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Dedication

To my beloved parents "Mom and Dad" for their patience, love, and encouragement to me.

To my dear brothers and sisters, with whom I have lived the most beautiful days, and to my whole family "**the Mazouzi family**".

To all my teachers from the beginning of my teaching career until today.

To all my friends and classmates with whom I have beautiful relationships, and everyone dear to my heart.

This thesis is dedicated to you.

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Just wanted to say Thank you

Abstract

In this thesis, we will explore the theory and applications of Bezier curves and Bernstein polynomials. We will study their basic properties and explain the de-Casteljau algorithm for Bezier curves. We will offer an example to illustrate the use of Bernstein polynomials in approximation theory and how Bezier curve approaches are used in many fields.

Notations and symbols

$\mathbb{N}$	:	the set of all natural numbers
$B_{k,n}(t)$	:	Bernstein polynomial
$k!$	:	the factorial of the number k
$\binom{n}{k}$	:	binomial coefficient
$:=$	:	equals by definition
$i.e$	:	(id est) means "that is" or "in other word"
δ, ϵ	:	small positive real numbers
$\mathbb{R}$	:	the set of all real numbers
$\mathbb{C}$	:	the set of all complex numbers
I	:	interval
$C(t)$	:	Bezier curve
p_i	:	control points
$\frac{d}{dt}$	:	derivative
$\mathcal{T}$	:	affine transformation
A	:	a Matrix
$\mathbb{M}_n(\mathbb{R})$	:	space of matrices
$CH\{X\}$	:	convex hull of set of points

p_i^j	:	De-casteljau algorithm
E^3	:	3-demensional euclidien space
$\mathcal{L}$	:	Bounded linear operator
$R_n[x]$	:	the space of polynomials
ODE	:	ordinary differential equation
SVG	:	scalable vector graphics
CAD	:	computer aided design

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Introduction

Before the discovery of the Bezier curve, it was not possible to draw curves on a computer. This mathematical equation led to the creation of computer software that profoundly changed how graphic design was done. The discovery of Bezier curve revolutionized various fields, including animation, designing cars, and robotics.

Historically, in 1960, the french engineer and one of the founders in the field of engineering and physical solid modeling Pierre Bezier developed the curve named after him (Bezier curve) while working at the automobile company known as Renault and also used these curves to create the distinctive designs of Peugeot and Renault cars, long long before other manufacturers used CAD.

He realized it was really hard to draw curves with computers so he decided to come up with an algorithm, and in doing so changed the course of history.

This was an incredible achievement in the field of software engineering and took polynomials to a whole new level. **Mathematically**, A curve is an infinitely large set of points. Each point has two neighbors except for endpoints.

Curves can be broadly classified into three categories: **explicit**, **implicit** and **parametric curves.** The latter is the subject of our study, as Bezier curves are parametric curves that use Bernstein polynomials as their basis. Curves having a parametric form are called parametric curves. The explicit and implicit curve representations can be used only when the function is known. A two-dimensional parametric curve has the following form

$$p(t) = (f(t), g(t)) \qquad \text{or} \quad p(t) = (x(t), y(t))$$

The functions f and g become the x, y coordinates of any point on the curve, and the points are obtained when the parameter t is varied over a certain interval $[a, b]$, normally $[0, 1]$

This thesis presents a study on Bezier curves, this work is organized in the following way :

In the first chapter, we present generalities and basic concepts about Bernstein polynomials and their properties, as well as their usefulness in approximating continuous functions in an interval.

The second chapter is devoted to the study of Bezier curves and their properties, derivations, as well as their geometric construction.

In the end, we provide a simple illustration of Bernstein polynomial approximation with additional examples of Bezier curve applications in various fields.

Chapter 1

Bernstein polynomials

In this chapter we will present the Bernstein polynomials thus named in the honor of the Ukrainian mathematician **Sergei Natanovich Bernstein** (1880-1968).

At an early stage of our studies in the field of mathematics, we got acquainted with polynomials, we probably remember them in this form below

$$p(t) = a_n t^n + a_{n-1} t^{n-1} + \cdots + a_1 t + a_0$$

Which represents a polynomial as a linear combination of certain elementary polynomials, i.e,$1, t, t^2, \ldots t^n$.

–The set of polynomials of degree less than or equal to n forms a vector space: polynomials can be added together, can be multiplied by a scalar, and all the vector space properties hold.

–The set of functions $\{1, t, t^2, \ldots, t^n\}$ form a basis for this vector space – that is, any polynomial of degree less than or equal to n can be uniquely written as a linear combination of these functions.

There are an unlimited number of bases for the space of polynomials, including this basis, which is frequently referred to as the canonical basis.

We will talk about the Bernstein basis, one more popular base for the space of polynomials, and go through all of its beneficial properties.

1.1 Bernstein polynomials

Definition 1.1.1 *for $n \in \mathbb{N}$ the Bernstein polynomials $\{B_{k,n}\}_{k=0}^{n}$ of degree n on $[0,1]$ are defined as*

$$B_{k,n}(t) = \binom{n}{k} t^{k} (1-t)^{n-k} \tag{1.1}$$

where

$$\binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

Remark 1.1.1 *The value of the Bernstein polynomials $B_{k,n}(t)$ at the endpoints of the interval $[0,1]$ are*

$$B_{0,n}(0) = 1,\ \ B_{k,n}(0) = 0 \text{ for } k \neq 0$$

and

$$B_{n,n}(1) = 1,\ \ B_{k,n}(1) = 0 \text{ for } k \neq n$$

To illustrate the definition of Bernstein polynomials, we give below the formula of these polynomials for degree 1, 2 and 3

1) The Bernstein polynomials of degree 1 are

$$B_{0,1}(t) = 1 - t$$

$$B_{1,1}(t) = t$$

and can be plotted for $0 \leq t \leq 1$ as

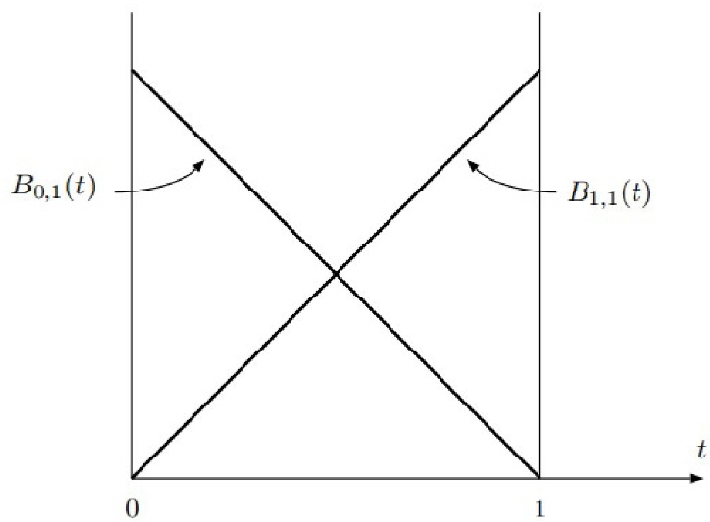


Figure 1.1: Bernsrein polynomials n=1

2) The Bernstein polynomials of degree 2 are

$$B_{0,2}(t) = (1 - t)^2$$

$$B_{1,2}(t) = 2t(1 - t)$$

$$B_{2,2}(t) = t^2$$

and can be plotted for $0 \leq t \leq 1$ as

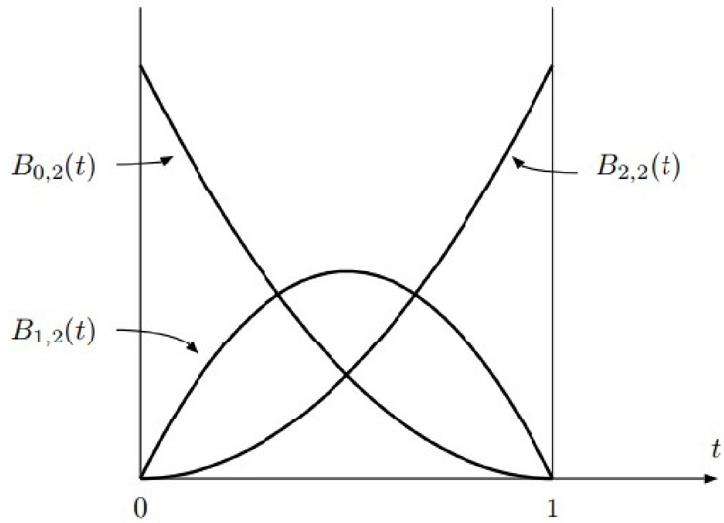


Figure 1.2: Bernstein polynomials n=2

3) The Bernstein polynomials of degree 3 are

$$B_{0,3}(t) = (1-t)^3$$

$$B_{1,3}(t) = 3t(1-t)^2$$

$$B_{2,3}(t) = 3t^2(1-t)$$

$$B_{3,3}(t) = t^3$$

and can be plotted for $0 \leq t \leq 1$ as

1.2 Properties of Bernstein polynomials

1.2.1 Recurrence

The Bernstein polynomials of degree n can be defined by blending together two Bernstein polynomials of degree $(n-1)$. That is, the kth nth-degree Bernstein

