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The stochastic maximum principle for an impulsive optimal control problem

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Dedication

To my parents,

To all my family,

To all those who are dear to me.

BEZZIOU EZZOBIR

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First of all, I thank Allah, the Almighty, for having given me the strength, to survive, as well as the audacity to overcome all the difficulties.

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Thanks

Bezziou Ezzobir

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Introduction

Stochastic impulse control has gained significant attention in research due to its broad applications in various areas. A system involving impulse control has been studied in [3], [9] and [10]. The stochastic control problems have two primary methods for solving them, which are the stochastic maximum principle and dynamic programming principle. The former is a stochastic generalization of the Pontryagin maximum principle. Typically, the preferred method for investigating the optimal impulse control problem is through the dynamic programming principle. The results demonstrate that the value function resolves certain quasi-variational inequalities. The optimal impulse moments may be determined by a verification theorem, such as Korn [5]. Numerous authors have investigated singular control problems. Cadenillas and Haussmann [2] derived the stochastic maximum principle of singular control, which examines linear dynamics and convex value function. This paper assume that the singular control terms are a process of bounded variation. Nonetheless, impulse control is a piecewise process that may not always be increasing. Wu and Zhang [9] suggested a piecewise impulse control process that is not increasing for a forward-backward system. Stochastic systems in the studies mentioned above are represented by stochastic differential equations. Nevertheless, there are also some cases where the system is reliant on the expected value of the system.

Mean-field models illustrate the intricate responses of particles within a medium. Nevertheless, for a typical mean-field controlled jump-diffusion, where the configuration is non-Markovian, the characteristic of Hamilton-Jacobi-Belman equations relies on the law of iterated expectations on value function. In this scenario, the principle of dynamic programming may not generally be applicable. The stochastic maximum principle presents a good perspective for resolving such problem. Lasry and Lions [6] first introduced the mean-field model in physics and statistical mechanics and demonstrated that the mean-field system could be decomposed into a sequence of nonlinear equations. Mean-field type problems are utilized in Andersson and Djehiche [1] demonstrated that the relevant principles of backward stochastic differential equations could be utilized as a basis for resolving the optimal control of mean-field type problems. Shen and Siu [8] examined the maximum principle for a jump-diffusion mean-field model.

In [3], conditions for near-optimal in mean-field control models involving continuous and impulse control were investigated. The authors formulated an inequality equation using spike variation technique to obtain the absolutely continuous part of the near-optimal while the near-optimal impulse controls were derived through convex perturbation. We establish a necessary and sufficient stochastic maximum principle using the duality and the convex analysis. In this memory the dynamics of the controlled system is driven by

$$\begin{cases} dX(t) = b(t, X(t), E[X(t)], u(t)dt + \sigma(t, X(t), E[(X(t)], u(t)dB(t) \\ \qquad\qquad + C(t)d\xi(t) \\ X(0) = X_0 \end{cases}$$

where $\xi(t) = \sum_{i\geq 0} \xi_i \mathbf{1}_{[\tau_i,T]}$ is a piecewise consumption process. $\{\tau_i\}$ is a fixed sequence of increasing $\mathcal{F}_t$-stopping time. Each ξ_i is an $\mathcal{F}_{\tau_i}$-measurable random variable.

The goal of the controller is to minimize the expected cost function, which relies on the control inputs to the system

$$J(u(\cdot), \xi(\cdot)) \\ = E\left[\int_0^T f(t, X(t), E[X(t)], u(t))dt + g(X(T), E[X(T)]) + \sum_{i\geq 0} l\left(\tau_i, \xi_i\right)\right].$$

In this memory, we examine a stochastic optimal control problem in which the governed state process is characterized by a diffusion mean-field model involving impulse controls. Furthermore, the existence and uniqueness of the solution to this equation have also been studied. These findings are then applied to the continuous-time Markowitz's mean-variance portfolio selection model incorporating a piecewise consumption process.

The remaining sections of the memory are structured as follows. Chapter 1 is a recall on the stochastic calculation. Chapter 2 presents the the existence and uniqueness of solutions with respect to stochastic differential equations (SDEs) of mean-field involving impulse control. In Chapter 3, we discuss the stochastic maximum principle for the optimal control problem and a verification theorem and our results are applied to a Markowitz's mean-variance portfolio selection model.

Chapter 1

Recall on the stochastic calculation

In this introductory chapter, we give some basic definitions and the more often elementary, concerning the results of stochastic calculation. We limit ourselves to what is strictly necessary for the following chapters.

1.1 Stochastic processes

Definition 1.1.1 (Stochastic processes) *A stochastic process is a collection of random variables* $\mathbb{X} = \{X(t), t \in \mathcal{T}\}$ *defined on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *and assuming valeus in* $\mathbb{R}^n$

Remark 1.1.1

1. *We often interpret* t *as time.*
2. *If the index set* $\mathcal{T}$ *is countable set, we call* $\mathbb{X}$ *a discrete-time stochastic process, and if* $\mathcal{T}$ *is a continuous, we call it a continuous-time stochastic process.*
3. *The index set* $\mathcal{T}$ *is usually the halfline* $[0, \infty)$*,*

4. *For each* t *in the index set* $\mathcal{T}$, $X(t)$ *is a random variable.*

$$\omega \longrightarrow X(t,\omega); \quad \omega \in \Omega.$$

5. *If we fix* $\omega \in \Omega$, $X(\omega)$ *is a function*

$$t \longrightarrow X(t,\omega); \quad t \in \mathcal{T}.$$

which is called a path of $X(t)$.

Definition 1.1.2 (Modification of process) *We say that a stochastic process* $(X(t))_{t\in\mathcal{T}}$ *is a modification of another process* $(Y(t))_{t\in\mathcal{T}}$ *if*

$$\mathbb{P}\left(X(t) = Y(t)\right) = 1, \quad \forall\, t \in \mathcal{T}.$$

Definition 1.1.3 (Indistinguishable processes) *We say that two stochastic processes* $(X(t))_{t\in\mathcal{T}}$ *and* $(Y(t))_{t\in\mathcal{T}}$ *are indistinguishable if*

$$\mathbb{P}\left(X(t) = Y(t), \forall\, t \in \mathcal{T}\right).$$

Remark 1.1.2 *If two stochastic processes* $(X(t))_{t\in\mathcal{T}}$ *and* $(Y(t))_{t\in\mathcal{T}}$ *are indistinguishable, then they are modifications of each other.But Conversely not always true*

Definition 1.1.4 (Measurable stochastic process) *a stochastic process* $\mathbb{X} = \{X(t), t \in \mathcal{T}\}$ *is measurable if the mapping* $\mathbb{X} : [0,T]\times\Omega \to \mathbb{R}^n$ *is* $(\mathcal{B}\left([0,T]\right) \otimes \mathcal{F}, \mathcal{B})$ *measurable.*

Definition 1.1.5 (Filtration) *A filtration is a family of σ-algebras $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, such that $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$ for all $0 \leq s \leq t$, where $\mathcal{F}$ is the σ-algebra of all events in Ω. And we call $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{R}^+}, \mathbb{P})$ a filtered probability space.*

Intuitively, the filtration represents the increasing amount of information available to an observer as time passes, with $\mathcal{F}_t$ being the set of events that the observer can distinguish up to time t.

Remark 1.1.3

1. *A filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathbb{R}^+}$ is right continuous if $\cap_{\epsilon>0}\mathcal{F}_{t+\epsilon} = \mathcal{F}_t$ for all $t \geq 0$*
2. *A filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathbb{R}^+}$ is complete if each $\mathcal{F}_t$ contains every nigligible set.*
3. *A filtration that is right continuous and complete is said to satisfy the usual conditions.*

Definition 1.1.6 (Adapted stochastic process) *We say that a stochastic process $\mathbb{X} = \{X(t), t \in \mathcal{T}\}$ is adapted to a filtration $\mathbb{F} = (\mathcal{F}_t)_{t\in\mathbb{R}^+}$ if $X(t)$ is $\mathcal{F}_t$-measurable for each t*

Remark 1.1.4

1. *The collection of σ-algebras $\{\mathcal{G}(t)\}_{t\geq 0}$ where*

$$\mathcal{G}(t) = \sigma\{X(s) : 0 \leq s \leq t\}$$

   for all $t \geq 0$ we call it the natural filtration of a stochastic process $\mathbb{X} = \{X(t), t \in \mathcal{T}\}$

2. *We define the minimal augmented filtration generated by $\mathbb{X} = \{X(t), t \in \mathcal{T}\}$ to be the smallest filtration that is right continuous and complete and with respect to which the process $\mathbb{X} = \{X(t), t \in \mathcal{T}\}$ is adapted*

