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On some Properties of Forward and Backward Stochastic Differential Equations with Jumps

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Dedication

I dedicate this work:

to the late memory of my father,

my beloved mother,

my good husband,

my dear brothers,

and to my precious daughter.

Acknowledgment

First, I should be so grateful and thankful to Allah, the most gracious and the most merciful for giving me the patience, health, power, and will to complete this work.

I would like to take this opportunity to express my sincere gratitude to my supervisor, Pr. Nabil KHELFALLAH, for devoting much of his time to the development of my Ph.D. study, for the many suggestions, as well as for his continuous support and patience with me. All words of thanks are not enough for you. Thank you so much.

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ملخص

نقدم في أطروحة الدكتوراه هذه موضوعين بحثيين مهمين يتعلقان بمشكلة وجود و وحدانية الحل من خلال إضعاف شرط ليبشيتز على مولدات المعادلات التفاضلية العشوائية التراجعية المولدة بواسطة قياس بواسون العشوائي و الحركة البراونية المستقلة.

يتناول الموضوع الأول من هذه الأطروحة جملة من المعادلات التفاضلية العشوائية التراجعية من نوع ماركوف. درسنا وجود و وحدانية حل هذا النوع من المعادلات ذات القفزات عندما يكون المولد ليبشيزي و مربع الشرط النهائي قابل للتكامل. بعد ذلك، في ظل شرط ليبشيتز، نوضح أن حل هذا الشكل من المعادلات يمكن كتابته على شكل دوال عددية. هذه الأخيرة تستخدم في إضعاف شرط ليبشيتز لإثبات وجود حلول لهذا النوع من المعادلات في الحالتين التاليتين: في الحالة الاولى عندما يكون المولد مستمر بالنسبة للمتغير الأول و الثاني و يحقق شرط ليبشيتز بالنسبة للمتغير الثالث، و في الحالة الثانية عندما يكون المولد مستمر بالنسبة للمتغيرات الثلاث. تتمثل طريقة البرهان في تكوين متتالية من المعادلات التي تتمتع بخاصية وجود و وحدانية الحل لإثبات أن المعادلة الأصلية تقبل على الأقل حلا و ذلك بالمرور الى النهاية آخذين بعين الاعتبار شرط الهيمنة.

يتعلق الجزء الثاني من أطروحتنا بموضوع آخر و هو وجود أو وحدانية حلول المعادلات ذات القفزات، حيث تتميز المولدات بنمو تربيعي في المتغير البراوني. استنادا على هذه النتيجة، قدمنا حل عدة أمثلة لمعادلات مختلفة المولدات بالإضافة الى مبرهنات المقارنة و المقارنة الصارمة دون شرط الرتابة في المتغير الثالث للمولد. الفكرة الرئيسية في هذا الجزء هي استخدام تحويل زفونكين، لتحويل معادلاتنا المدروسة الى معادلات بدون الطرف التربيعي. أخيرا، ناقشنا العلاقة بين هذا النوع من المعادلات و المعادلات التفاضلية التكاملية الجزئية التربيعية عندما يكون المولد قابلا للقياس. يوفر لنا هذا الترابط تمثيلا احتماليا لحلول اللزوجة لبعض المعادلات ذات المشتقات الجزئية التربيعية و التي أثبتناها بمساعدة صيغة فايمان كاك.

الكلمات المفتاحية: المعادلات التفاضلية العشوائية التراجعية؛ عملية القفز؛ قياس بواسون العشوائي؛ الحركة البراونية؛ عملية ماركوف.

Résumé

Le but de cette thèse de doctorat est d'étudier le problème de l'existence et l'unicité en affaiblissant la condition de Lipschitz sur les générateurs d'équations différentielles stochastiques rétrogrades avec sauts (EDSRSs en abrégé).

La première partie traite les EDSRs Markoviennes multidimensionnelles dirigées par une mesure aléatoire de Poisson et un mouvement Brownien indépendant. Des résultats d'existence pour ce type d'équations avec des générateurs continus qui satisfont la condition de croissance linéaire habituelle sont prouvés.

La deuxième partie de cette thèse concerne une classe des EDSRs quadratiques avec sauts (EDSRQSs en abrégé) où les générateurs admettent une croissance quadratique par rapport au terme de la composante Brownienne et une forme fonctionnelle non linéaire par rapport au terme de saut. Nous établissons l'existence (et parfois l'unicité) des solutions aussi bien que des principes de comparaison et de comparaison stricte sans condition de monotonicité sur le générateur. Des représentations probabilistes de solutions à certaines classes des équations différentielles aux dérivées partielles quadratiques (EDPQs en abrégé) sont données au moyen de solutions de ces EDSRQSs.

Ce travail sera présent en trois chapitres. Le premier chapitre port sur l'existence et l'unicité de la solution d'une EDSRS Markovienne multidimensionnelle avec un générateur Lipschitz et une condition terminale de carrée intégrable. Nous montrons aussi que la solution de l'EDSRS peut être représentée en termes d'un processus de Markov donné et de certaines fonctions déterministes.

Le deuxième chapitre s'intéresse aux EDSRSs Markoviennes multidimensionnelles dans deux cas. Dans le premier cas, lorsque le générateur est continu par rapport aux première et deuxième variables et satisfait la condition de Lipschitz par rapport à la troisième variable. Dans le deuxième cas, lorsque le générateur est continu par rapport aux trois variables. L'existence d'une solution (pas nécessairement unique) de ce type d'équations est prouvée par approximer le générateur par une suite convenable des fonctions Lipchitziennes et d'utiliser la condition de $L^{\mathbf{2}}$–domination, sur la loi du processus de Markov, pour faire un passage à la limite. Ensuite, certains cas particuliers sur les conditions de croissance linéaires et sous-linéaires et la régularité du générateur sont discutés. Nous concluons ce chapitre avec plusieurs exemples de processus de Markov ayant la propriété de $L^{\mathbf{2}}$–domination.

Le troisième chapitre est consacré à l'existence et/ou l'unicité des solutions à plusieurs types d'EDSRQSs. De plus, deux théorèmes de comparaison sont établis. L'idée principale des preuves est d'utiliser la transformation de Zvonkin [72], pour transformer l'EDSRS quadratique initiale en une EDSRS standard avec un générateur continu ou globalement Lipschizien. Grâce à la formule de Feynman-Kac, nous prouvons une relation entre les EDSRQSs et les EDPQs. Cette connexion fournit une représentation probabiliste des solutions de viscosité des EDPQs sous considération.

Mots-clés: Équations différentielles stochastiques rétrogrades; Processus de saut; Mesure aléatoire de Poisson; Mouvement Brownien; Processus de Markov. .

Abstract

The aim of this Ph.D. thesis is to study the problem of existence and uniqueness by relaxing the Lipschitz condition on generators of backward stochastic differential equations with jumps.

The first topic deals with multidimensional Markovian BSDEs driven by a Poisson random measure and independent Brownian motion. Existence results for such equations with continuous generators that satisfy the usual linear growth condition are proved.

The second topic is concerned with a class of quadratic BSDEs with jumps where the generators show quadratic growth in the Brownian component and non-linear functional form with respect to the jump term. We establish the existence (and sometimes the uniqueness) of solutions as well as a comparison and strict comparison principles under no monotonicity condition in the third argument of the generator. Probabilistic representations of solutions to some classes of quadratic partial integral differential equations are given by means of solutions of these QBSDEJs.

This thesis presents three chapters. The first chapter focuses on the existence and uniqueness of the solution to multidimensional Markovian BSDEJs under the global Lipschitz property of the generator and square-terminal value. We prove, under the Lipschitz condition, that the BSDEJ's adapted solution can be represented in terms of a given Markov process and some deterministic functions.

The second chapter is concerned with multidimensional Markovian BSDEJs in two cases. In the first case, when the generator is continuous with respect to the first and second variables and satisfies the Lipschitz condition with respect to the third variable.

In the second case, when the generator is continuous with respect to the three variables. The existence of a solution (not necessarily unique) to BSDEJs under study is proved by using the so-called L^2-domination technique and some regularization and approximations arguments. Furthermore, some special cases of linear and sub-linear growth conditions and the regularity of the generator are discussed. We conclude this chapter with several examples of the Markov process having the L^2-domination property.

The third chapter is devoted to the existence and/or uniqueness of the solutions to a variety of types of QBSDEJs. More precisely, the solvability of some QBSDEJs via several examples dealing with different generators of other quadratic forms. Furthermore, two comparison theorems are established. Finally, this chapter deals with the relationship between quadratic BSDEJs and QPIDEs with measurable generators. This connection provides a probabilistic representation of viscosity solutions of some QPIDEs, which is proved by means of the Feynman-Kac formula.

Key-words: Backward stochastic differential equations; Jump process; Poisson random measure; Brownian motion; Markov process.

List of Symbols and Abbreviations

These are the different symbols and abbreviations used in this thesis.

Symbols:

- $(\Omega, \mathcal{F}, \mathbb{P})$: probability space.
- $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$: filtration.
- $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$: filtered probability space.
- $W = \{W_t\}_{t\in[0,T]}$: Brownian motion.
- N: Poisson random measure.
- $\nu(\mathrm{d}e)\mathrm{d}s$: the compensator of N.
- $\tilde{N}$: the compensated Poisson measure.
- $\mathbb{R}$: real numbers.
- $\mathbb{R}^q$: $q-$dimensional real Euclidean space.
- $\mathbb{R}^{q\times d}$: the set of all $(q \times d)$ real matrixes.
- $E = \mathbb{R}^q \setminus \{0_{\mathbb{R}^q}\}$.
- $\mathbb{G} = (\mathcal{G}_s)_{s\in[0,T]}$: the filtration generated by the deterministic functions $\int_t^T \mathbb{E}\psi(r, X_r^{t,y})\mathrm{d}r$ where ψ is a continuous $\mathbb{R}^q$-valued function.
- $\bar{A}$: the closure of the set A.
- $1\!\!1_A$: the indicator function of the set A.
- $\sigma(A)$: $\sigma-$algebre generated by A.
- (a, b): the inner product in $\mathbb{R}^q$.
- $|a| = \sqrt{(a, a)}$: the norm of $\mathbb{R}^q$.

- (A, B): the inner product in $\mathbb{R}^{q\times d}$.

- $|A| = \sqrt{(A, A)}$: the norm of $\mathbb{R}^{q\times d}$.

- $L^1(\mathbb{R})$: the space of the functions whose absolute value is integrable.

- $L^2(\Omega)$: the Banach space of $\mathbb{R}^q$-valued, square integrable random variables on $(\Omega, \mathcal{F}, \mathbb{P})$.

- $\mathcal{L}^p_\nu$: the set real-valued measurable functions u defined on $[0, T] \times E$ such that:
$$\|u(\cdot)\|_{p,\nu} = (\int_E |u(e)|^p \, \nu(\mathrm{d}e))^{\frac{1}{p}} < \infty.$$

- $\mathcal{L}^{2,q}_\nu$: the Banach space of $\mathbb{R}^q$-valued deterministic functions $(\varphi(e))_{e\in E}$ such that:
$$\|\varphi(\cdot)\|^2_{q,\nu} = \int_E |\varphi(e)|^2 \, \nu(\mathrm{d}e) < \infty.$$

- $\mathcal{S}^2_{\mathcal{F}}\,(0, T; \mathbb{R}^q)$: the Banach space of $\mathbb{R}^q$-valued, $\mathcal{F}_t-$adapted and càdlàg processes $(Y_t)_{0\le t\le T}$ such that:
$$\mathbb{E}[\sup_{0\le t\le T} |Y_t|^2] < \infty.$$

- $\mathcal{M}^2_{\mathcal{F}}(0, T, \mathbb{R}^q)$: the Banach space of $\mathbb{R}^q$-valued $\mathcal{F}_t-$predictable processes $(\varphi_t)_{0\le t\le T}$ such that:
$$\int_0^T \mathbb{E}\,|\varphi_t|^2 \,\mathrm{d}t < \infty.$$

- $\mathcal{M}^2_{\mathcal{F}}([0, T] \times E, \mathbb{R}^q, \mathrm{d}t\nu(\mathrm{d}e))$: the Banach space of $\mathbb{R}^q$-valued $\mathcal{F}_t-$predictable processes $(\psi_t(e))_{0\le t\le T, e\in E}$ satisfying:
$$\mathbb{E}\left[\int_0^T \int_E |\psi_t(e)|^2 \, \nu(\mathrm{d}e)\mathrm{d}t\right] < \infty.$$

- $\mathcal{W}^2_1\,(\mathbb{R})$: the space of continuous functions g from $\mathbb{R}$ to $\mathbb{R}$ such that g' is continuous and g'' is integrable on $\mathbb{R}$.

- $\mathcal{W}^2_{p,\,loc}\,(\mathbb{R})$: the Sobolev space of functions g defined on $\mathbb{R}$ such that both g and its generalized derivatives g' and $g''$are locally integrable on $\mathbb{R}$.

- $\mathbb{M}^2 = \mathcal{M}^2_{\mathcal{F}}(0, T, \mathbb{R}^q) \otimes \mathcal{M}^2_{\mathcal{F}}(0, T, \mathbb{R}^{q\times d}) \otimes \mathcal{M}^2_{\mathcal{F}}([0, T] \times E, \mathbb{R}^q, \mathrm{d}t\nu(\mathrm{d}e))$.

- $\mathbb{M}^2_{\mathcal{S}} = \mathcal{S}^2_{\mathcal{F}}\,(0, T; \mathbb{R}^q) \otimes \mathcal{M}^2_{\mathcal{F}}(0, T, \mathbb{R}^{q\times d}) \otimes \mathcal{M}^2_{\mathcal{F}}([0, T] \times E, \mathbb{R}^q, \mathrm{d}t\nu(\mathrm{d}e))$.

Abbreviations:

- a.e.: almost everywhere.
- a.s.: almost surely.
- i.e.: that's to say.
- càdlàg: right continuous with left limits.
- w.r.t: with respect to.
- e.g: for example.
- SDEs: stochastic differential equations.
- BSDEs: backward stochastic differential equations.
- BSDEJs: backward stochastic differential equations with jumps.
- PDEs: partial differential equations.
- QBSDEJs: quadratic backward stochastic differential equations with jumps.
- PIDEs: partial integral differential equations.
- QPIDEs: quadratic partial integral differential equations.
- BDG: Burkholder–Davis–Gundy inequalities.

Contents

General Introduction

Backward stochastic differential equations have been extensively studied from different viewpoints due to their applications in many fields such as mathematical finance El Karoui, Peng & Quenez (1997) [30], finance and insurance Dos Reis (2010) [26], insurance reserve Delong (2013) [25], optimal control theory [6, 17, 35, 46, 61, 69] , stochastic differential games and stochastic control Hamadène & Lepltier (1995) [38, 39], Hamadène (1998) [36] and Baghery *et al.* (2014) [7] and their strong connection with partial differential equations [11, 12, 60, 62]. Applications on utility optimization and dynamic risk measures were discussed by Becherer (2006) [14], Morlais (2009) [53] and Quenez & Sulem (2013) [66].

In 1990, the theory of BSDE was greatly developed by many authors, and there were a large number of published articles devoted to the theory of BSDE and its applications. Among these academic researchers, the most famous ones are Pardoux and Peng [58] who showed, based on the martingale representation theorem, the existence and uniqueness of solutions to BSDEs under the square-integrability on the terminal data and the Lipschitz continuity condition on the driver, which is considered the powerful condition that ensures the well-posedness of BSDEs.

