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Record Statistics and Theory of Entropy

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Dedication

This master's thesis is dedicated to :

My seewet heart mother who always by my side ,supported me , inspired me and bilieve in me .

To my dear father who sacrificed and worked hard to see me here today .

To my second father "uncle Sadek" who I wished to be by my side to day before he left this world .

To my dear brothers : Khaled and Ahmed .

To my dear sisters : Djanna and Salsabil .

To my beloved family .

And to all my friends near and far .

To all my colleague from 2018– 2023 *promotion .*

To all those who are dear to me .

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Thank's

Abstract

In recent years, attention has been paid to the study of records statistics in terms of Rényi extropy which was used widely in reliability and information studies.

In this master's thesis, an explained representation of records statistics is expressed and some properties results are exploited in terms of continuous distributions. In addition, some characterization results are explored based on the records values, and related properties are addressed for the Rényi of record statistics (ordinary record and k-records). Simplified expressions of the Rényi entropy of records are derived which allows us to investigate monotonicity properties and characterizations of some probability distributions. Moreover. These results are based on applying the result obtained by Jitto Jose E. I. Abdul Sathar (2022) [12].

Notations and symbols

iid	Independent and identically distributed
cdf	Cumulative distribution function note $F(.)$
pdf	Probability density function also density function note $f(.)$
$h(.)$	Hazard or failure rate function
$H(.)$	Cumulative hazard function
$U(n)$	Upper record times
U_n^X	Upper record values
$U_{n,k}^X$	nth value of upper k-records
$L(n)$	Lower record times
L_n^X	Lower record values
$L_{n,k}^X$	nth value of lower k-records
$f_{U_n^X}(x)$	Probability density function of nth Upper record values
$f_{U_m^X,U_n^X}(x,y)$	The joint probability density function of mth and nth Upper records
$f_{U_n^X\|U_m^X}(y\|x)$	The conditional probability density function of nth upper given mth upper record

$f_{U_{n,k}^X}(x)$	Probability density function of nth value of upper k-records
$f_{U_{m,k}^X,U_{n,k}^X}(x,y)$	The joint probability density function of mth and nth Upper k-records
$f_{U_{n,k}^X \mid U_{m,k}^X}(y \mid x)$	The conditional probability density function of nth upper given mth upper k record
X_i	Random variables with i=1,2,...
$f_{L_n^X}(x)$	Probability density function of nth lower record values
$f_{L_m^X,L_n^X}(x,y)$	The joint probability density function of mth and nth lower records
$f_{L_m^X,L_n^X}(x,y)$	The joint probability density function of mth and nth lower records
$f_{L_n^X \mid L_m^X}(y \mid x)$	The conditional probability density function of nth lower given mth lower record
et al	and others
$f_{L_{n,k}^X}(x)$	Probability density function of nth value of lower k-records
$f_{L_{m,k}^X,L_{n,k}^X}(x,y)$	The joint probability density function of mth and nth lower k-records
$f_{L_{n,k}^X \mid L_{m,k}^X}(y \mid x)$	The conditional probability density function of nth lower given mth lower k-record
$H_\alpha(X)$	Reiny entropy of X
$X \leq_{lr} Y$	X is smaller than Y in the likelihood ratio ordre
$X \leq_{st} Y$	X is smaller than Y in the usual stochastic order

Contents

List of Figures

List of Tables

Introduction

The interest of Reliability theory has garnered significant importance in various fields due to its practical applications and benefits specially in statistics that deals with the analysis and interpretation of data related to reliability and information.

The concept of Shannon entropy, introduced by Shannon in 1948 [21], has been extensively employed in the fields of reliability theory and information studies. From an engineering standpoint, the Shannon entropy or Rènyi information may be more suited to employ as measures to quantify the information content of an a sample than the Fisher information, which is important in statistical inference and information theoretic studies.

The concept of entropy is essential in record statistics as it provides insights into the randomness and predictability of data sets. Entropy measurement allows for the assessment of the level of uncertainty and randomness present in a given data set. Higher entropy indicates a greater degree of randomness and unpredictability, while lower entropy suggests a higher level of predictability within the data set.

In data compression, the objective is to minimize data redundancy and encode it in the most efficient manner possible. By analyzing the entropy of a data set, it becomes possible to determine which compression method is most suitable for achieving optimal results. Understanding the entropy of the data helps in selecting the appropriate compression technique that can effectively reduce the data size while preserving its essential information.

In this master's thesis, the Rènyi entropy is addressed in the context of record statistics, which has witnessed a continuous rise since Chandler introduced the theory of record statistics in 1952 [6]. It is noteworthy that certain scenarios, such as those encountered in metrology, hydrology, seismology, and mining, involve the observation of only record values. Consequently, numerous researchers have dedicated their attention to the study of record values and associated statistical properties.

Many scientists have studied the subject of entropy properties of record statistics like Madai and Tata [14, 15] present results on the shannon information contained in classical record values and in k-record values ,Abbasnejad and Arghami [1] have discussed about the information contained in classical record values in detail and have derived some important properties as well and Ahmadi and Fashandi [9] showed that the equality of the entropy of the endpoints of record converage is characteristic property for symmetric distribution.

The goal from this work is to highlight the most important characteristics of Rényi entropy of k-records arising from any continuous distribution .

To achieve our objective we divided this master's thesis in tow chapter . In the first one, we present the classical record and the k-record from any continuous distribution and introduce the terme of upper and lower records. Also, we extract the record form specific continuous distribution . In the second chapter we present the Rényi entropy of k-record which is a measure of the degree of predictability or regularity in the data. And apply this important measure in the sense of record statistics. We try to explain and investigate specific characterizations and distributional properties. This study is based on analyzing and explaining the work of Jitto Jose , El Adbul sathar [12], which have studied the general concept of Rènyi entropy in the concept of records 2022.

We are hoping that the readers of this master's thesis find all that is useful about record statistics and the theory of entropy.

Chapter 1

Record Statistics

In this chapter, we introduce the record statistics for continuous random variables. It constitutes of four sections, section 1.1 is introductive about the record times and values, and section 1.2 we present some characterization of record statistics from a sequence of continuous random variables of the ordinary (classical) record statistics which is a special case of k-record when ($k = 1$), the section 1.3 we also present the characterization of k-record statistics, finally, in section 1.4 we derive the density of lower and upper record form specific continuous distribution.

1.1 Introduction

In 1952, Chandler [6] introduced the concept of record times and values and gave groundwork for a mathematical theory of records. For six decades beginning his pioneering work, about 500 papers and some monographs focusing on different aspects of the theory of records appeared. Let X_1,X_2,... be an infinite sequence of independent and identically distributed *iid* random variables having the same distribution as the (population) random variable X, we called about X_j an upper record value (or simply a record) if its value exceeds that of all previous observations. Thus X_j is a record if $X_j > X_i$ for every $i < j$. A similar definition pertains to the analysis of lower record values.

In statistics, a record value or record statistic is the largest or smallest value obtained from a sequence of random variables. The theory is closely related to that used in order statistics. Chandler defined a record as successive extremes occurring in a sequence of independent and identically distributed (*iid*) random variables. Record keeping is very important for some real-world problems related to weather and the economy, studies, sports, etc, and record are observed such as in metrology, hydrology, and seismology,...Prediction of the next record value is an interesting problem in many real-life situations, For example, prediction of the lowest share value in stock markets is essential to plan for investment strategies.

Records play an essential role in situations where observations are challenging to acquire or when observations are prone to destruction during experimental testing. Illustrating the statistical definition of the theory of records is best accomplished through examples.

Let's consider a scenario where we are conducting a sequence of product tests that are likely to fail under stress. Our objective is to identify the minimum failure stress of the products in a sequential manner. To illustrate this, we can denote the strength at which a product fails as X_1. We stress the first product and record its failure stress.

As we proceed with testing subsequent products, we record the failure stress X_m of the mth product only if it is lower than the minimum failure stress observed so far, which is represented by $min(X_1, X_2, ..., X_{m-1})$ for $m > 1$. In other words, we compare the failure stress of each product to the minimum failure stress observed among the previous products. If the current product's failure stress is lower, we consider it as a record and record the value.

These recorded failure stresses represent the lower record values because they are the minimum failure stresses observed at each step of the testing process. By capturing the lower record values, we can gain insights into the minimum stress threshold at which the products fail, helping us understand their strength characteristics and make informed decisions in product testing and development.

1.2 Ordinary Record Statistics

Let $X_1,\ X_2,\ .\ .\ .$ be a sequence of independent and identically distributed (*iid*) random variables and $X_{1,n} \leq ... \leq X_{n,n}$ be the corresponding order statistics where $X_{1,n} = \min\{X_1, X_2, ..., X_n\}$ and $X_{n,n} = \max\{X_1, X_2, ..., X_n\}$.

Definition 1.2.1 *the classical upper record times $U(n)$ and upper record values U_n^X as follows:*

$$U(1) = 1, U(n+1) = \min\{j : j \geq U(n), X_j \geq U_n^X\}, U_n^X = X_{U(n)}, n = 1, 2, ...$$

Definition 1.2.2 *the sequences of lower record times $L(n)$ $(n \geq 1)$ and lower record values L_n^X as follows :*

$$L(1) = 1, L(n+1) = \min\{j : j \geq L(n), X_j \leq L_n^X\}, L_n^X = X_{L(n)}, n = 1, 2, ...$$

1.2.1 Record Probability Density Function

In this subsection, we first provide an expression of the density function of U_n^X and L_n^X. In addition, we present the joint *pdf* of upper and lower records. Ending it by defining the conditional *pdf* of the upper and the lower record.

