other hand, i also

want to dedicate this thesis to all my other family members.

Acknowledgements

I would like to express my thanks to **Pr. Zouhir MOKHTARI** for supervising this thesis. I am really grateful to him for all his advices, and for his encouragement and for his support.

I want to express my gratitude to **Dr. Hanane KABOUL** for her contribution to this work. I am thankful to her for her support and for her time.

I express my thanks to all the members of the jury of my doctoral thesis, **Pr. Khaled MELKEMI**, **Pr. Yahia DJABRANE**, **Dr. El Amir DJEFAL** and **Pr. Boubakeur LABED**.

I would like to express my gratitude to **Pr. Kamel MEFTAH** for his support. I am really grateful to him for his encouragement.

I am thankful to all teachers in the faculty of Exact Sciences, Natural and Life Sciences of Biskra university and to everyone in the Applied Mathematics Laboratory.

Contents

Symbols

I_y	the identity operator
$\mathcal{L}(Y)$	the set of continuous linear operators from Y into itself
$\|Y\|_2$	the Euclidean norm of $Y \in \mathbb{R}^{n+1}$ such that $\|Y\|_2 = \sqrt{\sum_{i=1}^{n+1} y_i^2}$
$C[a,b]$	the space of continuous functions given on an interval $[a,b]$
$\|\gamma\|_\infty$	the uniform norm of $\gamma \in C[a,b]$ such that $\|\gamma\|_\infty = \sup_{t\in[a,b]} \|\gamma(t)\|$
dG	the derivative of a differentiable function G
$\|.\|$	a norm
$\det[M]$	the determinant of a matrix M
P^T	the transpose of a vector P
M^{-1}	the inverse of a matrix M
I	the identity matrix
$\lvert . \rvert$	the absolute value

Abstract

Abstract: This thesis is intended to solve Volterra integral equations. More precisely, it focuses on the cases of a weakly singular kernel. These integral equations can be solvable when we use the product integration method that plays an important role. For simplicity, we begin this thesis by giving some elementary concepts and basic theories.

keywords: Volterra integral equation, product integration method, weakly singular kernel.

الملخص :

هذه الاطروحة تهدف الى حل المعادلات التكاملية لفولتيرا. لأكثر دقة, هي تركز على الحالات ذات نواة ضعيفة.

هذه المعادلات التكاملية تستطيع ان تكون قابلة للحل عند استخدام طريقة تكامل المنتج التي تلعب دورا مهما. لأجل التسهيل , نبتدأ هذه الاطروحة بتقديم بعض المبادئ الاولية و النظريات الاساسية.

الكلمات المفتاحية :

المعادلات التكاملية لفولتيرا. طريقة تكامل المنتج. نواة ضعيفة.

Résumé: Cette thèse a pour but de résoudre les équations intégrales de Volterra, plus précisément, elle s'intéresse aux cas du noyau faiblement singulier. Ces équations intégrales sont possible à résoudre en utilisant la méthode d'intégration produit qui joue un rôle important. Pour simplifier, on commence cette thèse en donnant quelques concepts élémentaires et des théories de base.

Mots-clés: Équation intégrale de Volterra, méthode d'intégration produit, noyau faiblement singulier.

Introduction

The challenge of our work is to treat the more general cases of linear and nonlinear weakly singular Volterra integral equations, for $z, \tau \in [a, b]$,

$$x(z) = \gamma(z) + \int_a^z k(z,\sigma)x(\sigma)d\sigma, \tag{1}$$

and

$$z(\tau) = g(\tau) + \int_a^\tau \nu(\tau,\sigma)\kappa(\tau,\sigma,z(\sigma))d\sigma, \tag{2}$$

when their kernels are subject to certain conditions. More specifically, in [59], Nemer, Mokhtari and Kaboul apply a product integration method to solve (2) which certainly represents the nonlinear cases of Volterra intgral equations. This method enables us to cover general forms of weakly singular Volterra integral equations and to obtain precise results. In addition, Nemer, Kaboul and Mokhtari treat the linear Volterra integral equation (1) in [58] that is based on the techniques of a product integration method to get best solutions. This reflects the fact that the application of a product integration method makes it possible to solve the former integral equations. For both equations (1) and (2), we base on a piecewise linear approximation. This leads to formulate linear and nonlinear systems of integral equations that are solved by classical methods. Specifically, we use Broyden's method for nonlinear case. The convergence of Broyden's method cannot be realized without satisfying specific conditions. One of these conditions is an appropriate choice of an initial guess $Y^{(0)}$ and of an approximation of the Jacobian matrix at a root Y^*, see [40, 21, 83, 55] for more information. Indeed, the way of analysing (1) and (2) is inspired by the paper [29] that investigates the Fredholm integral equation of the form

$$x(s) - \int_a^b H(s,t)L(s,t)F(t,x(t))dt = y(s),\ s \in [a,b], \tag{3}$$

see [39] for more details.

On the other hand, practical techniques of a product integration method help us to skip insurmountable obstacles. This reveals the fact that a product integration method is of importance in our work.

Over the years, several methods are used to solve different types of integral equations. As an example, a Legendre-collocation method emerged in [78] that is devoted to the Volterra integral equation of the second kind

$$y(t) + \int_0^t R(t,s)y(s)ds = f(t),\ t \in [0,T], \tag{4}$$

where R and f are known. For some other methods, we can mention a Nyström type method which is applied in [6] to the nonlinear Volterra integral equation

$$y(t) = \int_0^t h(t,s)f(s,y(s))ds + g(t),\ t \in [0,T], \tag{5}$$

where h is weakly singular.

Indeed, we begin this thesis by reviewing various classes of Volterra integral equations, see [5, 50, 65, 66, 26, 48]. We can see that this part pays attention to first-kind and second-kind Volterra integral equations. To distinguish between them, it suffices to observe their sides. For an overview of integral equations, Section 1.1 includes some examples as Volterra-Hammerstein integral equations. We then proceed to discuss the convergence of Broyden's method, and to study polynomial approximations that involve Hermite−Fejér interpolation polynomials, piecewise linear interpolation and Bernstein polynomials. On the other hand, we present a review of bounded linear operators to be able to deal with the operator equation of the form

$$Hy = p. \tag{6}$$

For a deeper understanding of bounded linear operators, we can resort to [4, 1, 44] which enables us to get an exact knowledge about the existence and uniqueness of the solution of (6). Moreover, we give some important theorems like the theorem of the convergence of the Neumann series.

In the sequel, this thesis involves four chapters which are summarized as follows.

In Chapter 1, we present some elementary concepts and basic theories. Specifically, we begin with a review of various classes of Volterra integral equations. We proceed to describe the convergence of Broyden's method, and then we discuss polynomial approximations. In addition, we give a review of bounded linear operators.

In Chapter 2, we study the application of a product integration method to nonlinear weakly singular Volterra integral equations. More specifically, we begin with the solvability of these integral equations and then show the techniques of a product integration method in detail. Of course, we conclude Chapter 2 by discussing the convergence of approximate solutions.

We devote Chapter 3 to linear weakly singular Volterra integral equations which can be solved by resorting to a product integration method that allows us to get an optimal solution.

To illustrate the convergence of approximate solutions, we need to show the numerical performance of a product integration method. Chapter 4 provides numerical applications involving some examples of weakly singular Volterra integral equations.

Chapter 1

Discussion of basic theories for Volterra integral equations

In this chapter, we present some elementary concepts related to Volterra integral equations. This enables us to avoid complications that can happen in the later chapters. We also provide fundamental theories which play an important role, and we try to cover all basic notions that are required to get a complete work. Therefore, we can say that this chapter represents a significant guide to Volterra integral equations.

1.1 Review of various classes of Volterra integral equations

This section gives a brief summary of various types of integral equations that involves Volterra integral equations of the first and second kind, and we discuss linear and nonlinear cases. This illustrates the differences between these integral equations. To acquire an elementary knowledge about the types of weakly singular Volterra integral equations, we review several examples like Abel-Volterra integral equations.

1.1.1 First−kind Volterra integral equations

To study integral equations, it is possible to begin with the linear Volterra integral equation of the following form

$$0 = \gamma(z) + \Upsilon x(z), \tag{1.1}$$

where Υx, the so-called Volterra integral operator, is determined by

$$\Upsilon x(z) = -\int_a^z k(z,\sigma)x(\sigma)d\sigma,\ z \geqslant a, \tag{1.2}$$

such that the functions k and γ are known (See [50, 5, 65, 66]). Moreover, one can seek an approximate solution for the former integral equation by employing various methods, for example a product integration method. In other words, a product integration method represents one of the best methods that produces an optimal approximation to the exact solution x.

On the other hand, it is important to recall Abel-Volterra integral equations of the first kind. This class of integral equations is given by

$$0 = \gamma(z) - \int_0^z k(z,\sigma)x(\sigma)d\sigma,\ z > 0, \tag{1.3}$$

where the part k is defined as, for a smooth function φ,

$$k(z,\sigma) = (z^\ell - \sigma^\ell)^{-\alpha}\varphi(z,\sigma), \tag{1.4}$$

such that the value of ℓ is strictly positive ($\ell > 0$) and $\alpha \in (0,1)$, for details see [5]. To conclude this subsequent section, we must define nonlinear Volterra integral equations of the first kind. For this, it suffices to take the equation (1.1), but Υx now represents the integral operator of the form

$$\Upsilon x(z) = -\int_a^z k(z,\sigma,x(\sigma))d\sigma,\ z \geqslant a. \tag{1.5}$$

1.1.2 Second−kind Volterra integral equations

In what follows, we provide a rather comprehensive account of second-kind Volterra integral equations that involves linear and nonlinear cases. We then talk about smooth and weakly singular kernels. More precisely, we start with the general form of the nonlinear Volterra integral equation

$$x(z) = \gamma(z) + \Upsilon x(z), \tag{1.6}$$

where

$$\Upsilon x(z) = -\int_a^z k(z,\sigma,x(\sigma))d\sigma,\ z \geqslant a, \tag{1.7}$$

with x denotes the unknown function.