A few years later, further research weakens the Lipschitz hypothesis on the generator. Besides, the former mentioned works, we would like to make a synopsis of papers that focused on two directions: the BSDE with continuous as well as with quadratic generators. In the context of the first direction, Lepeltier and Martin (1997) [49] proved the existence of a solution for such BSDEs, where the generator is continuous, has a linear growth, and the terminal condition is square integrable. Later, Jia and Peng (2007) [43], based on the result found in [49] showed that underlying BSDE has either one or uncountably many solutions. They also provided the structure of those solutions. Then, Kobylanski (2000) [47] provided existence, comparison, and stability results for one-dimensional BSDEs when the coefficient is continuous and has a quadratic growth in the Brownian component and the terminal condition is bounded. Contrariwise, there are a few papers that studied the existence problem for continuous BSDE with jumps. Yin and Mao (2008) [71] dealt with a class of one-dimensional BSDE with Poisson jumps and with random terminal times. They showed the existence and uniqueness of a minimal solution when the BSDE coefficient is continuous and has a linear growth condition. Then, Qin and Xia (2013) [65] proved the existence of a minimal solution for BSDEs driven by Poisson processes where the coefficient is continuous and satisfies an improved linear growth assumption. They also extended the result to the case where the coefficient is left or right continuous. More recently, Madoui *et al.* (2022) [52] and Abdelhadi *et al.* (2022) [1] provided some examples that ensure the connection between one type of quadratic BSDEs with jumps and standard BSDEs with continuous drivers. It is worth pointing out that all the former mentioned results are given for one-dimensional BSDE and the main tools in the proofs are the approximating technique and the comparison theorem. On the other hand, limited results have been obtained about the multidimensional BSDE with continuous generator, for example, Hamadène (2003) [37] and Hamadène and Mu (2015) [40].

The second direction is concerned with an interesting subclass of BSDEs that shows a quadratic growth w.r.t respect to the Brownian component. Motivated by the study of utility maximization problems and some particular forms of PDEs or PIDEs, this genre of equations appeared and loomed large in BSDE's theory. Quadratic BSDEs have been extensively studied by many authors in various fields of applications. Among them, in the continuous framework, Kobylanski [47] for QBSDEs and their connections with viscosity and Sobolev solutions of PIDEs when the non-linearity has a quadratic growth in the gradient. Limit theorems of QBSDEs and non-linear PIDEs by Eddahbi and Ouknine (2002) [28]. Later, Briand and Hu (2006, 2008) [18, 19] have analyzed QBSDEs with convex generators and unbounded but exponentially integrable terminal conditions. Classical and variational differentiability of quadratic BSDEs has been addressed by Ankirchner, Imkeller, and Dos Reis (2007) in [2]. Tevzadze (2008) initiated a fixed-point approach for QBSDEs in [70]. Delbaen, Hu, and Richou (2011) [24] have established the uniqueness of solutions to QBSDEs with convex generators and unbounded terminal conditions. BSDEs with stochastic quadratic growth have been studied by Essaky and Hassani (2013) in [31]. Based on the former paper a domination method has been introduced by Bahlali, Eddahbi, and Ouknine (2013, 2017) [9, 10] and Bahlali (2020) [8].

In the jumps setting, quadratic BSDEs with jumps have been also studied, in particular, motivated by exponential utility optimization Morlais (2010) [54] and Jeanblanc *et al.* (2015) [41] considered quadratic BSDE with jumps. Kazi-Tani, Possamai, and Zhou (2015, 2016) [44, 45] have adopted the fixed-point approach initiated by Tevzadze [70]. Antonelli and Mancini (2016) [4] studied the same class under different assumptions on local Lipschitz generators. Based on the stability of quadratic semi-martingales, Barrieu and El Karoui (2013) [13] introduced the so-called quadratic structure condition and have proved the existence of a non-necessary unique solution in the continuous setting when the terminal value is unbounded. Some extended results to a quadratic-exponential structure condition have been conducted by Jeanblanc, Matoussi and Ngoupeyou (2013) [42] and El Karoui, Matoussi and Ngoupeyou (2018) [29]. Fujii and Takahashi (2018) [32] have studied Malliavin's differentiability of solutions of BSDEs with jumps under the quadratic-exponential growth condition. Recent extensions on the monotone stability approach to BSDEs with jumps have been developed by Becherer, Büttner and Klebert (2019) in [15].

This present dissertation focuses on the study of the existence and/or the uniqueness problem of two types of BSDE driven by both a Wiener and a Poisson random measure. We first study the existence of multidimensional Markovian BSDE with a jump with a continuous generator (not necessarily Lipschitz). To do this, we show, under the Lipschitz condition, that the BSDEJ's adapted solution can be represented in terms of a given Markov process and some deterministic functions. Then, the second type focuses on the study of the existence and/or uniqueness of the solutions to a variety of types of BSDEJs when the generators show quadratic growth in the Brownian component and non-linear functional form w.r.t the jump term. Lastly, we study the relationship between QBSDEJs and QPIDEs with measurable generators.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, $(\mathcal{F}_t)_{t\in[0,T]}$ be a non-decreasing family of sub-σ-algebras of $\mathcal{F}$, on which are defined two fundamental time-homogeneous independent stochastic processes: a standard $\mathbb{R}^q$-valued Wiener $\{W_t : t \in [0,T]\}$ and a real-valued Poisson random measure $N(\mathrm{d}s, \mathrm{d}e)$ defined in $[0,T]\times E$, where $E = \mathbb{R}^q \backslash \{0_{\mathbb{R}^q}\}$. We denote by $\nu(\mathrm{d}e)\mathrm{d}s$ the compensator of N where $\widetilde{N}(\mathrm{d}s, \mathrm{d}e) = N(\mathrm{d}s, \mathrm{d}e) - \nu(\mathrm{d}e)\mathrm{d}s$ is a martingale with mean zero called the compensated Poisson random measure. For the theory of stochastic differential equations with Poisson's measure, we refer to [34].
We consider $\mathbb{F} = (\mathcal{F}_t)_{t\in[0,T]}$ to be the filtration generated by the two processes W and $\widetilde{N}$. For a given $\mathbb{R}^q$-valued random variable ξ defined on $(\Omega, \mathcal{F}_T, \mathbb{P})$ and an $\mathbb{R}^p$-valued càdlàg Markov process $(X_t)_{t\in[0,T]}$ on $(\Omega, \mathcal{F}_T, \mathbb{F}, \mathbb{P})$, we are interested in the following multidimensional BSDEJ: for any $t \in [0,T]$,

$$Y_t = \xi + \int_t^T f(r, X_r, Y_r, Z_r, K_r(\cdot))\mathrm{d}r - \int_t^T Z_r \,\mathrm{d}W_r - \int_t^T \int_E K_r(e)\widetilde{N}(\mathrm{d}r, \mathrm{d}e), \tag{0.1}$$

where the generator $f : [0,T] \times \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^{q\times q} \times \mathcal{L}^{2,q}_\nu \longrightarrow \mathbb{R}^q$ is of linear growth and Lipschitz, w.r.t y, z and $k(\cdot)$ uniformly in s.

The purpose of this thesis is to focus in two research themes. Our first topic [27] come to complete the studies of Hamadène [37], Hamadène and Mu [40] and Mu and Wu (2015) [55] without jump part in the setting of BSDEs with jumps and continuous generators.

As the first result, we prove, without using the connection with PIDE, a new representation theorem in the setting of BSDEJs with globally Lipschitz generators. Then, based on the seminal papers Çinlar *et al.* (1980) [23] and Çinlar & Jacod (1981) [22], we represent the components of the BSDEJ's adapted solution in terms of the Markov process $(X_s^{t,x})_{s\in[t,T]}$. In other words, we prove the existence of three measurable and deterministic functions $u : [0,T] \times \mathbb{R}^p \longrightarrow \mathbb{R}^q$, $v : [0,T] \times \mathbb{R}^p \longrightarrow \mathbb{R}^{q\times q}$ and $\theta : [0,T] \times \mathbb{R}^p \times E \longrightarrow \mathcal{L}^{2,q}_\nu$ such that for any $(s, e) \in [t,T] \times E$

$$Y_s^{t,x} = u(s, X_s^{t,x}), \quad Z_s^{t,x} = v(s, X_s^{t,x}) \text{ and } \; K_s^{t,x}(e) = \theta(s, X_{s-}^{t,x}, e).$$

In fact, this result generalizes the one obtained by El Karoui *et al.* (1997) [30, Theorem 4.1 p. 46] to the jump case. Compared with the representation by the well-known Feynman–Kac formula using PIDEs. Our method does not require smoothness on the coefficients.

As the second result, we study the BSDEJ (0.1) with a continuous generator in y, z, and globally Lipschitz in $k(\cdot)$. For the case when f is of linear growth in x, y, z and $k(\cdot)$, we prove the existence of at least one solution to BSDEJ (0.1) which belongs to the Banach space $\mathbb{M}^2$. The later result is obtained by utilizing the so-called $L^{\mathbf{2}}$–domination technique related to the existence, with some lower and upper bounds, of the density of the law of the transition probability of the underlying Markov process, as well as some regularization and approximation arguments. Then, for the case where f satisfies the sub-linear growth condition we can prove that the solution of BSDEJ (0.1) is in fact in $\mathbb{M}^2_{\mathcal{S}}$ which is a subspace of $\mathbb{M}^2$.
Finally, by assuming that the generator f depends on x, y, z and $\int_0^T k\,(e)\,\nu\,(\mathrm{d}e)$ rather than $k(\cdot)$, we get the existence of at least one solution to BSDEJ whose generator is continuous in y, z and k. Notice that our results do not use either a comparison theorem or a representation based on the partial integral differential equations. We conclude our results with several examples of Markov processes having the $L^{\mathbf{2}}$–domination property.

The second topic [52] is a natural continuation and extension to the jump case of the recent papers of Bahlali, Eddahbi & Ouknine (2017) [10] and Bahlali (2020) [8]. We prove the existence and/or uniqueness of the solution of the $\mathbb{R}$-valued BSDEJs of quadratic type of the form:

$$Y_t = \xi + \int_t^T H(Y_s, Z_s, K_s(\cdot))\mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s - \int_t^T \int_E K_s(e)\tilde{N}(\mathrm{d}s, \mathrm{d}e), \tag{0.2}$$

herein, the terminal data ξ is assumed to be square integrable. Throughout this work, we shall refer to the equation (0.2) as $\mathrm{Eq}(\xi, H)$.

Our study covers the following cases:

$$H(y,z,k(\cdot)) = \begin{cases} f(y)\,|z|^2 + [k]_f(y) =: H_f(y,z,k(\cdot))) \\ h\,(y,k(\cdot)) + cz + H_f(y,z,k(\cdot))) \\ a + b\,|y| + c\,|z| + d\,\|k(\cdot)\|_{1,\nu} + H_f(y,z,k(\cdot)) \\ cz + f(y)\,|z|^2 - \int_E k(e)\nu(\mathrm{d}e) \\ cz + f(y)\,|z|^2 \\ h\,(y,k(\cdot)) + cz + f(y)\,|z|^2 \\ H_0\,(r,X_r) + H_f(y,z,k(\cdot)),\ (X_r)_{r\geq 0} \text{ is a Markov process,} \end{cases}$$

where f is a measurable and integrable function, h and H_0 enjoy some classical assumptions and $[k]_f\,(\cdot)$ is a functional defined as follows

$$[k]_f\,(y) = \int_E \frac{F(y+k(e)) - F(y) - F'(y)k(e)}{F'(y)}\nu(\mathrm{d}e),$$

and the function F is defined, for every $x \in \mathbb{R}$, by

$$F(x) = \int_0^x \exp\left(2\int_0^y f(t)\mathrm{d}t\right)\mathrm{d}y.$$

The generators show quadratic growth in the Brownian component and non-linear functional form w.r.t the jump term. Using Zvonkin transformation to eliminate the drift or a part of the drift. Furthermore, comparison and strict comparison theorems (Theorem 3.4.1 in chapter 3) are established to compare the solutions for QBSDEJs of type $\mathrm{Eq}(\xi, F)$. The novelty is that the comparison of solutions holds whenever we can compare the generators a.e. in the y–variable and both the generators can be non-Lipschitz, non-continuous, and non-convex. Moreover, let (Y^1, Z^1, K^1), (Y^2, Z^2, K^2) be respectively the solution of $\mathrm{Eq}(\xi_1, H_{f_1})$ and $\mathrm{Eq}(\xi_2, H_{f_2})$ and if $\xi_1 \leq \xi_2$ $\mathbb{P}$–a.s. and $f_1 \leq f_2$–a.e. Then $Y_t^1 \leq Y_t^2$ $\mathbb{P}$–a.s. In addition, if $\mathbb{P}\,(\xi_2 > \xi_1) > 0$ then $\mathbb{P}(Y_t^2 > Y_t^1$ for all $t \in [0,T]) > 0$. In particular, we have, $Y_0^2 > Y_0^1$.

Our goal in this topic is to investigate a class of BSDEJs and related PIDEs of a quadratic type associated with a Brownian component and independent Poisson random measure. We introduce the forward SDE with jumps that will generate a Markov process to be used to solve some QPIDEs. For a given t as the initial time and $\zeta \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R})$ as the initial state, let $(X_s^{t.\zeta})_{s\in[t,T]}$ be the solution of the following SDE with jumps:

$$X_s^{t.\zeta} = \zeta + \int_t^s b(r, X_r^{t,\zeta})\mathrm{d}r + \int_t^s \sigma(r, X_r^{t,\zeta})\mathrm{d}W_r + \int_t^s \int_E \varphi(r, X_{r-}^{t,\zeta}, e)\tilde{N}(\mathrm{d}r, \mathrm{d}e), \tag{0.3}$$

where $X_s^{t.\zeta} = \zeta$ for all $0 \leq s \leq t$ and the mappings $b : [0,T]\times\mathbb{R} \longrightarrow \mathbb{R}$, $\sigma : [0,T]\times\mathbb{R} \longrightarrow \mathbb{R}$ and $\varphi : [0,T] \times \mathbb{R} \times E \longrightarrow \mathbb{R}$ satisfy the assumptions $(\mathbf{A}_{5.5}), (\mathbf{A}_{5.6})$ and $(\mathbf{A}_{5.7})$ which will be mentioned in the third chapter.