Lemma 1.2.1 *The hazard (function also known as the failure rate, hazard rate, or force of function)* $h(x)$ *is the ratio of the probability density function* $f(x)$ *to the survival function* $1 - F(x)$*, given by:*

$$h(x) = \frac{f(x)}{1 - F(x)}.$$

The cumulative hazard function

$$H(x) = -\log[1 - F(x)]. \tag{1.1}$$

Theorem 1.2.1 *The pdf of* U_n^X *is obtained to be* (*Arnold et al.* (1998))

$$f_{U_n^X}(x) = \frac{[H(x)]^{n-1}}{(n-1)!} f(x), \quad -\infty < x < \infty, n = 1, 2, ... \tag{1.2}$$

Proof. To prove the theorem, we are going to consider exponential observations since this distribution has the lack of memory property then the differences between successive records will be *iid* standard exponential random variable. Let $\{X_j^*, j > 1\}$ be a sequence of *iid* $Exp(1)$ random variables, and consequently, it follows that the nth upper record U_n^* has a gamma distribution with shape $n+1$ and rate 1.

$$U_n^* \sim Gamma(n+1, 1), n = 1, 2, ... \tag{1.3}$$

These results will be useful to obtain the distribution of the nth record corresponding to an *iid* sequence of random variables $\{X_j\}$ with common continuous *cdf* F.

If X has a continuous *cdf* F, then the cumulative hazard function has a standard exponential distribution. we have

$$H(X) \equiv -\log[1 - F(X)].$$

then

$$X \stackrel{d}{=} F^{-1}(1 - e^{-X^*})$$

$\{X_j^*\}$ follows standard exponential, therefore, we have:

$$U_n \stackrel{d}{=} F^{-1}(1 - e^{-U_n^*}),\ n = 1, 2... \tag{1.4}$$

Repeated integration by parts can be used to justify the following expression for the survival function of U_n^*(a $Gamma(n+1, 1)$ random variable):

$$P(U_n^* > x) = e^{x^*} \sum_{k=0}^{n} \frac{(x^*)^k}{(k-1)!}, x^* > 0.$$

We may then use the relation 1.4 to immediately derive the survival function of the nth record corresponding to an *iid* F sequence.

$$P(U_n > x) = 1 - F_{U_n^X}(x) = [1 - F(x)] \sum_{k=0}^{n} \frac{[-\log(1 - F(x))]^k}{(k-1)!}.$$

which is equivalent to:

$$P(U_n < x) = \int_0^{-\log(1 - F(x))} y^n e^{-y}/(n-1)!\, dy$$

If F is absolutely continuous with corresponding probability density function *pdf*, we may differentiate either of the above expressions to obtain the *pdf* for U_n^X. In fact, we have:

$$f_{U_n^X}(x) = \frac{[H(x)]^n}{(n-1)!} f(x).$$

(for more details see Arnold et al page (31) [3]) ■

Theorem 1.2.2 *The joint pdf of* U_m^X *and* U_n^X *,where* $1 \leq m \leq n$*, is given by* (*Arnold et al.* (1998))

$$f_{U_m^X, U_n^X}(x,y) = \frac{[H(x)]^{m-1}}{(m-1)!} \frac{[H(y) - H(x)]^{n-m-1}}{(n-m-1)!} h(x) f(y), \quad -\infty < x < y < \infty. \tag{1.5}$$

Proof. to start the prove we know that the joint *pdf* of a set of $Exp(1)$ records $(U_0^*, U_1^*, ..., U_n^*)$ is easily written down, and since the record spacings $U_n^* - U_{n-1}^*$ are *iid* $Exp(1)$. Thus, we find:

$$f_{U_0^*, U_1^*, ..., U_n^*}(x_0^*, x_1^*, ..., x_n^*) = e^{x^*}, 0 < x_0^* < x_1^* < ... < x_n^*.$$

Now,we apply the transformation eq.(1.4) coordinatewise to obtain the joint *pdf* of the set of records $U_0, U_1, ..., U_n$ corresponding to an *iid* F sequence which is given by

$$f_{U_0, U_1, ..., U_n}(x_0, x_1, ..., x_n) = \frac{\prod_{i=0}^{n} f(x_i)}{\prod_{i=1}^{n-1} [1 - F(x_i)]}, \tag{1.6}$$

$$= f(x_n) \prod_{i=1}^{n-1} h(x_i), -\infty < x_0 < ... < x_n.$$

The joint *pdf* of any pair of records U_m, U_n can of course be obtained from eq.(1.6) by integration. It is perhaps easier to first find the joint distribution for two $Exp(1)$ records (U_m^*, U_n^*) and then use the transformation eq.(1.4) where $m < n$, since $(U_m^*, U_n^*) = (Y_1, Y_1 + Y_2)$ where $Y_1 \backsim Gamma(m+1,1)$ and $Y_2 \backsim Gamma(n-m,1)$ are independent, we may use a simple Jacobian argument beginning with the joint pdf of (Y_1, Y_2) to obtain:

$$f_{U_m^*,U_n^*}\left(x_m^*, x_n^*\right) = \frac{1}{m!\,(n-m-1)!} x_m^{*m} \left(x_n^* - x_m^*\right)^{n-m-1} e^{-x_n^*}, 0 < x_m^* < x_n^* < \infty.$$

With eq.(1.4) applied U_m^* to U_n^*, we obtain

$$f_{U_m,U_n}\left(x_m, x_n\right) = \frac{\left[-\log\left(1 - F\left(x_m\right)\right)\right]^m}{m!} \frac{\left[-\log\left(\frac{1-F(xn)}{1-F(x_m)}\right)\right]}{(n-m-1)!} \frac{f\left(x_m\right) h\left(x_n\right)}{1 - F\left(x_m\right)},$$

$$-\infty < x_m < x_n < \infty.$$

■

Corollary 1.2.1 *The conditional pdf* U_n^X *given* U_m^X $(1 < n < m)$ *using eq.(1.2) and eq.(1.5) is as follows:*

$$f_{U_n^X|U_m^X}\left(y|x\right) = \frac{f_{U_m^X,U_n^X}\left(x,y\right)}{f_{U_m^X}\left(x\right)} = \frac{\left[H\left(y\right) - H\left(x\right)\right]^{n-m-1}}{(n-m-1)!} \frac{f\left(y\right)}{1 - F\left(x\right)}, \quad x < y. \quad (1.7)$$

Remark 1.2.1 *The survival function of the nth upper record can be represented as:*

$$1 - F_{U_n^X}\left(x\right) = \left[1 - F\left(x\right)\right] \sum_{j=0}^{n-1} \frac{\left[-\log(1 - F\left(x\right))\right]^j}{j!} \quad (1.8)$$

Where

$$F_{U_n^X}\left(x\right) = \int_0^{-\log(1-F(x))} \frac{u^{n-1}}{(n-1)!} e^{-u} du, n = 1, 2, ...$$

The following corollary presents the pdf of lower records and the joint and conditional pdf of L_m^X and L_n^X.

Corollary 1.2.2 *We have:*

- *the pdf of* L_n^X *is given by :*

$$f_{L_n^X}(x) = \frac{[-\log F(x)]^{n-1}}{(n-1)!} f(x), -\infty < x < \infty, n = 1, 2, ... \tag{1.9}$$

- *The joint pdf of* L_m^X *and* L_n^X*,where* $1 \leq m \leq n$*, can be written as:*

$$\begin{aligned} f_{L_m^X, L_n^X}(x, y) &= \frac{[-\log F(x)]^{m-1}}{(m-1)!} \frac{[\log F(x) - \log F(y)]^{n-m-1}}{(n-m-1)!} \\ & \frac{f(x)}{F(x)} f(y), -\infty < x < y < \infty. \end{aligned} \tag{1.10}$$

- *The conditional pdf of* L_n^X *given* L_m^X *where* $1 < m < n$*,can be written as:*

$$f_{L_n^X|L_m^X}(y|x) = \frac{[\log F(x) - \log F(y)]^{n-m-1}}{(n-m-1)!} \frac{f(y)}{F(x)}, x < y. \tag{1.11}$$

Example 1.2.1 *Let's consider the first 10 observations* $n = 10$ *from data given by David and Nagaraja[[7]] where the data is derived from a normal distribution for 20 observations. We have:*

$$0.464,\ 0.060,\ 1.486,\ 1.022,\ 1.394,\ 0.906,\ 1.179,\ -1.501,\ -0.690,\ 1.372.$$

Using the definition of upper record 1.2.1, we have:

- *For* $n = 1$*, the first upper record times is* $U(1) = 1$ *where the corresponding first upper record value should be* $X_{U(1)} = U_1^X = X_1 = 0.464$.

