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Chaotic Systems of Fractional Order

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Dedication

I dedicate this dissertation to those who have had great credit, my sister Hadda, my husband, and my daughter, my brothers, my sisters, my stepmother, my parents, may god have mercy on them, my nieces and nephews, my husband's family.

B.Radhia

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Appendix A: Abbreviaions and Noaions

Γ :	Gamma Function
B :	Beta Function
$n!$:	Factorial function
$C\left(a,b\right]$:	The space of continuous functions
$L^1\left(a,b\right]$:	The space of integrable functions
$\|\|$:	Absolute value or long is a complex number
$\|\|\|\|$:	The norme
E_α :	The one-parameter Mittag-Leffler function
$E_{\alpha,\beta}$:	The two-parameter Mittag-Leffler function
$\mathcal{L}\left\{f(t)\right\}$:	The Laplace transform of $f(t)$
${}_aI_t^\alpha$:	The right fractional integral of order α
${}_tI_a^\alpha$:	The right fractional integral of order α
${}_tD_a^\alpha$:	Fractional derivative of Riemann-Liouville order α
${}_t^CD_a^\alpha$:	Fractional derivative of Caputo order α
${}_t^GD_a^\alpha$:	Fractional derivative of Grünwald-Letnikov order α
$Re(z)$:	Partie réelle de z

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Introduction

In a letter to L'Hospital in 1695, Leibniz asked the following question: "Can the meaning of integer-order derivatives be generalized to non-integer-order derivatives?" L'Hospital was very curious about that question and replied to Leibniz by asking what would happen to the term $\frac{d^n x(t)}{dt^n}$ if $n = \frac{1}{2}$. Leibniz wrote a letter dated 30 September 1695, in order to explain the answer to the query raised by L'Hospital, known as the birthday of fractional calculus, which mentioned "It will lead to a paradox, from which one-day useful consequences will be drawn." This was the beginning of fractional calculus. Many famous mathematicians, namely Liouville, Riemann, Weyl, Fourier, Abel, Lacroix, Leibniz, Grünwald, and Letnikov, contributed to fractional calculus over the years. Recently, various types of fractional differential and integral operators have been developed, namely the Riemann-Liowville fractional integral and derivative, Caputo fractional derivative, Grünwald-Letnikov fractional derivative, and Riesz fractional derivative [33, 40, 12, 17, 2, 15, 45, 14, 49].

Fractional calculus is a mathematical subject dating back over 300 years. But throughout these centuries, this topic developed slowly and was even neglected for some time due to the lack of a geometric interpretation of these derivatives and the lack of realistic applications for them. A few years ago, the subject received more attention from researchers and developed rapidly. Recently, more considerable interest has been attributed to the applications of fractional derivatives in several fields. Due to the availability of computers for numerical computations, fractional-order differential equations have attracted the attention of many researchers because of their interesting potential applications in many fields of science and engineering [11, 25, 27]. As many physical phenomena cannot be correctly modeled by integer differential equations, there is a need for fractional-order differential equations.

Another reason stands in the non-local property of fractional differential equations, for which the next state of a system not only depends on its current state but also on its previous states. Therefore, fractional differential equations have become popular and have been applied to many actual dynamic systems.

In addition, applications of fractional calculus have been reported in several fields such as: signal processing, image processing, automatic control and robotics, these examples, and many other similar samples perfectly clarify the importance of consideration and analysis of dynamical systems with fractional order models.

Another theory that is developing in parallel with fractional derivation, such is the theory of dynamical systems, which aims to study physical systems..., that change over time. It has its origins in the work of Henri Poincaré (1854-1912). There are important differences in many aspects between ordinary differential systems and the corresponding fractional differential systems, for example, in qualitative properties. Most of the properties of ordinary systems cannot be simply extended to the case of fractional order systems. The purpose of this thesis is study is the stability of fixed points for two discrete chaotic systems extracted from continuous chaotic systems.

The first chapter, we give some essential preliminary notions, used in the fractional derivation and contain several definitions and properties of the integration and the derivation of different types of fractional order, which are necessary in the following chapters of this work.

The second chapter, we mentioned what is related to chaotic behavior and the two mathematical principles that explain chaotic behavior, namely Devaney [20] and Li-Yorke [34], and properties of the chaotic attractor and types of attractor. We have provided some definitions of discrete dynamic system and bifurcation, local stability of fixed points, and stability of periodic orbits, exponents of Lyapunov, because dynamic behaviors depend on the instability and non-linearity of dynamic systems.

In the third chapter, we mentioned some discretization methods used to construct the discrete time model of a continuous time differential equation, such as explicit and implicit Euler's scheme [51], Taylor method [43], Runge-Kutta [8], predictor-corrector [19, 18, 21, 6], and nonstandard finite difference methods [28, 52, 29, 54, 41, 42, 44, 10, 7, 32, 39, 38], and piecewise constant approximation method [4]. Some of these are based on the approximation technique of the derivatives, and conversely, some others on the integral.

In the fourth chapter, we presented two examples of the fractional chaotic continuous system, extracted from them discrete systems using the method mentioned in the articles [23, 1, 24, 22, 3, 31, 53, 30, 35], studied the stability of the fixed points, and confirmed the validity of the theoretical results numerically.

Chapter 1

Fractional Derivatives and Fractional Integrals

1.1 Fractional Derivatives and Fractional Integrals

1.1.1 Specific Functions for Fractional Derivation

In this section, we present the Gamma and Mittag-Leffler functions, which will be used later in this work. These functions play a very important role in the theory of fractional calculus.

Gamma Function

We start by considering the Gamma function (or second order Euler integral), denoted $\Gamma(.)$ represented in Figure 1.1.

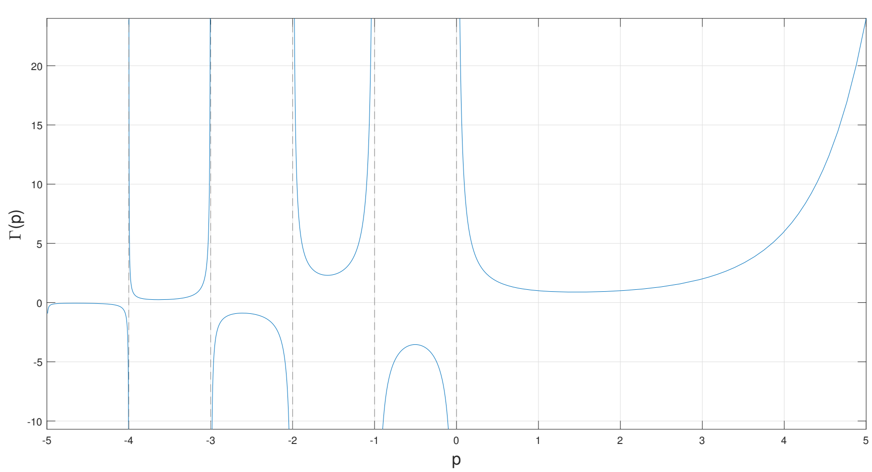


Figure 1.1: Gamma Function.

The gamma function is defined as:

$$\Gamma(p) = \int_0^\infty e^{-x} x^{p-1} dx, \tag{1.1}$$

where $p > 0$.

The basic properties of the Gamma function are:

1. The function $\Gamma(p)$ is continuous for $p > 0$.
2. The function $\Gamma(p)$ obeys the property:
$$\Gamma(p+1) = p\Gamma(p).$$
3. The following relations are also valid:
$$\begin{aligned} \Gamma(p+n) &= (p+n-1)\cdots(p+1)\,p\Gamma(p), \qquad n \in \mathbb{N}, \\ \Gamma(1) &= 1, \\ \Gamma(n+1) &= n!, \\ \Gamma(0) &= +\infty. \end{aligned}$$

Beta Function

Here we consider the Beta function, denoted Beta function B.

The Beta function, or the first order Euler function, can be defined as:

$$B(p,q) = \int_0^1 x^{p-1}(1-x)^{q-1}\,dx, \tag{1.2}$$

where $Re(p) > 0$ and $Re(q) > 0$.

The basic properties of the Beta function are:

1. For every $p > 0$ and $q > 0$ we have:
$$B(p,q) = B(q,p).$$
2. For every $p > 0$ and $q > 1$, the Beta function satisfies the property:
$$B(p,q) = \frac{q-1}{p+q-1} B(p,q-1).$$
3. For every $p > 0$ and $q > 0$, it is valid the identity:
$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$
4. For every $p > 0$, and for the natural number n, it can be proved
$$\begin{aligned} B(p,n) &= \frac{1 \cdot 2 \cdot 3 \cdots (n-1)}{p(p+1)\cdots(p+n)}, \\ B(p,1) &= \frac{1}{p}. \end{aligned}$$

Mittag-Leffler Function

The exponential function e^z, plays a very important role in the theory of integer order differential equations. Its one-parameter generalization, is the Mittag-Leffter function.

We introduce the one- and two-parameter Mittag-Leffler functions, denoted as $E_p(\cdot)$ and $E_{p,q}(\cdot)$, respectively.

- The one-parameter Mittag-Leffler function (E_p), is defined as:

$$E_p(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(pk+1)}, \qquad Re\,(p) > 0.$$

- The two-parameter Mittag-Leffler function $(E_{p,q})$, is defined as:

$$E_{p,q}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(pk+q)}, \qquad Re\,(p) > 0, Re\,(q) > 0,\ q \in \mathbb{C}.$$

For particular values of p and q it results:

$$\begin{aligned}
E_{0,1}(z) &= \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(pk+1)} = \sum_{k=0}^{\infty} z^k, \\
E_{1,1}(z) &= \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)} = e^z, \\
E_{1,2}(z) &= \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+2)} = \frac{1}{z}\sum_{k=0}^{\infty} \frac{z^{k+1}}{(k+1)!} = \frac{e^z-1}{z}, \\
E_{1,3}(z) &= \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+3)} = \frac{1}{z^2}\sum_{k=0}^{\infty} \frac{z^{k+2}}{(k+2)!} = \frac{e^z-1-z}{z^2}, \\
E_{1,0}(z) &= \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k)} = ze^z.
\end{aligned}$$

Laplace Transforms

Let $f(t)$ be a real or complex-valued function of the (time) variable $t > 0$ and s is a real or complex parameter. Then the Laplace transform of $f(t)$, denoted by $\mathcal{L}\{f(t)\}$, is defined by

$$\begin{aligned}
F(s) = \mathcal{L}\{f(t)\}(s) &= \int_0^{\infty} e^{-st} f(t)\,dt \\
&= \lim_{\tau \longrightarrow \infty} \int_0^{\tau} e^{-st} f(t)\,dt. \qquad (1.3)
\end{aligned}$$

If the integral in (1.3) is convergent at $s_0 \in \mathbb{C}$, then it converges absolutely for $s \in \mathbb{C}$ such that $Re\,(s) > Re\,(s_0)$.

The inverse Laplace transform is denoted by

$$\mathcal{L}^{-1}\{F(s)\}(t) = f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{st}F(s)\,dt, \qquad c = Re(s). \tag{1.4}$$

In which $\mathcal{L}$ and $\mathcal{L}^{-1}$ are linear integral operators.

Proposition 1.1.1 *[33, 48] If the Laplace transforms of f and g exist, then the Laplace transform of the convolution product verifie*

$$\mathcal{L}(f * g)(s) = \mathcal{L}(f)(s).\mathcal{L}(g)(s), \tag{1.5}$$

with the convolution product of f and g is defined by

$$(f * g)(t) = \int_0^t f(t-x)\,g(x)\,dx. \tag{1.6}$$

Proposition 1.1.2 *The Laplace transform of the derivative of order $n \in \mathbb{N}^*$ of the function f is given by*

$$\mathcal{L}\left(f^{(n)}\right)(s) = s^n\mathcal{L}f(s) - \sum_{k=0}^{n-1} s^{n-k-1}\left[f^{(k)}(t)\right]_{t=0} = s^n\mathcal{L}f(s) - \sum_{k=0}^{n-1} s^{k}\left[f^{(n-k-1)}(t)\right]_{t=0}. \tag{1.7}$$

Proof 1.1.3 *From relation (1.3) and by integration by parts, hence the result.*

1.1.2 Fractional Derivatives and Fractional Integrals

Definition 1.1.4 *(Fractional Integral of Order α)*

For every $\alpha > 0$ and a local integrable function $f(t)$, the left fractional integral of order α is defined:

$${}_aI_t^{\alpha}\ f(t) = \frac{1}{\Gamma(\alpha)}\int_a^t (t-x)^{\alpha-1} f(x)\,dx, \qquad -\infty \le a < t < \infty. \tag{1.8}$$

Alternatively, it can be defined also the right fractional integral

$${}_tI_b^{\alpha}\ f(t) = \frac{1}{\Gamma(\alpha)}\int_t^b (x-t)^{\alpha-1} f(x)\,dx, \qquad -\infty < t < b \le \infty. \tag{1.9}$$

For particular values of the a and b parameters, the following cases are known:

- Riemann: $a = 0$, $b = +\infty$.
- Liouville: $a = -\infty$, $b = 0$.

Riemann-Liouville Fractional Derivatives

Definition 1.1.5 *(Fractional Derivative of Riemann-Liouville Order α)*

Let f be an integrable function. For every α, the Riemann-Liouville fractional derivative of ordre α can be defined as:

$$ {}_aD_t^{\alpha}\ f(t) = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^t (t-x)^{n-\alpha-1} f(x)\,dx = D^n\left[\ {}_aI_t^{n-\alpha}\ f(t)\right], \qquad n-1 \leq \alpha < n,\ t > a, \tag{1.10}$$

hence $D^n = \dfrac{d^n}{dt^n}$.

In particular

- *for $\alpha = 0$, we have:*

$$ {}_aD_t^{0}\ f(t) = D\left[\ {}_aI_t^{1-0}\ f(t)\right] = f(t), \tag{1.11}$$

- *for $\alpha = m \in \mathbb{N}$, we have:*

$$ {}_aD_t^{m}\ f(t) = D^{m+1}\left[\ {}_aI_t^{m+1-m}\ f(t)\right] = D^{m+1}\left[\ {}_aI_t^{1}\ f(t)\right] = D^mD\left[\ {}_aI_t^{1}\ f(t)\right] = D^m f(t). \tag{1.12}$$

Remark 1.1.6 *For $\alpha \in \mathbb{N}$, the fractional derivative of Riemann-Liouville coincides with the usual derivative.*

Proposition 1.1.7 *If C is a constant, then:*

$$ {}_0D_t^{\alpha} C = \frac{Cx^{-\alpha}}{\Gamma(1-\alpha)}, \qquad \text{where } 0 < \alpha < 1.$$

Example 1.1.8 *For $\alpha > 0$, $n-1 < \alpha < n$, $\beta > n-1$ the Riemann–Liouville derivative of the function $f(t) = t^{\beta}$, we can write:*

$$ I = {}_0D_t^{\alpha}\ t^{\beta} = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-x)^{n-\alpha-1} x^{\beta} dx,$$

and we take:

$$x = yt, \qquad dx = tdy.$$

It follows:

$$\begin{aligned} I &= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^1 (t-yt)^{n-\alpha-1}(yt)^{\beta}\,tdy, \\ &= \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n} t^{n-\alpha+\beta}\int_0^1 y^{\beta}(1-y)^{n-\alpha-1}\,dy, \end{aligned}$$

but

$$\begin{aligned} \frac{d^n}{dt^n}t^p &= \frac{\Gamma(p+1)}{\Gamma(p-n+1)}t^{p-n}, \\ B(p,q) &= \int_0^1 y^{p-1}(1-y)^{q-1}\,dy = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}, \end{aligned}$$

so that it results:

$$
\begin{aligned}
{}_0D_t^\alpha \ t^\beta &= I = \frac{1}{\Gamma(n-\alpha)} \frac{\Gamma(\beta+1)\Gamma(n-\alpha)}{\Gamma(n-\alpha+\beta+1)} \frac{\Gamma(n-\alpha+\beta+1)}{\Gamma(-\alpha+\beta+1)} t^{\beta-\alpha}, \\
&= \frac{\Gamma(\beta+1)}{\Gamma(-\alpha+\beta+1)} t^{\beta-\alpha}.
\end{aligned}
$$

Example 1.1.9 *For $\alpha > 0$, $n-1 < \alpha < n$, $\beta > n-1$, $a \in \mathbb{R}$ the Riemann–Liouville derivative of the function $f(t) = (t-a)^\beta$, we can write:*

$$
I = {}_aI_t^\alpha \ (t-a)^\beta = \frac{1}{\Gamma(\alpha)} \int_a^t (t-x)^{\alpha-1} (x-a)^\beta \, dx.
$$

The following change of variable

$$
\frac{x-a}{t-a} = y, \qquad dx = (t-a)\, dy,
$$

allows to calculate:

$$
\begin{aligned}
I &= \frac{(t-a)^{\alpha+\beta}}{\Gamma(\alpha)} \int_0^1 y^\beta (1-y)^{\alpha-1} \, dy = \frac{(t-a)^{\alpha+\beta}}{\Gamma(\alpha)} B(\alpha, \beta+1), \\
&= \frac{\Gamma(\beta+1)}{\Gamma(\alpha+\beta+1)} (t-a)^{\alpha+\beta},
\end{aligned}
$$

and

$$
{}_aD_t^\alpha \ (t-a)^\beta = \frac{d^n}{dt^n} \left[\ {}_aI_t^{n-\alpha} \ (t-a)^\beta \right],
$$

and finally:

$$
{}_aD_t^\alpha (t-a)^\beta = \frac{\Gamma(\beta+1)}{\Gamma(n-\alpha+\beta+1)} \frac{d^n}{dt^n} (t-a)^{n-\alpha+\beta} = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)} (t-a)^{\beta-\alpha}.
$$

Theorem 1.1.10 *For $0 < \alpha < 1$, $\phi, \psi \in C[a,b]$ the following integration rules are valid:*

$$
\int_a^b \phi(x) \ {}_aI_x^\alpha \ \psi(x)\, dx = \int_a^b \psi(t) \ {}_tI_b^\alpha \ \phi(t)\, dt, \tag{1.13}
$$

$$
\int_a^b f(t) \ {}_aD_x^\alpha \ g(x)\, dt = \int_a^b g(x) \ {}_tD_b^\alpha \ f(t)\, dt, \tag{1.14}
$$

where ${}_aI_x^\alpha \ {}_aD_x^\alpha \ f(x) = f(x)$.

Proof 1.1.11 *We use the Dirichlet theorem, written in the form:*

$$
\int_a^b dx \int_a^x f(x,t)\, dt = \int_a^b dt \int_t^b f(x,t)\, dx
$$

- *For the case (1.13), we introduce the following notation:*

$$
f(x,t) = \frac{1}{\Gamma(\alpha)} \frac{\phi(x)\psi(t)}{(x-t)^{1-\alpha}},
$$

in the Dirichlet theorem. It results:

$$\int_a^b \phi(x)\,dx \frac{1}{\Gamma(\alpha)} \int_a^x \psi(t)(x-t)^{\alpha-1}\,dt = \int_a^b \psi(t)\,dt \frac{1}{\Gamma(\alpha)} \int_t^b \phi(x)(x-t)^{\alpha-1}\,dx,$$

$$\int_a^b \phi(x) \ {}_aI_x^\alpha \ \psi(x)\,dx = \int_a^b \psi(t) \ {}_tI_b^\alpha \ \phi(t)\,dt.$$

- *For the case (1.14), we introduce in (1.13):*

$${}_tD_b^\alpha \ f(t) = \phi(t), \quad {}_aD_x^\alpha \ g(x) = \psi(x), \quad {}_aI_x^\alpha \quad {}_aD_x^\alpha \ f(x) = f(x).$$

Theorem 1.1.12 *For $n-1<\alpha<n$, p is a positive integer and ${}_aI_t^{\alpha+p}$, D^p, ${}_aD_t^{\alpha+p}$ exists. The following integration and derivation rules are valid:*

a- ${}_aI_t^{\alpha+1} \ [Df(t)] = {}_aI_t^\alpha \ f(t) - \dfrac{(t-a)^\alpha}{\Gamma(\alpha+1)} f(a).$

b- ${}_aI_t^\alpha \ \left[{}_aD_t^\alpha \ f(t)\right] = f(t) - \sum\limits_{k=1}^{n} {}_aD_t^{\alpha-k} \ f(t)\Big|_{t=a} \dfrac{(t-a)^{\alpha-k}}{\Gamma(\alpha-k+1)}.$

c- $D\left[{}_aI_t^\alpha \ f(t)\right] = {}_aI_t^\alpha \ [Df(t)] + \dfrac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} f(a).$

d- ${}_aI_t^\alpha \ f(t) = {}_aI_t^{\alpha+p} \ [D^p f(t)] + \sum\limits_{k=0}^{p-1} \dfrac{(t-a)^{\alpha+k}}{\Gamma(\alpha+k+1)} \ D^k f(t)\big|_{t=a}.$

e- $D^p\left[{}_aI_t^\alpha \ f(t)\right] = {}_aI_t^\alpha \ [D^p f(t)] + \sum_{k=0}^{p-1} D^k f(a) \dfrac{(t-a)^{\alpha-k}}{\Gamma(\alpha-k+1)}.$

Proof 1.1.13 **a-** *Integrating by parts, it results:*

$$\begin{aligned}
{}_aI_t^{\alpha+1}[Df(t)] &= \frac{1}{\Gamma(\alpha+1)} \textstyle\int_a^t (t-x)^\alpha f'(x)\,dx,\\
&= \frac{1}{\Gamma(\alpha+1)} (t-x)^\alpha f(x)\Big|_a^t + \frac{\alpha}{\Gamma(\alpha+1)} \textstyle\int_a^t (t-x)^{\alpha-1} f(x)\,dx,\\
&= -\frac{(t-a)^\alpha}{\Gamma(\alpha+1)} f(a) + \frac{\alpha}{\alpha\Gamma(\alpha)} \textstyle\int_a^t (t-x)^{\alpha-1} f(x)\,dx,\\
&= -\frac{(t-a)^\alpha}{\Gamma(\alpha+1)} f(a) + {}_aI_t^\alpha \ f(t).
\end{aligned}$$

b- *We using a in Theorem (*1.1.12*), it results:*

$$\begin{aligned}
I = {}_aI_t^\alpha\left[{}_aD_t^\alpha f(t)\right] &= -\frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} \ {}_aD_t^{\alpha-1} f(t)\big|_{t=a} + {}_aI_t^{\alpha-1}\left[{}_aD_t^{\alpha-1} f(t)\right],\\
&= -\frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} \ {}_aD_t^{\alpha-1} f(t)\big|_{t=a} - \frac{(t-a)^{\alpha-2}}{\Gamma(\alpha-1)} \ {}_aD_t^{\alpha-2} f(t)\big|_{t=a} + {}_aI_t^{\alpha-2}\left[{}_aD_t^{\alpha-2} f(t)\right],\\
&\vdots\\
&= -\sum_{k=1}^{n} \frac{(t-a)^{\alpha-k}}{\Gamma(\alpha-k+1)} \ {}_aD_t^{\alpha-k} f(t)\Big|_{t=a} + f(t).
\end{aligned}$$

c- *We have:*

$$ {}_aI_t^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-x)^{\alpha-1} f(x)\, dx, $$

and

$$ \begin{aligned} D\left[\ {}_aI_t^{\alpha} f(t)\ \right] = D\left[\ {}_aD_t^{-\alpha} f(t)\ \right] = {}_aD_t^{1-\alpha} f(t) &= {}_aI_t^{\alpha-1} f(t) \\ &= {}_aI_t^{\alpha}\left[Df(t)\right] + \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} f(a). \end{aligned} $$

d- *We using a in Theorem (*1.1.12*), it results:*

$$ {}_aI_t^{\alpha+1}\left[Df(t)\right] = {}_aI_t^{\alpha} f(t) - \frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)} f(a), $$

we have

$$ \begin{aligned} {}_aI_t^{\alpha+2}\left[D^2 f(t)\right] &= {}_aI_t^{\alpha+1}\left[Df(t)\right] - \frac{(t-a)^{\alpha+1}}{\Gamma(\alpha+2)} Df(t)\big|_{t=a}, \\ &= {}_aI_t^{\alpha} f(t) - \frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)} f(a) - \frac{(t-a)^{\alpha+1}}{\Gamma(\alpha+2)} Df(t)\big|_{t=a}, \\ {}_aI_t^{\alpha+3}\left[D^3 f(t)\right] &= {}_aI_t^{\alpha+2}\left[D^2 f(t)\right] - \frac{(t-a)^{\alpha+2}}{\Gamma(\alpha+3)} D^2 f(t)\big|_{t=a}, \\ &= {}_aI_t^{\alpha} f(t) - \frac{(t-a)^{\alpha}}{\Gamma(\alpha+1)} f(a) - \frac{(t-a)^{\alpha+1}}{\Gamma(\alpha+2)} Df(t)\big|_{t=a} - \frac{(t-a)^{\alpha+2}}{\Gamma(\alpha+3)} D^2 f(t)\big|_{t=a}, \\ &\vdots \\ {}_aI_t^{\alpha+p}\left[D^p f(t)\right] &= {}_aI_t^{\alpha} f(t) - \sum_{k=0}^{p-1} \frac{(t-a)^{\alpha+k}}{\Gamma(\alpha+k+1)} D^k f(t)\Big|_{t=a}. \end{aligned} $$

Hence

$$ {}_aI_t^{\alpha} f(t) = {}_aI_t^{\alpha+p}\left[D^p f(t)\right] + \sum_{k=0}^{p-1} \frac{(t-a)^{\alpha+k}}{\Gamma(\alpha+k+1)} D^k f(t)\Big|_{t=a}. $$

e- *This formula can be verified by induction, using c in Theorem (*1.1.12*), or:*

$$ D\left[{}_aI_t^{\alpha} f(t)\right] = {}_aI_t^{\alpha}\left[Df(t)\right] + \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} f(a). $$

We have

$$\begin{aligned}
D^2\left[{}_aI_t^\alpha f(t)\right] &= D\ {}_aI_t^\alpha\ \left[Df(t)\right] + Df(a)\frac{(t-a)^{\alpha-2}}{\Gamma(\alpha-1)}, \\
&= {}_aI_t^\alpha\ \left[D^2f(t)\right] + Df(a)\frac{(t-a)^{\alpha-2}}{\Gamma(\alpha-1)} + Df(a)\frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)}, \\
D^3\left[{}_aI_t^\alpha f(t)\right] &= D\ {}_aI_t^\alpha\ \left[D^2f(t)\right] + D^2f(a)\frac{(t-a)^{\alpha-3}}{\Gamma(\alpha-2)} + D^2f(a)\frac{(t-a)^{\alpha-2}}{\Gamma(\alpha-1)}, \\
&= D\ {}_aI_t^\alpha\ \left[D^2f(t)\right] + D^2f(a)\frac{(t-a)^{\alpha-3}}{\Gamma(\alpha-2)} + D^2f(a)\frac{(t-a)^{\alpha-2}}{\Gamma(\alpha-1)}, \\
&= {}_aI_t^\alpha\ \left[D^3f(t)\right] + D^2f(a)\frac{(t-a)^{\alpha-3}}{\Gamma(\alpha-2)} + D^2f(a)\frac{(t-a)^{\alpha-2}}{\Gamma(\alpha-1)} + D^2f(a)\frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)}, \\
&\ \ \vdots \\
D^p\left[{}_aI_t^\alpha f(t)\right] &= {}_aI_t^\alpha\ \left[D^pf(t)\right] + \sum_{k=0}^{p-1} D^kf(a)\frac{(t-a)^{\alpha-k}}{\Gamma(\alpha-k+1)}.
\end{aligned}$$

Theorem 1.1.14 *For $f \in C[a,b]$. The exponents property:*

$$ {}_aI_t^\alpha\ \ {}_aI_t^\beta\ f(t) = {}_aI_t^{\alpha+\beta}\ f(t), \qquad \forall\alpha,\beta > 0. \tag{1.15}$$

Proof 1.1.15 *For α, $\beta > 0$, it results:*

$$\begin{aligned}
I = {}_aI_t^\alpha\ \ {}_aI_t^\beta\ f(t) &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t (t-x)^{\alpha-1}\int_a^x (x-y)^{\beta-1} f(y)\,dxdy, \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t\int_a^x (t-x)^{\alpha-1}(x-y)^{\beta-1} f(y)\,dxdy,
\end{aligned}$$

we apply the Dirichlet equality

$$\int_a^t\int_a^x f(y)\,dxdy = \int_a^t\int_y^t f(y)\,dxdy.$$

We obtien

$$I = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_a^t\int_y^t (t-x)^{\alpha-1}(x-y)^{\beta-1} f(y)\,dxdy,$$

and we apply the change of variable

$$x = y + z(t-y),$$

$$dx = (t-y)\,dz, \qquad t-x = (1-z)(t-y),$$

$$
\begin{aligned}
I &= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t \int_0^1 (1-z)^{\alpha-1} (t-y)^{\alpha-1} z^{\beta-1} (t-y)^{\beta-1} (t-y) f(y)\, dzdy, \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t (t-y)^{\alpha+\beta-1} f(y) \int_0^1 (1-z)^{\alpha-1} z^{\beta-1} dzdy, \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t (t-y)^{\alpha+\beta-1} f(y)\, B(\alpha,\beta)\, dy, \\
&= \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_a^t (t-y)^{\alpha+\beta-1} f(y) \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} dy, \\
&= \frac{1}{\Gamma(\alpha+\beta)} \int_a^t (t-y)^{\alpha+\beta-1} f(y)\, dy, \\
&= {}_aI_t^{\alpha+\beta} .
\end{aligned}
$$

Theorem 1.1.16 *For $n-1<\alpha<n$, $m-1<\beta<m$, and ${}_aI_t^{\alpha}$, ${}_aD_t^{\alpha}$, ${}_aD_t^{\beta}$ exists. The following integration and derivation rules are valid:*

a- ${}_aD_t^{\alpha}\left[{}_aI_t^{\beta} f(t)\right] = {}_aD_t^{\alpha-\beta} f(t)$.

b- ${}_aI_t^{\alpha}\left[{}_aD_t^{\beta} f(t)\right] = {}_aI_t^{\alpha-\beta} f(t) - \sum_{k=1}^{m} {}_aD_t^{\beta-k} f(t) \Big|_{t=a} \frac{(t-a)^{\alpha-k}}{\Gamma(\alpha+1-k)}$.

c- ${}_aD_t^{\alpha}\left[{}_aD_t^{\beta} f(t)\right] = {}_aD_t^{\alpha+\beta} f(t) - \sum_{k=1}^{m} {}_aD_t^{\beta-k} f(t) \Big|_{t=a} \frac{(t-a)^{-\alpha-k}}{\Gamma(1-\alpha-k)}$.

Proof 1.1.17 a-

$$
{}_aD_t^{\alpha}\left[{}_aI_t^{\beta} f(t)\right] = \frac{d^n}{dt^n}\left[{}_aI_t^{n-\alpha}\left[{}_aI_t^{\beta} f(t)\right]\right] = \frac{d^n}{dt^n}\left[{}_aI_t^{n-(\alpha-\beta)} f(t)\right] = {}_aD_t^{\alpha-\beta} f(t) .
$$

b-

$$
\begin{aligned}
I &= {}_aI_t^{\alpha}\left[{}_aD_t^{\beta} f(t)\right] = \frac{1}{\Gamma(\alpha)} \int_a^t (t-x)^{\alpha-1} \left[{}_aD_x^{\beta} f(x)\right] dx, \\
&= -\left[{}_aD_t^{\beta-1} f(t)\right]\Big|_{t=a} \frac{(t-a)^{\alpha-1}}{\Gamma(\alpha)} + \frac{1}{\Gamma(\alpha-1)} \int_a^t (t-x)^{\alpha-2} \left[{}_aD_x^{\beta-1} f(x)\right] dx, \\
&\ \ \vdots \\
&= {}_aI_t^{\alpha-\beta} f(t) - \sum_{k=1}^{m} {}_aD_t^{\beta-k} f(t) \Big|_{t=a} \frac{(t-a)^{\alpha-k}}{\Gamma(\alpha+1-k)}.
\end{aligned}
$$

c-

$$\begin{aligned} I &= {}_aD_t^{\alpha}\left[{}_aD_t^{\beta}f(t)\right] = \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dt}\right)^n \int_a^t (t-x)^{n-\alpha-1}\left[{}_aD_x^{\beta}f(x)\right]dx, \\ &= -\left[{}_aD_t^{\beta-1}f(t)\right]\Big|_{t=a}\left(\frac{(t-a)^{-\alpha-1}}{\Gamma(-\alpha)}\right) + \\ &\quad \frac{1}{\Gamma(n-\alpha-1)}\left(\frac{d}{dt}\right)^n \int_a^t (t-x)^{n-\alpha-2}\left[{}_aD_x^{\beta-1}f(x)\right]dx, \\ &\vdots \\ &= {}_aD_t^{\alpha+\beta}f(t) - \sum_{k=1}^{m} {}_aD_t^{\beta-k}f(t)\Big|_{t=a}\frac{(t-a)^{-\alpha-k}}{\Gamma(1-\alpha-k)}. \end{aligned}$$

Theorem 1.1.18 *Let $f(t)$, $g(t)$ functions defined on $[a,b]$, as ${}_aD_t^{\alpha}\ f(t)$, ${}_aD_t^{\alpha}\ g(t)$ exists almost everywhere. Moreover, $C_i \in \mathbb{R}$, $i = 1, 2$, so:*

$$ {}_aI_t^{\alpha}\left[C_1f(t) + C_2g(t)\right] = C_1\ {}_aI_t^{\alpha}\ f(t) + C_2\ {}_aI_t^{\alpha}\ g(t) \tag{1.16}$$

Proof 1.1.19

$$\begin{aligned} {}_aI_t^{\alpha}\left[C_1f(t) + C_2g(t)\right] &= \frac{1}{\Gamma(\alpha)}\int_a^t (t-x)^{\alpha-1}\left[C_1f(x) + C_2g(x)\right]dx, \\ &= C_1\frac{1}{\Gamma(\alpha)}\int_a^t (t-x)^{\alpha-1}f(x)\,dx + C_2\frac{1}{\Gamma(\alpha)}\int_a^t (t-x)^{\alpha-1}g(x)\,dx, \\ &= C_1\ {}_aI_t^{\alpha}\ f(t) + C_2\ {}_aI_t^{\alpha}\ g(t). \end{aligned}$$

Lemma 1.1.20 *That is $\alpha > 0$, $n = [\alpha]+1$ and $f \in L^1(0,b)$, the Laplace transform of the Riemann-Liouville fractional integral is formulated as follows:*

$$\mathcal{L}\left({}_0I_t^{\alpha}f\right)(s) = s^{-\alpha}\mathcal{L}f(s). \tag{1.17}$$

Proof 1.1.21 *We can ${}_0I_t^{\alpha}f$ write as a convolution of two functions $g(x) = \dfrac{x^{\alpha-1}}{\Gamma(\alpha)}$ and $f(t)$ so*

$$ {}_0I_t^{\alpha}f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-t)^{\alpha-1}f(t)\,dt = \frac{x^{\alpha-1}}{\Gamma(\alpha)} * f(t),$$

then

$$\mathcal{L}\left({}_0I_t^{\alpha}f\right)(s) = \mathcal{L}\left(\frac{x^{\alpha-1}}{\Gamma(\alpha)}\right)(s).\mathcal{L}f(s),$$

by the integral by part we have

$$\mathcal{L}\left(x^{\alpha-1}\right)(s) = \Gamma(\alpha)\,s^{-\alpha},$$

hence the result.

Theorem 1.1.22 *If $f \in L^1(0,b)$, the Laplace transform of the Riemann-Liouville fractional derivative is formulated as follows:*

$$\mathcal{L}\left[\ {}_0D_t^{\alpha} f\ \right](s) = s^{\alpha}\mathcal{L}f(s) - \sum_{k=0}^{n-1} s^k \left[\ {}_0D_t^{\alpha-k-1}\ f(t)\right]_{t=0}, \tag{1.18}$$

with $n-1<\alpha<n$.

Proof 1.1.23

$$\begin{aligned}
\mathcal{L}\left[\ {}_0D_t^{\alpha}\ f\right](s) &= \mathcal{L}\left[D^n\ {}_0I_t^{n-\alpha}\ f\right](s), \qquad \textit{(using (1.10))}\\
&= s^n\mathcal{L}\left(\ {}_0I_t^{n-\alpha}\ f\right)(s) - \sum_{k=0}^{n-1} s^{n-k-1}\left[D^{(k)}\left(\ {}_0I_t^{n-\alpha}\ f\right)(t)\right]_{t=0}, \qquad \textit{(using (1.7))}\\
&= s^n s^{-(n-\alpha)}\mathcal{L}(f)(s) - \sum_{k=0}^{n-1} s^{n-k-1}\left[D^{(k)}\left(\ {}_0I_t^{n-\alpha}\ f\right)(t)\right]_{t=0}, \qquad \textit{(using (1.17))}\\
&= s^{\alpha}\mathcal{L}(f)(s) - \sum_{k=0}^{n-1} s^{n-k-1}\left[D^{(k)}\left(\ {}_0I_t^{n-\alpha}\ f\right)(t)\right]_{t=0},\\
&= s^{\alpha}\mathcal{L}(f)(s) - \sum_{k=0}^{n-1} s^{k}\left[D^{(n-k-1)}\left(\ {}_0I_t^{n-\alpha}\ f\right)(t)\right]_{t=0},\\
&= s^{\alpha}\mathcal{L}(f)(s) - \sum_{k=0}^{n-1} s^{k}\left[D^{(n-k-1)}\left(\ {}_0I_t^{n-\alpha-(k+1)+(k+1)}\ f\right)(t)\right]_{t=0},\\
&= s^{\alpha}\mathcal{L}(f)(s) - \sum_{k=0}^{n-1} s^{k}\left[\left(\ {}_0D_t^{\alpha-k-1}\ f\right)(t)\right]_{t=0}.
\end{aligned}$$

Caputo Fractional Derivatives

In this subsection, we present the definitions and some properties of the Caputo fractional derivative, with the relation between the Caputo and Riemann-Liouville fractional derivatives.