Now, we shall introduce the next Markovian BSDEJ:

$$\begin{aligned} Y_t \;&=\; g(X_T^{t.x}) + \int_t^T (H_0(r, X_r^{t,x}) + H(Y_r^{t,x}, Z_r^{t,x}, K_r^{t,x}(\cdot)))\mathrm{d}r - \int_t^T Z_r^{t,x}\mathrm{d}W_r \\ &\quad - \int_t^T \int_E K_r^{t,x}(e)\tilde{N}(\mathrm{d}r, \mathrm{d}e), \end{aligned} \tag{0.4}$$

where

$$g : \mathbb{R} \longrightarrow \mathbb{R}, \quad H_0 : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R},$$

satisfies the following conditions:

There exists $L, C > 0$, such that for all $r \in [0,T]$ and $x,\ \acute{x} \in \mathbb{R}$

$$|g(x) - g(\acute{x})| + |H_0(r,x) - H_0(r,\acute{x})| \leq L\,|x - \acute{x}| \quad \text{and} \quad |H_0(r,x)| \leq C.$$

For any $h : \mathbb{R} \times \mathcal{L}_\nu^2 \longrightarrow \mathbb{R}$ and f satisfying $(\mathbf{A}_{3.1}) - (\mathbf{A}_{3.2})$ from chapter 3, we set

$$\begin{aligned} \mathcal{G} = \{&H_f(y,z,k(\cdot)), \quad h(y,k(\cdot)) + cz + H_f(y,z,k(\cdot)), \\ &a + b\,|y| + c\,|z| + d\,\|k(\cdot)\|_{1,\nu} + H_f(y,z,k(\cdot)) \\ &cz + f(y)\,|z|^2 - \int_E k(e)\nu(\mathrm{d}e), \quad cz + f(y)\,|z|^2, \quad h(y,k(\cdot)) + cz + f(y)\,|z|^2\}. \end{aligned}$$

It is clear from all the results in the section 4 (from chapter 3) that the BSDEJ $\mathrm{Eq}(g(X_T^{t,x}), H_0 + H)$ has at least one solution for H in $\mathcal{G}$. We show that this solution can be represented by as a deterministic function of the Markov process X. which is the solution of the following PIDE:

$$\begin{cases} (\mathcal{L}(\theta) + \mathcal{I}(\theta,\varphi) + H_0 + H(\theta, \sigma\frac{\partial\theta}{\partial x}, \Delta\theta_\varphi(\cdot)))(t,x) = 0, \\ \theta(T,x) = g(x), \end{cases} \tag{0.5}$$

where θ be the $\mathcal{C}^{1,2}$ classical solution, H is one of the elements of $\mathcal{G}$ and the operators $\mathcal{L}(\theta)$ and $\mathcal{I}(\theta,\varphi)$ will be defined in chapter 3. The solution of (0.5) can be represented by

$$Y_s^{t,x} = \theta(s, X_s^{t,x}), \quad Z_s^{t,x} = \sigma(s, X_{s-}^{t,x})\frac{\partial\theta}{\partial x}(s, X_{s-}^{t,x}),$$

and

$$\begin{aligned} K_s^{t,x}(e) &= \theta(s, X_{s-}^{t,x} + \varphi(s, X_{s-}^{t,x}, e)) - \theta(s, X_{s-}^{t,x}) \\ &= \Delta\theta_\varphi(e)(s, X_{s-}^{t,x}), \end{aligned}$$

for $t \le s \le T$ and $e \in E$. Furthermore, we have the representation

$$\begin{aligned} Y_t^{t,x} &= \mathbb{E}\left[g(X_T^{t,x})\right] + \mathbb{E}\left[\int_t^T (H_0(r, X_r^{t,x}) + H(Y_r^{t,x}, Z_r^{t,x}, K_r^{t,x}(\cdot)))\mathrm{d}r\right] \\ &:= \theta(t,x). \end{aligned}$$

These connections provide a probabilistic representation of viscosity solutions of some QPIDEs by means of the Feynman-Kac formula. This work is a continuation and extension of Barles, Buckdahn, and Pardoux [12]. We consider the following QPIDE with a non-linear functional term

$$\begin{cases} (\mathcal{L}(\theta) + \mathcal{I}(\theta,\varphi) + H_0 + H_f(\theta, \sigma\frac{\partial\theta}{\partial x}, \Delta\theta_\varphi(\cdot)))(t,x) = 0, \\ \theta(T,x) = g(x). \end{cases} \tag{0.6}$$

Let $(Y_s^{t,x})_{0\leq s\leq T}$ be the unique solution of equation (0.5) where $Y_s^{t,x} = Y_t^{t,x}$ for all $0 \leq s \leq t$ and $t \leq T$. We prove that the continuous function $\theta(t,x) = Y_t^{t,x}$, for all $(t,x) \in [0,T] \times \mathbb{R}$, is a viscosity solution of the QPIDE (0.6) if and only if the continuous function $\alpha(t,x) = F(Y_t^{t,x})$ is a viscosity solution of the following PIDE

$$\begin{cases} \left(\mathcal{L}\left(\alpha\right) + \mathcal{I}(\alpha,\varphi) + F'(F^{-1}(\alpha))H_0\right)(t,x) = 0 \\ \alpha(T,x) = F(g(x)). \end{cases}$$

Finally, we get that the function θ given by $\theta(t,x) := Y_t^{t,x}$, for all $(t,x) \in [0,T] \times \mathbb{R}$ is a viscosity solution to the QPIDE (0.6).

The content of this thesis was the subject of the following relevant papers [27, 52]:

1. Mhamed Eddahbi, Anwar Almualim, Nabil Khelfallah, and Imène Madoui. Multidimensional Markovian BSDEs with jumps and continuous generators. Axioms, 12(1):26, 2022.

2. Imène Madoui, Mhamed Eddahbi, and Nabil Khelfallah. Quadratic BSDEs with jumps and related PIDEs. Stochastics, 94(3):386-414, 2022.

Chapter 1

Multidimensional BSDEJs with Lipschitz Generators

1.1 Introduction

In this present chapter, we study the existence and uniqueness of the solution to multidimensional BSDEs with jumps where the generator achieves the global Lipschitz property and square-integrable terminal value ξ. Note that this proof can be found in many papers on BSDEJs with slightly different techniques in particular in dimension one see [25, 67] and the references therein or in Hilbert valued BSDEs [69], but we give complete proof here for the sake of the content of this work. We prove a new representation theorem without using PIDE. Our objective is to generalize to the jump case in El Karoui *et al.* [30]. That is to represent the components of the BSDEJ's adapted solution in the Lipschitz framework in terms of X and some deterministic functions.

We start this chapter by giving definitions and some technical results that will be needed in the proofs of our main results. Section three is devoted to studying the existence and uniqueness result of a multidimensional BSDEJ in the globally Lipschitz setting. Finally, we focus, together with the representation of additive functionals of Markov processes, on the most important result, which will be used in the second chapter that is to establish a Markovian structure of the solution of a BSDEJ in terms of three deterministic and measurable functions.

1.2 Definitions and Auxiliary Results

1.2.1 Predictable Representation Theorem

Proposition 1.2.1

(Kunita-Watanabe [48, Theorem 2.9]) Let $(M_t)_{t\in[0,T]}$ be an $\mathbb{R}^q$-valued square integrable $(\mathcal{F}_t)_{t\in[0,T]}$–martingale. Then there exist, predictable processes $Z_.$ and $K_.(\cdot)$ taking values respectively in $\mathbb{R}^{q\times d}$ and $\mathcal{L}^{2,q}_\nu$ such that:

$$\int_0^T \mathbb{E}\left[|Z_s|^2\right] \mathrm{d}s < \infty \text{ and } \int_0^T \mathbb{E}\left[\|K_s(\cdot)\|^2_{q,\nu}\right] \mathrm{d}s < \infty,$$

where $|Z_s|^2 = \sum_{i=1}^q \sum_{j=1}^d (Z_s^{ij})^2$ and $|K_s(e)|^2 = \sum_{i=1}^q (K_s^i(e))^2$. The martingale $(M_t)_{t\in[0,T]}$ is represented by

$$M_t = M_0 + \int_0^t Z_s \mathrm{d}W_s + \int_0^t \int_E K_s(e)\widetilde{N}(\mathrm{d}s, \mathrm{d}e), \tag{1.1}$$

or equivalently for all $i = 1, \ldots, q$

$$M_t^i = M_0^i + \sum_{j=1}^d \int_0^t Z_s^{ij} \mathrm{d}W_s^j + \int_0^t \int_E K_s^i(e)\widetilde{N}(\mathrm{d}s, \mathrm{d}e).$$

The couple $(Z_s, K_s(e))$ is uniquely determined from $(M_t)_{t\in[0,T]}$, i.e., if $(M_t)_{t\in[0,T]}$ is represented by (1.1) with another $(Z'_s, K'_s(e))$ satisfying (1.1), then we have $Z_. = Z'_.$ a.e. $\mathrm{d}s \otimes \mathrm{d}\mathbb{P}$ and $K_.(\cdot) = K'_.(\cdot)$ a.e. $\mathrm{d}s \otimes \nu(\mathrm{d}e) \otimes \mathrm{d}\mathbb{P}$ where $\mathrm{d}s$ is the Lebesgue measure on $[0, T]$ and $\nu(\mathrm{d}e)$ is the Lévy measure on E.

Corollary 1.2.1

The spaces of continuous martingales $\mathcal{M}^c$ and that of purely discontinuous martingales $\mathcal{M}^d$ are characterized as follows:

$$\mathcal{M}^c = \left\{ \int_0^t Z_s \mathrm{d}W_s : \sum_{i,j=1}^d \int_0^T \mathbb{E}\left[Z_s^i Z_s^j\right] a_{i,j} \mathrm{d}s < \infty \right\},$$

and

$$\mathcal{M}^d = \left\{ \int_0^t \int_E K_s(e)\widetilde{N}(\mathrm{d}s, \mathrm{d}e) : \int_0^T \mathbb{E}\left[\|K_s(\cdot)\|^2_{q,\nu}\right] \mathrm{d}s < \infty \right\},$$

where $a_{i,j}$ is a non-negative definite symmetric matrix.

Proof. See [48].

Corollary 1.2.2

Let M, N be $\mathbb{R}^q$-valued square-integrable martingales represented by

$$M_t = \sum_{j=1}^{d} \int_0^t Z_s^j \mathrm{d}W_s^j + \int_0^t \int_E K_s(e) \widetilde{N}(\mathrm{d}s, \mathrm{d}e),$$

as well as,

$$N_t = \sum_{j=1}^{d} \int_0^t H_s^j \mathrm{d}W_s^j + \int_0^t \int_E J_s(e) \widetilde{N}(\mathrm{d}s, \mathrm{d}e).$$

Then,

$$\langle M_., N_. \rangle_t = \sum_{j=1}^{d} \int_0^t Z_s^j H_s^j \mathrm{d}s + \int_0^t \int_E K_s(e) J_s(e) \nu(\mathrm{d}e) \mathrm{d}s,$$

and

$$[M_., N_.]_t = \sum_{j=1}^{d} \int_0^t Z_s^j H_s^j \mathrm{d}s + \int_0^t \int_E K_s(e) J_s(e) N(\mathrm{d}s, \mathrm{d}e).$$

Moreover,

$$\begin{aligned} [M_., N_.]_t - \langle M_., N_. \rangle_t &= \int_0^t \int_E K_s(e) J_s(e) \left(N(\mathrm{d}s, \mathrm{d}e) - \nu(\mathrm{d}e)\mathrm{d}s \right) \\ &= \int_0^t \int_E K_s(e) J_s(e) \widetilde{N}(\mathrm{d}s, \mathrm{d}e) \end{aligned}$$

is a martingale.

Proof. See [48].

Remark 1.2.1

Let $K_.(\cdot)$ be a predictable process such that

$$\mathbb{E} \int_0^T \int_E |K_s(e)| \, \nu(\mathrm{d}e) \mathrm{d}s < \infty.$$

We may define $\int_0^t \int_E K_s(e) N(\mathrm{d}s, \mathrm{d}e)$ and $\int_0^t \int_E K_s(e) \nu(\mathrm{d}e) \mathrm{d}s$ as integrable processes of finite variation. Further,

$$\int_0^t \int_E K_s(e) \widetilde{N}(\mathrm{d}s, \mathrm{d}e) = \int_0^t \int_E K_s(e) \left(N(\mathrm{d}s, \mathrm{d}e) - \nu(\mathrm{d}e)\mathrm{d}s \right)$$

is a martingale.

In what follows, we state a version of useful Itô's formula that was proven in Protter [64] or Applebaum [5].

Proposition 1.2.2

Let $Y_t = (Y_t^1, \ldots, Y_t^q)$ be a q-dimensional semi-martingale and $F(y_t^1, \ldots, y_t^q)$ be a real-valued $\mathcal{C}^2$ function on $\mathbb{R}^q$. Then, we have

$$F(Y_t) = F(Y_0) + \sum_{i=1}^{q} \int_0^t \frac{\partial F}{\partial y_i}(Y_{s-}) \mathrm{d}Y_s^i + \frac{1}{2} \sum_{i,j=1}^{q} \int_0^t \frac{\partial^2 F}{\partial y_i \partial y_j}(Y_{s-}) \mathrm{d}[Y_\cdot^i, Y_\cdot^j]_s^c \tag{1.2}$$
$$+ \sum_{0<s\leq t} \left(F(Y_s) - F(Y_{s-}) - \sum_{i=1}^{q} \frac{\partial F}{\partial y_i}(Y_{s-}) \Delta Y_s^i\right),$$

where $[Y_\cdot^i, Y_\cdot^j]^c$ denotes the continuous part of $[Y_\cdot^i, Y_\cdot^j]$. In particular, in the proofs, we shall make use of the following two forms of the Itô's formula (1.2). For $F(t,y) = e^{\alpha t} |y|^2$, we have

$$e^{\alpha T} |Y_T|^2 = e^{\alpha t} |Y_t|^2 + \alpha \int_t^T e^{\alpha s} |Y_s|^2 \mathrm{d}s + \sum_{i=1}^{q} \int_t^T 2e^{\alpha s} Y_{s-}^i \mathrm{d}Y_s^i + \sum_{i=1}^{q} \int_t^T e^{\alpha s} \mathrm{d}[Y^i]_s. \tag{1.3}$$

Now, take $\alpha = 0$ in (1.3), we obtain

$$|Y_t|^2 = |Y_0|^2 + \int_0^t 2Y_{s-} \mathrm{d}Y_s + \sum_{i=1}^{q} \int_0^t \mathrm{d}[Y^i]_s, \tag{1.4}$$

which is equivalent to

$$\sum_{i=1}^{q} |Y_t^i|^2 = \sum_{i=1}^{q} |Y_0^i|^2 + \sum_{i=1}^{q} \int_0^t 2Y_{s-}^i \mathrm{d}Y_s^i + \sum_{i=1}^{q} \int_0^t \mathrm{d}[Y^i]_s.$$

Lemma 1.2.1

(Burkholder–Davis–Gundy Inequalities [63, Theorem 48, p.193)] Let $M.$ be a martingale with càdlàg paths and let $p \geq 1$ be fixed. Let $M_t^* = \sup_{0\leq s\leq t} |M_s|$. Then, there exist constants c_p and C_p such that

$$c_p \left(\mathbb{E}\left[[M,M]_t^{\frac{p}{2}}\right]\right)^{\frac{1}{p}} \leq \left(\mathbb{E}\left[(M_t^*)^p\right]\right)^{\frac{1}{p}} \leq C_p \left(\mathbb{E}\left[[M,M]_t^{\frac{p}{2}}\right]\right)^{\frac{1}{p}},$$

for all $0 \leq t < \infty$. The constants c_p and C_p are universal: they do not depend on the choice of $M.$.