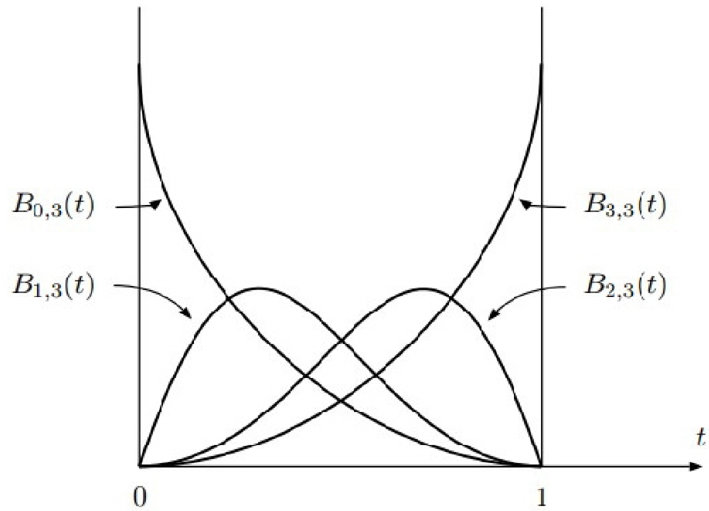


Figure 1.3: Bernstein polynomials n=3

polynomial can be written as

$$B_{k,n}(t) = (1-t)B_{k,n-1}(t) + tB_{k-1,n-1}(t)$$

Proof. To show this, we need only use the definition of the Bernstein polynomials and some simple algebra calculus.

$$\begin{aligned}
(1-t)B_{k,n-1}(t) + tB_{k-1,n-1}(t) &= (1-t)\binom{n-1}{k} t^k (1-t)^{n-1-k} \\
&\quad + t\binom{n-1}{k-1} t^{k-1}(1-t)^{n-1-(k-1)} \\
&= \binom{n-1}{k} t^k (1-t)^{n-k} + \binom{n-1}{k-1} t^k (1-t)^{n-k} \\
&= \left[\binom{n-1}{k} + \binom{n-1}{k-1}\right] t^k (1-t)^{n-k}
\end{aligned}$$

we simplify

$$\begin{aligned}
\binom{n-1}{k}+\binom{n-1}{k-1} &= \frac{(n-1)!}{k!(n-1-k)!}+\frac{(n-1)!}{(k-1)!(n-1-k+1)!}\\
&= \frac{(n-1)!}{k!(n-1-k)!}+\frac{(n-1)!}{(k-1)!(n-k)!}\\
&= \frac{(n-k)(n-1)!}{(n-k)k!(n-k-1)!}+\frac{k(n-1)!}{k(k-1)!(n-k)!}\\
&= \frac{(n-k)(n-1)!}{k!(n-k)!}+\frac{k(n-1)!}{k!(n-k)!}\\
&= \frac{(n-k)(n-1)!+k(n-1)!}{k!(n-k)!}\\
&= \frac{n!}{k!(n-k)!}=\binom{n}{k}
\end{aligned}$$

we arrive at

$$\begin{aligned}
(1-t)B_{k,n-1}(t)+tB_{k-1,n-1}(t) &= \left[\binom{n-1}{k}+\binom{n-1}{k-1}\right]t^k(1-t)^{n-k}\\
&= \binom{n}{k}t^k(1-t)^{n-k}
\end{aligned}$$

this gives the result. ■

1.2.2 Partition of Unity

the sum of Bernstein polynomials is the constant 1 function,i.e, they form what is called a partition of unity

$$\sum_{k=0}^{n} B_{k,n}(t)=1$$

In fact by using Newton's binomial theorem, we get

$$\begin{aligned}
1 &= [(1-t)+t]^n \\
&= \sum_{k=0}^{n} \binom{n}{k} t^k (1-t)^{n-k} \\
&= \sum_{k=0}^{n} B_{k,n}(t).
\end{aligned}$$

1.2.3 Positivity

Bernstein polynomials are non- negative on $[0, 1]$

$$\forall t \in [0,1], B_{k,n}(t) \geq 0$$

To prove this we use the definition of Bernstein polynomials

$$B_{k,n}(t) = \binom{n}{k} t^k (1-t)^{n-k}$$

$\binom{n}{k} \geq 0$ and for $t \in [0,1], t \geq 0$ and $1-t \geq 0 \Longrightarrow t^k \geq 0$ and $(1-t)^{n-k} \geq 0$ so $\forall t \in [0,1], B_{k,n}(t) \geq 0$

so the Bernstein polynomials are positive when $0 \leq t \leq 1$.

1.2.4 Symmetry

if $0 \leq k \leq 1$ so $B_{k,n}(1-t) = B_{n-k,n}(t)$.

We have

$$\forall k \in \{0, ..., 1\}, B_{k,n}(t) = \binom{n}{k} t^k (1-t)^{n-k}$$

so

$$\begin{aligned} B_{k,n}(1-t) &= \binom{n}{k}(1-t)^k\left[1-(1-t)\right]^{n-k} \\ &= \binom{n}{k} t^{n-k}(1-t)^k \end{aligned}$$

or

$$\begin{aligned} \binom{n}{k} &= \frac{n!}{k!(n-k)!} \\ &= \frac{n!}{(n-k)!\left[n-(n-k)\right]!} \\ &= \binom{n}{n-k} \end{aligned}$$

from where

$$\begin{aligned} B_{k,n}(1-t) &= \binom{n}{n-k} t^{n-k}(1-t)^k \\ &= B_{n-k,n}(t). \end{aligned}$$

1.2.5 Degree Raising

A linear combination of Bernstein polynomials of degree n can be used to express any of the lower-degree Bernstein polynomials (degree$\prec n$). The degree $(n-1)$ of Bernstein polynomial, for example, can be expressed as a linear combination of degree n Bernstein polynomials. We first note that

$$\begin{aligned}
tB_{k,n}(t) &= \binom{n}{k} t^{k+1}(1-t)^{n-k} \\
&= \binom{n}{k} t^{k+1}(1-t)^{n+1-(k+1)} \\
&= \frac{\binom{n}{k}}{\binom{n+1}{k+1}} B_{k+1,n+1}(t) \\
&= \frac{k+1}{n+1} B_{k+1,n+1}(t)
\end{aligned}$$

and

$$\begin{aligned}
(1-t)B_{k,n}(t) &= \binom{n}{k} t^{k}(1-t)^{n-k+1} \\
&= \frac{\binom{n}{k}}{\binom{n+1}{k}} B_{k,n+1}(t) \\
&= \frac{n-k+1}{n+1} B_{k,n+1}(t)
\end{aligned}$$

and finally

$$\begin{aligned}
\frac{1}{\binom{n}{k}} B_{k,n}(t) + \frac{1}{\binom{n}{k+1}} B_{k+1,n}(t) &= t^{k}(1-t)^{n-k} + t^{i+1}(1-t)^{n-(i+1)} \\
&= t^{i}(1-t)^{n-k-1}((1-t)+1) \\
&= t^{k}(1-t)^{n-k-1} \\
&= \frac{1}{\binom{n-1}{k}} B_{k,n-1}(t)
\end{aligned}$$

We can write every Bernstein polynomial in terms of higher degree Bernstein polynomials using the final equation.

That is,

$$\begin{aligned} B_{k,n-1}(t) &= \binom{n-1}{k}\left[\frac{1}{\binom{n}{k}} B_{k,n}(t) + \frac{1}{\binom{n}{k+1}} B_{k+1,n}(t)\right] \\ &= \left(\frac{n-k}{n}\right) B_{k,n}(t) + \left(\frac{k+1}{n}\right) B_{k+1,n}(t) \end{aligned}$$

It translates a degree $(n-1)$ Bernstein polynomial into a linear combination of degree n Bernstein polynomials. This can be extended to demonstrate that any Bernstein polynomial of degree k (less than n) can be expressed as a linear combination of Bernstein polynomials of degree n. For instance, a Bernstein polynomial of degree $(n-2)$ can be expressed as a linear combination of two Bernstein polynomials of degree $(n-1)$, each of which can be expressed as a linear combination of two Bernstein polynomials of degree n, etc.