In Chapter 2, we deal with the nonlinear weakly singular Volterra integral equation of the form

$$z(\tau) = g(\tau) + \int_a^\tau \nu(\tau,\sigma)\kappa(\tau,\sigma,z(\sigma))d\sigma,\ \tau \in [a,b], \tag{1.8}$$

where κ and g are sufficiently smooth functions. Concerning the weakly singular kernel ν, it is enough to claim that

$$\sup_{\tau\in[a,b]} \int_{[a,b]} |\nu(\tau,\sigma)|\, d\sigma < +\infty, \tag{1.9}$$

which is also employed in [28, 29] for the convergence proofs.

We proceed to the linear Volterra integral equation that can be expressed by

$$x(z) = \gamma(z) - \int_a^z \phi(z,\sigma)x(\sigma)d\sigma,\ z \in [a,b], \tag{1.10}$$

for given functions ϕ and γ. Comparing (1.10) with (1.6), the difference between them appears in the right-hand sides of these equations. More precisely, we can note that the function k of the equation (1.6) is turned into

$$k(z,\sigma,x(\sigma)) = \phi(z,\sigma)x(\sigma), \tag{1.11}$$

which leads to get linear Volterra integral equations. The third chapter is devoted to the treatment of the linear weakly singular Volterra integral equation of the expression

$$x(z) = \gamma(z) + \int_a^z k(z,\sigma)x(\sigma)d\sigma,\ z \in [a,b], \tag{1.12}$$

the kernel k can be rewritten in the following form

$$k(z,\sigma) = \psi(z,\sigma)\varphi(z,\sigma),\ a \leqslant \sigma \leqslant z \leqslant b, \tag{1.13}$$

while ψ is a singular function. This means that we have, for sufficiently smooth functions γ and φ,

$$x(z) = \gamma(z) + \int_a^z \psi(z,\sigma)\varphi(z,\sigma)x(\sigma)d\sigma,\ z \in [a,b]. \tag{1.14}$$

To deal with the former integral equation, we need to assume that

1) $\sup_{z\in[a,b]} \int_{[a,b]} |\psi(z,\sigma)|\, d\sigma < +\infty,$

2) $\lim_{\delta\to 0} \sup_{z,z'\in[a,b],|z-z'|\leq\delta} \int_{[a,b]} |\psi(z,\sigma) - \psi(z',\sigma)|\, d\sigma = 0,$

which represent the principal assumptions for analysing Fredholm integral equations in [28, 29].

As a particular case, in (1.6), if the function k is given by

$$k(z,\sigma,x(\sigma)) = -(z-\sigma)^{-\alpha}\varphi(z,\sigma)p(\sigma,x(\sigma)),\ \alpha \in (0,1), \tag{1.15}$$

or

$$k(z,\sigma,x(\sigma)) = -\log(z-\sigma)\varphi(z,\sigma)p(\sigma,x(\sigma)), \tag{1.16}$$

for $0 \leqslant \sigma \leqslant z \leqslant b$, then the equation (1.6) becomes the so-called weakly singular Volterra-Hammerstein integral equation, as shown in [13].

1.2 Convergence of Broyden's method

To solve the nonlinear system of the form

$$G(Y) = 0,\ Y \in D^{n+1}, \tag{1.17}$$

it suffices to apply Broyden's method which makes it possible to find $n+1$ roots, see [40, 21, 83, 55]. More specifically, the algorithm of this method is defined by

$$Y^{(k+1)} = Y^{(k)} - B_k\, G(Y^{(k)}),\ k = 0, 1, \dots \tag{1.18}$$

where

$$B_{k+1} = B_k + \frac{(\delta_k - B_k\theta_k)\delta_k^T B_k}{\delta_k^T B_k \theta_k}, \tag{1.19}$$

for

$$\theta_k = G(Y^{(k+1)}) - G(Y^{(k)}) \text{ and } \delta_k = Y^{(k+1)} - Y^{(k)}. \tag{1.20}$$

Before discussing the convergence of Broyden's method, we give some information concerning vector spaces.

Definition 1.2.1 *[16]*

We say that a mapping $\|\cdot\| : Y \to \mathbb{R}$ is a norm on Y, where Y denotes a vector space over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$, if for all $y, z \in Y$ and all $\alpha \in \mathbb{K}$ we have

- $\|y\| = 0 \; iff \; y = 0,$
- $\|y\| \geqslant 0,$
- $\|\alpha y\| = |\alpha| \, \|y\|,$
- $\|y + z\| \leqslant \|y\| + \|z\|.$

Remark 1.2.1 *(see [16, 24] for details)*

1. *The pair of a vector space Y and a norm $\|\cdot\|$ constructs a normed vector space $(Y, \|\cdot\|)$.*
2. *Let $Y = C[a, b]$, where a and b belong to $\mathbb{R}$, and claim that, for $\gamma \in Y$,*

$$\|\gamma\| = \max_{t \in [a,b]} |\gamma(t)|. \tag{1.21}$$

Then a pair $(Y, \|\cdot\|)$ is said to be a normed vector space.

Definition 1.2.2 *([24] A convex subset)*

Assume that D is a subset of a vector space Y and take $y, z \in D$. If we let $ty + (1-t)z \in D$, for $t \in [0,1]$, then we call D a convex subset.

Theorem 1.2.1 *[57]*

Assume that

(a) *D is an open convex set.*

(b) *For $\ell, i, j = 1, \cdots, n+1$, and for $Y \in D^{n+1}$,*

$$\left|\partial^2_{\ell j} G_i(Y)\right| \leqslant K,\ K > 0. \tag{1.22}$$

(c) *For $Y \in D^{n+1}$, the Jacobian matrix, denoted by $J_G(Y)$, is continuously differentiable.*

Then we can say that the Jacobian matrix of the mapping $G : D^{n+1} \subset \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is Lipschitz continuous on D^{n+1}.

Under certain assumptions on the nonlinear mapping G, we can obtain $n+1$ optimal roots. In other words, when G is subject to some conditions that are mentioned in the following theorem, we can say that we have $n+1$ precise roots. Indeed, this leads to achieve the convergence of Broyden's method.

Theorem 1.2.2 *[21, 55]*

Assume that, for an open convex set D,

(1) *the function* $G : D^{n+1} \subset \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ *is continuously differentiable,*

(2) $\exists Y^* \in D^{n+1} : \; G(Y^*) = 0$,

(3) *the Jacobian matrix* J_G *satisfies*

$$J_G \in Lip_{\chi}(Y^*), \tag{1.23}$$

(4) *the inverse of the Jacobian matrix* J_G *exists at* Y^*,

and choose the approximation of the Jacobian matrix at Y^* *and the initial guess, denoted respectively by,* B_0^{-1} *and* $Y^{(0)}$*, satisfying*

(i) $\exists\, \varepsilon > 0 \; : \|B_0^{-1} - J_G(Y^{(*)})\| < \varepsilon.$

(ii) $\exists\, \rho > 0 \; : \|Y^{(0)} - Y^*\| < \rho.$

Then Broyden's method converges to the root of the function G*, denoted by* Y^*.

1.3 Polynomial approximations

This section is intended to review certain interpolation polynomials that represent some of the few best approximations. More precisely, we focus on Hermite-Fejér interpolation polynomials which are followed by piecewise linear interpolation. We also present a detailed description of Bernstein polynomials. To complete this section, we add some important theorems which play a role in the convergence proofs of the second and third chapters.

1.3.1 Hermite–Fejér interpolation polynomials

Let γ be a given function on an interval $[-1,1]$. The Chebyshev polynomial of the first kind is defined as

$$T_{n+1}(s) = \cos\left((n+1)\arccos s\right),\ s \in [-1,1], \tag{1.24}$$

and the zeros of the Chebyshev polynomial of the first kind T_{n+1} are given by

$$t_i = \cos\left(\frac{\pi}{2(n+1)}(2i-1)\right),\ for\ i = 1,\dots,n+1. \tag{1.25}$$

The Hermite-Fejér interpolation polynomial of γ is written as

$$H_{n+1}(\gamma, s) = \sum_{i=1}^{n+1} \gamma(t_i)(1-st_i)h_{n+1}^2(s), \tag{1.26}$$

where

$$h_{n+1}(s) = \frac{T_{n+1}(s)}{(n+1)(s-t_i)}, \tag{1.27}$$

while the degree of H_{n+1} is m with $m \leqslant 2n+1$.