We state now the definition of the solution of multidimensional BSDEJs with Lipschitz generators and prove some technical lemmas that will be used in the proofs of our results. Let us consider the BSDEJ on $\mathbb{R}^q$, for all $0 \leq s \leq T$

$$Y_s = \xi + \int_s^T f(r, Y_r, Z_r, K_r(\cdot))\mathrm{d}r - \int_s^T Z_r \mathrm{d}W_r - \int_s^T \int_E K_r(e)\widetilde{N}(\mathrm{d}r, \mathrm{d}e). \tag{1.5}$$

In component-wise for any $i = 1, \ldots, q$

$$dY_r^i = -f_i(r, Y_r, Z_r, K_r(\cdot))\mathrm{d}r + \sum_{j=1}^{d} Z_r^{ij}\mathrm{d}W_r^j + \int_E K_r^i(e)\widetilde{N}(\mathrm{d}r, \mathrm{d}e),$$

where $Y_T = \xi$ is an $\mathbb{R}^q$–valued terminal value and the generator f is defined as

$$f : [0, T] \times \Omega \times \mathbb{R}^q \times \mathbb{R}^{q\times d} \times \mathcal{L}_\nu^{2,q} \longrightarrow \mathbb{R}^q,$$

such that the following two hypotheses are investigated:

($\mathbf{H}_{3.1}$) ξ and $\{f(t, 0, 0, 0)\}_{0\leq t\leq T}$ are square-integrable in the sense:

$$\mathbb{E}\left[|\xi|^2 + \int_0^T |f(t, 0, 0, 0)|^2 \,\mathrm{d}t\right] < +\infty.$$

($\mathbf{H}_{3.2}$) There exists $L \geq 0$, such that $\mathbb{P}$-a.s. for all $r \in [0, T]$, $\forall\,(y, y') \in \mathbb{R}^q$, $\forall\,(z, z') \in \mathbb{R}^{q\times d}$ and $(k(\cdot), k'(\cdot)) \in \mathcal{L}_\nu^{2,q}$:

$$|f(r, y, z, k(\cdot)) - f(r, y', z', k'(\cdot))| \leq L(|y - y'| + |z - z'| + \|(k - k')\,(\cdot)\|_{q,\nu}).$$

Definition 1.2.1

The triple $(Y_\cdot, Z_\cdot, K_\cdot(\cdot))$ is solution of equation (1.5) which belongs to $\mathbb{M}_\mathcal{S}^2$.

Lemma 1.2.2

For $Z \in \mathcal{M}_\mathcal{F}^2(0, T, \mathbb{R}^{q\times d})$ and $K_\cdot(\cdot) \in \mathcal{M}_\mathcal{F}^2([0, T] \times E, \mathbb{R}^q, \mathrm{d}t\nu(\mathrm{d}e))$, we set $M_t^c = \int_0^t Z_r \mathrm{d}W_r$ and $M_t^d = \int_0^t \int_E K_r(e)\widetilde{N}(\mathrm{d}r, \mathrm{d}e)$. Let $(Y_\cdot, Z_\cdot, K_\cdot(\cdot))$ solution of the following equation

$$dY_r = -f(r, Y_r, Z_r, K_r(\cdot))\mathrm{d}r + \mathrm{d}M_r^c + \mathrm{d}M_r^d \text{ and } Y_T = \xi.$$

Then, under ($\mathbf{H}_{3.1}$) there exists a constant $C_L > 0$ such that for

$$\chi = |Y_0| + \int_0^T \left(|f(r, 0, 0, 0)| + L\,|Z_r| + L\,\|K_r(\cdot)\|_{q,\nu}\right)\mathrm{d}r + \sup_{0\leq t\leq T}|M_t^c| + \sup_{0\leq t\leq T}|M_t^d|,$$

we have

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|Y_t|^2\right] \leq C_L e^{2LT}\mathbb{E}\left[\chi^2\right].$$

Proof. We can write for any solution

$$Y_t = Y_0 - \int_0^t f(r, Y_r, Z_r, K_r(\cdot))\mathrm{d}r + M_t^c + M_t^d.$$

Using the L–Lipschitz assumption on f,

$$\begin{aligned} |Y_t| \leq |Y_0| + L\int_0^t |Y_r|\,\mathrm{d}r + |M_t^c| + |M_t^d| \\ + \int_0^t \left(|f(r,0,0,0)| + L\,|Z_r| + L\,\|K_r(\cdot)\|_{q,\nu}\right)\mathrm{d}r. \end{aligned}$$

Therefore, taking the supremum over $[0, T]$, we get

$$|Y_t| \leq \chi + L\int_0^T |Y_r|\,\mathrm{d}r.$$

Gronwall's Lemma yields that

$$\sup_{0\leq t\leq T} |Y_t| \leq \chi e^{LT}.$$

By assumptions in Lemma 1.2.2 one can check using Lemma 1.2.1 that $\chi \in L^2(\Omega, \mathcal{F}_T, \mathbb{P})$, this leads to the result. □

Remark 1.2.2

The above estimate remains true if f has a linear growth in $y, z, k(\cdot)$

$$|f(r, y, z, k(\cdot))| \leq \varphi_r + L(|y| + |z| + \|k(\cdot)\|_{q,\nu}),$$

where $\varphi_.$ is an $(\mathcal{F}_t)$–adapted process belonging to $L^2([0, T] \times \Omega, \mathrm{d}t \otimes \mathbb{P})$.

Lemma 1.2.3

Let us denote by $|Z_r|^2 = \sum_{i=1}^q \sum_{j=1}^d (Z_r^{ij})^2$. If $Y \in \mathcal{S}^2_{\mathcal{F}}(0, T; \mathbb{R}^q)$ and $Z \in \mathcal{M}^2_{\mathcal{F}}(0, T, \mathbb{R}^{q\times d})$, then

$$M_t = \sum_{i=1}^q \int_0^t 2Y_{r-}^i \sum_{j=1}^d Z_r^{ij}\mathrm{d}W_r^j = \sum_{j=1}^d \int_0^t 2(\sum_{i=1}^q Y_{r-}^i Z_r^{ij})\mathrm{d}W_r^j$$

is a uniformly integrable martingale and for $\varepsilon > 0$,

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |M_t|\right] \leq 3\varepsilon\mathbb{E}\left[\sup_{0\leq t\leq T} |Y_t|^2\right] + \frac{3q}{\varepsilon}\int_0^T \mathbb{E}\left[|Z_r|^2\right]\mathrm{d}r.$$

Proof. We know that

$$[M]_T = \sum_{j=1}^{d} 4\int_0^T (\sum_{i=1}^{q} Y_{r-}^i Z_r^{ij})^2 \mathrm{d}r = \sum_{j=1}^{d} 4\int_0^T (\sum_{i=1}^{q} Y_r^i Z_r^{ij})^2 \mathrm{d}r.$$

By BDG inequality (Lemma (1.2.1)), we have

$$\begin{aligned}\mathbb{E}\left[\sup_{0\le t\le T} |M_t|\right] &\le 3\mathbb{E}\left[\sqrt{[M]_T}\right] \le 6\mathbb{E}\left[\sqrt{\sum_{j=1}^{d}\int_0^T (\sum_{i=1}^{q} Y_r^i Z_r^{ij})^2 \mathrm{d}r}\right] \\ &\le 6\mathbb{E}\left[\max_{1\le i\le q}\sup_{0\le t\le T} |Y_t^i| \sqrt{\sum_{j=1}^{d}\int_0^T (\sum_{i=1}^{q} Z_r^{ij})^2 \mathrm{d}r}\right].\end{aligned}$$

Use the inequality $2ab \le \varepsilon a^2 + \frac{b^2}{\varepsilon}$ to get

$$\mathbb{E}\left[\sup_{0\le t\le T} |M_t|\right] \le 3\varepsilon\mathbb{E}\left[\max_{1\le i\le q}\sup_{0\le t\le T} |Y_t^i|^2\right] + \frac{3}{\varepsilon}\sum_{j=1}^{d}\int_0^T \mathbb{E}\left[(\sum_{i=1}^{q} Z_r^{ij})^2\right] \mathrm{d}r,$$

by convex inequality, we have

$$(\sum_{i=1}^{q} Z_r^{ij})^2 = q^2(\sum_{i=1}^{q} \frac{1}{q} Z_r^{ij})^2 \le q^2 \sum_{i=1}^{q} \frac{1}{q}(Z_r^{ij})^2 = q\sum_{i=1}^{q} (Z_r^{ij})^2. \tag{1.6}$$

Hence,

$$\begin{aligned}\sum_{j=1}^{d}\int_0^T \mathbb{E}\left[(\sum_{i=1}^{q} Z_r^{ij})^2\right] \mathrm{d}r &\le q\sum_{j=1}^{d}\sum_{i=1}^{q}\int_0^T \mathbb{E}\left[(Z_r^{ij})^2\right] \mathrm{d}r \\ &= q\int_0^T \mathbb{E}\left[|Z_r|^2\right] \mathrm{d}r.\end{aligned}$$

Finally,

$$\mathbb{E}\left[\sup_{0\le t\le T} |M_t|\right] \le 3\varepsilon\mathbb{E}\left[\sup_{0\le t\le T} |Y_t|^2\right] + \frac{3q}{\varepsilon}\int_0^T \mathbb{E}\left[|Z_r|^2\right] \mathrm{d}r,$$

which end the proof. □

Lemma 1.2.4

If $Y_{\cdot} \in \mathcal{S}_{\mathcal{F}}^2(0,T;\mathbb{R}^q)$ and $K_{\cdot}(\cdot) \in \mathcal{M}_{\mathcal{F}}^2([0,T]\times E, \mathbb{R}^q, \mathrm{d}t\nu(\mathrm{d}e))$, then

$$N_t = \int_0^t \int_E 2\left(\sum_{i=1}^{q} Y_{r-}^i K_r^i(e)\right) \widetilde{N}(\mathrm{d}r, \mathrm{d}e)$$

is a uniformly integrable real–valued martingale and for $\varepsilon > 0$,

$$\mathbb{E}\left[\sup_{0\le t\le T} |N_t|\right] \le 3\varepsilon\mathbb{E}\left[\sup_{0\le t\le T} |Y_t|^2\right] + \frac{3q}{\varepsilon}\int_0^T \mathbb{E}\left\|K_r(\cdot)\right\|_{q,\nu}^2 \mathrm{d}r.$$

Proof. We know that

$$\begin{aligned}[N]_t &= 4\int_0^t \int_E (\sum_{i=1}^q Y_{r-}^i K_r^i(e))^2 N(\mathrm{d}r, \mathrm{d}e) \\ &= 4\int_0^t \int_E (\sum_{i=1}^q Y_{r-}^i K_r^i(e))^2 \nu(\mathrm{d}e)\mathrm{d}r + P_t,\end{aligned}$$

where $(P_t)_{0\leq t\leq T}$ is a real–valued martingale given by

$$P_t = 4\int_0^t \int_E (\sum_{i=1}^q Y_{r-}^i K_r^i)^2 \widetilde{N}(\mathrm{d}r, \mathrm{d}e).$$

Thanks to BDG inequality,

$$\begin{aligned}\mathbb{E}\left[\sup_{0\leq t\leq T}|N_t|\right] &\leq 3\mathbb{E}\left[\sqrt{[N]_T}\right] \leq 6\mathbb{E}\left[\sqrt{\int_0^T (\sum_{i=1}^q Y_{r-}^i K_r^i(e))^2 \mathrm{d}r + P_T}\right] \\ &\leq 6\mathbb{E}\left[\max_{1\leq i\leq q}\sup_{0\leq t\leq T}|Y_t^i|\sqrt{\int_0^T \int_E (\sum_{i=1}^q K_r^i(e))^2 \nu(\mathrm{d}e)\mathrm{d}r + P_T}\right].\end{aligned}$$

Again using the inequality $2ab \leq \varepsilon a^2 + \frac{b^2}{\varepsilon}$ to get

$$\begin{aligned}\mathbb{E}\left[\sup_{0\leq t\leq T}|N_t|\right] \leq 3\varepsilon\mathbb{E}\left[\sup_{0\leq t\leq T}|Y_t|^2\right] + \frac{3}{\varepsilon}\mathbb{E}\left[P_T\right] \\ + \frac{3}{\varepsilon}\int_0^T \mathbb{E}\left[\int_E (\sum_{i=1}^q K_r^i(e))^2 \nu(\mathrm{d}e)\right]\mathrm{d}r.\end{aligned}$$

Similarly to (1.6), we obtain

$$(\sum_{i=1}^q K_r^i(e))^2 = q^2(\sum_{i=1}^q \frac{1}{q} K_r^i(e))^2 \leq q^2 \sum_{i=1}^q \frac{1}{q}(K_r^i(e))^2 = q\sum_{i=1}^q (K_r^i(e))^2.$$

Finally,

$$\mathbb{E}\left[\sup_{0\leq t\leq T}|N_t|\right] \leq 3\varepsilon\mathbb{E}\left[\sup_{0\leq t\leq T}|Y_t|^2\right] + \frac{3q}{\varepsilon}\int_0^T \mathbb{E}\left\|K_r(\cdot)\right\|_{q,\nu}^2 \mathrm{d}r,$$

which implies the desired result. □

1.3 Existence and Uniqueness of Adapted Solutions

Starting with this section, we derive the so-called a priori estimates, which are crucial in the study of multidimensional BSDEJs. Then, we study the existence and uniqueness result for multidimensional BSDEs with jumps.

1.3.1 A Priori Estimates

Lemma 1.3.1

Let $(Y_\cdot, Z_\cdot, K_\cdot(\cdot))$ be a solution to equation (1.5) then, for any $\kappa > 0$ and $\alpha > 0$

$$\mathbb{E}\left[\sup_{0\le t\le T} e^{\alpha t}\left|Y_t\right|^2 + \int_0^T e^{\alpha s}\left|Z_s\right|^2 \mathrm{d}s + \int_0^T e^{\alpha s}\left\|K_s(\cdot)\right\|_{q,\nu}^2 \mathrm{d}s\right]$$
$$\le (2+576q)\,\mathbb{E}\left[e^{\alpha T}\left|\xi\right|^2 + \int_0^T e^{\alpha t}\left|f(t,0,0,0)\right|^2 \mathrm{d}t\right],$$

provided that $\alpha \ge \frac{1}{\kappa} + 2L + 4L^2$, L is the Lipschitz constant of f appearing in assumption $(\mathbf{H}_{3.2})$.