1.3 Derivation of Bernstein polynomials

Proposition 1.3.1 *Derivatives of the nth-degree Bernstein polynomials are polynomials of degree $(n-1)$. Using the definition of the Bernstein polynomial. We can demonstrate that this derivative can be represented as a linear combination of Bernstein polynomials use only its definition. In particular*

$$\frac{d}{dt} B_{k,n}(t) = n\left(B_{k-1,n-1}(t) - B_{k,n-1}(t)\right)$$

for $0 \leq k \leq n$.

Proof. This can be shown by direct differentiation

$$
\begin{aligned}
\frac{d}{dt}B_{k,n}(t) &= \frac{d}{dt}\binom{n}{k}t^k\,(1-t)^{n-k} \\
&= \frac{kn!}{k!(n-k)!}t^{k-1}(1-t)^{n-k} + \frac{(n-k)n!}{k!(n-k)!}t^k(1-t)^{n-k-1} \\
&= \frac{n(n-1)!}{(k-1)!(n-k)!}t^{k-1}(1-t)^{n-k} + \frac{n(n-1)!}{k!(n-k-1)!}t^k(1-t)^{n-k-1} \\
&= n\left(\frac{(n-1)!}{(k-1)!(n-1)!}t^{k-1}(1-t)^{n-k} + \frac{(n-1)!}{k!(n-k-1)!}t^k(1-t)^{n-k-1}\right) \\
&= n\left(B_{k-1,n-1}(t) - B_{k,n-1}\left(t\right)\right)
\end{aligned}
$$

■

That is, the derivative of a Bernstein polynomial can be expressed as the degree of the polynomial multiplied by the difference of two Bernstein polynomials of degree $(n-1)$.

1.4 The basis of Bernstein polynomials

1.4.1 converting from the Bernstein basis to the canonical basis

Any Bernstein polynomial of degree n can be expressed in terms of the canonical basis$\{1, t, t^2, \ldots, t^n\}$ because it serves as the basis for polynomials with degrees less than or equal to n. The definition of the Bernstein polynomials and the binomial theorem can be used to directly calculate this, as follows

$$\begin{aligned}
B_{k,n}(t) &= \binom{n}{k} t^k (1-t)^{n-k} \\
&= \binom{n}{k} t^k \sum_{i=0}^{n-k} (-1)^i \binom{n-k}{i} t^i \\
&= \sum_{i=0}^{n-k} (-1)^i \binom{n}{k} \binom{n-k}{i} t^{i+k}
\end{aligned}$$

we put $i = i - k$

$$\begin{aligned}
B_{k,n}(t) &= \sum_{i=k}^{n} (-1)^{i-k} \binom{n}{k} \binom{n-k}{i-k} t^i \\
&= \sum_{i=k}^{n} (-1)^{i-k} \binom{n}{i} \binom{i}{k} t^i
\end{aligned}$$

where we have used the binomial theorem to expand $(1-t)^{n-k}$.

We use the degree elevation formulas and induction to demonstrate that each canonical basis element may be represented as a linear combination of Bernstein polynomials to calculate

$$
\begin{aligned}
t^k &= t\left(t^{k-1}\right) \\
&= t\sum_{i=k-1}^{n} \frac{\binom{i}{k-1}}{\binom{n}{k-1}} B_{i,n-1}(t) \\
&= \sum_{i=k}^{n} \frac{\binom{i-1}{k-1}}{\binom{n-1}{k-1}} t B_{i-1,n-1}(t) \\
&= \sum_{i=k-1}^{n-1} \frac{\binom{i}{k-1}}{\binom{n}{k-1}} \frac{i}{n} B_{i,n}(t) \\
&= \sum_{i=k-1}^{n-1} \frac{\binom{i}{k}}{\binom{n}{k}} B_{i,n}(t)
\end{aligned}
$$

where the induction hypothesis was used in the second step.

1.4.2 The Bernstein polynomials as a basis

Why do the Bernstein polynomials of order n form a basis for the space of polynomials of degree less than or equal to n?

1) They span the space of polynomials, any polynomial of degree less than or equal to n can be written as a linear combination of the Bernstein polynomials.

   This is easily seen if one realizes that the canonical basis spans the space of polynomials and any member of the canonical basis can be written as a linear combination of Bernstein polynomials.

2) They are linearly independent, that is, if there exist constants $c_0, c_1, c_2, \ldots, c_n$ so that the identity

$$0 = c_0 B_{0,n}(t) + c_1 B_{1,n}(t) + \ldots + c_n B_{n,n}(t)$$

holds for all t, then all the c_i's must be zero.

If this were true, then we could write

$$\begin{aligned}
0 &= c_0 B_{0,n}(t) + c_1 B_{1,n}(t) + \cdots + c_n B_{n,n}(t) \\
&= c_0 \sum_{i=0}^{n} (-1)^i \binom{n}{i}\binom{i}{0} t^i + c_1 \sum_{i=1}^{n} (-1)^{i-1} \binom{n}{i}\binom{i}{1} t^i + \cdots \\
&+ c_n \sum_{i=n}^{n} (-1)^{i-n} \binom{n}{i}\binom{i}{n} t^i \\
&= c_0 + \left[\sum_{i=0}^{1} c_i \binom{n}{1}\binom{1}{1}\right] t^1 + \cdots + \left[\sum_{i=0}^{n} c_i \binom{n}{n}\binom{n}{n}\right] t^n
\end{aligned}$$

Since the canonical basis is a linearly independent set, we must have that

$$\begin{aligned}
c_0 &= 0 \\
\sum_{i=1}^{1} c_i \binom{n}{1}\binom{1}{1} &= 0 \\
&\vdots \\
\sum_{i=0}^{n} c_i \binom{n}{n}\binom{n}{n} &= 0
\end{aligned}$$

which implies that $c_0 = c_1 = \cdots = c_n = 0$ (c_0 is clearly zero, substituting this in the second equation gives $c_1 = 0$, substituting these two into the third equation gives $c_2 = 0$, etc ...)

1.5 A Matrix Representation for Bernstein Polynomials

A matrix formulation for the Bernstein polynomials is helpful in numerous applications, represented as a linear combination of the Bernstein basis functions for the given polynomial

$$B(t) = c_0 B_{0,n}(t) + c_1 B_{1,n}(t) + \cdots + c_n B_{n,n}(t)$$

It is easy to write this as a dot product of two vectors

$$B(t) = \begin{bmatrix} B_{0,n}(t) & B_{1,n}(t) & \cdots & B_{n,n}(t) \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix}$$

We can convert this to

$$B(t) = \begin{bmatrix} 1 & t & t^2 & \cdots & t^n \end{bmatrix} \begin{bmatrix} b_{0,0} & 0 & 0 & \cdots & 0 \\ b_{1,0} & b_{1,1} & 0 & \cdots & 0 \\ b_{2,0} & b_{2,1} & b_{2,2} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ b_{n,0} & b_{n,1} & b_{n,2} & \cdots & b_{n,n} \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

where the $b_{i,j}$ are the coefficients of the canonical basis that are used to determine the respective Bernstein polynomials. We note that the matrix in this case is lower triangular.

Example 1.5.1 *In the quadratic case* $(n = 2)$*, the matrix representation is*

$$B(t) = \begin{bmatrix} 1 & t & t^2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -2 & 2 & 0 \\ 1 & -2 & 0 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix}$$

and in the cubic case $(n = 3)$, the matrix representation is

$$B(t) = \begin{bmatrix} 1 & t & t^2 & t^3 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 3 & -6 & 3 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

1.6 Approximation by Bernstein polynomials

S. Bernstein provided a beautiful proof for the Weierstrass approximation theorem. We will demonstrate that polynomials can uniformly approximate any continuous function on the closed interval $[0, 1]$. Beginning with the fundamental units, the Bernstein polynomials provided by the equations

$$B_{n,k}(x) = \binom{n}{k} x^k (1-x)^{n-k}, k = 0, 1, \ldots, n$$

Lemma 1.6.1 *we have the following formulas*

$$\sum_{k=0}^{n} B_{n,k}(x) = 1, \tag{1.2}$$

$$\sum_{k=0}^{n} k B_{n,k}(x) = nx, \tag{1.3}$$

and

$$\sum_{k=0}^{n} k\left(k-1\right) B_{n,k}\left(x\right) = n\left(n-1\right) x^{2}. \tag{1.4}$$

Proof. the first formula 1.2 is just which yields $\left(x+1-x\right)^{n} = 1$ (which we have previously proved).