Theorem 1.3.1 *[27, 54]*

Suppose that γ, defined on $[-1,1]$, is a continuous function, and let $H_{n+1}(\gamma)$ be the Hermite-Fejér interpolation polynomials (1.26). Then we can get

$$\lim_{n\to+\infty} \|\gamma - H_{n+1}(\gamma)\|_\infty = 0.$$

1.3.2 Piecewise linear interpolation

Firstly, we begin by setting

$$F_n := \{t_i,\ 1 \leqslant i \leqslant n+1\},$$

which represents the set of a mesh on the interval $[a, b]$ such that

$$a = t_1 < t_2 < \cdots < t_{n+1} = b.$$

To get a uniform mesh, it suffices to put

$$t_i = a + (i-1)h_n,\ i = 1, \ldots, n+1, \tag{1.28}$$

where

$$h_n = \frac{b-a}{n}. \tag{1.29}$$

As mentioned in [8], the piecewise linear interpolation of a function γ, defined on $[a, b]$, is written as, $\forall t \in [t_i, t_{i+1}]$,

$$[\gamma(t)]_n = \frac{t_{i+1}-t}{t_{i+1}-t_i}\gamma(t_i) + \frac{t-t_i}{t_{i+1}-t_i}\gamma(t_{i+1}),\ for\ i = 1, \ldots, n+1. \tag{1.30}$$

Lemma 1.3.1 *If $F_n := \{t_i,\ 1 \leqslant i \leqslant n+1\}$ is the set of a uniform mesh on $[a, b]$, and if $[v(s,t)]_n$ represents the piecewise linear interpolation of a continuous function $v(s,t)$. Then*

$$\Big|[v(s,t)]_n - v(s,t)\Big| \leq w_2(v, h_n),$$

where

$$w_2(v, h_n) = \sup_{s\in[a,b]} w(v(s,\cdot), h_n),$$

and

$$w(v(s,\cdot),h_n) = \sup_{s\in[a,b],|\cdot\,-t|\leq h_n} |v(s,\cdot)-v(s,t)|,$$

and where, for $t \in [t_i, t_{i+1}]$,

$$[v(s,t)]_n = \frac{1}{h_n}\left[(t_{i+1}-t)v(s,t_i) + (t-t_i)v(s,t_{i+1})\right],\ i = 1,\dots,n+1.$$

Proof : See [28] ∎

1.3.3 Bernstein polynomials

In what follows, we take the set of a uniform mesh, denoted by S_n, which is determined by

$$S_n := \{t_i = (i-1)h_n,\ 1 \leqslant i \leqslant n+1\},$$

such that

$$0 = t_1 < t_2 < \cdots < t_{n+1} = 1,$$

while the formula of h_n is given by

$$h_n = \frac{1}{n},\ for\ n = 1,2,\dots.$$

Let γ be a continuous function defined on $[0,1]$, the expression of Bernstein polynomials is

$$(B_n\gamma)(s) = \sum_{i=1}^{n+1} l_{i,n+1}(s)\gamma(t_i),\ for\ s \in [0,1], \tag{1.31}$$

where

$$l_{i,n+1}(s) = \frac{n!}{(i-1)!(n-i+1)!}s^{i-1}(1-s)^{n-i+1},\ i = 1,\dots,n+1. \tag{1.32}$$

The following theorem, the so-called Bernstein's theorem, is given and proved in detail in [16, 64, 72].

Theorem 1.3.2 *The function γ is continuous on $[0,1]$ and $B_n\gamma$ is the Bernstein polynomial, given by (1.31) and (1.32). Then, the uniform norm of the difference $\gamma - B_n\gamma$ satisfies*

$$\|\gamma - B_n\gamma\|_\infty \longrightarrow 0 \textit{ as } n \longrightarrow +\infty.$$

1.4 Review of bounded linear operators

We devote this section to bounded linear operators. We present some interesting definitions and theorems. In particular, we show the theorem of the convergence of the Neumann series. This theorem gives us information on the existence and uniqueness of the solution of the equation of the second kind

$$y - Qy = p, \tag{1.33}$$

which can also be written as

$$(I_y - Q)y = p, \tag{1.34}$$

where I_y denotes the identity operator.

Definition 1.4.1 *[11]*

Let Y and Z be Banach spaces, and let Q be a mapping from Y to Z.

1. *Assume that*

(a) $Q(y_1 + y_2) = Qy_1 + Qy_2$, *for* $y_1\ y_2 \in Y$,

(b) $Q(\alpha y_1) = \alpha Q y_1$, *for* $\alpha \in \mathbb{R}$.

Then, we can say that Q is a linear operator.

2. *Claim that the assumptions* $(a) - (b)$ *hold, and* Q *satisfies,* $\forall y_1 \in Y$,

$$\|Qy_1\|_Z \leqslant \chi \|y_1\|_Y .$$

Then, Q is a bounded linear operator.

3. *For* $y_1 \in Y$ *and* $\{y_n\} \subset Y$, *if we have, as* $n \longrightarrow +\infty$,

$$\|y_n - y_1\|_Y \longrightarrow 0 \Longrightarrow \|Qy_n - Qy_1\|_Z \longrightarrow 0.$$

Then, the linear operator Q becomes continuous.

The norm of the bounded linear operator Q is defined as

$$\|Q\| = \sup_{\|y_1\|_Y = 1} \|Qy_1\|_Z .$$

Theorem 1.4.1 *[32]*

Let Y *and* Z *be normed spaces, and let* $Q : Y \longrightarrow Z$ *be a bounded linear operator. Then, for the case* $Y \neq 0$,

$$\|Q\| = \sup_{\|y\| \leqslant 1} \|Qy\| = \sup_{y \neq 0} \frac{\|Qy\|}{\|y\|} = \sup_{\|y\|=1} \|Qy\| .$$

Theorem 1.4.2 *([16] Convergence of the Neumann series)*

Suppose that

(i) $(Y, \|.\|)$ *is a Banach space,*

(ii) $Q \in \mathcal{L}(Y)$,

(iii) *the norm of* Q *satisfies* $\|Q\| < 1$.

Then, we have that $(I_y - Q)$ *is bijective and that* $(I_y - Q)^{-1} \in \mathcal{L}(Y)$ *with*

$$\left\|(I_y - Q)^{-1}\right\| \leqslant \frac{1}{1 - \|Q\|},$$

where

$$(I_y - Q)^{-1} = \sum_{k=0}^{\infty} Q^k.$$

Subsequently, we investigate the operator equation of the form

$$Hy = p, \tag{1.35}$$

where H is a mapping from Y to Z such that Y and Z are Banach spaces. To solve the former operator equation, it suffices to resort to

$$H_n y_n = p, \tag{1.36}$$

which certainly represents an approximation of (1.35) (see [44, 5]). The following theorem proves the existence of H_n^{-1}, in the case where $H_n = \lambda - Q_n$ and $Z = Y$, and the convergence of y_n to y as $n \longrightarrow +\infty$.

Theorem 1.4.3 *If Q is a bounded linear operator,*

$$\|Q - Q_n\| \longrightarrow 0 \text{ as } n \longrightarrow +\infty,$$

where Q_n is a sequence of bounded linear operators and if $\lambda - Q : Y \underset{onto}{\overset{1-1}{\longrightarrow}} Y$. Then, for $n \geqslant N$, $\lambda - Q_n$ are invertible with

$$\left\|(\lambda - Q_n)^{-1}\right\| \leqslant \frac{\left\|(\lambda - Q)^{-1}\right\|}{1 - \|(\lambda - Q)^{-1}\| \, \|Q - Q_n\|},$$

and the error between the solutions of (1.35) and (1.36) satisfies, for $H = \lambda - Q$,

$$\|y - y_n\| \leqslant \left\|(\lambda - Q_n)^{-1}\right\| \|Qy - Q_n y\| .$$

Proof : See [5] ∎

Chapter 2

Numerical analysis of integral equations

In this chapter, the focus is on the numerical study of nonlinear Volterra integral equations. We begin this study by investigating the solvability of these integral equations. More specifically, we try to cover the Volterra integral equations when their kernels are weakly singular. We then discuss the techniques of a product integration method in detail. We conclude this chapter by presenting the convergence of the product integration method.

2.1 Solvability of nonlinear Volterra integral equations

The nonlinear weakly singular Volterra integral equation is, for $\tau \in [a, b]$,

$$\forall a, b \in \mathbb{R} \,:\, z(\tau) = g(\tau) + \int_a^\tau \nu(\tau, \sigma)\kappa(\tau, \sigma, z(\sigma))\, d\sigma, \tag{2.1}$$

while the functions κ and g are known and ν is a weakly singular kernel. More specifically, we study (2.1) in the case where, see [13, 46],

$$\kappa(\tau, \sigma, z(\sigma)) = \phi(\tau, \sigma)\, p(\sigma, z(\sigma)). \tag{2.2}$$

In addition, the functions ν and p are supposed to be subject to the following two conditions that are also given in [45, 29, 28, 30] :

1) The weakly singular kernel ν satisfies

$$\sup_{\tau \in [a,b]} \int_{[a,b]} |\nu(\tau, \sigma)|\, d\sigma < +\infty. \tag{2.3}$$