- *For $n = 2$, we need to find the upper record time and then we obtain the upper record value. Therefore, we search for j that satisfies the following*

$$U(2) = \min\left\{j : j \geq U(1), X_j \geq U_1^X\right\}.$$

For $j = 2$ we have:

$X_2 = 0.060 < 0.464.$, *so, X_2 does not give the required results. For $j = 3$ we have:*

$X_3 = 1.486 > 0.464$ *therefore :the corresponding second upper record value should be* $U(2) = 3, X_{U(2)} = U_2^X = X_3 = 1.486.$

- *On the other hand, if we continue to search the upper record values for all the reminder n we will find that $X_{U(i)} = U_i^X = X_3 = 1.486,\ \forall i = 3, 4, ..., 10$. Because X_3 is the biggest value in $X_j, 1 \leq j \leq 10$,we finished at $n = 2$ because it reached the maximum value of the data.*

Therefore, the upper record values of this data are $0.464,\ 1.486.$

By the same procedure, the lower record values are: $0.464,\ 0.060,\ -1.501$

1.3 K-Record Statistics

Definition 1.3.1 *Let $\{X_n, n \geq 1\}$be a sequence of iid random variables with cdf $F(X)$ and pdf $f(x)$. Let $X_{1,n} \leq X_{2,n} \leq ... \leq X_{n,n}$, be the order statistics of $X_1, ..., X_n$. Let's define*

$$U_{1,k} = k$$

and

$$U_{n,k} = \min\left\{j : j > U_{n-1,k}, X_j > X_{U_{n-1,K}-k+1:U_{n-1,k}}\right\},$$

for fixed integer $k \geq 1$ and $n \geq 2$. where, $U_{n,k}^{X} = X_{U_{n,K}-k+1:U_{n,k}}$ is the sequences of upper k-record values and $U_{n,k}$ $(n \geq 1)$ is sequences of upper k-record times.

Remark 1.3.1 *An analogous definition can be given respectively for lower k-record values and lower k-record times.*

$$L_{1,k} = k,$$

and

$$L_{n,k} = \min\left\{j : j > L_{n-1,k}, X_j < X_{L_{n-1,K}-k+1:L_{n-1,k}}\right\}$$

and $L_{n,k}^{X} = X_{L_{n,K}-k+1:L_{n,k}}$ is the sequences of lower k-record values and $L_{n,k}$ $(n \geq 1)$ is sequences of lower k-record times.

1.3.1 K-Record Probability Density Function

The model of k-record values is proposed at first by (Dziubdziela and Kopocinski 1976 [[8]]) .

Theorem 1.3.1 *The pdf of $U_{n,k}^{X}$ is obtained to be:*

$$f_{U_{n,k}^{X}}(x) = k^n \frac{[H(x)]^{n-1}}{(n-1)!} [1 - F(x)]^{k-1} f(x), \quad -\infty < x < \infty. \tag{1.12}$$

Proof. By induction ,the densities $f_{U_n^X}(x_1, ...x_k)$,$n = 1, 2, ...$, satisfy the equations

$$f_{U_n^X}(x_1, ...x_k) = \begin{cases} k! g_{U_n^X}(x_1) f(x_1) f(x_2) ... f(x_k), x_1 < x_2 < ... < x_k \\ 0, \text{otherwise} \end{cases}$$

where

$$g_{U_1^X}(x) = 1$$

$$g_{U_{n+1}^X}(x) = k \int_{-\infty}^{x} g_{U_n^X}(y) \frac{f(y)}{1-F(y)} dy, n = 1, 2, ...$$

It is easy to verify that

$$g_{U_n^X}(x) = \frac{1}{(n-1)!} \left[-k \log\left(1 - F(x)\right)\right]^{n-1}, n = 1, 2, ...$$

Then, we have

$$\begin{aligned} f_{U_{n,k}^X}(x) &= \int_x^\infty \int_{x_2}^\infty ... \int_{x_{k-1}}^\infty \frac{k!}{(n-1)!} \left[-k \log\left(1 - F(x)\right)\right]^{n-1} f(x) f(x_2) ... f(x_k) dx_k ... dx_2 \\ &= \frac{k}{(n-1)!} \left[-k \log\left(1 - F(x)\right)\right]^{n-1} \left[1 - F(x)\right]^{k-1} f(x). \end{aligned}$$

■

Remark 1.3.2 *The probability density of the record value is of the form*

$$f_{U_{n,1}^X}(x) = f_{U_n^X}(x) = \frac{1}{(n-1)!} \left[-\log\left(1 - F(x)\right)\right]^{n-1} f(x).$$

(see Karlin ([13])).

Corollary 1.3.1 *The cumulative distribution function of the k-record value is of the form*

$$F_{U_{n,k}^X}(x) = \int_0^{-k \log(1-F(x))} \frac{u^{n-1}}{(n-1)!} e^{-u} du, n = 1, 2, ...$$

Corollary 1.3.2 *Let X be a continuous random variable. we have*

- *The joint pdf of $U_{m,k}^X$ and $U_{n,k}^X$,where $1 \leq m \leq n$, is given by (see,Grudzien*

1982([11]))

$$f_{U_{m,k}^X, U_{n,k}^X}(x,y) = \frac{k^n}{(m-1)!\,(n-m-1)!}[H(x)]^{m-1}[H(y)-H(x)]^{n-m-1}$$
$$[1-F(y)]^{k-1}\, h(x)\, f(y)\,, -\infty < x < y < \infty \tag{1.13}$$

- *The conditional pdf of* $U_{n,k}^X$ *given* $U_{m,k}^X$ $(1 \leq m \leq n)$ *can be written as:*

$$f_{U_{m,k}^X | U_{n,k}^X}(y|x) = \frac{k^{n-m}}{(n-m-1)!}[H(y)-H(x)]^{n-m-1}[1-F(y)]^{k-1}\frac{f(y)}{[1-F(x)]^k},$$
$$x < y. \tag{1.14}$$

Proposition 1.3.1 *An analogous pdf 's can be given respectively for lower k-record*

- *The pdf of* $L_{n,k}^X$ *is given by:*

$$f_{L_{n,k}^X}(x) = k^n \frac{[-\log F(x)]^{n-1}}{(n-1)!}[F(x)]^{k-1} f(x)\,, \quad -\infty < x < \infty. \tag{1.15}$$

- *The joint pdf of* $L_{m,k}^X$ *and* $L_{n,k}^X$ *,where* $1 \leq m \leq n$*, can be written as* (*see,Pawlas and Szynal* (1998, 2000)) *[16] and [18].*

$$f_{L_{m,k}^X, L_{n,k}^X}(x,y) = \frac{[-\log F(x)]^{m-1}}{(m-1)!}\frac{[\log F(x) - \log F(y)]^{n-m-1}}{(n-m-1)!}[F(y)]^{k-1}$$
$$\frac{f(x)}{F(x)} f(y)\,, -\infty < x < y < \infty. \tag{1.16}$$

- *The conditional pdf of* $L_{n,k}^X$ *given* $L_{m,k}^X$ *,where* $1 \leq m \leq n$*, can be written* *as:*

$$f_{L^X_{n,k}|L^X_{m,k}}(y|x) = \frac{k^{n-m}}{(n-m-1)!}\left[\log F(x) - \log F(y)\right]^{n-m-1}\left[F(y)\right]^{k-1}$$
$$\frac{f(y)}{\left[F(x)\right]^k}, x < y. \tag{1.17}$$

Remark 1.3.3 *We obtain ordinary upper/lower record values in the case of* $k = 1$.

Example 1.3.1 *For the same data in Example 1.2.1. Let* $X_{1,10} \leq ... \leq X_{10,10}$ *be the order statistics of* $X_1, ..., X_{10}$ *as follows :*

$$-1.501 \leq -0.690 \leq 0.060 \leq 0.464 \leq 0.906 \leq 1.022 \leq 1.179 \leq 1.372 \leq 1.394 \leq 1.486.$$

we construct upper k-records from the data where $k = 2,\ 3,\ 4$ *as given below: Let* $k = 2$ *and* $n = 1, 2, ..., 10$*. Using definition 1.3.1, we have :*

- *For* $n = 1$:

$$U_{1,2} = 2, U^X_{1,2} = X_{U_{1,2}-2+1:U_{1,2}} = X_{1,2} = 0.060.$$

- *For* $n = 2$:

$$U_{2,2} = \min\left\{j : j > U_{2-1,2} \text{ and } X_j > X_{U_{2-1,2}-2+1:U_{2-1,2}}\right\}$$
$$= \min\left\{j : j > 2 \text{ and } X_j > X_{1,2}\right\}. \tag{1.18}$$

Now, we search the value of j *that satisfies eq.(1.18). Thus, we find for*

$j = 3: \ U_{2,2} = \min\{j : 3 > 2 \ and \ X_3 > 0.060\}$. *Therefore,*

$$U_{2,2} = 3, U_{2,2}^X = X_{U_{2,2}-2+1:U_{2,2}} = X_{2,3} = 0.464.$$

- *For* $n = 3$:

$$U_{3,2} = \min\left\{j : j > U_{3-1,2} \ and \ X_j > X_{U_{3-1,2}-2+1:U_{3-1,2}}\right\}$$
$$= \min\{j : j > 3 \ and \ X_j > X_{2,3}\}.$$

Thus, $j = 4$ *satisfies the condition, and therefore we have:*

$$U_{3,2} = 4, U_{3,2}^X = X_{U_{3,2}-2+1:U_{3,2}} = X_{3,4} = 1.022.$$

We continue in the same way to get the sequence of the upper 2-record which is given as follows:

$$0.60, \ 0.464, \ 1.022, \ 1.394$$

As a consequence, the following table presents the upper k-record for $k = 2, 3, 4$.