Proof: The Itô's formula (1.3) can be written for this particular process $Y_\cdot$ satisfying (1.5), as

$$e^{\alpha t}\left|Y_t\right|^2 + \alpha\int_t^T e^{\alpha s}\left|Y_s\right|^2 \mathrm{d}s + \int_t^T e^{\alpha s}\left|Z_s\right|^2 \mathrm{d}s + \int_t^T e^{\alpha s}\left\|K_s(\cdot)\right\|_{q,\nu}^2 \mathrm{d}s \tag{1.7}$$
$$= e^{\alpha T}\left|\xi\right|^2 + \sum_{i=1}^{q}\int_t^T 2e^{\alpha s}Y_{s-}^i f_i(s,Y_s,Z_s,K_s(\cdot))\mathrm{d}s - \left(M_T^\alpha - M_t^\alpha\right) - \left(N_T^\alpha - N_t^\alpha\right),$$

where

$$M_t^\alpha = \sum_{i=1}^{q}\int_0^t 2e^{\alpha s}Y_{s-}^i\sum_{j=1}^{d} Z_s^{ij}\mathrm{d}W_s^j + \sum_{i=1}^{q}\int_0^t\int_E 2e^{\alpha s}Y_{s-}^i K_s^i(e)\widetilde{N}(\mathrm{d}s,\mathrm{d}e), \tag{1.8}$$

and

$$N_t^\alpha = \int_0^t\int_E e^{\alpha s}\left|K_s(e)\right|^2\widetilde{N}(\mathrm{d}s,\mathrm{d}e) = \int_0^t\int_E e^{\alpha s}\left|K_s(e)\right|^2\left(N(\mathrm{d}s,\mathrm{d}e) - \nu(\mathrm{d}e)\mathrm{d}s\right) \tag{1.9}$$

are real–valued martingales thanks to Remark 1.2.1. From the Lipschitz property of f and the Young inequality $2ab \le \kappa\left|a\right|^2 + \frac{1}{\kappa}\left|b\right|^2$,

$$\sum_{i=1}^{q} 2y_i f_i(s,y,z,k(\cdot)) = 2y\cdot f(s,y,z,k(\cdot)) \le 2\left|y\right|\left|f(s,y,z,k(\cdot))\right|$$
$$\le 2\left|y\right|\left|f(s,0,0,0)\right| + 2L\left|y\right|^2 + 2L\left|y\right|\left|z\right| + 2L\left|y\right|\left\|k(\cdot)\right\|_{q,\nu}$$
$$\le \kappa\left|f(s,0,0,0)\right|^2 + \left(\frac{1}{\kappa} + 2L + 4L^2\right)\left|y\right|^2 + \frac{1}{2}\left|z\right|^2 + \frac{1}{2}\left\|k(\cdot)\right\|_{q,\nu}^2.$$

Set $\zeta = e^{\alpha T}\left|\xi\right|^2 + \kappa\int_0^T e^{\alpha s}\left|f(s,0,0,0)\right|^2 \mathrm{d}s$, if $\alpha \ge \frac{1}{\kappa} + 2L + 4L^2$, then for all $0 \le t \le T$,

$$e^{\alpha t}\left|Y_t\right|^2 + \frac{1}{2}\int_t^T e^{\alpha s}\left|Z_s\right|^2 \mathrm{d}s + \frac{1}{2}\int_t^T e^{\alpha s}\left\|K_s(\cdot)\right\|_{q,\nu}^2 \mathrm{d}s \le \zeta - \left(M_T^\alpha - M_t^\alpha\right) - \left(N_T^\alpha - N_t^\alpha\right). \tag{1.10}$$

Taking the conditional expectation of inequality (1.10), we find

$$\mathbb{E}\left[e^{\alpha t}\left|Y_t\right|^2\right]+\frac{1}{2}\mathbb{E}\left[\int_t^T e^{\alpha s}\left|Z_s\right|^2\mathrm{d}s\mid\mathcal{F}_t\right]+\frac{1}{2}\mathbb{E}\left[\int_t^T e^{\alpha s}\left\|K_s(\cdot)\right\|_{q,\nu}^2\mathrm{d}s\mid\mathcal{F}_t\right]\leq\mathbb{E}\left[\zeta\mid\mathcal{F}_t\right]. \tag{1.11}$$

Now, we take the expectation in (1.11), hence for $t=0$,

$$\int_0^T e^{\alpha s}\mathbb{E}\left[\left|Z_s\right|^2\right]\mathrm{d}s+\int_0^T e^{\alpha s}\mathbb{E}\left[\left\|K_s(\cdot)\right\|_{q,\nu}^2\right]\mathrm{d}s\leq 2\mathbb{E}\left[\zeta\right]. \tag{1.12}$$

Taking the supremum over $[0,T]$ in (1.10), we obtain

$$\sup_{0\leq t\leq T} e^{\alpha t}\left|Y_t\right|^2\leq\zeta+2\sup_{0\leq t\leq T}\left|M_t^\alpha\right|+2\sup_{0\leq t\leq T}\left|N_t^\alpha\right|.$$

Therefore,

$$\mathbb{E}\left[\sup_{0\leq t\leq T} e^{\alpha t}\left|Y_t\right|^2\right]\leq\mathbb{E}\left[\zeta\right]+2\mathbb{E}\left[\sup_{0\leq t\leq T}\left|M_t^\alpha\right|\right]+2\mathbb{E}\left[\sup_{0\leq t\leq T}\left|N_t^\alpha\right|\right].$$

Thanks to Lemmas 1.2.3 and 1.2.4, corresponding to $\alpha=0$, we get with similar estimations

$$\begin{aligned}\mathbb{E}\left[\sup_{0\leq t\leq T} e^{\alpha t}\left|Y_t\right|^2\right]&\leq\mathbb{E}\left[\zeta\right]+6\varepsilon\mathbb{E}\left[\sup_{0\leq t\leq T} e^{\alpha t}\left|Y_t\right|^2\right]+\frac{6q}{\varepsilon}\left[\int_0^T e^{\alpha s}\mathbb{E}\left[\left|Z_s\right|^2\right]\mathrm{d}s\right]\\&\quad+6\varepsilon\mathbb{E}\left[\sup_{0\leq t\leq T} e^{\alpha t}\left|Y_t\right|^2\right]+\frac{6q}{\varepsilon}\int_0^T e^{\alpha s}\mathbb{E}\left[\left\|K_s(\cdot)\right\|_{q,\nu}^2\right]\mathrm{d}s.\end{aligned}$$

Choosing $\varepsilon=\frac{1}{24}$ leads to:

$$\frac{1}{2}\mathbb{E}\left[\sup_{0\leq t\leq T} e^{\alpha t}\left|Y_t\right|^2\right]\leq\mathbb{E}\left[\zeta\right]+144q\left[\int_0^T e^{\alpha s}\mathbb{E}\left[\left|Z_s\right|^2\right]\mathrm{d}s\right]+144q\int_0^T e^{\alpha s}\mathbb{E}\left[\left\|K_s(\cdot)\right\|_{q,\nu}^2\right]\mathrm{d}s.$$

Finally, taking into account the estimation (1.12)

$$\mathbb{E}\left[\sup_{0\leq t\leq T} e^{\alpha t}\left|Y_t\right|^2+\int_0^T e^{\alpha s}\left|Z_s\right|^2\mathrm{d}s+\int_0^T e^{\alpha s}\left\|K_s(\cdot)\right\|_{q,\nu}^2\mathrm{d}s\right]\leq(2+576q)\,\mathbb{E}\left[\zeta\right],$$

this archives the proof. □

We are now in a position to state our main result of this section.

Theorem 1.3.1

Under the assumptions $(\mathbf{H}_{3.1})$ *and* $(\mathbf{H}_{3.2})$ *the equation (1.5) has a unique solution in* $\mathbb{M}^2_{\mathcal{S}}$.

Proof: Let $(Y_\cdot, Z_\cdot, K_\cdot(\cdot))$ and $(\widetilde{Y}_\cdot, \widetilde{Z}_\cdot, \widetilde{K}_\cdot(\cdot))$ be solutions of equation (1.5) respectively with inputs (ξ, f) and $(\widetilde{\xi}, \widetilde{f})$. Apply Lemma 1.3.1 to $(\xi - \widetilde{\xi})$ and $(f - \widetilde{f})$ and $(Y_\cdot - \widetilde{Y}_\cdot, Z_\cdot - \widetilde{Z}_\cdot, (K_\cdot - \widetilde{K}_\cdot)(\cdot))$ to get

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\leq t\leq T} e^{\alpha t}|Y_t - \widetilde{Y}_t|^2\right] + \int_0^T e^{\alpha s}\mathbb{E}\left|Z_s - \widetilde{Z}_s\right|^2 \mathrm{d}s + \int_0^T e^{\alpha s}\mathbb{E}\|(K_s - \widetilde{K}_s)(\cdot)\|^2_{q,\nu}\mathrm{d}s \\
&\leq (2 + 576q)\,\mathbb{E}\left[e^{\alpha T}\left|\xi - \widetilde{\xi}\right|^2 + \int_0^T e^{\alpha t}\left|\left(f - \widetilde{f}\right)(t, 0, 0, 0)\right|^2 \mathrm{d}t\right].
\end{aligned}$$

The uniqueness follows directly for the same data.

The existence result is divided into two steps and makes use of the following version of Itô's formula. To simplify the notations we shall write $f_i(r) = f_i(r, Y_r, Z_r, K_r(\cdot))$.

Step 1: For a given $\mathbb{R}^q$–valued square integrable random variable ξ and a drift $f(r)$ independent of y, z and $k(\cdot)$ our BSDEJ becomes

$$dY_r = -f(r)\mathrm{d}r + Z_r\mathrm{d}W_r + \int_E K_r(e)\widetilde{N}(\mathrm{d}r, \mathrm{d}e),\ Y_T = \xi, \tag{1.13}$$

or equivalently for any $i = 1, \ldots, q$

$$dY_r^i = -f_i(r)\mathrm{d}r + \sum_{j=1}^d Z_r^{ij}\mathrm{d}W_r^j + \int_E K_r^i(e)\widetilde{N}(\mathrm{d}r, \mathrm{d}e),\quad Y_T^i = \xi^i, \tag{1.14}$$

or in integral form

$$Y_t^i = \xi^i + \int_t^T f_i(r)\mathrm{d}r - \sum_{j=1}^d \int_t^T Z_r^{ij}\mathrm{d}W_r^j - \int_t^T \int_E K_r^i(e)\widetilde{N}(\mathrm{d}r, \mathrm{d}e).$$

Consider the $\mathbb{R}^q$–valued square integrable martingale

$$M_t = \mathbb{E}\left[\xi + \int_0^T f(r)\mathrm{d}r \mid \mathcal{F}_t\right],$$

by the Proposition 1.2.1 there exist unique predictable processes $Z_\cdot$ and $K_\cdot(\cdot)$ taking values respectively in $\mathbb{R}^{q\times d}$ and $\mathcal{L}_\nu^{2,q}$ such that

$$\int_0^T \mathbb{E}\left[|Z_s|^2\right]\mathrm{d}s < \infty \quad \text{and} \quad \int_0^T \mathbb{E}\left[\|K_s(\cdot)\|^2_{q,\nu}\right]\mathrm{d}s < \infty,$$

and

$$M_t = M_0 + \int_0^t Z_s\mathrm{d}W_s + \int_0^t \int_E K_s(e)\widetilde{N}(\mathrm{d}s, \mathrm{d}e) = \mathbb{E}\left[\xi + \int_t^T f(s)\mathrm{d}s \mid \mathcal{F}_t\right] + \int_0^t f(s)\mathrm{d}s.$$

Therefore,

$$M_T - M_t = \int_t^T Z_s\mathrm{d}W_s + \int_t^T \int_E K_s(e)\widetilde{N}(\mathrm{d}s, \mathrm{d}e).$$

On the other hand,

$$\begin{aligned} M_T - M_t &= \mathbb{E}\left[\xi + \int_0^T f(s)\mathrm{d}s \mid \mathcal{F}_T\right] - \mathbb{E}\left[\xi + \int_t^T f(s)\mathrm{d}s \mid \mathcal{F}_t\right] - \int_0^t f(s)\mathrm{d}s \\ &= \xi + \int_t^T f(s)\mathrm{d}s - \mathbb{E}\left[\xi + \int_t^T f(s)\mathrm{d}s \mid \mathcal{F}_t\right]. \end{aligned}$$

Then, the equation becomes

$$\mathbb{E}\left[\xi + \int_t^T f(s)\mathrm{d}s \mid \mathcal{F}_t\right] = \xi + \int_t^T f(s)\mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s - \int_t^T \int_E K_s(e)\widetilde{N}(\mathrm{d}s, \mathrm{d}e).$$

So, $Y_t = \mathbb{E}[\xi + \int_t^T f(s)\mathrm{d}s \mid \mathcal{F}_t]$.

Step 2: Now, let us define $(Y^n, Z^n, K^n(\cdot))$ by the classical Picard's iteration scheme:

$Y^0 = Z^0 = K^0 = 0$ and $(Y^{n+1}, Z^{n+1}, K^{n+1}(\cdot))$ is the unique solution to the BSDEJ:

$$Y_t^{n+1} = \xi + \int_t^T f(s, Y_{s-}^n, Z_s^n, K_s^n(\cdot))\mathrm{d}s - \int_t^T Z_s^{n+1}\mathrm{d}W_s - \int_t^T \int_E K_s^{n+1}(e)\widetilde{N}(\mathrm{d}s, \mathrm{d}e).$$

We shall prove that $(Y^n, Z^n, K^n(\cdot))$ is a Cauchy sequence in the Banach space $\mathbb{M}^2_{\mathcal{S}}$.

To simplify the notations, we put:

$$\bar{Y}_s^{n,m} := Y_s^n - Y_s^m, \quad \bar{Z}_s^{n,m} := Z_s^n - Z_s^m \quad \text{and} \quad \bar{K}_s^{n,m}(\cdot) := K_s^n(\cdot) - K_s^m(\cdot),$$

and

$$\bar{f}^{n,m}(s) := f(s, Y_{s-}^n, Z_s^n, K_s^n(\cdot)) - f(s, Y_{s-}^m, Z_s^m, K_s^m(\cdot)).$$

Itô's formula (1.7), shows that for every $n < m$,

$$\begin{aligned} &e^{\alpha t}|\bar{Y}_t^{n+1,m+1}|^2 + \int_t^T e^{\alpha s}|\bar{Z}_s^{n+1,m+1}|^2\mathrm{d}s \\ &+ \int_t^T e^{\alpha s}\|\bar{K}_s^{n+1,m+1}(\cdot)\|^2_{q,\nu}\mathrm{d}s + \alpha\int_t^T e^{\alpha s}|\bar{Y}_s^{n+1,m+1}|^2\mathrm{d}s \\ &= 2\int_t^T e^{\alpha s}\bar{Y}_{s-}^{n+1,m+1}\bar{f}^{n,m}(s)\mathrm{d}s - \left(M_T^{n+1,m+1} - M_t^{n+1,m+1}\right) - \left(N_T^{n+1,m+1} - N_t^{n+1,m+1}\right), \end{aligned}$$

where the processes $M_t^{n+1,m+1}$ and $N_t^{n+1,m+1}$ are defined similarly as in (1.8) and (1.9) by

$$M_t^{n+1,m+1} = 2\int_0^t e^{\alpha s}\bar{Y}_{s-}^{n+1,m+1}\bar{Z}_s^{n+1,m+1}\mathrm{d}W_s - 2\int_0^t \int_E e^{\alpha s}\bar{Y}_{s-}^{n+1,m+1}\bar{K}_s^{n+1,m+1}(e)\widetilde{N}(\mathrm{d}s, \mathrm{d}e),$$

and

$$N_t^{n+1,m+1} = \int_0^t \int_E e^{\alpha s}|\bar{K}_s^{n+1,m+1}(e)|^2\widetilde{N}(\mathrm{d}s, \mathrm{d}e)$$

are real-valued martingales. Taking the expectation, we get

$$\begin{aligned}
&\mathbb{E}e^{\alpha t}|\bar{Y}_t^{n+1,m+1}|^2+\mathbb{E}\int_t^T e^{\alpha s}|\bar{Z}_s^{n+1,m+1}|^2\mathrm{d}s\\
&+\mathbb{E}\int_t^T e^{\alpha s}\|\bar{K}_s^{n+1,m+1}(\cdot)\|_{q,\nu}^2\mathrm{d}s+\alpha\mathbb{E}\int_t^T e^{\alpha s}|\bar{Y}_{s-}^{n+1,m+1}|^2\mathrm{d}s\\
&=2\mathbb{E}\int_t^T e^{\alpha s}\bar{Y}_{s-}^{n+1,m+1}\bar{f}^{n,m}(s)\mathrm{d}s.
\end{aligned}$$