2) For an open convex set D in $\mathbb{R}$, we assume that $R : D \to \mathbb{R}$ is two times continuously differentiable where, for $x \in D$,

$$R(x) = p(\sigma, x),\ \sigma \in [a, \tau]. \tag{2.4}$$

For the existence and uniqueness of a continuous solution, it is necessary that κ and g are sufficiently smooth (see [63, 62, 13, 6]). Before advancing further, we need to get an equivalent equation defined on an interval $[-1, 1]$. As in [7], we use the following substitutions

$$\tau = \frac{b}{2}(s+1) - \frac{a}{2}(s-1),\ s \in [-1, 1], \tag{2.5}$$

and

$$\sigma = \frac{b}{2}(t+1) - \frac{a}{2}(t-1),\ t \in [-1, s]. \tag{2.6}$$

Thus, the nonlinear Volterra integral equation

$$z(\tau) = g(\tau) + \Upsilon z(\tau),\ \tau \in [a, b], \tag{2.7}$$

where the integral operator Υz is defined as

$$\Upsilon z(\tau) = \int_a^\tau \nu(\tau, \sigma)\phi(\tau, \sigma)\, p(\sigma, z(\sigma))\, d\sigma, \tag{2.8}$$

is transformed into

$$y(s) = f(s) + \mathit{\Gamma} y(s),\ s \in [-1, 1], \tag{2.9}$$

where $\mathit{\Gamma} y$ represents the integral operator given by

$$\mathit{\Gamma} y(s) = \frac{b-a}{2} \int_{-1}^{s} \upsilon(s, t)\, \omega(s, t)\, q(t, y(t))\, dt. \tag{2.10}$$

More precisely, the functions q, υ, and ω are

$$q(t, y(t)) = p\Big(\frac{b(t+1)}{2} - \frac{a(t-1)}{2}, z(\frac{b(t+1)}{2} - \frac{a(t-1)}{2})\Big), \tag{2.11}$$

$$\upsilon(s, t) = \nu\Big(\frac{b(s+1)}{2} - \frac{a(s-1)}{2}, \frac{b(t+1)}{2} - \frac{a(t-1)}{2}\Big), \tag{2.12}$$

and

$$\omega(s, t) = \phi\Big(\frac{b(s+1)}{2} - \frac{a(s-1)}{2}, \frac{b(t+1)}{2} - \frac{a(t-1)}{2}\Big). \tag{2.13}$$

From (2.3), we can deduce that

$$\sup_{s \in [-1,1]} \frac{b-a}{2} \int_{-1}^{1} |\upsilon(s, t)|\, dt < +\infty. \tag{2.14}$$

Clearly, the functions q, ω, y and f are continuous from the properties of the composition of continuous functions. The functions f and y are determined, respectively, by

$$f(s) = g\left(\frac{b}{2}(s+1) - \frac{a}{2}(s-1)\right), \tag{2.15}$$

and

$$y(s) = z\left(\frac{b}{2}(s+1) - \frac{a}{2}(s-1)\right). \tag{2.16}$$

From [8], the piecewise linear approximation is defined by, for $i = 1, \cdots, n+1$,

$$[y(t)]_n = \frac{t_{i+1} - t}{t_{i+1} - t_i} y(t_i) + \frac{t - t_i}{t_{i+1} - t_i} y(t_{i+1}), \text{ for } t \in [t_i, t_{i+1}], \tag{2.17}$$

which is employed to approximate $\omega(s,t)\, q(t, y(t))$ in the same way as in [29]. As we will see later, we need to support the convergence proof of the product integration method by adding some numerical examples which present comparison results by using mean absolute and mean squared errors that are defined in [10]. To solve integral equations, it is possible to apply different methods, see for example [5, 46, 22, 79, 15, 78, 7]. For more details of our work, we can check the series of references [60, 3, 35, 41, 17, 74, 49, 51].

2.2 Techniques of a product integration method

In this section, we investigate the nonlinear system, for an open convex set D,

$$G(Y) = 0,\ Y \in D^{n+1}, \tag{2.18}$$

which arises from the application of a product integration method to nonlinear weakly singular Volterra integral equations. In order to be able to solve (2.18), we need to employ Broyden's method (see [40, 21, 83, 55]). The nonlinear mapping G formed by the product integration method is defined as

$$G(Y) = Y - F - \frac{b-a}{2}\, E\, Q(Y), \tag{2.19}$$

where, for $i, j = 1, \cdots, n+1$,

$$E_{ij} := \begin{cases} \frac{1}{t_2 - t_1}\Big[\int_{t_1}^{t_2} \upsilon(t_i, t)(t_2 - t)dt\Big]\omega(t_i, t_1) & j = 1,\ 2 \leqslant i \leqslant n+1, \\ \frac{1}{t_i - t_{i-1}}\Big[\int_{t_{i-1}}^{t_i} \upsilon(t_i, t)(t - t_{i-1})dt\Big]\omega(t_i, t_i) & j = i,\ 2 \leqslant i \leqslant n+1, \\ [\eta_{ij} + \mu_{ij}]\omega(t_i, t_j) & 2 \leqslant j \leqslant i-1 \leqslant n, \\ 0 & \text{otherwise}, \end{cases}$$

such that

$$\begin{aligned} \eta_{ij} &= \frac{1}{t_j - t_{j-1}} \int_{t_{j-1}}^{t_j} \upsilon(t_i, t)(t - t_{j-1})dt, \\ \mu_{ij} &= \frac{1}{t_{j+1} - t_j} \int_{t_j}^{t_{j+1}} \upsilon(t_i, t)(t_{j+1} - t)dt, \end{aligned}$$

and the vector F is defined by $F = \Big[f(t_1), f(t_2), \cdots, f(t_{n+1})\Big]^T$. Also, the vectors $Q(Y)$ and Y are given, respectively, by

$$Q(Y) = \begin{pmatrix} q(t_1, y(t_1)) \\ q(t_2, y(t_2)) \\ \vdots \\ q(t_{n+1}, y(t_{n+1})) \end{pmatrix},$$

and

$$Y = \begin{pmatrix} y(t_1) \\ y(t_2) \\ \vdots \\ y(t_{n+1}) \end{pmatrix},$$

where, for $i = 1, \cdots, n+1$,

$$t_i = \cos\left(\frac{\pi}{2(n+1)}(2i-1)\right), \tag{2.20}$$

which represent the zeros of a Chebyshev polynomial, see [54, 27]. As in [29], the nonlinear system (2.18) is obtained by approximating Γy by the piecewise linear interpolation. This leads to

$$\begin{aligned} \tilde{\Gamma} y(t_i) &= \frac{b-a}{2} \int_{-1}^{t_i} \upsilon(t_i, t)[\,\omega(t_i, t)\, q(t, y(t))]_n dt \\ &= \frac{b-a}{2} \sum_{j=1}^{i-1} \int_{t_j}^{t_{j+1}} \upsilon(t_i, t)[\,\omega(t_i, t)\, q(t, y(t))]_n dt \\ &= \frac{b-a}{2} \sum_{j=1}^{i-1} E_{ij}\, q\left(t_{j+1}, y(t_{j+1})\right) \end{aligned}$$

where

$$[\omega(t_i,t)q(t,y(t))]_n = \frac{t_{j+1}-t}{t_{j+1}-t_j}\,\omega(t_i,t_j)\,q(t_j,y(t_j)) + \frac{t-t_j}{t_{j+1}-t_j}\,\omega(t_i,t_{j+1})\,q(t_{j+1},y(t_{j+1})), \tag{2.21}$$

and $\tilde{\Gamma} y(t_i)$ represents the approximation of the integral operator Γy at the zeros of a Chebyshev polynomial. On the other hand, the Jacobian matrix of the nonlinear mapping G is given by

$$J_G(Y) = I - \frac{b-a}{2}\, E\, L, \tag{2.22}$$

where L represents the diagonal matrix with the entries, for $i = 1, \cdots, n+1$,

$$L_{ii} = \partial_i Q_i(Y). \tag{2.23}$$

More precisely, we have

$$J_G(Y) = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ \varphi_{2,1} & \varphi_{2,2} & \cdots & \cdots & 0 \\ \varphi_{3,1} & \varphi_{3,2} & \varphi_{3,3} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \varphi_{n+1,1} & \varphi_{n+1,2} & \cdots & \varphi_{n+1,n} & \varphi_{n+1,n+1} \end{pmatrix},$$

where

$$\varphi_{ij} := \begin{cases} 1 - \frac{b-a}{2} E_{ii}\, \partial_i Q_i(Y) & j = i,\ 2 \leqslant i \leqslant n+1, \\ -\frac{b-a}{2} E_{ij}\, \partial_j Q_j(Y) & 1 \leqslant j \leqslant i-1 \leqslant n. \end{cases}$$

2.3 Convergence details of the product integration method

This section is devoted to the convergence proof of the product integration method. More specifically, we present some definitions and some theorems that illustrate the convergence of approximate solutions.

Definition 2.3.1 *([47] Differentiability)*

Let G be a given function on an open set $D^n \subseteq \mathbb{R}^n$, let $Y \in D^n$ and assume that

$$\lim_{\|H\|_2 \to 0} \frac{\|G(Y+H) - G(Y) - dG(Y)H\|_2}{\|H\|_2} = 0$$

Then, we can say that the function $G : D^n \longrightarrow \mathbb{R}^n$ is differentiable at Y.

Theorem 2.3.1 *Let D be an open convex set, and let $G : D^{n+1} \subset \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ be a nonlinear mapping defined by*

$$G(Y) = Y - F - \frac{b-a}{2}\, E\, Q(Y),\ Y \in D^{n+1}. \tag{2.24}$$

Then G is continuously differentiable.