Definition 1.1.7 (Stopping time) *A stopping time τ with respect to a continuous-time stochastic process $(X(t))_{t\in\mathcal{T}}$ is a random variable $\tau : \Omega \to [0, +\infty]$ such that for all $t \in \mathcal{T}$, the event $\{\tau \leq t\} = \{\omega \in \Omega : \tau(\omega) \leq t\} \in \mathcal{F}_t$*

If τ is a stopping time. The σ-algebra $\mathcal{F}_\tau = \{A \in \mathcal{F},\, A \cap \{\tau \leq t\} \in \mathcal{F}_t,\, \forall t \in \mathcal{T}\}$ is called the σ-algebra of events prior to $\mathcal{T}$

1.1.1 Brownian Motion

Definition 1.1.8 (Standard Brownian Motion) *The Standard Brownian Motion, also known as Wiener process, is a stochastic process $B(t)_{t\geq 0}$ with independent and identically distributed increments such that:*

1. *$B(0) = 0$ almost surely.*
2. *For all $0 \leq s < t$, the increment $B(t) - B(s)$ is normally distributed with mean 0 and variance $t - s$, (i.e., $B(t) - B(s) \sim \mathcal{N}(0, t - s)$).*
3. *The sample paths of $B(t)$ are almost surely continuous.*

Remark 1.1.5

1. *We call $B = (B^{(1)}, B^{(2)}, \ldots, B^{(d)})$ a d-dimensional Brownian motion if $B^{(i)}$ are independent standard Brownian motions , for $i = 1, 2, \ldots, d$.*
2. *The filtration $\{\mathcal{F}_t),\, t \geq 0\}$ generated by a Brownian motion B is defined as follows : $\mathcal{F}_t = \sigma\left(B(s) : s \leq t\right), \quad t \geq 0$, and we call it the natural filtration of B or Brownian filtration.*

1.1.2 Martingales

Definition 1.1.9 (Martingale) *A continuous time martingale is a stochastic process $\mathbb{X} = \{X(t), t \geq 0\}$ that satisfies the following conditions:*

(i) *$X(t)$ is adapted to a filtration $\mathbb{F} = \{\mathcal{F}_t, t \geq 0\}$, (i.e., $X(t)$ is measurable with respect to $\mathcal{F}_t$ for all $t \geq 0$).*

(ii) *$X(t)$ is integrable for all $t \geq 0$, (i.e., $\mathbb{E}[|X(t)|] < \infty$ for all $t \geq 0$).*

(iii) *For all $0 \leq s \leq t$, $\mathbb{E}[X(t)|\mathcal{F}_s] = X(s)$ almost surely.*

Remark 1.1.6 1. *$\mathbb{X}$ is **submartingale** (resp **supermartingale**) if it satisfies (i), (ii) and if moreover for all $s, t \geq 0$ such that $s < t$, we have $\mathbb{E}\left(X(t)\,|\,\mathcal{F}_s\right) \geq X(t)$ (resp $\mathbb{E}\left(X(t)\,|\,\mathcal{F}_s\right) \leq X(t)$)*

2. *$\mathbb{X}$ is a martingale if it is submartingale and supermartingale at the same time*

3. *if $\mathbb{X}$ is martingale then $\mathbb{E}\left(X(t)\right) = \mathbb{E}\left(X(0)\right)$ for all $t \in \mathcal{T}$*

Proposition 1.1.1 *if B is a Brownian motion then B, $\left((B(t))^2 - t\right)_{t\in\mathcal{T}}$ and $\left\{\exp\left(\sigma B(t) - \frac{\sigma^2 t}{2}\right)\right\}$ are martingales. Reciprocally if $\mathbb{X}$ is a continuous process such that $\mathbb{X}$ and $\left\{(X(t))^2 - t\right\}_{t\geq 0}$ are martingales, $\mathbb{X}$ is a Brownian motion*

Definition 1.1.10 (Local Martingale) *A stochastic process $\{M(t)\}_{t\in\mathbb{R}^+}$ adapted caglad (right-continuous with left limits), is a local martingale if there exists an increasing sequence of stopping times (τ_n) such that $\tau_n \to +\infty$ as $n \to \infty$ and $M(t \wedge \tau_n)$ is a martingale for all n.*

A positive local martingale is a supermartingale. A locally uniformly integrable martingale is a martingale.

Definition 1.1.11 (Semimartingale) *A semimartingale is a cadlag adapted process $\mathbb{X}$ admitting a decomposition of the form:*

$$\mathbb{X} = A + M \tag{1.1}$$

where M is a cadlag local martingale null at 0 *and A is an adapted process of finite variation and null at* 0.

A continuous semimartingale is a semimartingale such that in the decomposition (1.1), M and A are continuous. Such a decomposition where M and A are continuous is unique.

1.2 Stochastic integration and Itô's formula

In this section, T is a positive real number, and we are seeking to define the integral

$$I(\theta) = \int_0^T \theta(t)\, dB(t) \tag{1.2}$$

Where $(\theta(t))_{t\geq 0}$ is any process and $(B(t))_{t\geq 0}$ is a Brownian motion. The problem is to give meaning to the differential element $dW(s)$ since the function $s \to W(s)$ is not differentiable.

1.2.1 Wiener integral

Let

$$L^2([0,T],\mathbb{R}) = \left\{\theta : [0,T] \to \mathbb{R} \text{ such that, } \int_0^T |\theta(s)|^2\, ds < \infty\right\}.$$

The Wiener integral is an integral of the form (1.2) with θ being a deterministic function, meaning it does not depend on the random w.

If θ^n is a deterministic step function of the form

$$\theta^n(t) = \sum_{i=1}^{p_n} \alpha_i(t)\, \mathbf{1}_{\left[t_i^{(n)}, t_{i+1}^{(n)}\right]},$$

where $p_n \in \mathbb{N}$, the α_i are real numbers, and $\left\{t_i^{(n)}\right\}$ is an increasing sequence in $\mathcal{T} = [0, T]$. The Wiener integral is defined as

$$I\left(\theta^n\right) = \int_0^T \theta^n(s)\, dB(t) = \sum_{i=1}^{p_n} \alpha_i \left(B(t_{i+1}) - B(t_i)\right).$$

Due to the Gaussian nature of Brownian motion and the independence of its increments, the random variable $I(\theta^n)$ is a Gaussian variable with zero mean and variance

$$\begin{aligned} Var\left(I\left(\theta^n\right)\right) &= \sum_{i=1}^{p_n} \alpha_i^2 Var\left(W(t_{i+1}) - W(t_i)\right) \\ &= \sum_{i=1}^{p_n} \alpha_i^2 \left(t_{i+1} - t_i\right) \\ &= \int_0^T \left(\theta^n(s)\right)^2 ds \end{aligned}$$

Remark 1.2.1 *We notice that $\theta \rightarrow I(\theta)$ is a linear function. Moreover, if f and g are two step functions, we have*

$$\mathbb{E}\left[I\left(f\right) I\left(g\right)\right] = \int_0^T f\left(s\right) g\left(s\right) ds$$

Then we speak of the isometry property of the Wiener integral. Now let $\theta \in L^2([0,T], \mathbb{R})$. Therefore, there exists a sequence of step functions $\{\theta^n, n \geq 0\}$ that converges in $L^2([0,T], \mathbb{R})$ to θ. According to the previous paragraph, we can construct the Wiener integrals $I(\theta^n)$, which are centered Gaussians that form a Cauchy sequence by isometry. Since the space $L^2([0,T], \mathbb{R})$ is complete, this sequence converges to a Gaussian random variable denoted by $I(\theta)$. It can be shown that the limit does not depend on the choice of the sequence $\theta^n, n \geq 0$. $I(\theta)$ is called the Wiener integral of θ with respect to $(B(t))_{t \in \mathbb{R}}$.

1.2.2 The stochastic integral or Itô's integral

We now aim to define the integral (1.2), and the construction of $I(\theta)$ is done through discretization, as in the case of the Wiener integral.