Since f is L-Lipschitz, we get

$$\begin{aligned}
&\mathbb{E}e^{\alpha t}|\bar{Y}_t^{n+1,m+1}|^2+\mathbb{E}\int_t^T e^{\alpha s}|\bar{Z}_s^{n+1,m+1}|^2\mathrm{d}s\\
&+\mathbb{E}\int_t^T e^{\alpha s}\|\bar{K}_s^{n+1,m+1}(\cdot)\|_{q,\nu}^2\mathrm{d}s+\alpha\mathbb{E}\int_t^T e^{\alpha s}|\bar{Y}_{s-}^{n+1,m+1}|^2\mathrm{d}s\\
&\leq 2L\mathbb{E}\int_t^T e^{\alpha s}|\bar{Y}_{s-}^{n+1,m+1}|\left[|\bar{Y}_{s-}^{n,m}|+|\bar{Z}_s^{n,m}|+\|\bar{K}_s^{n,m}(\cdot)\|_{q,\nu}\right]\mathrm{d}s.
\end{aligned}$$

The inequality $2xy\leq\beta^2x^2+\frac{1}{\beta^2}y^2$ shows that

$$\begin{aligned}
&\mathbb{E}e^{\alpha t}|\bar{Y}_t^{n+1,m+1}|^2+\mathbb{E}\int_t^T e^{\alpha s}|\bar{Z}_s^{n+1,m+1}|^2\mathrm{d}s+\mathbb{E}\int_t^T e^{\alpha s}\|\bar{K}_s^{n+1,m+1}(\cdot)\|_{q,\nu}^2\mathrm{d}s\\
&+(\alpha-L^2\beta^2)\mathbb{E}\int_t^T e^{\alpha s}|\bar{Y}_s^{n+1,m+1}|^2\mathrm{d}s\\
&\leq\frac{3}{\beta^2}\mathbb{E}\int_t^T e^{\alpha s}\left(|\bar{Y}_s^{n,m}|^2+|\bar{Z}_s^{n,m}|^2+\|\bar{K}_s^{n,m}(\cdot)\|_{q,\nu}^2\right)\mathrm{d}s.
\end{aligned}$$

Choosing β and α such that $\frac{3}{\beta^2}=\frac{1}{2}$ and $\alpha-6L^2=1$, we get

$$\begin{aligned}
&\mathbb{E}e^{\alpha t}|\bar{Y}_t^{n+1,m+1}|^2+\mathbb{E}\int_t^T e^{\alpha s}|\bar{Z}_s^{n+1,m+1}|^2\mathrm{d}s+\mathbb{E}\int_t^T e^{\alpha s}\|\bar{K}_s^{n+1,m+1}(\cdot)\|_{q,\nu}^2\mathrm{d}s\\
&\leq\frac{1}{2}\mathbb{E}\int_t^T e^{\alpha s}\left(|\bar{Y}_s^{n,m}|^2+|\bar{Z}_s^{n,m}|^2+\|\bar{K}_s^{n,m}(\cdot)\|_{q,\nu}^2\right)\mathrm{d}s.
\end{aligned}$$

It follows immediately, for all $m>n$, that

$$\mathbb{E}\int_0^T e^{\alpha s}|\bar{Y}_s^{n,m}|^2\mathrm{d}s+\mathbb{E}\int_0^T e^{\alpha s}|\bar{Z}_s^{n,m}|^2\mathrm{d}s+\mathbb{E}\int_0^T e^{\alpha s}\|\bar{K}_s^{n,m}(\cdot)\|_{q,\nu}^2\mathrm{d}s\leq\frac{C}{2^n}.$$

Using again Itô's formula, BDG inequality and Gronwall's Lemma, it is follows that there exists a universal constant C such that

$$\mathbb{E}[\sup_{0\leq s\leq T}e^{\alpha s}|\bar{Y}_s^{n,m}|^2]+\mathbb{E}\int_0^T e^{\alpha s}|\bar{Z}_s^{n,m}|^2\mathrm{d}s+\mathbb{E}\int_0^T e^{\alpha s}\|\bar{K}_s^{n,m}(\cdot)\|_{q,\nu}^2\mathrm{d}s\leq\frac{C}{2^n}.$$

Consequently, $(Y_\cdot^n,Z_\cdot^n,K_\cdot^n(\cdot))$ is a Cauchy sequence in the Banach space $\mathbb{M}_{\mathcal{S}}^2$, and

$$(Y_\cdot,Z_\cdot,K_\cdot(\cdot))=\lim_{n\longrightarrow\infty}(Y_\cdot^n,Z_\cdot^n,K_\cdot^n(\cdot))$$

solves our BSDEJ. □

1.4 Deterministic Representation for Markovian BSDEJ

The aim of this section is to investigate a deterministic representation theorem for the solution of the equation (0.1). We first recall some existing results in the literature that study regularity and representation of the viscosity solution of partial differential equations via the solution of forward-backward stochastic differential equation driven by continuous Brownian motion. The first result that went in this direction was established by Pardoux & Peng in [59] which claims the backward components Y can be determined in terms of the forward component X, when the coefficients satisfy the Lipschitz continuity condition. Then, they proved under more strong smoothness conditions on the coefficients (e.g. of class $\mathcal{C}^3$ in their spacial variables), that the Brownian component Z has continuous paths. Two years later, under the previous smoothness conditions, Ma *et al.* [50] proved the following explicit representation for all s in $[t,T]$, $Y_s^{t,x} = u(s, X_s^{t,x})$ and $Z_s^{t,x} = \partial_x u(s, X_s^{t,x})\sigma(s, X_s^{t,x})$. Subsequently, this result has been weakened by Ma & Zhang [51] where they relaxed the smoothness condition on the coefficients by assuming that they are only $\mathcal{C}^1$ and the diffusion coefficient of the forward component is uniformly elliptic. Later, N'zi *et al.* [56] studied the regularity of the viscosity solution of a quasi-linear parabolic partial differential equation with merely Lipschitz coefficients. The main results are obtained by using Krylov's inequality in the case, where the diffusion coefficient of the forward equation is uniformly elliptic. In the degenerate case, they exploited the idea used by Bouleau-Hirsch on the absolute continuity of probability density measures. On the other hand, when the Markov process is a solution of some SDE with jumps it is shown in Barles *et al.* [12] that the solution $Y_s^{t,x}$ of a class of BSDE with jumps provides a viscosity solution of PIDE by meaning the deterministic function $u(t,x) = Y_t^{t,x}$ but no representation has been given for the $Z_s^{t,x}$ and $K_s^{t,x}(\cdot)$.

1.4.1 Representation of Additive Functional of Markov Processes

Let $X = (\Omega, \mathcal{F}, \mathcal{F}_t, \theta_t, X_t, \mathbb{P}_x)$ be a right–continuous left–hand limited strong Markov process with an infinite lifetime, with state space $\mathbb{R}^p$. The operators θ_t, $t \geq 0$, are called the shift operators defined by

$$X_s(\theta_t(\omega)) = X_{t+s}(\omega),$$

where as usual X is the coordinate process. Assume further, that X is a right process in the sense of Getoor see [33, (9.7) Terminology p. 55].

Definition 1.4.1

(i) An additive locally square integrable martingale on the space $(X, (\mathcal{F}_t)_{0\leq t\leq T})$ is an $\mathbb{R}^p$–valued process Y that is adapted to $(\mathcal{F}_t)_{0\leq t\leq T}$, is right-continuous, is a locally square integrable local martingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P}_x)$ for every $x \in \mathbb{R}^p$ and is additive w.r.t (θ_t) (vanishing at 0), and for every pair (t, u),

$$Y_{t+u} = Y_t + Y_u \circ \theta_t$$

a.s.

(ii) We say that Y is quasi–left–continuous if $Y_{T_n} \longrightarrow Y_T$ a.s. for every increasing sequence $(T_n)_{n\geq 0}$ of stopping times with finite limit T.

Now, we recall some more facts about semi-martingales which are defined on the probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}_x)$. We consider a q–dimensional semi-martingale $Y = (Y^i)_{1\leq i\leq q}$. We define the q–dimensional process $Y^e_\cdot = ((Y^e_\cdot)^i)_{1\leq i\leq q}$

$$Y^e_t = \sum_{0<s\leq t} \Delta Y_s \mathbb{1}_{\{|\Delta Y_s|\geq 1\}},$$

where $\mathbb{1}_G$ stands for the indicator function of the set G and $\Delta Y_s = Y_s - Y_{s-}$. It is well known that $Y^e_\cdot$ is a right continuous pure jump process having finitely many jumps in any finite interval. Thus, the semi-martingale $Y_\cdot - Y^e_\cdot$ has bounded jumps and can be decomposed uniquely in the following form:

$$Y_t - Y^e_t = Y_0 + Y^b_t + Y^c_t + Y^d_t,$$

where $Y^b_\cdot$ is a predictable process of bounded variation on every finite interval, $Y^c_\cdot$ is a continuous local martingale and $Y^d_\cdot$ is a purely discontinuous local martingale (corresponding to the compensated sum of jumps). Furthermore, $Y^c_0 = Y^d_0 = 0$. The canonical decomposition of the q–dimensional special semi-martingale $Y_\cdot$ is

$$Y_t = Y_0 + Y^b_t + Y^c_t + Y^d_t + Y^e_t. \tag{1.15}$$

The decomposition (1.15) is unique up to a $\mathbb{P}_x$–null set. All the above processes are q–dimensional, for example, the i^{th} component of Y^c_t is simply Y^{ic}_t.
We define the following integer–valued random measure Γ on $\mathbb{R}_+ \times \mathbb{R}^q$ as follows:

$$\Gamma(\omega, \mathrm{d}t, \mathrm{d}y) = \sum_{s>0} 11_{\{\Delta Y_s(\omega) \neq 0\}} \delta_{(s, \Delta Y_s(\omega))}(\mathrm{d}t, \mathrm{d}y),$$

Γ is called the jump measure of Y.
Let $B_t = Y^b_t$ in the decomposition (1.15), $C_t = (C^{ij}_t)_{1\leq i,j\leq q} = (\langle Y^{ic}_\cdot, Y^{jc}_\cdot \rangle_t)_{1\leq i,j\leq q}$ and γ is the dual predictable projection of Γ (called also the compensator). The triplet (B, C, γ) is called the local characteristics of Y which is unique, up to a $\mathbb{P}$–null set. In fact, one can choose a version of (B, C, γ) which satisfies:

a. for all $t \geq s \geq 0$, $C_t - C_s$ is a non–negative symmetric matrix;

b. $\Gamma(\omega, \mathbb{R}_+ \times \{0_{\mathbb{R}^q}\}) = 0$;

c. $\int_{\mathbb{R}^q} (1 \wedge |y|^2) \gamma(\omega, [0, t], \mathrm{d}y) < \infty$ for every $t \geq 0$.

According to [23, Theorem 2.43], a q-dimensional additive semi-martingale has the decomposition (1.15). Moreover, $B_\cdot$ and $C_\cdot$ are $\mathbb{F}$-predictable additive processes and γ is an $\mathbb{F}$-predictable additive random measure.

Lemma 1.4.1 (*[23], Theorem 2.43*)

Let Y be a q–dimensional additive semi-martingale on $\left(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\right)$ which is quasi–left–continuous. Then, there exist:

(i) *an $(\mathcal{F}_t)$–adapted continuous increasing additive functional A;*

(ii) *a $\mathcal{B}(\mathbb{R}^q)$–measurable $\mathbb{R}^q$-valued function $b = (b_1, \ldots, b_q)$;*

(iii) *a $\mathcal{B}(\mathbb{R}^{q\times q})$-measurable $\mathbb{R}^{q\times q}$–valued function lower triangular matrix–valued function $c = (c_{ij})_{1\le i,j\le q}$ of measurable functions such that $c_{ij} = 0$ if $j > i$, or if $c_{jj} = 0$;*

(iv) *a positive kernel $\Theta(x, \mathrm{d}y)$ from $(\mathbb{R}^q, \mathcal{B}(\mathbb{R}^q))$ to $(\mathbb{R}^q, \mathcal{B}(\mathbb{R}^q))$ having $\Theta(x, \{0_{\mathbb{R}^q}\}) = 0$ for all $x \in \mathbb{R}^q$ such that*

$$\int_{\mathbb{R}^q} f(x,y)\Theta(x,\mathrm{d}y) < \infty \text{ for all } x \in E, \tag{1.16}$$

for $(\mathcal{B}(\mathbb{R}^q) \otimes \mathcal{B}(E))$–measurable strictly positive function f.

Such that

$$B_t = \int_0^t b(X_s)\mathrm{d}A_s, \quad C_t = \int_0^t cc^*(X_s)\mathrm{d}A_s \;\; \text{and} \;\; \gamma(\mathrm{d}s,\mathrm{d}y) = \Theta(X_s,\mathrm{d}y)\mathrm{d}A_s,$$

define a version (B, C, γ) of the triplet of local characteristics of Y under every $\mathbb{P}_x$, $x \in \mathbb{R}^p$.

We consider the following assumptions:

$(\mathbf{A_1})$ Let $Y = (Y^i)_{1\le i\le q}$ be a collection of continuous additive local martingales on $\left(\Omega, (\mathcal{F}_t)_{t\ge 0}\right)$ such that $\mathrm{d}\left\langle Y^i_\cdot, Y^i_\cdot\right\rangle_t \ll \mathrm{d}t$ a.s., for all $1 \le i \le q$.

Let $c = (c_{ij})_{1\le i,j\le q}$ be the collection of $\mathcal{B}(\mathbb{R}^{q\times q})$-measurable functions whose existence and properties are given by the Lemma 1.4.1 with $A_t = t$.

$(\mathbf{A_2})$ Let Γ be an additive integer-valued random measure on $\mathbb{R}_+ \times \mathbb{R}^q$ defined over $\left(\Omega, (\mathcal{F}_t)_{t\ge 0}\right)$. Let γ be its dual predictable projection. For each G in $\mathcal{B}(\mathbb{R}^q)$ and $t > 0$ set $\gamma_t^G = \gamma\left([0,t] \times G\right)$. Assume that $\mathrm{d}\gamma_t^G \ll \mathrm{d}t$ *a.s.* such that the mapping $t \longmapsto \gamma_t^G$ is locally integrable. This is equivalent to the existence of a positive kernel $\Theta(x, \mathrm{d}y)$ on $(\mathbb{R}^q, \mathcal{B}(\mathbb{R}^q))$ satisfying (1.16) and $\gamma(\mathrm{d}s, \mathrm{d}y) = \Theta(X_s, \mathrm{d}y)\mathrm{d}s$ a.s.