2-records	0.60	0.464	1.022	1.394	
3-records	0.60	0.464	1.022	1.179	1.372
4-records	0.60	0.464	0.906	1.022	1.179

Table 1.1: the upper k-record for $k = 2, 3, 4$.

1.4 Record From Specific Continuous Distribution

In this section, we derive some record values from a specific continuous distribution like the uniform, exponential, Weibull, and Pareto.

1.4.1 Uniform Distribution

Let X be a Uniformly distributed random variable on interval $[0, \theta]$ .with the following *cdf* and *pdf* :

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{x}{\theta} & 0 < x < \theta \\ 1 & x > \theta \end{cases} \quad ; \quad f(x) = \begin{cases} 0 & x < 0 \\ \frac{1}{\theta} & 0 < x < \theta \end{cases}$$

$\forall k > 1$ and $\forall n > 2$,we are going to use eq (1.12) to define the *pdf* of upper k record valeus:

$$f_{U_{n,k}^X}(x) = k^n \frac{\left[-\log\left(1 - \frac{x}{\theta}\right)\right]^{n-1}}{(n-1)!} \left[1 - \frac{x}{\theta}\right]^{k-1} \frac{1}{\theta}$$

$$f_{U_{n,k}^X}(x) = \frac{k^n}{\theta} \frac{\left[-\log(\theta - x) + \log(\theta)\right]^{n-1}}{(n-1)!} \left[\frac{\theta - x}{\theta}\right]^{k-1}$$

if $k = 1$ we obtain ordinary upper record values:

$$f_{U_n^X}(x) = \frac{1}{\theta} \frac{\left[-\log(\theta - x) + \log(\theta)\right]^{n-1}}{(n-1)!}$$

We use eq (1.15) to define the *pdf* of lower k record values :

$$f_{L_{n,k}^{X}}(x) = k^{n}\frac{\left[-\log\left(\frac{x}{\theta}\right)\right]^{n-1}}{(n-1)!}\left[\frac{x}{\theta}\right]^{k-1}\frac{1}{\theta}$$

$$f_{L_{n,k}^{X}}(x) = \frac{k^{n}}{\theta^{k}}\frac{\left[-\log\left(\frac{x}{\theta}\right)\right]^{n-1}}{(n-1)!}(x)^{k-1}$$

if $k = 1$ we obtain ordinary lower record values:

$$f_{L_{n}^{X}}(x) = \frac{1}{\theta}\frac{\left[-\log\left(\frac{x}{\theta}\right)\right]^{n-1}}{(n-1)!}$$

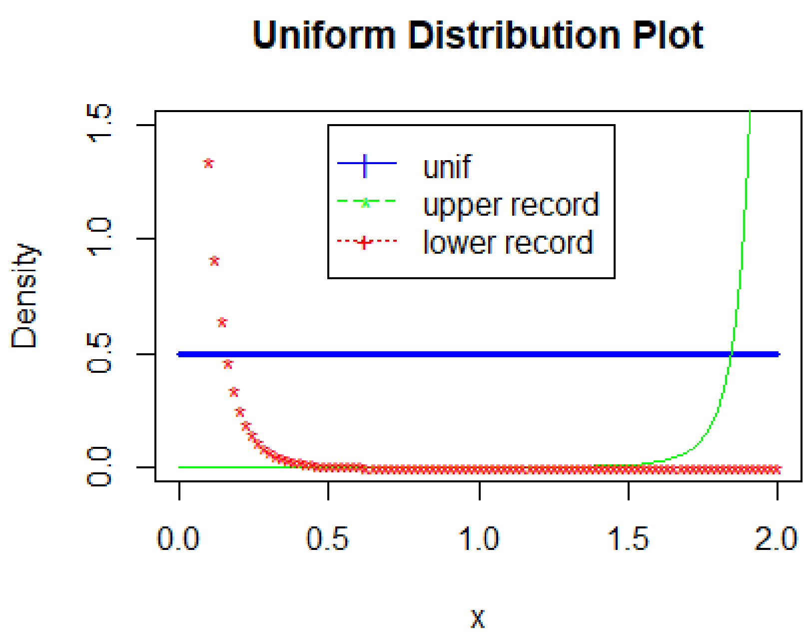

Figure 1.1: Pdf of plots of uniform(0,2) distribution and upper and lower 3-record for n=10

1.4.2 Exponenntial Distribution

Let X be an Exponentially distributed random variable, with the following *cdf* and *pdf* :

$$F(x) = 1 - \exp(-\lambda x), x > 0$$
$$f(x) = \lambda \exp(-\lambda x)$$

The *pdf* of upper k record values:

$$\begin{aligned}
f_{U_{n,k}^X}(x) &= k^n \lambda \exp(-\lambda x) \frac{[-\log[1-(1-\exp(-\lambda x))]]^{n-1}}{(n-1)!} (1-(1-\exp(-\lambda x)))^{k-1} \\
&= k^n \lambda \exp(-\lambda x) \frac{[-\log(\exp(-\lambda x))]^{n-1}}{(n-1)!} \exp(-\lambda x (k-1)) \\
f_{U_{n,k}^X}(x) &= k^n \frac{(\lambda x)^{n-1}}{(n-1)!} \lambda \exp(-\lambda k x)
\end{aligned}$$

if $k = 1$ we obtain ordinary upper record values:

$$f_{U_n^X}(x) = \frac{(\lambda x)^{n-1}}{(n-1)!} \lambda \exp(-\lambda x)$$

The *pdf* of lower k record values:

$$f_{L_{n,k}^X}(x) = \lambda k^n \frac{[-\log(1-\exp(-\lambda x))]^{n-1}}{(n-1)!} (1-\exp(-\lambda x))^{k-1} \exp(-\lambda x)$$

if $k = 1$ we obtain ordinary lower record values:

$$f_{L_n^X}(x) = \lambda \exp(-\lambda x) \frac{[-\log(1-\exp(-\lambda x))]^{n-1}}{(n-1)!}$$

1.4.3 Weibull Distribution

Let X be a Weibull distributed random variable, with the following *cdf* and *pdf*:

$$F(x) = 1 - \exp -\lambda x^{\alpha}$$

$$f(x) = \lambda\alpha \exp -\lambda x^{\alpha-1}$$

By eq (1.12) the *pdf* of upper k record values :

$$f_{U_{n,k}^{X}}(x) = k^{n}\lambda\alpha \exp -\lambda x^{\alpha-1}\frac{\left[-\log\left(1-\left(1-\exp\left(-\lambda x\right)\right)\right)\right]^{n-1}}{(n-1)!}\exp -\lambda\left(k-1\right)x^{\alpha}$$

$$f_{U_{n,k}^{X}}(x) = k^{n}\lambda\alpha \exp -\lambda x^{\alpha-1}\frac{\left[\lambda x^{\alpha}\right]}{(n-1)!}\exp -\lambda\left(k-1\right)x^{\alpha}$$

if $k = 1$ we obtain ordinary upper record values:

$$f_{U_{n}^{X}}(x) = \lambda\alpha \exp -\lambda x^{\alpha-1}\frac{\left[\lambda x^{\alpha}\right]}{(n-1)!}$$

By eq (1.15) the *pdf* of lower k record values:

$$f_{L_{n,k}^{X}}(x) = \lambda k^{n}\alpha \exp -\lambda x^{\alpha-1}\frac{\left[-\log\left(1-\exp -\lambda x^{\alpha}\right)\right]^{n-1}}{(n-1)!}\left[1-\exp -\lambda x^{\alpha}\right]^{k-1}$$

if $k = 1$ we obtain ordinary lower record values:

$$f_{L_{n}^{X}}(x) = \lambda\alpha \exp -\lambda x^{\alpha-1}\frac{\left[-\log\left(1-\exp -\lambda x^{\alpha}\right)\right]^{n-1}}{(n-1)!}$$

1.4.4 Pareto Distribution

Let X be a Pareto distributed random variable, with the following *cdf* and *pdf* :

$$F(x) = 1 - \left(\frac{\lambda}{x}\right)^{\alpha}, x > 1, \alpha > 0.$$
$$f(x) = \alpha \lambda^{\alpha} x^{-\alpha-1}.$$

By eq (1.12) the *pdf* of upper k record values :

$$\begin{aligned} f_{U_{n,k}^{X}}(x) &= k^{n} \frac{\left[-\log\left[1-\left(1-\left(\frac{\lambda}{x}\right)^{\alpha}\right)\right]\right]^{n-1}}{(n-1)!} \left[1-\left(1-\left(\frac{\lambda}{x}\right)^{\alpha}\right)\right]^{k-1} \alpha\lambda^{\alpha} x^{-\alpha-1}. \\ &= k^{n} \alpha \lambda^{\alpha} \frac{\left[-\alpha \log\left(\frac{\lambda}{x}\right)\right]^{n-1}}{(n-1)!} \left(\frac{\lambda}{x}\right)^{\alpha(k-1)} x^{-\alpha-1}. \end{aligned}$$

if $k = 1$ we obtain ordinary upper record values:

$$f_{U_{n}^{X}}(x) = \alpha\lambda^{\alpha} x^{-\alpha-1} \frac{\left[-\alpha \log\left(\frac{\lambda}{x}\right)\right]^{n-1}}{(n-1)!}.$$

By eq (1.15) the *pdf* of lower k record values:

$$f_{L_{n,k}^{X}}(x) = k^{n} \frac{\left[-\log\left(1-\left(\frac{\lambda}{x}\right)^{\alpha}\right)\right]^{n-1}}{(n-1)!} \left[1-\left(\frac{\lambda}{x}\right)^{\alpha}\right]^{k-1} \alpha\lambda^{\alpha} x^{-\alpha-1}.$$

if $k = 1$ we obtain ordinary lower record values:

$$f_{L_{n}^{X}}(x) = \frac{\left[-\log\left(1-\left(\frac{\lambda}{x}\right)^{\alpha}\right)\right]^{n-1}}{(n-1)!} \alpha\lambda^{\alpha} x^{-\alpha-1}.$$

Chapter 2

Entropy of K- Record Statistics

In this chapter, we start it by giving some background about entropy in section 2.1 and we present the Rényi entropy in the sense of k-records in section 2.2, 2.3. In section 2.4, 2.5, and 2.6, we discuss some basic results and characterisations using Rényi entropy (monotone properties and ordering).