the second one **??**: for $k \geq 1$, $k\binom{n}{k} = \frac{n!}{(k-1)!(n-k)!} = n\binom{n-1}{k-1}$ (the statement assumes $n > 1$)

so

$$\begin{aligned}\sum_{k=0}^{n} kB_{n,k}\left(x\right) &= \sum_{k=0}^{n} k\binom{n}{k} x^{k}\left(1-x\right)^{n-k} \\ &= \sum_{k=1}^{n} k\binom{n}{k} x^{k}\left(1-x\right)^{n-k}\end{aligned}$$

because the term for $k = 0$ is zero, we obtain

$$\begin{aligned}\sum_{k=0}^{n} kB_{n,k}\left(x\right) &= \sum_{k=1}^{n} n\binom{n-1}{k-1} x^{k}\left(1-x\right)^{n-k} \\ &= nx\sum_{k=1}^{n} \binom{n-1}{k-1} x^{k-1}\left(1-x\right)^{n-k} \\ &= nx\left(x+1-x\right)^{n-1} = nx\end{aligned}$$

the third one 1.4: $k \geq 2$ and $n \geq 2$, $k\left(k-1\right)\binom{n}{k} = \frac{n!}{(k-2)!(n-k)!} = n\left(n-1\right)\binom{n-2}{k-2}$

so

$$\sum_{k=0}^{n} k\,(k-1)\,B_{n,k}\,(x) = \sum_{k=0}^{n} k\,(k-1)\binom{n}{k} x^{k}\,(1-x)^{n-k}$$
$$= \sum_{k=2}^{n} k\,(k-1)\binom{n}{k} x^{k}\,(1-x)^{n-k}$$

because the term for $k = 0, 1$ is zero, we obtain

$$\sum_{k=0}^{n} k\,(k-1)\,B_{n,k}\,(x) = \sum_{k=2}^{n} n\,(n-1)\binom{n-2}{k-2} x^{k}\,(1-x)^{n-k}$$
$$= n\,(n-1)\,x^{2}\sum_{k=2}^{n}\binom{n-2}{k-2} x^{k-2}\,(1-x)^{n-k}$$
$$= n\,(n-1)\,x^{2}\,(x+1-x)^{n-2} = n\,(n-1)\,x^{2}$$

■

Lemma 1.6.2 *we have that*

$$\sum_{k=0}^{n} \frac{k}{n} B_{n,k}\,(x) = x,$$

and

$$\sum_{k=0}^{n} \left(\frac{k}{n} - x\right)^{2} B_{n,k}\,(x) = \frac{x\,(1-x)}{n}.$$

If we consider the $B_{n,k}\,(x)$ as probabilities assigned to the 'events' $k = n$, the first statement states that the expectation value of $k = n$ is x and the second formula states that the variance is $x\,(1-x)\,n$ which is small for n large. The proof is simple and derives from the prior lemma 1.6.1 by straightforward manipulations. Now we are ready to state

Theorem 1.6.1 *let f be a continuous function. Define the polynomial*

$$B_n(f)(x) := \sum f\left(\frac{k}{n}\right) B_{n,k}(x).$$

Then for any $\epsilon > 0$ there exists N such that for all $n > N$ and all $x \in [0,1]$,

$$|B_n(f)(x) - f(x)| < \epsilon.$$

Note that the polynomial is a Riemann sum but with weights $B_{n,k}(x)$.

Proof. Since f is continuous on the closed interval, it is uniformly continuous, i.e., for every $\epsilon > 0$ the exists $\delta > 0$ which depends only on ϵ such that

$$|f(x) - f(t)| \leq \frac{\epsilon}{2}$$

whenever $|x - y| < \delta$. We may write, because of the first Lemma

$$B_n(f)(x) - f(x) = \sum_{k=0}^{n} \left[f\left(\frac{k}{n}\right) - f(x) \right] B_{n,k}(x).$$

thus

$$|B_n(f)(x) - f(x)| \leq \sum_{\left|\frac{k}{n} - x\right| < \delta} \left| f\left(\frac{k}{n}\right) - f(x) \right| B_{n,k}(x) + \sum_{\left|\frac{k}{n} - x\right| \geq \delta} \left| f\left(\frac{k}{n}\right) - f(x) \right| B_{n,k}(x).$$

Note that we have sued the fact that the Bernstein polynomials are non-negative. Hence, we may estimate the right side by

$$|B_n(f)(x) - f(x)| \leq \frac{\epsilon}{2} \sum_{\left|\frac{k}{n} - x\right| < \delta} B_{n,k}(x) + 2\max|f(x)| \sum_{\left|\frac{k}{n} - x\right| \geq \delta} B_{n,k}(x)$$

in the second term, we write

$$\sum_{\left|\frac{k}{n}-x\right|\geq\delta} B_{n,k}(x) = \sum_{\left|\frac{k}{n}-x\right|\geq\delta} \frac{\left|\frac{k}{n}-x\right|^2}{\left|\frac{k}{n}-x\right|^2} B_{n,k}(x) \leq \frac{1}{\delta^2} \sum_{\left|\frac{k}{n}-x\right|\geq\delta} \left|\frac{k}{n}-x\right|^2 B_{n,k}(x) \leq \frac{x(1-x)}{n\delta^2}$$

thus, we have

$$|B_n(f)(x) - f(x)| \leq \frac{\epsilon}{2} + 2\max|f(x)| \frac{x(1-x)}{n\delta^2} \leq \frac{\epsilon}{2} + \frac{\max|f(x)|}{2n\delta^2}$$

and choosing

$$n > \frac{\max|f(x)|}{\epsilon\delta^2}$$

yields the theorem. ■

Chapter 2

Bezier Curves

In this chapter, we will present the Bezier curves, which are parametric curves that are based on a polynomial function with one parameter. They are named after the engineer **Pierre Bezier**, who developed them to model the stylized shapes of cars while working at Renault in the 1960s. Interestingly enough, they were also independently developed by **Paul De Casteljau** at Citroen, likely before Bezier, but it appears that he was not allowed to publish them.

2.1 Parametric curves

Definition 2.1.1 *Let $C \subset \mathbb{R}^2$ be some curve in the plane. A parametrization of the curve C is pair of functions $f, g : I \subseteq \mathbb{R} \to \mathbb{R}$ such that $C = \{(f(t), g(t)) / t \in I\}$. In other words, a parametric curve is a mapping from $I \subset \mathbb{R} \to \mathbb{R}^2$ given by the rule $t \to (f(t), g(t))$ for each $t \in I$. We say that t is parameter and that the parametric equations for the curve are $x = f(t)$ and $y = g(t)$. In case $I = [a, b]$ we say that $(f(a), g(a))$ is the initial point and $(f(b), g(b))$ is the terminal point.*

2.2 Bezier Curves

Definition 2.2.1 *Given $(n+1)$ control points $p_0, p_1, \ldots, p_n$ in space, the Bezier curve (of degree n) defined from these points is the parametric curve $C(t)$ defined by*

$$C(t) = \sum_{i=0}^{n} B_{i,n}(t)\, p_i \qquad \forall t \in [0,1].$$

The points $(p_i)_{i=0,\ldots,n}$ are called the control points and the line segments $p_0p_1, p_1p_2, \ldots, p_{n-1}p_n$ form in this order called the control polygon.

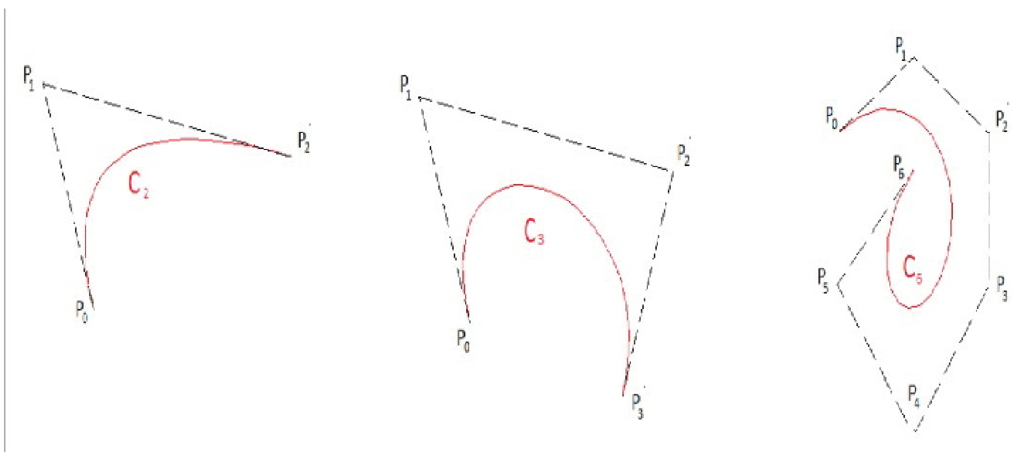


Figure 2.1: Bezier curves of various degrees and their control polygons.