First, let us consider the step processes of the form

$$\theta^n(t) = \sum_{i=0}^{p_n} \theta_i \mathbf{1}_{\left[t_i^{(n)}, t_{i+1}^{(n)}\right]}(t), \tag{1.3}$$

where $p_n \in \mathbb{N}$, $\left\{t_i^{(n)}\right\}$ is an increasing sequence in $\mathcal{T} = [0, T]$, and $\theta_i \in L^2(\Omega, \mathcal{F}_{t_i}, \mathbb{P})$ for all $i = 0, \cdots, p_n$. We define $I(\theta^n)$ as

$$I(\theta^n) = \sum_{i=1}^{p_n} \theta_i \left(B(t_{i+1}) - B(t_i)\right)$$

It can be verified that

$$\mathbb{E}\left[I(\theta^n)\right] = 0, \text{ and } Var\left(I(\theta^n)\right) = \mathbb{E}\left[\int_0^T \left(\theta^n(s)\right)^2 ds\right]$$

Let H be the space of caglad (i.e., left-continuous and right-limited), $\mathcal{F}_t$-adapted processes θ such that

$$\|\theta\|^2 = \mathbb{E}\left[\int_0^T |\theta(s)|^2 ds\right] < \infty$$

We can define $I(\theta)$ for any $\theta \in H$. We approximate θ by a sequence of step processes given by (1.3), and the limit is in $L^2(\Omega, [0, T])$. The integral $I(\theta)$ is then defined as $\lim_{n \to +\infty} I(\theta^n)$, where

$$\mathbb{E}\left[I(\theta)\right] = 0$$

and

$$Var\left(I\left(\theta\right)\right) = \mathbb{E}\left[\int_0^T \theta^2(s)ds\right]$$

Definition 1.2.1 (Itô Process) *A real-valued process $\mathbb{X} = (X(t))_{t\in\mathcal{T}}$ is an Itô process if $\mathbb{P}.p.s$*

$$X(t) = X(0) + \int_0^t b(s)ds + \int_0^t \sigma(s)dB(s),\ \forall\, 0 \leq t \leq T \tag{1.4}$$

Where $X(0)$ is $\mathcal{F}_0$-measurable, b and σ are two progressively measurable processes satisfying the conditions $\mathbb{P}.a.s$

$$\int_0^T |b(s)|\, ds < \infty \text{ and } \int_0^T |\sigma(s)|^2\, ds < \infty,$$

In other words, $b \in L^1_{loc}(\mathbb{F})$ and $\sigma \in L^2_{loc}(\mathbb{F})$.

The coefficient b is the drift or derivative and σ is the diffusion coefficient.

Proposition 1.2.1 (Integration by parts) *If X and Y are two Itô processes,*

$$X(t) = X(0) + \int_0^t b_1(s)ds + \int_0^t \sigma_1(s)dB(s),$$
$$Y(t) = Y(0) + \int_0^t b_2(s)ds + \int_0^t \sigma_2(s)dB_s,$$

then,

$$X(t)Y(t) = X(0)Y(0) + \int_0^t X(s)dY(s) + \int_0^t Y(s)dX(s) + \langle X, Y\rangle_t$$

such that,

$$\langle X, Y\rangle_t = \int_0^t \sigma_1(s)\sigma_2(s)ds, \text{ and } dX(t) = b(t)dt + \sigma(t)dB(t).$$

Theorem 1.2.1 (Itô's formula) *Let $b \in L^1_{loc}(\mathbb{F})$, $\sigma \in L^2_{loc}(\mathbb{F})$ and let $\mathbb{X}$ be an Itô process defined as in (1.4), let $\langle X(t) \rangle := \int_0^t |\sigma(s)|^2 \, ds$.*
Let $f \in C^{1,2}(\mathcal{T} \times \mathbb{R}, \mathbb{R})$, then

$$\begin{aligned} df(t, X(t)) &= \partial_t f(t, X(t))\, dt + \partial_x f(t, X(t))\, dX(t) + \frac{1}{2}\partial_{xx} f(t, X(t))\, d\langle X \rangle_t \\ &= \left[\partial_t f + \partial_x f b(t) + \frac{1}{2}\partial_{xx} f |\sigma(t)|^2\right](t, X(t))\, dt \\ &+ \partial_x f(t, X(t))\, \sigma(t) dB(t). \end{aligned}$$

Or alternatively, in integral form

$$\begin{aligned} f(t, X(t)) = f(0, X(0)) &+ \int_0^t \left[\partial_t f + \partial_x f b(t) + \frac{1}{2}\partial_{xx} f |\sigma(t)|^2\right](s, X(s)) ds \\ &+ \int_0^t \partial_x f(s, X(s))\, \sigma(s) dB(s). \end{aligned}$$

We finish this paragraph by extending the previous formula to the case of a d-dimensional Brownian motion.

Theorem 1.2.2 *Let $B = \left(B^1, \cdots, B^d\right)^\top$ be an BM d-dimensional,*

$$b^i \in L^1_{loc}(\mathbb{F}),\ \sigma^{i,j} \in L^2_{loc}(\mathbb{F}),\ 1 \le i \le n,\ 1 \le j \le d$$

We denote $b = (b^1, \cdots, b^n)^\top$ and $\sigma := (\sigma^{i,j})_{1 \le i \le n, 1 \le j \le d}$, which take values in $\mathbb{R}^n$ and $\mathbb{R}^{n \times d}$ respectively. Let $\mathbb{X} = (X^1, \cdots, X^n)$ be an Itô's process taking values in $\mathbb{R}^n$ such that

$$dX^i(t) := b^i(t)dt + \sum_{j=1}^d \sigma^{i,j}(t) dB^j(t), \quad i = 1, \cdots, n.$$

Chapter 2

Existence and uniqueness of the solution of the stochastic deferential equation

2.1 Diffusion mean-field SDE involving impulse control

We choose $T > 0$ and represent $\mathcal{T} = [0,T]$, $\mathbb{R}^* = \mathbb{R} - \{0\}$ and $\mathcal{B}(\mathbb{R}^*)$ as the Borel σ-field generated by open subsets of $\mathbb{R}^*$, where the closure of such subsets does not include the point 0. Let $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ be a filtered probability space satisfying the usual conditions, i.e., the right-continuity and the $\mathbb{P}$-completeness of the filtration $\mathbb{F} := \{\mathcal{F}_t \mid t \in \mathcal{T}\}$. $\mathbb{F}$ is the right-continuous, $\mathbb{P}$-complete, natural filtration generated by Brownian motion. We assume that $\mathcal{F}_T = \mathcal{F}$ for convenience.

- $S^2(\mathcal{T}, \mathbb{R}^n)$: The space of $\mathbb{R}^n$-valued $\mathbb{F}$-adapted càdlàg processes $\{X(t) : t \in [0,T]\}$ such that $\mathbb{E}\left[\sup_{0\leq t\leq T} |X^2(t)|\right] < \infty$.

- $L^2(\mathcal{F}_T, \mathbb{R}^n)$: The space of all the $\mathcal{F}_T$-measurable random variables $X : \Omega \longrightarrow \mathbb{R}^n$ such that $\mathbb{E}\left[|X(t)|^2\right] < \infty$.

- $L^2_\beta\left(\mathcal{T}, \mathbb{R}^{n\times d}\right)$: The space of all $\mathbb{R}^{n\times d}$-value $\mathbb{F}$-progressively measurable processes $\{v(t) : t \in [0,T]\}$ such that $\mathbb{E}\left[\int_0^T \mathrm{e}^{-\beta t}|v(t)|^2 dt\right] < \infty, \forall\beta > 0$.

- $L^2\left(\mathcal{T}, \mathbb{R}^{n\times d}\right)$: The space of all $\mathbb{R}^{n\times d}$-value $\mathbb{F}$-progressively measurable processes $\{v(t) : t \in [0,T]\}$ such that $\mathbb{E}\left[\int_0^T |v(t)|^2 dt\right] < \infty$.

- $L^\infty(\Omega, \mathcal{F}_t, \mathbb{P}, \mathbb{R}^{n\times m})$: The space of $\mathbb{R}^{n\times m}$-value $\mathcal{F}_t$-measurable bounded processes.

$\mathcal{I}(\mathcal{T}, \mathbb{R}^n)$: The class of processes $\xi(\cdot) = \sum_{i\geq 0}\xi_i \mathbf{1}_{[\tau_i,T]}$ such that each ξ_i is $\mathbb{R}^n$-valued $\mathcal{F}_{\tau_i}$ measurable random variable, $\mathbb{E}\left[\sum_{i=0}^\infty |\eta_i|^2\right] < \infty$. Assuming $\tau_i \to \infty$ implies that at most finitely many impulses may occur on $\mathcal{T}$.

$$M^2\left(\mathcal{T}, \mathbb{R}^n \times \mathbb{R}^{n\times d}\right) = S^2\left(\mathcal{T}, \mathbb{R}^k\right) \times L^2\left(\mathcal{T}, \mathbb{R}^{k\times d}\right).$$

In the remaining part of this chapter, we present certain conditions that are necessary for the existence and uniqueness of a solution to a diffusion mean-field stochastic differential equation with impulse control. b, σ are mappings measurable with respect to $\mathcal{F}_t$.

Assumption H$_1$ b, σ are Lipschitz with respect to $x, \bar{x}$ and exhibit linear growth in $(x, \bar{x})$, i.e., $\exists\, c > 0$ such that

$$|b(t, x_1, \bar{x}_1) - b(t, x_2, \bar{x}_2)| + |\sigma(t, x_1, \bar{x}_1) - \sigma(t, x_2, \bar{x}_2)| \leq c\left(|x_1 - x_2| + |\bar{x}_1 - \bar{x}_2|\right),$$

and

$$|b(t, x, \bar{x})| + |\sigma(t, x, \bar{x})| \leq c(1 + |x| + |\bar{x}|).$$

Given the following stochastic differential equation:

$$\begin{cases} dX(t) = b(t, X(t), \mathbb{E}[X(t)])dt + \sigma(t, X(t), \mathbb{E}[(X(t)])dB(t) + C(t)d\xi(t) \\ X(0) = X_0 \end{cases} \tag{2.1}$$

Here b, σ are $\mathbb{F}$-measurable functions satisfying

$$b : \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n, \sigma : \Omega \times [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n \times d},$$
$$C : \Omega \times [0, T] \to \mathbb{R}^{n \times m}.$$

$B(t) = (B_1(t), B_2(t), \cdots, B_d(t))$ is d-dimensional standard Brownian motion.