Definition 1.1.24 *Let $\alpha>0$, $n-1<\alpha\leq n$, $n\in\mathbb{N}^*$, the Caputo derivative operator of order α is defined as:*

$${}_a^CD_t^{\alpha}\ f(t) = \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-x)^{n-\alpha-1} f^{(n)}(x)\,dx = \ {}_aI_t^{n-\alpha}\ f^{(n)}(t). \tag{1.19}$$

Theorem 1.1.25 *That is $\alpha\geq 0$, $n-1<\alpha<n$, if f has $n-1$ derivatives at a and if ${}_aD_t^{\alpha}f$ exists, then:*

$${}_a^CD_t^{\alpha}\ f(t) = \ {}_aD_t^{\alpha}\ \left[f(t) - \sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(t-a)^k\right],\quad \forall t\in[a,b]. \tag{1.20}$$

Proof 1.1.26 *We have*

$$ {}_aD_t^{\alpha}\left[f(t)-\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(t-a)^k\right]=D^n\ {}_aI_t^{n-\alpha}\left[f(t)-\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(t-a)^k\right]. $$

Using integration by parts, we get

$$ \begin{aligned} & {}_aI_t^{n-\alpha}\left[f(t)-\sum_{k=0}^{n-1}\tfrac{f^{(k)}(a)}{k!}(t-a)^k\right] \\ &= \frac{1}{\Gamma(n-\alpha)}\int_a^t (t-x)^{n-\alpha-1}\left[f(x)-\sum_{k=0}^{n-1}\tfrac{f^{(k)}(a)}{k!}(x-a)^k\right]dx, \\ &= \frac{1}{\Gamma(n-\alpha)}\left[\begin{array}{c} -\frac{1}{n-\alpha}(t-x)^{n-\alpha}\left(f(x)-\sum_{k=0}^{n-1}\tfrac{f^{(k)}(a)}{k!}(x-a)^k\Big|_{x=a}^{x=t}\right)+ \\ \frac{1}{n-\alpha}\int_a^t (t-x)^{n-\alpha}\left(Df(x)-D\sum_{k=0}^{n-1}\tfrac{f^{(k)}(a)}{k!}(x-a)^k\right)dx \end{array}\right], \\ &= \frac{1}{\Gamma(n-\alpha+1)}\int_a^t (t-x)^{n-\alpha}\left[Df(x)-D\sum_{k=0}^{n-1}\tfrac{f^{(k)}(a)}{k!}(x-a)^k\right]dx, \\ &= {}_aI_t^{n-\alpha+1}\ D\left[f(t)-\sum_{k=0}^{n-1}\tfrac{f^{(k)}(a)}{k!}(t-a)^k\right], \\ &\ \vdots \\ &= {}_aI_t^{n-\alpha+n}\ D^n\left[f(t)-\sum_{k=0}^{n-1}\tfrac{f^{(k)}(a)}{k!}(t-a)^k\right], \end{aligned} $$

and $\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(t-a)^k$ *is a polynomial of degree* $n-1$ *then*

$$ D^n\left[\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(t-a)^k\right]=0, $$

so

$$ {}_aI_t^{n-\alpha}\left[f(t)-\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(t-a)^k\right]={}_aI_t^{n-\alpha+n}\ D^nf(t) $$

and

$$ \begin{aligned} {}_aD_t^{\alpha}\left[f(t)-\sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(t-a)^k\right] &= D^n\ {}_aI_t^{n-\alpha+n}\ D^nf(t)=D^n\ {}_aI_t^{n}\ \ {}_aI_t^{n-\alpha}\ D^nf(t), \\ &= {}_aI_t^{n-\alpha}\ D^nf(t)={}_a^CD_t^{\alpha}\ f(t). \end{aligned} $$

Corollary 1.1.27 *That is* $\alpha\geq 0$, $n=[\alpha]+1$, *if* ${}_aD_t^{\alpha}f$ *and* ${}_a^CD_t^{\alpha}f$ *exists, we suppose que* $D^Kf(a)=0$ *for* $k=0,1,\dots,n-1$, *so:*

$$ {}_a^CD_t^{\alpha}\ f(t)={}_aD_t^{\alpha}\ f(t). \tag{1.21} $$

Theorem 1.1.28 *If* $f\in C[a,b]$ *and if* $\alpha>0$ $(n-1<\alpha\leq n)$, *we suppose que* $D^Kf(a)=0$ *for* $k=0,1,\dots,n-1$, *so:*

$$ {}_a^CD_t^{\alpha}\ \ {}_aI_t^{\alpha}\ f(t)=f(t). \tag{1.22} $$

and

$$ {}_aI_t^\alpha \ {}_a^CD_t^\alpha \ f(t) = f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (t-a)^k. \tag{1.23} $$

Proof 1.1.29 *We suppose that* ${}_aI_t^\alpha \ f(t) = g(t)$ *and by corollary* (1.21) $D^K g(a) = 0$ *for* $k = 0, 1, \ldots, n-1$ *so*

$$ {}_a^CD_t^\alpha \ {}_aI_t^\alpha \ f(t) = {}_a^CD_t^\alpha \ g(t) = {}_aD_t^\alpha \ g(t) = {}_aD_t^\alpha \ {}_aI_t^\alpha \ f(t) = f(t). $$

and

$$ \begin{aligned} {}_aI_t^\alpha \ {}_a^CD_t^\alpha \ f(t) &= {}_aI_t^\alpha \ {}_aD_t^\alpha \left[f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (t-a)^k \right] \\ &= f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{k!} (t-a)^k. \end{aligned} $$

Theorem 1.1.30 *Let* f_i, $i = 1,2$ *functions defined on* $[a,b]$, *as* ${}_a^CD_t^\alpha \ f_i$, $i = 1,2$ *exist almost everywhere. Moreover,* $c_i \in \mathbb{R}$, $i = 1,2$, *so* ${}_a^CD_t^\alpha \ (c_1 f_1 + c_2 f_2)$ *exist almost everywhere on* $[a,b]$, *and we have*

$$ {}_a^CD_t^\alpha \ (c_1 f_1 + c_2 f_2) = c_1 \ {}_a^CD_t^\alpha \ f_1 + c_2 \ {}_a^CD_t^\alpha \ f_2. \tag{1.24} $$

Proof 1.1.31

$$ \begin{aligned} {}_a^CD_t^\alpha \ (c_1 f_1 + c_2 f_2) &= {}_aI_t^{n-\alpha} \ D^n (c_1 f_1 + c_2 f_2), \\ &= c_1 \ {}_aI_t^{n-\alpha} \ D^n f_1 + c_2 \ {}_aI_t^{n-\alpha} \ D^n f_2, \\ &= c_1 \ {}_a^CD_t^\alpha \ f_1 + c_2 \ {}_a^CD_t^\alpha \ f_2. \end{aligned} $$

Proposition 1.1.32 *If* C *is a constant, then:*

$$ {}_a^CD_t^\alpha C = 0. $$

Theorem 1.1.33 *If* $n-1 < \alpha < n$, *where* $n \in \mathbb{N}^*$, *then:*

$$ \begin{aligned} \lim_{\alpha \longrightarrow n} {}_0^CD_t^\alpha f(t) &= f^{(n)}(t), \\ \lim_{\alpha \longrightarrow n-1} {}_0^CD_t^\alpha f(t) &= f^{(n-1)}(t) - f^{(n-1)}(0). \end{aligned} $$

Proof 1.1.34 *In the formula*

$$ {}_0^CD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-x)^{n-\alpha-1} f^{(n)}(x)\, dx, $$

We will use the integration by parts. It results:

$$ {}_0^CD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \left[-\frac{(t-x)^{n-\alpha}}{n-\alpha} f^{(n)}(x) \Big|_0^t + \int_0^t \frac{(t-x)^{n-\alpha}}{n-\alpha} f^{(n+1)}(x)\, dx \right]. $$

Using the property of Γ function

$$\Gamma(n-\alpha+1) = (n-\alpha)\,\Gamma(n-\alpha),$$

it results:

$$\begin{aligned}
{}_0^C D_t^\alpha f(t) &= \frac{1}{\Gamma(n-\alpha+1)}\left[t^{n-\alpha} f^{(n)}(0) + \int_0^t (t-x)^{n-\alpha} f^{(n+1)}(x)\,dx\right],\\
\lim_{\alpha\longrightarrow n} {}_0^C D_t^\alpha f(t) &= f^{(n)}(0) + \int_0^t f^{(n+1)}(x)\,dx = f^{(n)}(0) + f^{(n)}(x)\Big|_0^t = f^{(n)}(t),
\end{aligned}$$

and

$$\begin{aligned}
\lim_{\alpha\longrightarrow n-1} {}_0^C D_t^\alpha f(t) &= t f^{(n)}(0) + \int_0^t (t-x)\, f^{(n+1)}(x)\,dx,\\
&= t f^{(n)}(0) + t\, f^{(n)}(x)\Big|_0^t - x f^{(n)}(x)\Big|_0^t + \int_0^t f^{(n)}(x)\,dx,\\
&= f^{(n-1)}(x)\Big|_0^t = f^{(n-1)}(t) - f^{(n-1)}(0).
\end{aligned}$$

Example 1.1.35 *For $\alpha > 0$, $n-1 < \alpha < n$, $\beta > n-1$ Caputo derivative of the function $f(t) = t^\beta$, we can write:*

$$\begin{aligned}
I \;=\; {}_0^C D_t^\alpha t^\beta &= \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-x)^{n-\alpha-1}\,\frac{d^n}{dx^n}x^\beta dx\\
&= \frac{1}{\Gamma(n-\alpha)}\int_0^t (t-x)^{n-\alpha-1}\,\frac{\Gamma(\beta+1)}{\Gamma(\beta-n+1)}x^{\beta-n}dx
\end{aligned}$$

and we take:

$$x = yt, \qquad dx = tdy.$$

It follows:

$$\begin{aligned}
I &= \frac{1}{\Gamma(n-\alpha)}\frac{\Gamma(\beta+1)}{\Gamma(\beta-n+1)}t^{\beta-\alpha}\int_0^1 (1-y)^{n-\alpha-1}\,y^{\beta-n}dx,\\
&= \frac{1}{\Gamma(n-\alpha)}\frac{\Gamma(\beta+1)}{\Gamma(\beta-n+1)}t^{\beta-\alpha}\frac{\Gamma(\beta-n+1)\,\Gamma(n-\alpha)}{\Gamma(\beta-\alpha+1)},\\
&= \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}t^{\beta-\alpha}.
\end{aligned}$$

Theorem 1.1.36 *For $n-1 < \alpha < n$, $n \in \mathbb{N}$ and a function $f(t)$ which obey the conditions of Taylor theorem, then:*

$${}_0^C D_t^\alpha f(t) = \sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{\Gamma(k-\alpha+1)}t^{k-\alpha}.$$

Proof 1.1.37 *We can apply the Taylor expansion, because $f(t)$ satisfy the conditions of the Taylor theorem:*

$$f(t) = \sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{k!}t^k = \sum_{k=0}^{\infty}\frac{f^{(k)}(0)}{\Gamma(k+1)}t^k.$$

The Caputo derivative of $f(t)$:

$$ {}_0^C D_t^\alpha f(t) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{\Gamma(k+1)} \, {}_0^C D_t^\alpha t^k = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{\Gamma(k+1)} \frac{\Gamma(k+1)}{\Gamma(k-\alpha+1)} t^{k-\alpha}, $$

and finally:

$$ {}_0^C D_t^\alpha f(t) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{\Gamma(k-\alpha+1)} t^{k-\alpha}. $$

Theorem 1.1.38 *If $f \in C[0,b]$, the Laplace transform of the Caputo fractional derivative is formulated as follows:*

$$ \mathcal{L}\left[{}_0^C D_t^\alpha f\right](s) = s^\alpha \mathcal{L} f(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} \left[f^{(k)}(t)\right]_{t=0}, \tag{1.25} $$

with $n-1 < \alpha \leq n$.

Proof 1.1.39 *We have*

$$ {}_0^C D_t^\alpha f(t) = {}_0 I_t^{n-\alpha} f^{(n)}(t), $$

so

$$ \begin{aligned} \mathcal{L}\left({}_0^C D_t^\alpha f\right)(s) &= \mathcal{L}\left({}_0 I_t^{n-\alpha} f^{(n)}\right)(s), \\ &= s^{-(n-\alpha)} \mathcal{L}\left(f^{(n)}\right)(s), \qquad \textit{(using (1.17))} \\ &= s^{-(n-\alpha)} \left(s^n \mathcal{L} f(s) - \sum_{k=0}^{n-1} s^{n-k-1} \left[f^{(k)}(t)\right]_{t=0} \right), \qquad \textit{(using (1.7))} \\ &= s^{-(n-\alpha)} \left(s^n \mathcal{L} f(s) - \sum_{k=0}^{n-1} s^{n-k-1} \left[f^{(k)}(t)\right]_{t=0} \right) \\ &= s^\alpha \mathcal{L} f(s) - \sum_{k=0}^{n-1} s^{\alpha-k-1} \left[f^{(k)}(t)\right]_{t=0}. \end{aligned} $$

Grünwald-Letnikov Fractional Derivative

Differintegral Ordinary derivatives are defined in terms of backward differences as

$$ f'(t) = \frac{df}{dt}(t) = \lim_{h \longrightarrow 0} \frac{f(t) - f(t-h)}{h}, \tag{1.26} $$

$$ \begin{aligned} f''(t) &= \frac{d^2 f}{dt^2}(t) = \lim_{h \longrightarrow 0} \frac{1}{h} \left(\frac{f'(t) - f'(t-h)}{h} \right) = \lim_{h \longrightarrow 0} \frac{1}{h} \left(\frac{f(t) - f(t-h)}{h} - \frac{f(t-h) - f(t-2h)}{h} \right), \\ &= \lim_{h \longrightarrow 0} \frac{f(t) - 2f(t-h) + f(t-2h)}{h^2}. \end{aligned} \tag{1.27} $$

In general, for $n \in \mathbb{N}$, and $f \in C^m[a,b]$, $a < t < b$

$$D^n f(t) = \lim_{h \longrightarrow 0} \frac{\sum\limits_{k=0}^{n} \left[(-1)^k \binom{n}{k} f(t-kh)\right]}{h^n}. \tag{1.28}$$

Consider base point a and define for $t > a$, $h = \dfrac{t-a}{n}$, n is a positive integer and the symbol $\binom{n}{k}$, can be generalized for negative integers as

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} = \frac{n(n-1)\dots(n-k+1)}{k!}.$$

The idea of Grünwald-Letnikov approach is based on the remark that one can express the derivative of the integer order α (if α is positive) and the repeated integral $(-\alpha)$ fois (if α is negative) of a function f by the following general formula:

$$D^\alpha f(t) = \lim_{h \longrightarrow 0} h^{-\alpha} \sum_{k=0}^{n} (-1)^k \binom{\alpha}{k} f(t-kh), \tag{1.29}$$

which represents the derivative of integer order α if $0 < \alpha < n$ and the repeated integral (α) fois if $-n < -\alpha < 0$ with $nh = t - a$.

The generalization of this formula for non-integer α ($0 \leq n-1 < \alpha < n$ and $(-1)^k \binom{\alpha}{k} = \dfrac{-\alpha(1-\alpha)(2-\alpha)\dots(k-\alpha-1)}{k!} = \dfrac{\Gamma(k-\alpha)}{\Gamma(k+1)\Gamma(-\alpha)}$) gives us:

$${}_a^G D_t^\alpha \ f(t) = \lim_{h \longrightarrow 0} h^{-\alpha} \sum_{k=0}^{n} \frac{\Gamma(k-\alpha)}{\Gamma(k+1)\Gamma(-\alpha)} f(t-kh), \tag{1.30}$$

and

$${}_a^G D_t^{-\alpha} \ f(t) = \lim_{h \longrightarrow 0} h^{\alpha} \sum_{k=0}^{n} \frac{\Gamma(k+\alpha)}{\Gamma(k+1)\Gamma(\alpha)} f(t-kh) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-\tau)^{\alpha-1} f(\tau)\, d\tau. \tag{1.31}$$

Using integration by parts of (1.31) we get:

$${}_a^G D_t^{-\alpha} \ f(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)(t-a)^{k+\alpha}}{\Gamma(k+\alpha+1)} + \frac{1}{\Gamma(n+\alpha)} \int_a^t (t-\tau)^{n+\alpha-1} f^{(n)}(\tau)\, d\tau, \tag{1.32}$$

we substitute $-\alpha$ with α we find

$${}_a^G D_t^{\alpha} \ f(t) = \sum_{k=0}^{n-1} \frac{f^{(k)}(a)(t-a)^{k-\alpha}}{\Gamma(k-\alpha+1)} + \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} f^{(n)}(\tau)\, d\tau. \tag{1.33}$$

Proposition 1.1.40 *If C is a constant, then:*

$${}_a^G D_t^{\alpha} \ C = \frac{C}{\Gamma(1-\alpha)} (t-a)^{-\alpha}.$$

Proposition 1.1.41 *For m positive integer and α non-integer we have:*

1. $\dfrac{d^m}{dt^m}\left({}_a^G D_t^{\alpha}\ f(t)\right) = {}_a^G D_t^{\alpha+m}\ f(t)\,.$

2. ${}_a^G D_t^{\alpha}\ \left(\dfrac{d^m}{dt^m} f(t)\right) = {}_a^G D_t^{\alpha+m}\ f(t) - \sum_{k=0}^{m-1} \dfrac{f^{(k)}(a)\,(t-a)^{k-\alpha-m}}{\Gamma(k-\alpha-m+1)}.$

Proof 1.1.42 1. $\dfrac{d^m}{dt^m}\left({}_a^G D_t^{\alpha}\ f(t)\right)$

$$= \sum_{k=0}^{n-1} \frac{f^{(k)}(a)\,\frac{d^m}{dt^m}(t-a)^{k-\alpha}}{\Gamma(k-\alpha+1)} + \frac{1}{\Gamma(n-\alpha)} \int_a^t f^{(n)}(\tau)\,\frac{d^m}{dt^m}(t-\tau)^{n-\alpha-1}\,d\tau,$$

$$= \sum_{k=0}^{n-1} \frac{f^{(k)}(a)\,(t-a)^{k-\alpha-m}}{\Gamma(k-\alpha-m+1)} + \frac{1}{\Gamma(n-\alpha-m)} \int_a^t (t-\tau)^{n-\alpha-m-1} f^{(n)}(\tau)\,d\tau,$$

$$= {}_a^G D_t^{\alpha+m}\ f(t)\,.$$

2. ${}_a^G D_t^{\alpha}\ \left(\dfrac{d^m}{dt^m} f(t)\right)$

$$= \sum_{k=0}^{n-1} \frac{f^{(k+m)}(a)\,(t-a)^{k-\alpha}}{\Gamma(k-\alpha+1)} + \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} f^{(n+m)}(\tau)\,d\tau,$$

$$= \sum_{k=m}^{n+m-1} \frac{f^{(k)}(a)\,(t-a)^{k-\alpha-m}}{\Gamma(k-\alpha-m+1)} + \frac{1}{\Gamma(n'-m-\alpha)} \int_a^t (t-\tau)^{n'-m-\alpha-1} f^{(n')}(\tau)\,d\tau,$$

$$= \sum_{k=0}^{n'-1} \frac{f^{(k)}(a)\,(t-a)^{k-\alpha-m}}{\Gamma(k-\alpha-m+1)} + \frac{1}{\Gamma(n'-m-\alpha)} \int_a^t (t-\tau)^{n'-m-\alpha-1} f^{(n')}(\tau)\,d\tau$$
$$- \sum_{k=0}^{m-1} \frac{f^{(k)}(a)\,(t-a)^{k-\alpha-m}}{\Gamma(k-\alpha-m+1)},$$

$$= {}_a^G D_t^{\alpha+m}\ f(t) - \sum_{k=0}^{m-1} \frac{f^{(k)}(a)\,(t-a)^{k-\alpha-m}}{\Gamma(k-\alpha-m+1)}.$$

Proposition 1.1.43 • *If $\beta < 0$ and $\alpha \in \mathbb{R}$ then:*

$${}_a^G D_t^{\alpha}\ \left({}_a^G D_t^{\beta}\ f(t)\right) = {}_a^G D_t^{\alpha+\beta}\ f(t)\,.$$

• *If $0 \le m-1 < \beta < m$ and $\alpha < 0$ then:*

$${}_a^G D_t^{\alpha}\ \left({}_a^G D_t^{\beta}\ f(t)\right) = {}_a^G D_t^{\alpha+\beta}\ f(t)\,,$$

only if $f^{(k)}(a) = 0\ \forall k = 0, 1, \ldots, m-2$.

• *If $0 \le m-1 < \beta < m$ and $0 \le n-1 < \alpha < n$ then:*

$${}_a^G D_t^{\alpha}\ \left({}_a^G D_t^{\beta}\ f(t)\right) = {}_a^G D_t^{\beta}\ \left({}_a^G D_t^{\alpha}\ f(t)\right) = {}_a^G D_t^{\alpha+\beta}\ f(t)\,,$$

only if $f^{(k)}(a) = 0\ \forall k = 0, 1, \ldots, r-2$ with $r = \max(n, m)$.

Proof 1.1.44 • *If $\beta < 0$ and $\alpha < 0$ then:*

$$\begin{aligned}
&{}_a^G D_t^{\alpha}\left({}_a^G D_t^{\beta}\ f(t)\right) = \frac{1}{\Gamma(-\alpha)}\int_a^t (t-\tau)^{-\alpha-1}\left({}_a^G D_t^{\beta}\ f(\tau)\right)d\tau,\\
&= \frac{1}{\Gamma(-\alpha)\Gamma(-\beta)}\int_a^t (t-\tau)^{-\alpha-1}\int_a^\tau (\tau-s)^{-\beta-1} f(s)\,dsd\tau,\\
&= \frac{1}{\Gamma(-\alpha)\Gamma(-\beta)}\int_a^t \int_s^t (t-\tau)^{-\alpha-1}(\tau-s)^{-\beta-1} f(s)\,dsd\tau, \qquad \textit{(using the Dirichlet equality)}\\
&= \frac{1}{\Gamma(-\alpha)\Gamma(-\beta)}\int_a^t (t-s)^{-\alpha-\beta-1} f(s)\int_0^1 (1-z)^{-\alpha-1} z^{-\beta-1}dzds,
\end{aligned}$$

(using the change of variable $\tau = s + z(t-s)$)

$$\begin{aligned}
&= \frac{1}{\Gamma(-\alpha)\Gamma(-\beta)}\int_a^t (t-s)^{-\alpha-\beta-1} f(s)\frac{\Gamma(-\alpha)\Gamma(-\beta)}{\Gamma(-\alpha-\beta)}ds,\\
&= \frac{1}{\Gamma(-\alpha-\beta)}\int_a^t (t-s)^{-\alpha-\beta-1} f(s)\,ds = {}_a^G D_t^{\alpha+\beta}\ f(t).
\end{aligned}$$

• *If $\beta < 0$ and $0 \le n-1 < \alpha < n$ so $\alpha = \alpha - n + n$, $\alpha - n < 0$ then:*

$$\begin{aligned}
{}_a^G D_t^{\alpha}\left({}_a^G D_t^{\beta}\ f(t)\right) &= \frac{d^n}{dt^n}\left\{{}_a^G D_t^{\alpha-n}\left({}_a^G D_t^{\beta}\ f(t)\right)\right\},\\
&= \frac{d^n}{dt^n}\left({}_a^G D_t^{\alpha-n+\beta}\ f(t)\right),\\
&= {}_a^G D_t^{\alpha-n+\beta+n}\ f(t) = {}_a^G D_t^{\alpha+\beta}\ f(t).
\end{aligned}$$

• *If $0 \le m-1 < \beta < m$ and $\alpha < 0$ then:*

$$ {}_a^G D_t^{\beta}\ f(t) = \sum_{k=0}^{m-1}\frac{f^{(k)}(a)(t-a)^{k-\beta}}{\Gamma(k-\beta+1)} + \frac{1}{\Gamma(m-\beta)}\int_a^t (t-\tau)^{m-\beta-1} f^{(m)}(\tau)\,d\tau, $$

if $f^{(k)}(a) = 0\ \forall k = 0, 1, \ldots, m-2$ we have

$$ {}_a^G D_t^{\beta}\ f(t) = \frac{f^{(m-1)}(a)(t-a)^{m-\beta-1}}{\Gamma(m-\beta)} + \frac{1}{\Gamma(m-\beta)}\int_a^t (t-\tau)^{m-\beta-1} f^{(m)}(\tau)\,d\tau, $$

and $(t-a)^{k-\beta}$ have non-integrable singularities so ${}_a^G D_t^{\alpha+\beta}\ f(t)$ only exists if $f^{(k)}(a) = 0\ \forall k = 0, 1, \ldots, m-2$ so

$$\begin{aligned}
{}_a^G D_t^{\alpha}\left({}_a^G D_t^{\beta}\ f(t)\right) &= \frac{f^{(m-1)}(a)(t-a)^{m-\alpha-\beta-1}}{\Gamma(m-\alpha-\beta)} + {}_a^G D_t^{\alpha+\beta-m}\ f^{(m)}(t)\\
&= \frac{f^{(m-1)}(a)(t-a)^{m-\alpha-\beta-1}}{\Gamma(m-\alpha-\beta)} + \frac{1}{\Gamma(m-\alpha-\beta)}\int_a^t (t-\tau)^{m-\alpha-\beta-1} f^{(m)}(\tau)\,d\tau,\\
&= {}_a^G D_t^{\alpha+\beta}\ f(t).
\end{aligned}$$

•*If $0 \le m-1 < \beta < m$ and $0 \le n-1 < \alpha < n$ then:*

$$ {}_a^G D_t^{\alpha}\left({}_a^G D_t^{\beta}\ f(t)\right) = \frac{d^n}{dt^n}\left\{{}_a^G D_t^{\alpha-n}\left({}_a^G D_t^{\beta}\ f(t)\right)\right\} $$

only if $f^{(k)}(a) = 0 \ \forall k = 0, 1, \ldots, m-2$ *so:*

$$\begin{aligned} {}_{a}^{G}D_{t}^{\alpha}\left({}_{a}^{G}D_{t}^{\beta}\ f(t)\right) &= \frac{d^{n}}{dt^{n}}\left\{ {}_{a}^{G}D_{t}^{\alpha+\beta-n}\ f(t)\right\} \\ &= {}_{a}^{G}D_{t}^{\alpha+\beta}\ f(t). \end{aligned}$$

Theorem 1.1.45 *If* $f \in L^{1}(0, b)$, *the Laplace transform of the Grünwald-Letnikov fractional derivative is formulated as follows:*

$$\mathcal{L}\left({}_{0}^{G}D_{t}^{\alpha}f \ \right)(s) = s^{\alpha}\mathcal{L}f(s), \tag{1.34}$$

with $0 \leq \alpha < 1$.

Proof 1.1.46 *We have*

$${}_{0}^{G}D_{t}^{\alpha}\ f(t) = \frac{f(0)\,t^{-\alpha}}{\Gamma(1-\alpha)} + \frac{1}{\Gamma(1-\alpha)}\int_{0}^{t}(t-x)^{-\alpha}f'(x)\,dx,$$

so

$$\begin{aligned} \mathcal{L}\left({}_{0}^{G}D_{t}^{\alpha}\ f\right)(s) &= \frac{f(0)}{\Gamma(1-\alpha)}\mathcal{L}\left(t^{-\alpha}\right)(s) + \mathcal{L}\left({}_{0}I_{t}^{1-\alpha}f'\right)(s), \\ &= \frac{f(0)}{\Gamma(1-\alpha)}\int_{0}^{\infty}e^{-st}t^{-\alpha}dt + s^{\alpha-1}\mathcal{L}f'(s), \qquad (\textit{using}\,(1.17\,)) \\ &= \frac{f(0)}{s^{1-\alpha}\Gamma(1-\alpha)}\Gamma(1-\alpha) + s^{\alpha-1}\left(s\mathcal{L}f(s) - f(0)\right), \qquad (\textit{using}\,(1.7)) \\ &= s^{\alpha}\mathcal{L}f(s). \end{aligned}$$

Remark 1.1.47 *For* $\alpha \geq 1$ *does not exist the Laplace transform in the classical sense but in the sense of the distributions we also have:*

$$\mathcal{L}\left({}_{0}^{G}D_{t}^{\alpha}f \ \right)(s) = s^{\alpha}\mathcal{L}f(s). \tag{1.35}$$

Chapter 2

Chaos and Local Stability of Fixed Points

2.1 Chaos

Chaotic behavior [47, 46, 5, 9, 50, 47] is related to instability and non-linearity in deterministic dynamic systems. The system manifests a very high sensitivity to changes in conditions, which is affirmed by Poincaré in a chapter in his book entitled "Science et Méthode "[5]. There is no standard definition of chaos hence the publication of several slightly different definitions. Two important mathematical principles explain the chaotic behavior, those of Devaney [20] and Li-Yorke [34].

2.1.1 Chaos in the Sense of Devaney

Devaney proposed the following definition of chaos: a dynamic system is chaotic if and only if

- it is topologically transitive.
- it has a dense set of periodic orbits.
- it presents the phenomenon of sensitivity to initial conditions.

The first two assumptions imply the third without the converse being true [30].

2.1.2 Chaos in the Sense of Li-Yorke

Li and Yorke introduced the first mathematical definition of chaos. They established a very simple criterion: "The presence of three periods implies chaos". This criterion plays a very important role in the analysis of chaotic dynamical systems.

2.2 Attractors

2.2.1 Types of Attractors

There are four main types of attractors:

1. fixed point: is a point in phase space towards which the trajectories tend, it is therefore a constant stationary solution.

2. cycle limit: is a closed trajectory in the phase space towards which the trajectories tend. It is therefore a periodic solution to the system.

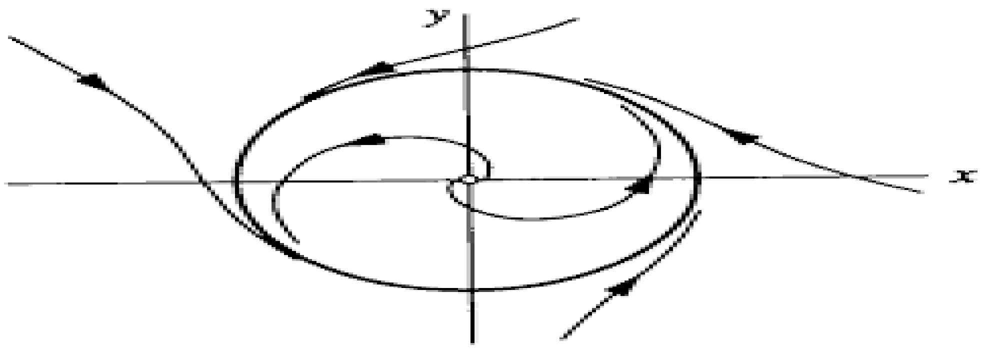


Figure 2.1: Cycle limit.

3. torus: represents the movements resulting from two or more independent oscillations, which are sometimes called "quasi-periodic movements".

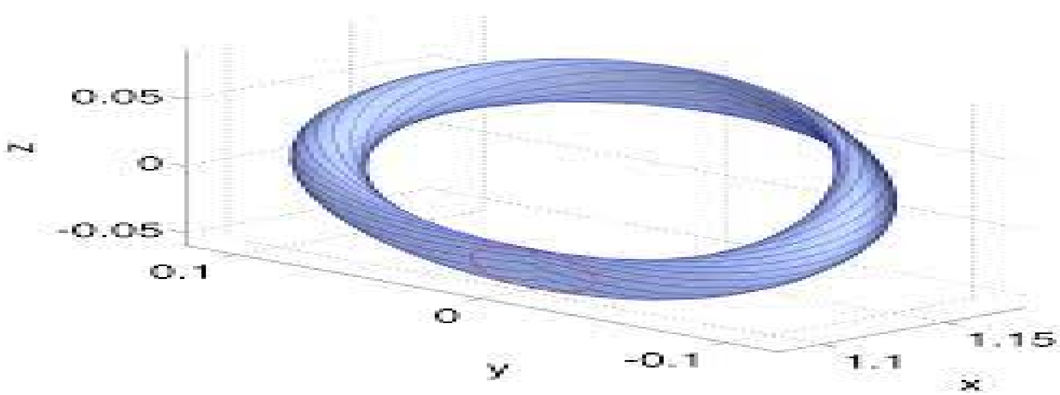


Figure 2.2: Torus.

4. Chaotic attractors (strange attractor): we observe that the trajectory in phase space to remains confined in a well-defined region, after a transitional period of variable duration.

2.2.2 Properties of a Chaotic Attractors

Strange attractor manifest the following properties:

Self-similarity: the geometric pattern that repeats on scales smaller, and smaller whatever the scale where we look at this structure, the appearance looks identical in Figure 2.3.

Sensitivity to initial conditions: two trajectories initially very close to the phase space diverge from each other and diverge exponentially over time, but this divergence cannot be indefinite because the attractor has a finite diameter in Figure 2.4.

The fractal dimension: the dimension "d " of the strange attractor is not integer. It must be strictly greater than the dimension of the phase space.

Stretching and folding: The strange attractor is invariant by stretching and folding for multiple iterations. If the system is three-dimensional, stretching the attractor by the flow is in one direction, folding in another direction, and periodic behavior in the third direction.

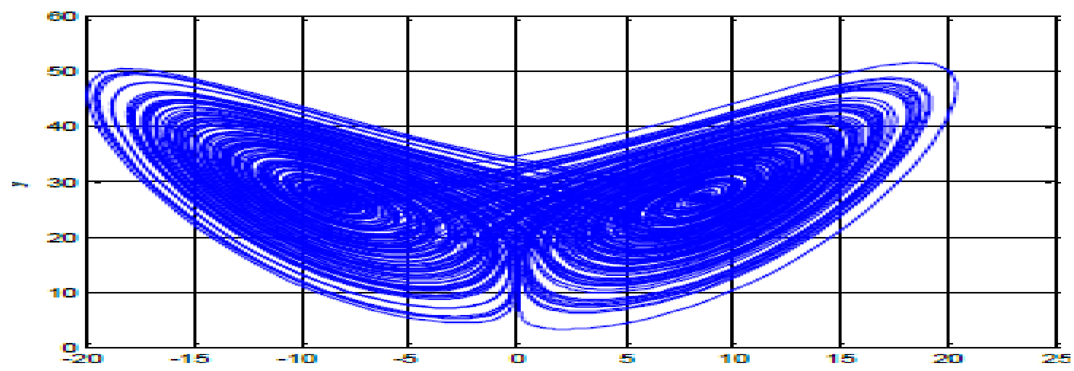


Figure 2.3: Lorenz attractor.

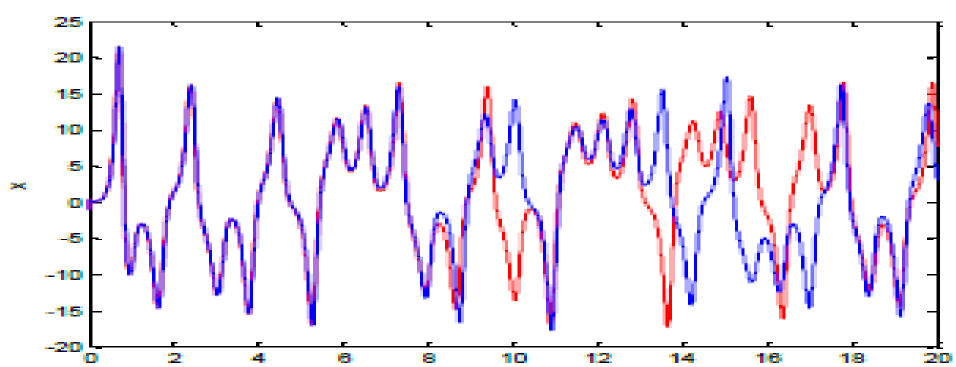


Figure 2.4: Sensitivity to initial conditions.

2.3 General Notions of Discrete Dynamical Systems

The objective of this part is to introduce many basic concepts and techniques of discrete dynamic systems theory.

2.3.1 Definitions

Definition 2.3.1 *[46] A dynamical system is one whose state changes with time* (t)*. Two main types of dynamical system are encountered in applications: those for which the time variable is discrete* ($t \in \mathbb{Z}$ *or* $\mathbb{N}$) *and those for which it is continuous* ($t \in \mathbb{R}$)*.*

Definition 2.3.2 *[46] (Discrete dynamic system)*
Let f be a function of class C^1 defined on an open $D \subset \mathbb{R}^m$. A discrete dynamic system noted $(D, \mathbb{N}, f)$ is a relation of the form

$$x_{n+1} = f(x_n). \tag{2.1}$$

Thus, if $\circ$ represents the composition of the applications, we have

$$x_n = f^n(x_0), \tag{2.2}$$

or

$$f^n(x) = f \circ f \circ \cdots \circ f(x), \forall n \in \mathbb{N} \text{ and } f^0 = Id,\ x_0 \text{ is the initial condition.}$$

The application f is called recurrence, iteration or point transformation. If the discrete dynamic system is invertible, the equality (2.2) remains true for $n \in \mathbb{Z}$.