Linear, Quadratic, and Cubic Bezier curves, are given as follows:

1) Linear Bezier curves $(n=1)$ are thus given by

$$\begin{aligned} C(t) &= \sum_{i=0}^{1} B_{i,1}(t)\, p_i \\ &= (1-t)\, p_0 + t p_1 \qquad \text{,for } 0 \le t \le 1. \end{aligned}$$

2) Quadratic Bezier curves ($n = 2$) are given by

$$\begin{aligned} C(t) &= \sum_{i=0}^{2} B_{i,2}(t)\, p_i \\ &= (1-t)^2 p_0 + 2t(1-t)\, p_1 + t^2 p_2 \qquad \text{,for } 0 \le t \le 1. \end{aligned}$$

3) Cubic Bezier curves ($n = 3$) are given by

$$\begin{aligned} C(t) &= \sum_{i=0}^{3} B_{i,3}(t)\, p_i \\ &= (1-t)^3 p_0 + 3t(1-t)^2 p_1 + 3t^2(1-t)\, p_2 + t^3 p_3 \qquad \text{,for } 0 \le t \le 1. \end{aligned}$$

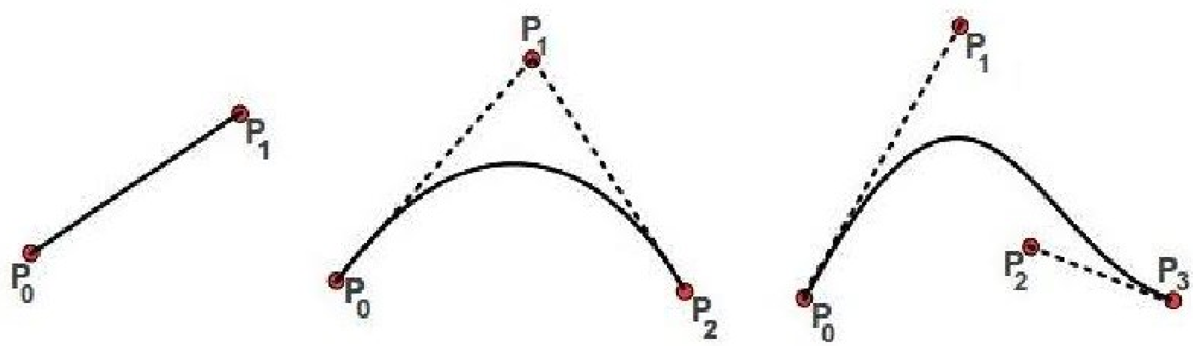


Figure 2.2: Linear,Quadratic and Cubic Bezier curve

2.2.1 Derivatives of a Bezier Curves

The derivative of a Bezier curve of degree n is another Bezier curve of degree $(n-1)$ given by

$$\frac{d}{dt} C(t) = n \sum_{k=0}^{n-1} B_{k,n-1}(t)\,(p_{k+1} - p_k)\,,$$

More generally, derivatives of higher order are given by

$$\frac{d^r}{dt^r}C(t) = \frac{n!}{(n-r)!}\sum_{k=0}^{n-r} B_{k,n-r}(t)\left(p_{k+1}-p_k\right)^r \qquad \forall r \leq n,$$

This property has important consequences. In particular, the curve at extremities is tangent to the first and last control points lines. This could be used to apply Neumann boundary conditions for instance.

Proof. For the first derivative, we have

$$\begin{aligned}
\frac{d}{dt}C(t) &= \frac{d}{dt}\left(\sum_{k=0}^{n} B_{k,n}(t)\,p_k\right) \\
&= \sum_{k=0}^{n}\left(\frac{dB_{k,n}(t)}{dt}\right)p_k \\
&= \sum_{k=0}^{n}\left(n\left(B_{k-1,n-1}(t) - B_{k,n-1}(t)\right)\right)p_k \\
&= \left(n\sum_{k=0}^{n} B_{k-1,n-1}(t)\,p_k\right) - \left(n\sum_{k=0}^{n} B_{k,n-1}(t)\,p_k\right) \\
&\text{(note that } B_{n-1,n-1}(t) = B_{n,n-1}(t) = 0) \\
&= \left(n\sum_{k=0}^{n-1} B_{k,n-1}(t)\,p_{k+1}\right) - \left(n\sum_{k=0}^{n-1} B_{k,n-1}(t)\,p_k\right) \\
&= n\sum_{k=0}^{n-1} B_{k,n-1}(t)\left(p_{k+1}-p_k\right).
\end{aligned}$$

■

this can be written as

$$\begin{aligned}
\frac{d}{dt}C(t) &= n\sum_{k=0}^{n-1} B_{k,n-1}(t)\,p_k \\
&= \frac{n!}{(n-1)!}\sum_{k=0}^{n-1} B_{k,n-1}(t)\,p_k \qquad (2.1)
\end{aligned}$$

Now assume that 2.1 is true up to the order r. Let us prove it for $r+1$

$$
\begin{aligned}
\frac{d^{r+1}}{dt^{r+1}}C(t) &= \frac{d}{dt}\left(\frac{d^r}{dt^r}c(t)\right)\\
&= \frac{d}{dt}\left(\frac{n!}{(n-r)!}\left(\sum_{k=0}^{n-r}B_{k,n-1}(t)\,(p_{k+1}-p_k)^r\right)\right)\\
&= \frac{n!}{(n-r)!}\left(\sum_{k=0}^{n-r}\frac{d}{dt}B_{k,n-1}(t)\,(p_{k+1}-p_k)^r\right)\\
&= \frac{n!}{(n-r)!}\left(\sum_{k=0}^{n-r}((n-r)\,(B_{k-1,n-r-1}(t)-B_{k,n-r-1}(t)))\,(p_{k+1}-p_k)^{r+1}\right)\\
&= \frac{n!}{(n-r-1)!}\left(\sum_{k=0}^{n-r}(B_{k-1,n-r-1}(t)-B_{k,n-r-1}(t))\,(p_{k+1}-p_k)^{r+1}\right)\\
&= \frac{n!}{(n-r-1)!}\left(\sum_{k=1}^{n-r}(p_{k+1}-p_k)^{r+1}B_{k-1,n-r-1}(t)-\sum_{k=0}^{n-r-1}(p_{k+1}-p_k)^{r+1}B_{k,n-r-1}(t)\right)\\
&= \frac{n!}{(n-r-1)!}\left(\sum_{k=0}^{n-r-1}(p_{k+1}-p_k)^{r+1}B_{k,n-r-1}(t)-\sum_{k=0}^{n-r-1}(p_{k+1}-p_k)^{r+1}B_{k,n-r-1}(t)\right)\\
&= \frac{n!}{(n-(r+1))!}\left(\sum_{k=0}^{n-(r+1)}(p_{k+1}-p_k)^{r+1}B_{k,n-(r+1)}(t)\right).
\end{aligned}
$$

Therefore, the result is true for $(r+1)$.

2.3 Properties of Bezier Curves

let's see some of the important properties of the Bezier curves

2.3.1 Interpolation at the extremities

$$C(0) = p_0 \text{ and } C(1) = p_n$$

and that it means that the Bezier curve of degree n always starts from the first control points p_0 and ends at the last control poins p_n.

Proof.

$$\begin{aligned}
C(0) &= \sum_{i=0}^{n} B_{i,n}(0)\, p_i \\
&= \sum_{i=0}^{n} \binom{n}{i} 0^i 1^{n-i} p_i \\
&= p_0.1 \qquad\qquad \left(0^0 = 1 \ \text{ by convention}\right) \\
&= p_0
\end{aligned}$$

in the same way, we have

$$\begin{aligned}
C(1) &= \sum_{i=0}^{n} B_{i,n}(1)\, p_i \\
&= \sum_{i=0}^{n} \binom{n}{i} 1^i 0^{n-i} p_i \\
&= p_0.1 \qquad\qquad \left(0^0 = 1 \ \text{ by convention}\right) \\
&= p_0
\end{aligned}$$

■

2.3.2 Invariance under affine transformation

Let $\mathcal{T}$ be an affine transformation (for example, a rotation, reflection, translation, or scaling).

Then
$$\mathcal{T}\left(\sum_{i=0}^{n} B_{i,n}(t)\, p_i\right) = \sum_{i=0}^{n} B_{i,n}(t)\, \mathcal{T}(p_i)$$

As shown, it consists in applying the Affine transformation only to control points.