Lemma 2.1.1 *Let* $C(\cdot) \in L^\infty (\Omega, \mathcal{F}_t, \mathbb{P}, \mathbb{R}^{n \times m})$ *be continuous. Under Assumption* H_1 *the SDE*(2.1) *has a unique solution* $X(\cdot) \in S^2 (\mathcal{T}, \mathbb{R}^n)$

Proof. If we assume that $\eta(\cdot) \equiv 0$ for all $t \in [0, T]$, the equation (2.1) transforms into a classical SDE without impulses, and the result remains valid based on the diffusion theory (see [7]).

Let

$$h(t) = \int_0^t C(s)d\xi(s) = \sum_{\tau_i \le t} C(\tau_i)\, \xi_i, \quad Y(t) = X(t) - h(t),$$
$$\widetilde{b}(t, X(t), \mathbb{E}[X(t)]) = b(t, X(t) + h(t), \mathbb{E}[X(t) + h(t)]),$$
$$\widetilde{\sigma}(t, X(t), \mathbb{E}[X(t)]) = \sigma(t, X(t) + h(t), \mathbb{E}[X(t) + h(t)]).$$

Then we have

$$\begin{cases} dY(t) = \widetilde{b}(t, Y(t), \mathbb{E}[Y(t)])dt + \widetilde{\sigma}(t, Y(t), \mathbb{E}[Y(t)])dB(t), \\ Y(0) = X_0. \end{cases} \tag{2.2}$$

It is easy to verify that $\widetilde{b}, \widetilde{\sigma}$ satisfy Assumption H_1.

Step 1 For arbitrary $y(\cdot) \in L^2(\mathcal{T}, \mathbb{R}^n)$, consider the following SDE

$$dY(t) = \widetilde{b}(t, Y(t), \mathbb{E}[y(t)])dt + \widetilde{\sigma}(t, Y(t), \mathbb{E}[y(t)])dB(t). \tag{2.3}$$

The existence and uniqueness of the solution of SDE (2.3) are provided by Theorem 1.19 in [6].

Step 2 We construct a mapping from SDE (2.3) into itself, i.e., $l(y(\cdot)) \to Y(\cdot)$ and l is a contractive mapping. Indeed, for any $y_1(\cdot), y_2(\cdot) \in L^2(\mathcal{T}, \mathbb{R}^n), Y_1(\cdot) = l(y_1(\cdot)), Y_2(\cdot) = l(y_2(\cdot)), \widehat{y}(\cdot) = y_1(\cdot) - y_2(\cdot), \widehat{Y}(\cdot) = Y_1(\cdot) - Y_2(\cdot)$. Applying Itô's formula to $\mathrm{e}^{-\beta t}|\widehat{Y}(t)|^2$, yielding

$$\begin{aligned}
\mathrm{e}^{-\beta t}|\widehat{Y}(t)|^2 = & \int_0^t \mathrm{e}^{-\beta s}\widehat{Y}^{\mathrm{T}}(s)dY(s) + \frac{1}{2}\int_0^t \mathrm{e}^{-\beta s}\mathrm{Tr}\left(\widehat{Y}^{\mathrm{T}}(s)\widehat{Y}^{\mathrm{T}}(s)\right)ds \\
& + \int_0^t \mathrm{e}^{-\beta s}\widehat{Y}^{\mathrm{T}}(s)\left|\widetilde{b}(s, Y_1(s), \mathbb{E}[y_1(s)]) - \widetilde{b}(s, Y_2(s), \mathbb{E}[y_2(s)])\right| ds \\
& + \int_0^t \mathrm{e}^{-\beta s}\left|\widetilde{\sigma}(s, Y_1(s), \mathbb{E}[y_1(s)]) - \widetilde{\sigma}(s, Y_2(s), \mathbb{E}[y_2(s)])\right|^2 ds,
\end{aligned}$$

taking the mathematical expectation of both sides of the equation obtained, the result follows from Lipschitz condition

$$\begin{aligned}
& \mathbb{E}\left[\mathrm{e}^{-\beta t}|\widehat{Y}(t)|^2\right] + \mathbb{E}\left[\int_0^t \beta \mathrm{e}^{-\beta s}|\widehat{Y}(s)|^2\right] ds \\
= & 2\mathbb{E}\left[\int_0^t \mathrm{e}^{-\beta s}\widehat{Y}^{\mathrm{T}}(s)\left|\widetilde{b}(s, Y_1(s), \mathbb{E}[y_1(s)]) - \widetilde{b}(s, Y_2(s), \mathbb{E}[y_2(s)])\right| ds\right] \\
& + \mathbb{E}\left[\int_0^t \mathrm{e}^{-\beta s}\left|\widetilde{\sigma}(s, Y_1(s), \mathbb{E}[y_1(s)]) - \widetilde{\sigma}(s, Y_2(s), \mathbb{E}[y_2(s)])\right|^2 ds\right] \\
\leq & \left(6c^2 + 1\right)\mathbb{E}\left[\int_0^t \mathrm{e}^{-\beta s}|\widehat{Y}(s)|^2 ds\right] + 6c^2\mathbb{E}\left[\int_0^t \mathrm{e}^{-\beta s}|\widehat{y}(s)|^2 ds\right].
\end{aligned}$$

We get

$$\left(\beta - 6c^2 - 1\right)\mathbb{E}\left[\int_0^T \mathrm{e}^{-\beta s}|\widehat{Y}(s)|^2 ds\right] \leq 6c^2\mathbb{E}\left[\int_0^T \mathrm{e}^{-\beta s}|\widehat{y}(s)|^2 ds\right].$$

Let β be $18c^2+1$. We have

$$\mathbb{E}\left[\int_0^T \mathrm{e}^{-\beta s}|\widehat{Y}(s)|^2 ds\right] \leq \frac{6c^2}{\beta-6c^2+1}\mathbb{E}\left[\int_0^T \mathrm{e}^{-\beta s}|\widehat{y}(s)|^2 ds\right],$$

since $\frac{6c^2}{\beta-6c^2+1}<1$, l is a contractive mapping in $L^2_\beta(\mathcal{T},\mathbb{R}^n)$. We define $\mathcal{L}_\beta[0,T]$ as the Banach space $\mathcal{L}_\beta[0,T]=L^2_\beta(\mathcal{T},\mathbb{R}^n)$, with the norm $E\left[\int_0^T \mathrm{e}^{-\beta t}|v(t)|^2 dt\right]$. Since $0<T<\infty$, all the norms $|\cdot|_{\mathcal{L}_\beta[0,T]}$ with different β are equivalent. By applying the fixed-point theorem, the mapping has a unique fixed point $Y(\cdot)=l(Y(\cdot))$. The existence and uniqueness of the solution of SDE (2.3) imply the existence and uniqueness of the solution of SDE (2.2). Since $Y(t)=X(t)-h(t)$ is invertible, SDE (2.1) has a unique solution.

Step 3 Using Cauchy-Schwartz inequality, we have the following inequality

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\int_0^t b(s,X(s),\mathbb{E}[X(s)])ds\right|^2\right] \leq T\,\mathbb{E}\left[\int_0^T |b(s,X(s),\mathbb{E}[X(s)])|^2\, ds\right].$$

Using Doob martingale inequality, Itô's isometry and Burkhölder-Davis-Gundy inequality, we obtain the following results respectively

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\int_0^t \sigma(s,X(s),\mathbb{E}[X(s)])dB(s)\right|^2\right] \\
&\leq 4\mathbb{E}\left[\left|\int_0^T \sigma(s,X(s),\mathbb{E}[X(s)])dB(s)\right|^2\right], \\
&= 4\mathbb{E}\left[\int_0^T |\sigma(s,X(s),\mathbb{E}[X(s)])|^2\, ds\right].
\end{aligned}$$

The result follows from Lipschitz condition and Jensen's inequality

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T |b(s,X(s),\mathbb{E}[X(s)])|^2\,ds\right]+\mathbb{E}\left[\int_0^T |\sigma(s,X(s),\mathbb{E}[X(s)])|^2\,ds\right]\\
\leq &c\mathbb{E}\left[\int_0^T |X(s)|^2 ds\right]+c\mathbb{E}\left[\int_0^T |b(s,0,0)|^2 ds\right]+c\mathbb{E}\left[\int_0^T |\sigma(s,0,0)|^2\right],\\
&+\mathbb{E}\sum_{\tau_i\leq T}\left[C(\tau_i)\,\xi_i\right]^2\Bigg\}<\infty,
\end{aligned}$$