Definition 2.3.3 *[46](Autonomous and non-autonomous discrete dynamic systems)*
When the function f in (2.2) *depends explicitly on time t, the system is known as non-autonomous. Otherwise, the system is known to be autonomous.*

Definition 2.3.4 *[46] (Trajectories)*
Given the initial point x_0, we called the orbit or trajectory of the system (2.2) *the sequence*

$$\begin{aligned} \mathcal{O}(x_0) &= \{x(0) = x_0,\ x(1) = f(x(0)), \ldots, x(n+1) = f(x(n)), \ldots\} \\ &= \{x_0,\ x_1,\ x_2, \ldots, x_n, \ldots\}. \end{aligned}$$

Example 2.3.5 *The fixed point is a simple trajectory.*

2.3.2 Exponents of Lyapunov

The Lyapunov exponent denoted λ can be considered a quantitative measure of sensitivity to initial conditions ([9]). For two slightly different initial states x_0 and $x_0 + \epsilon$, their divergence after n iterations can be characterized as follows:

$$|f^n(x_0 + \epsilon) - f^n(x_0)| = \epsilon e^{n\lambda(x_0)}$$

As shown in figure (2.5), the Lyapunov exponent gives the average rate of divergence of two trajectories and depends on the initial conditions.

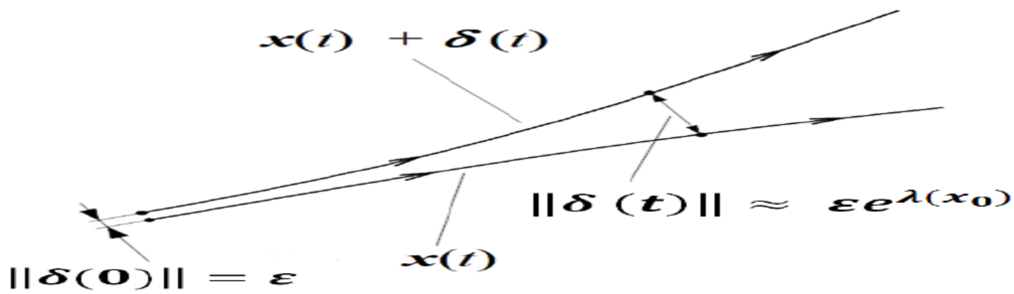


Figure 2.5: Quantification of the divergence of trajectories by Lyapunov exponents.

Case of a One-Dimensional Discrete Map

$|f^n(x_0+\varepsilon) - f^n(x_0)| \simeq \varepsilon e^{n\lambda}$ where $n\lambda \simeq \ln \dfrac{|f^n(x_0+\varepsilon) - f^n(x_0)|}{\varepsilon}$

and for $\varepsilon \to 0$ we have:

$$\begin{aligned}
\lambda &\simeq \frac{1}{n}\ln\left|\frac{df^n(x_0)}{dx_0}\right|, \\
&\simeq \frac{1}{n}\ln\left|\frac{df^n(x_0)}{df^{n-1}(x_0)}\frac{df^{n-1}(x_0)}{df^{n-2}(x_0)}\cdots\frac{df^1(x_0)}{dx_0}\right|, \\
&\simeq \frac{1}{n}\ln\left|\frac{df(x_{n-1})}{dx_{n-1}}\frac{df(x_{n-2})}{dx_{n-2}}\cdots\frac{df(x_0)}{dx_0}\right|, \\
&\simeq \frac{1}{n}\sum_{i=1}^{n-1}\ln\left|\frac{df(x_i)}{dx_i}\right|,
\end{aligned}$$

finally for $n \to 0$ we have:

$$\lambda = \lim_{n\to 0}\frac{1}{n}\sum_{i=1}^{n-1}\ln\left|f'(x_i)\right|$$

λ is called Lyapunov exponent, he indicates the average rate of divergence.

* If $\lambda > 0$ so there is a sensitivity to the initial conditions.
* If $\lambda < 0$ trajectories are approaching and we lose information on the initial conditions.

Case of a Multidimensional Discrete Map

Let f be a discrete map from $\mathbb{R}^m$ to $\mathbb{R}^m$:

$$x_{n+1} = f(x_n).$$

An $m-$dimensional system has m Lyapunov exponents, each of them measures the rate of divergence along one of the axes of the system. As before we are interested in:

$$|f^n(x_0+\varepsilon) - f^n(x_0)| \simeq \varepsilon e^{n\lambda}.$$

Pose $x_0^{'} = x_0 + \varepsilon$, we have the development in limited series of order 1 of $f^n\left(x_0^{'}\right)$ in the vicinity of x_0 following:

$$\begin{aligned} x_n^{'} - x_n &= f^n\left(x_0^{'}\right) - f^n\left(x_0\right), \\ &= f^n\left(x_0\right) + \frac{df^n\left(x_0\right)}{dx_0}\left(x_0^{'} - x_0\right) - f^n\left(x_0\right), \\ &= \frac{df^n\left(x_0\right)}{dx_0}\left(x_0^{'} - x_0\right), \\ &= J^n\left(x_0\right)\left(x_0^{'} - x_0\right), \end{aligned}$$

$J^n\left(x_0\right)$ denotes the Jacobian matrix of f^n at point x_0. This is an $m \times m$ square matrix, if it is diagonalizable, that is to say if there exist two matrices P_m invertible and D_m diagonal of the eigenvalues $u_i\left(f^n\left(x_0\right)\right),\ i = 1, \ldots, m$ of J^n such that:

$$J^n = P_m D_m P_m^{-1}.$$

We then define the m Lyapunov exponents as follows:

$$\lambda_i = \lim_{n \to 0} \frac{1}{n} \ln\left|u_i\left(f^n\left(x_0\right)\right)\right|,\ \ i = 1, \ldots, m. \tag{2.3}$$

For x^* the equilibrium point the formula (2.3) becomes

$$\lambda_i = \lim_{n \to 0} \frac{1}{n} \ln\left|u_i\left(x^*\right)\right|,\ \ i = 1, \ldots, m. \tag{2.4}$$

The following table summarizes the different Lyapunov exponent configurations discussed earlier:

Type of attractor	Lyapunov exponent
Fixed point	$\lambda_n \leq \cdots \leq \lambda_1 \leq 0$
Periodic	$\lambda_1 = 0,\ \lambda_n \leq \cdots \leq \lambda_2 \leq 0$
Periodic of 2 order	$\lambda_1 = \lambda_2 = 0,\ \lambda_n \leq \cdots \leq \lambda_3 \leq 0$
Periodic of k order	$\lambda_1 = \cdots = \lambda_k = 0,\ \lambda_n \leq \cdots \leq \lambda_{k+1} \leq 0$
Chaotic	$\lambda_1 > 0,\ \sum\limits_{i=1}^{n} \lambda_i < 0$
Hyper-chaotic	$\lambda_1, \lambda_2 > 0,\ \sum\limits_{i=1}^{n} \lambda_i < 0$

2.3.3 Limit Points, Limit Sets and a Periodic Orbits

Definition 2.3.6 *[37] A point z is known a limit point of the orbit $\mathcal{O}(x_0)$ if there is a subsequence $\{x_{n_k} : k = 0.1, \ldots\}$ of $\mathcal{O}(x_0)$ such that*

$$\lim_{k \longrightarrow +\infty} \|x_{n_k} - z\| = 0.$$

Definition 2.3.7 *[37] The point x_0 has period n or is a periodic point of period n if $f^n(x_0) = x_0$. If x_0 has period n. then the orbit of x_0, which is $\left\{x_0, f(x_0), f^2(x_0), \ldots, f^{n-1}(x_0)\right\}$ is a periodic orbit and is called an $n-$cycle.*

Remark 2.3.8 *1. A stationary orbit has a single limit point (fixed point).*

2. $\mathcal{O}(x_0)$ a p-periodic orbit has exactly p limit points.

Definition 2.3.9 *[37] The limit set $L(x_0)$ of the orbit $\mathcal{O}(x_0)$ in the set of all limit points of the orbit.*

The following is a fundamental equality between $L(x_0)$ and its image under the continuous function f which governs the dynamical system:

$$f(L(x_0)) = L(x_0), \quad \text{with } f(L(x_0)) = \{f(z) \,:\, z \in L(x_0)\}.$$

Definition 2.3.10 *[37]*

An orbit $\mathcal{O}(x_0)$ is said to be asymptotically stationary if its limit set is a stationary state, and asymptotically periodic if its limit set a periodic orbit.

An orbit $\mathcal{O}(x_0)$ such that $x_{n+p} = x_n$ for some $n \geq 1$ and some $p \geq 1$ is said to be eventually stationary if $p = 1$ and eventually periodic if $p > 1$.

Hence, every eventually stationary or eventually periodic orbit is respectively, asymptotically stationary or asymptotically periodic. The converse is not true.

Definition 2.3.11 *[37]*

An orbit $\mathcal{O}(x_0)$ is stable if for every $r > 0$ there exists $\delta > 0$ such that $\|x_0 - y_0\| \leq \delta$ implies that $\|x_n - y_n\| \leq r$ for every $n \geq 1$.

Local Stability of Fixed Points

The notion of stability plays a role in mathematics and mechanics, numerical algorithms, quantum mechanics, economic models, nuclear physics, etc. J_{eq} the Jacobian matrix at the fixed point x^* of the map f, to simplify the notions of the local stability of the fixed point x^*. To characterize the nature of this fixed point, we give the following definitions:

Definition 2.3.12 *[26](fixed points)*

The point x^* is a fixed point of $f(x)$ if and only if $f(x^*) = x^*$.

Remark 2.3.13 *Sometimes, these points are called stationary points or equilibrium points.*

Definition 2.3.14 *[26] A point $\overline{x}$ is said to be an eventually equilibrium or stationary point for equation (2.1) or an eventually fixed point for f if there exists a positive integer r and a fixed point $x^* \in \mathbb{R}$ such that*

$$f^{\,r}(\overline{x}) = x^*, \quad f^{\,r-1}(\overline{x}) \neq x^*.$$

Definition 2.3.15 *[26] A fixed point x^* is stable if for all $\epsilon > 0$ there exists $\delta > 0$ such that*

$\|x_0 - x^*\| \leq \delta$ *involved* $\|x_n - x^*\| \leq \epsilon$ *for all* $n \geq 1$.

Definition 2.3.16 *[26] A fixed point x^* is not stable is known as unstable.*

Definition 2.3.17 *[26] If all of eigenvalues of jacobian matrix strictly less than 1, then the fixed point x^* of f is said to be asymptotically stable.*

Definition 2.3.18 *[26] If one of eigenvalues of jacobian matrix strictly greater than 1, then the fixed point x^* of f is said to be unstable.*

Remark 2.3.19 *This conditions of stability and instability on the eigenvalues is not necessary, it is only sufficient.*

Stability of Perodic Orbits

Definition 2.3.20 *[37] A periodic orbit of period n $\{x_0, \ldots, x_{n-1}, \ldots\}$ is stable if each point x_j, $j = 0, 1, \ldots, n-1$ is a fixed point of the dynamical system governed by f^n.*

Remark 2.3.21 *The stability of x_0 as a fixed point of f^n guarantees(it is actually equivalent to) the stability of all remaining points $x_1, \ldots, x_{n-1}$ whenever f is continuous.*

Definition 2.3.22 *[37] A periodic orbit of period n $\{x_0, \ldots, x_{n-1}, \ldots\}$ which is not stable is said to be unstable.*

Remark 2.3.23 *The orbit is unstable if one of its points x_j, $j = 0, 1, \ldots, n-1$ is an unstable fixed point of f^n. the instability of x_0 as a fixed point of f^n guarantees(it is actually equivalent to) the instability of all remaining points $x_1, \ldots, x_{n-1}$ whenever f is continuous.*

2.3.4 Bifurcation

Definition 2.3.24 *[37] Consider the following nonlinear dynamic system:*

$$x(n+1) = f(x(n), \mu) \tag{2.5}$$

from where $x(n) \in \mathbb{R}^n$, $\mu \in \mathbb{R}^m$, $n \in \mathbb{N}$ *and* $f : \mathbb{R}^n \times \mathbb{R}^m \longrightarrow \mathbb{R}^n$.

Definition 2.3.25 *[37] A bifurcation is a qualitative change of the solution x of the system* (2.5) *when we modify the control parameter μ, i.e. the disappearance or change of stability and the appearance of new solutions.*

Definition 2.3.26 *[37] A bifurcation diagram is a portion of the parameter space on which all the bifurcation points are represented.*

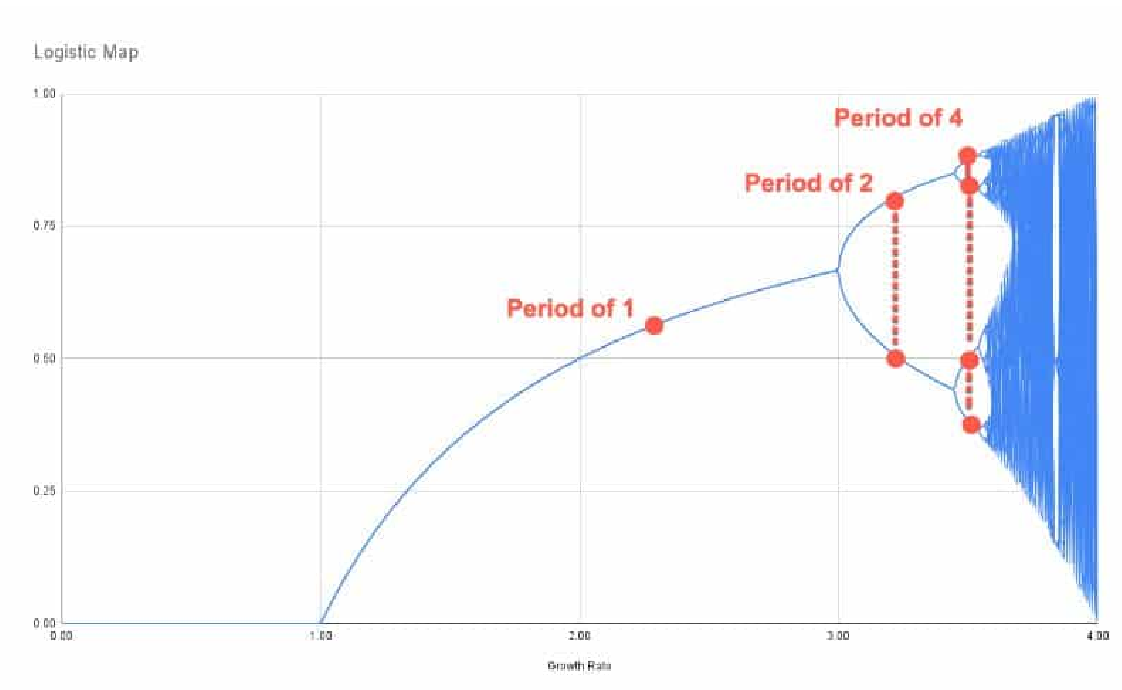


Figure 2.6: Bifurcation diagram.

Chapter 3

Discretization Methods

There are many discretization methods that have been used to construct the discrete time model using continuous-time methods such as Euler's method, Runge–Kutta method, Taylor method, predictor-corrector method, non standard finite difference methods, and piecewise constant approximation method. Some of them are approximations for the derivative and some for the integral. The methods are used because, for most fractional differential equations, obtaining an exact analytical solution is very complicated; thus, it is necessary to turn to numerical methods.

3.1 Presentation of the Initial-Value Problem

Let us consider the following initial-value problem:

$$ {}_0\mathcal{D}_t^{\sigma_n} x(t) + \sum_{j=1}^{n-1} p_j(t)\, \mathcal{D}_t^{\sigma_{n-1}} x(t) + p_n(t)\, x(t) = f(t), \qquad 0 < t < T < \infty \tag{3.1} $$

$$ \left[{}_0\mathcal{D}_t^{\sigma_{k-1}} x(t)\right]_{t=0} = b_k, \qquad k = 1, 2, \ldots, n, \tag{3.2} $$

where

$$ \begin{aligned} {}_a\mathcal{D}_t^{\sigma_k} &\approxeq {}_aD_t^{\alpha_k}\ {}_aD_t^{\alpha_{k-1}}\ \ldots\ {}_aD_t^{\alpha_1}\ ; \\ {}_a\mathcal{D}_t^{\sigma_{k-1}} &\approxeq {}_aD_t^{\alpha_{k-1}}\ {}_aD_t^{\alpha_{k-2}}\ \ldots\ {}_aD_t^{\alpha_1}\ ; \\ \sigma_k &= \sum_{j=1}^{k} \alpha_j, \quad k = 1, 2, \ldots, n, \\ 0 &< \alpha_j \le 1, \quad j = 1, 2, \ldots, n, \end{aligned} $$

$p_k(t)$, $j = 1, 2, \ldots, n$ are continuous functions on $[0, T]$, and $f(t) \in L^1(0, T)$, i.e.

$$ \int_0^T |f(t)|\, dt < \infty. $$

An equation that is not linear is called nonlinear.

3.2 Euler's Method

The initial value problem of fractional differential equations

$$\begin{cases} {}_aD_t^{\alpha} x(t) = f(t, x(t)), \\ x(a) = x_0. \end{cases} \tag{3.3}$$

The fractional derivative is in Riemann-Liouville sense with the order $0 < \alpha < 1$ and $t \in [a, b]$, $f \in \mathbb{R}$. If we apply the Riemann-Liouville fractional derivative of $1-\alpha$ order on (3.3), and use the properties of fractional integration and fractional derivative, we get the following equation:

$$\begin{cases} x'(t) = F(t, x(t)) = \ {}_aD_t^{1-\alpha}\ f(t, x(t)), \\ x(a) = x_0. \end{cases} \tag{3.4}$$

$F(t,x) = \ {}_aD_t^{1-\alpha}\ f(t,x) = \dfrac{1}{\Gamma(1-\alpha)} \dfrac{d}{dt} \int_a^t (t-\tau)^{-\alpha} f(\tau, x)\, d\tau.$

We cut the time domain according to step h, where $h = (t_{n+1} - t_n)$.

3.2.1 Explicit Euler Method

$$\begin{cases} x_{n+1} = x_n + h\left[F(t, x(t))\,|_{t=t_n}\right], \\ x(a) = x_0. \end{cases}$$

3.2.2 Implicit Euler Method

$$\begin{cases} x_{n+1} = x_n + h\left[F(t, x(t))\,|_{t=t_{n+1}}\right], \\ x(a) = x_0. \end{cases}$$

3.3 Taylor Method

3.3.1 Generalized Taylor's Formula

Theorem 3.3.1 *(Generalized mean value theorem)*

Suppose that $f(t) \in C[a,b]$ and ${}_a^CD_t^{\alpha} \in C(a,b]$, for $0 < \alpha \leq 1$, then we have

$$f(t) = f(a) + \frac{1}{\Gamma(\alpha+1)} \left({}_a^CD_t^{\alpha} f \right)(\xi) . (t-a)^{\alpha}, \tag{3.5}$$

with $a \leq \xi \leq t$, $\forall t \in (a, b]$.

Proof 3.3.2 *From* (1.8) *we have*

$$\left({}_aI_t^{\alpha}\left({}_a^CD_t^{\alpha} f \right)\right)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-x)^{\alpha-1} \left({}_a^CD_t^{\alpha} f \right)(x)\, dx.$$

Using the integral mean value theorem, we get

$$\begin{aligned} \left({}_aI_t^{\alpha}\left({}_a^CD_t^{\alpha} f \right)\right)(t) &= \frac{1}{\Gamma(\alpha)} \left({}_a^CD_t^{\alpha} f \right)(\xi) \int_a^t (t-x)^{\alpha-1}\, dx, \\ &= \frac{1}{\Gamma(\alpha+1)} \left({}_a^CD_t^{\alpha} f \right)(\xi)\, (t-a)^{\alpha}, \end{aligned}$$

with $a \leq \xi \leq t$.
From (1.23) *we have*

$$\left({}_aI_t^{\alpha} \left({}_a^CD_t^{\alpha} f \right) \right)(t) = f(t) - f(a),$$

so

$$f(t) = f(a) + \frac{1}{\Gamma(\alpha+1)} \left({}_a^CD_t^{\alpha} f \right)(\xi)(t-a)^{\alpha}.$$

Remark 3.3.3 *If* $\alpha = 1$, *the generalized mean value theorem reduces to the classical mean value theorem.*

Theorem 3.3.4 *Suppose that* ${}_a^CD_t^{n\alpha} f(t)$, ${}_a^CD_t^{(n+1)\alpha} f(t) \in C(a,b]$, *and* $0 < \alpha \leq 1$ *then we have*

$$\left({}_aI_t^{n\alpha} \left({}_a^CD_t^{n\alpha} f \right) \right)(t) - \left({}_aI_t^{(n+1)\alpha} \left({}_a^CD_t^{(n+1)\alpha} f \right) \right)(t) = \frac{(t-a)^{n\alpha}}{\Gamma(n\alpha+1)} \left({}_a^CD_t^{n\alpha} f \right)(a), \tag{3.6}$$

where

$${}_a^CD_t^{n\alpha} = {}_a^CD_t^{\alpha} \, . \, {}_a^CD_t^{\alpha} \cdots {}_a^CD_t^{\alpha} \qquad \text{(n-times)}.$$

Proof 3.3.5 *We have*

$$\begin{aligned}
& \left({}_aI_t^{n\alpha} \left({}_a^CD_t^{n\alpha} f \right) \right)(t) - \left({}_aI_t^{(n+1)\alpha} \left({}_a^CD_t^{(n+1)\alpha} f \right) \right)(t) \\
= \; & \left({}_aI_t^{n\alpha} \left({}_a^CD_t^{n\alpha} f \right) \right)(t) - \left({}_aI_t^{n\alpha} \; {}_aI_t^{\alpha} \left({}_a^CD_t^{(n+1)\alpha} f \right) \right)(t) \quad \text{(using (1.15))} \\
= \; & {}_aI_t^{n\alpha} \left(\left({}_a^CD_t^{n\alpha} f \right)(t) - \left({}_aI_t^{\alpha} \left({}_a^CD_t^{(n+1)\alpha} f \right) \right)(t) \right) \quad \text{(using (1.16))} \\
= \; & {}_aI_t^{n\alpha} \left(\left({}_a^CD_t^{n\alpha} f \right)(t) - \left({}_aI_t^{\alpha} \; {}_a^CD_t^{\alpha} \right) \left({}_a^CD_t^{n\alpha} f \right)(t) \right) \\
= \; & {}_aI_t^{n\alpha} \left(\left({}_a^CD_t^{n\alpha} f \right)(t) - \left[\left({}_a^CD_t^{n\alpha} f \right)(t) - \left({}_a^CD_t^{n\alpha} f \right)(a) \right] \right) \\
= \; & {}_aI_t^{n\alpha} \left[\left({}_a^CD_t^{n\alpha} f \right)(a) \right] \\
= \; & \frac{1}{\Gamma(n\alpha)} \int_a^t (t-x)^{n\alpha-1} \left({}_a^CD_t^{\alpha} f \right)(a)\, dx \\
= \; & \frac{(t-a)^{n\alpha}}{\Gamma(n\alpha+1)} \left({}_a^CD_t^{n\alpha} f \right)(a).
\end{aligned}$$

Theorem 3.3.6 *(Generalized Taylor's formula)*
Suppose that ${}_a^CD_t^{k\alpha} f(t) \in C(a,b]$ *for* $k = 0, 1, \ldots, n+1$, *where* $0 < \alpha \leq 1$, *then we have*

$$f(t) = \frac{\left({}_a^CD_t^{(n+1)\alpha} f \right)(\xi)}{\Gamma((n+1)\alpha+1)} . (t-a)^{(n+1)\alpha} + \sum_{i=0}^{n} \frac{(t-a)^{i\alpha}}{\Gamma(i\alpha+1)} \left({}_a^CD_t^{i\alpha} f \right)(a), \tag{3.7}$$

with $a \leq \xi \leq t$, $\forall t \in (a,b]$, *and* ${}_a^CD_t^{n\alpha} = {}_a^CD_t^{\alpha} \, . \, {}_a^CD_t^{\alpha} \cdots {}_a^CD_t^{\alpha} \qquad$ *(n-times).*

Proof 3.3.7 *From (3.6), we have*

$$\sum_{i=0}^{n}\left(\left({}_aI_t^{i\alpha}\ {}_a^CD_t^{i\alpha}f\ \right)(t)-\left({}_aI_t^{(i+1)\alpha}\ {}_a^CD_t^{(i+1)\alpha}f\ \right)(t)\right)=\sum_{i=0}^{n}\frac{(t-a)^{i\alpha}}{\Gamma(i\alpha+1)}\left(\ {}_a^CD_t^{i\alpha}f\ \right)(a),$$

and

$$\sum_{i=0}^{n}\left(\left({}_aI_t^{i\alpha}\ {}_a^CD_t^{i\alpha}f\ \right)(t)-\left({}_aI_t^{(i+1)\alpha}\ {}_a^CD_t^{(i+1)\alpha}f\ \right)(t)\right)=f(t)-\left({}_aI_t^{(n+1)\alpha}\ {}_a^CD_t^{(n+1)\alpha}f\ \right)(t),$$

so

$$f(t)-\left({}_aI_t^{(n+1)\alpha}\ {}_a^CD_t^{(n+1)\alpha}f\ \right)(t)=\sum_{i=0}^{n}\frac{(t-a)^{i\alpha}}{\Gamma(i\alpha+1)}\left(\ {}_a^CD_t^{i\alpha}f\ \right)(a).$$

Using the integral mean value theorem, we get

$$\begin{aligned}\left({}_aI_t^{(n+1)\alpha}\ {}_a^CD_t^{(n+1)\alpha}f\ \right)(t) &= \frac{1}{\Gamma((n+1)\alpha)}\int_a^t (t-x)^{(n+1)\alpha-1}\ \left({}_a^CD_t^{(n+1)\alpha}f\right)(x)\ dx,\\ &= \frac{\left(\ {}_a^CD_t^{(n+1)\alpha}f\ \right)(\xi)}{\Gamma((n+1)\alpha+1)}.(t-a)^{(n+1)\alpha}.\end{aligned}$$

So

$$f(t)=\frac{\left(\ {}_a^CD_t^{(n+1)\alpha}f\ \right)(\xi)}{\Gamma((n+1)\alpha+1)}.(t-a)^{(n+1)\alpha}+\sum_{i=0}^{n}\frac{(t-a)^{i\alpha}}{\Gamma(i\alpha+1)}\left(\ {}_a^CD_t^{i\alpha}f\ \right)(a).$$

Approximation of functions through the generalized Taylor's formula

Theorem 3.3.8 *Suppose that* ${}_a^CD_t^{k\alpha}f(t)\ \in C(a,b]$ *for* $k=0,1,\ldots,n+1$, *where* $0<\alpha\leq 1$, *then*

$$f(t)\simeq P_N^{\alpha}(t)=\sum_{i=0}^{N}\frac{(t-a)^{i\alpha}}{\Gamma(i\alpha+1)}\left(\ {}_a^CD_t^{i\alpha}f\ \right)(a),$$

and the error term $R_N^{\alpha}(t)$ *has the form*

$$R_N^{\alpha}(t)=\frac{\left(\ {}_a^CD_t^{(N+1)\alpha}f\ \right)(\xi)}{\Gamma((N+1)\alpha+1)}.(t-a)^{(N+1)\alpha},$$

with $a\leq\xi\leq t$.

Proof 3.3.9 *The larger the* N, *the greater the* $P_N^{\alpha}(t)$ *and the lower the error* $R_N^{\alpha}(t)$, *because* t *approaches* a *in this case.*

3.3.2 Series Solutions of Fractional Differential Equations

In this part, we use the generalized Taylor's formula to solve fractional differential equations. This method is very useful and can be applied to solve many important fractional differential equations with non constant coefficients.

To solve a fractional differential equation using the generalized Taylor formula, we write the solution as a fractional series of the form:

$$\sum_{n\geq 0} c_n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)},$$

and writing each term in the differential equation as a fractional power series, equate the coefficients of the resulting series on both sides of the equation, and finally find the unknown coefficients in the series representation of the assumed solution.

Example 3.3.10 *Consider the initial value problem*

$$ {}_0^C D_t^{\alpha} x(t) = \lambda x(t), \quad x(0) = x_0, \tag{3.8}$$

where $0 < \alpha \leq 1$, $\lambda \in \mathbb{R}$ *and* $t > 0$.

Using the generalized Taylor's formula, assuming that the solution $x(t)$ *can be written as*

$$x(t) = \sum_{n\geq 0} c_n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}, \tag{3.9}$$

From the definition of Caputo fractional derivative (1.19), *we obtain*

$$\begin{aligned}
{}_0^C D_t^{\alpha} x(t) &= \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-y)^{-\alpha} x'(y)\, dy, \\
&= \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-y)^{-\alpha} \sum_{n\geq 1} c_n \frac{t^{n\alpha-1}}{\Gamma(n\alpha)} dy, \\
&= \frac{c_1}{\Gamma(1-\alpha)\Gamma(\alpha)} \int_0^t t^{\alpha-1} (t-y)^{-\alpha} dy + \frac{c_2}{\Gamma(1-\alpha)\Gamma(2\alpha)} \int_0^t t^{2\alpha-1} (t-y)^{-\alpha} dy + \cdots,
\end{aligned}$$

we pose $z = \dfrac{y}{t}$ *then*

$$\begin{aligned}
{}_0^C D_t^{\alpha} x(t) &= \frac{c_1}{\Gamma(1-\alpha)\Gamma(\alpha)} \int_0^1 z^{\alpha-1} (1-z)^{-\alpha} dz + \frac{c_2 t^{\alpha}}{\Gamma(1-\alpha)\Gamma(2\alpha)} \int_0^1 z^{2\alpha-1} (1-z)^{-\alpha} dz + \cdots, \\
&= \frac{c_1 \Gamma(\alpha)\Gamma(1-\alpha)}{\Gamma(1-\alpha)\Gamma(\alpha)\Gamma(1)} + \frac{c_2 \Gamma(2\alpha)\Gamma(1-\alpha) t^{\alpha}}{\Gamma(1-\alpha)\Gamma(2\alpha)\Gamma(1+\alpha)} + \cdots, \\
&= \sum_{n\geq 1} c_n \frac{t^{(n-1)\alpha}}{\Gamma((n-1)\alpha+1)}.
\end{aligned}$$

And equation (3.8) *becomes*

$$\sum_{n\geq 0} c_{n+1} \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} - \lambda \sum_{n\geq 0} c_n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} = 0,$$

we get $\dfrac{t^{n\alpha}}{\Gamma(n\alpha+1)} \neq 0$ *so* $c_{n+1} - \lambda c_n = 0$ *with* $c_0 = x(0) = x_0$,
that is

$$c_n = \lambda^n x_0. \tag{3.10}$$

Substituting (3.10) *into* (3.9), *we obtain the solution*

$$x(t) = x_0 \sum_{n\geq 0} \lambda^n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} = x_0 \sum_{n\geq 0} \frac{(\lambda t^\alpha)^n}{\Gamma(n\alpha+1)} = x_0 E_\alpha(\lambda t^\alpha),$$

where $E_\alpha(t)$ *is the Mittag–Leffler function.*

Example 3.3.11 *Consider the fractional differential equation*

$${}_0^C D_t^{2\alpha} x(t) + {}_0^C D_t^{\alpha} x(t) - 2x(t) = 0, \tag{3.11}$$

where ${}_0^C D_t^{2\alpha} = {}_0^C D_t^{\alpha} \cdot {}_0^C D_t^{\alpha}$, *and* $t > 0$.

Using the generalized Taylor's formula, assuming that the solution $x(t)$ *can be written as*

$$x(t) = \sum_{n\geq 0} c_n \frac{t^{n\alpha}}{\Gamma(n\alpha+1)}, \tag{3.12}$$

From the definition of Caputo fractional derivative (1.19), *we obtain*

$${}_0^C D_t^{\alpha} x(t) = \sum_{n\geq 1} c_n \frac{t^{(n-1)\alpha}}{\Gamma((n-1)\alpha+1)}, \tag{3.13}$$

and

$${}_0^C D_t^{2\alpha} x(t) = \sum_{n\geq 2} c_n \frac{t^{(n-2)\alpha}}{\Gamma((n-2)\alpha+1)}. \tag{3.14}$$

And equation (3.11) *becomes*

$$\sum_{n\geq 0} (c_{n+2} + c_{n+1} - 2c_n) \frac{t^{n\alpha}}{\Gamma(n\alpha+1)} = 0,$$

we get $\dfrac{t^{n\alpha}}{\Gamma(n\alpha+1)} \neq 0$ *so* $c_{n+2} + c_{n+1} - 2c_n = 0$.
This gives

$$c_2 = 2c_0 - c_1,\ c_3 = -2c_0 + 3c_1, c_4 = 6c_0 - 5c_1, \cdots \tag{3.15}$$

Substituting (3.15) *into* (3.12), *we obtain the solution*

$$\begin{aligned} x(t) &= c_0 \left(1 + 2\frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - 2\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + 6\frac{t^{4\alpha}}{\Gamma(4\alpha+1)} + \cdots\right) + \\ & c_1 \left(\frac{t^{\alpha}}{\Gamma(\alpha+1)} - \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} + 3\frac{t^{3\alpha}}{\Gamma(3\alpha+1)} - 5\frac{t^{4\alpha}}{\Gamma(4\alpha+1)} + \cdots\right). \end{aligned}$$

3.4 Non Standard Finite Difference Methods

NSFD schemes, which were first proposed by Mickens ([38]). This class of schemes and their formulations center on two issues. First, how should discrete representations for derivatives be determined, and second, what are the proper forms to be used for nonlinear terms.

We can discretize the fractional differential equation

$$\begin{cases} D_t^{\alpha} x(t) = f(t, x(t)), & t \in [0, T] \\ x(0) = x_0, \end{cases} \tag{3.16}$$

To apply Mickens' scheme, we have chosen the Grünwald–Letnikov method approximation for the fractional derivative as follows:

$$D_t^{\alpha} x(t) = \lim_{h \longrightarrow 0} h^{-\alpha} \sum_{j=0}^{n} (-1)^j \binom{\alpha}{j} x(t - jh), \tag{3.17}$$

where $n = \dfrac{t}{h}$ and $[t]$ denotes the integer part of t and h is the step size.

Therefore , equation (3.16) is discretized in the next form,

$$\sum_{j=0}^{n} c_j^{\alpha} x_{n-j} = f(t_n, x_n), \quad n = 1, 2, 3, \ldots, \tag{3.18}$$

in this expression, $n - 1 < \alpha \leq n$, $n \in \mathbb{N}$, T is the final time, x_n is the approximation of $x(t_n)$, $t_n = nh$, c_j^{α} is the Grünwald–Letnikov coefficient defined as:

$$c_0^{\alpha} = h^{-\alpha}, \qquad c_j^{\alpha} = h^{-\alpha} (-1)^j \left(\begin{array}{c} \alpha \\ j \end{array} \right) = \left(1 - \frac{1+\alpha}{j}\right) c_{j-1}^{\alpha}, \quad j = 1, 2, 3, \ldots, \tag{3.19}$$

3.4.1 Non Standard Finite Difference Scheme

The non-standard finite difference (NSFD) schemes were first proposed by Mickens. A scheme is called nonstandard if at least one of the following two conditions is satisfied:

1. a nonlocal approximation is used for nonlinear terms appearing in the system.

2. the discretization of derivatives is not traditional and uses a non negative function.