Proof : Using the Leibniz integral rule (see [67]), we obtain

$$dG(Y) = I - \frac{b-a}{2}\, E\, dQ(Y). \tag{2.25}$$

From the definition and properties of differentiability (see [47, 56, 85]), we define, choosing H small enough and $Y \in D^{n+1}$,

$$A_G(H) = \|G(Y+H) - G(Y) - d\,G(Y)\,H\| \,/\, \|H\|\,,\ Y + H \in D^{n+1}. \tag{2.26}$$

Then, we have, for $Z = \Big[z(\sigma_1), z(\sigma_2), \cdots, z(\sigma_{n+1})\Big]^T$ and for $P(Z) = \Big[p(\sigma_1, z(\sigma_1)), p(\sigma_2, z(\sigma_2)), \cdots, p(\sigma_{n+1}, z(\sigma_{n+1}))\Big]^T$ with $\sigma_i = \frac{b}{2}(t_i+1) - \frac{a}{2}(t_i-1)$, $i = 1, \cdots, n+1$,

$$\begin{aligned} A_G(H) &= \left\| -\left(\frac{b-a}{2}\right) E\Big[Q(Y+H) - Q(Y) - dQ(Y)\,H\Big]\right\| / \|H\| \\ &= \left\| -\left(\frac{b-a}{2}\right) E\Big[P(Z+H) - P(Z) - dP(Z)\,H\Big]\right\| / \|H\|\,. \end{aligned}$$

Now, we need to prove the differentiability of G_i for $i = 1, \cdots, n+1$. This leads to

$$\begin{aligned} A_G^i(H) &= \left| -\left(\frac{b-a}{2}\right) \sum_{j=1}^{n+1} E_{ij} \left(P_j(Z+H) - P_j(Z) - \sum_{\ell=1}^{n+1} \partial_\ell P_j(Z)\, H_\ell \right) \right| / \|H\| \\ &\leqslant \frac{b-a}{2} \sum_{j=1}^{n+1} |E_{ij}|\, A_P^j(H). \end{aligned}$$

From the differentiability of each component of $P(Z)$, we can have, for $j = 1, \cdots, n+1$,

$$\lim_{\|H\| \to 0} A_P^j(H) = 0. \tag{2.27}$$

Thus

$$\lim_{\|H\| \to 0} A_G^i(H) = 0. \tag{2.28}$$

Then, we deduce that G is differentiable. Clearly, $\partial_j G_i$ is continuous for each $i, j = 1, \cdots, n+1$.

We can finally say that G is continuously differentiable. ∎

Definition 2.3.2 *([57] Lipschitz-continuity)*

Suppose that a matrix function $J : D^n \subset \mathbb{R}^n \longrightarrow \mathbb{R}^{n \times n}$ *satisfies, for each* $Y, Z \in D^n$,

$$\|J(Y) - J(Z)\| \leqslant \chi \|Y - Z\|, \text{ with } \chi > 0.$$

Then, the matrix function J *is Lipschitz continuous.*

Theorem 2.3.2 *Let* G *be a nonlinear mapping defined on an open convex set* D^{n+1}, *where*

$$G(Y) = Y - F - \frac{b-a}{2} E\, Q(Y), \; Y \in D^{n+1}. \tag{2.29}$$

Then, for $\ell, i, j = 1, \cdots, n+1$,

$$\left|\partial^2_{\ell j} G_i(Y)\right| \leqslant K, \; K > 0. \tag{2.30}$$

Proof : Define S_υ and S_P, respectively, by

$$S_\upsilon = \sup_{s \in [-1,1]} \frac{b-a}{2} \int_{-1}^{1} |\upsilon(s,t)| \, dt, \tag{2.31}$$

$$S_P = \sup_{Z \in D^{n+1}} \left| \frac{\partial^2 P_j}{\partial Z_j^2}(Z) \right|, \tag{2.32}$$

and let

$$M_\omega = \max_{s,t \in [-1,1]} |\omega(s,t)| \,. \tag{2.33}$$

If $2 \leqslant j \leqslant i-1 \leqslant n$ and $\ell = j$:

$$\begin{aligned}
|\eta_{ij}| &= \left| \frac{1}{t_j - t_{j-1}} \int_{t_{j-1}}^{t_j} \upsilon(t_i,t)(t-t_{j-1})dt \right| \\
&\leqslant \frac{1}{|t_j - t_{j-1}|} \int_{t_{j-1}}^{t_j} |\upsilon(t_i,t)| \, |t-t_{j-1}| \, dt \\
&\leqslant \int_{t_{j-1}}^{t_j} |\upsilon(t_i,t)| \, dt \\
&\leqslant \sup_{s \in [-1,1]} \int_{-1}^{1} |\upsilon(s,t)| \, dt \\
&\leqslant \frac{2}{b-a} S_\upsilon
\end{aligned}$$

and $|\mu_{ij}| \leqslant \frac{2}{b-a} S_\upsilon$ can be proved in the same way. This leads to

$$\begin{aligned}
\left|\partial^2_{\ell j} G_i(Y)\right| &= \left| -\Big(\frac{b-a}{2}\Big) E_{ij} \, \frac{\partial^2 Q_j}{\partial Y_j^2}(Y) \right| \\
&= \left| -\Big(\frac{b-a}{2}\Big) [\eta_{ij} + \mu_{ij}] \omega(t_i,t_j) \frac{\partial^2 P_j}{\partial Z_j^2}(Z) \right| \\
&\leqslant \frac{b-a}{2} \Big(\, |\eta_{ij}| + |\mu_{ij}| \, \Big) \, |\omega(t_i,t_j)| \left| \frac{\partial^2 P_j}{\partial Z_j^2}(Z) \right| \\
&\leqslant 2 \, S_\upsilon M_\omega S_P.
\end{aligned}$$

If $j = 1$ or $j = i$ for $2 \leqslant i \leqslant n+1$ such that $\ell = j$:

$$\begin{aligned}
\left|\partial^2_{\ell j} G_i(Y)\right| &= \left| -\Big(\frac{b-a}{2}\Big) E_{ij} \, \frac{\partial^2 Q_j}{\partial Y_j^2}(Y) \right| \\
&\leqslant \frac{b-a}{2} \, |E_{ij}| \left| \frac{\partial^2 P_j}{\partial Z_j^2}(Z) \right| \\
&\leqslant S_\upsilon M_\omega S_P.
\end{aligned}$$

Clearly, the boundedness of $\partial^2_{\ell j} G_i(Y) = 0$ is satisfied in the remaining cases. Then we can say that, for $\ell, i, j = 1, \cdots, n+1$,

$$\left|\partial^2_{\ell j} G_i(Y)\right| \leqslant K, \; K > 0. \tag{2.34}$$

∎

Theorem 2.3.3 *Consider the Jacobian matrix of G*

$$J_G(Y) = I - \frac{b-a}{2}\, E\, L, \; Y \in D^{n+1}, \tag{2.35}$$

such that L denotes the diagonal matrix with the entries, for $i = 1, \cdots, n+1$,

$$L_{ii} = \partial_i Q_i(Y). \tag{2.36}$$

Then J_G is continuously differentiable.

Proof : By resorting to the definition and properties of the differentiability of matrices (see [57, 47, 80, 42]), we have, for $i, j = 1, \cdots, n+1$,

$$A^{ij}_{J_G}(H) = \left|\partial_j G_i(Y+H) - \partial_j G_i(Y) - \sum_{\ell=1}^{n+1} \partial^2_{\ell j} G_i(Y) H_\ell \right| / \left\|H\right\|, \; Y \in D^{n+1}, \tag{2.37}$$

in the cases where $2 \leqslant j = i \leqslant n+1$ and $1 \leqslant j \leqslant i-1 \leqslant n$, we can get

$$\begin{aligned} A^{ij}_{J_G}(H) &= \left|\partial_j G_i(Y+H) - \partial_j G_i(Y) - \left(-\frac{b-a}{2} E_{ij}\, \frac{\partial^2 Q_j}{\partial Y_j^2}(Y)\right) H_j \right| / \left\|H\right\| \\ &= \frac{b-a}{2}\left|E_{ij}\right| \left|\partial_j Q_j(Y+H) - \partial_j Q_j(Y) - \frac{\partial^2 Q_j}{\partial Y_j^2}(Y) H_j \right| / \left\|H\right\| \\ &= \frac{b-a}{2}\left|E_{ij}\right| \left|\partial_j P_j(Z+H) - \partial_j P_j(Z) - \frac{\partial^2 P_j}{\partial Z_j^2}(Z) H_j \right| / \left\|H\right\|. \end{aligned}$$

Then, we obtain

$$\lim_{\|H\| \to 0} A_{J_G}^{ij}(H) = 0, \tag{2.38}$$

because each component of $P(Z)$ is two times continuously differentiable. In the other cases, we have

$$\partial_{\ell j}^2 G_i(Y) = 0, \tag{2.39}$$

since $\partial_j G_i(Y)$ are constants, the differentiability of J_G is clear. Clearly, we deduce that, for $\ell, i, j = 1, \cdots, n+1$, $\partial_{\ell j}^2 G_i(Y)$ are continuous. This reflects the fact that J_G is continuously differentiable. ∎

Remark 2.3.1 *For the existence of J_G^{-1} at the root Y^*, it is enough to take*

$$\frac{b-a}{2} E_{ii}\, \partial_i Q_i(Y) \neq 1, \tag{2.40}$$

for $i = 2, \cdots, n+1$.

Chapter 3

Application of the product integration method

To be able to treat weakly singular Volterra integral equations, we can resort to the product integration method studied in this chapter. In other words, this chapter analyses Volterra integral equations by employing the product integration method that is needed to formulate a linear system. Before this, we first investigate the solvability of the linear Volterra integral equations. On the other hand, we conclude this chapter by stating the convergence of the product integration method.