Then

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\leq t\leq T}|X(t)|^2\right]\leq L\left\{\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\int_0^t b(s,X(s),\mathbb{E}[X(s)])ds\right|^2\right]\right.\\
&+\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\int_0^t \sigma(s,X(s),\mathbb{E}[X(s)])dB(s)\right|^2\right]\\
&\left.+\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\int_0^t C(s)d\xi(s)\right|^2+\mathbb{E}\left[X_0\right]^2\right]\right\},\\
\leq &L\left\{\mathbb{E}\left|X_0\right|^2+T\mathbb{E}\left[\int_0^T |b(s,X(s),\mathbb{E}[X(s)])|^2 ds\right]\right.\\
&\left.+4\mathbb{E}\left[\int_0^T |\sigma(s,X(s),\mathbb{E}[X(s)])|^2 ds\right]+\mathbb{E}\sum_{\tau_i\leq T}\left[C(\tau_i)\,\xi_i\right]^2\right\},\\
&\leq L\left\{\mathbb{E}\left|X_0\right|^2+\mathbb{E}\left[\int_0^T\left(|X(s)|^2+|b(s,0,0)|^2+|\sigma(s,0,0)|^2\right)ds\right]\right.,
\end{aligned}$$

where L is a constant which will change from line to line. We obtain $X(\cdot)\in S^2(\mathcal{T},\mathbb{R}^n)$. ■

Assumption H_2

$\mathrm{H}_{2.1}:\phi(\cdot,0,0,0,0,0,0)\in L^2(\mathcal{F},R^n)$

$\mathrm{H}_{2.2}:\phi$ is uniformly Lipschitz, i.e., $\exists c>0$ such that $\forall t,\chi^1=(y^1,z^1,v^1)$, $\chi^2=(y^2,z^2,v^2)$, $\bar{\chi}^1=(\bar{y}^1,\bar{z}^1,\bar{v}^1)$, $\bar{\chi}^2=(\bar{y}^2,\bar{z}^2,\bar{v}^2)$,

$|\phi(t,\chi^1,\bar{\chi}^1)-\phi(t,\chi^2,\bar{\chi}^2)|\leq c\,(|y^1-y^2|+|z^1-z^2|+|v^1-v^2|_\nu+|\bar{y}^1-\bar{y}^2|$ $+|\bar{z}^1-\bar{z}^2|+|\bar{v}^1-\bar{v}^2|_\nu)$.

Lemma 2.1.2 *Under Assumption* H_2*, the given BSDE (2.4) has a unique* *solution* $(Y(\cdot), Z(\cdot)) \in M^2\left(\mathcal{T}, R^n \times R^{n\times d}\right)$.

$$\begin{cases} dY(t) = -\phi(t, Y(t), Z(t), E[Y(t)], E[Z(t)])dt + Z(t)dB(t) + C(t)d\xi(t) \\ Y(T) = \zeta \end{cases} \tag{2.4}$$

where ζ *is a square-integrable and* $\mathcal{F}_T$*-measurable random variable.*

Proof. Lemma (2.1.2) can be proved by the fixed-point theorem.

Let

$$\begin{cases} d\tilde{Y}(t) = \widetilde{\phi}(t, \tilde{Y}(t), Z(t), E[\tilde{Y}(t)], E[Z(t)])dt + Z(t)dB(t) \\ \tilde{Y}(T) = \zeta - h(T). \end{cases} \tag{2.5}$$

where $h(t) = \int_0^t C(s)d\xi(s) = \sum_{\tau_i \leq t} C(\tau_i)\xi_i, \quad \widetilde{Y}(t) = Y(t) - h(t)$, and

$$\widetilde{\phi}(t, Y(t), Z(t), E[Y(t)], E[Z(t)]) = \phi(t, Y(t)+h(t), Z(t), E[Y(t)+h(t)], E[Z(t)])$$

According to Theorem 3.1 in [6], the mean-field BSDE (2.5) has a unique solution. We note that $\tilde{Y}(t) = Y(t) - h(t)$ is an invertible function. Consequently, we obtain the solution $(Y(\cdot), Z(\cdot)) \in M^2\left(\mathcal{T}, R^n \times R^{n\times d}\right)$. ■

Chapter 3

The stochastic maximum principle for an impulsive optimal control problem

3.1 Stochastic maximum principle

We refer to the following problem as an optimal control problem of a diffusion mean-field model with impulse control. The state process $X(\cdot) := \{X(t) : t \in [0,T]\}$ is described by a diffusion mean-field stochastic differential equation with impulse control

$$\begin{cases} dX(t) = b(t, X(t), \mathbb{E}[X(t)], u(t))dt + \sigma(t, X(t), \mathbb{E}[X(t)], u(t))dB(t) \\ \qquad\qquad + C(t)d\xi(t) \\ X(0) = X_0. \end{cases} \tag{3.1}$$

The goal of the controller is to minimize the expected cost function, which is determined by the control inputs to the system

$$J(u(\cdot),\xi(\cdot)) \\ = \mathbb{E}\left[\int_0^T f(t,X(t),\mathbb{E}[X(t)],u(t))dt + g(X(T),\mathbb{E}[X(T)]) + \sum_{i\geq 0} l(\tau_i,\xi_i)\right]. \tag{3.2}$$

Here $f:[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\times U\to\mathbb{R}, g:\mathbb{R}^n\times\mathbb{R}^n\to\mathbb{R}, l:[0,T]\times U\to\mathbb{R}$. Suppose that the controller aims to minimize the cost functional J by selecting an appropriate admissible control $(u(\cdot),\xi(\cdot))$ such that

$$J(u(\cdot),\xi(\cdot))) = \inf_{(v,\eta)\in\mathcal{A}} J(v(\cdot),\eta(\cdot)),$$

where $\mathcal{A}=\mathcal{U}\times\mathcal{I}$ is called an admissible control set. $u:\mathcal{T}\times\Omega\to U$ is defined within a non-empty, closed, and convex set $\mathcal{U}$, ($\mathcal{U}$: A non-empty convex set of $\mathbb{R}^n$). We require that the control process $\{u(t)\mid t\in\mathcal{T}\}$ is $\mathbb{F}$-predictable and has right limits.

Assumption $\mathbf{H}_3$

$\mathrm{H}_{3.1}$: b,σ are continuously differentiable, Lipschitz in $(x,\bar{x},u)$, and exhibit a linear growth in $(x,\bar{x},u)$, i.e., for any $(x,\bar{x}_1,u_1),(x_2,\bar{x}_2,u_2)$, $\exists\, c$ such that:

$$\begin{aligned}|b(t,x_1,\bar{x}_1,u_1)-b(t,x_2,\bar{x}_2,u_2)|+|\sigma(t,x_1,\bar{x}_1,u_1)-\sigma(t,x_2,\bar{x}_2,u_2)|\\ \leq c\left(|x_1-x_2|+|\bar{x}_1-\bar{x}_2|+|u_1-u_2|\right),\end{aligned}$$

and

$$|b(t,x,\bar{x},u)|+|\sigma(t,x,\bar{x},u)|\leq c(1+|x|+|\bar{x}|+|u|).$$

$\mathrm{H}_{3.2}$: l is continuously differentiable in ξ and $l(\tau,\xi)\leq c(1+|\xi|)$.

The objective of the rest of this section is to present the stochastic maximum principle for an optimal control problem of diffusion mean-field model involving impulse control. Since $\mathcal{A}$ is convex, a convex perturbation technique will be used to establish a necessary condition for the above optimal control problem. We will provide a verification theorem for the necessary condition.

Lemma 3.1.1 *Under Condition* H_3*, the mean-field SDE (3.1) has a unique solution* $X(\cdot) \in S^2(\mathcal{T}, \mathbb{R}^n)$.

Proof. The result follows from Lemma (2.1.1) ■

The Hamiltonian function is defined as follows $H : \Omega \times \mathcal{T} \times \mathbb{R}^n \times \mathbb{R}^n \times U \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \longrightarrow \mathbb{R}$

$$H(t, x, \bar{x}, u, p, q) = f(t, x, \bar{x}, u) + b^{\mathrm{T}}(t, x, \bar{x}, u)p + \operatorname{tr}\left[\sigma^{\mathrm{T}}(t, x, \bar{x}, u)q\right]. \quad (3.3)$$

Assumption H$_4$

$\mathrm{H}_{4.1}$: f, b, σ are continuously differentiable in $(x, \bar{x}, u)$, g is continuously differentiable in $(x, \bar{x})$, while l is continuously differentiable in ξ.