Lemma 1.4.2

[23, Lemma 3.4 and Theorem 3.7] Under the assumptions $(\mathbf{A_1})$ and $(\mathbf{A_2})$, there exist a Wiener process and a Poisson random measure both still denoted by $W = (W^i)_{1\le i\le q}$

and N on $\mathbb{R}_+ \times \mathbb{R}^q$ with compensator $\mathrm{d}s\nu(\mathrm{d}e)$ (by extending the probability space if necessary by usual product spaces) such that

$$Y_t^i = \sum_{j=1}^{q} \int_0^t c_{ij}(X_s)\mathrm{d}W_s^j \quad \text{for all} \quad i = 1, 2, \dots, q,$$

and

$$\Gamma(G) = \int\int_{\mathbb{R}_+ \times E} \mathbb{1}_G(s, \theta(X_{s-}, e))N(\mathrm{d}s, \mathrm{d}e) \quad \text{for all} \quad G \in \mathcal{B}(\mathbb{R}_+) \otimes \mathcal{B}(\mathbb{R}^q),$$

where θ is a measurable function satisfying

$$\Theta(x, H) = \int_E \mathbb{1}_H(\theta(x, e))\nu(\mathrm{d}e) \quad \text{for all} \quad x \in \mathbb{R}^p \text{ and for all } H \in \mathcal{B}(\mathbb{R}^q).$$

Remark 1.4.1

Suppose $(X_t)_{0 \le t \le T}$ is a semi-martingale Markov process on $\mathbb{R}^p$ that is not time-homogeneous, then the time-homogeneous process $(t, X_t - X_0)$ is an $\mathbb{R}^{p+1}$-valued semi-martingale additive functional. Thus, the measurable functions $b_i(x)$, $c_{ij}(x)$, and $\theta(x, e)$ become $b_i(s, x)$, $c_{ij}(s, x)$, and $\theta(s, x, e)$.

Now, we are interested in a class of multidimensional BSDE with jumps for which the generator f and the random terminal value ξ at time T are both functions of a right process X on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ for $x \in \mathbb{R}^p$. Notice that the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ is generated by the Markov process X and two processes obtained in the Lemma 1.4.2, still denoted W and $\widetilde{N}$. We shall deal with the following Markovian BSDEJ: for all $t \le s \le T$ and $x \in \mathbb{R}^p$,

$$\begin{aligned} Y_s^{t,x} = g(X_T^{t,x}) &+ \int_s^T f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}, K_r^{t,x}(\cdot))\mathrm{d}r \\ &- \int_s^T Z_r^{t,x}\mathrm{d}W_r - \int_s^T \int_E K_r^{t,x}(e)\widetilde{N}(\mathrm{d}r, \mathrm{d}e), \end{aligned} \tag{1.17}$$

where $(X_s^{t,x})_{s\ge 0}$ is an $\mathbb{R}^p$–valued right process, $X_s^{t,x} = x$ if $s \le t$ and

$$\begin{aligned} &f : [0, T] \times \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^{q\times q} \times \mathcal{L}_\nu^{2,q} \longrightarrow \mathbb{R}^q, \\ &g : \mathbb{R}^p \longrightarrow \mathbb{R}^q \end{aligned}$$

are measurable functions satisfy the following hypotheses:

($\mathbf{H}_{4.1}$) $\sup_{0\leq s\leq T}\mathbb{E}[|X_s^{t,x}|^2]<\infty$.

($\mathbf{H}_{4.2}$) For any $(r,x,y,z)\in[0,T]\times\mathbb{R}^p\times\mathbb{R}^q\times\mathbb{R}^{q\times q}$ and $k\in\mathcal{L}_\nu^{2,q}$, $|g(x)|\leq C(1+|x|)$ and

$$|f(r,x,y,z,k(\cdot))|\leq C(1+|x|+|y|+|z|+\|k(\cdot)\|_{q,\nu}).$$

($\mathbf{H}_{4.3}$) There exists a constant $L\geq 0$, such that for all $r\in[0,T]$, $\forall\ x\in\mathbb{R}^p$, $\forall\ (y,y')\in\mathbb{R}^q$, $\forall\ (z,z')\in\mathbb{R}^{q\times q}$ and $(k(\cdot),k'(\cdot))\in\mathcal{L}_\nu^{2,q}$

$$|f(r,x,y,z,k(\cdot))-f(r,x,y',z',k'(\cdot))|\leq L(|y-y'|+|z-z'|+\|(k-k')(\cdot)\|_{q,\nu}).$$

Thanks for the hypotheses ($\mathbf{H}_{4.1}$), ($\mathbf{H}_{4.2}$) and ($\mathbf{H}_{4.3}$) , the Theorem 1.3.1 (see also [57, 67, 69]) among others, confirms that the BSDEJ (1.17) admits a unique solution $(Y_s^{t,x},Z_s^{t,x},K_s^{t,x}(\cdot))_{s\leq T}$ which belongs to $\mathbb{M}_{\mathcal{S}}^2$ where:
$\mathbb{M}_{\mathcal{S}}^2=\mathcal{S}_{\mathcal{F}}^2\,(0,T;\mathbb{R}^q)\otimes\mathcal{M}_{\mathcal{F}}^2(0,T,\mathbb{R}^{q\times q})\otimes\mathcal{M}_{\mathcal{F}}^2([0,T]\times E,\mathbb{R}^q,\mathrm{d}t\nu(\mathrm{d}e))$.

The following lemma is found to be useful.

Lemma 1.4.3

Under the assumptions ($\mathbf{H}_{4.1}$) $-$ ($\mathbf{H}_{4.3}$). *There exists a constant C such that for any $t\leq s\leq T$, we have*

$$\mathbb{E}\left[|Y_s^{t,x}|^2+\int_0^T\left(|Z_r^{t,x}|^2+\|K_r^{t,x}(\cdot)\|_{q,\nu}^2\right)\mathrm{d}r\right]\leq C(1+|x|^2). \tag{1.18}$$

Proof. Applying Itô's formula from s to T, to $|y|^2$ with the equation (1.17), we get

$$\begin{aligned}&|Y_s^{t,x}|^2+\int_s^T\left(|Z_r^{t,x}|^2+\|K_r^{t,x}(\cdot)\|_{q,\nu}^2\right)\mathrm{d}r\\&=|g(X_T^{t,x})|^2+2\int_s^T Y_r^{t,x}\ f(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x},K_r^{t,x}(\cdot))\mathrm{d}r\\&-(M_T^{t,x}-M_s^{t,x})-(N_T^{t,x}-N_s^{t,x}),\end{aligned}$$

where

$$M_s^{t,x}=2\int_0^s Y_r^{t,x}\ Z_r^{t,x}\mathrm{d}W_r+2\int_0^s\int_E Y_r^{t,x}\ K_r^{t,x}(e)\widetilde{N}(\mathrm{d}r,\mathrm{d}e),$$

and

$$N_s^{t,x}=\int_0^s\int_E|K_r^{t,x}\,(e)\,|^2\widetilde{N}(\mathrm{d}r,\mathrm{d}e)$$

are real-valued martingales. Taking the expectation in each member, we obtain

$$\begin{aligned}&\mathbb{E}\left[|Y_s^{t,x}|^2+\int_s^T\left(|Z_r^{t,x}|^2+\|K_r^{t,x}(\cdot)\|_{q,\nu}^2\right)\mathrm{d}r\right]\\&=\mathbb{E}|g(X_T^{t,x})|^2+2\mathbb{E}\left[\int_s^T Y_r^{t,x}\ f(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x},K_r^{t,x}(\cdot))\mathrm{d}r\right].\end{aligned}$$

Making use the linear growth of g and f and $|ab|\leq\varepsilon\,|a|^2+\dfrac{1}{\varepsilon}\,|b|^2$ for any $\varepsilon>0$ and $a,b\in\mathbb{R}^q$, we obtain, by the usual techniques for BSDEJs,

$$\begin{aligned}&\mathbb{E}\left[|Y_s^{t,x}|^2+\frac{1}{2}\int_s^T\left(|Z_r^{t,x}|^2+\|K_r^{t,x}(\cdot)\|_{q,\nu}^2\right)\mathrm{d}r\right]\\&\leq C\left(1+\mathbb{E}|X_T^{t,x}|^2+\int_s^T\mathbb{E}|X_r^{t,x}|^2dr+\int_s^T\mathbb{E}|Y_r^{t,x}|^2\mathrm{d}r\right).\end{aligned}\tag{1.19}$$

Thanks to Gronwall's Lemma and assumption $(\mathbf{H}_{4.1})$, we find that

$$\mathbb{E}|Y_s^{t,x}|^2\leq C\left(1+\sup_{0\leq s\leq T}\mathbb{E}|X_s^{t,x}|^2\right)\leq C(1+|x|^2).$$

Likewise, from (1.19), we can get

$$\mathbb{E}\left[\int_s^T\left(|Z_r^{t,x}|^2+\|K_r^{t,x}(\cdot)\|_{q,\nu}^2\right)\mathrm{d}r\right]\leq C(1+|x|^2).$$

Finally, a combination of the two above inequalities leads to (1.18) which achieves the proof. □

In the following theorem, we are interested in establishing a Markovian structure of the solution $(Y_s^{t,x},Z_s^{t,x},K_s^{t,x}(\cdot))_{s\leq T}$ of a BSDEJ in terms of some deterministic measurable functions evaluated at $(s,X_s^{t,x})$.

Theorem 1.4.1

Under the assumptions $(\mathbf{H}_{4.1})-(\mathbf{H}_{4.3})$, *there exist three measurable and deterministic functions* $u:[0,T]\times\mathbb{R}^p\longrightarrow\mathbb{R}^q$, $v:[0,T]\times\mathbb{R}^p\longrightarrow\mathbb{R}^{q\times q}$ *and* $\theta:[0,T]\times\mathbb{R}^p\times E\longrightarrow\mathcal{L}_\nu^{2,q}$ *such that for any* $(s,e)\in[t,T]\times E$

$$Y_s^{t,x}=u(s,X_s^{t,x}),\quad Z_s^{t,x}=v(s,X_s^{t,x})\ \text{ and }\ K_s^{t,x}(e)=\theta(s,X_{s-}^{t,x},e).$$

Furthermore, $\forall\ (s,x)\in[t,T]\times\mathbb{R}^p$,

$$u(s,x)=\mathbb{E}\left[g(X_T^{s,x})+\int_s^T f(r,X_r^{s,x},Y_r^{s,x},Z_r^{s,x},K_r^{s,x}(\cdot))\mathrm{d}r\right],$$

and is continuous such that $|u(t,x)|\leq C(1+|x|)\ \forall\ (t,x)\in[0,T]\times\mathbb{R}^p$.

Proof. We divided the proof into two steps.

Step 1. We suppose that f does not depend on y, z and $k(\cdot)$ in which case, the equation (1.17) becomes

$$Y_s^{t,x} = g(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x})\mathrm{d}r - \int_s^T Z_r^{t,x}\mathrm{d}W_r - \int_s^T \int_E K_r^{t,x}(e)\widetilde{N}(\mathrm{d}r, \mathrm{d}e). \tag{1.20}$$

Taking the conditional expectation w.r.t $\mathcal{F}_s$, we get for all $t \leq s \leq T$

$$\begin{aligned} Y_s^{t,x} &= \mathbb{E}\left[g(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x})\mathrm{d}r \mid \mathcal{F}_s\right] \\ &= \mathbb{E}\left[g(X_T^{t,x}) + \int_t^T f(r, X_r^{t,x})\mathrm{d}r \mid \mathcal{F}_s\right] - \int_t^s f(r, X_r^{t,x})dr. \end{aligned} \tag{1.21}$$

According to the Markov property of $(s, X_s^{t,x} - X_t^{t,x}) = (s, X_s^{t,x} - x)$ for all $s \geq t$, we can write $Y_s^{t,x} = u\left(s, X_s^{t,x}\right)$ where

$$u\left(s, y\right) = \mathbb{E}\left[g(X_T^{s,y}) + \int_s^T f(r, X_r^{s,y})\mathrm{d}r\right].$$

The regularity of u can be checked similarly as in Proposition 2.5 in [12].

Define $\mathbb{G} = (\mathcal{G}_s)_{s\in[0,T]}$ the filtration generated by the functions $\int_t^T \mathbb{E}\psi(r, X_r^{t,y})\mathrm{d}r$ where ψ is a continuous $\mathbb{R}^q$–valued function. Consequently, for any $\mathbb{G}$–measurable f and g such that

$$\mathbb{E}|g(X_T^{t,x})|^2 + \int_0^T \mathbb{E}|f(r, X_r^{t,x})|^2\mathrm{d}r < \infty.$$

Remark that we do not change the filtration here; we have just introduced the appropriate filtration to guarantee the measurability of the deterministic function u. The process $(Y_s^{t,x})_{s\in[0,T]}$ admits a càdlàg version given by $Y_s^{t,x} = u(s, X_s^{t,x})$ thanks to the decomposition (1.21) as the sum of an absolutely continuous process and a martingale which can be chosen to be given càdlàg. Clearly, the stochastic process $(\widetilde{Y}_s)_{s\in[t,T]}$

$$\widetilde{Y}_s := \int_t^s f(r, X_r^{t,x})\mathrm{d}r + Y_s^{t,x} = \mathbb{E}\left[g(X_T^{t,x}) + \int_t^T f(r, X_r^{t,x})\mathrm{d}r \mid \mathcal{F}_s\right] \tag{1.22}$$

is an additive square-integrable martingale and thus, by Lemma 4.1 [30] p. 45, or by the above Lemma 1.4.2, with X starting at x at time t, it admits the following representation:

$$\widetilde{Y}_s = \int_t^s v(r, X_r^{t,x})\mathrm{d}W_r + \int_t^s \int_E \theta(r, X_{r-}^{t,x}, e)\widetilde{N}(\mathrm{d}r, \mathrm{d}e)$$

where $v(r,x) \in \mathbb{R}^{q\times q}$ and $\theta(r,x,\cdot) \in \mathcal{L}^{2,q}_{\nu}$ are two measurable functions. Further, for $s=T$, we have

$$\widetilde{Y}_T = \int_t^T v(r, X_r^{t,x})\mathrm{d}W_r + \int_t^T \int_E \theta(r, X_{r-}^{t,x}, e)\widetilde{N}(\mathrm{d}r, \mathrm{d}e).$$

On one hand, we have

$$\widetilde{Y}_T - \widetilde{Y}_s = \int_s^T v(r, X_r^{t,x})\mathrm{d}W_r + \int_s^T \int_E \theta(r, X_{r-}^{t,x}, e)\widetilde{N}(\mathrm{d}r, \mathrm{d}e).$$

On the other hand, in view of the equality (1.22),

$$\begin{aligned}\widetilde{Y}_T - \widetilde{Y}_s &= \int_t^T f(r, X_r^{t,x})\mathrm{d}r + Y_T^{t,x} - \int_t^s f(r, X_r^{t,x})\mathrm{d}r - Y_s^{t,x}\\ &= g(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x})\mathrm{d}r - Y_s^{t,x}.\end{aligned}$$

Hence,

$$\begin{aligned}Y_s^{t,x} &= g(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x})\mathrm{d}r\\ &\quad - \int_s^T v(r, X_r^{t,x})\ \mathrm{d}W_r - \int_s^T \int_E \theta(r, X_{r-}^{t,x}, e)\widetilde{N}(\mathrm{d}r, \mathrm{d}e)\\ &= g(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x})\mathrm{d}r\\ &\quad - \int_s^T Z_r^{t,x}\ \mathrm{d}W_r - \int_s^T \int_E K_r^{t,x}(e)\widetilde{N}(\mathrm{d}r, \mathrm{d}e).\end{aligned}$$

Due to the uniqueness of the solution of equation (1.20), we conclude

$$Z_r^{t,x} = v(r, X_r^{t,x}) \quad \text{and} \quad K_r^{t,x}(e) = \theta(r, X_{r-}^{t,x}, e),$$

that is exactly the solution of our BSDEJ.