2.1 Background

When exploring the field of reliability theory, one is inevitably confronted with inquiries about its origins and importance. The credit for introducing the concept of entropy in this context goes to Claud Shannon, often regarded as the father of the mathematical theory of communication. Shannon's groundbreaking contribution can be traced back to his influential paper titled "A Mathematical Theory of Communication," which was published in the Bell System Technical Journal in 1948. This paper serves as the seminal work that initially introduced the notion of entropy and its application within the realm of reliability theory.

Entropy plays a crucial role in statistical inference as it provides a measure of the uncertainty or information content within a probability distribution. In statistical inference, the goal is to make inferences or draw conclusions about population parameters based on observed data samples. Entropy provides a quantitative measure of the information contained in the data, which aids in making informed decisions and drawing reliable conclusions.

One key application of entropy in statistical inference is in estimating and comparing probability distributions. Entropy measures, such as Shannon entropy or Rényi entropy, can be used to assess the diversity or complexity of distribution. By comparing the entropies of different distributions, statisticians can evaluate which distribution provides a better fit to the observed data.

Entropy also plays a role in model selection and hypothesis testing. When comparing different statistical models, entropy can help determine which model best captures the underlying structure of the data. Models with higher entropy values indicate a greater complexity or information content, suggesting a better fit to the data. Entropy-based criteria, such as Akaike Information Criterion (AIC) or Bayesian Information Criterion (BIC), are commonly used for model selection.

Moreover, entropy is employed in information theory to quantify the amount of information gained or lost in statistical inference. Measures like mutual information, which is derived from entropy, are used to assess the dependence or correlation between variables. Mutual information helps identify relevant features or variables that contribute the most to the inference process. In summary, entropy is a powerful tool in statistical inference that allows for quantifying uncertainty, comparing distributions, selecting models, and measuring information gain. It helps statisticians make reliable inferences and draw meaningful conclusions based on observed data. Let's now define the entropy for a continuous random variable

Definition 2.1.1 *The Shannon entropy measure of uncertainty is inversely related to the occurrence random probability of the event.For a non-negative and absolutely continuous random variable* X *with cumulative distribution function (cdf)* $F(X)$ *and density function (pdf)* $f(x)$*, the Shannon entropy of* X *is defined as*

$$\begin{aligned} H(x) &= E\left[-\ln f(x)\right] \\ &= -\int_{0}^{\infty} f(x)\ln f(x)\,dx. \end{aligned}$$

Example 2.1.1 *Let* X *be a uniformly distributed random variable on interval* $[2,4]$ *.then*

$$f_X(x) = \begin{cases} \frac{1}{2}, x \in [2\ 4]\,. \\ 0, \textit{otherwise.} \end{cases}$$

The Shannon entropy

$$\begin{aligned} H(X) &= -\int_{2}^{4} \frac{1}{2}\ln\frac{1}{2}dx \\ &= \ln 2. \end{aligned}$$

Example 2.1.2 *Let* X *be an exponentially distributed random variable with a hazard rate parameter* $\lambda > 0$*.Then*

$$f_X(x) = \lambda\exp(-\lambda x).$$

The Shannon entropy

$$\begin{aligned}
H\left(X\right) &= -\int_{0}^{\infty} \lambda e^{-\lambda x}\ln(\lambda e^{-\lambda x})dx \\
&= -\lambda\left[\ln\lambda\int_{0}^{\infty} e^{-\lambda x}dx - \int_{0}^{\infty} x\lambda e^{-\lambda x}dx\right] \\
&= -\ln\lambda + E\left[X\right] \\
&= -\ln\lambda + \frac{1}{\lambda}.
\end{aligned}$$

Remark 2.1.1 *It is not easy to study the entropy properties of Shannon in the sense of record statistics, this is why Rényi developed the Shannon entropy.*

2.2 Rényi Entropy

In this section, we present the Rényi Entropy for a continuous random variable and we provide some properties.

Definition 2.2.1 *The information-theoretic concept of Rényi entropy (which is a generalization of the entropy of Shannon) may play a somewhat similar role. Rényi introduced one parameter of Shannon entropy by defining an entropy of order α called Rényi entropy.The Rényi entropy of continuous random variable X with pdf $f\left(x\right)$ is defined by:*

$$H_{\alpha}\left(X\right) = \frac{1}{1-\alpha}\ln\int_{-\infty}^{\infty} f^{\alpha}\left(x\right)dx, \alpha > 0, \left(\alpha \neq 1\right). \tag{2.1}$$

Remark 2.2.1 *It can be easily shown that*:

$$\lim_{\alpha\longrightarrow 1} H_\alpha(x) = H(x).$$

Example 2.2.1 *Let* X *be a uniformly distributed random variable on interval* $[2,4]$ *The Rényi entropy*

$$\begin{aligned} H_\alpha(X) &= \frac{1}{1-\alpha}\ln\int_2^4 \frac{1}{2^\alpha}dx \\ &= \frac{1}{1-\alpha}\ln(\frac{1}{2^\alpha}2) \\ &= \ln 2. \end{aligned}$$

Example 2.2.2 *Let* X *be an exponentially distributed random variable with a hazard rate parameter* $\lambda > 0$. *The Rényi entropy*

$$\begin{aligned} H_\alpha(X) &= \frac{1}{1-\alpha}\ln\int_0^\infty \left(\lambda e^{-\lambda x}\right)^\alpha dx \\ &= \frac{1}{1-\alpha}\ln(\lambda^\alpha\frac{1}{-\alpha\lambda}) \\ &= -\ln(\lambda) + \frac{1}{1-\alpha}\ln(\alpha). \end{aligned}$$

Property 2.2.1 *Some important properties of Rényi entropy are as follows:*

1. $H_\alpha(X)$ *can be negative.*
2. $H_\alpha(X)$ *is invariant under a location transformation.*
3. $H_\alpha(X)$ *is not in variant under a scale transformation.*
4. *Any* $\alpha_1 < \alpha_2$ *we have* $H_{\alpha_1}(X) > H_{\alpha_2}(X)$ *,this equality come true if and only if* X *is uniformly distributed.*

5. *The Rényi divergence of order α between two random variables X and Y with density function $f(x)$ and $g(y)$,respectively, given by:*

$$D_{\alpha}(f,g)=\frac{1}{\alpha-1}\int_{-\infty}^{\infty}\left[\frac{f(X)}{g(X)}\right]^{\alpha-1}f(X)\,dx$$

For more details, see Golshain and Pasha and Contreras-Reyes

2.3 The Rényi Entropy of k-Record Statistics

In this section, we first provide the expression for the entropy of $U_{n,k}^{X}$ and $L_{n,k}^{X}$.to this end ,subststituting2.1 into 1.12 .

Theorem 2.3.1 *The Rényi entropy of the nth value of the upper k-record is given by:*

$$H_{\alpha}\left(U_{n,k}^{X}\right)=H_{\alpha}(U_{n(k)}^{\star})+\frac{1}{1-\alpha}\ln\left\{E\left[f^{\alpha-1}\left(F^{-1}\left(1-\exp\left(-u\right)\right)\right)\right]\right\}.$$

Proof. we have

$$\begin{aligned}H_{\alpha}\left(U_{n,k}^{X}\right)&=\frac{1}{1-\alpha}\ln\int_{-\infty}^{\infty}k^{\alpha n}\frac{\left[-\log\left(1-F\left(x\right)\right)\right]^{\alpha(n-1)}}{\left((n-1)!\right)^{\alpha}}\left[1-F\left(x\right)\right]^{\alpha(k-1)}f^{\alpha}\left(x\right)dx\\&=\frac{1}{1-\alpha}\ln\frac{k^{\alpha n}}{\Gamma^{\alpha}\left(n\right)}\int_{-\infty}^{\infty}\left[-\log\left(1-F\left(x\right)\right)\right]^{\alpha(n-1)}\left[1-F\left(x\right)\right]^{\alpha(k-1)}f^{\alpha}\left(x\right)dx.\end{aligned}$$