$\mathrm{H}_{4.2}$: The derivative of b and σ are bounded. The derivative of f, g and l are bounded by $C(1+ |x| + |\bar{x}| + |u|)$, $C(1 + |x| + |\bar{x}|)$ and $C(1 + |\xi|)$, respectively.

We denote $\psi(t) = \psi(t.X(t), \mathbb{E}[X(t)], u(t))$, for $\psi = b, \sigma, b_x, b_{\bar{x}}, b_u, \sigma_x, \sigma_{\bar{x}}, \sigma_u, f$, $f_x, f_u, H(t) = H(t, x, \bar{x}, u, p, q, r)$. We introduce the adjoint equation

$$\begin{cases} dp(t) = -\left(\nabla_x H(t) + \mathbb{E}\left[\nabla_{\bar{x}} H(t)\right]\right) dt + q(t)dB(t), \\ p(T) = \nabla_x g(X(T), \mathbb{E}[X(T)]) + \mathbb{E}\left[\nabla_{\bar{x}} g(X(T), \mathbb{E}[X(T)])\right]. \end{cases} \quad (3.4)$$

It is evident that (3.4) has a unique solution $(p(\cdot), q(\cdot)) \in M^2\left(\mathcal{T}, \mathbb{R}^n \times \mathbb{R}^{n\times d}\right)$ under Assumption H_4.

Let $\left(u(\cdot), \xi(\cdot) = \sum_{i\geq 0} \xi_i \mathbf{1}_{[\tau_i,T]}\right)$ be the optimal control for the considered stochastic optimal control problem. The control $\left(v(\cdot), \eta(\cdot) = \sum_{i\geq 0} \eta_i \mathbf{1}_{[\tau_i,T]}\right)$ ensures $u(\cdot) + v(\cdot) \in \mathcal{U}, \xi(\cdot) + \eta(\cdot) \in \mathcal{I}$. due to the convexity of $\mathcal{U}$ and $\mathcal{I}$. By selecting an arbitrary $\varepsilon > 0$, $u^{\varepsilon}(\cdot) = u(\cdot) + \varepsilon v(\cdot) \in \mathcal{U}$ and $\xi^{\varepsilon}(\cdot) = \xi(\cdot) + \varepsilon\eta(\cdot) \in \mathcal{I}$. $X^{\varepsilon}(\cdot)$ represents the corresponding trajectory of $(u^{\varepsilon}(\cdot), \xi^{\varepsilon}(\cdot))$. We introduce the following variational equation:

$$\begin{cases} dX^1(t)^{\mathrm{T}} = \left[X^1(t)^{\mathrm{T}} b_x(t) + \mathbb{E}\left[X^1(t)^{\mathrm{T}}\right] b_{\bar{x}}(t) + v(t)^{\mathrm{T}} b_u(t)\right] dt \\ \qquad\qquad + \left[X^1(t)^{\mathrm{T}} \sigma_x(t) + \mathbb{E}\left[X^1(t)^{\mathrm{T}}\right] \sigma_{\bar{x}}(t) + v(t)^{\mathrm{T}} \sigma_u(t)\right] dB(t) \\ \qquad\qquad + C(t) d\eta(t), \\ X^1(0) = 0. \end{cases} \tag{3.5}$$

By Assumption H_4, Equation (3.5) has a unique solution. Denote $\widetilde{X}(t) = \frac{X^{\varepsilon}(t) - X(t)}{\varepsilon} - X^1(t)$.

Lemma 3.1.2

$$\lim_{\varepsilon \to 0} \sup_{0 \le t \le T} \mathbb{E}\left[|\widetilde{X}(t)|^2\right] = 0.$$

Proof. Since $\widetilde{X}(t)$ is independent of the impulse term, the corresponding result follows from Lemma 4.3 in [8]. ■

Since $(u(\cdot), \xi(\cdot))$ is an optimal control, it is clear that

$$\varepsilon^{-1}\left[J\left(u^{\varepsilon}(\cdot), \xi^{\varepsilon}(\cdot)\right) - J(u(\cdot), \xi(\cdot))\right] \ge 0.$$

We find the following variational inequality.

Lemma 3.1.3 *Suppose $(u(\cdot), \xi(\cdot))$ is an optimal control, then*

$$\mathbb{E}\Bigg[X^1(T)^{\mathrm{T}}\left(\nabla_x g(X(T), \mathbb{E}[X(T)]) + \mathbb{E}\left[\nabla_{\bar{x}} g(X(T), \mathbb{E}[X(T)])\right]\right) + \int_0^T \left[X^1(t)^{\mathrm{T}} f_x(t)\right.$$
$$\left. + \mathbb{E}\left[X^1(t)^{\mathrm{T}}\right] f_{\bar{x}}(t) + f_u(t) v(t)\right] dt + \sum_{i \ge 0} l_{\xi}\left(\tau_i, \xi_i\right) \eta_i\Bigg] \ge 0$$

Proof. From Lemma (3.1.2), it is straightforward to observe that as $\varepsilon \to 0$,

$$\begin{aligned}&\varepsilon^{-1}\mathbb{E}\left[\sum_{i\geq 0} l\left(\tau_i,\xi_i^{\varepsilon}\right)-l\left(\tau_i,\xi_i\right)\right]\\&=\mathbb{E}\left[\sum_{i\geq 0}\int_0^1 l_{\xi}\left(\tau_i,\xi_i+\varepsilon\lambda\eta_i\right)\right]\eta_i d\lambda \to \mathbb{E}\left[\sum_{i\geq 0} l_{\xi}\left(\tau_i,\xi_i\right)\eta_i\right],\end{aligned}$$

also we have

$$\begin{aligned}&\varepsilon^{-1}\mathbb{E}\left[\int_0^T \left(f\left(t,X^{\varepsilon}(t),\mathbb{E}\left[X^{\varepsilon}(t)\right],u^{\varepsilon}(t)\right)-f(t,X(t),\mathbb{E}[X(t)],u(t))\right)dt\right]\\&=\mathbb{E}\left[\int_0^T\int_0^1\left\{\tfrac{(X^{\varepsilon}(t)-X(t))^{\mathrm{T}}}{\varepsilon}f_x\left(t,X(t)+\lambda\varepsilon\left(\tilde{X}(t)+X^1(t)\right),\mathbb{E}[X(t)],u(t)\right)\right.\right.\\&\quad+\tfrac{(\mathbb{E}[X^{\varepsilon}(t)]-\mathbb{E}[X(t)])^{\mathrm{T}}}{\varepsilon}f_{\bar{x}}\left(t,X(t),E\left[X(t)+\lambda\varepsilon\left(\widetilde{X}(t)+X^1(t)\right)\right],u(t)\right)\\&\quad+f_u(t,X(t),\mathbb{E}[X(t)],u(t)+\lambda\varepsilon v(t))v(t)\}\,d\lambda dt]\\&\to\mathbb{E}\left[\int_0^T\left[X^1(t)^{\mathrm{T}}f_x(t)+\mathbb{E}\left[X^1(t)^{\mathrm{T}}\right]f_{\bar{x}}(t)+f_u(t)v(t)\right]dt\right],\end{aligned}$$

and

$$\begin{aligned}&\varepsilon^{-1}\mathbb{E}\left[g\left(X^{\varepsilon}(T),\mathbb{E}\left[X^{\varepsilon}(T)\right]\right)-g\left(X^(T),\mathbb{E}\left[X(T)\right]\right)\right]\\&\quad=\mathbb{E}\left[\tfrac{(X^{\varepsilon}(t)-X(t))^{\mathrm{T}}}{\varepsilon}g_x(X(T),E[X(T)])+\tfrac{(\mathbb{E}X^{\varepsilon}(t)-\mathbb{E}[X(t)])^{\mathrm{T}}}{\varepsilon}\right]g_x(T,X(T))\Big]\\&\to\mathbb{E}X^1(T)^{\mathrm{T}}g_x(X(T),\mathbb{E}[X(T)]+\mathbb{E}\left[g_x(X(T),\mathbb{E}[X(T)])\right]\end{aligned}$$

Due to the optimality of $(u(\cdot), \xi(\cdot))$, we conclude that

$$
\begin{aligned}
0 \leq & \varepsilon^{-1}\left[J\left(u^{\varepsilon}(\cdot), \xi^{\varepsilon}(\cdot)\right)-J(u(\cdot), \xi(\cdot))\right] \\
= & \varepsilon^{-1} \mathbb{E}\left[\int_0^T f\left(t, X^{\varepsilon}(t), \mathbb{E}\left[X^{\varepsilon}(t)\right], u^{\varepsilon}(t)\right) d t+g\left(X^{\varepsilon}(T), \mathbb{E}\left[X^{\varepsilon}(T)\right]\right)+\sum_{i \geq 0} l\left(\tau_i, \xi_i^{\varepsilon}\right)\right] \\
& -\varepsilon^{-1} \mathbb{E}\left[\int_0^T f(t, X(t), \mathbb{E}[X(t)], u(t)) d t+g(X(T), \mathbb{E}[X(T)])+\sum_{i \geq 0} l\left(\tau_i, \xi_i\right)\right] \\
\rightarrow & \mathbb{E}\left[X^1(T)^{\mathrm{T}}\left(\nabla_x g(X(T), \mathbb{E}[X(T)])+\mathbb{E}\left[\nabla_{\bar{x}} g(X(T), \mathbb{E}[X(T)])\right]\right)\right. \\
& \left.+\int_0^T\left[X^1(t)^{\mathrm{T}} f_x(t)+\mathbb{E}\left[X^1(t)^{\mathrm{T}}\right] f_{\bar{x}}(t)+f_u(t) v(t)\right] d t+\sum_{i \geq 0} l_{\xi}\left(\tau_i, \xi_i\right) \eta_i\right]
\end{aligned}
$$