In standard discretization, the derivative term $\dfrac{dx}{dt}$ is replaced by $\dfrac{x(t+h) - x(t)}{h}$. However, in the Mickens schemes this term is replaced by $\dfrac{x(t+h) - x(t)}{\phi(h)}$, where $\phi(h)$ is a continuous function of step size h, and this function has the following properties:

$$\phi(h) = h + o\left(h^2\right), \quad 0 < \phi(h) < 1 \text{ where } h \longrightarrow 0. \tag{3.20}$$

For example, of the function $\phi(h)$ that satisfy these conditions are $[Mickens, 2000]$

$$\phi(h) = h, \quad e^h - 1, \quad \sin(h), \quad \sinh(h), \quad \ln(1+h), \quad 1 - e^{-h} \text{ etc.} \tag{3.21}$$

The nonlinear terms $f(x(t))$ on the right-hand side in (3.16) can be in general replaced by nonlocal discrete representations[$Mickens$, 2000]. For example

$$\begin{aligned}
x^2 &\longrightarrow x_n x_{n+1}, \quad x_{n-1}x_n, \quad x_{n-1}x_{n+1}, \quad \left(\frac{x_{n-1}+x_n}{2}\right)x_{n-1}, \\
x^3 &\longrightarrow \left(\frac{x_{n+1}+x_{n-1}}{2}\right)x_n^2, \quad \left(\frac{x_{n-1}+x_n}{2}\right)x_{n-1}^2, \quad x_{n-1}^2 x_n, \quad x_{n-2}x_{n-1}x_n, \\
xy &= 2xy - xy \longrightarrow 2x_{n+1}y_n - x_{n+1}y_{n+1}, \quad x_{n+1}y_n, \quad x_n y_{n+1}.
\end{aligned}$$

Now we extend the above-mentioned approach to the fractional differential equation

$$\begin{cases} D_t^{\alpha} x(t) = f(t, x(t)), & t \in [0, T] \\ x(0) = x_0, \end{cases} \tag{3.22}$$

For this purpose, we employ the discretized Grünwald–Letnikov approximation formula from relation (3.18). Therefore, we can provide a modification of the NSFD scheme in the fractional sense as follows:

$$x_{n+1} = \frac{-\sum_{j=1}^{n+1} c_j^{\alpha} x_{n+1-j} + f(t_{n+1}, x_{n+1})}{c_0^{\alpha}}, \qquad n = 0, 1, 2, \ldots,$$

where

$$c_0^{\alpha} = (\phi(h))^{-\alpha}, \qquad c_j^{\alpha} = (-1)^j \begin{pmatrix} \alpha \\ j \end{pmatrix} = \left(1 - \frac{1+\alpha}{j}\right) c_{j-1}^{\alpha}, \quad j = 1, 2, 3, \ldots,$$

Example 3.4.1 *Consider a fractional-order generalization of the Rössler system [47]. In this system, the integer-order derivatives are replaced by fractional-order derivatives, as follows:*

$$\begin{cases} D^{\alpha_1} x(t) = -y(t) - z(t), \\ D^{\alpha_2} y(t) = x(t) + a y(t), \\ D^{\alpha_3} z(t) = b + z(t)(x(t) - c), \end{cases} \tag{3.23}$$

where (x, y, z) are the state variables and (a, b, c) are positive constants. Applying Mickens' scheme by replacing the step size h by a function $\phi(h)$ and using the Grünwald–Letnikov discretization method, yields the following equations:

$$\begin{cases} \sum_{j=0}^{n+1} c_j^{\alpha_1} x(t_{n+1-j}) = -y(t_n) - z(t_n), \\ \sum_{j=0}^{n+1} c_j^{\alpha_2} y(t_{n+1-j}) = x(t_{n+1}) + a(2y(t_n) - y(t_{n+1})), \\ \sum_{j=0}^{n+1} c_j^{\alpha_3} z(t_{n+1-j}) = b + 2x(t_{n+1}) z(t_n) - x(t_{n+1}) z(t_{n+1}) - c z(t_n), \end{cases}$$

to simplify, we have the following relations:

$$\begin{cases} x(t_{n+1}) = \dfrac{-\sum_{j=1}^{n+1} c_j^{\alpha_1} x(t_{n+1-j}) - y(t_n) - z(t_n)}{c_0^{\alpha_1}}, \\ y(t_{n+1}) = \dfrac{-\sum_{j=1}^{n+1} c_j^{\alpha_2} y(t_{n+1-j}) + x(t_{n+1}) + 2a y(t_n)}{c_0^{\alpha_2} + a}, \\ z(t_{n+1}) = \dfrac{-\sum_{j=1}^{n+1} c_j^{\alpha_3} z(t_{n+1-j}) + b + 2x(t_{n+1}) z(t_n) - c z(t_n)}{c_0^{\alpha_3} + x(t_{n+1})}, \end{cases}$$

where $c_0^{\alpha_1} = (\phi(h))^{-\alpha_1}$, $c_0^{\alpha_2} = (\phi(h))^{-\alpha_2}$, $c_0^{\alpha_3} = (\phi(h))^{-\alpha_3}$, $\phi(h) = \sin(h)$.

3.5 Piecewise Constant Method

In the following, we consider an initial value fractional differential equation of the form:

$$\begin{cases} {}_0^C D_t^\alpha x(t) = f(x(t)), \\ x(0) = x_0. \end{cases} \tag{3.24}$$

where $0 < \alpha < 1$ and $f(x(t))$ is potentially a nonlinear vector field.

The corresponding equation with a piecewise constant argument

$${}_0^C D_t^\alpha x(t) = f\left(x\left(h\left[\frac{t}{h}\right]\right)\right). \tag{3.25}$$

With time step h. The piecewise constant function is chosen so that in the limit of small h we recover the original equation, i.e.

$$\lim_{h\to 0} f\left(x\left(h\left[\frac{t}{h}\right]\right)\right) = f(x(t)). \tag{3.26}$$

Has been used a unit step function, defined by

$$u(t) = \begin{cases} 0 & t < 0, \\ 1 & t \geq 0, \end{cases} \tag{3.27}$$

we rewrite the right-hand side equation (3.25) as a sum, giving

$${}_0^C D_t^\alpha x(t) = \sum_{m=0}^{\infty} f(x(mh))(u(t-mh) - u(t-(m+1)h)). \tag{3.28}$$

The in finite sum on the right hand side of this equation is convergent, as for any value t the difference between the step functions is 0 for all bar one term. Equation (3.28) can be expressed as a sum of single unit step functions,

$${}_0^C D_t^\alpha x(t) = f(x_0) + \sum_{m=1}^{\infty} (f(x(mh)) - f(x((m-1)h)))\, u(t-mh). \tag{3.29}$$

The Laplace transform of equation (3.29) yields,

$$s^\alpha \mathcal{L}\{x(t)\} - s^{\alpha-1}x_0 = s^{-1} f(x_0) + s^{-1} \sum_{m=1}^{\infty} (f(x(mh)) - f(x((m-1)h)))\, e^{-smh}, \tag{3.30}$$

the solution of equation (3.29) can be found utilising inverting the Laplace transform gives,

$$x(t) = x_0 + \frac{t^\alpha}{\Gamma(1+\alpha)} f(x_0) + \sum_{m=1}^{\infty} \frac{(t-mh)^\alpha}{\Gamma(1+\alpha)} (f(x(mh)) - f(x((m-1)h)))\, u(t-mh). \tag{3.31}$$

where $x(0) = x_0$.

This is a solution in continuous t. can be simpli ed to an $n-th$ order difference equation by setting $t = nh$,

$$x(nh) = x_0 + \frac{(nh)^\alpha}{\Gamma(1+\alpha)} f(x_0) + \sum_{m=1}^{n-1} \frac{((n-m)h)^\alpha}{\Gamma(1+\alpha)} \left(f(x(mh)) - f(x((m-1)h)) \right). \quad (3.32)$$

Example 3.5.1 *The fractional order Riccati equation,*

$$ {}_0^C D_t^\alpha x(t) = 1 - \rho (x(t))^2, \quad (3.33)$$

the initial condition $x(0) = x_0$, *and* $\tau_n = nh$. *Our discretization of equation* (3.33) *can be found from* (3.32)

$$x(\tau_n) = x_0 + \frac{(\tau_n)^\alpha}{\Gamma(1+\alpha)} \left(1 - \rho x_0^2\right) + \sum_{m=1}^{n-1} \rho \frac{(\tau_n - \tau_m)^\alpha}{\Gamma(1+\alpha)} \left(\left((x(\tau_{m-1}))^2 - (x(\tau_m))^2 \right) \right).$$

3.6 Predictor-Corrector Method

Consider the following initial value problem:

$$\begin{cases} {}_0^C D_t^\alpha x(t) = f(t, x(t)) & 0 \le t \le T \\ x^{(k)}(0) = x_0^{(k)} & , k = 0, 1, \ldots, n-1, \end{cases} \quad (3.34)$$

where the $x_0^{(k)}$ may be arbitrary real numbers and where $n - 1 < \alpha < n$.

Lemma 3.6.1 *If the function* f *is* C^1 , *then the initial value problem* (3.34) *is equivalent to Volterra integral equations:*

$$x(t) = \sum_{k=0}^{n-1} x_0^{(k)} \frac{t^k}{k!} + \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau, x(\tau))\, d\tau. \quad (3.35)$$

Proof 3.6.2 *Suppose* $x(t)$ *satisfies* (3.35). *Then observe that* $D^n x$ *exists and is integrable, because*

$$\begin{aligned} x^{(n)}(t) &= D^n \left(\sum_{k=0}^{n-1} x_0^{(k)} \frac{t^k}{k!} + {}_0I_t^\alpha\ f(t, x(t)) \right), \\ &= D^n\ {}_0I_t^\alpha\ f(t, x(t)), \\ &= DD^{n-1}\ {}_0I_t^{n-1}\ {}_0I_t^{\alpha-n+1}\ f(t, x(t)), \\ &= D\ {}_0I_t^{1-(n-\alpha)}\ f(t, x(t)) = {}_0D_t^{n-\alpha}\ f(t, x(t)), \end{aligned}$$

which exists and is integrable as f' *is continuous. Thus* ${}_0I_t^{n-\alpha}\ x^{(n)}(t) = {}_0^C D_t^\alpha\ x(t)$ *exists. So*

$$ {}_0^C D_t^\alpha\ x(t) = {}_0I_t^{n-\alpha}\ x^{(n)}(t) = {}_0I_t^{n-\alpha}\ {}_0D_t^{n-\alpha}\ f(t, x(t)) = f(t, x(t)),$$

as f *is continuous and* $0 < n - \alpha < 1$. *Hence* $x(t)$ *satisfies* (3.34).

Let f be a continuous function that satisfies a Lipschitz condition with respect to the second argument:

$$\left|f\left(t,x_1\right)-f\left(t,x_2\right)\right|\leq L\left|x_1-x_2\right|,\quad L\in\mathbb{R}_*^+,$$

then the initial value problem (3.34) has a unique solution on the interval $[0,T]$, the interval is divided into N subintervals. Let $h=\dfrac{T}{N}$, $t_n=nh$, $n=0,1,2,\cdots,N$.

The principle of this method is to replace the original equation (3.34) by the Volterra integral equation (3.35) and we use the product trapezoidal quadrature formula to replace the integral with the nodes t_j; $j=0,1,\ldots,n$ which are taken respectively from the function $(t_{n+1}-.)^{\alpha-1}$ that is to say:

$$\int_0^{t_{n+1}}\left(t_{n+1}-\tau\right)^{\alpha-1}g\left(\tau\right)d\tau\simeq\int_0^{t_{n+1}}\left(t_{n+1}-\tau\right)^{\alpha-1}\overset{\sim}{g_{n+1}}\left(\tau\right)d\tau \tag{3.36}$$

where $\overset{\sim}{g_{n+1}}$ is the piecewise linear interpolant for g with nodes and knots chosen at the t_j. We write the integral on the right-hand side of (3.36) as

$$\int_0^{t_{n+1}}\left(t_{n+1}-\tau\right)^{\alpha-1}\overset{\sim}{g_{n+1}}\left(\tau\right)d\tau=\sum_{j=0}^{n+1}a_{j,n+1}g\left(t_j\right), \tag{3.37}$$

where

$$a_{j,n+1}=\frac{h^\alpha}{\alpha\left(\alpha+1\right)}\times\begin{cases}\left(n^{\alpha+1}-\left(n-\alpha\right)\left(n+1\right)^\alpha\right) & \text{if } j=0\\ \left(\left(n-j+2\right)^{\alpha+1}+\left(n-j\right)^{\alpha+1}-2\left(n-j+1\right)^{\alpha+1}\right) & \text{if } 1\leq j\leq n\\ 1 & \text{if } j=n+1.\end{cases}$$

This then gives us our corrector formula (i.e., the fractional variant of the Adams–Moulton method), which is

$$x\left(t_{n+1}\right)=\sum_{k=0}^{n-1}x_0^{(k)}\frac{t_{n+1}^k}{k!}+\frac{h^\alpha}{\Gamma\left(\alpha+2\right)}\sum_{j=0}^{n+1}a_{j,n+1}f\left(t_j,x\left(t_j\right)\right)+\frac{h^\alpha}{\Gamma\left(\alpha+2\right)}f\left(t_{n+1},x^p\left(t_{n+1}\right)\right),$$

where we have used $\alpha\left(\alpha+1\right)\Gamma\left(\alpha\right)=\Gamma\left(\alpha+2\right)$ and $a_{n+1,n+1}=1$.

To determine the prediction formula (predictor), which gives $x^p\left(t_{n+1}\right)$, we proceed in the same way as before but this time the integral will be replaced using the rectangle method

$$\int_0^{t_{n+1}}\left(t_{n+1}-\tau\right)^{\alpha-1}g\left(\tau\right)d\tau\simeq\sum_{j=0}^{n}b_{j,n+1}g\left(t_j\right),$$

where now

$$b_{j,n+1}=\frac{h^\alpha}{\alpha}\left(\left(n+1-j\right)^\alpha-\left(n-j\right)^\alpha\right),$$

therefore we have:

$$x^p\left(t_{n+1}\right)=\sum_{k=0}^{n-1}x_0^{(k)}\frac{t_{n+1}^k}{k!}+\frac{1}{\Gamma\left(\alpha\right)}\sum_{j=0}^{n}b_{j,n+1}f\left(t_j,x\left(t_j\right)\right).$$

3.7 Runge-Kutta Method

The initial value problem with time-fractional derivative in Caputo's sense of the form:

$$\begin{cases} {}_0^C D_t^\alpha x \ (t) = f\left(t, x\left(t\right)\right), \\ x\left(0\right) = x_0, \end{cases} \tag{3.38}$$

where $\alpha \in (0,1]$. Let $[0,a]$ be an interval for which we are finding the solution of the problem in equation (3.38). The collection of points $(t_i, x(t_i))$ are used to find the approximation. The interval $[0,a]$ is subdivided into r subintervals $[t_i, t_{i+1}]$ of equal step size $h = \dfrac{a}{r}$ using the nodal points $t_i = ih$ for $i = 0, 1, 2, ..., r$.

Suppose that $x(t)$, ${}_0^C D_t^\alpha x \ (t)$ and ${}_0^C D_t^{2\alpha} x \ (t)$ are continuous functions on the interval $[0,a]$, and applying Taylor's formula involving fractional derivatives, we have

$$x(t+h) = x\left(t\right) + \frac{h^\alpha}{\Gamma\left(1+\alpha\right)} \ {}_0^C D_t^\alpha x \ (t) + \frac{h^{2\alpha}}{\Gamma\left(1+2\alpha\right)} \ {}_0^C D_t^{2\alpha} x \ (t) + \cdots \tag{3.39}$$

and using the formula ${}_0^C D_t^{2\alpha} x \ (t) = \ {}_0^C D_t^\alpha f \ (t, x\left(t\right)) + f\left(t, x\left(t\right)\right) \ {}_0^C D_x^\alpha f \ (t, x\left(t\right))$ in equation (3.39) gives

$$x(t+h) = x\left(t\right) + \frac{h^\alpha}{\Gamma\left(1+\alpha\right)} f\left(t, x\left(t\right)\right) + \frac{h^{2\alpha}}{\Gamma\left(1+2\alpha\right)} \left[{}_0^c D_t^\alpha f \ (t, x\left(t\right)) + f\left(t, x\left(t\right)\right) \ {}_0^c D_x^\alpha f \ (t, x\left(t\right)) \right] + \cdots . \tag{3.40}$$

And we have

$$x(t+h) = x\left(t\right) + \frac{h^\alpha}{2\Gamma\left(1+\alpha\right)} f\left(t, x\left(t\right)\right) + \frac{h^\alpha}{2\Gamma\left(1+\alpha\right)} \left[\begin{array}{l} f\left(t, x\left(t\right)\right) + \dfrac{2h^\alpha \Gamma\left(1+\alpha\right)}{\Gamma\left(1+2\alpha\right)} \ {}_0^C D_t^\alpha f \ (t, x\left(t\right)) \\ + \dfrac{2h^\alpha \Gamma\left(1+\alpha\right)}{\Gamma\left(1+2\alpha\right)} f\left(t, x\left(t\right)\right) \ {}_0^C D_x^\alpha f \ (t, x\left(t\right)) \end{array} \right] + \cdots . \tag{3.41}$$

It can also be written as

$$x(t+h) = x\left(t\right) + \frac{h^\alpha}{2\Gamma\left(1+\alpha\right)} f\left(t, x\left(t\right)\right) + \frac{h^\alpha}{2\Gamma\left(1+\alpha\right)} f\left(t + \frac{2h^\alpha \Gamma\left(1+\alpha\right)}{\Gamma\left(1+2\alpha\right)}, x\left(t\right) + \frac{2h^\alpha \Gamma\left(1+\alpha\right)}{\Gamma\left(1+2\alpha\right)} f\left(t, x\left(t\right)\right)\right). \tag{3.42}$$

The following formula is the 2−stage fractional Runge–Kutta method.

We have

$$x_{n+1} = x_n + \frac{h^\alpha}{2\Gamma\left(1+\alpha\right)} \left[K_1 + K_2\right], \tag{3.43}$$

where

$K_1 = f\left(t_n, x_n\right)$,

$K_2 = f\left(t_n + \dfrac{2h^\alpha \Gamma\left(1+\alpha\right)}{\Gamma\left(1+2\alpha\right)}, x_n + \dfrac{2h^\alpha \Gamma\left(1+\alpha\right)}{\Gamma\left(1+2\alpha\right)} f\left(t_n, x_n\right)\right)$.

Example 3.7.1 *Consider the nonlinear fractional differential equation*

$$\begin{cases} {}_0^C D_t^\alpha x \ (t) = \left(x\left(t\right)\right)^2 - \dfrac{2}{\left(t+1\right)^2}, \\ x\left(0\right) = -2, \qquad 0 < \alpha \leq 1. \end{cases} \tag{3.44}$$

By using the fractional Runge-Kutta method, we get the iterative

$$x_{n+1} = x_n + \frac{h^\alpha}{2\Gamma\left(1+\alpha\right)}\left[K_1 + K_2\right], \tag{3.45}$$

where

$$\begin{aligned} K_1 &= x_n^2 - \frac{2}{\left(t_n+1\right)^2}, \\ K_2 &= \left(x_n + \frac{2h^\alpha\Gamma\left(1+\alpha\right)}{\Gamma\left(1+2\alpha\right)}K_1\right)^2 - \frac{2}{\left(t_n + \frac{2h^\alpha\Gamma\left(1+\alpha\right)}{\Gamma\left(1+2\alpha\right)} + 1\right)^2}. \end{aligned}$$

Chapter 4

Main Results

In this chapter, we derive a new discrete chaotic dynamical systems from the fractional order differential Arneodo's system and the fractional-order differential finance system. the asymptotic stability of the fixed points of this new systems, are thoroughly analyzed in the three-dimensional space of parameters. One can prove analytically the asymptotic stability of each of the three fixed points. And we brighten some dynamical behaviors, such as the chaotic attractor, bifurcation for different values of parameters. Moreover, the numerical simulations confirm the validity of our theories.

4.1 Bifurcation and Stability in a New Discrete System Induced from Fractional Order Continuous Chaotic Finance System

The fractional-order finance model [16] is given by the following dynamical system:

$$\begin{cases} D^{\alpha}x(t) = (y(t) - a)\,x(t) + z(t), \\ D^{\alpha}y(t) = 1 - x^2(t) - by(t), \\ D^{\alpha}z(t) = -(x(t) + cz(t)), \end{cases} \tag{4.1}$$

where x is the interest rate, y is the investment demand, z is the price index, $a > 0$ denotes saving amount, $b > 0$ denotes cost per investment, and $c > 0$ denotes elasticity of demand of commercial markets. Where $t > 0$, and α is the fractional-order satisfying $\alpha \in (0, 1]$. The equilibrium points of system (4.1) are given as follow:
$E_0 = \left(0, \frac{1}{b}, 0\right)$, $E_1 = \left(\sqrt{1 - b\left(a + \frac{1}{c}\right)}, \left(a + \frac{1}{c}\right), -\frac{1}{c}\sqrt{1 - b\left(a + \frac{1}{c}\right)}\right)$, and $E_2 = \left(-\sqrt{1 - b\left(a + \frac{1}{c}\right)}, \left(a + \frac{1}{c}\right), \frac{1}{c}\sqrt{1 - b\left(a + \frac{1}{c}\right)}\right)$.

Assume that $x(0) = x_0$, $y(0) = y_0$ and $z(0) = z_0$ are the initial conditions of system (4.1).

Following [24] a transformation process from a continuous system of fractional order to a discrete system is proposed as follows:

$$\begin{cases} D^{\alpha}x\left(t\right)=\left(y\left(\left[\dfrac{t}{h}\right]h\right)-a\right)x\left(\left[\dfrac{t}{h}\right]h\right)+z\left(\left[\dfrac{t}{h}\right]h\right), \\ D^{\alpha}y\left(t\right)=1-x^{2}\left(\left[\dfrac{t}{h}\right]h\right)-by\left(\left[\dfrac{t}{h}\right]h\right), \\ D^{\alpha}z\left(t\right)=-\left(x\left(\left[\dfrac{t}{h}\right]h\right)+cz\left(\left[\dfrac{t}{h}\right]h\right)\right). \end{cases} \tag{4.2}$$

First, let $t\in[0,h)$, so $t/h\in[0,1)$. Thus, we obtain

$$\begin{cases} D^{\alpha}x\left(t\right)=\left(y_{0}-a\right)x_{0}+z_{0}, \\ D^{\alpha}y\left(t\right)=1-x_{0}^{2}-by_{0}, \\ D^{\alpha}z\left(t\right)=-\left(x_{0}+cz_{0}\right), \end{cases} \tag{4.3}$$

and the solution of (4.3) is reduced to

$$\begin{cases} x_{1}(t)=x_{0}+J^{\alpha}\left(\left(y_{0}-a\right)x_{0}+z_{0}\right)=x_{0}+\dfrac{t^{\alpha}}{\Gamma\left(1+\alpha\right)}\left[\left(y_{0}-a\right)x_{0}+z_{0}\right)], \\ y_{1}(t)=y_{0}+J^{\alpha}\left(1-x_{0}^{2}-by_{0}\right)=y_{0}+\dfrac{t^{\alpha}}{\Gamma\left(1+\alpha\right)}\left[1-x_{0}^{2}-by_{0}\right], \\ z_{1}(t)=z_{0}+J^{\alpha}\left(-\left(x_{0}+cz_{0}\right)\right)=z_{0}+\dfrac{t^{\alpha}}{\Gamma\left(1+\alpha\right)}\left[-\left(x_{0}+cz_{0}\right)\right]. \end{cases} \tag{4.4}$$

Second, let $t\in[h,2h)$, which makes $t/h\in[1,2)$. Hence, we get

$$\begin{cases} D^{\alpha}x(t)=\left(y_{1}-a\right)x_{1}+z_{1}, \\ D^{\alpha}y(t)=1-x_{1}^{2}-by_{1}, \\ D^{\alpha}z(t)=-\left(x_{1}+cz_{1}\right), \end{cases} \tag{4.5}$$

which have the following solution

$$\begin{cases} x_{2}(t)=x_{1}\left(h\right)+J_{h}^{\alpha}\left(\left(y_{1}-a\right)x_{1}+z_{1}\right)=x_{1}\left(h\right)+\dfrac{\left(t-h\right)^{\alpha}}{\Gamma\left(1+\alpha\right)}\left[\left(y_{1}-a\right)x_{1}+z_{1}\right], \\ y_{2}(t)=y_{1}\left(h\right)+J_{h}^{\alpha}\left(1-x_{1}^{2}-by_{1}\right)=y_{1}\left(h\right)+\dfrac{\left(t-h\right)^{\alpha}}{\Gamma\left(1+\alpha\right)}\left[1-x_{1}^{2}-by_{1}\right], \\ z_{2}(t)=z_{1}\left(h\right)+J_{h}^{\alpha}\left(-\left(x_{1}+cz_{1}\right)\right)=z_{1}\left(h\right)+\dfrac{\left(t-h\right)^{\alpha}}{\Gamma\left(1+\alpha\right)}\left[-\left(x_{1}+cz_{1}\right)\right], \end{cases} \tag{4.6}$$

where $J_{h}^{\alpha}=\int_{h}^{t}\dfrac{\left(t-s\right)^{\alpha-1}}{\Gamma\left(\alpha\right)}ds$, $\alpha>0$. Thus, after repeating the discretization process n times, we obtain

$$\begin{cases} x_{n+1}(t)=x_{n}\left(nh\right)+\dfrac{\left(t-nh\right)^{\alpha}}{\Gamma\left(1+\alpha\right)}\left[\left(y_{n}\left(nh\right)-a\right)x_{n}\left(nh\right)+z_{n}\left(nh\right)\right], \\ y_{n+1}(t)=y_{n}\left(nh\right)+\dfrac{\left(t-nh\right)^{\alpha}}{\Gamma\left(1+\alpha\right)}\left[1-x_{n}^{2}\left(nh\right)-by_{n}\left(nh\right)\right], \\ z_{n+1}(t)=z_{n}\left(nh\right)+\dfrac{\left(t-nh\right)^{\alpha}}{\Gamma\left(1+\alpha\right)}\left[-\left(x_{n}\left(nh\right)+cz_{n}\left(nh\right)\right)\right], \end{cases} \tag{4.7}$$

where $t \in [nh, (n+1)h)$. For $t \to (n+1)h$, system (4.7) is reduced to

$$\begin{cases} x_{n+1} = x_n + \dfrac{h^\alpha}{\Gamma(1+\alpha)} \left[(y_n - a) x_n + z_n \right], \\ y_{n+1} = y_n + \dfrac{h^\alpha}{\Gamma(1+\alpha)} \left[1 - x_n^2 - b y_n \right], \\ z_{n+1} = z_n + \dfrac{h^\alpha}{\Gamma(1+\alpha)} \left[-(x_n + c z_n) \right], \end{cases} \tag{4.8}$$

which can be expressed as

$$\begin{cases} x_{n+1} = x_n + s \left[(y_n - a) x_n + z_n \right], \\ y_{n+1} = y_n + s \left[1 - x_n^2 - b y_n \right], \\ z_{n+1} = z_n + s \left[-(x_n + c z_n) \right], \end{cases} \tag{4.9}$$

in which $s = \dfrac{h^\alpha}{\Gamma(1+\alpha)}$, and h is a new positive parameter in the discrete system.

For the following values $a = 1.63, b = 0.418, c = 1.98, \alpha = 0.99$ the system (4.9) is in a chaotic state because one of the Lyapunov exponent is positive, which is considered as one of the characteristics of the existence of chaos (see Figure 4.1).

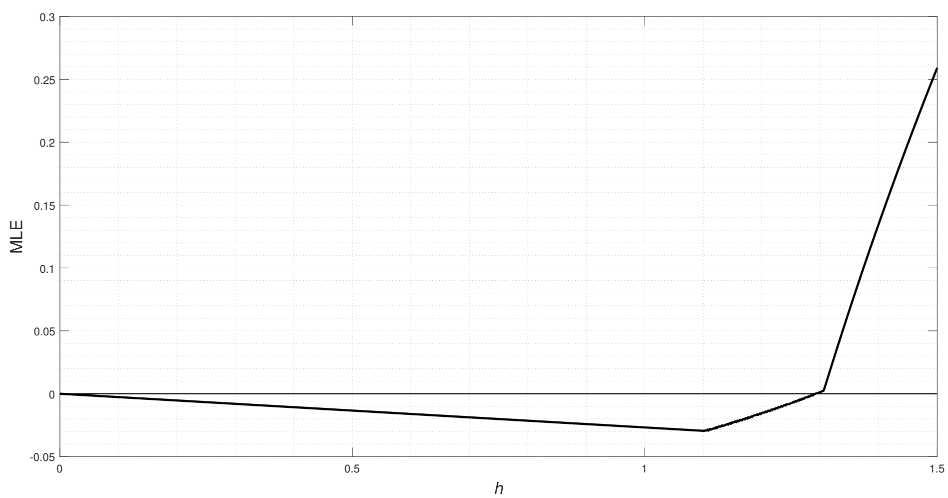


Figure 4.1: Maximal Lyapunov exponent of model (4.9) for $h \in [0, 1.5]$.

4.1.1 Stability of the Fixed Points of Discrete System

In this subsection, one discusses the local stability of the fixed points of system (4.9), which is determined by the eigenvalues of the Jacobian matrices corresponding to its fixed points. The Jacobian matrix of system (4.9) is:

$$J_{E_{eq}} = \begin{pmatrix} 1 + s(y_n - a) & s x_n & s \\ -2 s x_n & 1 - bs & 0 \\ -s & 0 & 1 - cs \end{pmatrix}. \tag{4.10}$$

Stability of Fixed Point E_0

In order to study the stability of the fixed point E_0 of system (4.9), we recall the two lemmas.

Lemma 4.1.1 *Let $F(\lambda) = \lambda^2 + B\lambda + C$. Suppose that $F(1) > 0$, λ_1 and λ_2 are two roots of $F(\lambda) = 0$. Then*

1. $|\lambda_1| < 1$ *and* $|\lambda_2| < 1$ *if and only if* $F(-1) > 0$ *and* $C < 1$.
2. $|\lambda_1| < 1$ *and* $|\lambda_2| > 1$ *(or* $|\lambda_1| > 1$ *and* $|\lambda_2| < 1$*) if and only if* $F(-1) < 0$.
3. $|\lambda_1| > 1$ *and* $|\lambda_2| > 1$ *if and only if* $F(-1) > 0$ *and* $C > 1$.
4. λ_1 *and* λ_2 *are complex and* $|\lambda_1| = |\lambda_2| = 1$ *if and only if* $B^2 - 4C < 0$ *and* $C = 1$.

Lemma 4.1.2 *When the associated Jacobian matrix has three real eigenvalues λ_i, $i = 1$, 2, 3.*

1. *The fixed point E_{eq} is called a locally asymptotically stable (sink) if $|\lambda_i| < 1$ for all $i = 1$, 2, 3.*
2. *The fixed point E_{eq} is called an unstable (source) if $|\lambda_i| > 1$ for all $i = 1$, 2, 3.*
3. *The fixed point E_{eq} is called a one-dimensional saddle if one $|\lambda_i| < 1$.*
4. *The fixed point E_{eq} is called a two-dimensional saddle if one $|\lambda_i| > 1$.*
5. *The fixed point E_{eq} if is called non-hyperbolic if one $|\lambda_i| = 1$.*

The jacobian matrix of E_0 of system (4.9) is

$$J_{E_0} = \begin{pmatrix} 1 + s\left(\frac{1}{b} - a\right) & 0 & s \\ 0 & 1 - bs & 0 \\ -s & 0 & 1 - cs \end{pmatrix}, \tag{4.11}$$

The characteristic equation of the Jacobian matrix (4.11) is

$$P(\lambda) = (\lambda + bs - 1) F(\lambda) = (\lambda + bs - 1)\left(\lambda^2 + B\lambda + C\right), \tag{4.12}$$

where $B = -\frac{1}{b}(2b + s - abs - bcs)$, $C = \frac{1}{b}\left(b + s + bs^2 - cs^2 - abs - bcs + abcs^2\right)$. By calculating, one further has

$F(1) = \frac{1}{b}s^2(b - c + abc)$, $F(-1) = \frac{1}{b}\left((b - c + abc)s^2 + (2 - 2bc - 2ab)s + 4b\right)$, and $B^2 - 4C = \frac{1}{b^2}s^2\left(a^2b^2 - 2ab^2c - 2ab + b^2c^2 - 4b^2 + 2bc + 1\right)$.

Let $a_1 = \frac{1}{b}(-2b + bc + 1)$, $a_2 = \frac{1}{b}(2b + bc + 1)$, and $a_3 = -\frac{b-c}{bc}$, $a_4 = -\frac{1}{b}(bc - 1)$

$s_1 = \frac{1}{b - c + abc}\left(\sqrt{-(2b + ab - bc - 1)(2b - ab + bc + 1)} + ab + bc - 1\right)$,

$$s_2 = -\frac{1}{b-c+abc}\left(\sqrt{-\left(2b+ab-bc-1\right)\left(2b-ab+bc+1\right)}-ab-bc+1\right),$$
and $s_3 = \dfrac{ab+bc-1}{b-c+abc}$, $\lambda_3 = 1-bs$.

Remark 4.1.3 *$F(1) > 0$ if and only if $a > a_3$ and $b,\ c \in \mathbb{R}_*^+$.*

Remark 4.1.4 *$a_3 > 0$ if $b < c$.*

Remark 4.1.5 *If $a \in \left]-\infty, a_1\right] \cup \left[a_2, +\infty\right[$ then $-\left(2b+ab-bc-1\right)\left(2b-ab+bc+1\right) \geq 0$.*

Remark 4.1.6 *$s_1 s_2 = 4\dfrac{b}{b-c+abc} > 0$ so $s_1, s_2 > 0$.*

Remark 4.1.7 *If $a > a_4$ so $ab+bc-1 > 0$.*

Remark 4.1.8 *If $ab+bc-1 > 0$, $b-c+abc > 0$, $b > 0$ then $s_1,\ s_2,\ s_3 > 0$.*

Lemma 4.1.9 *The sign of $s_1 - s_2$, $s_3 - s_1$, and $s_3 - s_2$ depends on the sign of $b-c+abc$.*

Proof 4.1.10 *We have:*

$$s_1 - s_2 = \frac{2}{b-c+abc}\sqrt{-\left(2b+ab-bc-1\right)\left(2b-ab+bc+1\right)}, \text{ and}$$
$$s_3 - s_1 = -\frac{1}{b-c+abc}\sqrt{-\left(2b+ab-bc-1\right)\left(2b-ab+bc+1\right)},$$
$$s_3 - s_2 = \frac{1}{b-c+abc}\sqrt{-\left(2b+ab-bc-1\right)\left(2b-ab+bc+1\right)}.$$

Lemma 4.1.11 *If $c \in \left]1, +\infty\right[$ then $a_4 < a_3 < a_1 < a_2$.*

Proof 4.1.12 *We have $a_3 - a_2 = -\dfrac{1}{c}\left(c+1\right)^2$ and $a_3 - a_1 = -\dfrac{1}{c}\left(c-1\right)^2$, $a_3 - a_4 = \dfrac{1}{c}\left(c^2-1\right)$.*

Remark 4.1.13 *$F(-1)$ is a polynomial of degree 2 of the variable s.*

1. *$F(-1) < 0$ if $\left]s_2, s_1\right[$.*
2. *$F(-1) = 0$ if $s = s_1, s_2$.*
3. *$F(-1) > 0$ if $\left]-\infty, s_2\right[\cup \left]s_1, +\infty\right[$.*

Remark 4.1.14 *$C - 1$ is a polynomial of degree 2 of the variable s.*

1. *$C - 1 < 0$ if $s \in \left]-\infty, s_3\right[$.*
2. *$C - 1 = 0$ if $s = s_3$.*
3. *$C - 1 > 0$ if $s \in \left]s_3, +\infty\right[$.*

Remark 4.1.15 *$B^2 - 4C < 0$ if $a \in \left]a_1, a_2\right[$.*

Lemma 4.1.16 *The sign of* $|\lambda_3| - 1$ *depends on the sign of* $b\,(bs-2)$.

Proof 4.1.17 *Calculating* $|\lambda_3|^2 - 1$ *one find*

$$(|\lambda_3| - 1)\,(|\lambda_3| + 1) = sb\,(bs-2)\,.$$

Now, relatively to the dynamical properties of the fixed point E_0, one has the following results.

Theorem 4.1.18 *If the fixed point* E_0 *exists with the following assumptions* $a \in \left]a_3, a_1\right] \cup \left[a_2, +\infty\right[$, $b \in \mathbb{R}_*^+$ *and* $c \in \left]1, +\infty\right[$ *then:*

a- E_0 *is a asymptotically stable if* $0 < s < \min\left(s_2, \dfrac{2}{b}\right)$.

b- E_0 *is a unstable if* $s > \max\left(s_1, \dfrac{2}{b}\right)$.

c- E_0 *is a one-dimensional saddle if* $s_2 < s < s_1$ *or* $\dfrac{2}{b} < s < s_1$.

d- E_0 *is a two-dimensional saddle if* $\dfrac{2}{b} < s < s_2$.

e- E_0 *is a non-hyperbolic if* $s = \dfrac{2}{b}$.

Proof 4.1.19 *For* $b \in \mathbb{R}_*^+$:

1. *If* $a \in \left]a_3, a_1\right] \cup \left[a_2, +\infty\right[$, $c \in \left]1, +\infty\right[$:

The condition $F(1) > 0$ *is verified, then (Remark* 4.1.3*), and according to Remark* 4.1.5*, Remark* 4.1.7 *and Lemma* 4.1.11 *so* $a \in (\left]-\infty, a_1\right] \cup \left[a_2, +\infty\right[) \cap \left]a_3, +\infty\right[\cap \left]a_4, +\infty\right[= \left]a_3, a_1\right] \cup \left]a_2, +\infty\right[$.

$F(-1) > 0$ *if* $s \in \left]-\infty, s_2\right[\cup \left]s_1, +\infty\right[$.*(*3 *Remark* 4.1.13*),* $C < 1$ *if* $s \in \left]-\infty, s_3\right[$ *(*1 *Remark* 4.1.14*) then* $s \in (\left]-\infty, s_2\right[\cup \left]s_1, +\infty\right[) \cap \left]-\infty, s_3\right[= \left]-\infty, s_2\right[$*(Lemma* 4.1.9*).*

$C > 1$ *if* $s \in \left]s_3, +\infty\right[$ *(*3 *Remark* 4.1.14*) then* $s \in (\left]-\infty, s_2\right[\cup \left]s_1, +\infty\right[) \cap \left]s_3, +\infty\right[= \left]s_1, +\infty\right[$*(Lemma* 4.1.9*).*

$F(-1) < 0$ *if* $s \in \left]s_2, s_1\right[$ *(*1 *in Remark* 4.1.13*).*

$B^2 - 4C < 0$ *if* $a \in \left]a_1, a_2\right[$ (*Remark* 4.1.15), *and* $C = 1$ *if* $s = s_3$ *(*2 *in Remark* 4.1.14*).*

And using the sign of $|\lambda_1| - 1$ *(Lemma* 4.1.16*).*

Using Lemma 4.1.1*, hence applying the stability conditions using Lemma* 4.1.2*. One can obtain the results.*

Stability of Fixed Points E_1 and E_2

We recall the lemma 1 of [31], the definition and the theorem 4 of [23].