We use the transformation $u = -\log(1 - F(x))$ Thus $\exp(-u) = (1 - F(x))$, $\Psi(u) = 1 - \exp(-u)$,we immediately have

$$H_\alpha\left(U_{n,k}^X\right) = \frac{1}{1-\alpha} \ln \frac{k^{\alpha n}}{\Gamma^\alpha(n)} \int_0^\infty u^{\alpha(n-1)} \exp(-u\alpha(k-1)) f^{\alpha-1}(\Psi(u)) \exp(-u) du.$$

$$= \frac{1}{1-\alpha} \ln \frac{k^{\alpha n}}{\Gamma^\alpha(n)} \frac{\Gamma(\alpha(n-1)+1)}{(\alpha(k-1)+1)^{\alpha(n-1)+1}} \int_0^\infty u^{\alpha(n-1)} e^{-u\alpha(k-1)}$$

$$f^{\alpha-1}(\Psi(u)) e^{-u} du.$$

Where,

- U follows gamma distribution with parameters $\alpha(n-1)+1$ and $\alpha(k-1)+1$ and we denote it by $U \sim Gamma(\alpha(n-1)+1, \alpha(k-1)+1)$.Then ,from the Rényi entropy of $U_{n,k}^X$is given by:

$$H_\alpha\left(U_{n,k}^X\right) = H_\alpha(U_{n(k)}^\star) + \frac{1}{1-\alpha} \ln\left\{E\left[f^{\alpha-1}\left(F^{-1}(1 - \exp(-u))\right)\right]\right\}.$$

- $U_{n(k)}^\star$ denote the nth upper k-record value arising form the sequence $\{X_i, i > 1\}$, we get:

$$H_\alpha(U_{n(k)}^\star) = \frac{1}{1-\alpha} \ln\left[\frac{k^{\alpha n}}{\Gamma^\alpha(n)} \frac{\Gamma(\alpha(n-1)+1)}{(\alpha(k-1)+1)^{\alpha(n-1)+1}}\right].$$

■

Theorem 2.3.2 *The Rényi entropy of nth lower k-record value arising from any continuous distribution can be expressed in terms of Rényi entropy of nth lower k-record value arising from $U(0,1)$ .Let $L_{n,k}^X$ denote the lower k-record value of the* *sequence* $\{X_i\}$*. Then , the Rényi entropy of* $L^X_{n,k}$ *is given by:*

$$H_\alpha\left(L^X_{n,k}\right) = H_\alpha(L^\star_{n(k)}) + \ln\left\{E\left[f^{\alpha-1}\left(F^{-1}(\exp\left(-u\right))\right)\right]\right\}.$$

Proof. The proof of the theorem 2.3.2 is similar to the proof of the Theorem 2.3.1.

- Where $H_\alpha\left(L^X_{n,k}\right)$ denote the Rényi entropy of nth lower k-record value arising from $U\left(0,1\right)$ and $U \sim Gamma(\alpha\left(n-1\right)+1, \alpha\left(k-1\right)+1)$.
- Where $L^\star_{n(k)}$ denote the nth lower k-record value arising form the sequence $\{X_i, i > 1\}$, we get:

$$H_\alpha(L^\star_{n(k)}) = \frac{1}{1-\alpha}\ln\left[\frac{k^{\alpha n}}{\Gamma^\alpha\left(n\right)}\frac{\Gamma\left(\alpha\left(n-1\right)+1\right)}{\left(\alpha\left(k-1\right)+1\right)^{\alpha(n-1)+1}}\right].$$

■

Proposition 2.3.1 *The entropy of nth value of upper k-records* $U^X_{n,k}$ *and nth value of lower k-records* $L^X_{n,k}$ *can be expressed , respectively, as*

$$H_\alpha\left(U^X_{n,k}\right) = H_\alpha(U^\star_{n(k)}) + \frac{1}{1-\alpha}\ln\left\{E\left[f^{\alpha-1}\left(F^{-1}(1-\exp\left(-u\right))\right)\right]\right\}.$$

and

$$H_\alpha\left(L^X_{n,k}\right) = H_\alpha(L^\star_{n(k)}) + \frac{1}{1-\alpha}\ln\left\{E\left[f^{\alpha-1}\left(F^{-1}(\exp\left(-u\right))\right)\right]\right\}.$$

Remark 2.3.1 *If we put k = 1, we can easily obtain the classical records from the sequence of k-records.*

Example 2.3.1 *Let* $\{X_i, i > 1\}$ *be a sequence of iid random variables having a common Uniformly distribution on the interval* $[0, \theta]$ *.Then*

$$f\left(F^{-1}(1-e^{-u})\right) = f\left(F^{-1}\left(e^{-u}\right)\right) = \frac{1}{\theta}, u > 0.$$

where

$$E\left[f^{\alpha-1}\left(F^{-1}(1-e^{-u})\right)\right] = \frac{1}{\theta}.$$

we have from Proposition 2.3.1 that the entropy of $U_{n,k}^X$ *and* $L_{n,k}^X$*is given by :*

$$H_\alpha\left(U_{n,k}^X\right) = \frac{1}{1-\alpha}\ln\left\{\frac{k^{\alpha n}}{\Gamma^\alpha(n)}\frac{\Gamma(\alpha(n-1)+1)}{(\alpha(k-1)+1)^{\alpha(n-1)+1}}\theta^{1-\alpha}\right\}$$
$$H_\alpha\left(U_{n,k}^X\right) = \frac{1}{1-\alpha}\ln\left\{\frac{k^{\alpha n}}{\Gamma^\alpha(n)}\frac{\Gamma(\alpha(n-1)+1)}{(\alpha(k-1)+1)^{\alpha(n-1)+1}}\theta^{1-\alpha}\right\} = H_\alpha\left(L_{n,k}^X\right)$$

The following figure 2.3.1 presents the Rényi entropy for X *and for the upper k-record values, we chose 10 observations* $n = 10$ *and also* $k = 3, 4, 8$*.*

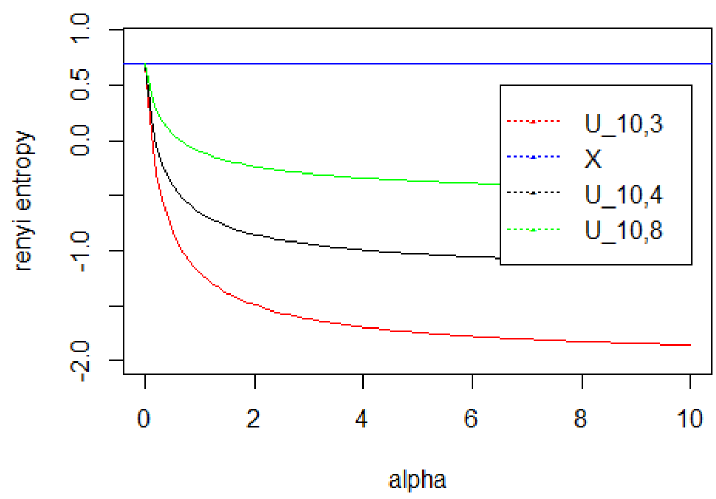

Figure 2.1: Rényi entropy of uniform(0,2) distribution and upper and lower k-record for $n = 10$ and $k = 3, 4, 8$

Example 2.3.2 *Let $\{X_i, i > 1\}$ be a sequence of iid random variables having a common Exponential distribution with failure rate parameter $\lambda > 0$. Then*

$$F^{-1}(x) = -\frac{1}{\lambda}\log(1-x).$$

$$f\left(F^{-1}(x)\right) = \lambda(1-x), 0 \leq x \leq 1.$$
$$F^{-1}\left(1-e^{-x}\right) = \frac{1}{\lambda}x, x > 0.$$

Now, we have

$$E\left[f^{\alpha-1}(F^{-1}(1-e^{-u}))\right] = \left[\frac{\alpha(k-1)+1}{\alpha k}\right]^{\alpha(n-1)+1}\lambda^{\alpha-1}.$$

we have from Proposition 2.3.1 that the entropy of $U_{n,k}^{X}$ is given by:

$$H_{\alpha}\left(U_{n,k}^{X}\right)=\frac{1}{1-\alpha}\ln\left\{\frac{k^{\alpha n}}{\Gamma^{\alpha}(n)}\frac{\lambda^{\alpha-1}\Gamma\left(\alpha\left(n-1\right)+1\right)}{\left(\alpha k\right)^{\alpha(n-1)+1}}\right\}.$$

Example 2.3.3 *Let $\{X_i, i>1\}$ be a sequence of iid random variables having a common Pareto distribution with density function given by*

$$f\left(x\right)=\frac{\beta}{\sigma}\left(\frac{x}{\sigma}\right)^{-\beta-1},x>\sigma.$$

Here

$$F^{-1}\left(x\right)=\sigma\left[1-x\right]^{-\frac{1}{\beta}}.$$

Now, we have

$$E\left[f(F^{-1}(1-e^{-u}))\right]=\frac{\beta^{\alpha n}}{\sigma^{\alpha-1}}\left[\frac{\alpha\left(k-1\right)+1}{\alpha\left(\beta k+1\right)-1}\right]^{\alpha(n-1)+1}.$$

we have from Proposition 2.3.1 that the entropy of $U_{n,k}^{X}$ is given by:

$$H_{\alpha}\left(U_{n,k}^{X}\right)=\frac{1}{1-\alpha}\ln\left\{\frac{k^{\alpha n}}{\Gamma^{\alpha}(n)}\frac{\beta^{\alpha n}\Gamma\left(\alpha\left(n-1\right)+1\right)}{\sigma^{\alpha-1}\left[\alpha\left(\beta k+1\right)-1\right]^{\alpha(n-1)+1}}\right\}.$$

2.4 Some Results on Rényi Entropy of k-Record statistics

In this section, we provide some results on the entropy of k- record statistics and state symmetric properties between the entropy of the nth value of the upper and the nth value of the lower form k-record.