The proof is complete. ∎

Theorem 3.1.1 *Under Assumptions* $\mathrm{H}_1, \mathrm{H}_2, \mathrm{H}_3, \mathrm{H}_4, (u(\cdot), \xi(\cdot))$ *is an optimal control;* $(p(\cdot),\ q(\cdot))$ *is the solution of (3.4) and* $X(\cdot)$ *is the corresponding trajectory. Then* $\forall v \in \mathcal{U}, \eta \in \mathcal{I}$,

$$
\nabla_u H(t, X(t), \mathbb{E}[X(t)], u(t), p(t), q(t))(v-u(t))^{\mathrm{T}} \geq 0, \quad a.e.\ t \in \mathcal{T}, \mathbb{P}-\quad a.s. \tag{3.6}
$$

$$
\mathbb{E}\left[\sum_{i \geq 0}\left[l_{\xi}\left(\tau_i, \xi_i\right)+p\left(\tau_i\right) C\left(\tau_i\right)\right] \mathbf{1}_{0 \leq \tau_i \leq T}\left(\eta_i-\xi_i\right)\right] \geq 0 \tag{3.7}
$$

Proof. Applying Itô's formula to $p(t)^{\mathrm{T}} X^1(t)$ and combining with Lemma

(3.1.3), we obtain

$$\begin{aligned}
&\mathbb{E}\big[X^1(T)^{\mathrm{T}}p(T)-X^1(0)^{\mathrm{T}}p(0)\big]\\
&=\mathbb{E}\bigg[\int_0^T X^1(t)^{\mathrm{T}}\left[-\nabla_x H(t,X(t),\mathbb{E}[X(t)],u(t),p(t),q(t)\right.\\
&\quad+\mathbb{E}\left[\nabla_{\bar{x}}H(t,X(t),\mathbb{E}[X(t)],u(t),p(t),q(t))\right]\big)\big]\,dt\\
&\quad+p(t)\left\{\left[X^1(t)^{\mathrm{T}}b_x(t)+\mathbb{E}\left[X^1(t)^{\mathrm{T}}\right]b_{\bar{x}}(t)+v(t)^{\mathrm{T}}b_u(t)\right]dt+C(t)d\eta(t)\right\}\\
&\quad+q(t)\left[X^1(t)^{\mathrm{T}}\sigma_x(t)+\mathbb{E}\left[X^1(t)^{\mathrm{T}}\right]\sigma_{\bar{x}}(t)+v(t)^{\mathrm{T}}\sigma_u(t)\right]dt\\
&=\bigg[\int_0^T\left[\nabla_u H(t,X(t),\mathbb{E}[X(t)],u(t),p(t),q(t))v(t)-X^1(t)^{\mathrm{T}}f_x(t)-\mathbb{E}\left[X^1(t)^{\mathrm{T}}\right]f_{\bar{x}}(t)\right.\\
&\qquad\left.-f_u(t)v(t)\right]dt+\sum_{i\geq 0}p\left(\tau_i\right)C\left(\tau_i\right)\eta_i\bigg].
\end{aligned}$$

Adding the identical term

$$\mathbb{E}\left[\int_0^T\left[X^1(t)^{\mathrm{T}}f_x(t)+\mathbb{E}\left[X^1(t)^{\mathrm{T}}\right]f_{\bar{x}}(t)+f_u(t)v(t)\right]dt+\sum_{i\geq 0}l_\xi\left(\tau_i,\xi_i\right)\eta_i\right]$$

to both sides and utilizing Lemma (3.1.3), we obtain

$$\begin{aligned}
\mathbb{E}\bigg[&\int_0^T\nabla_u H(t,X(t),\mathbb{E}[X(t)],u(t),p(t),q(t))v(t)dt\\
&+\sum_{i\geq 0}\left[p\left(\tau_i\right)C\left(\tau_i\right)+l_\xi\left(\tau_i,\xi_i\right)\right]\mathbf{1}_{\{0\leq\tau_i\leq T\}}\eta_i\bigg]\geq 0
\end{aligned}$$

By the freedom of choice for $v(\cdot)$ and $\eta_i, i=1,2,\cdots$, we can set $v(\cdot)\equiv 0$ and $\eta_i\equiv 0$ to obtain

$$\mathbb{E}\left[\int_0^T\nabla_u H(t,X(t),\mathbb{E}[X(t)],u(t),p(t),q(t))(\widetilde{v}(t)-u(t))dt\right]\geq 0 \qquad (3.8)$$

for all $\widetilde{v}(\cdot)\in\mathcal{U}$, and

$$\mathbb{E}\left[\sum_{i\geq 0}\left[p\left(\tau_i\right)C\left(\tau_i\right)+l_\xi\left(\tau_i,\xi_i\right)\right]\mathbf{1}_{\{0\leq\tau_i\leq T\}}\left(\eta_i-\xi_i\right)\right]\geq 0$$

for all $\eta(\cdot)=\sum_{i\geq 0}\eta_i \mathbf{1}_{[\tau_i,T]}\in\mathcal{I}$. Next we refer to [9] for the proof. For $v\in\mathcal{U}$, defined by B^v the set of $(t,\omega)\in[0,T]\times\Omega$ such that

$$\nabla_u H(t,X(t),\mathbb{E}[X(t)],u(t),p(t),q(t))(v(t)-u(t))dt<0.$$

Obviously, for each $t\in[0,T]$, $B_t^v\in\mathcal{F}_t$. Let us consider $\widetilde{v}\in\mathcal{U}$ defined by

$$\widetilde{v}(t,\omega)=\begin{cases} v, & \text{if } (t,\omega)\in B^v,\\ u(t,\omega), & \text{otherwise.}\end{cases}$$

If $(Leb\otimes\mathbb{P})(B^v)>0$, then it follows that

$$\mathbb{E}\left[\int_0^T \nabla_u H(t,X(t),\mathbb{E}[X(t)],u(t),p(t),q(t))(\widetilde{v}(t)-u(t))dt\right]<0,$$

which contradicts (3.8). Hence, we conclude that $(\text{Leb}\otimes\mathbb{P})(B^v)=0$, and the statements (3.6), (3.7) remain valid. ■

Theorem 3.1.2 *Under Assumptions* $\mathrm{H}_1,\mathrm{H}_2,\mathrm{H}_3$*, we assume that* l,g,H *are convex with respect to* $\xi,(x,\bar{x})$ *and* $(x,\bar{x},u)$*, respectively.* $X(\cdot)$ *is the corresponding trajectory of* $(u(\cdot),\xi(\cdot))\in\mathcal{A}$ *and* $(p(\cdot),q(\cdot))$ *is a unique solution of (3.4). If (3.6), (3.7) are satisfied, then* $(u(\cdot),\xi(\cdot))$ *is an optimal control process.*

Proof. For any $(v(\cdot),\eta(\cdot))\in\mathcal{U}\times\mathcal{I}$, $X^v(\cdot)$ denotes the corresponding trajectory of $(v(\cdot),\eta(\cdot))$. Consider $J(v(\cdot),\eta(\cdot))-J(u(\cdot),\xi(\cdot))$, by the convexity of $g(\cdot),l(\cdot)$, we obtain.