Definition 4.1.20 *When the associated Jacobian matrix has one real eigenvalue λ_1 and a pair of complex eigenvalues $\lambda_{2,3} = \rho \pm \omega i$, then the following definitions are valid*

1. *The fixed point E_{eq} is called a locally asymptotically stable (sink) if $|\lambda_i| < 1$ for all $i = 1, 2, 3$.*
2. *The fixed point E_{eq} is called an unstable (source) if $|\lambda_i| > 1$ for all $i = 1, 2, 3$.*
3. *The fixed point E_{eq} is called a one-dimensional saddle if $|\lambda_1| < 1$ and $|\lambda_{2,3}| > 1$.*
4. *The fixed point E_{eq} is called a two-dimensional saddle if $|\lambda_1| > 1$ and $|\lambda_{2,3}| < 1$.*
5. *The fixed point E_{eq} is called non-hyperbolic if $|\lambda_1| = 1$ or $|\lambda_2| = 1$ or $|\lambda_3| = 1$.*

The Jacobian matrix associated with the fixed point E_1 and E_2 of the system (4.9) is given by

$$J_{E_1} = \begin{pmatrix} \frac{1}{c}(c+s) & s\sqrt{-\frac{1}{c}(b-c+abc)} & s \\ -2s\sqrt{-\frac{1}{c}(b-c+abc)} & 1-bs & 0 \\ -s & 0 & 1-cs \end{pmatrix}, \tag{4.13}$$

and

$$J_{E_2} = \begin{pmatrix} \frac{1}{c}(c+s) & -s\sqrt{-\frac{1}{c}(b-c+abc)} & s \\ 2s\sqrt{-\frac{1}{c}(b-c+abc)} & 1-bs & 0 \\ -s & 0 & 1-cs \end{pmatrix}, \tag{4.14}$$

then, the characteristic polynomial of J_{E_1} and J_{E_2} is

$$P_1(\lambda) = \lambda^3 + b_1\lambda^2 + b_2\lambda + b_3, \tag{4.15}$$

where $b_1 = -\eta_3 s - 3$, $b_2 = \eta_2 s^2 + 2\eta_3 s + 3$, and $b_3 = \eta_1 s^3 - s^2\eta_2 - \eta_3 s - 1$,
and $\eta_1 = 2(c - b - abc)$, $\eta_2 = \frac{1}{c}\left(2c - 3b + bc^2 - 2abc\right)$, and $\eta_3 = -\frac{1}{c}\left(c^2 + bc - 1\right)$.
By calculating, one further has
$A = b_1^2 - 3b_2 = -s^2\left(3\eta_2 - \eta_3^2\right)$,
$B = b_1 b_2 - 9b_3 = -s^3\left(9\eta_1 + \eta_2\eta_3\right) + s^2\left(6\eta_2 - 2\eta_3^2\right)$,
$C = b_2^2 - 3b_1 b_3 = \left(\eta_2^2 + 3\eta_1\eta_3\right)s^4 + \left(9\eta_1 + \eta_2\eta_3\right)s^3 + \left(\eta_3^2 - 3\eta_2\right)s^2$,
$\Delta = B^2 - 4AC = 3s^6((27\eta_1^2 + 18\eta_1\eta_2\eta_3 - 4\eta_1\eta_3^3 + 4\eta_2^3 - \eta_2^2\eta_3^2) = s^6\Delta^*$.
The derivative of $P_1(\lambda)$ is $P_1'(\lambda) = 3\lambda^2 + 2b_1\lambda + b_2$, and the equation $P_1'(\lambda) = 0$ has two roots:

$$\lambda_{1,2}^* = \frac{1}{3}\left(-b_1 \pm \sqrt{b_1^2 - 3b_2}\right) = \frac{1}{3}s\left(\eta_3 \pm \sqrt{\eta_3^2 - 3\eta_2}\right) + 1. \tag{4.16}$$

When $\Delta^* \leq 0$, i.e. $\Delta \leq 0$, by Lemma 1 page 6 in [31], equation (4.15) has three real roots λ_1, λ_2 and λ_3. From this, one can easily prove that both roots $\lambda_{1,2}^*$ (let $\lambda_1^* \leq \lambda_2^*$) of equation $P_1'(\lambda) = 0$ also are real.

When $\Delta^* > 0$, i.e. $\Delta > 0$, by Lemma 1 page 6 in [31], one has that equation (4.15) has one real root $\lambda_1 = -\dfrac{b_1 + y_1^{\frac{1}{3}} + \ y_2^{\frac{1}{3}}}{3}$,

and a pair of conjugate complex roots $\lambda_{2,3}$:

$$\lambda_{2,3} = \frac{1}{6}\left(\left(-2b_1 + y_1^{\frac{1}{3}} + y_2^{\frac{1}{3}}\right) \pm i\sqrt{3}\left(y_1^{\frac{1}{3}} - y_2^{\frac{1}{3}}\right)\right),$$

where

$$y_{1,2} = \frac{s^3}{2}\left(\left(-2\eta_3^3 + 9\eta_2\eta_3 + 27\eta_1\right) \pm \sqrt{\Delta^*}\right), y_1 > y_2,$$

and

$$(\eta_1)_1 = \frac{2}{27}\left(\left(\eta_3^2 - \frac{9}{2}\eta_2\right)\eta_3 - \sqrt{-\left(3\eta_2 - \eta_3^2\right)^3}\right), (\eta_1)_2 = \frac{2}{27}\left(\left(\eta_3^2 - \frac{9}{2}\eta_2\right)\eta_3 + \sqrt{-\left(3\eta_2 - \eta_3^2\right)^3}\right).$$

Let

$a_3 = -\dfrac{b-c}{bc}$, $\varphi_1 = \dfrac{y_1^{\frac{1}{3}} + y_2^{\frac{1}{3}}}{s} - \eta_3$, then $\varphi_1^3 = -3\eta_3\varphi_1^2 - 9\eta_2\varphi_1 + 27\eta_1$, and

$(\varphi_1)_1 = \dfrac{1}{2}\left(-3\eta_3 - \sqrt{3}\sqrt{3\eta_3^2 - 8\eta_2}\right)$, $(\varphi_1)_2 = \dfrac{1}{2}\left(-3\eta_3 + \sqrt{3}\sqrt{3\eta_3^2 - 8\eta_2}\right)$,

and $P_1(1) = \eta_1 s^3$, $P_1(-1) = \eta_1 s^3 - 2\eta_2 s^2 - 4\eta_3 s - 8$.

Remark 4.1.21 $-\dfrac{1}{c}(b - c + abc) > 0$ *if* $b, c \in \mathbb{R}_*^+$, $b < c$ *and* $0 < a < a_3$.

Remark 4.1.22 *If* $b \in \mathbb{R}_*^+$, $c \in [1, +\infty[$, $b < c$ *and* $0 < a < a_3$, *then* η_1, $\eta_2 > 0$, $\eta_3 < 0$.

Remark 4.1.23 Δ^* *is a polynomial of degree* 2 *of the variable* η_1.

1. $\Delta^* > 0$ *if* $3\eta_2 < \eta_3^2$, *and* $\eta_1 \in \left]-\infty, (\eta_1)_1\right[\cup \left](\eta_1)_2, +\infty\right[$ *or if* $3\eta_2 > \eta_3^2$ *or if* $3\eta_2 = \eta_3^2$, $\eta_1 \in \mathbb{R} - \{-\frac{1}{3}\eta_2\eta_3 - \frac{2}{27}\eta_3^3\}$.

2. $\Delta^* \leq 0$ *if* $3\eta_2 < \eta_3^2$, $\eta_1 \in [(\eta_1)_1, (\eta_1)_2]$ *or if* $3\eta_2 = \eta_3^2$, *and* $\eta_1 = -\frac{1}{3}\eta_2\eta_3 - \frac{2}{27}\eta_3^3$.

Lemma 4.1.24

1. *If* $4\eta_2 < \eta_3^2$ *then* $(\eta_1)_1 < 0$, $(\eta_1)_2 > 0$.

2. *If* $\eta_2 > 0, \eta_3 < 0$ *and* $3\eta_2 < \eta_3^2 < 4\eta_2$, *then* $(\eta_1)_1$, $(\eta_1)_2 > 0$.

3. *If* $\eta_2, \eta_3 > 0$ *and* $3b < c^2 < 4b$ *then* $(\eta_1)_1$, $(\eta_1)_2 < 0$.

Proof 4.1.25 *As* $(\eta_1)_1.(\eta_1)_2 = \dfrac{1}{27}\eta_2^2\left(4\eta_2 - \eta_3^2\right)$, *then the sign of* $(\eta_1)_1$, $(\eta_1)_2$ *depends on the sign of* $4\eta_2 - \eta_3^2$.

Lemma 4.1.26 *If* $4\eta_2 < \eta_3^2$ *or* $3\eta_2 < \eta_3^2 < 4\eta_2$, *and* $\eta_1 > (\eta_1)_2 > 0$, *then* $y_1 > 0$.

Proof 4.1.27 *Either* $4\eta_2 < \eta_3^2$ *or* $3\eta_2 < \eta_3^2 < 4\eta_2$, *and* $\eta_1 \in \left](\eta_1)_2, +\infty\right[$, *one has* $3\eta_2 < \eta_3^2$, *and* Δ^* *positive (1 in Remark 4.1.23),*

and $\left(-\eta_3^3 + \frac{9}{2}\eta_2\eta_3 + \frac{27}{2}\eta_1\right) > 0$ *because*

$$\left(-\eta_3^3 + \frac{9}{2}\eta_2\eta_3 + \frac{27}{2}\eta_1\right) > \left(-\eta_3^3 + \frac{9}{2}\eta_2\eta_3 + \frac{27}{2}(\eta_1)_2\right) = \sqrt{-\left(3\eta_2 - \eta_3^2\right)^3} > 0,$$

as

$$y_1 = \frac{s^3}{2}\left(\left(-2\eta_3^3 + 9\eta_2\eta_3 + 27\eta_1\right) + \sqrt{\Delta^*}\right),$$

then $y_1 > 0$.

Lemma 4.1.28 *If* $3\eta_2 < \eta_3^2$, *then* y_1 *and* y_2 *have the same sign.*

Proof 4.1.29 *If* $3\eta_2 < \eta_3^2$, *one has* $y_1 y_2 = s^6\left(-3\eta_2 + \eta_3^2\right)^3 > 0$, *then* y_1, y_2 *have the same sign.*

Lemma 4.1.30 *If* $\eta_2 \in \mathbb{R}^+$, $\eta_3 < -\sqrt{3\eta_2}$ *then* $(\eta_1)_2 + \eta_2\eta_3 < 0$.

Proof 4.1.31 *The difference between* $(\eta_1)_2$ *and* $-\eta_2\eta_3$, *is:*

$$(\eta_1)_2 + \eta_2\eta_3 = \frac{2}{27}\left(\sqrt{-\left(3\eta_2 + \eta_3^2\right)^3} + \eta_3^3\right) + \frac{2}{3}\eta_2\eta_3,$$

then $\left(\sqrt{-\left(3\eta_2 + \eta_3^2\right)^3} + \eta_3^3\right) < 0$ *because*

$$\left(\sqrt{-\left(3\eta_2 + \eta_3^2\right)^3} + \eta_3^3\right)\left(\sqrt{-\left(3\eta_2 + \eta_3^2\right)^3} - \eta_3^3\right) = -27\eta_2^3 + 9\eta_2\eta_3^2\left(3\eta_2 - \eta_3^2\right) < 0,$$

and $\left(\sqrt{-\left(3\eta_2 + \eta_3^2\right)^3} - \eta_3^3\right) > 0$. *Therefore,* $(\eta_1)_2 + \eta_2\eta_3 < 0$.

Remark 4.1.32 *If* $\eta_2 \in \mathbb{R}^+$ *and* $3\eta_2 < \eta_3^2$, *then* $8\eta_2 < 3\eta_3^2$.

Lemma 4.1.33 *If* $3\eta_2 < \eta_3^2$, $\eta_1 \in \left]-\infty, (\eta_1)_1\right[\cup \left](\eta_1)_2, +\infty\right[$, *then*

1. *the sign of* $|\lambda_{2,3}| - 1$ *depends on the sign of* $\left(\varphi_1^2 + 3\eta_3\varphi_1 + 9\eta_2\right)s + 3\left(\varphi_1 + 3\eta_3\right)$.
2. *the sign of* $|\lambda_1| - 1$ *depends on the sign of* $\varphi_1\left(s\varphi_1 - 6\right)$.

Proof 4.1.34 $\Delta^* > 0$ *if* $3\eta_2 < \eta_3^2$ *and* $\eta_1 \in \left]-\infty, (\eta_1)_1\right[\cup\left](\eta_1)_2, +\infty\right[$ *(1 in Remark 4.1.23). Calculating* $|\lambda_{2,3}|^2 - 1$ *and* $|\lambda_1|^2 - 1$ *one finds:*

$$\begin{aligned}\left(|\lambda_{2,3}| - 1\right)\left(|\lambda_{2,3}| + 1\right) &= \frac{1}{9}\left(\left(\varphi_1^2 + 3\eta_3\varphi_1 + 9\eta_2\right)s^2 + 3\left(\varphi_1 + 3\eta_3\right)s\right),\\ \left(|\lambda_1| - 1\right)\left(|\lambda_1| + 1\right) &= \frac{1}{9}s\varphi_1\left(s\varphi_1 - 6\right).\end{aligned}$$

The above result is true because $|\lambda_{2,3}| + 1$, $|\lambda_1| + 1$, s *are positive.*

Remark 4.1.35 *$\varphi_1 > 0$ if $y_1,\ y_2, -\eta_3 > 0$ and $\varphi_1 < 0$ if $y_1,\ y_2, -\eta_3 < 0$.*

Lemma 4.1.36 *For $\eta_2 \in \mathbb{R}^+$, $\eta_3 \in \left]-\infty, -2\sqrt{\eta_2}\right[$, then $\dfrac{2}{-\eta_3} < \dfrac{6}{-\eta_3 + \sqrt{-3\eta_2 + \eta_3^2}}$.*

Proof 4.1.37 *$\dfrac{2}{-\eta_3} - \dfrac{6}{-\eta_3 + \sqrt{-3\eta_2 + \eta_3^2}} = \dfrac{2}{\eta_2\eta_3}\left(\eta_3^2 - \eta_2 + \eta_3\sqrt{\eta_3^2 - 3\eta_2}\right)$, $\eta_2 < \eta_3^2$ and $\eta_3^2 > 3\eta_2$ because $4\eta_2 < \eta_3^2$, and*

$\left(\eta_3^2 - \eta_2 + \eta_3\sqrt{\eta_3^2 - 3\eta_2}\right)\left(\eta_3^2 - \eta_2 - \eta_3\sqrt{\eta_3^2 - 3\eta_2}\right) = \eta_2\left(\eta_3^2 + \eta_2\right) > 0$ one has

$\left(\eta_3^2 - \eta_2 + \eta_3\sqrt{\eta_3^2 - 3\eta_2}\right) > 0$ because $\left(\eta_3^2 - \eta_2 - \eta_3\sqrt{\eta_3^2 - 3\eta_2}\right) > 0$,

then $\dfrac{2}{-\eta_3} < \dfrac{6}{-\eta_3 + \sqrt{-3\eta_2 + \eta_3^2}}$.

Lemma 4.1.38 *For $\eta_2 \in \mathbb{R}^+$, $\eta_3 \in \left]-\infty, -2\sqrt{\eta_2}\right[$ or $\eta_3 \in \left]-2\sqrt{\eta_2}, -\sqrt{3\eta_2}\right[$, the sign of $\lambda_1^* + 1$ and $\lambda_2^* + 1$ are positive if $s \in \left]0, \dfrac{6}{-\eta_3 + \sqrt{-3\eta_2 + \eta_3^2}}\right[$.*

Proof 4.1.39 *$\lambda_1^* + 1 = \dfrac{1}{3}s\left(\eta_3 - \sqrt{\eta_3^2 - 3\eta_2}\right) + 2$ and $\lambda_2^* + 1 = \dfrac{1}{3}s\left(\eta_3 + \sqrt{\eta_3^2 - 3\eta_2}\right) + 2$ are a polynomial of degree 1 of the variable s, and $\left(\eta_3 - \sqrt{\eta_3^2 - 3\eta_2}\right)\left(\eta_3 + \sqrt{\eta_3^2 - 3\eta_2}\right) = 3\eta_2 > 0$, then $\left(\eta_3 + \sqrt{\eta_3^2 - 3\eta_2}\right) < 0$, and $\eta_3^2 > 3\eta_2$.*

The sign of $\lambda_1^* + 1$ is positive if $s \in \left]0, \dfrac{6}{-\eta_3 + \sqrt{-3\eta_2 + \eta_3^2}}\right[$, and the sign of $\lambda_2^* + 1$ is positive if $s \in \left]0, \dfrac{6}{-\eta_3 - \sqrt{-3\eta_2 + \eta_3^2}}\right[$.

One has $\dfrac{6}{-\eta_3 + \sqrt{-3\eta_2 + \eta_3^2}} < \dfrac{6}{-\eta_3 - \sqrt{-3\eta_2 + \eta_3^2}}$.

Finally $\lambda_1^* + 1,\ \lambda_2^* + 1$ are positive if $s \in \left]0, \dfrac{6}{-\eta_3 + \sqrt{-3\eta_2 + \eta_3^2}}\right[$.

Now, relatively to the dynamical properties of the fixed points E_1 and E_2, one has the following results.

For theorems 4.1.40 to 4.1.42, let: $\kappa_1 = -3\dfrac{\varphi_1 + 3\eta_3}{\varphi_1^2 + 3\eta_3\varphi_1 + 9\eta_2}$, $\kappa_2 = \dfrac{2}{\sqrt{-\eta_2}}$ and $\kappa_3 = \dfrac{6}{\varphi_1}$.

Theorem 4.1.40 *If the fixed points $E_i, i = 1, 2$ exist with the following assumptions $\eta_2 \in \mathbb{R}^+$, $\eta_3 < -2\sqrt{\eta_2}$.*

1. *When $\eta_1 = 0$ then $\Delta \leq 0$ and E_i is a non-hyperbolic.*
2. *When $\eta_1 \in \left]0, (\eta_1)_2\right]$ then $\Delta \leq 0$ and E_i is a asymptotically stable if $0 < s < \dfrac{2}{-\eta_3}$.*
3. *When $\eta_1 \in \left](\eta_1)_2, -\eta_2\eta_3\right[$ then $\Delta > 0$ and:*
   (a) *E_i is a asymptotically stable if $0 < s < \min(\kappa_1, \kappa_3)$.*
   (b) *E_i is a unstable if $s > \max(\kappa_1, \kappa_3)$.*
   (c) *E_i is a one-dimensional saddle if $\kappa_1 < s < \kappa_3$.*
   (d) *E_i is a two-dimensional saddle if $\kappa_3 < s < \kappa_1$.*
   (e) *E_i is a non-hyperbolic if $s = \dfrac{6}{(\varphi_1)_1}$ or $s = \dfrac{6}{(\varphi_1)_2}$.*
4. *When $\eta_1 \in \left[-\eta_2\eta_3, +\infty\right[$ then $\Delta > 0$ and:*
   (a) *E_i is a unstable if $s > \kappa_3$.*
   (b) *E_i is a one-dimensional saddle if $0 < s < \kappa_3$.*
   (c) *E_i is a non-hyperbolic if $s = \kappa_3$.*

Proof 4.1.41 *For $\eta_2 \in \mathbb{R}^+$, $\eta_3 < -2\sqrt{\eta_2}$:*

1. *If $a = 0$:*
   the condition $4\eta_2 < \eta_3^2$ is verified, and $3\eta_2 < \eta_3^2$.
   $\Delta \leq 0$ because $\Delta^* \leq 0$ (2 in Remark 4.1.23). One has $P_1(1) = 0$ (a null).
   Hence, applying the stability conditions using Theorem 4 (5.i) of [23], one obtains the result.
2. *If $\eta_1 \in \left]0, (\eta_1)_2\right]$:*
   the condition $4\eta_2 < \eta_3^2$ is verified, then $(\eta_1)_2 > 0$ (1 in Lemma 4.1.24), and $3\eta_2 < \eta_3^2$.
   $\Delta \leq 0$ because $\Delta^* \leq 0$ (2 in Remark 4.1.23). One has $P_1(1) > 0$ (η_1 positive) and $P_1(-1) < 0$ if $s \in \left]0, \dfrac{2}{-\eta_3}\right[$ because
   $$\begin{aligned}&\eta_1 s^3 - 2\eta_2 s^2 - 4\eta_3 s - 8 < -\eta_2\eta_3 s^3 - 2\eta_2 s^2 - 4\eta_3 s - 8 \ \left((\eta_1)_2 + \eta_2\eta_3 \text{ negative (Lemma 4.1.30)}\right),\\ &\eta_1 s^3 - 2\eta_2 s^2 - 4\eta_3 s - 8 < (-\eta_3 s - 2)\left(\eta_2 s^2 + 4\right),\end{aligned}$$
   hence $P_1(-1) < 0$ if $(-\eta_3 s - 2) < 0$ because $\left(\eta_2 s^2 + 4\right) > 0$.
   And
   $$\begin{aligned}&\left(\eta_3 - \sqrt{\eta_3^2 - 3\eta_2}\right)\left(\eta_3 + \sqrt{\eta_3^2 - 3\eta_2}\right) = 3\eta_2 > 0,\\ &\text{so } \lambda_1^*, \lambda_2^* < 1 \text{ because } \left(\eta_3 + \sqrt{\eta_3^2 - 3\eta_2}\right) < 0.\end{aligned}$$
   Hence, applying the stability conditions using Lemma 4.1.36, Lemma 4.1.38 and Theorem 4 (1.i) of [23], one obtains the result.

3. *If* $a \in \left](\eta_1)_2, +\infty\right[$:

   The condition $4\eta_2 < \eta_3^2$ *is verified, then* $(\eta_1)_2 > 0$ *(*1 *in Lemma* 4.1.24*), and* $3\eta_2 < \eta_3^2$.

   $\Delta > 0$ *because* $\Delta^* > 0$ *(*1 *in Remark* 4.1.23*). And according to Lemma* 4.1.26*, Lemma* 4.1.28 *and Remark* 4.1.35*, one has* $\varphi_1 > 0$.

   One study the sign of $\left(\varphi_1^2 + 3\eta_3\varphi_1 + 9\eta_2\right), \left(\varphi_1 + 3\eta_3\right), \varphi_1\left(s\varphi_1 - 6\right).$

   The sign of $\varphi_1^2 + 3\eta_3\varphi_1 + 9\eta_2$ *is positive because*
   $\left(\varphi_1^2 + 3\eta_3\varphi_1 + 9\eta_2\right)\varphi_1 = 27\eta_1$ *is positive, and* $\varphi_1 + 3\eta_3$ *is negative if* $\eta_1 + \eta_2\eta_3$ *is negative because* $\left(\varphi_1 + 3\eta_3\right)\left(\varphi_1^2 + 9\eta_2\right) = 27\left(\eta_1 + \eta_2\eta_3\right).$

   Hence $a \in \left](\eta_1)_2, +\infty\right[\cap \left]0, -\eta_2\eta_3\right[= \left](\eta_1)_2, -\eta_2\eta_3\right[$ *(*$(\eta_1)_2 < -\eta_2\eta_3$ *using Lemma* 4.1.30*),*

   and $\varphi_1 + 3\eta_3$ *is positive or null if* $\eta_1 + \eta_2\eta_3$ *is positive or null because*
   $\left(\varphi_1 - 3c\right)\left(\varphi_1^2 + 9\eta_2\right) = 27\left(\eta_1 + \eta_2\eta_3\right).$

   Hence $a \in \left](\eta_1)_2, +\infty\right[\cap \left[-\eta_2\eta_3, +\infty\right[= \left[-\eta_2\eta_3, +\infty\right[$ *(*$(\eta_1)_2 < -\eta_2\eta_3$ *using Lemma* 4.1.30*).*

   Using Lemma 4.1.33*, hence applying the stability conditions using Definition* 4.1.20*. One can obtain the results.*

Theorem 4.1.42 *If the fixed points* $E_i, i = 1, 2$ *exist with the following assumptions* $\eta_2 \in \mathbb{R}^+$, $\eta_3 \in \left]-2\sqrt{\eta_2}, -\sqrt{3\eta_2}\right[$.

1. *When* $\eta_1 \in \left[0, (\eta_1)_2\right]$ *then* $\Delta \leq 0$ *and* E_i *is a asymptotically stable if* $0 < s < \dfrac{2}{-\eta_3}$.

2. *When* $\eta_1 \in \left](\eta_1)_2, -\eta_2\eta_3\right[$ *then* $\Delta > 0$ *and:*

   (a) E_i *is a asymptotically stable if* $0 < s < \min\left(\kappa_1, \kappa_3\right).$

   (b) E_i *is a unstable if* $s > \max\left(\kappa_1, \kappa_3\right)$.

   (c) E_i *is a one-dimensional saddle if* $\kappa_1 < s < \kappa_3$.

   (d) E_i *is a two-dimensional saddle if* $\kappa_3 < s < \kappa_1$.

   (e) E_i *is a non-hyperbolic if* $s = \dfrac{6}{(\varphi_1)_1}$ *or* $s = \dfrac{6}{(\varphi_1)_2}$.

3. *When* $\eta_1 \in \left[-\eta_2\eta_3, +\infty\right[$ *then* $\Delta > 0$ *and:*

   (a) E_i *is a unstable if* $s > \kappa_3$.

   (b) E_i *is a one-dimensional saddle if* $0 < s < \kappa_3$.

   (c) E_i *is a non-hyperbolic if* $s = \kappa_3$.

Proof 4.1.43 *For* $\eta_2 \in \mathbb{R}^+$, $\eta_3 \in \left]-2\sqrt{\eta_2}, -\sqrt{3\eta_2}\right[$*:*

1. *When* $\eta_1 \in [0, (\eta_1)_2]$:

   the condition $3\eta_2 < \eta_3^2 < 4\eta_2$ *is verified, hence* $(\eta_1)_2 > 0$ *(2 in Lemma 4.1.24).*

   $\Delta \leq 0$ *because* $\Delta^* \leq 0$ *(2 in Remark 4.1.23). One has* $P_1(1) > 0$ *(*η_1 *positive) and* $P_1(-1) < 0$ *if* $s \in \left]0, \dfrac{2}{-\eta_3}\right[$ *because*

   $\eta_1 s^3 - 2\eta_2 s^2 - 4\eta_3 s - 8 < -\eta_2\eta_3 s^3 - 2\eta_2 s^2 - 4\eta_3 s - 8$*(*$(\eta_1)_2 + \eta_2\eta_3$ *negative (Lemma 4.1.30)),*
   $\eta_1 s^3 - 2\eta_2 s^2 - 4\eta_3 s - 8 < (-\eta_3 s - 2)\left(\eta_2 s^2 + 4\right),$

   so $P_1(-1) < 0$ *if* $(-\eta_3 s - 2) < 0$ *because* $\left(\eta_2 s^2 + 4\right) > 0$.

   And

   $\left(\eta_3 - \sqrt{\eta_3^2 - 3\eta_2}\right)\left(\eta_3 + \sqrt{\eta_3^2 - 3\eta_2}\right) = 3\eta_2 > 0,$
   then $\lambda_1^*, \lambda_2^* < 1$ *because* $\left(\eta_3 + \sqrt{\eta_3^2 - 3\eta_2}\right) < 0.$

   Hence, applying the stability conditions using Lemma 4.1.36, Lemma 4.1.38 and Theorem 4 (1.i) of [23], one obtains the result.

2. *If* $a \in \left](\eta_1)_2, +\infty\right[$:

   the condition $3\eta_2 < \eta_3^2 < 4\eta_2$ *is verified, then* $(\eta_1)_2 > 0$ *(2 in Lemma 4.1.24).*

   $\Delta > 0$ *because* $\Delta^* > 0$ *(1 in Remark 4.1.23). And according to Lemma 4.1.26, Lemma 4.1.28 and Remark 4.1.35, one has* $\varphi_1 > 0$.

   One study the sign of $\left(\varphi_1^2 + 3\eta_3\varphi_1 + 9\eta_2\right), (\varphi_1 + 3\eta_3), \varphi_1(s\varphi_1 - 6).$

   The sign of $\varphi_1^2 + 3\eta_3\varphi_1 + 9\eta_2$ *is positive because* $\left(\varphi_1^2 + 3\eta_3\varphi_1 + 9\eta_2\right)\varphi_1 = 27\eta_1$ *is positive, and* $\theta_1 - 3c$ *is negative if* $\eta_1 + \eta_2\eta_3$ *is negative because* $(\varphi_1 + 3\eta_3)\left(\varphi_1^2 + 9\eta_2\right) = 27(\eta_1 + \eta_2\eta_3).$

   Hence $\eta_1 \in \left](\eta_1)_2, +\infty\right[\cap \left]0, bc\right[= \left](\eta_1)_2, bc\right[$ *(*$(\eta_1)_2 < -\eta_2\eta_3$ *using Lemma 4.1.30),*

   and $\varphi_1 + 3\eta_3$ *is positive or null if* $\eta_1 + \eta_2\eta_3$ *is positive or null because* $(\varphi_1 + 3\eta_3)\left(\varphi_1^2 + 9\eta_2\right) = 27(\eta_1 + \eta_2\eta_3).$

   Hence $\eta_1 \in \left](\eta_1)_2, +\infty\right[\cap \left[-\eta_2\eta_3, +\infty\right[= \left[-\eta_2\eta_3, +\infty\right[$ *(*$(\eta_1)_2 < -\eta_2\eta_3$ *using Lemma 4.1.30).*

   Using Lemma 4.1.33, hence applying the stability conditions using Definition 4.1.20. One can obtain the results.

4.1.2 Numerical Simulations

In this subsection, we present bifurcation diagrams, phase portraits of the model (4.9), which confirm the analytical results above and illustrate the dynamic behaviors of our model numerically relay. A bifurcation occurs when the stability of a point of equilibrium changes [13].

Based on the previous analysis, the parameters of the model (4.9) can be examined by: varying h in the range $1.25 \leq h \leq 1.4$ and fixing $a = 1.63$, $b = 0.418$, $c = 1.98$, $\alpha = 0.99$, with the initial conditions $(x_0, y_0, z_0) = (0.3, 2.11, -0.1)$. The resulting points are plotted versus the parameter h (see Figure 4.7).

According to Theorem 4.1.40, we have $\eta_2 = 0.83163 \in \mathbb{R}^+$, $\eta_3 = -1.8929 < -2\sqrt{\eta_2} = -1.8239$, and $\eta_1 = 0.425\,89 \in]\left(\eta_1\right)_2, -\eta_2\eta_3[=]0.106\,42, 1.574\,2[$; we have $\left(\varphi_1\right)_2 \simeq 4.5921$, $s = \dfrac{6}{\left(\varphi_1\right)_2} = -3\dfrac{\left(\varphi_1\right)_2 + 3\eta_3}{\left(\varphi_1\right)_2^2 + 3\eta_3\left(\varphi_1\right)_2 + 9\eta_2} \simeq 1.3066$, $h = \sqrt[\alpha]{s\Gamma\left(1+\alpha\right)} \simeq 1.3046$ and E_1 is asymptotically stable if $0 < h < 1.3046$, (see (a) and (b) in Figure 4.3), all trajectories converge to the point E_1. If $h \simeq 1.3046$, system (4.9) undergoes a bifurcation as mentioned above (see (c) in Figure 4.3); the fixed point E_1 becomes unstable if $h > 1.3046$ (see (d) in Figure 4.3).

In this second part of numerical results, varying b in the range $0.417 \leq b \leq 0.5$, and fixing $a = 1.63$, $c = 1.98$, $h = 1.399$, $\alpha = 0.99$, the resulting points are plotted versus the parameter b (see Figure 4.4). Attracting invariant circles and chaos appear when decreasing b in such way that the parameter remains in the interval $[0.417, 0.5]$. The phase portraits for various $b-$values corresponding to Figure 4.4 are plotted. Furthermore, the period-2 orbits ($b = 0.44$) are shown (a) in Figure 4.5, and for the attracting invariant circles ($b = 0.4395, b = 0.428$) see (b) and (c) in Figure 4.5. Attracting chaotic sets are also observed if $b = 0.418$ and are plotted (d) in Figure 4.5.

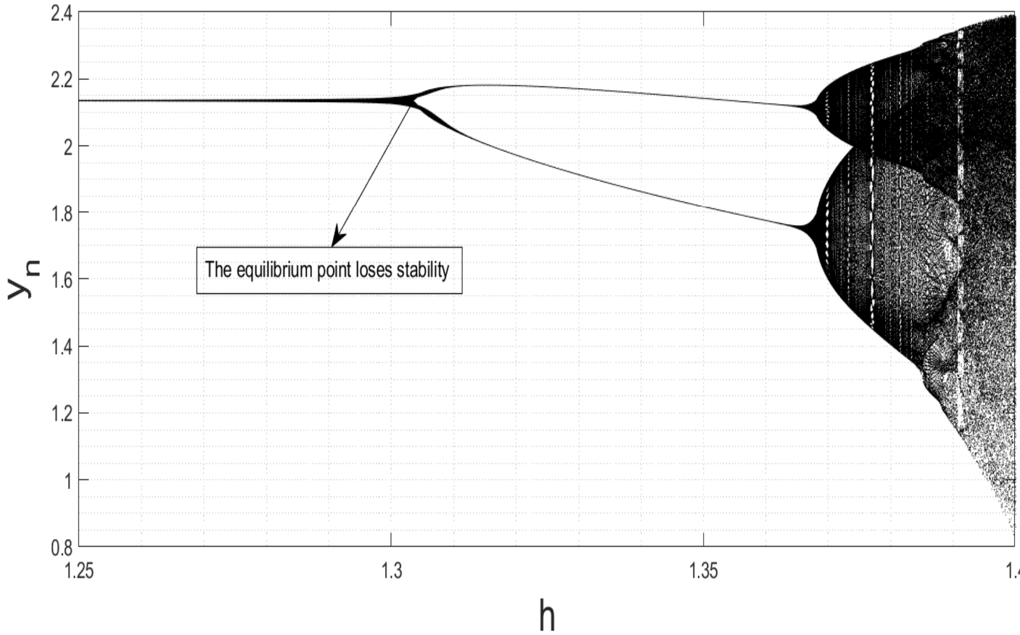
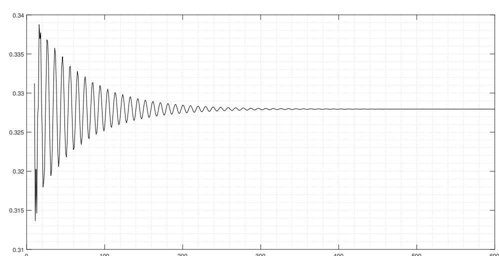
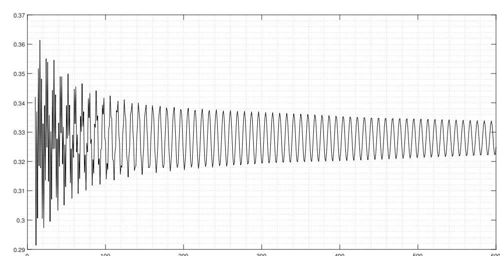


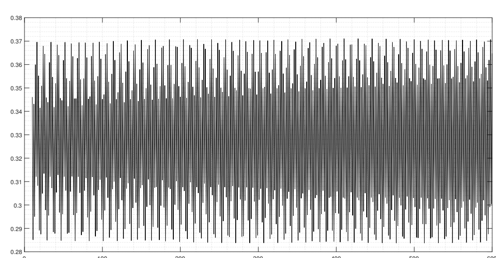
Figure 4.2: Bifurcation Diagram of model (4.9) for $h \in [1.25, 1.4]$.



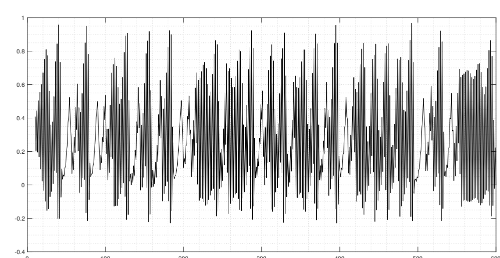
(a) Stable fixed point at $h = 1.2$.



(b) Stable fixed point at $h = 1.3046$.



(c) The fixed point loses stability at $h = 1.3046$.



(d) Unstable fixed point at $h = 1.4$.

Figure 4.3: The trajectory diagrams of model (4.9) for Various h Corresponding to Figure 4.2.