Proof. Let $\mathcal{T}$ be an affine transformation in $\mathbb{R}^n$

$$\mathcal{T}(X) = AX + b \qquad \text{with } A \in \mathbb{M}_n(\mathbb{R}) \quad \text{and} \quad b \in \mathbb{R}^n.$$

The affine transformation of a Bezier curve $C(t)$ of degree n is

$$\begin{aligned}
\mathcal{T}(C(t)) &= \mathcal{T}\left(\sum_{i=0}^{n} B_{i,n}(t)\, p_i\right) \\
&= A\left(\sum_{i=0}^{n} B_{i,n}(t)\, p_i\right) + b \\
&= \sum_{i=0}^{n} AB_{i,n}(t)\, p_i + \sum_{i=0}^{n} bB_{i,n}(t) \\
&= \sum_{i=0}^{n} (Ap_i + b)\, B_{i,n}(t) \\
&= \sum_{i=0}^{n} \mathcal{T}(p_i)\, B_{i,n}(t).
\end{aligned}$$

So $\mathcal{T}(C(t))$ is a Bezier curve. ■

2.3.3 Convex hull

This property indicates that Bezier curves always lie within the convex hull of their control points. The convex hull of the control points is also referred to as the convex hull of the Bezier curve

Definition 2.3.1 *Let $X = \{x_0, x_1, \ldots, x_n\}$ be a set of points. the convex hull of X denoted $CH\{X\}$defined by*

$$CH\{X\} = \left\{ a_0 x_0 + \ldots + a_n x_n / \sum_{i=0}^{n} a_i = 1, a_i \geq 0 \right\}$$

Proposition 2.3.1 *All the points of the bezier curve are located inside the convex envelope of all the control points* $\{p_0, p_1, \ldots, p_n\}$ *ie*

$$\forall t \in [0,1], C(t) \in CH\{p_0, p_1, \ldots, p_n\}$$

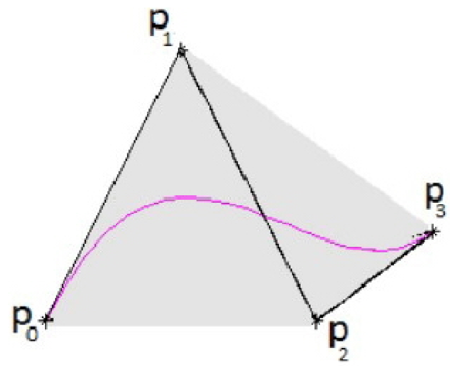


Figure 2.3: Cubic Bézier curve, its control polygon and the convex hull

Proof. we have

$$C(t) = \sum_{i=0}^{n} B_{i,n}(t)\, p_i \qquad \forall t \in [0,1]$$

as the Bernstein polynomials verify the properties (positivity and partition of unity), then according to the definition of the convex envelope, we get

$$C(t) \in CH\{p_0, p_1, \ldots, p_n\}$$

■

2.3.4 Symmetry

It is clear that it does not matter if the Bezier points are labeled $p_0, p_1, \ldots, p_n$ or $p_n, p_{n-1}, \ldots, p_0$. The curves that correspond to the two different orderings look the same; they differ only in the direction in which they are traversed. Written as a formula

$$\sum_{i=0}^{n} B_{i,n}(t)\, p_i = \sum_{i=0}^{n} B_{i,n}(1-t)\, p_{n-i}$$

Proof. This follows from the property of symmetry of Bernstein polynomials

$$B_{i,n}(t) = B_{i,n}(1-t)$$

hence$\sum_{i=0}^{n} B_{i,n}(t)\, p_i = \sum_{i=0}^{n} B_{i,n}(1-t)\, p_{n-i}$

by reindexing, we will have

$$\sum_{i=0}^{n} B_{i,n}(t)\, p_i = \sum_{i=0}^{n} B_{i,n}(1-t)\, p_{n-i}$$

■

2.4 Geometric construction of Bezier curves

At citroen, **Mr.Paul De Casteljau** started developing the algorithm that bears his name in 1959. Citroen kept them very quiet and then they were released as technical reports, prior to 1975, when **W. Bohm** first learned of these works and made them widely known, they were unheard of. The development of the usage of **Pierre Bezier** curves would not have been possible without the aid of this algorithm, which has been extremely helpful for computer science, which frequently employs Bezier curves (drawing , modeling, software, etc.).

2.4.1 The De-Casteljau Algorithm

Given: $p_0, p_1, \dots, p_n \in \mathbb{E}^3, t \in \mathbb{R}$,

let

$$p_i^j = (1-t)\, p_i^{j-1} + t p_{i+1}^{j-1} \qquad \begin{cases} j = 1, \dots, n \\ i = 0, \dots, n-j \end{cases}$$

and $p_i^0 = p_i$, then p_i^n is the point with parameter value t on the Bezier curve.

The formula generates a triangular set of values, for example in the cubic case

$$\begin{array}{llll} p_0^0 & p_1^0 & p_2^0 & p_3^0 \\ p_0^1 & p_1^1 & p_2^1 & \\ p_0^2 & p_1^2 & & \\ p_0^3 & & & \end{array}$$

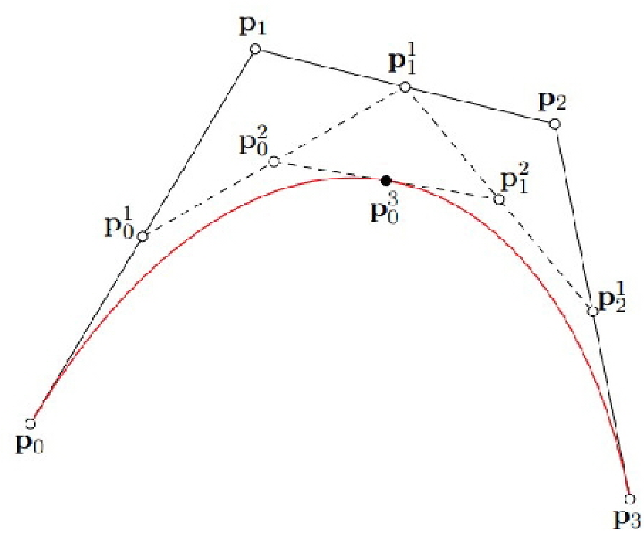


Figure 2.4: De-casteljau algorithm

Example 2.4.1 *for example, a cubic Bezier curve has four control points* $p_0\,(1,1)$, $p_1\,(2,3)$, $p_2\,(8,1)$ *and* $p_3\,(4,7)$. *The point* $C\,(0,25)$ *is determined by applying the De-Casteljau* *algorithm with* $t = 0,25$. *Then*

$$
\begin{aligned}
p_0^1 &= \frac{3}{4}(1,1) + \frac{1}{4}(2,3) = (1.25, 1.5)\,, \\
p_1^1 &= \frac{3}{4}(2,3) + \frac{1}{4}(8,1) = (3.5, 2.5)\,, \\
p_2^1 &= \frac{3}{4}(8,1) + \frac{1}{4}(4,7) = (7, 2.5)\,, \\
p_0^2 &= \frac{3}{4}(1.25, 1.5) + \frac{1}{4}(3.5, 2.5) = (1.8125, 1.57)\,, \\
p_1^2 &= \frac{3}{4}(3.5, 2.5) + \frac{1}{4}(7, 2.5) = (4.375, 2.5)\,, \\
p_0^3 &= \frac{3}{4}(1.8125, 1.57) + (4.375, 2.5) = (2.4531, 1.8025)\,,
\end{aligned}
$$

The algorithm gives the following table of points

$$
\begin{array}{ccccccc}
(1,1) & & (2,3) & & (8,1) & & (4,7) \\
\frac{3}{4}\downarrow & \swarrow \frac{1}{4} & & \frac{3}{4}\downarrow \quad \swarrow \frac{1}{4} & & \frac{3}{4}\downarrow \quad \swarrow \frac{1}{4} & \\
(1.25, 1.5) & & (3.5, 2.5) & & (7, 2.5) & & \\
\frac{3}{4}\downarrow & \swarrow \frac{1}{4} & & \frac{3}{4}\downarrow \quad \swarrow \frac{1}{4} & & & \\
(1.8125, 1.57) & & (4.375, 2.5) & & & & \\
\frac{3}{4}\downarrow & \swarrow \frac{1}{4} & & & & & \\
(2.4531, 1.8025) & & & & & &
\end{array}
$$

The algorithm yields $C(0.25) = (2.4531, 1.8025)$.