3.1 Solvability of linear Volterra integral equations

In fact, several numerical methods are applied in analysing different types of integral equations (for example a Legendre-collocation method, a Chebyshev-collocation method, a Jacobi-collocation spectral method and a product integration method, see [78, 14, 15, 22, 8, 62, 38, 29, 28, 7]. Here, the focus is on the product integration method which is used to deal with the weakly singular Volterra integral equations of the form

$$\forall a, b \in \mathbb{R} \,:\, x(z) = \gamma(z) + \int_a^z k(z,\sigma)x(\sigma)\, d\sigma,\ z \in [a,b], \tag{3.1}$$

where the functions γ and k are given, more precisely, the function k represents a weakly singular kernel. For the existence and uniqueness of a continuous solution, we can take that γ is sufficiently smooth (see [3, 62, 13]). By applying a product integration method, we can solve the former integral equation. Before this, we recast (3.1) in the form, for $z \in [a,b]$,

$$x(z) = \gamma(z) + \int_a^z \psi(z,\sigma)\varphi(z,\sigma)x(\sigma)\, d\sigma, \tag{3.2}$$

where ψ is a singular function and φ is sufficiently smooth (see [46, page 472]). To be able to deal with (3.2), we can take the following assumptions from [28, 29, 5]

$$\sup_{z\in[a,b]} \int_{[a,b]} |\psi(z,\sigma)|\, d\sigma < +\infty, \tag{3.3}$$

and

$$\lim_{\delta\to 0} \sup_{z,z'\in[a,b],|z-z'|\le\delta} \int_{[a,b]} |\psi(z,\sigma) - \psi(z',\sigma)|\, d\sigma = 0. \tag{3.4}$$

In this chapter, the main idea of the product integration method is to transform (3.2) into a linear system by applying the linear interpolation in the same way as in [28, 29]. This can be achieved after a change of the variables of (3.2). This means that we take, see [75, 37],

$$z = bs - a(s-1), \; s \in [0,1], \tag{3.5}$$

and

$$\sigma = bt - a(t-1), \; t \in [0,s]. \tag{3.6}$$

This change enables us to get the equivalent equation of the form, for $s \in [0,1]$,

$$y(s) = p(s) + \int_0^s u(s,t)v(s,t)y(t)\,dt. \tag{3.7}$$

From (3.3) and (3.4), we can deduce that

$$\sup_{s\in[0,1]} \int_{[0,1]} |u(s,t)|\,dt < +\infty, \tag{3.8}$$

and

$$\lim_{\delta\to 0} \sup_{s,s'\in[0,1],|s-s'|\le\delta} \int_{[0,1]} |u(s,t) - u(s',t)|\,d\sigma = 0. \tag{3.9}$$

Also, we can conclude from the continuity properties mentioned in [17, 18, 56](see also [60, 53, 23] for more details) that y, p and v are continuous functions.

3.2 Linear systems formed by the product integration method

The linear system formed by applying the product integration method is given by

$$H_n Y_n = P_n, \tag{3.10}$$

where

$$H_n = I - (1/h_n)M_n, \tag{3.11}$$

which represents a lower triangular matrix, while Y_n and P_n are column vectors. The solution of (3.10) is well known by

$$Y_n = H_n^{-1}P_n, \tag{3.12}$$

in the case where $\det[H_n] \neq 0$, for the remaining details see [57, 74]. More precisely, the matrix M_n is given by

$$M_n = \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 \\ \beta_{2,1} & \beta_{2,2} & \cdots & \cdots & 0 \\ \beta_{3,1} & \beta_{3,2} & \beta_{3,3} & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \beta_{n+1,1} & \beta_{n+1,2} & \cdots & \beta_{n+1,n} & \beta_{n+1,n+1} \end{pmatrix},$$

and the column vectors P_n and Y_n are defined respectively by

$$P_n = \begin{pmatrix} p(t_1) \\ \vdots \\ p(t_{n+1}) \end{pmatrix},$$

$$Y_n = \begin{pmatrix} y(t_1) \\ \vdots \\ y(t_{n+1}) \end{pmatrix}.$$

Moreover, we get the former linear system after using the linear interpolation which is also employed in [28, 29, 30] such that, for $i = 2, \cdots, n+1$,

$$\begin{aligned}
\beta_{i1} &= \Big[\int_{t_1}^{t_2} u(t_i,t)(t_2 - t)dt\Big] v(t_i,t_1),\\
\beta_{ii} &= \Big[\int_{t_{i-1}}^{t_i} u(t_i,t)(t - t_{i-1})dt\Big] v(t_i,t_i),\\
\beta_{ij} &= \Big[\int_{t_j}^{t_{j+1}} u(t_i,t)(t_{j+1} - t)dt + \int_{t_{j-1}}^{t_j} u(t_i,t)(t - t_{j-1})dt\Big] v(t_i,t_j),\ 2 \leq j \leq n+1,
\end{aligned}$$

where, for $i = 1, \cdots, n+1$,

$$t_i = (i-1)h_n, \tag{3.13}$$

and where

$$h_n = 1/n. \tag{3.14}$$

From [28, page 775], the linear interpolation that is used to approximate $v(s,t)y(t)$ is written in the following form

$$[v(s,t)y(t)]_n := \frac{1}{h_n}\Big[(t_{i+1} - t)v(s,t_i)y(t_i) + (t - t_i)v(s,t_{i+1})y(t_{i+1})\Big],\ for\ t \in [t_i, t_{i+1}]. \tag{3.15}$$

As an example of some other interpolation polynomials, there is the Lagrange interpolation (see [36, 52, 9, 73]). On the other hand, we can resort to Cramer's rule to deal with (3.10), see [57, 77, 2]. For more information about our work, we have to see the references [31, 19, 76, 71, 82, 61, 12, 70, 20, 69, 68, 43, 81, 25].

3.3 Convergence analysis

This section comprises some basic theorems that are used to show the convergence analysis of the product integration method. One of these theorems is the Cauchy criterion that plays a significant role in this section, more precisely, in the proof of Theorem 3.3.3. This proof is presented to demonstrate the continuity of the operator Qy which is given by

$$Qy(s) = \int_0^s u(s,t)v(s,t)y(t)dt, \tag{3.16}$$

where v and y are continuous for $0 \leqslant t \leqslant s \leqslant 1$.

Theorem 3.3.1 *Assume that*

(i) $\sup\limits_{s\in[0,1]} \int_{[0,1]} |u(s,t)|\, dt < +\infty,$

(ii) $\eta(\delta) = \sup\limits_{s,s'\in[0,1],|s-s'|\leq\delta} \int_{[0,1]} |u(s,t) - u(s',t)|\, dt,$

with

$$\lim_{\delta\to 0} \eta(\delta) = 0. \tag{3.17}$$

Then we can get

$$\lim_{n\to+\infty} \max_{1\leq i\leq n+1} |\tilde{Q}y(t_i) - Qy(t_i)| = 0. \tag{3.18}$$

where

$$\tilde{Q}y(t_i) = \frac{1}{h_n}\sum_{j=1}^{i-1} \beta_{ij} y(t_j). \tag{3.19}$$

Proof : Let

$$S = \sup_{s\in[0,1]} \int_{[0,1]} |u(s,t)|\, dt. \tag{3.20}$$

On the other hand, we have from [28, 30] that

$$\lim_{h_n\to 0} w_2(v,h_n) = 0,\ for\ v \in C([0,1]^2).$$

From the continuity properties of the product of continuous functions (see [56, 17, 34]), we can obtain

$$\lim_{n\to+\infty} w_2(vy,h_n) = 0. \tag{3.21}$$

Now, we can proceed to prove (3.18). Then, we have

$$\begin{aligned}
|\tilde{Q}y(t_i) - Qy(t_i)| &= \Big| \int_0^{t_i} u(t_i,t)[v(t_i,t)y(t)]_n\, dt - \int_0^{t_i} u(t_i,t)v(t_i,t)y(t)\, dt \Big| \\
&= \Big| \int_0^{t_i} u(t_i,t)\{[v(t_i,t)y(t)]_n - v(t_i,t)y(t)\}\, dt \Big| \\
&= \sum_{j=1}^{i-1} \int_{t_j}^{t_{j+1}} u(t_i,t)\{[v(t_i,t)y(t)]_n - v(t_i,t)y(t)\}\, dt \\
&\leq w(v(s,\cdot)y(\cdot),h_n) \sum_{j=1}^{i-1} \int_{t_j}^{t_{j+1}} u(t_i,t)\, dt \\
&\leq w_2(vy,h_n) \sum_{j=1}^{i-1} \int_{t_j}^{t_{j+1}} |u(t_i,t)|\, dt \\
&\leq w_2(vy,h_n)S.
\end{aligned}$$

Then, we can deduce that

$$\lim_{n\to+\infty} \max_{1\leq i\leq n+1} |\tilde{Q}y(t_i) - Qy(t_i)| = 0. \tag{3.22}$$

∎

Theorem 3.3.2 *([33] The Cauchy criterion)*

If a function γ is defined on an interval $[a,b)$ for $-\infty < a < b \leqslant +\infty$, and if γ is Riemann integrable on $[a,c]$, where $c \in (a,b)$, then the following statements are equivalent.