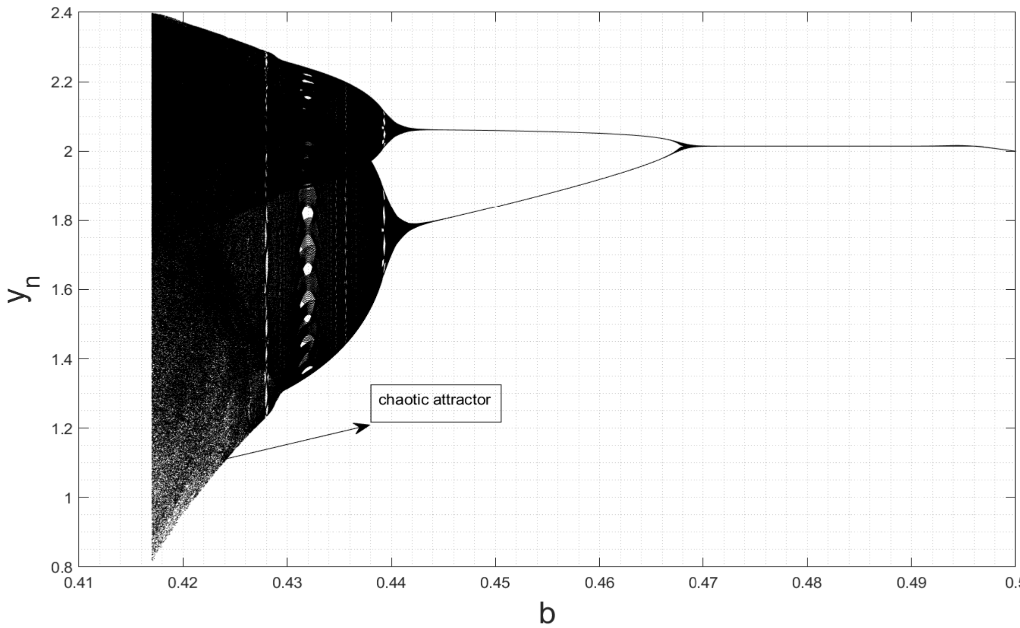
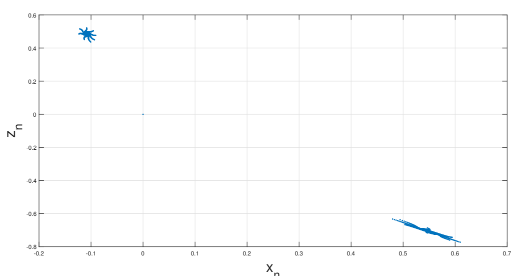
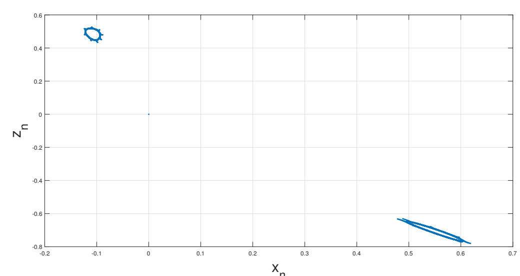


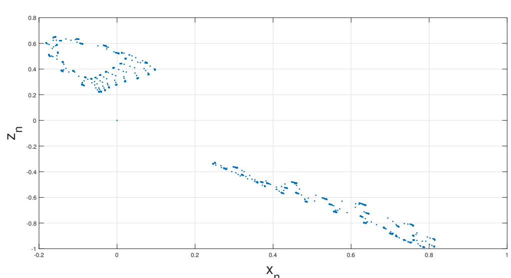
Figure 4.4: Bifurcation Diagram of Model (4.9) for $b \in [0.417, 0.5]$.



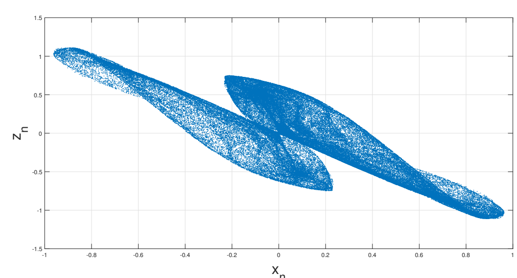
(a) Period-2 orbits at $b = 0.44$.



(b) Attracting invariant circles at $b = 0.4395$.



(c) Attracting invariant circles at $b = 0.428$.



(d) Chaotic attractor at $b = 0.418$

Figure 4.5: Phase Portrait Diagrams of model (4.9) for Various b Corresponding to Figure 4.4.

4.2 Bifurcation and Stability in a New Discrete System Induced from Fractional Order Continuous Chaotic Arneodo's System

The fractional-order Arneodo's system [36] is given by:

$$\begin{cases} D^{\alpha}x(t) = y(t), \\ D^{\alpha}y(t) = z(t), \\ D^{\alpha}z(t) = -ax(t) - by(t) - cz(t) + dx^3(t), \end{cases} \tag{4.17}$$

where a, b, c and d are constant parameters, $t > 0$ and α is the fractional order satisfying $\alpha \in (0,1)$. Assume that $x(0) = x_0$; $y(0) = y_0$ and $z(0) = z_0$ are the initial conditions of system (4.17).

Following [24] a transformation process from a continuous system of fractional order to a discrete system is proposed as follows

$$\begin{cases} x_{n+1} = x_n + sy_n, \\ y_{n+1} = y_n + sz_n, \\ z_{n+1} = z_n + s\left(-ax_n - by_n - cz_n + dx_n^3\right), \end{cases} \tag{4.18}$$

where $s = \dfrac{h^{\alpha}}{\Gamma(1+\alpha)}$, and h is a new positive parameter in the discrete system.

For the following values $a = 0.7437$, $b = 1.523$, $c = 2.158$, $d = 2$, $\alpha = 0.99$, the system (4.18) is in a chaotic state because one of the Lyapunov exponent is positive, which is considered as one of the characteristics of the existence of chaos (see Figure 4.6).

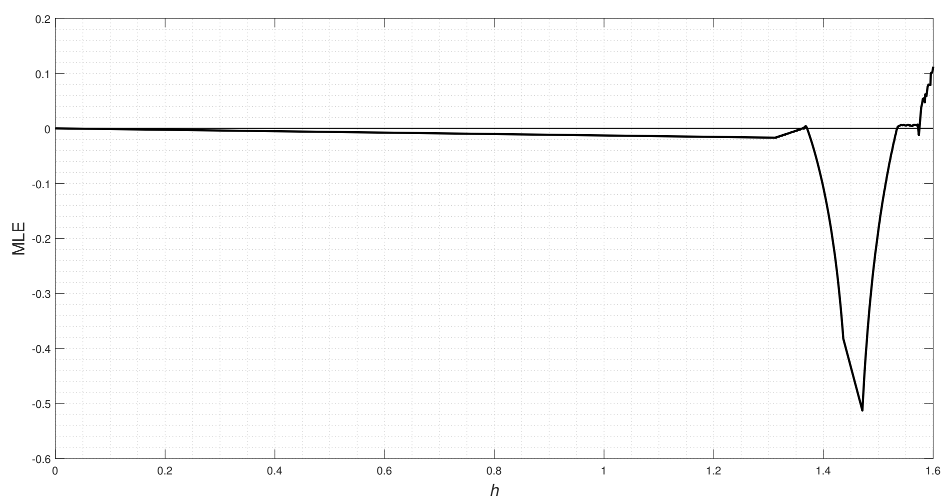


Figure 4.6: Maximal Lyapunov exponent of model (4.18) for $h \in [0, 1.6]$.

4.2.1 Stability of the Fixed Points of Discrete System

In this subsection, one discusses the local stability of the fixed points of system (4.18), which is determined by the eigenvalues of the Jacobian matrices corresponding to its fixed points. The Jacobian matrix of system (4.18) is:

$$J_{E_{eq}} = \begin{pmatrix} 1 & s & 0 \\ 0 & 1 & s \\ s\left(-a+3dx_n^2\right) & -bs & 1-cs \end{pmatrix}. \tag{4.19}$$

The fixed points of system (4.18) are: $E_0 = (0,0,0)$, $E_1 = \left(\sqrt{\frac{a}{d}},0,0\right)$, and $E_2 = \left(-\sqrt{\frac{a}{d}},0,0\right)$. In order to study the stability of the fixed points of system (4.18), we recall the lemma 1 of [31], the definition and the theorem 4 of [23].

Stability of Fixed Point E_0

The Jacobian matrix associated with the fixed point E_0 of the system (4.18) is given by

$$J_{E_0} = \begin{pmatrix} 1 & s & 0 \\ 0 & 1 & s \\ -as & -bs & 1-cs \end{pmatrix}, \tag{4.20}$$

then, the characteristic polynomial of J_{E_0} is

$$P_1(\lambda) = \lambda^3 + \sigma_1\lambda^2 + \sigma_2\lambda + \sigma_3, \tag{4.21}$$

where $\sigma_1 = cs-3$, $\sigma_2 = bs^2 - 2cs + 3$, and $\sigma_3 = as^3 - bs^2 + cs - 1$. By calculating, one further has

$A_1 = \sigma_1^2 - 3\sigma_2 = -s^2\left(3b - c^2\right),$
$B_1 = \sigma_1\sigma_2 - 9\sigma_3 = -s^2\left(2c^2 - bsc - 6b + 9as\right),$
$C_1 = \sigma_2^2 - 3\sigma_1\sigma_3 = s^2\left(b^2s^2 - bcs - 3b + c^2 - 3acs^2 + 9as\right),$
$\Delta_1 = B_1^2 - 4A_1C_1 = 3s^6\left(27a^2 - 18abc + 4ac^3 + \left(4b - c^2\right)b^2\right) = s^6\Delta_1^*.$

The derivative of $P_1(\lambda)$ is $P_1'(\lambda) = 3\lambda^2 + 2\sigma_1\lambda + \sigma_2$, and the equation $P_1'(\lambda) = 0$ has two roots:

$$\lambda_{1,2}^* = \frac{1}{3}\left(-\sigma_1 \pm \sqrt{\sigma_1^2 - 3\sigma_2}\right) = \frac{1}{3}s\left(-c \pm \sqrt{c^2 - 3b}\right) + 1.$$

When $\Delta_1^* \le 0$, i.e. $\Delta_1 \le 0$, by Lemma 1 page 6 in [31], equation (4.21) has three real roots λ_1, λ_2 and λ_3. From this, one can easily prove that both roots $\lambda_{1,2}^*$ (let $\lambda_1^* \le \lambda_2^*$) of equation $P_1'(\lambda) = 0$ are also real. When $\Delta_1^* > 0$, i.e. $\Delta_1 > 0$, by Lemma 1 page 6 in [31], one has that equation (4.21) has one real root $\lambda_1 = -\frac{\sigma_1 + y_1^{\frac{1}{3}} + y_2^{\frac{1}{3}}}{3}$, and a pair of conjugate complex roots $\lambda_{2,3}$:

$$\lambda_{2,3} = \frac{1}{6}\left(\left(y_1^{\frac{1}{3}} + y_2^{\frac{1}{3}} - 2\sigma_1\right) \pm i\sqrt{3}\left(y_1^{\frac{1}{3}} - y_2^{\frac{1}{3}}\right)\right),$$

where

$$y_{1,2} = \frac{s^3}{2}\left(\left(2c^3 - 9bc + 27a\right) \pm 3\sqrt{\Delta_1^*}\right), y_1 > y_2,$$

and

$$a_1 = \frac{2}{27}\left(\left(\frac{9}{2}b - c^2\right)c - \sqrt{-\left(3b - c^2\right)^3}\right), a_2 = \frac{2}{27}\left(\left(\frac{9}{2}b - c^2\right)c + \sqrt{-\left(3b - c^2\right)^3}\right).$$

Let $\theta_1 = \dfrac{y_1^{\frac{1}{3}} + y_2^{\frac{1}{3}}}{s} + c$, then $\theta_1^3 = 3c\theta_1^2 - 9b\theta_1 + 27a$, and

$(\theta_1)_1 = \dfrac{1}{2}\left(3c - \sqrt{3}\sqrt{3c^2 - 8b}\right)$, $(\theta_1)_2 = \dfrac{1}{2}\left(3c + \sqrt{3}\sqrt{3c^2 - 8b}\right)$,

and $P_1(1) = as^3$, $P_1(-1) = as^3 - 2bs^2 + 4cs - 8$.

Remark 4.2.1 *Δ_1^* is a polynomial of degree 2 of the variable a.*

1. *$\Delta_1^* > 0$ if $3b < c^2$, and $a \in]-\infty, a_1[\cup]a_2, +\infty[$ or if $3b > c^2$ or if $3b = c^2$, $a \in \mathbb{R} - \{\frac{1}{3}bc - \frac{2}{27}c^3\}$.*

2. *$\Delta_1^* \leq 0$ if $3b < c^2$, $a \in [a_1, a_2]$ or if $3b = c^2$ and $a = \frac{1}{3}bc - \frac{2}{27}c^3$.*

Lemma 4.2.2

1. *If $4b < c^2$ then $a_1 < 0$, $a_2 > 0$.*

2. *If $b, c > 0$ and $3b < c^2 < 4b$, then $a_1, a_2 > 0$.*

3. *If $b > 0$, $c < 0$ and $3b < c^2 < 4b$ then $a_1, a_2 < 0$.*

Proof 4.2.3 *As $a_1 a_2 = \frac{1}{27}b^2\left(4b - c^2\right)$, then the sign of a_1, a_2 depends on the sign of $4b - c^2$.*

Lemma 4.2.4 *If $4b < c^2$ or $3b < c^2 < 4b$, and $a < a_1 < 0$, then $y_2 < 0$.*

Proof 4.2.5 *Either $4b < c^2$ or $3b < c^2 < 4b$, and $a \in]-\infty, a_1[$, one has $3b < c^2$, and Δ_1^* positive (1 in Remark 4.2.1),*

and $\left(c^3 - \frac{9}{2}bc + \frac{27}{2}a\right) < 0$ because

$$\left(c^3 - \frac{9}{2}bc + \frac{27}{2}a\right) < \left(c^3 - \frac{9}{2}bc + \frac{27}{2}a_1\right) = -\sqrt{-\left(3b - c^2\right)^3} < 0,$$

as

$$y_2 = \frac{s^3}{2}\left(\left(2c^3 - 9bc + 27a\right) - 3\sqrt{\Delta_1^*}\right),$$

then $y_2 < 0$.

Lemma 4.2.6 *If $4b < c^2$ or $3b < c^2 < 4b$, and $a > a_2 > 0$, then $y_1 > 0$.*

Proof 4.2.7 *Either* $4b < c^2$ *or* $3b < c^2 < 4b$, *and* $a \in]a_2, +\infty[$, *one has* $3b < c^2$, *and* Δ_1^* *positive (*1 *in Remark* 4.2.1*),*

and $\left(c^3 - \frac{9}{2}bc + \frac{27}{2}a\right) > 0$ *because*

$$\left(c^3 - \frac{9}{2}bc + \frac{27}{2}a\right) > \left(c^3 - \frac{9}{2}bc + \frac{27}{2}a_2\right) = \sqrt{-\left(3b - c^2\right)^3} > 0,$$

as

$$y_1 = \frac{s^3}{2}\left(\left(2c^3 - 9bc + 27a\right) + 3\sqrt{\Delta_1^*}\right),$$

then $y_1 > 0$.

Lemma 4.2.8 *If* $3b < c^2$, *then* y_1 *and* y_2 *have the same sign.*

Proof 4.2.9 *If* $3b < c^2$, *one has* $y_1 y_2 = s^6\left(-3b + c^2\right)^3 > 0$, *then* y_1, y_2 *have the same sign.*

Lemma 4.2.10 *If* $b \in \mathbb{R}^+$, $c < -\sqrt{3b}$ *then* $a_1 - bc > 0$.

Proof 4.2.11 *The difference between* a_1 *and* bc, *is:*

$$a_1 - bc = -\frac{2}{27}\left(\sqrt{-\left(3b - c^2\right)^3} + c^3\right) - \frac{2}{3}bc,$$

then $\left(\sqrt{-\left(3b - c^2\right)^3} + c^3\right) < 0$ *because*

$$\left(\sqrt{-\left(3b - c^2\right)^3} + c^3\right)\left(\sqrt{-\left(3b - c^2\right)^3} - c^3\right) = -27b^3 + 9bc^2\left(3b - c^2\right) < 0,$$

and $\left(\sqrt{-\left(3b - c^2\right)^3} - c^3\right) > 0$. *Therefore,* $a_1 - bc > 0$.

Lemma 4.2.12 *If* $b \in \mathbb{R}^+$, $\sqrt{3b} < c$ *then* $a_2 - bc < 0$.

Proof 4.2.13 *The difference between* a_2 *and* bc, *is:*

$$a_2 - bc = \frac{2}{27}\left(\sqrt{-\left(3b - c^2\right)^3} - c^3\right) - \frac{2}{3}bc,$$

then $\left(\sqrt{-\left(3b - c^2\right)^3} - c^3\right) < 0$ *because*

$$\left(\sqrt{-\left(3b - c^2\right)^3} - c^3\right)\left(\sqrt{-\left(3b - c^2\right)^3} + c^3\right) = -27b^3 + 9bc^2\left(3b - c^2\right) < 0,$$

and $\left(\sqrt{-\left(3b - c^2\right)^3} + c^3\right) > 0$. *Therefore,* $a_2 - bc < 0$.

Remark 4.2.14 *If $b \in \mathbb{R}$ and $3b < c^2$, then $8b < 3c^2$.*

Lemma 4.2.15 *If $3b < c^2$, $a \in]-\infty, a_1[\cup]a_2, +\infty[$, then*

1. *the sign of $|\lambda_{2,3}| - 1$ depends on the sign of* $\left(\theta_1^2 - 3c\theta_1 + 9b\right) s + 3\left(\theta_1 - 3c\right).$
2. *the sign of $|\lambda_1| - 1$ depends on the sign of $\theta_1\left(s\theta_1 - 6\right).$*

Proof 4.2.16 *$\Delta_1^* > 0$ if $3b < c^2$ and $a \in]-\infty, a_1[\cup]a_2, +\infty[$ (1 in Remark 4.2.1). Calculating $|\lambda_{2,3}|^2 - 1$ and $|\lambda_1|^2 - 1$ one finds:*

$$\begin{aligned}
\left(|\lambda_{2,3}| - 1\right)\left(|\lambda_{2,3}| + 1\right) &= \frac{1}{9}\left(\left(\theta_1^2 - 3c\theta_1 + 9b\right) s^2 + 3\left(\theta_1 - 3c\right) s\right), \\
\left(|\lambda_1| - 1\right)\left(|\lambda_1| + 1\right) &= \frac{1}{9} s\theta_1\left(s\theta_1 - 6\right).
\end{aligned}$$

The above result is true because $|\lambda_{2,3}| + 1$, $|\lambda_1| + 1$, s are positive.

Remark 4.2.17 *$\theta_1 > 0$ if y_1, $y_2, c > 0$ and $\theta_1 < 0$ if y_1, $y_2, c < 0$.*

Lemma 4.2.18 *For $b \in \mathbb{R}^+$, $c \in \left]2\sqrt{b}, +\infty\right[$, then $\dfrac{2}{c} < \dfrac{6}{c + \sqrt{-3b + c^2}}$.*

Proof 4.2.19 *$\dfrac{2}{c} - \dfrac{6}{c + \sqrt{-3b + c^2}} = \dfrac{2}{bc}\left(b - c^2 + c\sqrt{c^2 - 3b}\right)$, $b < c^2$ and $c^2 > 3b$ because $4b < c^2$, and*

$$\left(b - c^2 + c\sqrt{c^2 - 3b}\right)\left(b - c^2 - c\sqrt{c^2 - 3b}\right) = b\left(c^2 + b\right) > 0, \quad \text{one has}$$

$b - c^2 + c\sqrt{c^2 - 3b} < 0$ because $b - c^2 - c\sqrt{c^2 - 3b} < 0$,

then $\dfrac{2}{c} < \dfrac{6}{c + \sqrt{-3b + c^2}}$.

Lemma 4.2.20 *For $b \in \mathbb{R}^+$, $c \in \left]2\sqrt{b}, +\infty\right[$ or $c \in \left]\sqrt{3b}, 2\sqrt{b}\right[$, the sign of $\lambda_1^* + 1$ and $\lambda_2^* + 1$ are positive if $s \in \left]0, \dfrac{6}{c + \sqrt{-3b + c^2}}\right[$.*

Proof 4.2.21 *$\lambda_1^* + 1 = \dfrac{1}{3} s\left(-c - \sqrt{c^2 - 3b}\right) + 2$ and $\lambda_2^* + 1 = \dfrac{1}{3} s\left(-c + \sqrt{c^2 - 3b}\right) + 2$ are a polynomial of degree 1 of the variable s, and $\left(-c - \sqrt{c^2 - 3b}\right)\left(-c + \sqrt{c^2 - 3b}\right) = 3b > 0$, then $\left(-c + \sqrt{c^2 - 3b}\right) < 0$, and $c^2 > 3b$.*

The sign of $\lambda_1^* + 1$ is positive if $s \in \left]0, \dfrac{6}{c + \sqrt{-3b + c^2}}\right[$, and the sign of $\lambda_2^* + 1$ is positive if $s \in \left]0, \dfrac{6}{c - \sqrt{-3b + c^2}}\right[$.

One has $\dfrac{6}{c + \sqrt{-3b + c^2}} < \dfrac{6}{c - \sqrt{-3b + c^2}}$.

Finally $\lambda_1^* + 1$, $\lambda_2^* + 1$ are positive if $s \in \left]0, \dfrac{6}{c + \sqrt{-3b + c^2}}\right[$.

Now, relatively to the dynamical properties of the fixed point E_0, one has the following results.

For theorems 4.2.22 to 4.2.36, let: $\nu_1 = -3\dfrac{\theta_1 - 3c}{\theta_1^2 - 3c\theta_1 + 9b}$, $\nu_2 = \dfrac{2}{\sqrt{-b}}$ and $\nu_3 = \dfrac{6}{\theta_1}$.

Theorem 4.2.22 *If the fixed point E_0 exists with the following assumptions $b \in \mathbb{R}_*^-$, $c \in \mathbb{R}^-$ and $d \in \mathbb{R}$.*

1. *When $a \in \left]-\infty, a_1\right[$ then $\Delta_1 > 0$ and:*
   - *(a) E_0 is a unstable if $s > \nu_1$.*
   - *(b) E_0 is a two-dimensional saddle if $0 < s < \nu_1$.*
   - *(c) E_0 is a non-hyperbolic if $s = \nu_1$.*
2. *When $a \in \left[a_1, 0\right[$ then $\Delta_1 \leq 0$ and E_0 is a source if $0 < s < \nu_2$.*
3. *When $a = 0$ then $\Delta_1 \leq 0$ and E_0 is a non-hyperbolic.*

Proof 4.2.23 *For $b \in \mathbb{R}_*^-$, $c \in \mathbb{R}^-$ and $d \in \mathbb{R}$:*

1. *If $a \in \left]-\infty, a_1\right[$:*

   the condition $4b < c^2$ is verified, hence $a_1 < 0$ (1 in Lemma 4.2.2), and $3b < c^2$.

   $\Delta_1 > 0$ because $\Delta_1^* > 0$ (1 in Remark 4.2.1). And according to Lemma 4.2.4, Lemma 4.2.8 and Remark 4.2.17, one has $\theta_1 < 0$.

   One study the sign of $\left(\theta_1^2 - 3c\theta_1 + 9b\right)$, $\left(\theta_1 - 3c\right)$, $\theta_1\left(s\theta_1 - 6\right)$.

   The sign of $\theta_1^2 - 3c\theta_1 + 9b$ is positive because $\left(\theta_1^2 - 3c\theta_1 + 9b\right)\theta_1 = 27a$ is negative, and $\theta_1 - 3c$ is negative because $\left(\theta_1 - 3c\right)\theta_1^2 = -9b\theta_1 + 27a$ is negative, and $\theta_1\left(s\theta_1 - 6\right)$ is positive.

   Using Lemma 4.2.15, hence applying the stability conditions using Definition 4.1.20. One can obtain the results.

2. *If $a \in \left[a_1, 0\right[$:*

   the condition $4b < c^2$ is verified, then $a_1 < 0$ is negative (1 in Lemma 4.2.2), and $3b < c^2$.

   $\Delta_1 \leq 0$ because $\Delta_1^* \leq 0$ (2 in Remark 4.2.1). One has $P_1(1) < 0$ (a negative) and $P_1(-1) < 0$ if $s \in \left]0, \nu_2\right[$ because

   $$as^3 - 2bs^2 + 4cs - 8 < bcs^3 - 2bs^2 + 4cs - 8 \text{ (a negative, bc positive)},$$
   $$as^3 - 2bs^2 + 4cs - 8 < (cs - 2)\left(bs^2 + 4\right),$$

   hence $P_1(-1) < 0$ if $\left(bs^2 + 4\right) > 0$ because $(cs - 2) < 0$.

   $\lambda_2^* > 1$ *because* $\left(-c + \sqrt{c^2 - 3b}\right) > 0.$

   Hence, applying the stability conditions using Theorem 4 (2.i.b) of [23], one obtains the result.

3. *If* $a = 0$:

   the condition $4b < c^2$ *is verified, and* $3b < c^2$.

   $\Delta_1 \leq 0$ *because* $\Delta_1^* \leq 0$ *(*2 in Remark 4.2.1*). One has* $P_1(1) = 0$ *(a null).*

   Hence, applying the stability conditions using Theorem 4 *(5.i) of [23], one obtains the result.*

Theorem 4.2.24 *If the fixed point* E_0 *exists with the following assumptions* $b \in \mathbb{R}_*^-$, $c \leq -3\sqrt{-b}$ *and* $a \in \left]bc, a_2\right]$, $d \in \mathbb{R}$ *then* E_0 *is a source if* $s > \nu_2$.

Proof 4.2.25 *For* $b \in \mathbb{R}_*^-$, $c \leq -3\sqrt{-b}$, *and* $a \in \left]bc, a_2\right]$ *the condition* $4b < c^2$ *is verified, hence* $a_2 > 0$ *(1 in Lemma* 4.2.2*), and* $3b < c^2$.

$-\frac{2}{3}bc - \frac{2}{27}c^3 \geq 0$ *if* $c \leq -3\sqrt{-b}$ *then* $a_2 - bc = \frac{2}{27}\sqrt{-\left(3b - c^2\right)^3} - \frac{2}{3}bc - \frac{2}{27}c^3 > 0$.

$\Delta_1 \leq 0$ *because* $\Delta_1^* \leq 0$ *(2 in Remark* 4.2.1*). One has* $P_1(1) > 0$ *(a positive) and* $P_1(-1) > 0$ *if* $s \in \left]\nu_2, +\infty\right[$ *because*

$as^3 - 2bs^2 + 4cs - 8 > bcs^3 - 2bs^2 + 4cs - 8$ *(*$a \in \left]bc, a_2\right]$*)*,

$as^3 - 2bs^2 + 4cs - 8 > (cs - 2)\left(bs^2 + 4\right)$,

hence $P_1(-1) > 0$ *if* $\left(bs^2 + 4\right) < 0$ *because* $(cs - 2) < 0$.

$\lambda_2^* > 1$ *because* $\left(-c + \sqrt{c^2 - 3b}\right) > 0$.

Hence, applying the stability conditions using Theorem 4 *(2.i.a) of [23], one obtains the result.*

Theorem 4.2.26 *If the fixed point* E_0 *exists with the following assumptions* $b \in \mathbb{R}_*^-$, $c \geq 3\sqrt{-b}$ *and* $a \in \left[a_1, bc\right[$, $d \in \mathbb{R}$ *then* E_0 *is a source if* $0 < s < \frac{2}{c}$ *or* $\nu_2 < s$.

Proof 4.2.27 *For* $b \in \mathbb{R}_*^-$, $c \geq 3\sqrt{-b}$, *and* $a \in \left[a_1, bc\right[$ *the condition* $4b < c^2$ *is verified, then* $a_1 < 0$ *(1 in Lemma* 4.2.2*), and* $3b < c^2$.

$-\frac{2}{3}bc - \frac{2}{27}c^3 \leq 0$ *if* $c \geq 3\sqrt{-b}$ *hence* $a_1 - bc = -\frac{2}{27}\sqrt{-\left(3b - c^2\right)^3} - \frac{2}{3}bc - \frac{2}{27}c^3 < 0$.

$\Delta_1 \leq 0$ *because* $\Delta_1^* \leq 0$ *(2 in Remark* 4.2.1*). One has* $P_1(1) < 0$ *(a negative) and* $P_1(-1) < 0$ *if* $s \in \left]0, \frac{2}{c}\right[$ *or* $s \in \left]\nu_2, +\infty\right[$ *because*

$as^3 - 2bs^2 + 4cs - 8 < bcs^3 - 2bs^2 + 4cs - 8$ *(*$a \in \left[a_1, bc\right[$*)*,

$as^3 - 2bs^2 + 4cs - 8 < (cs - 2)\left(bs^2 + 4\right)$,

then $P_1(-1) < 0$ *if* $\left(bs^2 + 4\right) < 0$, $(cs - 2) > 0$ *or* $\left(bs^2 + 4\right) > 0$, $(cs - 2) < 0$, *and* $\frac{2}{c} < \frac{2}{\sqrt{-b}}$ $\left(\text{using } c \geq 3\sqrt{-b}\right)$.

One has $\left(-c + \sqrt{c^2 - 3b}\right)\left(-c - \sqrt{c^2 - 3b}\right) = 3b < 0$.

$\lambda_2^* > 1$ *because* $\left(-c + \sqrt{c^2 - 3b}\right) > 0$.

Hence, applying the stability conditions using Theorem 4 *(2.i.b) of [23], one obtains the result.*

Theorem 4.2.28 *If the fixed point E_0 exists with the following assumptions $b \in \mathbb{R}^-$, $c \in \mathbb{R}_*^+$ and $d \in \mathbb{R}$.*

1. *When $a = 0$ then $\Delta_1 \leq 0$ and E_0 is a non-hyperbolic.*
2. *When $a \in]0, a_2]$ then $\Delta_1 \leq 0$ and E_0 is a source if $s > \dfrac{2}{c}$.*
3. *When $a \in]a_2, +\infty[$ then $\Delta_1 > 0$ and:*
   (a) *E_0 is a unstable if $s > \nu_3$.*
   (b) *E_0 is a one-dimensional saddle if $0 < s < \nu_3$.*
   (c) *E_0 is a non-hyperbolic if $s = \nu_3$.*

Proof 4.2.29 *For $b \in \mathbb{R}^-$, $c \in \mathbb{R}_*^+$ and $d \in \mathbb{R}$:*

1. *If $a = 0$:*

   the condition $4b < c^2$ is verified, and $3b < c^2$.

   $\Delta_1 \leq 0$ because $\Delta_1^* \leq 0$ (2 in Remark 4.2.1). One has $P_1(1) = 0$ (a null).

   Hence, applying the stability conditions using Theorem 4 (5.i) of [23], one obtains the result.

2. *If $a \in]0, a_2]$:*

   the condition $4b < c^2$ is verified, then $a_2 > 0$ (1 in Lemma 4.2.2), and $3b < c^2$.

   $\Delta_1 \leq 0$ because $\Delta_1^* \leq 0$ (2 in Remark 4.2.1). One has $P_1(1) > 0$ (a positive) and $P_1(-1) > 0$ if $s \in \left]\dfrac{2}{c}, +\infty\right[$ because

   $as^3 - 2bs^2 + 4cs - 8 > 4cs - 8$ (a positive, b negative),

   then $P_1(-1) > 0$ if $(4cs - 8) > 0$.

   One has $\left(-c + \sqrt{c^2 - 3b}\right)\left(-c - \sqrt{c^2 - 3b}\right) = 3b < 0$, hence

   $\lambda_2^* > 1$ because $\left(-c + \sqrt{c^2 - 3b}\right) > 0$.

   Hence, applying the stability conditions using Theorem 4 (2.i.a) of [23], one obtains the result.

3. *If $a \in]a_2, +\infty[$:*

   the condition $4b < c^2$ is verified, then $a_2 > 0$ (1 in Lemma 4.2.2), and $3b < c^2$.

   $\Delta_1 > 0$ because $\Delta_1^* > 0$ (1 in Remark 4.2.1). And according to Lemma 4.2.6, Lemma 4.2.8 and Remark 4.2.17, one has $\theta_1 > 0$.

   One study the sign of $\left(\theta_1^2 - 3c\theta_1 + 9b\right)$, $(\theta_1 - 3c)$, $\theta_1(s\theta_1 - 6)$.

   The sign of $\theta_1^2 - 3c\theta_1 + 9b$ is positive because $\left(\theta_1^2 - 3c\theta_1 + 9b\right)\theta_1 = 27a$ is positive, and $\theta_1 - 3c$ is positive because $(\theta_1 - 3c)\theta_1^2 = -9b\theta_1 + 27a$ is positive.

   Using Lemma 4.2.15, hence applying the stability conditions using Definition 4.1.20. One can obtain the results.

Theorem 4.2.30 *If the fixed point E_0 exists with the following assumptions $b \in \mathbb{R}^+$, $c < -2\sqrt{b}$ and $d \in \mathbb{R}$.*

1. *When $a \in \left]-\infty, bc\right[$ then $\Delta_1 > 0$ and:*

   (a) *E_0 is a unstable if $s > \nu_1$.*

   (b) *E_0 is a two-dimensional saddle if $0 < s < \nu_1$.*

   (c) *E_0 is a non-hyperbolic if $s = \nu_1$.*

2. *When $a \in \left[bc, a_1\right[$ then $\Delta_1 > 0$ and E_0 is a unstable.*

3. *When $a \in \left[a_1, 0\right[$ then $\Delta_1 \leq 0$ and E_0 is a source.*

4. *When $a = 0$ then $\Delta_1 \leq 0$ and E_0 is a non-hyperbolic.*

5. *When $a \in \left]0, a_2\right]$ then $\Delta_1 \leq 0$ and E_0 is a one-dimensional if $0 < s < 2\dfrac{b}{a}$.*

Proof 4.2.31 *For $b \in \mathbb{R}^+$, $c < -2\sqrt{b}$ and $d \in \mathbb{R}$:*

1. *If $a \in \left]-\infty, a_1\right[$:*

   the condition $4b < c^2$ is verified, then $a_1 < 0$ (1 in Lemma 4.2.2), and $3b < c^2$.

   $\Delta_1 > 0$ because $\Delta_1^* > 0$ (1 in Remark 4.2.1). And according to Lemma 4.2.4, Lemma 4.2.8 and Remark 4.2.17, one has $\theta_1 < 0$.

   One study the sign of $\left(\theta_1^2 - 3c\theta_1 + 9b\right), \left(\theta_1 - 3c\right), \theta_1\left(s\theta_1 - 6\right).$

   The sign of $\theta_1^2 - 3c\theta_1 + 9b$ is positive because $\left(\theta_1^2 - 3c\theta_1 + 9b\right)\theta_1 = 27a$ is negative, and $\theta_1 - 3c$ is negative if $a - bc$ is negative because $\left(\theta_1 - 3c\right)\left(\theta_1^2 + 9b\right) = 27\left(a - bc\right),$ and $\theta_1\left(s\theta_1 - 6\right)$ is positive.

   Hence $a \in \left]-\infty, a_1\right[\cap \left]-\infty, bc\right[= \left]-\infty, bc\right[$ ($a_1 > bc$ using Lemma 4.2.10).

   And $\theta_1 - 3c$ is positive or null if $a - bc$ is positive or null because $\left(\theta_1 - 3c\right)\left(\theta_1^2 + 9b\right) = 27\left(a - bc\right),$ and $\theta_1\left(s\theta_1 - 6\right)$ is positive.

   Hence $a \in \left]-\infty, a_1\right[\cap \left[bc, 0\right[= \left[bc, a_1\right[$ ($a_1 > bc$ using Lemma 4.2.10).

   Using Lemma 4.2.15, hence applying the stability conditions using Definition 4.1.20. One can obtain the results.

2. *If $a \in \left[a_1, 0\right[$:*

   the condition $4b < c^2$ is verified, then $a_1 < 0$ (1 in Lemma 4.2.2), and $3b < c^2$.

   $\Delta_1 \leq 0$ because $\Delta_1^* \leq 0$ (2 in Remark 4.2.1). One has $P_1(1) < 0$ (a negative) and $P_1(-1) < 0$ (a, c negative, b positive).

   $\lambda_2^* > 1$ because $\left(-c + \sqrt{c^2 - 3b}\right) > 0.$

   Hence, applying the stability conditions using Theorem 4 (2.i.b) of [23], one obtains the result.

3. *If $a = 0$:*

   the condition $4b < c^2$ is verified, and $3b < c^2$.

   $\Delta_1 \leq 0$ because $\Delta_1^* \leq 0$ (2 in Remark 4.2.1). One has $P_1(1) = 0$ (a null).

   Hence, applying the stability conditions using Theorem 4 (5.i) of [23], one obtains the result.

4. *If $a \in]0, a_2]$:*

   the condition $4b < c^2$ is verified, hence $a_2 > 0$ (1 in Lemma 4.2.2), and $3b < c^2$.

   $\Delta_1 \leq 0$ because $\Delta_1^* \leq 0$ (2 in Remark 4.2.1). One has $P_1(1) > 0$ (a positive) and $P_1(-1) < 0$ if $s \in \left]0, 2\frac{b}{a}\right[$ because

   $as^3 - 2bs^2 + 4cs - 8 < s^2(as - 2b)$ *(c negative),*

   then $P_1(-1) < 0$ if $(as - 2b) < 0$.

   $\lambda_2^* > 1$ *because* $\left(-c + \sqrt{c^2 - 3b}\right) > 0$.

   Hence, applying the stability conditions using Theorem 4 (3.i.a) of [23], one obtains the result.

Theorem 4.2.32 *If the fixed point E_0 exists with the following assumptions $b \in \mathbb{R}^+$, $c \in \left]-2\sqrt{b}, -\sqrt{3b}\right[$ and $d \in \mathbb{R}$.*

1. *When $a \in]-\infty, bc[$ then $\Delta_1 > 0$ and:*

   (a) E_0 is a unstable if $s > \nu_1$.

   (b) E_0 is a two-dimensional saddle if $0 < s < \nu_1$.

   (c) E_0 is a non-hyperbolic if $s = \nu_1$.