Proof. (De-casteljau algorithm)

The De-Casteljau algorithm follows from the recursion property of the Bernstein polynomials

$$B_{i,n}\left(t\right)=\left(1-t\right)B_{i,n-1}\left(t\right)+tB_{i-1,n-1}\left(t\right)$$

then

$$\begin{aligned}C\left(t\right)&=\sum_{i=0}^{n}p_iB_{i,n}\left(t\right)=\sum_{i=0}^{n}p_i\left(\left(1-t\right)B_{i,n-1}\left(t\right)+tB_{i-1,n-1}\left(t\right)\right)\\&=\sum_{i=0}^{n}p_i\left(1-t\right)B_{i,n-1}\left(t\right)+\sum_{i=0}^{n}p_itB_{i-1,n-1}\left(t\right).\end{aligned}$$

since $B_{n,n-1}\left(t\right)=0$, and $B_{-1,n-1}\left(t\right)=0$ it follows that

$$C\left(t\right)=\sum_{i=0}^{n-1}p_i\left(1-t\right)B_{i,n-1}\left(t\right)+\sum_{i=1}^{n}p_itB_{i-1,n-1}\left(t\right).$$

Next renumber the second summation by replacing i by $i+1$,

$$\begin{aligned}C\left(t\right)&=\sum_{i=0}^{n-1}p_i\left(1-t\right)B_{i,n-1}\left(t\right)+\sum_{i=0}^{n-1}p_{i+1}tB_{i,n-1}\left(t\right)\\&=\sum_{i=0}^{n-1}(p_i\left(1-t\right)+p_{i+1}t)B_{i,n-1}\left(t\right)\end{aligned}$$

set $p_i^1=\left(1-t\right)p_i+tp_{i+1}=\left(1-t\right)p_i^0+tp_{i+1}^0\quad$ for $i=0,\ldots,n-1$, then

$$C\left(t\right)=\sum_{i=0}^{n-1}p_i^1B_{i,n-1}\left(t\right)$$

the last equation expresses $C\left(t\right)$ as a Bezier curve of degree $n-1$ with control points $p_0^1,\ldots,p_{n-1}^1$. Applying a similar argument yields

$$C\left(t\right)=\sum_{i=0}^{n-2}p_i^2B_{i,n-2}\left(t\right),$$

where $p_i^2 = (1-t)\,p_i^1 + tp_{i+1}^1$ for $i = 0, \ldots, n-2$. In general

$$C(t) = \sum_{i=0}^{n-j} p_i^j B_{i,n-j}(t),$$

where $p_i^j = (1-t)\,p_i^j + tp_{i+1}^j$ for $i = 0, \ldots, n-j$. In particular, $j = n$ gives

$$C(t) = \sum_{i=0}^{0} p_i^n B_{n,n-n}(t) = p_0^n.$$

■

Chapter 3

Applications

3.1 Collocation method

3.1.1 General principle

In general, the collocation method's basic tenets were used to roughly resolve operator equations

$$\mathcal{L}u = f$$

entails searching for a rough solution in a finite-dimensional subspace by limiting the verification of the equation to a small set of points known as collocation points

In actuality, we select a set of finite-dimensional subspaces of the form $X_n \subset X, n \geq 1$ which are typically $L^2(G)$ or $C(G)$.

Let $\{\phi_1, \ldots, \phi_n\}$ be a basis of X_n we are looking for a function $u_n \in X$, of the form

$$u_n = \sum_{j=1}^{n} c_j \phi_j (t) \quad , t \in G$$

to determine the coefficients (c_j), we substitute, this function in the equation, and we require that the equation be exact in the sense that the residual

$$\begin{aligned} R_n (t) &= \mathcal{L} u_n - f (t) \\ &= \mathcal{L} \sum_{j=1}^{n} c_j \phi_j (t) - f (t) \quad , t \in G \\ &= \sum_{j=1}^{n} c_j \mathcal{L}(\phi_j (t)). \end{aligned}$$

Let be zero on a system of nodes $t_1, t_2, \ldots, t_n \in G$, (ie, at the collocation points). Which leads systematically to the resolution of the linear system

$$\sum_{j=1}^{n} c_j \phi_j = f (t_i) , \quad i = 1, 2, \ldots, n$$

of the form $\phi_n X = f_n$ obviously, this system admits a unique solution if the det is non-zero, which moreover depends on the choice of collocation points.

In the collocation method, we force the remainder $R_n (t_j)$ for the collocation points.

3.2 Solving an ODE by Bernstein polynomials

The collocation method is a projection method that uses Bernstein polynomials to approximate the solution of an ordinary differential equation.

3.2.1 Discretization of an ODE

In this section, we will look at the first and second order ordinary differential equations described by

$$\mathcal{L}u = f\left(t\right), \quad t \in [a, b] \tag{3.1}$$

Where $f(t)$ is a continuous function in $[a; b]$ We convert this equation into a system of linear equations. For this result, we need some basic functions to be the ordinary differential equations solution. Of fate, we choose Bernstein polynomials as the basic functions.

Now we use the **Collocation method** technique. For this, we estimate the unknown function $u\left(t\right)$ as follows

$$u\left(t\right) = \sum_{i=0}^{n} c_i B_{i,n}\left(t\right)$$

Where $B_{i,n}\left(t\right)$ are Bernstein polynomials and, $c_i \quad i = 0, 1, \ldots, n$ are unknown parameters, to be determined, we replace in the equation 3.1, we get

$$\mathcal{L} \sum_{i=0}^{n} c_i B_{i,n}\left(t\right) = \sum_{i=0}^{n} a_i B_{i,n}\left(t\right)$$

or

$$\sum_{i=0}^{n} c_i \left[\mathcal{L} B_{i,n}\left(t\right)\right] = \sum_{i=0}^{n} a_i B_{i,n}\left(t\right)$$

The Collocation equations are thus obtained, and we then choose a linear equation with the unknowns $(n + 1)$ $(c_i = 0, 1, \ldots, n)$ for each $(j = 0, 1, \ldots, n)$.

Finally, a system of linear equations representing the unknowns $(n+1)$ are given.

$$A_{i,j} X_i = b_j \qquad i, j = 0, 1, \ldots, n$$

where

$$\begin{aligned} A_{i,j} &= \mathcal{L} B_{i,n} (t_j) \\ b_j &= \sum_{i=0}^{n} a_i B_{i,n} (t_j) \qquad , j = 0, 1, \ldots, n \\ x_i &= (c_i) \quad , \qquad i = 0, 1, \ldots n \end{aligned}$$

3.3 Illustration with a numerical example

Example 3.3.1 *Let the equation be*

$$\begin{cases} u' = u; \quad t \in [0, 1] \\ \quad u(0) = 1, \end{cases}$$

The exact solution of this equation is

$$u(t) = \exp(t).$$

The solution of the system of linear equations for $n = 4$ yields the approximate solution $\widetilde{u}(t)$ of the exact solution $u(t)$

$ans =$

t	ex sol	app sol	error
0	1.0000	1.0074	0.0074
0.1000	1.1052	1.1128	0.0077
0.2000	1.2214	1.2297	0.0083
0.3000	1.3499	1.3590	0.0092
0.4000	1.4918	1.5020	0.0120
0.5000	1.6487	1.6600	0.0113
0.6000	1.8221	1.8346	0.0125
0.7000	2.0138	2.0275	0.0138
0.8000	2.2255	2.2407	0, 0152
0.9000	2.4596	2.4764	0.0168
1.0000	2.7183	2.7369	0.0186

Table 3.1: table of solutions

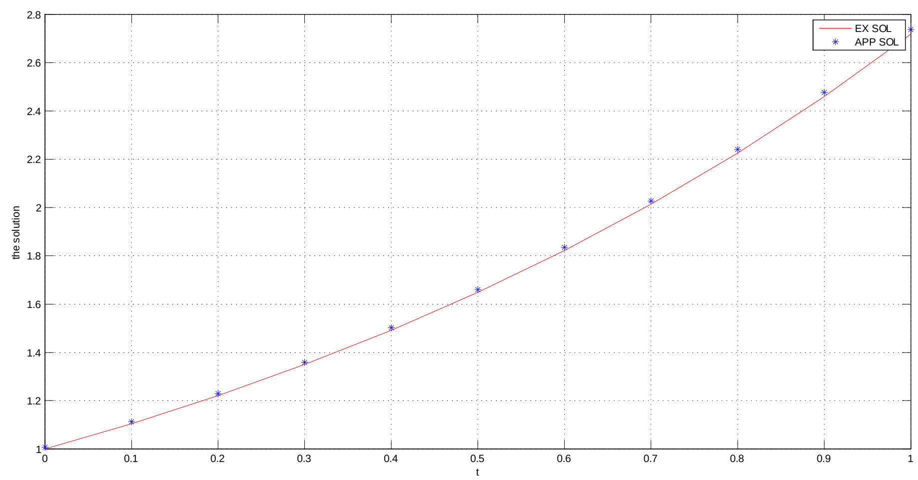


Figure 3.1: Approximation of the equation u’=u

3.4 Applications of Bezier Curves

Applications for Bezier curves are numerous and extremely significant since they allow visual designers and engineers to model actual objects. Let's examine a few of them.