(i) $\int_a^b \gamma(y)dy$ *is convergent.*

(ii) *There are $\eta \in (a,b)$ and $\varepsilon > 0$ such that*

$$|\int_{\eta'}^{\eta''} \gamma(y)dy| < \varepsilon, \tag{3.23}$$

for $\eta < \eta' < b$ and $\eta < \eta'' < b$.

The proof of the following theorem is largely based on the proof of [15, Lemma 3.5] and on, of course, some other references.

Theorem 3.3.3 *Consider the operator, for $s \in [0,1]$ and for $y \in C([0,1])$,*

$$Qy(s) = \int_0^s u(s,t)v(s,t)y(t)dt. \tag{3.24}$$

Suppose that

$$\sup_{s\in[0,1]} \int_{[0,1]} |u(s,t)|\, dt < +\infty, \tag{3.25}$$

and that

$$\eta(\delta) = \sup_{s,s'\in[0,1],|s-s'|\leq\delta} \int_{[0,1]} |u(s,t) - u(s',t)|\, dt, \tag{3.26}$$

with

$$\lim_{\delta\to 0} \eta(\delta) = 0. \tag{3.27}$$

Then the operator Qy satisfies, for a continuous function v,

$$Qy \in C([0,1]). \tag{3.28}$$

Proof : We take

$$\phi(\delta) = \sup_{s,s',t\in[0,1],|s-s'|\leq\delta} |v(s,t) - v(s',t)| \quad , \tag{3.29}$$

such that

$$\lim_{\delta\to 0} \phi(\delta) = 0, \tag{3.30}$$

and define

$$S = \sup_{s\in[0,1]} \int_{[0,1]} |u(s,t)|\, dt. \tag{3.31}$$

Choose $s \leq s'$, and let

$$T_v = \max_{s,t\in[0,1]} |v(s,t)| \; and \; T_y = \max_{s\in[0,1]} |y(s)|. \tag{3.32}$$

We have that $|Qy(s) - Qy(s')|$ satisfies, for $s, s' \in [0,1]$,

$$\begin{aligned}
|Qy(s) - Qy(s')| &= \Big| \int_0^s u(s,t)v(s,t)y(t)\, dt - \int_0^{s'} u(s',t)v(s',t)y(t)\, dt \Big| \\
&= \Big| \int_0^s u(s,t)(v(s,t) - v(s',t))y(t)\, dt - \int_0^{s'} u(s',t)v(s',t)y(t)\, dt \\
&\quad + \int_0^s u(s,t)v(s',t)y(t)\, dt \Big| \\
&= \Big| \int_0^s u(s,t)(v(s,t) - v(s',t))y(t)\, dt + \int_0^s (u(s,t) \\
&\quad - u(s',t))v(s',t)y(t)\, dt - \int_s^{s'} u(s',t)v(s',t)y(t)\, dt \Big| \\
&\leq T_y \Big(S\phi(\delta) + T_v\eta(\delta) + T_v\mu(\delta) \Big),
\end{aligned}$$

where

$$\mu(\delta) = \sup_{s,s' \in [0,1], |s-s'| \leq \delta} \int_s^{s'} |u(s',t)|\, dt. \tag{3.33}$$

By resorting to the Cauchy criterion (see [33, 84]), we can get

$$\lim_{\delta \to 0} \mu(\delta) = 0. \tag{3.34}$$

Then, we deduce that

$$\lim_{\delta \to 0} \sup_{s,s' \in [0,1], |s-s'| \leq \delta} |Qy(s) - Qy(s')| = 0. \tag{3.35}$$

Finally, we can say that

$$Qy \in C([0,1]). \tag{3.36}$$

∎

Chapter 4

Numerical applications

This chapter is intended to present several numerical examples that are solved by applying the techniques of the product integration method which are described in detail in Chapter 2 and 3. For the sake of comparison, we try to cover two cases of weakly singular Volterra integral equations i.e., linear and nonlinear cases. By dealing with these two cases, we can show the fact that the product integration method is one of the efficient methods.

4.1 Examples of nonlinear Volterra integral equations

It is possible to solve many complicated models of nonlinear weakly singular Volterra integral equations by applying the product integration method which is studied extensively in Chapter 2. Although we prove the convergence of this method, it is advantageous to add some numerical examples. These examples provide us more information about the approximate solutions.

Before proving the accuracy of the product integration method, we need to define the expressions of the mean absolute error, denoted by E_1, and of the mean squared error, denoted by E_2, which are used to compare between the approximate and the exact solutions. Then, we have

$$E_1 = \frac{1}{n+1} \sum_{i=1}^{n+1} \left| y(t_i) - y^{(k+1)}(t_i) \right|, \tag{4.1}$$

and

$$E_2 = \frac{1}{m+1} \sum_{i=1}^{m+1} \left(y(t_i) - H_{n+1}(y^{(k+1)}, t_i) \right)^2, \ for \ m > n. \tag{4.2}$$

It remains to give some numerical examples. For this, we can take the following examples.

Example 4.1.1 *Consider the integral equation*

$$z(\tau) = |-\frac{1}{5\tau} + 0.75| + \Upsilon z(\tau), \ for \ \tau \in [2.25, 2.5], \tag{4.3}$$

where the integral operator Υz *is defined by*

$$\Upsilon z(\tau) = \frac{1}{280} \int_{2.25}^{\tau} \ln|\tau - \sigma| \left(z(\sigma) - |3.75 - \frac{1}{\sigma}| \right) d\sigma. \tag{4.4}$$

Here, the exact solution of (4.3) *is written in the following form*

$$z(\tau) = |0.75 - \frac{1}{5\tau}|. \tag{4.5}$$

Example 4.1.2 *Suppose that the function* g *is equal to*

$$g(\tau) = -\tau + \exp(-4\tau + 10), \ \tau \in [2, 3], \tag{4.6}$$

and that the nonlinear Volterra integral equation is given by, for $\alpha \in (0, 1)$,

$$z(\tau) = g(\tau) + \int_{2}^{\tau} |\tau - \sigma|^{-\alpha} (\sigma - 1)^2 \ (\sigma + z(\sigma) - \exp(-4\sigma + 10)) \ d\sigma. \tag{4.7}$$

Then, we have

$$z(\tau) = -\tau + \exp(10 - 4\tau). \tag{4.8}$$

Example 4.1.3 *We claim that*

$$z(\tau) = 1/4\tau + \Upsilon z(\tau),\ \tau \in [3, 3.25], \tag{4.9}$$

where

$$\Upsilon z(\tau) = \frac{1}{23}\int_3^\tau \frac{\sigma^{1/6}}{\sqrt{|\tau - \sigma|}}\left(-1/8\sigma + 0.5z(\sigma)\right)\, d\sigma. \tag{4.10}$$

The exact solution z can then be determined by

$$z(\tau) = \frac{1}{4}\tau. \tag{4.11}$$

After calculating the mean absolute and the mean squared errors, we can compare between the exact and the approximate solutions. In the first example, we can get the following table which involves the results of the mean absolute error E_1, in the case where $y^{(0)}(t_i) = 0.75$ and where B_0 is the diagonal matrix with the entries $B_0^{ii} = 0.9$ for $i = 1, \cdots, n+1$,

k	$n = 14$
2	2.6792 e-06
4	5.6173 e-10
5	8.5873 e-13
7	8.8818 e-17

Table 1: Mean absolute errors E_1 for (4.3).

We now want to study, in Example 4.1.1, the convergence of the Hermite-Fejér interpolation polynomial. To do this, we can compute the mean squared error E_2. This leads to

n	$k = 7$ and $m = 39$
9	4.9451 e-08
12	2.9261 e-08
15	1.9317 e-08
25	7.3153 e-09

Table 2: Mean squared errors E_2 for (4.3) .

In what follows, we want to present the results of E_1, in the second example. By choosing $y^{(0)}(t_i) = -7 + \exp(2)$, $B_0^{ii} = \exp(-0.15)$ for $i = 1, \cdots, n+1$, and $\alpha = 0.27$, we can have

k	$\alpha = 0.27$
6	5.6931e-04
8	1.6381e-05
11	6.0413e-08
14	1.5430e-10
16	1.6128e-12
19	1.9384e-15
21	8.8818e-18

Table 3: Mean absolute errors E_1 for (4.7), in the case where $n = 24$.

If we take $\alpha = 0.47$, then we obtain

k	$\alpha = 0.47$
10	1.9296e-04
14	7.1872e-07
16	3.7506e-08
19	1.9363e-10
21	1.4792e-12
24	8.5221e-15
27	0

Table 4: Mean absolute errors E_1 for (4.7), in the case where $n = 24$.