2. *When $a \in [bc, a_1[$ then $\Delta_1 > 0$ and E_0 is a unstable.*

3. *When $a \in [a_1, a_2]$ then $\Delta_1 \leq 0$ and E_0 is a source.*

Proof 4.2.33 *For $b \in \mathbb{R}^+$, $c \in \left]-2\sqrt{b}, -\sqrt{3b}\right[$ and $d \in \mathbb{R}$:*

1. *If $a \in]-\infty, a_1[$:*

   the condition $3b < c^2 < 4b$ is verified, then $a_1 < 0$ (3 in Lemma 4.2.2).

   $\Delta_1 > 0$ because $\Delta_1^* > 0$ (1 in Remark 4.2.1). And according to Lemma 4.2.4, Lemma 4.2.8 and Remark 4.2.17, one has $\theta_1 < 0$.

   One study the sign of $\left(\theta_1^2 - 3c\theta_1 + 9b\right)$, $(\theta_1 - 3c)$, $\theta_1(s\theta_1 - 6)$.

   The sign of $\theta_1^2 - 3c\theta_1 + 9b$ is positive because $\left(\theta_1^2 - 3c\theta_1 + 9b\right)\theta_1 = 27a$ is negative, and $\theta_1 - 3c$ is negative if $a - bc$ is negative because $(\theta_1 - 3c)\left(\theta_1^2 + 9b\right) = 27(a - bc)$, and $\theta_1(s\theta_1 - 6)$ is positive.

   Then $a \in]-\infty, a_1[\cap]-\infty, bc[=]-\infty, bc[$ ($a_1 > bc$ using Lemma 4.2.10), *and* $\theta_1 - 3c$ *is positive or null if* $a - bc$ *is positive or null because* $(\theta_1 - 3c)\left(\theta_1^2 + 9b\right) = 27\,(a - bc)$, *and* $\theta_1\,(s\theta_1 - 6)$ *is positive.*

Hence $a \in \left]-\infty, a_1\right[\cap \left[bc, 0\right[= \left[bc, a_1\right[$ *(*$a_1 > bc$ *using Lemma 4.2.10).*

Using Lemma 4.2.15, hence applying the stability conditions using Definition 4.1.20. One can obtain the results.

2. *If* $a \in [a_1, a_2]$:

   the condition $3b < c^2 < 4b$ *is verified, then* $a_1, a_2 < 0$ *(3 in Lemma 4.2.2).*

   $\Delta_1 \leq 0$ *because* $\Delta_1^* \leq 0$ *(2 in Remark 4.2.1). One has* $P_1(1) < 0$ *(a negative) and* $P_1(-1) < 0$ *(a, c negative, b positive).*

   $\lambda_2^* > 1$ *because* $\left(-c + \sqrt{c^2 - 3b}\right) > 0$.

   Hence, applying the stability conditions using Theorem 4 (2.i.b) of [23], one obtains the result.

Theorem 4.2.34 *If the fixed point* E_0 *exists with the following assumptions* $b \in \mathbb{R}^+$, $c \in \left]\sqrt{3b}, 2\sqrt{b}\right[$ *and* $d \in \mathbb{R}$.

1. *When* $a \in [a_1, a_2]$ *then* $\Delta_1 \leq 0$ *and* E_0 *is a asymptotically stable if* $0 < s < \dfrac{2}{c}$.

2. *When* $a \in \left]a_2, bc\right[$ *then* $\Delta_1 > 0$ *and:*

   (a) E_0 *is a asymptotically stable if* $0 < s < \min(\nu_1, \nu_3)$.
   (b) E_0 *is a unstable if* $s > \max(\nu_1, \nu_3)$.
   (c) E_0 *is a one-dimensional saddle if* $\nu_1 < s < \nu_3$.
   (d) E_0 *is a two-dimensional saddle if* $\nu_3 < s < \nu_1$.
   (e) E_0 *is a non-hyperbolic if* $s = \dfrac{6}{(\theta_1)_1}$ *or* $s = \dfrac{6}{(\theta_1)_2}$.

3. *When* $a \in \left[bc, +\infty\right[$ *then* $\Delta_1 > 0$ *and:*

   (a) E_0 *is a unstable if* $s > \nu_3$.
   (b) E_0 *is a one-dimensional saddle if* $0 < s < \nu_3$.
   (c) E_0 *is a non-hyperbolic if* $s = \nu_3$.

Proof 4.2.35 *For* $b \in \mathbb{R}^+$, $c \in \left]\sqrt{3b}, 2\sqrt{b}\right[$ *and* $d \in \mathbb{R}$:

1. *When* $a \in [a_1, a_2]$:

   the condition $3b < c^2 < 4b$ *is verified, hence* $a_1, a_2 > 0$ *(2 in Lemma 4.2.2).*

   $\Delta_1 \leq 0$ *because* $\Delta_1^* \leq 0$ *(2 in Remark 4.2.1). One has* $P_1(1) > 0$ *(a positive) and* $P_1(-1) < 0$ *if* $s \in \left]0, \dfrac{2}{c}\right[$ *because*

$as^3 - 2bs^2 + 4cs - 8 < bcs^3 - 2bs^2 + 4cs - 8$*($a_2 - bc$ negative (Lemma* 4.2.12*)),*
$as^3 - 2bs^2 + 4cs - 8 < (cs - 2)\left(bs^2 + 4\right),$

so $P_1(-1) < 0$ *if* $(cs - 2) < 0$ *because* $\left(bs^2 + 4\right) > 0$.

And
$\left(-c - \sqrt{c^2 - 3b}\right)\left(-c + \sqrt{c^2 - 3b}\right) = 3b > 0,$
then $\lambda_1^*, \lambda_2^* < 1$ *because* $\left(-c + \sqrt{c^2 - 3b}\right) < 0$.

Hence, applying the stability conditions using Lemma 4.2.18*, Lemma* 4.2.20 *and Theorem 4 (1.i) of [23], one obtains the result.*

2. *If* $a \in]a_2, +\infty[$:

   the condition $3b < c^2 < 4b$ *is verified, then* $a_2 > 0$ *(2 in Lemma* 4.2.2*).*

   $\Delta_1 > 0$ *because* $\Delta_1^* > 0$ *(1 in Remark* 4.2.1*). And according to Lemma* 4.2.6*, Lemma* 4.2.8 *and Remark* 4.2.17*, one has* $\theta_1 > 0$.

   One study the sign of $\left(\theta_1^2 - 3c\theta_1 + 9b\right), (\theta_1 - 3c), \theta_1(s\theta_1 - 6)$.

   The sign of $\theta_1^2 - 3c\theta_1 + 9b$ *is positive because* $\left(\theta_1^2 - 3c\theta_1 + 9b\right)\theta_1 = 27a$ *is positive, and* $\theta_1 - 3c$ *is negative if* $a - bc$ *is negative because* $(\theta_1 - 3c)\left(\theta_1^2 + 9b\right) = 27(a - bc)$.

   Hence $a \in]a_2, +\infty[\cap]0, bc[=]a_2, bc[$ *(*$a_2 < bc$ *using Lemma* 4.2.12*),*

   and $\theta_1 - 3c$ *is positive or null if* $a - bc$ *is positive or null because* $(\theta_1 - 3c)\left(\theta_1^2 + 9b\right) = 27(a - bc)$.

   Hence $a \in]a_2, +\infty[\cap [bc, +\infty[= [bc, +\infty[$ *(*$a_2 < bc$ *using Lemma* 4.2.12*).*

   Using Lemma 4.2.15*, hence applying the stability conditions using Definition* 4.1.20*. One can obtain the results.*

Theorem 4.2.36 *If the fixed point* E_0 *exists with the following assumptions* $b \in \mathbb{R}^+$, $c > 2\sqrt{b}$ *and* $d \in \mathbb{R}$.

1. *When* $a \in [a_1, 0[$ *then* $\Delta_1 \leq 0$ *and* E_0 *is a two-dimensional saddle if* $0 < s < \dfrac{2}{c}$.
2. *When* $a = 0$ *then* $\Delta_1 \leq 0$ *and* E_0 *is a non-hyperbolic.*
3. *When* $a \in]0, a_2]$ *then* $\Delta_1 \leq 0$ *and* E_0 *is a asymptotically stable if* $0 < s < \dfrac{2}{c}$.
4. *When* $a \in]a_2, bc[$ *then* $\Delta_1 > 0$ *and:*

   (a) E_0 *is a asymptotically stable if* $0 < s < \min(\nu_1, \nu_3)$.

   (b) E_0 *is a unstable if* $s > \max(\nu_1, \nu_3)$.

   (c) E_0 *is a one-dimensional saddle if* $\nu_1 < s < \nu_3$.

   (d) E_0 *is a two-dimensional saddle if* $\nu_3 < s < \nu_1$.

   (e) E_0 *is a non-hyperbolic if* $s = \dfrac{6}{(\theta_1)_1}$ *or* $s = \dfrac{6}{(\theta_1)_2}$.

5. *When* $a \in [bc, +\infty[$ *then* $\Delta_1 > 0$ *and:*

   (a) E_0 *is a unstable if* $s > \nu_3$.

   (b) E_0 *is a one-dimensional saddle if* $0 < s < \nu_3$.

   (c) E_0 *is a non-hyperbolic if* $s = \nu_3$.

Proof 4.2.37 *For* $b \in \mathbb{R}^+$, $c > 2\sqrt{b}$ *and* $d \in \mathbb{R}$:

1. *If* $a \in [a_1, 0[$:

   the condition $4b < c^2$ *is verified, then* $a_1 < 0$ *(1 in Lemma* 4.2.2*), and* $3b < c^2$.

   $\Delta_1 \leq 0$ *because* $\Delta_1^* \leq 0$ *(2 in Remark* 4.2.1*). One has* $P_1(1) < 0$ *(a negative) and* $P_1(-1) < 0$ *if* $s \in \left]0, \dfrac{2}{c}\right[$ *because*

   $$as^3 - 2bs^2 + 4cs - 8 < bcs^3 - 2bs^2 + 4cs - 8 \text{ (a negative, bc positive)},$$
   $$as^3 - 2bs^2 + 4cs - 8 < (cs - 2)\left(bs^2 + 4\right),$$

   then $P_1(-1) < 0$ *if* $(cs - 2) < 0$ *because* $\left(bs^2 + 4\right) > 0$.

   And

   $$\lambda_1^* < 1 \text{ because } \left(-c - \sqrt{c^2 - 3b}\right) < 0.$$

   Hence, applying the stability conditions using Lemma 4.2.18*, Lemma* 4.2.20 *and Theorem* 4 *(4.i.b) of [23], one can obtains the result.*

2. *If* $a = 0$:

   the condition $4b < c^2$ *is verified, and* $3b < c^2$.

   $\Delta_1 \leq 0$ *because* $\Delta_1^* \leq 0$ *(2 in Remark* 4.2.1*). One has* $P_1(1) = 0$ *(a null).*

   Hence, applying the stability conditions using Theorem 4 *(5.i) of [23], one obtains the result.*

3. *If* $a \in]0, a_2]$:

   the condition $4b < c^2$ *is verified, then* $a_2 > 0$ *(1 in Lemma* 4.2.2*), and* $3b < c^2$.

   $\Delta_1 \leq 0$ *because* $\Delta_1^* \leq 0$ *(2 in Remark* 4.2.1*). One has* $P_1(1) > 0$ *(a positive) and* $P_1(-1) < 0$ *if* $s \in \left]0, \dfrac{2}{c}\right[$ *because*

   $$as^3 - 2bs^2 + 4cs - 8 < bcs^3 - 2bs^2 + 4cs - 8 \text{ } (a_2 - bc \text{ negative (Lemma 4.2.12)}),$$
   $$as^3 - 2bs^2 + 4cs - 8 < (cs - 2)\left(bs^2 + 4\right),$$

   hence $P_1(-1) < 0$ *if* $(cs - 2) < 0$ *because* $\left(bs^2 + 4\right) > 0$.

   And

   $$\left(-c - \sqrt{c^2 - 3b}\right)\left(-c + \sqrt{c^2 - 3b}\right) = 3b > 0,$$
   $$\text{so } \lambda_1^*, \lambda_2^* < 1 \text{ because } \left(-c + \sqrt{c^2 - 3b}\right) < 0.$$

   Hence, applying the stability conditions using Lemma 4.2.18*, Lemma* 4.2.20 *and Theorem* 4 *(1.i) of [23], one obtains the result.*

4. If $a \in]a_2, +\infty[$:

The condition $4b < c^2$ is verified, then $a_2 > 0$ (1 in Lemma 4.2.2), and $3b < c^2$.

$\Delta_1 > 0$ because $\Delta_1^* > 0$ (1 in Remark 4.2.1). And according to Lemma 4.2.6, Lemma 4.2.8 and Remark 4.2.17, one has $\theta_1 > 0$.

One study the sign of $\left(\theta_1^2 - 3c\theta_1 + 9b\right)$, $(\theta_1 - 3c)$, $\theta_1 (s\theta_1 - 6)$.

The sign of $\theta_1^2 - 3c\theta_1 + 9b$ is positive because $\left(\theta_1^2 - 3c\theta_1 + 9b\right)\theta_1 = 27a$ is positive, and $\theta_1 - 3c$ is negative if $a - bc$ is negative because $(\theta_1 - 3c)\left(\theta_1^2 + 9b\right) = 27(a - bc)$.

Hence $a \in]a_2, +\infty[\cap]0, bc[=]a_2, bc[$ ($a_2 < bc$ using Lemma 4.2.12),

and $\theta_1 - 3c$ is positive or null if $a - bc$ is positive or null because $(\theta_1 - 3c)\left(\theta_1^2 + 9b\right) = 27(a - bc)$.

Hence $a \in]a_2, +\infty[\cap [bc, +\infty[= [bc, +\infty[$ ($a_2 < bc$ using Lemma 4.2.12).

Using Lemma 4.2.15, hence applying the stability conditions using Definition 4.1.20. One can obtain the results.

Stability of Fixed Points E_1 and E_2

The Jacobian matrix associated with the fixed points E_1 and E_2 of the system (4.18) is given by

$$J_{E_1} = J_{E_2} = \begin{pmatrix} 1 & s & 0 \\ 0 & 1 & s \\ 2as & -bs & 1-cs \end{pmatrix}, \tag{4.22}$$

then, the characteristic polynomial of J_{E_1} and J_{E_2} is

$$P_2(\lambda) = \lambda^3 + \mu_1\lambda^2 + \mu_2\lambda + \mu_3, \tag{4.23}$$

where $\mu_1 = cs - 3$, $\mu_2 = bs^2 - 2cs + 3$ and $\mu_3 = -2as^3 - bs^2 + cs - 1$. By calculating, one further has

$A_2 = \mu_1^2 - 3\mu_2 = -s^2\left(3b - c^2\right)$,
$B_2 = \mu_1\mu_2 - 9\mu_3 = s^2\left(-2c^2 + bsc + 6b + 18as\right)$,
$C_2 = \mu_2^2 - 3\mu_1\mu_3 = s^2\left(b^2s^2 - bcs - 3b + c^2 + 6acs^2 - 18as\right)$,
$\Delta_2 = B_2^2 - 4A_2C_2 = 3s^6\left(108a^2 + 36abc - 8ac^3 + \left(4b - c^2\right)b^2\right) = s^6\Delta_2^*$.

The derivative of $P_2(\lambda)$ is $P_2'(\lambda) = 3\lambda^2 + 2\mu_1\lambda + \mu_2$, and the equation $P_2'(\lambda) = 0$ has two roots:

$$\lambda_{1,2}^* = \frac{1}{3}\left(-\mu_1 \pm \sqrt{\mu_1^2 - 3\mu_2}\right) = \frac{1}{3}s\left(-c \pm \sqrt{c^2 - 3b}\right) + 1.$$

When $\Delta_2^* \leq 0$, i.e. $\Delta_2 \leq 0$, by Lemma 1 page 6 in [31], equation (4.21) has three real roots λ_1, λ_2 and λ_3. From this, one can easily prove that both roots $\lambda_{1,2}^*$ (let $\lambda_1^* \leq \lambda_2^*$) of equation $P_2'(\lambda) = 0$ are also real. When $\Delta_2^* > 0$, i.e. $\Delta_2 > 0$, by Lemma 1 page 6 in [31], one has that equation (4.21) has one real root $\lambda_1 = -\dfrac{\mu_1 + z_1^{\frac{1}{3}} + z_2^{\frac{1}{3}}}{3}$, and a pair of conjugate complex roots $\lambda_{2,3}$:

$$\lambda_{2,3} = \frac{1}{6}\left(\left(z_1^{\frac{1}{3}} + z_2^{\frac{1}{3}} - 2\mu_1\right) \pm i\sqrt{3}\left(z_1^{\frac{1}{3}} - z_2^{\frac{1}{3}}\right)\right),$$

where

$$z_{1,2} = \frac{s^3}{2}\left(\left(-54a + 2c^3 - 9bc\right) \pm 3\sqrt{\Delta_2}\right), z_1 > z_2,$$

and

$$a_3 = \frac{1}{27}\left(\left(c^2 - \frac{9}{2}b\right)c - \sqrt{-\left(3b - c^2\right)^3}\right), a_4 = \frac{1}{27}\left(\left(c^2 - \frac{9}{2}b\right)c + \sqrt{-\left(3b - c^2\right)^3}\right).$$

Let $\rho_1 = \dfrac{z_1^{\frac{1}{3}} + z_2^{\frac{1}{3}}}{s} + c$, then $\rho_1^3 = 3c\rho_1^2 - 9b\rho_1 - 54a$ and
$(\rho_1)_1 = \frac{1}{2}\left(3c - \sqrt{3}\sqrt{3c^2 - 8b}\right)$, $(\rho_1)_2 = \frac{1}{2}\left(3c + \sqrt{3}\sqrt{3c^2 - 8b}\right)$,
and $P_2(1) = -2as^3, P_2(-1) = -2as^3 - 2bs^2 + 4cs - 8$.

Remark 4.2.38 *Δ_2^* is a polynomial of degree 2 of the variable a.*

1. *$\Delta_2^* > 0$ if $3b < c^2$ and $a \in]-\infty, a_3[\cup]a_4, +\infty[$ or if $3b > c^2$ or if $3b = c^2$ and $a \in \mathbb{R} - \{\frac{2}{27}c^3 - \frac{1}{3}bc\}$.*

2. *$\Delta_2^* \leq 0$ if $3b < c^2$ and $a \in [a_3, a_4]$ or if $3b = c^2$ and $a = \dfrac{2}{27}c^3 - \dfrac{1}{3}bc$.*

Lemma 4.2.39 1. *If $4b < c^2$ then $a_3 < 0$, $a_4 > 0$.*

2. *If $b, c > 0$ and $3b < c^2 < 4b$ then $a_3, a_4 < 0$.*

3. *If $b > 0$, $c < 0$ and $3b < c^2 < 4b$ then $a_3, a_4 > 0$.*

Proof 4.2.40 *As $a_3a_4 = \dfrac{1}{108}b^2\left(4b - c^2\right)$, then the sign of a_3, a_4 depends on the sign of $4b - c^2$.*

Lemma 4.2.41 *If $4b < c^2$ or $3b < c^2 < 4b$ and $a < a_3 < 0$, then $z_1 > 0$.*

Proof 4.2.42 *Either $4b < c^2$ or $3b < c^2 < 4b$ and $a \in]-\infty, a_3[$, one has $3b < c^2$, and Δ_2^* positive (1 in Remark 4.2.38),*
and $\left(c^3 - \frac{9}{2}bc - 27a\right) > 0$ because

$$\left(c^3 - \frac{9}{2}bc - 27a\right) > \left(c^3 - \frac{9}{2}bc - 27a_3\right) = \sqrt{-\left(3b - c^2\right)^3} > 0,$$

as

$$z_1 = \frac{s^3}{2}\left(\left(-54a + 2c^3 - 9bc\right) + 3\sqrt{\Delta_2}\right),$$

then $z_1 > 0$.

Lemma 4.2.43 *If $4b < c^2$ or $3b < c^2 < 4b$ and $a > a_4 > 0$, then $z_2 < 0$.*

Proof 4.2.44 *Either $4b < c^2$ or $3b < c^2 < 4b$ and $a \in]a_4, +\infty[$, one has $3b < c^2$, and Δ_2^* positive (1 in Remark 4.2.38),*

and $\left(c^3 - \frac{9}{2}bc - 27a\right) < 0$ because

$$\left(c^3 - \frac{9}{2}bc - 27a\right) < \left(c^3 - \frac{9}{2}bc - 27a_4\right) = -\sqrt{-\left(3b - c^2\right)^3} < 0,$$

as

$$z_2 = \frac{s^3}{2}\left(\left(-54a + 2c^3 - 9bc\right) - 3\sqrt{\Delta_2}\right),$$

then $z_2 < 0$.

Lemma 4.2.45 *If $3b < c^2$, then z_1 and z_2 have the same sign.*

.

Proof 4.2.46 *If $3b < c^2$, one has $z_1 z_2 = s^6\left(-3b + c^2\right)^3 > 0$, then z_1, z_2 have the same sign.*

Lemma 4.2.47 *If $b \in \mathbb{R}^+$, $\sqrt{3b} < c$ then $a_3 + \frac{bc}{2} > 0$.*

Proof 4.2.48 *The difference between a_3 and $-\frac{bc}{2}$, is:*

$$a_3 + \frac{bc}{2} = -\frac{1}{27}\left(\sqrt{-\left(3b - c^2\right)^3} - c^3\right) + \frac{1}{3}bc,$$

then $\left(\sqrt{-\left(3b - c^2\right)^3} - c^3\right) < 0$ because

$$\left(\sqrt{-\left(3b - c^2\right)^3} - c^3\right)\left(\sqrt{-\left(3b - c^2\right)^3} + c^3\right) = -27b^3 + 9bc^2\left(3b - c^2\right) < 0,$$

and $\left(\sqrt{-\left(3b - c^2\right)^3} + c^3\right) > 0$. Therefore $a_3 + \frac{bc}{2} > 0$.

Lemma 4.2.49 *If $b \in \mathbb{R}^+$, $c < -\sqrt{3b}$ then $a_4 + \frac{bc}{2} < 0$.*

Proof 4.2.50 *The difference between a_4 and $-\frac{bc}{2}$, is:*

$$a_4 + \frac{bc}{2} = \frac{1}{27}\left(\sqrt{-\left(3b - c^2\right)^3} + c^3\right) + \frac{1}{3}bc,$$

then $\left(\sqrt{-\left(3b - c^2\right)^3} + c^3\right) < 0$ because

$$\left(\sqrt{-\left(3b - c^2\right)^3} - c^3\right)\left(\sqrt{-\left(3b - c^2\right)^3} + c^3\right) = -27b^3 + 9bc^2\left(3b - c^2\right) < 0,$$

and $\left(\sqrt{-\left(3b - c^2\right)^3} - c^3\right) > 0$. Therefore $a_4 + \frac{bc}{2} < 0$.

Lemma 4.2.51 *If $3b < c^2$ and $a \in]-\infty, a_3[\cup]a_4, +\infty[$:*

1. *the sign of $|\lambda_{2,3}| - 1$ depends on the sign of $\left(\rho_1^2 - 3c\rho_1 + 9b\right) s + 3\left(\rho_1 - 3c\right)$.*

2. *the sign of $|\lambda_1| - 1$ depends on the sign of $\rho_1\left(s\rho_1 - 6\right)$.*

Proof 4.2.52 *$\Delta_2^* > 0$ if $3b < c^2$ and $a \in]-\infty, a_3[\cup]a_4, +\infty[$ (1 in Remark 4.2.38). Calculating $|\lambda_{2,3}|^2 - 1$ and $|\lambda_1|^2 - 1$ one finds:*

$$
\begin{aligned}
\left(|\lambda_{2,3}| - 1\right)\left(|\lambda_{2,3}| + 1\right) &= \frac{1}{9}\left(\left(\rho_1^2 - 3c\rho_1 + 9b\right) s^2 + 3\left(\rho_1 - 3c\right) s\right), \\
\left(|\lambda_1| - 1\right)\left(|\lambda_1| + 1\right) &= \frac{1}{9} s\rho_1\left(s\rho_1 - 6\right).
\end{aligned}
$$

The above results are true because $|\lambda_{2,3}| + 1, |\lambda_1| + 1$, s are positive.

Remark 4.2.53 *$\rho_1 > 0$ if $z_1, z_2, c > 0$ and $\rho_1 < 0$ if $z_1, z_2, c < 0$.*

Now, relatively to the dynamical properties of the fixed points E_1 and E_2, one has the following results.

For theorems 4.2.54 to 4.2.68, let: $\gamma_1 = -3\dfrac{\rho_1 - 3c}{\rho_1^2 - 3c\rho_1 + 9b}$, $\gamma_2 = \dfrac{2}{\sqrt{-b}}$ and $\gamma_3 = \dfrac{6}{\rho_1}$.

Theorem 4.2.54 *If the fixed points E_i, $i = 1, 2$ exist with the following assumptions $b \in \mathbb{R}_*^-$, $c \in \mathbb{R}^-$ and $d \in \mathbb{R}$.*

1. *When $a = 0$ then $\Delta_2 \leq 0$ and E_i are non-hyperbolic.*

2. *When $a \in]0, a_4]$ then $\Delta_2 \leq 0$ and E_i are source if $0 < s < \gamma_2$.*

3. *When $a \in]a_4, +\infty[$ then $\Delta_2 > 0$ and:*

   (a) *E_i are unstable if $s > \gamma_1$.*
   (b) *E_i are a one-dimensional saddle if $0 < s < \gamma_1$.*
   (c) *E_i are non-hyperbolic if $s = \gamma_1$.*

Proof 4.2.55 *For $b \in \mathbb{R}_*^-$, $c \in \mathbb{R}^-$ and $d \in \mathbb{R}$:*

1. *If $a = 0$:*

   the condition $4b < c^2$ is verified, and $3b < c^2$.

   $\Delta_2 \leq 0$ because $\Delta_2^* \leq 0$ (2 in Remark 4.2.38). One has $P_2(1) = 0$ (a null).

   Hence, applying the stability conditions using Theorem 4 (5.i) of [23], one obtains the result.

2. *If* $a \in]0, a_4]$:

   the condition $4b < c^2$ *is verified, then* $a_4 > 0$ *(1 in Lemma* 4.2.39*), and* $3b < c^2$.

   $\Delta_2 \leq 0$ *because* $\Delta_2^* \leq 0$ *(2 in Remark* 4.2.38*). One has* $P_2(1) < 0$ *(a positive) and* $P_1(-1) < 0$ *if* $s \in]0, \gamma_2[$ *because*

   $$-2as^3 - 2bs^2 + 4cs - 8 < -2(-\frac{bc}{2})s^3 - 2bs^2 + 4cs - 8 \left(a, \frac{bc}{2} \text{ positive}\right),$$
   $$-2as^3 - 2bs^2 + 4cs - 8 < (cs - 2)\left(bs^2 + 4\right),$$

   then $P_2(-1) < 0$ *if* $\left(bs^2 + 4\right) > 0$ *because* $(cs - 2) < 0$.

   $\lambda_2^* > 1$ *because* $\left(-c + \sqrt{c^2 - 3b}\right) > 0.$

   Hence, applying the stability conditions using Theorem 4 *(2.i.b) of [23], one obtains the result.*

3. *If* $a \in]a_4, +\infty[$:

   the condition $4b < c^2$ *is verified, then* $a_4 > 0$ *(1 in Lemma* 4.2.39*), and* $3b < c^2$.

   $\Delta_2 > 0$ *because* $\Delta_2^* > 0$ *(1 in Remark* 4.2.38*). And according to Lemma* 4.2.43*, Lemma* 4.2.45 *and Remark* 4.2.53*, one has* $\rho_1 < 0$.

   One study the sign of $\left(\rho_1^2 - 3c\rho_1 + 9b\right), (\rho_1 - 3c), \rho_1(s\rho_1 - 6)$.

   The sign of $\rho_1^2 - 3c\rho_1 + 9b$ *is positive because* $\left(\rho_1^2 - 3c\rho_1 + 9b\right)\rho_1 = -54a$ *is negative, and* $\rho_1 - 3c$ *is negative because* $(\rho_1 - 3c)\rho_1^2 = -9b\rho_1 - 54a$ *is negative. And* $\rho_1(s\rho_1 - 6)$ *positive.*

   Using Lemma 4.2.51*, hence applying the stability conditions using Definition* 4.1.20*. One can obtain the results.*

Theorem 4.2.56 *If the fixed points* E_i, $i = 1, 2$ *exists with the following assumptions* $b \in \mathbb{R}_*^-$, $c \leq -3\sqrt{-b}$ *and* $a \in \left[a_3, -\frac{bc}{2}\right[$, $d \in \mathbb{R}$ *then* E_i *are source if* $s > \gamma_2$.

Proof 4.2.57 *For* $b \in \mathbb{R}_*^-$, $c \leq 3\sqrt{-b}$*, and* $a \in \left[a_3, -\frac{bc}{2}\right[$*, the condition* $4b < c^2$ *is verified, then* $a_3 < 0$ *(1 in Lemma* 4.2.39*), and* $3b < c^2$.

$\frac{1}{3}bc + \frac{1}{27}c^3 \leq 0$ *if* $c \leq -3\sqrt{-b}$ *hence* $a_3 + \frac{bc}{2} = -\frac{1}{27}\sqrt{-\left(3b - c^2\right)^3} + \frac{1}{3}bc + \frac{1}{27}c^3 < 0$.

$\Delta_2 \leq 0$ *because* $\Delta_2^* \leq 0$ *(2 in Remark* 4.2.38*). One has* $P_2(1) > 0$ *(a negative) and* $P_2(-1) > 0$ *if* $s \in]\gamma_2, +\infty[$ *because*

$$-2as^3 - 2bs^2 + 4cs - 8 > -2(-\frac{bc}{2})s^3 - 2bs^2 + 4cs - 8 \ (a \in \left[a_3, -\frac{bc}{2}\right[),$$
$$-2as^3 - 2bs^2 + 4cs - 8 > (cs - 2)\left(bs^2 + 4\right),$$

hence $P_2(-1) > 0$ *if* $\left(bs^2 + 4\right) < 0$ *because* $(cs - 2) < 0$.

$\lambda_2^* > 1$ *because* $\left(-c + \sqrt{c^2 - 3b}\right) < 0.$

Hence, applying the stability conditions using Theorem 4 *(2.i.a) of [23], one obtains the result.*

Theorem 4.2.58 *If the fixed points* E_i, $i = 1,2$ *exists with the following assumptions* $b \in \mathbb{R}_*^-$, $c \geq 3\sqrt{-b}$ *and* $a \in \left]-\frac{bc}{2}, a_4\right]$, $d \in \mathbb{R}$ *then* E_i *are source if* $0 < s < \frac{2}{c}$ *or* $\gamma_2 < s$.

Proof 4.2.59 *For* $b \in \mathbb{R}_*^-$, $c \geq 3\sqrt{-b}$, *and* $a \in \left]-\frac{bc}{2}, a_4\right]$ *the condition* $4b < c^2$ *is verified, then* $a_4 > 0$ *(1 in Lemma 4.2.39), and* $3b < c^2$.

$\frac{2}{3}bc + \frac{2}{27}c^3 \geq 0$ *if* $c \geq 3\sqrt{-b}$ *hence* $a_4 + \frac{bc}{2} = \frac{2}{27}\sqrt{-\left(3b - c^2\right)^3} + \frac{2}{3}bc + \frac{2}{27}c^3 > 0$.

$\Delta_2 \leq 0$ *because* $\Delta_2^* \leq 0$ *(2 in Remark 4.2.38). One has* $P_2(1) < 0$ *(a positive) and* $P_2(-1) < 0$ *if* $s \in \left]0, \frac{2}{c}\right[$ *or* $s \in \left]\gamma_2, +\infty\right[$ *because*

$-2as^3 - 2bs^2 + 4cs - 8 < -2(-\frac{bc}{2})s^3 - 2bs^2 + 4cs - 8$ *(*$a \in \left]-\frac{bc}{2}, a_4\right]$*),*

$-2as^3 - 2bs^2 + 4cs - 8 < (cs - 2)\left(bs^2 + 4\right)$,

hence $P_2(-1) < 0$ *if* $\left(bs^2 + 4\right) < 0$, $(cs - 2) > 0$ *or* $\left(bs^2 + 4\right) > 0$, $(cs - 2) < 0$, *and* $\frac{2}{c} < \gamma_2$ $\left(\textit{using } c \geq 3\sqrt{-b}\right)$.

One has $\left(-c + \sqrt{c^2 - 3b}\right)\left(-c - \sqrt{c^2 - 3b}\right) = 3b < 0$.

$\lambda_2^* > 1$ *because* $\left(-c + \sqrt{c^2 - 3b}\right) > 0$.

Hence, applying the stability conditions using Theorem 4 (2.i.b) of [23], one obtains the result.

Theorem 4.2.60 *If the fixed points* E_i, $i = 1,2$ *exist with the following assumptions* $b \in \mathbb{R}^-$, $c \in \mathbb{R}_*^+$ *and* $d \in \mathbb{R}$.

1. *When* $a \in \left]-\infty, a_3\right[$ *then* $\Delta_2 > 0$ *and:*
   (a) E_i *are unstable if* $s > \gamma_3$.
   (b) E_i *are a one-dimensional saddle if* $0 < s < \gamma_3$.
   (c) E_i *are non-hyperbolic if* $s = \gamma_3$.
2. *When* $a \in \left[a_3, 0\right[$ *then* $\Delta_2 \leq 0$ *and* E_i *are source if* $s > \frac{2}{c}$.
3. *When* $a = 0$ *then* $\Delta_2 \leq 0$ *and* E_i *are non-hyperbolic.*

Proof 4.2.61 *For* $b \in \mathbb{R}^-$, $c \in \mathbb{R}_*^+$ *and* $d \in \mathbb{R}$:

1. *If* $a \in \left]-\infty, a_3\right[$:

   the condition $4b < c^2$ *is verified, then* $a_3 < 0$ *(1 in Lemma 4.2.39), and* $3b < c^2$.

   $\Delta_2 > 0$ *because* $\Delta_2^* > 0$ *(1 in Remark 4.2.38). And according to Lemma 4.2.41, Lemma 4.2.45 and Remark 4.2.53, one has* $\rho_1 > 0$.

   One study the sign of $\left(\rho_1^2 - 3c\rho_1 + 9b\right)$, $(\rho_1 - 3c)$, $\rho_1(s\rho_1 - 6)$.

   The sign of $\rho_1^2 - 3c\rho_1 + 9b$ *is positive because* $\left(\rho_1^2 - 3c\rho_1 + 9b\right)\rho_1 = -54a$ *is positive, and* $\rho_1 - 3c$ *is positive because* $(\rho_1 - 3c)\rho_1^2 = -54a - 9b\rho_1$ *is positive.*

   Using Lemma 4.2.51, hence applying the stability conditions using Definition 4.1.20. One can obtain the results.

2. $a \in [a_3, 0[$:

   the condition $4b < c^2$ *is verified, then* $a_3 < 0$ *(*1 *in Lemma* 4.2.39*), and* $3b < c^2$.

   $\Delta_2 \leq 0$ *because* $\Delta_2^* \leq 0$ *(*2 *in Remark* 4.2.38*). One has* $P_2(1) > 0$ *(a negative) and* $P_2(-1) > 0$ *if* $s \in \left]\frac{2}{c}, +\infty\right[$ *because*

   $-2as^3 - 2bs^2 + 4cs - 8 > 4cs - 8$ *(a, b negative)*

   then $P_2(-1) > 0$ *if* $(4cs - 8) > 0$.

   One has $\left(-c + \sqrt{c^2 - 3b}\right)\left(-c - \sqrt{c^2 - 3b}\right) = 3b < 0, so$

   $\lambda_2^* > 1$ *because* $\left(-c + \sqrt{c^2 - 3b}\right) > 0$.

   Hence, applying the stability conditions using Theorem 4 *(2.i.a) of [23], one obtains the result.*

3. $a = 0$:

   the condition $4b < c^2$ *is verified, and* $3b < c^2$.

   $\Delta_2 \leq 0$ *because* $\Delta_2^* \leq 0$ *(*2 *in Remark* 4.2.38*). One has* $P_2(1) = 0$ *(a null).*

   Hence, applying the stability conditions using Theorem 4 *(5.i) of [23], one obtains the result.*

Theorem 4.2.62 *If the fixed points* E_i, $i = 1, 2$ *exist with the following assumptions* $b \in \mathbb{R}^+$, $c < -2\sqrt{b}$ *and* $d \in \mathbb{R}$.

1. *When* $a \in [a_3, 0[$ *then* $\Delta_2 \leq 0$ *and* E_i *are a one-dimensional saddle if* $0 < s < -\frac{b}{a}$.
2. *When* $a = 0$ *then* $\Delta_2 \leq 0$ *and* E_i *are non-hyperbolic.*
3. *When* $a \in]0, a_4]$ *then* $\Delta_2 \leq 0$ *and* E_i *are source.*
4. *When* $a \in \left]a_4, -\frac{bc}{2}\right]$ *then* $\Delta_2 > 0$ *and* E_i *are unstable.*
5. *When* $a \in \left]-\frac{bc}{2}, +\infty\right[$ *then* $\Delta_2 > 0$ *and:*
   (a) E_i *are unstable if* $s > \gamma_1$.
   (b) E_i *are a one-dimensional saddle if* $0 < s < \gamma_1$.
   (c) E_i *are non-hyperbolic if* $s = \gamma_1$.

Proof 4.2.63 *For* $b \in \mathbb{R}^+$, $c < -2\sqrt{b}$ *and* $d \in \mathbb{R}$:

1. $a \in [a_3, 0[$:

   the condition $4b < c^2$ *is verified, then* $a_3 < 0$ *(*1 *in Lemma* 4.2.39*), and* $3b < c^2$.