$$
\begin{aligned}
& J(v(\cdot),\eta(\cdot))-J(u(\cdot),\xi(\cdot)) \\
= & \mathbb{E}\bigg[\int_0^T f\left(t,X^v(t),\mathbb{E}\left[X^v(t)\right],v(t)\right)-f(t,X(t),\mathbb{E}[X(t)],u(t))dt \\
& +g\left(X^v(T),\mathbb{E}\left[X^v(T)\right]\right)-g(X(T),\mathbb{E}[X(T)])+\sum_{i\geq 0}\left(l\left(\tau_i,\eta_i\right)-l\left(\tau_i,\xi_i\right)\right)\bigg] \\
\geq & \mathbb{E}\bigg[\int_0^T f\left(t,X^v(t),\mathbb{E}\left[X^v(t)\right],v(t)\right)-f(t,X(t),\mathbb{E}[X(t)],u(t))dt \\
& +\left(X^v(T)-X(T)\right)^{\mathrm{T}}\left(\nabla_x g(X(T),\mathbb{E}[X(T)])+\mathbb{E}\left[\nabla_{\bar{x}} g(X(T),\mathbb{E}[X(T)])\right]\right) \\
& +\sum_{i\geq 0} l_\xi\left(\tau_i,\xi_i\right)\left(\eta_i-\xi_i\right)\bigg] \\
= & \mathbb{E}\bigg[\int_0^T \left[f\left(t,X^v(t),\mathbb{E}\left[X^v(t)\right],v(t)\right)-f(t,X(t),\mathbb{E}[X(t)],u(t))\right]dt \\
& +\left(X^v(T)-X(T)\right)^{\mathrm{T}} p(T)+\sum_{i\geq 0} l_\xi\left(\tau_i,\xi_i\right)\left(\eta_i-\xi_i\right)\bigg].
\end{aligned}
$$

Applying Itô's formula to $\left(X^v(T)-X(T)\right)^{\mathrm{T}} p(T)$, along with the definition of $H(\cdot)$, the convexity of $H(\cdot)$ and (3.7), we obtain.

$$
\begin{aligned}
& J(v(\cdot),\eta(\cdot))-J(u(\cdot),\xi(\cdot)) \\
\geq & \mathbb{E}\bigg[\int_0^T \left\{H\left(t,X^v(t),\mathbb{E}\left[X^v(t)\right],v(t),p(t),q(t),r(t)\right)\right. \\
& -H(t,X(t),\mathbb{E}[X(t)],u(t),p(t),q(t),r(t))) \\
& -\left(X^v(t)-X(t)\right)^{\mathrm{T}}\left(\nabla_x H(t,X(t),\mathbb{E}[X(t)],u(t),p(t),q(t),r(t))\right. \\
& \left.+\mathbb{E}\left[\nabla_{\bar{x}} H(t,X(t),\mathbb{E}[X(t)],u(t),p(t),q(t),r(t))\right]\right)\Big\}dt \\
& +\sum_{i\geq 0}\left[l_\xi\left(\tau_i,\xi_i\right)+p\left(\tau_i\right)C\left(\tau_i\right)\right]\left(\eta_i-\xi_i\right)\bigg] \geq 0.
\end{aligned}
$$

So $(u(\cdot),\xi(\cdot))$ is an optimal control. ■

3.2 Application

Suppose we have two types of securities in the market for potential investment options. A risk-free asset (e.g., a bond), whose price $S_0(t)$ at time t is provided by

$$\begin{cases} dS_0(t) = \rho(t)S_0(t)dt \\ S_0(0) = S_0. \end{cases}$$

A risky security (e.g., a stock), whose value $S(t)$ at time t is determined by

$$\begin{cases} dS(t) = S(t)\left[\mu(t)dt + \sigma(t)dB(t)\right] \\ S(0) = S \end{cases}$$

where $\rho(t) \leq \mu(t)$ and $\sigma : [0,T] \to \mathbb{R}$. The wealthy dynamics follows

$$\begin{cases} dX(t) = (\rho(t)(X(t) - u(t)) + u(t)\mu(t))dt + u(t)\sigma(t)dB(t) - d\xi(t), \\ X(0) = \beta. \end{cases}$$

$u(\cdot)$ is a portfolio strategy of agent and $X(t) = X^u(t)$ is the total wealth of the agent at time t corresponding to investment strategy $u(\cdot)$. $\xi(t) = \sum_{i\geq 0} \xi_i \mathbf{1}_{[\tau_i,T]}$ is a piecewise consumption process.

The investor chooses an investment strategy and a consumption strategy to minimize the variation and maximize the expected function. The cost functional is given by

$$J(u(\cdot),\xi(\cdot)) = \frac{a}{2}\mathrm{Var}[X(T)] - \mathbb{E}[X(T)] + \mathbb{E}\left[\frac{S}{2}\sum_{0\leq\tau_i\leq T}\xi_i^2 + \int_0^T \frac{1}{2}Q(t)u^2(t)dt\right],$$

where a is a constant, $Q(\cdot)$ is a deterministic function.

$$
\begin{aligned}
&J(u(\cdot),\xi(\cdot)) \\
&=\mathbb{E}\left[\int_0^T \frac{1}{2}Q(t)u^2(t)dt+\left[\frac{a}{2}X^2(T)-X(T)\right]-\frac{a}{2}[\mathbb{E}[X(T)]]^2+\frac{S}{2}\sum_{0\leq\tau_i\leq T}\xi_i^2\right].
\end{aligned}
$$

As we can see from (3.2), $g(x,\bar{x})=\frac{a}{2}x^2-x-\frac{a}{2}\bar{x}^2, l(\tau_i,\xi_i)=\frac{S}{2}\xi_i^2$ and $f(t,x,\bar{x},u)=\frac{1}{2}Q(t)u^2(t)$.

g is not convex in $\bar{x}$, but we have the following corollary.

Corollary 3.2.1 *If the convex condition is satisfied in expected sense, i.e., the following inequality holds, for any* $X_1,X_2\in L^2(\mathcal{F}_T,\mathbb{R}^n)$,

$$
\begin{aligned}
&\mathbb{E}\left[g\left(X_1,\mathbb{E}\left[X_1\right]\right)-g\left(X_2,\mathbb{E}\left[X_2\right]\right)\right] \\
&\leq\mathbb{E}\left[\left(X_1-X_2^{\mathrm{T}}\right)\left\{\nabla_x g\left(X_1,\mathbb{E}\left[X_1\right]\right)+\nabla_{\bar{x}} g\left(X_1,\mathbb{E}\left[X_1\right]\right)\right\}\right],
\end{aligned}
$$

the maximum principle is still valid.

Proof. This proof can refer to Corollary 4.1 in [8]. ∎

Define the Hamiltonian equation:

$$
H(t,x,u,p,q,r)=[\rho(t)(x-u)+u\mu(t)]p+u\sigma(t)q+\frac{1}{2}Q(t)u^2
$$

The associated adjoint equation is as follows

$$
\begin{cases}
dp(t)=-\rho(t)p(t)dt+q(t)dB(t)\\
p(T)=aX(T)-1-a\mathbb{E}[X(T)]
\end{cases}
$$

Then, according to (3.1.1) and Corollary (3.2.1) we have

$$u(t) = -\frac{[\mu(t) - \rho(t)]p(t) + \sigma(t)q(t)}{Q(t)}, \tag{3.9}$$

$$\xi(t) = \frac{1}{S} \sum_{0 \leq \tau_i \leq T} p(\tau_i). \tag{3.10}$$

Conclusion

In this memory, we focused on investigating diffusion mean-field stochastic differential equations with impulse control. Our primary objective was to establish the existence and uniqueness of solutions for these equations.

Additionally, we expanded our exploration to encompass the examination of maximum principles and a verification theorem, which are fundamental concepts in control theory. By formulating a necessary maximum principle, we identified the crucial conditions that must be met for optimal control in both continuous and impulse control settings.

Building upon our theoretical framework, we then addressed a mean-variance portfolio selection problem, we were able to tackle this practical problem and offer a comprehensive solution. Our findings provided a robust and reliable approach to selecting portfolios that strike a balance between expected returns and risk, taking into account the mean and variance of the portfolio's performance.

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Abstract

In this work, We have established a stochastic maximum principle controlling a mean-field model that is governed by an Itô process and utilizes both continuous and impulse control. Additionally, we demonstrate the existence and uniqueness of the solution to a diffusion mean-field stochastic differential equation that incorporates impulsive control. Concerning its practical implementation, we successfully tackle a problem related to selecting a portfolio with the aim of optimizing the balance between average returns and variance.

ملخص

في هذا العمل، قمنا بتطوير مبدأ حدي عشوائي للتحكم في نموذج متوسطي مستند إلى عملية إيتو ويستخدم كل من التحكم المستمر والتحكم المتقطع. بالإضافة إلى ذلك، نقوم بإثبات وجود ووحدانية الحل لمعادلة تفاضلية عشوائية تتضمن التحكم المتقطع. فيما يتعلق بتطبيقه العملي، تم بنجاح معالجة مشكلة تتعلق باختيار طريقة استثمار بغرض تحقيق التوازن بين متوسط العائدات والتشتت.

Résumé

Dans cette étude, nous avons établi un principe maximum stochastique pour contrôler un modèle de champ moyen qui est régi par un processus d'Itô et utilise à la fois un contrôle continu et impulsif. De plus, nous démontrons l'existence et l'unicité de la solution à une équation différentielle stochastique de diffusion de champ moyen qui intègre un contrôle impulsif. En ce qui concerne sa mise en œuvre pratique, nous résolvons avec succès un problème lié à la sélection d'un portefeuille dans le but d'optimiser l'équilibre entre les rendements moyens et la variance.