For uniformly distributed random variable on $[0,1]$, we observe from the example

2.3.1 that the entropy of the nth value of the upper k-record is equal to the entropy of the nth value of the lower k-record. This leads to the question of whether this result can be extended to a more general case. Proposition 2.4.1 provides a positive answer to this question. In order to prove Proposition 2.4.1, we require the following Lemma 2.4.1.

Lemma 2.4.1 (*Fashandi and Ahmadi*, (2012)) : *Let X be a random variable with cdf F,pdf f,and finite mean μ. Then*

$$f(\mu + x) = f(\mu - x).$$

for all $x > 0$ if and only if

$$f\left(F^{-1}(u)\right) = f\left(F^{-1}(1-u)\right).$$

for all $u \in [0,1]$

Proposition 2.4.1 *If the pdf of X is symmetric about its finite mean μ ,then*

$$H_\alpha\left(U_{n,k}^X\right) = H_\alpha\left(L_{n,k}^X\right).$$

Proof. By Proposition 2.4.1 and Lemma 2.4.1,it is easy to verify that

$$\begin{aligned} H_\alpha\left(U_{n,k}^X\right) &= H_\alpha(U_{n(k)}^\star) + \frac{1}{1-\alpha}\ln\left\{E\left[f^{\alpha-1}\left(F^{-1}(1-\exp(-u))\right)\right]\right\} \\ &= H_\alpha(L_{n(k)}^\star) + \frac{1}{1-\alpha}\ln\left\{E\left[f^{\alpha-1}\left(F^{-1}(\exp(-u))\right)\right]\right\} \\ &= H_\alpha\left(L_{n,k}^X\right). \end{aligned}$$

■

2.5 Monotone properties of entropy Of k-record statistic

In this section, we determine the monotonicity of Rényi entropy of the upper and lower k-records arising from any continuous distribution. Some of these definitions we use in proof :

Proposition 2.5.1 *The random variable X is said to :*

1. *have increasing (decreasing) failure rate IFR (DFR) if $\bar{F}$ is log-concave (log-convex);*

2. *have increasing (decreasing) failure rate in average $IFRA$ $(DFRA)$ if $-\log \bar{F}(x)$ is star-shaped (anti-star-shaped); that is, if $-\log \bar{F}(x)/x$ increasing (decreasing) in $x > 0$;*

3. *be a new better (worse) than used NBU (NWU) if $\bar{F}(x)\bar{F}(y) > (\leq) \bar{F}(x+y)$ for all $x, y > 0$.*

The relationships between them are:

$$IFR \subseteq IFRA \subseteq NBU$$

and

$$DFR \subseteq DFRA \subseteq NWU$$

For further details, see Barlow and Proschan(1981).

Definition 2.5.1 *Let X and Y be two non negative random variables with cdf F and G and with pdf f and g respectively,then X is said to be smaller than Y*

1. *In likelihood ratio order, denoted by* $X \leq_{lr} Y$, *if* $\frac{f(x)}{g(x)}$ *is decreasing in* $x > 0$.

2. *In the usual stochastic order , denoted by* $X <_{st} Y$,*if* $\bar{F}(x) \leq \bar{G}(x)$ *for all* $x > 0$, *where* $\bar{H}(.)$ *is the survival function.*

It is well known that $X \leq_{lr} Y \Longrightarrow X <_{st} Y$ *and* $X <_{st} Y$ *if and only if* $E[\Phi(X)] \leq E[\Phi(Y)]$ *for all increasing function* Φ.

Definition 2.5.2 *The random variable* X *is said to be less than or equal to the random variable* Y *in Rényi entropy ordering ,denoted by* $X <_{RE} Y$, *if* $H_\alpha(X) \leq H_\alpha(Y)$ *for all* $\alpha > 0$.

Remark 2.5.1 *we can conclude that* $H_\alpha(X) \leq H_\alpha(Y)$ *for all* $\alpha > 0$ *,which means that* X *is less uncertain that* Y *according to Rényi entropy.*

The following theorem reveals the monotone behavior of Rényi entropy of upper k-record.

Theorem 2.5.1 *Let* $\{X_i, i \geq 1\}$ *be a sequence of iid random variables with a common cdf* $F(x)$ *and pdf* $f(x)$. *Let* $U_{n,k}^X$ *denote the nth upper k-record value .*

$$f(x) \text{ is non-decreasing in } x \Rightarrow \text{for } n > k,\ H_\alpha\left(U_{n,k}^X\right) \text{ is non-increasing in } n.$$

Proof. see the proof of Theorem 2.1 of Abbasnejad and Arghami ([1]). ∎

In a similar way, we can state the monotone behavior of Rényi entropy of lower k-records as given in the following theorem.

Theorem 2.5.2 *Let* $\{Xi, i \geq 1\}$ *be a sequence of iid random variables with a common cdf* $F(x)$ *and pdf* $f(x)$. *Let* $L_{n,k}^X$ *denote the nth lower k-record value.*

$$f(x) \text{is non-increasing in } x \Rightarrow \text{for } n > k, H_\alpha\left(L_{n,k}^X\right) \text{ is non-increasing in } n$$

Next, we conduct some numerical computation for the Rényi entropy of the nth value of upper k-records for $\alpha = 2$ and for different choices of n and k from uniform $U(0,2)$. We have:

$$f(x) = \frac{1}{2} \text{ is non-decreasing in } x$$

. Also from example 2.3.1 and the proposition 2.3.1, we have:

$$H_\alpha\left(U_{n,k}^X\right) = \frac{1}{1-\alpha} \ln \left\{ \frac{k^{\alpha n}}{\Gamma^\alpha(n)} \frac{\Gamma(\alpha(n-1)+1)}{(\alpha(k-1)+1)^{\alpha(n-1)+1}} \theta^{1-\alpha} \right\}$$

is decreasing in n.

These numerical values are presented in Tables 2.1

Table 2.1: Rényi entropy of the nth upper k-records value for $\alpha = 2$ different choices n and k from $U(0,2)$

$\alpha = 2$			
	$k = 2$	$k = 3$	$k = 8$
$n = 1$	0.405465108108164	0.105360515657826	-0.757685701697516
$n = 2$	0.523248143764548	0.433864582629862	-0.193615563412713
$n = 3$	0.235566071312767	0.356903541493734	-0.035010533236075
$n = 4$	-0.15747651679684	0.174581984699779	0.0182339812827373
$n = 5$	-0.599309269075879	-0.0565297362636073	0.0226883316321177
$n = 6$	-1.06931289832162	-0.31581233419369	-0.00102819498519845
$n = 7$	-1.55766566623555	-0.59344407079197	-0.043093860270711
$n = 8$	-2.05892183898539	-0.883979212226157	-0.0980629303921314
$n = 9$	-2.56974746275138	-1.1840838046765	-0.162601451529703
$n = 10$	-3.08795319381499	-1.49156850442446	-0.234520079964897

2.6 Rényi entropy ordering

In this section, we will use Abbasnejad and Arghami to establish their Rényi entropy ordering of nth upper and lower k-record value.

Theorem 2.6.1 *Let X and Y be two continuous random variables with cdfs $F(x)$ and and pdfs $f(x)$ and $g(y)$ respectively. Suppose that $U_{n,k}^X$ and $L_{n,k}^X$ represents the nth upper k record value arising from X and Y respectively.Assume that:*

$$\Lambda_1 = \left\{u > 0, \frac{g\left(G^{-1}\left(1-e^{-u}\right)\right)}{f\left(F^{-1}\left(1-e^{-u}\right)\right)} \leq 1\right\},$$
$$\Lambda_2 = \left\{u > 0, \frac{g\left(G^{-1}\left(1-e^{-u}\right)\right)}{f\left(F^{-1}\left(1-e^{-u}\right)\right)} > 1\right\}$$

And $X <_{RE} Y$. $\inf \Lambda_1 \geq \sup \Lambda_2 \implies U_{n,k}^X \leq_{RE} U_{n,k}^Y, \forall n \geq 1$ and $n \geq k$.

Proof. see Theorem 2.3 in Abbasnejad and Arghami ([1]) . ■

Theorem 2.6.2 *Let X and Y be two continuous random variables with cdfs $F(x)$ and $G(y)$ and pdf $f(x)$ and $g(y)$ respectively. Suppose:*

$$\Lambda_1 = \left\{u > 0, \frac{g\left(G^{-1}\left(e^{-u}\right)\right)}{f\left(F^{-1}\left(e^{-u}\right)\right)} \leq 1\right\},$$
$$\Lambda_2 = \left\{u > 0, \frac{g\left(G^{-1}\left(e^{-u}\right)\right)}{f\left(F^{-1}\left(e^{-u}\right)\right)} > 1\right\}$$

And $X <_{RE} Y$. $\inf \Lambda_1 \geq \sup \Lambda_2 \implies L_{n,k}^X \leq_{RE} L_{n,k}^Y, \forall n \geq 1$ and $n \geq k$.