   $\Delta_2 \leq 0$ *because* $\Delta_2^* \leq 0$ *(*2 *in Remark* 4.2.38*). One has* $P_2(1) > 0$ *(a negative) and* $P_2(-1) < 0$ *if* $s \in \left]0, -\frac{b}{a}\right[$ *because*

$-2as^3 - 2bs^2 + 4cs - 8 < 2s^2(-as - b)$ *(c negative),*

hence $P_2(-1) < 0$ *if* $(-as - b) < 0$.

$\lambda_2^* > 1$ *because* $\left(-c + \sqrt{c^2 - 3b}\right) > 0$.

Hence, applying the stability conditions using Theorem 4 *(3.i.a) of [23], one obtains the result.*

2. *If* $a = 0$:

   the condition $4b < c^2$ *is verified, and* $3b < c^2$.

   $\Delta_2 \leq 0$ *because* $\Delta_2^* \leq 0$ *(2 in Remark* 4.2.38*). One has* $P_2(1) > 0$ *(a null).*

   Hence, applying the stability conditions using Theorem 4 *(5.i) of [23], one obtains the result.*

3. *If* $a \in \,]0, a_4]$:

   the condition $4b < c^2$ *is verified, then* $a_4 > 0$ *(1 in Lemma* 4.2.39*), and* $3b < c^2$.

   $\Delta_2 \leq 0$ *because* $\Delta_2^* \leq 0$ *(2 in Remark* 4.2.38*). One has* $P_2(1) < 0$ *(a positive) and* $P_2(-1) < 0$ *(c negative, b, a positive).*

   $\lambda_2^* > 1$ *because* $\left(-c + \sqrt{c^2 - 3b}\right) > 0$.

   Hence, applying the stability conditions using Theorem 4 *(2.i.b) of [23], one obtains the result.*

4. *If* $a \in \,]a_4, +\infty[$:

   the condition $4b < c^2$ *is verified, then* $a_4 > 0$ *(1 in Lemma* 4.2.39*), and* $3b < c^2$.

   $\Delta_2 > 0$ *because* $\Delta_2^* > 0$ *(1 in Remark* 4.2.38*). And according to Lemma* 4.2.43*, Lemma* 4.2.45 *and Remark* 4.2.53*, one has* $\rho_1 < 0$.

   One study the sign of $\left(\rho_1^2 - 3c\rho_1 + 9b\right)$, $(\rho_1 - 3c)$, $\rho_1(s\rho_1 - 6)$.

   The sign of $\rho_1^2 - 3c\rho_1 + 9b$ *is positive because* $\left(\rho_1^2 - 3c\rho_1 + 9b\right)\rho_1 = -54a$ *is negative, and* $\rho_1 - 3c$ *is positive or null if* $2a + bc$ *is negative or null because* $(\rho_1 - 3c)\left(\rho_1^2 + 9b\right) = -27(2a + bc)$.

   then $a \in \,]a_4, +\infty[\cap \left]0, -\dfrac{bc}{2}\right] = \left]a_4, -\dfrac{bc}{2}\right]$ *(*$a_4 < -\dfrac{bc}{2}$ *using Lemma* 4.2.49*), and* $\rho_1(s\rho_1 - 6)$ *positive.*

   And $\rho_1 - 3c$ *is negative if* $2a + bc$ *is positive because*
   $(\rho_1 - 3c)\left(\rho_1^2 + 9b\right) = -27(2a + bc)$. *And* $\rho_1(s\rho_1 - 6)$ *positive.*

   Then $a \in \,]a_4, +\infty[\cap \left]-\dfrac{bc}{2}, +\infty\right[= \left]-\dfrac{bc}{2}, +\infty\right[$ *(*$a_4 < -\dfrac{bc}{2}$ *using Lemma* 4.2.49*).*

   Using Lemma 4.2.51*, hence applying the stability conditions using Definition* 4.1.20*. One can obtain the results.*

Theorem 4.2.64 *If the fixed points E_i, $i = 1,2$ exist with the following assumptions $b \in \mathbb{R}^+$, $c \in \left]-2\sqrt{b}, -\sqrt{3b}\right[$ and $d \in \mathbb{R}$.*

1. *When $a \in [a_3, a_4]$ then $\Delta_2 \leq 0$ and E_i are source.*
2. *When $a \in \left]a_4, -\dfrac{bc}{2}\right]$ then $\Delta_2 > 0$ and E_i are unstable.*
3. *When $a \in \left]-\dfrac{bc}{2}, +\infty\right[$ then $\Delta_2 > 0$ and:*
   (a) *E_i are unstable if $s > \gamma_1$.*
   (b) *E_i are a one-dimensional saddle if $0 < s < \gamma_1$.*
   (c) *E_i are non-hyperbolic if $s = \gamma_1$.*

Proof 4.2.65 *For $b \in \mathbb{R}^+$, $c \in \left]-2\sqrt{b}, -\sqrt{3b}\right[$ and $d \in \mathbb{R}$:*

1. *If $a \in [a_3, a_4]$:*

   the condition $3b < c^2 < 4b$ is verified, then $a_3, a_4 > 0$ (3 in Lemma 4.2.39).

   $\Delta_2 \leq 0$ because $\Delta_2^* \leq 0$ (2 in Remark 4.2.38). One has $P_2(1) < 0$ (a positive) and $P_2(-1) < 0$ (c negative, b, a positive).

   $\lambda_2^* > 1$ because $\left(-c + \sqrt{c^2 - 3b}\right) > 0$.

   Hence, applying the stability conditions using Theorem 4 (2.i.b) of [23], one can obtains the result.

2. *If $a \in]a_4, +\infty[$:*

   the condition $3b < c^2 < 4b$ is verified, so $a_4 > 0$ (3 in Lemma 4.2.39).

   $\Delta_2 > 0$ because $\Delta_2^* > 0$ (1 in Remark 4.2.38). And according to Lemma 4.2.43, Lemma 4.2.45 and Remark 4.2.53, one has $\rho_1 < 0$.

   One study the sign of $\left(\rho_1^2 - 3c\rho_1 + 9b\right)$, $(\rho_1 - 3c)$, $\rho_1 (s\rho_1 - 6)$.

   The sign of $\rho_1^2 - 3c\rho_1 + 9b$ is positive because $\left(\rho_1^2 - 3c\rho_1 + 9b\right)\rho_1 = -54a$ is negative, and $\rho_1 - 3c$ is positive or null if $2a + bc$ is negative or null because $(\rho_1 - 3c)\left(\rho_1^2 + 9b\right) = -27(2a + bc)$.

   Hence $a \in]a_4, +\infty[\cap \left]0, -\dfrac{bc}{2}\right] = \left]a_4, -\dfrac{bc}{2}\right]$ ($a_4 < -\dfrac{bc}{2}$ using Lemma 4.2.49), and $\rho_1 (s\rho_1 - 6)$ positive,

   and $\rho_1 - 3c$ is negative if $2a + bc$ is positive because $(\rho_1 - 3c)\left(\rho_1^2 + 9b\right) = -27(2a + bc)$. And $\rho_1 (s\rho_1 - 6)$ positive.

   Hence $a \in]a_4, +\infty[\cap \left]-\dfrac{bc}{2}, +\infty\right[= \left]-\dfrac{bc}{2}, +\infty\right[$ ($a_4 < -\dfrac{bc}{2}$ using Lemma 4.2.49).

   Using Lemma 4.2.51, hence applying the stability conditions using Definition 4.1.20. One can obtain the results.

Theorem 4.2.66 *If the fixed points E_i, $i = 1,2$ exist with the following assumptions $b \in \mathbb{R}^+$, $c \in \left]\sqrt{3b}, 2\sqrt{b}\right[$ and $d \in \mathbb{R}$.*

1. *When $a \in \left]-\infty, -\frac{bc}{2}\right]$ then $\Delta_2 > 0$ and:*
   (a) *E_i are unstable if $s > \gamma_3$.*
   (b) *E_i are a one-dimensional saddle if $0 < s < \gamma_3$.*
   (c) *E_i are non-hyperbolic if $s = \gamma_3$.*
2. *When $a \in \left]-\frac{bc}{2}, a_3\right[$ then $\Delta_2 > 0$ and:*
   (a) *E_i are asymptotically stable if $0 < s < \min(\gamma_1, \gamma_3)$.*
   (b) *E_i are unstable if $s > \max(\gamma_1, \gamma_3)$.*
   (c) *E_i are a one-dimensional saddle if $\gamma_1 < s < \gamma_3$.*
   (d) *E_i are a two-dimensional saddle if $\gamma_3 < s < \gamma_1$.*
   (e) *E_i are non-hyperbolic if $s = \frac{6}{(\rho_1)_1}$ or $s = \frac{6}{(\rho_1)_2}$.*
3. *When $a \in [a_3, a_4]$ then $\Delta_2 \leq 0$ and E_i are asymptotically stable if $0 < s < \frac{2}{c}$.*

Proof 4.2.67 *For $b \in \mathbb{R}^+$, $c \in \left]\sqrt{3b}, 2\sqrt{b}\right[$ and $d \in \mathbb{R}$:*

1. *If $a \in \left]-\infty, a_3\right[$:*

   the condition $3b < c^2 < 4b$ is verified, then $a_3 < 0$ (2 in Lemma 4.2.39).

   $\Delta_2 > 0$ because $\Delta_2^* > 0$ (1 in Remark 4.2.38). And according to Lemma 4.2.43, Lemma 4.2.45 and Remark 4.2.53, one has $\rho_1 > 0$.

   One study the sign of $\left(\rho_1^2 - 3c\rho_1 + 9b\right)$, $(\rho_1 - 3c)$, $\rho_1 (s\rho_1 - 6)$.

   The sign of $\rho_1^2 - 3c\rho_1 + 9b$ is positive because $\left(\rho_1^2 - 3c\rho_1 + 9b\right)\rho_1 = -54a$ is positive, and $\rho_1 - 3c$ is positive or null if $2a + bc$ is negative or null because $(\rho_1 - 3c)\left(\rho_1^2 + 9b\right) = -27(2a + bc)$.

   Hence $a \in \left]-\infty, a_3\right[\cap \left]-\infty, -\frac{bc}{2}\right] = \left]-\infty, -\frac{bc}{2}\right]$ ($a_3 > -\frac{bc}{2}$ using Lemma 4.2.47),

   and $\rho_1 - 3c$ is negative if $a - bc$ is positive because $(\rho_1 - 3c)\left(\rho_1^2 + 9b\right) = -27(2a + bc)$.

   Hence $a \in \left]-\infty, a_3\right[\cap \left]-\frac{bc}{2}, 0\right[= \left]-\frac{bc}{2}, a_3\right[$ ($a_3 > -\frac{bc}{2}$ using Lemma 4.2.47).

   Using Lemma 4.2.51, hence applying the stability conditions using Definition 4.1.20. One can obtain the results.

2. *If $a \in [a_3, a_4]$:*
   the condition $3b < c^2 < 4b$ is verified, then $a_3, a_4 < 0$ (2 in Lemma 4.2.39).
   $\Delta_1 \leq 0$ because $\Delta_1^* \leq 0$ (2 in Remark 4.2.1). One has $P_1(1) > 0$ (a negative) and $P_1(-1) < 0$ if $s \in \left]0, \dfrac{2}{c}\right[$ because
   $-2as^3 - 2bs^2 + 4cs - 8 < -2(-\dfrac{bc}{2})s^3 - 2bs^2 + 4cs - 8$
   $(a_3 + \dfrac{bc}{2}$ *positive (Lemma 4.2.47)),*
   $-2as^3 - 2bs^2 + 4cs - 8 < (cs - 2)\left(bs^2 + 4\right),$
   hence $P_2(-1) < 0$ if $(cs-2) < 0$ because $\left(bs^2 + 4\right) > 0$.
   And
   $\left(-c - \sqrt{c^2 - 3b}\right)\left(-c + \sqrt{c^2 - 3b}\right) = 3b > 0,$
   then $\lambda_1^*, \lambda_2^* < 1$ because $\left(-c + \sqrt{c^2 - 3b}\right) < 0$.
   Hence, applying the stability conditions using Lemma 4.2.18, Lemma 4.2.20 and Theorem 4 (1.i) of [23], one obtains the result.

Theorem 4.2.68 *If the fixed points $E_i, i = 1, 2$ exist with the following assumptions $b \in \mathbb{R}^+$, $c > 2\sqrt{b}$ and $d \in \mathbb{R}$.*

1. *When $a \in \left]-\infty, -\dfrac{bc}{2}\right]$ then $\Delta_2 > 0$ and:*
   (a) *E_i are unstable if $s > \gamma_3$.*
   (b) *E_i are a one-dimensional saddle if $0 < s < \gamma_3$.*
   (c) *E_i are non-hyperbolic if $s = \gamma_3$.*
2. *When $a \in \left]-\dfrac{bc}{2}, a_3\right[$ then $\Delta_2 > 0$ and:*
   (a) *E_i are asymptotically stable if $0 < s < \min(\gamma_1, \gamma_3)$.*
   (b) *E_i are unstable if $s > \max(\gamma_1, \gamma_3)$.*
   (c) *E_i are a one-dimensional saddle if $\gamma_1 < s < \gamma_3$.*
   (d) *E_i are a two-dimensional saddle if $\gamma_3 < s < \gamma_1$.*
   (e) *E_i are non-hyperbolic if $s = \dfrac{6}{(\rho_1)_1}$ or $s = \dfrac{6}{(\rho_1)_2}$.*
3. *When $a \in [a_3, 0[$ then $\Delta_2 \leq 0$ and E_i are asymptotically stable if $0 < s < \dfrac{2}{c}$.*
4. *When $a = 0$ then $\Delta_2 \leq 0$ and E_i are non-hyperbolic.*
5. *When $a \in]0, a_4]$ then $\Delta_2 \leq 0$ and E_i are a two-dimensional saddle if $0 < s < \dfrac{2}{c}$.*

Proof 4.2.69 *For $b \in \mathbb{R}^+$, $c > 2\sqrt{b}$ and $d \in \mathbb{R}$:*

1. *If $a \in \left]-\infty, a_3\right[$:*

   the condition $4b < c^2$ is verified, then $a_3 < 0$ (1 in Lemma 4.2.39), and $3b < c^2$.

   $\Delta_2 > 0$ because $\Delta_2^* > 0$ (1 in Remark 4.2.38). And according to Lemma 4.2.41, Lemma 4.2.45 and Remark 4.2.53, one has $\rho_1 > 0$.

   One study the sign of $\left(\rho_1^2 - 3c\rho_1 + 9b\right)$, $(\rho_1 - 3c)$, $\rho_1 (s\rho_1 - 6)$.

   The sign of $\rho_1^2 - 3c\rho_1 + 9b$ is positive because $\left(\rho_1^2 - 3c\rho_1 + 9b\right)\rho_1 = -54a$ is positive,

   and $\rho_1 - 3c$ is positive or null if $2a + bc$ is negative or null because
   $(\rho_1 - 3c)\left(\rho_1^2 + 9b\right) = -27(2a + bc)$.

   Hence $a \in \left]-\infty, a_3\right[\cap \left]-\infty, -\frac{bc}{2}\right] = \left]-\infty, -\frac{bc}{2}\right]$ ($a_3 > -\frac{bc}{2}$ using Lemma 4.2.47),

   and $\rho_1 - 3c$ is negative if $2a + bc$ is positive because
   $(\rho_1 - 3c)\left(\rho_1^2 + 9b\right) = -27(2a + bc)$.

   Hence $a \in \left]-\infty, a_3\right[\cap \left]-\frac{bc}{2}, 0\right[= \left]-\frac{bc}{2}, a_3\right[$ ($a_3 > -\frac{bc}{2}$ using Lemma 4.2.47).

   Using Lemma 4.2.51, hence applying the stability conditions using Definition 4.1.20. One can obtain the results.

2. *If $a \in [a_3, 0[$:*

   the condition $4b < c^2$ is verified, then $a_3 < 0$ (1 in Lemma 4.2.39), and $3b < c^2$.

   $\Delta_2 \leq 0$ because $\Delta_2^* \leq 0$ (2 in Remark 4.2.38). One has $P_2(1) > 0$ (a negative) and $P_2(-1) < 0$ if $s \in \left]0, \frac{2}{c}\right[$ because

   $$-2as^3 - 2bs^2 + 4cs - 8 < -2\left(-\frac{bc}{2}\right)s^3 - 2bs^2 + 4cs - 8$$
   ($a_3 + \frac{bc}{2}$ positive (Lemma 4.2.47)),
   $$-2as^3 - 2bs^2 + 4cs - 8 < (cs - 2)\left(bs^2 + 4\right),$$

   then $P_2(-1) < 0$ if $(cs - 2) < 0$ because $\left(bs^2 + 4\right) > 0$.

   And
   $$\left(-c - \sqrt{c^2 - 3b}\right)\left(-c + \sqrt{c^2 - 3b}\right) = 3b > 0,$$
   so $\lambda_1^*, \lambda_2^* < 1$ because $\left(-c + \sqrt{c^2 - 3b}\right) < 0$.

   Hence, applying the stability conditions using Lemma 4.2.18, Lemma 4.2.20 and Theorem 4 (1.i) of [23], one can obtain the result.

3. *If $a = 0$:*

   the condition $4b < c^2$ is verified, and $3b < c^2$.

   $\Delta_2 \leq 0$ because $\Delta_2^* \leq 0$ (2 in Remark 4.2.38). One has $P_2(1) > 0$ (a null).

   Hence, applying the stability conditions using Theorem 4 (5.i) of [23], one obtains the result.

4. If $a \in]0, a_4]$:
the condition $4b < c^2$ is verified, then $a_4 > 0$ (1 in Lemma 4.2.39), and $3b < c^2$.
$\Delta_2 \leq 0$ because $\Delta_2^* \leq 0$ (2 in Remark 4.2.38). One has $P_2(1) < 0$ (a positive) and $P_2(-1) < 0$ if $s \in \left]0, \frac{2}{c}\right[$ because

$$-2as^3 - 2bs^2 + 4cs - 8 < -2(-\frac{bc}{2})s^3 - 2bs^2 + 4cs - 8 \ (a, \frac{bc}{2} \text{ positive}),$$
$$-2as^3 - 2bs^2 + 4cs - 8 < (cs-2)\left(bs^2+4\right),$$

hence $P_2(-1) < 0$ if $(cs-2) < 0$ because $\left(bs^2+4\right) > 0$.
And

$$\lambda_1^* < 1 \text{ because } \left(-c - \sqrt{c^2 - 3b}\right) < 0.$$

Hence, applying the stability conditions using Lemma 4.2.18, Lemma 4.2.20 and Theorem 4 (4.i.b) of [23], one obtains the result.

4.2.2 Numerical Simulations

In this subsection, we present bifurcation diagrams, phase portraits of the model (4.18), which confirm the analytical results above and illustrate the dynamic behaviors of our model numerically relay. A bifurcation occurs when the stability of a point of equilibrium changes [13].
As discussed earlier in Section 4.2.1, this paper focuses on varying new positive parameter h in the model (4.18). Based on the previous analysis, the parameters of the model (4.18) can be examined by: varying h in the range $1.31 \leq h \leq 1.6$ and fixing $a = 0.7437$, $b = 1.523$, $c = 2.158$, $d = 2$, $\alpha = 0.99$, with the initial conditions $(x_0, y_0, z_0) = (0.001, 0.001, 0.001)$. The resulting points are plotted versus the parameter h (see Figure 4.7).

According to Theorem 4.2.34, we have $b \in \mathbb{R}^+$, $c \in \left]\sqrt{3b}, 2\sqrt{b}\right[=]2.1375, 2.4682[$, and $a \in]a_2, bc[=]0.35305, 3.2866[$, $d \in \mathbb{R}$; we have $(\theta_1)_2 \simeq 4.3947$,
$s = \dfrac{6}{(\theta_1)_2} = -3\dfrac{(\theta_1)_2 - 3c}{(\theta_1)_2^2 - 3c(\theta_1)_2 + 9b} \simeq 1.3653$, $h = \sqrt[\alpha]{s\Gamma(1+\alpha)} \simeq 1.3638$ and E_0 is asymptotically stable if $0 < h < 1.3638$, (see (a) and (b) in Figure 4.8), all trajectories converge to the point E_0. If $h \simeq 1.3638$, system (4.18) undergoes a bifurcation as mentioned above (see (c) in Figure 4.8); the fixed point E_0 becomes unstable if $h > 1.3638$ (see (d) in Figure 4.8).

In this second part of numerical results, the fractional order is only considered as a parameter in the discrete system. Varying the fractional order (α) in the range $0.3 \leq \alpha < 1$, and fixing $a = 0.656$, $b = 1.64$, $c = 2.21$, $d = 2$, $h = 1.747$, the resulting points are plotted versus the fractional order (α) (see Figure 4.9). Attracting invariant circles and chaos appear when increasing (α) in such way that the fractional order remains in the interval $[0.3, 1[$. The phase portraits for various $\alpha-$values corresponding to Figure 4.9 are plotted. Furthermore, the period-2 orbits ($\alpha = 0.73$) are shown (a) in Figure 4.10, and for the attracting invariant circles ($\alpha = 0.77, \alpha = 0.83$) see (b) and (c) in Figure 4.10. Attracting chaotic sets are also observed if $\alpha = 0.99$ and are plotted (d) in Figure 4.10.

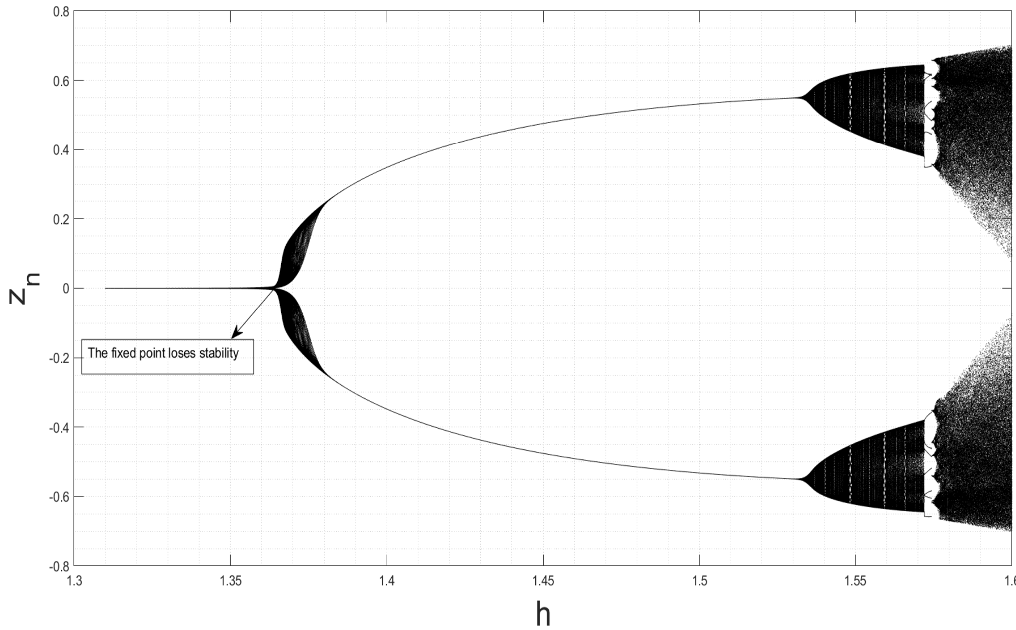
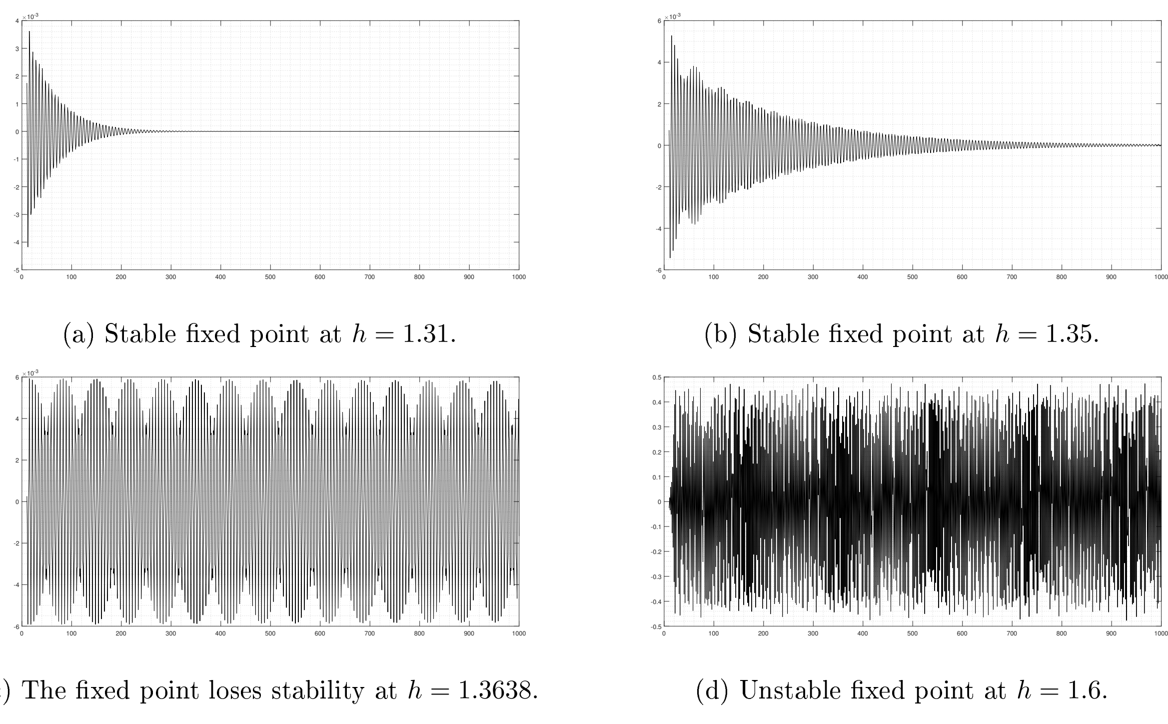


Figure 4.7: Bifurcation Diagram of model (4.18) for $h \in [1.31, 1.6]$.



(a) Stable fixed point at $h = 1.31$.

(b) Stable fixed point at $h = 1.35$.

(c) The fixed point loses stability at $h = 1.3638$.

(d) Unstable fixed point at $h = 1.6$.

Figure 4.8: The trajectory diagrams of model (4.18) for Various h Corresponding to Figure 4.7.

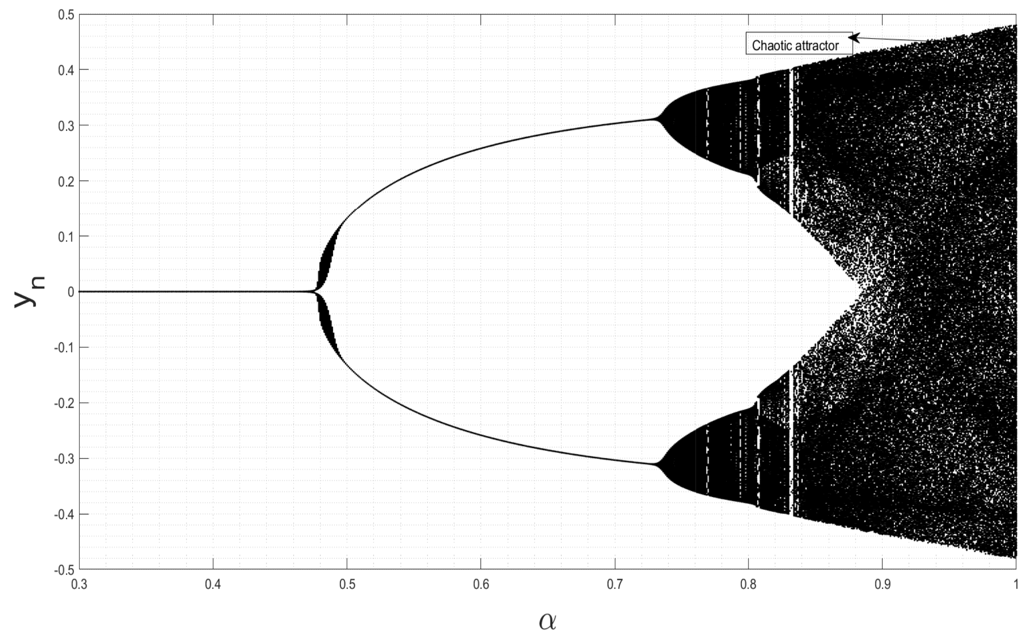
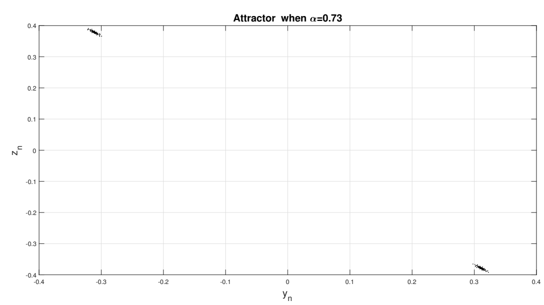
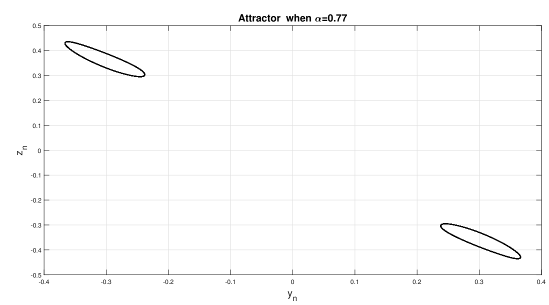


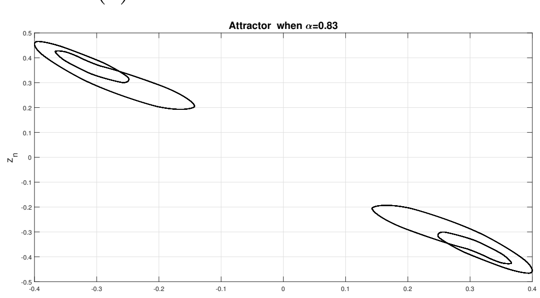
Figure 4.9: Bifurcation Diagram of model (4.18) for $\alpha \in [0.3, 1[$.



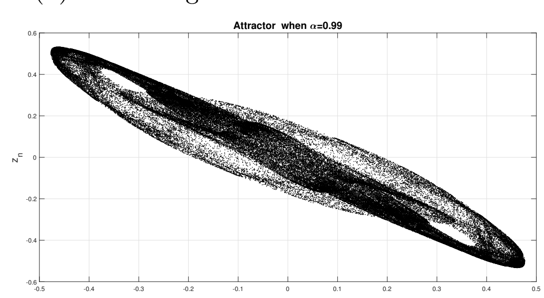
(a) Period-2 orbits at $a = 0.73$.



(b) Attracting invariant circles at $\alpha = 0.77$.



(c) Attracting invariant circles at $\alpha = 0.83$.



(d) Chaotic attractor at $\alpha = 0.99$.

Figure 4.10: Phase Portrait Diagrams of model (4.18) for Various α Corresponding to Figure 4.9.

Conclusion and Perspectives

This thesis has provided an in-depth exploration of fractional calculus and its application to chaotic systems. The preliminary chapters laid the foundation by introducing the necessary mathematical tools and concepts. Chaos theory was also discussed, highlighting its importance in understanding complex dynamical systems.

Chapter 3 focused on discretization methods for transforming continuous fractional-order chaotic systems into discrete counterparts. The various discretization techniques were analyzed, considering their impact on system dynamics and their ability to preserve the essential chaotic features during the discretization process.

Chapter 4 presented the main results of the thesis. Firstly, it investigated the bifurcation and stability properties of a new discrete system derived from a fractional-order continuous chaotic finance system. Through numerical simulations and mathematical analysis, the chapter revealed the existence of bifurcation points and characterized their influence on the system's stability. Secondly, a similar analysis was conducted on a discrete system induced from the chaotic Arneodo's system, exploring its bifurcation patterns and stability properties.

Overall, this research has contributed to the understanding of fractional-order chaotic systems and their discretization. The findings demonstrate the existence of bifurcation points and provide insights into the stability properties of the studied systems. These results have practical implications in various fields, such as finance and engineering, where chaotic dynamics are relevant.

Moving forward, there are several perspectives for further research. One avenue could involve investigating the control and synchronization of discrete fractional-order chaotic systems. Understanding how to control and synchronize these systems could have applications in secure communication and information processing. Additionally, exploring the effects of noise and uncertainties on the discretized chaotic systems would provide valuable insights into their robustness and practical implementation.

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Abstract:

This thesis investigates the bifurcation and stability properties of discrete systems induced by fractional-order continuous chaotic finance systems and Arneodo's chaotic system. The research is structured into four chapters, each focusing on different aspects related to fractional calculus, chaos theory, discretization methods, and the main results obtained from the analysis.

Chapter 1 and Chapter 2 provide a preliminary introduction to fractional calculus, presenting the necessary mathematical tools and concepts for understanding the fractional order systems. Additionally, it discusses the fundamentals of chaos theory, emphasizing the significance of chaos.

In Chapter 3, various discretization methods are examined to transform the continuous fractional-order chaotic systems into discrete counterparts. The discretization techniques are thoroughly analyzed, considering their impact on the system dynamics and preserving the essential chaotic features during the discretization process.

Chapter 4 presents the main results of the thesis. Firstly, it investigates the bifurcation and stability properties of a new discrete system induced from a fractional-order continuous chaotic finance system. Through numerical simulations and mathematical analysis, the chapter reveals the existence of bifurcation points and characterizes their influence on the system's stability. Secondly, a similar analysis is conducted on a discrete system induced from the chaotic Arneodo's system, exploring its bifurcation patterns and stability properties.

Key words:

Discrete dynamical system, Arneodo’s system, finance system, fixed point stability, bifurcation, chaotic behavior, discretization.

Résumé :

Cette thèse étudie les propriétés de bifurcation et de stabilité des systèmes discrets induits à partir de systèmes financiers chaotiques continus d'ordre fractionnaire et du système chaotique d'Arneodo. La recherche est structurée en quatre chapitres, chacun se concentrant sur différents aspects liés au calcul fractionnaire, à la théorie du chaos, aux méthodes de discrétisation et aux principaux résultats obtenus à partir de l'analyse.

Les chapitres 1 et 2 fournissent une introduction préliminaire au calcul fractionnaire, en présentant les outils mathématiques et les concepts nécessaires pour comprendre les systèmes d'ordre fractionnaire. De plus, ils abordent les fondamentaux de la théorie du chaos, en soulignant l'importance des phénomènes chaotiques.

Dans le chapitre 3, différentes méthodes de discrétisation sont examinées pour transformer les systèmes chaotiques continus d'ordre fractionnaire en systèmes discrets. Les techniques de discrétisation sont analysées en détail, en tenant compte de leur impact sur la dynamique du système et de leur capacité à préserver les caractéristiques chaotiques essentielles lors du processus de discrétisation.

Le chapitre 4 présente les principaux résultats de la thèse. Tout d'abord, il étudie les propriétés de bifurcation et de stabilité d'un nouveau système discret induit à partir d'un système financier chaotique continu d'ordre fractionnaire. Grâce à des simulations numériques et à une analyse mathématique, le chapitre révèle l'existence de points de bifurcation et caractérise leur influence sur la stabilité du système. Ensuite, une analyse similaire est menée sur un système discret induit à partir du système chaotique d'Arneodo, explorant ses motifs de bifurcation et ses propriétés de stabilité.

Mots clés:

Système dynamique discret, système d'Arneodo, système financier, stabilité du point fixe, bifurcation, comportement chaotique, discrétisation.

الخلاصة:

تدرس هذه الرسالة خصائص التشعب واستقرار الأنظمة المنفصلة الناتجة عن النظم المالية الفوضوية المستمرة ذات الترتيب الجزئي والنظام الفوضوي لأرنودو، يتكون البحث من أربعة فصول، يركز كل منها على جوانب مختلفة تتعلق بحساب التفاضل والتكامل الجزئي، ونظرية الفوضى، وطرق التقدير والنتائج الرئيسية التي تم الحصول عليها من التحليل.

يقدم الفصلان الأول والثاني مقدمة أولية لحساب التفاضل والتكامل الكسري، حيث يعرضان الأدوات والمفاهيم الرياضية اللازمة لفهم أنظمة الترتيب الكسري. بالإضافة إلى ذلك، تناولوا أساسيات نظرية الفوضى، مؤكدين على أهمية الظواهر الفوضوية.

في الفصل 3، تم فحص طرق تقديرية مختلفة لتحويل الأنظمة الفوضوية المستمرة من الترتيب الكسري إلى أنظمة منفصلة. يتم تحليل تقنيات التقدير بالتفصيل، مع الأخذ في الاعتبار تأثيرها على ديناميكيات النظام وقدرتها على الحفاظ على السمات الفوضوية الأساسية أثناء عملية التقدير.

يقدم الفصل الرابع النتائج الرئيسية للأطروحة. أولاً، يدرس خصائص التشعب والاستقرار لنظام منفصل جديد ناتج عن نظام مالي فوضوي مستمر من النظام الجزئي. من خلال المحاكاة العددية والتحليل الرياضي، يكشف الفصل عن وجود نقاط التشعب ويميز تأثيرها على استقرار النظام. بعد ذلك، يتم إجراء تحليل مماثل على نظام منفصل ناتج عن نظام ارنودو الفوضوي، واستكشاف أنماط التشعب وخصائص الاستقرار.

الكلمات الأساسية:

النظام الديناميكي المنفصل، نظام ارنودو، النظام المالي، استقرار النقطة الثابتة، التشعب، السلوك الفوضوي، تقديرية.