From the two preceding tables, we can see that the values of the error are decreased. To support the results of Tables 3 and 4, we present the following graph

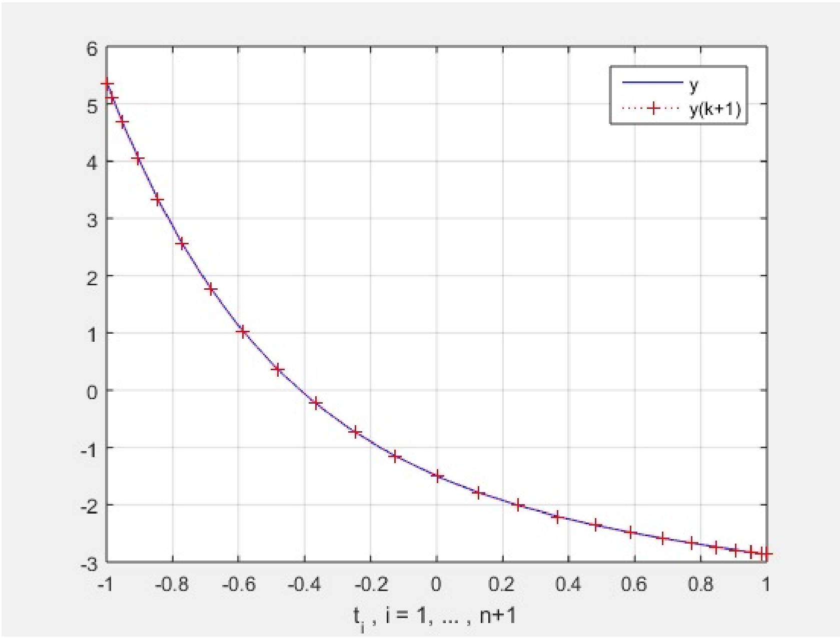

Figure 4.1: Approximate solutions where $n = 24$ and where $k = 27$ with $\alpha = 0.47$.

For more information about the convergence of the Hermite-Fejér interpolation polynomial, in Example 4.1.2 , it remains to discuss the results of the mean squared error E_2. Of course, we have two cases i.e., $\alpha = 0.27$ and $\alpha = 0.47$

n	$\alpha = 0.27$
24	0.0092
27	0.0073
30	0.0060
33	0.0050

Table 5: Mean squared errors E_2 for (4.7), in the case where $k = 16$ and where $m = 39$.

n	$\alpha = 0.47$
22	0.0108
25	0.0085
28	0.0068
31	0.0056

Table 6: Mean squared errors E_2 for (4.7), in the case where $k = 16$ and where $m = 39$.

In the last example, we can obtain the next two tables after computing the mean absolute error E_1 and the mean squared error E_2, in the case where $y^{(0)}(t_i) = 0.75$ and where $B_0^{ii} = 0.9$ for $i = 1, \cdots, n+1$,

k	$n = 14$
2	3.2307 e-06
4	2.2471 e-09
5	1.0940 e-11
7	4.2040 e-15

Table 7: Mean absolute errors E_1 for (4.9).

n	$k = 7$ and $m = 39$
9	2.4414 e-06
12	1.4446 e-06
15	9.5367 e-07
25	3.6115 e-07

Table 8: Mean squared errors E_2 for (4.9).

4.2 Examples of linear Volterra integral equations

To be able to solve linear weakly singular Volterra integral equations, we can use the product integration method, as shown in Chapter 3. This provides the best approximate solutions of different forms of integral equations. This reflects the fact that we can deal with many difficult problems by resorting to the product integration method. To state that the approximate solutions converge to the exact solutions, we can perform the following calculations, for the difference between the exact and the approximate solutions e_i at the points $t_i,\ i = 1, \cdots, n+1$,

$$E_{MA} = \frac{\sum_{i=1}^{n+1} |e_i(t_i)|}{n+1}, \tag{4.12}$$

and

$$E_{RMS} = \sqrt{\frac{\sum_{i=1}^{n+1} e_i^2(t_i)}{n+1}}, \tag{4.13}$$

where E_{MA} and E_{RMS} denote, respectively, the mean absolute error and the root mean squared error (see [10]). Moreover, we need to compute R, for $Z_m = [(B_n y)(t_1), (B_n y)(t_2), ..., (B_n y)(t_{m+1})]^T$ and $\|P_m\|_1 = \sum_{i=1}^{m+1} |p(t_i)|$,

$$R = \frac{\|(I - (1/h_m)M_m)Z_m - P_m\|_1}{\|P_m\|_1}, \tag{4.14}$$

while m is sufficiently bigger than n and R denotes the relative residual. In what follows, we take some Volterra integral equations as examples.

Example 4.2.1 *Assume that, for* $\alpha \in (0,1)$*,*

$$x(z) = \frac{z}{3\,|z + 1/7|} - \frac{z^{2-\alpha}}{(1-\alpha)(2-\alpha)} + \int_0^z k(z,\sigma)x(\sigma)\,d\sigma, \tag{4.15}$$

and that

$$k(z,\sigma) = (z-\sigma)^{-\alpha}\,|3\sigma + 3/7|\,,\ z \in [0,1]. \tag{4.16}$$

Then the exact solution of the former integral equation is determined by

$$x(z) = \frac{7z}{|3 + 21z|}. \tag{4.17}$$

Example 4.2.2 *Let, for* $z \in [1, 1.5]$,

$$x(z) = \gamma(z) + \int_1^z \log(z-\sigma)\,\frac{1}{99}\,(z-\sigma)^l\,x(\sigma)\,d\sigma, \tag{4.18}$$

be the linear Volterra integral equations and let γ *and* φ *be the functions which are defined by*

$$\gamma(z) = \frac{1}{3}\left(1 - q(z)\right), \tag{4.19}$$

where

$$q(z) = \frac{1}{99(l+1)}\,(z-1)^{l+1}\left(\log(z-1) - \frac{1}{l+1}\right),\ l \in \mathbb{N}^*, \tag{4.20}$$

and

$$\varphi(z,\sigma) = \frac{1}{99}\,(z-\sigma)^l\,. \tag{4.21}$$

Then, in this example, the expression of the exact solution x *is*

$$x(z) = \frac{1}{3}. \tag{4.22}$$

Example 4.2.3 *If we have, for* $\alpha \in (0,1)$,

$$x(z) = \gamma(z) + \int_3^z (z-\sigma)^{-\alpha}\varphi(z,\sigma)x(\sigma)\,d\sigma, \tag{4.23}$$

where, for $z \in [3, 4]$,

$$\gamma(z) = \exp(z) - \exp(1)\frac{(z-3)^{1-\alpha}}{1-\alpha}, \tag{4.24}$$

and

$$\varphi(z,\sigma) = \exp(1-\sigma), \tag{4.25}$$

then

$$x(z) = \exp(z). \tag{4.26}$$

The following tables show some important results that are obtained after computing E_{RMS}, E_{MA} and R. More specifically, Table 9 presents the numerical results of Example 4.2.1, and involves the root mean squared errors, in the case where $\alpha = 0.27$,

n	$\alpha = 0.27$
4	2.3313 e-14
5	2.4390 e-15
10	4.2974 e-16
11	1.1160 e-16

Table 9: Root mean squared errors E_{RMS} for (4.15).

In Example 4.2.3, the results of E_{RMS} are given in the following table, for $\alpha = 4/9$,

n	$\alpha = 4/9$
5	5.0243 e-15
18	4.3892 e-15
27	3.5527 e-15

Table 10: Root mean squared errors E_{RMS} for (4.23).

When we change the value of α in Examples 4.2.1 and 4.2.3, we can get the next two tables that present the results of the root mean squared error

n	$\alpha = 0.89$
4	9.6148 e-17
5	8.8499 e-17
10	1.0782 e-16
11	7.7683 e-17

Table 11: Root mean squared errors E_{RMS} for (4.15).

n	$\alpha = 8/9$
5	1.9242 e-14
18	3.0046 e-14
27	2.8334 e-14

Table 12: Root mean squared errors E_{RMS} for (4.23).

In view of the results of Tables 9, 10, 11 and 12, it is clear that the product integration method plays an important role in our work.

In Example 4.2.3, we compute the relative residual R. This leads to construct the following table, in the case where $\alpha = 4/9$ and where $m = 80$,

n	$\alpha = 4/9$ and $m = 80$
23	0.0035
37	0.0022
43	0.0019
57	0.0014
63	0.0013

Table 13: Relative residuals R for (4.23).

If we take $\alpha = 8/9$, and if we claim that m is of the same value as in the former table, then we can obtain

n	$\alpha = 8/9$ and $m = 80$
23	0.0033
37	0.0021
43	0.0018
57	0.0013
63	0.0012

Table 14: Relative residuals R for (4.23).

In the sequel of this section, the next three tables show some results of the mean absolute error E_{MA} for (4.18), for different values of l,

n	$l = 8$
3	8.2560 e-07
5	4.8209 e-07
8	3.4407 e-07
11	2.9448 e-07

Table 15: Mean absolute errors E_{MA} for (4.18).

n	$l = 14$
3	1.2892 e-08
5	6.2709 e-09
8	3.6720 e-09
11	2.8100 e-09

Table 16: Mean absolute errors E_{MA} for (4.18).

n	$l = 23$
3	2.5995 e-11
5	1.1859 e-11
8	5.9143 e-12
11	3.9772 e-12

Table 17: Mean absolute errors E_{MA} for (4.18).

By observing the results of the last three tables, we can deduce that we have the appropriate solutions.

Conclusion

We conclude this thesis by recalling the most important advantages of the application of the product integration method, and by giving some basic results that are related to this method. Before discussing these advantages and these results, we start with a brief summary of this thesis. More precisely, we first present some elementary concepts and basic theories, and then treat linear and nonlinear weakly singular Volterra integral equations. At the end of this thesis, we give several numerical examples that illustrate and state the importance of the product integration method. Now, we proceed to talk about the principal merits of our work. In this thesis, the focus is on the solvability of the weakly singular Volterra integral equations for any choice of their kernels. To solve these integral equations, the techniques of the product integration method represents one of the fastest ways which can achieves best numerical results.

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