3.4.1 Computer Graphics

Bezier curves enables us to model smooth curves since the curve is contained in the convex hull created by the control points. You can then apply affine transformations to the curve, such as rotation and translation, by applying these transforms to the control points.

Because higher-degree curves require more computation to analyze, cubic and quadratic curves are the most frequently utilized Bezier curves. These are used to create simple shapes.

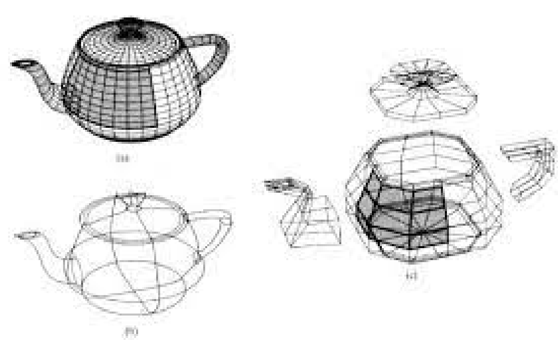


Figure 3.2: Bezier cuves in computer graphics.

3.4.2 Fonts

Because they had a significant impact on the precision and clarity of letters, bezier curves first appeared in the design world through printers.

For instance, composite Bezier curves formed of quadratic curves are used in True Type typefaces. Other linguistic and graphic design technologies, including PostScript, Asymptote, Metafont, and SVG, create curved shapes using composite Bezier curves made out of cubic curves. Depending on the technology that controls the OpenType wrapper (the encoding that instructs the system how to interpret the font), OpenType fonts can employ cubic or quadratic curves.

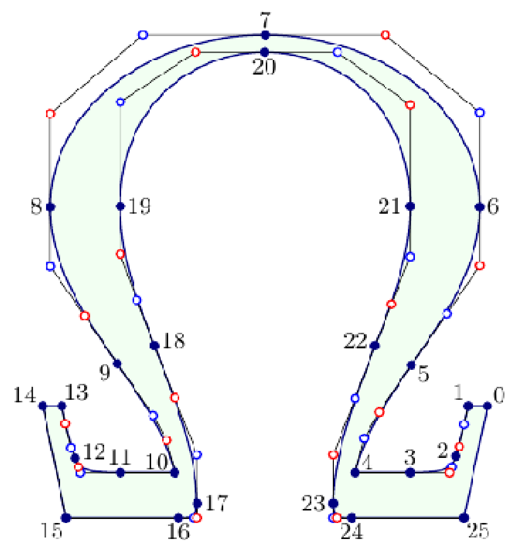


Figure 3.3: Font definition using Bezier curves.

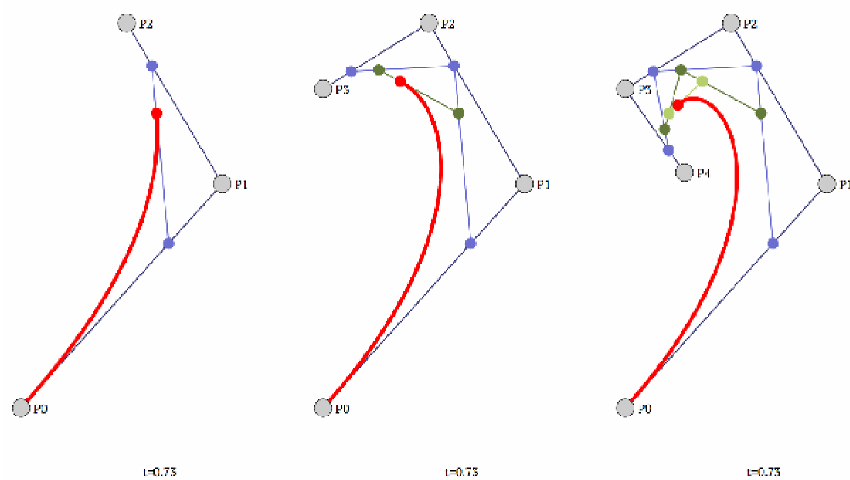
3.4.3 Animations

Bezier curves are used in programs like Synfig to outline movement. The application creates the necessary frames for the item to move along the path once the user draws the appropriate path using Bezier curves.

The "feel" or "physics" that motion designers and animators are after are created in this way. The Bezier curve indicates the movement's velocity over time in addition to where the object goes. The designer will use a Bezier curve to smooth the cursor trajectory and set the pace of travel if an icon needs to move from point A to point B.

Bezier curves are frequently used in $3D$ animation to construct $3D$ routes and 2D curves for keyframe interpolation. Bezier curves are currently extensively used in CSS, JavaScript, JavaFX, and Flutter SDK to govern animation ease.

(1)



$_1 2.png$

Figure 3.4: Animated Bezier curves.

3.4.4 Designing cars

Originally the Bezier curve was developed for geometry modeling, which, as mentioned earlier, Decasteljau on first at Citroen, and then later Pierre Bezier at Renault confirmed that the curve was originally developed to help in the design of cars.

The representation of an object by the three-dimensional space it occupies; this generation is known as the "Constructive Solid Geometry" CSG approach. the geometric description of a mechanical assembly that enables the modeling of objects in three dimensions. Using a bézier surface (Bezier tile) and CAD software, the example that follows shows how to model a vehicle body in three dimensions.

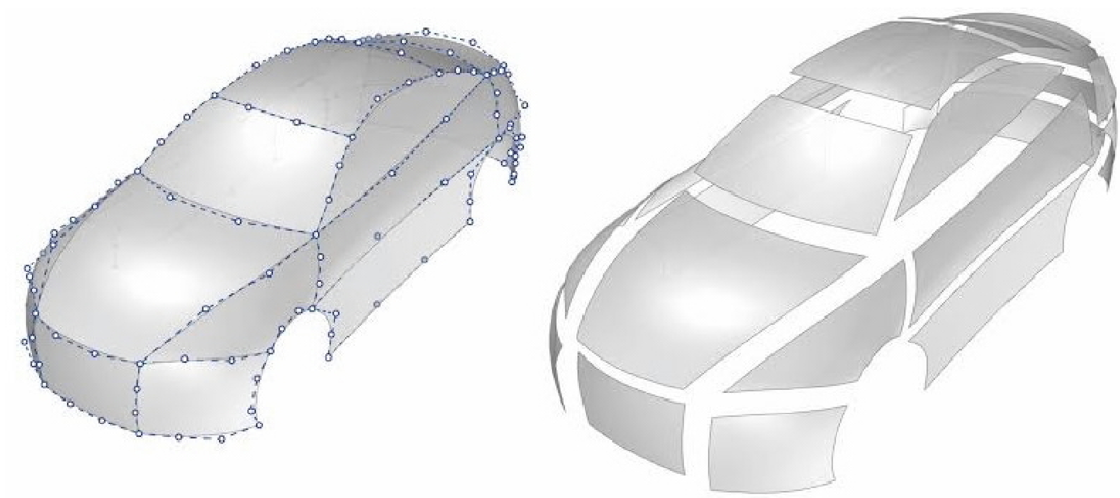


Figure 3.5: Designing body of car using Bezier curves.

Conclusion

In this thesis, we studied an important type of polynomial, namely Bernstein polynomials, where we presented their basic properties, and then we moved on to the study of Bezier curves, their properties, and geometric construction.

Finally, we presented a simple example of a method for solving ordinary differential equations(Collocation Method) using Bernstein polynomials, and we proved some applications of Bezier curves in the fields of computer design and data science.

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الملخص

في هذه المذكرة، سوف نتطرق لنظرية وتطبيقات منحنيات بيزيي وكثيرات حدود برنشتاين من خلال دراسة خصائصهما الأساسية موضحين خوارزمية دي-كاستيلجاو لمنحنيات بيزيي. سنقدم مثالاً لتوضيح استخدام كثيرات حدود برنشتاين في نظرية التقريب و نتطرق الى أمثلة عن استخدام منحنى بيزيي في العديد من المجالات

ABSTRACT

In this thesis, we will explore the theory and applications of Bézier curves and Bernstein polynomials. We will study their basic properties and explain the de-Casteljau algorithm for Bézier curves. We will offer an example to illustrate the use of Bernstein polynomials in approximation theory and how Bézier curve approaches are used in many fields.

RESUME

Dans ce mémoire, nous explorerons la théorie et les applications des courbes de Bézier et les polynômes de Bernstein. Nous étudierons leurs propriétés de base et nous expliquerons l'algorithme de de-Casteljau pour les courbes de Bézier. Nous offrirons un exemple pour illustrer l'utilisation des polynômes de Bernstein dans la théorie de l'approximation et comment les approches de courbe de Bézier sont utilisées dans de nombreux domaines.