Example 2.6.1 *Let X and Y be two random variables having a common exponential distribution with different scale parameters σ and λ respectively, where $\sigma > \lambda$. Then from (2.1), we get:*

$$H_\alpha(X) = \frac{1}{1-\alpha} \ln(\alpha) - \ln(\sigma).$$

It can be easily verified that $H_\alpha(X)$ *is a decreasing function of* σ. *Thus, we have* $H_\alpha(X) \leq H_\alpha(Y)$ *and thereby* $X <_{RE} Y$. *We have* $f(F(1-e^{-x})) = \frac{1}{\sigma}e^{-x}$ *and* $\inf \Lambda_1 = \sup \Lambda_2$.*Hence, by Theorem 2.6.1 we get* $U^X_{n,k} \leq_{RE} U^Y_{n,k}$.

Remark 2.6.1 *This passage discusses the relationship between two random variables,* X *and* Y *that share an exponential distribution but have different scale parameters.The author notes that* $H_\alpha(X)$ *,a measure of* X *entropy,decreases as* σ *(scale parameters) increases. The author then concludes that* X *is stochastically dominated by* Y *(i.e,*$X <_{RE} Y$ *) because* $H_\alpha(X) \leq H_\alpha(Y)$. *The author also mentions that (Theorem 2.6.1) applies to* X *and* Y *because their hazard functions are related, which further supports the conclusion that* $X <_{RE} Y$.

2.6.1 The effect of location-scale transformation

The location-scale transformation can affect the Rényi entropy of k-record random variables through a rescaling of the random variable. This effect is determined by the specific scaling factor of the transformation and is independent of any shifts in the random variable.

Lemma 2.6.1 *consider a non negative random variable* X *,with pdf* f *and cdf* F . *Let* $Z = aX + b$ *be a transformation on* X *,where* $a > 0$ *and* $b \geq 0$ *are constants .then*

$$H_\alpha\left(U^Z_{n,k}\right) = H_\alpha\left(U^X_{n,k}\right) + \ln a, \tag{2.2}$$

Where $U^Z_{n,k}$ *and* $U^X_{n,k}$ *are the nth k-record corresponding to* Z *and* X *respectively .*

Remark 2.6.2 *The Rényi entropy of k-records changes due to scale, but it does not change due to location.*

Theorem 2.6.3 *Consider two absolutely continuous random variables* X *and* Y*. Assume that* $U_{n,k}^{X}$ *and* $U_{n,k}^{Y}$ *are the nth upper k-record corresponding to* X *and* Y *respectively . Let* $U_{n,k}^{Z_1} = a_1 U_{n,k}^{X} + b_1$ *and* $U_{n,k}^{Z_2} = a_2 U_{n,k}^{Y} + b_2$ *,where* $a_1, a_2 > 0$ *and* $b_1, b_2 \geq 0$ *are constants .Then we have*

$$U_{n,k}^{X} \leq_{RE} U_{n,k}^{Y} \implies U_{n,k}^{Z_1} \leq_{RE} U_{n,k}^{Z_2}, \textit{for } a_1 \leq a_2.$$

Proof. if $U_{n,k}^{X} \leq_{RE} U_{n,k}^{Y}$, then

$$H_\alpha\left(U_{n,k}^{X}\right) \leq H_\alpha\left(U_{n,k}^{Y}\right).$$

Since $a_1 \leq a_2$, $\ln a_1 \leq \ln a_2$. Hence,

$$\ln a_1 + H_\alpha\left(U_{n,k}^{X}\right) \leq \ln a_2 + H_\alpha\left(U_{n,k}^{Y}\right).$$

Thus from 2.2 ,we get $U_{n,k}^{Z_1} \leq_{RE} U_{n,k}^{Z_2}$. ■

The following result removes the scale constant constraint.

Corollary 2.6.1 *Consider two absolutely continuous random variables* X *and* Y*. Assume that* $U_{n,k}^{X}$ *and* $U_{n,k}^{Y}$ *are the nth upper k-record corresponding to* X *and* Y *respectively.Let* $U_{n,k}^{Z_1} = aU_{n,k}^{X} + b$ *and* $U_{n,k}^{Z_2} = aU_{n,k}^{Y} + b$*, where* $a > 0$ *and* $b \geq 0$ *are constants.*

$$U_{n,k}^{X} \leq_{RE} U_{n,k}^{Y} \implies U_{n,k}^{Z_1} \leq_{RE} U_{n,k}^{Z_2}.$$

Conclusion

The work discussed in this master's thesis explains the relevance of Rényi entropy while using record statistics. Understanding historical trends, patterns, and outcomes requires the use of record statistics. They serve as a foundation for evaluating performance and establishing benchmarks and goals. They also help us to learn from the past, make educated decisions, and aim for continual progress across a variety of disciplines.

Jitto Jose E. I. Abdul Sathar(2022) [12] have expressed Rényi entropy for upper and lower records statistics arising from any continuous distribution. We have explained this study deeply by representing the expression used to provide the upper and lower k-record in terms of Rényi entropy of records arising from any continuous distribution, especially uniform distribution. And we have used the result to express the monotone behavior and ordering of Rényi entropy. In addition, we have used that representation to derive a numerical representation and how the Rényi entropy may have a negative value.

In the end, we can say that the record statistic and theory of entropy helped us to understand the reliability measure and its characteristics, as one of the benefits of entropy, it can help in compressing data by identifying and removing redundancy in the data, thus reducing the amount of storage space required and in predictive modeling, entropy can be used to measure the uncertainty or randomness of a dataset, this can help in improving the accuracy of predictive models by identifying the key variables that can impact the outcome. Also, entropy can be used to generate random keys and passwords, which can enhance security in various ways, for example, by making it more challenging for hackers to guess the password .in addition, Entropy is useful in making cryptographic systems more robust by generating unpredictable keys that cannot be deciphered by an attacker.

This does not prevent us from finding some weaknesses in entropy theory, and we mention the following calculating the entropy of a k-record requires significant computational power, especially when working with large and complex datasets. This can result in slower system performance and increased processing times, moreover to it can be influenced by the bias of the individuals collecting and analyzing the data. This can result in incorrect or incomplete analyses if the individuals gathering the data are not careful. Also, The entropy of a k-record can be challenging to interpret, especially if the results are not presented in a visually appealing way. This can lead to confusion or incorrect conclusions being drawn from the data.

Each theory has positive and negative points that were the reason for its development. Finally, as a future project, we propose to provide this work with a numerical illustration in real life to observe the result that Rényi entropy can have. in addition, we propose to study the measure of extropy discovered by Lad et al (2015) . which is a dual theory of entropy and it has more benefits that need to be explored.

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Abstract :

In recent years, attention has been paid to the study of records statistics in terms of Rényi entropy which was used widely in reliability and information studies.
In this master’s thesis, an explained representation of records statistics is expressed and some properties results are exploited in terms of continuous distributions of upper and lower records. In addition, some characterization results are explored based on the records values, and related properties are addressed for the Rényi of record statistics (ordinary record and k-records). Simplified expressions of the Rényi entropy of records are derived which allows us to investigate monotonicity properties and characterizations of some probability distributions. Moreover. These results are based on applying the result obtained by Jitto Jose E. I. Abdul Sathar (2022).

Résume :

Ces dernières années, l'attention a été portée sur l'étude des statistiques du record en termes l’entropie de Rényi qui a été largement utilisée dans les études de fiabilité et d'information. Dans ce mémoire, une représentation expliquée des statistiques du record est exprimée et certains résultats de propriétés sont exploités en termes de distributions continues pour les records supérieurs et inférieures. En outre, certains résultats de caractérisation sont explorés sur la base des valeurs des records, et les propriétés connexes sont abordées pour le Rényi entropie des statistiques du records (record ordinaire et k-records). Des expressions simplifiées de l'entropie de Rényi des records valeurs sont dérivées, ce qui nous permet d'étudier les propriétés de monotone et les caractérisations de certaines distributions de probabilité. De plus, ces résultats sont basés sur l'application du résultat de Rényi entropie du résultat obtenu par Jitto Jose E. I. Abdul Sathar (2022).

ملخص:

في السنوات الأخيرة، تم الاهتمام بدراسة الإحصائيات القياسية (السجل) من حيث Rényi Entropy، التي استخدمت على نطاق واسع في دراسات الموثوقية والمعلومات .
في هذه المذكرة، تم التعبير عن تمثيل موضح لإحصائيات القياسية ويتم استغلال بعض نتائج الخصائص من حيث التوزيعات المستمرة. بالإضافة إلى ذلك، يتم استكشاف بعض نتائج الوصف بناءا على قيم الإحصائيات القياسية، وتم مناقشة الخصائص ذات الصلة لـ Rényi Entropy بالنسبة للإحصائيات العادية او المتغيرة بدلالة K. و التي تسمح لنا بدراسة خصائص الرتابة و الترتيب و توصيفا ت التوزيعات الاحتمالية المستمرة. تستند هاته النتائج إلى تطبيق Rényi Entropy وذلك نتيجة لتحليل النتائج المتحصلة من مقال Jitto Jose E. I. Abdul Sathar (2022).