Step 2. We shall consider the general case where the generator f depends on r, x, y, z and $k(\cdot)$. Let us introduce the following sequence $(Y_\cdot^{t,x,n}, Z_\cdot^{t,x,n}, K_\cdot^{t,x,n}(\cdot))_{n\in\mathbb{N}}$ defined by $Y^{t,x,0}=0$, $Z^{t,x,0}=0$ and $K^{t,x,0}=0$ and

$$\begin{aligned}Y_s^{t,x,n+1} &= g(X_T^{t,x}) + \int_s^T f(r, X_r^{t,x}, Y_r^{t,x,n}, Z_r^{t,x,n}, K_r^{t,x,n}(\cdot))\mathrm{d}r\\ &\quad - \int_s^T Z_r^{t,x,n+1}\mathrm{d}W_r - \int_s^T \int_E K_r^{t,x,n+1}(e)\widetilde{N}(\mathrm{d}r, \mathrm{d}e).\end{aligned}$$

Under the Lipschitz condition of the generator, one can show exactly as in the proof of the Theorem 1.3.1 that $(Y_\cdot^{t,x,n}, Z_\cdot^{t,x,n}, K_\cdot^{t,x,n}(\cdot))_{n\in\mathbb{N}}$ is a Cauchy sequence in the Banach space $\mathbb{M}^2_{\mathcal{S}}$, hence

$$(Y^{t,x}_{\cdot}, Z^{t,x}_{\cdot}, K^{t,x}_{\cdot}(\cdot)) = \lim_{n\longrightarrow+\infty}(Y^{t,x,n}_{\cdot}, Z^{t,x,n}_{\cdot}, K^{t,x,n}_{\cdot}(\cdot)). \tag{1.23}$$

The previous step leads that, for any $r \in [t,T]$, there exist three measurable functions u^1, v^1 and θ^1 such that

$$(Y^{t,x,1}_r, Z^{t,x,1}_r, K^{t,x,1}_r(e)) = (u^1(r, X^{t,x}_r), v^1(r, X^{t,x}_r), \theta^1(r, X^{t,x}_r, e)),\ \mathbb{P}\text{–a.s.}$$

By recursion, we get, for any $n \in \mathbb{N}$, there exist measurable functions u^n, v^n and θ^n such that

$$(Y^{t,x,n}_r, Z^{t,x,n}_r, K^{t,x,n}_r(e)) = (u^n(r, X^{t,x}_r), v^n(r, X^{t,x}_r), \theta^n(r, X^{t,x}_r, e)). \tag{1.24}$$

Notice that these representations have been studied in the literature for smooth coefficients by Barles *et al.* [12], Bouchard & Elie [16, Section 4] and Delong [25].

Set

$$u(r, X^{t,x}_r) = \lim_{n\longrightarrow+\infty}\sup u^n(r, X^{t,x}_r),\ \ v(r, X^{t,x}_r) = \lim_{n\longrightarrow+\infty}\sup v^n(r, X^{t,x}_r),$$

and

$$\theta(r, X^{t,x}_r, e) = \lim_{n\longrightarrow+\infty}\sup \theta^n(r, X^{t,x}_r, e).$$

Then, by invoking (1.23) and (1.24), it follows that $\mathbb{P}$–a.s. $\forall\ r \in [t,T]$

$$\begin{aligned}
u(r, X^{t,x}_r) &= \lim_{n\longrightarrow+\infty}\sup u^n(r, X^{t,x}_r) = \lim_{n\longrightarrow+\infty} Y^{t,x,n}_r = Y^{t,x}_r;\\
v(r, X^{t,x}_r) &= \lim_{n\longrightarrow+\infty}\sup v^n(r, X^{t,x}_r) = \lim_{n\longrightarrow+\infty} Z^{t,x,n}_r = Z^{t,x}_r;\\
\theta(r, X^{t,x}_r, e) &= \lim_{n\longrightarrow+\infty}\sup \theta^n(r, X^{t,x}_r, e) = \lim_{n\longrightarrow+\infty} K^{t,x,n}_r(e) = K^{t,x}_r(e).
\end{aligned}$$

Finally, the linear growth condition on u is a simple consequence of the previous representation in step 2 and Lemma 1.4.3. This achieves the proof. $\square$

Chapter 2

Multidimensional Markovian BSDEJs with Continuous Generators

2.1 Introduction

The purpose of the current chapter is to discuss the problem of the existence of a solution to the multidimensional Markovian BSDEs driven by both a Wiener and a Poisson random measure when the generator f is continuous and satisfies some linear growth conditions and when the terminal value ξ is a function of the Markov process $(X_s)_{s\in[0,T]}$ at time T. Our principal aim is to complete the studies of Hamadène [38], Hamadène & Mu [40] and Mu & Wu [55] without jump part in the setting of BSDEs with jumps and continuous generators.
This chapter is divided as follows. Section one is devoted to prove the existence of a solution to our Markovian BSDEJ in two cases: the first one when f is continuous in y, z and Lipschitz in $k(.)$, and the second one when f is continuous in y, z and k. Moreover, in section two, we give some examples of Markov processes having the $L^{\mathbf{2}}$–domination property.

2.2 Existence Results of BSDEJs

In this section, we aim to handle the existence problem for multidimensional Markovian BSDE driven by both q-dimensional Brownian motion and compensated Poisson random measure. First, we study the case when the generator of the BSDEJ is only continuous w.r.t the state variable along with the Brownian component and Lipschitz in the jump component. The concept of the proof is to approximate this BSDEJ by a suitable sequence of BSDEJs having globally Lipschitz generators that guarantee the existence and uniqueness of the solution (this result is proved in Theorem 1.3.1 from Chapter 1) and then obtain the existence result of the original equation by using limit arguments in appropriate spaces. The drawback to relaxing the Lipschitz condition on $K(\cdot)$ is that it belongs to the functional space $\mathcal{L}_{\nu}^{2,q}$ and thus, the approximating technique does not work in this situation. However, if we allow the generator f to depend on $\int_E K_r(e)\,\nu(\mathrm{d}e)$ rather than $K_r(\cdot)$, as a particular case, we can prove an existence result in the case where f is also continuous in k. Finally, we mention here, that due to the lack of the comparison principle between solutions of multidimensional Markovian BSDE, the technique used in [49] cannot be applicable in our situation. As a trade-off, we shall use the relationship between the processes $X_\cdot^{t,x}$ and $(Y_\cdot^{t,x}, Z_\cdot^{t,x}, K_\cdot^{t,x}(\cdot))$ that is established in Theorem 1.4.1 and the $L^{\mathbf{2}}$-domination technique defined below.

2.2.1 Measure Domination

Before we state our main theorems, let us recall the precise definition of the $L^{\mathbf{2}}$-domination condition as given in Hamadène [40].

Definition 2.2.1

(L^2-domination condition) *For a given $t \in [0,T]$, a family of probability measures $\{\mu_1(s,\mathrm{d}x),\ s \in [t,T]\}$ defined on $\mathbb{R}^p$ is said to be L^2-dominated by another family of probability measures $\{\mu_0(s,\mathrm{d}x),\ s \in [t,T]\}$, if for any $\varepsilon \in (0, T-t]$, there exists an application $\phi_t : [t+\varepsilon, T] \times \mathbb{R}^p \longrightarrow \mathbb{R}_+$ such that*

(i) $\forall\ N \geq 1$, $\phi_t \in L^2([t+\varepsilon,T] \times [-N,N]^p;\ \mu_0(s,\mathrm{d}x)\mathrm{d}s)$.

(ii) $\mu_1(s,\mathrm{d}x)\mathrm{d}s = \phi_t(s,x)\mu_0(s,\mathrm{d}x)\mathrm{d}s$ on $[t+\varepsilon,T] \times \mathbb{R}^p$.

Let $x_0 \in \mathbb{R}^p$, $(t,x) \in [0,T] \times \mathbb{R}^p$, $s \in [t,T]$ and $\mu(t,x;s,\mathrm{d}y)$ the law of our Markov process $(X_s^{t,x})_{t \leq s \leq T}$, defined for each $A \in \mathcal{B}(\mathbb{R}^p)$ by $\mu(t,x;s,A) = \mathbb{P}(X_s^{t,x} \in A)$. We further assume the following assumption:

$(\mathbf{H}_{2.1})$ For each $t \geq 0$ and for each $x \in \mathbb{R}^p$ the family $\{\mu(t,x;s,\mathrm{d}y),\ s \in [0,T]\}$ is L^2-dominated by $\{\mu(0,x_0;s,\mathrm{d}y),\ s \in [0,T]\}$.

2.2.2 First Case: f is Continuous in y, z and Lipschitz in $k(.)$

We consider the BSDEJ of the following type for all $s \in [0, T]$,

$$\begin{aligned} Y_s^{0,x_0} &= g(X_T^{0,x_0}) + \int_s^T f(r, X_r^{0,x_0}, Y_r^{0,x_0}, Z_r^{0,x_0}, K_r^{0,x_0}(\cdot))\mathrm{d}r \\ &- \int_s^T Z_r^{0,x_0}\,\mathrm{d}W_r - \int_s^T \int_E K_r^{0,x_0}(e)\widetilde{N}(\mathrm{d}r, \mathrm{d}e), \end{aligned} \tag{2.1}$$

where $X_0^{0,x_0} = x_0 \in \mathbb{R}^p$, g is the same, in chapter 1, as in BSDEJ (1.17) and f satisfies the following assumptions:

($\mathbf{H}_{2.2}$) The mapping $(y, z) \mapsto f(s, x, y, z, k(\cdot))$ is continuous for any fixed $(s, x, k(\cdot)) \in [0, T] \times \mathbb{R}^p \times \mathcal{L}_\nu^{2,q}$.

($\mathbf{H}_{2.3}$) For any $(t, x, y, z) \in [0, T] \times \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^{q\times q}$ and $k, k' \in \mathcal{L}_\nu^{2,q}$

$$|f(t, x, y, z, k(\cdot)) - f(t, x, y, z, k'(\cdot))| \leq C \left\|(k - k')(\cdot)\right\|_{q,\nu}.$$

To prove our main result the upcoming lemma is useful.

Lemma 2.2.1

Let f satisfy ($\mathbf{H}_{4.1}$), ($\mathbf{H}_{4.2}$), ($\mathbf{H}_{2.2}$), and ($\mathbf{H}_{2.3}$). Then, there exists a sequence of functions $(f_n)_{n\geq 1}$ such that:

(a) $\sup_{t,x} |f_n(t, x, y, z, k(\cdot)) - f_n(t, x, y', z', k'(\cdot))|$
$\leq C \left(|y - y'| + |z - z'| + \|(k - k')(\cdot)\|_{q,\nu}\right)$, *for some positive constant C.*

(b) $|f_n(t, x, y, z, k(\cdot))| \leq C(1 + |x| + |y| + |z| + \|k(\cdot)\|_{q,\nu})$,
for all $(t, x, y, z, k(\cdot)) \in [0, T] \times \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^{q\times q} \times \mathcal{L}_\nu^{2,q}$.

(c) *For all $(t, x, y, z, k(\cdot)) \in [0, T] \times \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^{q\times q} \times \mathcal{L}_\nu^{2,q}$ and $n \geq 1$, there exists a positive constant C such that $|f_n(t, x, y, z, k(\cdot))| \leq C(1 + |x|)$.*

(d) *For any $(t, x, k(\cdot)) \in [0, T] \times \mathbb{R}^p \times \mathcal{L}_\nu^{2,q}$, and for any compact subset $S \subset \mathbb{R}^q \times \mathbb{R}^{q\times q}$*

$$\sup_{(y,z)\in S} |f_n(t, x, y, z, k(\cdot)) - f(t, x, y, z, k(\cdot))| \longrightarrow 0 \text{ as } n \to +\infty.$$

Proof. Let ψ be an element of $\mathcal{C}^\infty(\mathbb{R}^q \times \mathbb{R}^{q\times q}, \mathbb{R})$ with compact support and satisfies

$$\int_{\mathbb{R}^{q+q\times q}} \psi(\overrightarrow{u})\mathrm{d}\overrightarrow{u} = 1,$$

where $\overrightarrow{u} = (y,z) \in \mathbb{R}^{q+q\times q}$. For any $k \in \mathcal{L}^{2,q}_\nu$, we set

$$f(t,x,(\cdot),k) * \psi(n(\cdot))(\overrightarrow{u}) = \int_{\mathbb{R}^{q+q\times q}} f(t,x,\overrightarrow{v},k)\psi(n(\overrightarrow{u}-\overrightarrow{v}))\mathrm{d}\overrightarrow{v}$$

and $\bar{\psi} \in \mathcal{C}^\infty(\mathbb{R}^q \times \mathbb{R}^{q\times q}, \mathbb{R})$ such that

$$\bar{\psi}(\overrightarrow{u}) = \begin{cases} 1, & |\overrightarrow{u}|^2 \leq 1, \\ 0, & |\overrightarrow{u}|^2 \geq 2. \end{cases}$$

Then, we define the sequence of the measurable functions $\{f_n, n \geq 1\}$ as follows:

$$f_n(t,x,\overrightarrow{u},k) = n^2\bar{\psi}(\frac{\overrightarrow{u}}{n})\left(f(t,x,(\cdot),k) * \psi(n(\cdot))\right)(\overrightarrow{u}),$$

satisfies all the properties of Lemma 2.2.1. □

We are now ready to provide our main result in this subsection that is the following theorem which extends a part from the paper [40], from BSDEs driven by Brownian motion to BSDEs with jumps. However, our generator depends also on the state variable k which is not covered in [40].

Theorem 2.2.1

Assume that $(\mathbf{H}_{4.1}) - (\mathbf{H}_{4.2})$ *and* $(\mathbf{H}_{2.2}) - (\mathbf{H}_{2.1})$ *are in force. Then, there exists a triple of processes* $(Y_., Z_., K_.(\cdot))$ *belonging to* $\mathbb{M}^2$ *that solves the BSDEJ (2.1) where* $\mathbb{M}^2 = \mathcal{M}^2_{\mathcal{F}}(0,T,\mathbb{R}^q) \otimes \mathcal{M}^2_{\mathcal{F}}(0,T,\mathbb{R}^{q\times q}) \otimes \mathcal{M}^2_{\mathcal{F}}([0,T]\times E, \mathbb{R}^q, \mathrm{d}t\nu(\mathrm{d}e))$.

Proof. We first define the following family of approximating BSDEJs obtained by replacing the generator f in BSDEJ (1.17) by f_n defined in Lemma 2.2.1.

$$\begin{aligned} Y^{t,x;n}_s = g(X^{t,x}_T) &+ \int_s^T f_n(r, X^{t,x}_r, Y^{t,x;n}_r, Z^{t,x;n}_r, K^{t,x;n}_r(\cdot))\mathrm{d}r \\ &- \int_s^T Z^{t,x;n}_r \,\mathrm{d}W_r - \int_s^T \int_E K^{t,x;n}_r(e)\widetilde{N}(\mathrm{d}r,\mathrm{d}e). \end{aligned} \tag{2.2}$$

On one hand, since for each $n \geq 1$, f_n is uniformly Lipschitz w.r.t $(y,z,k(\cdot))$, so Theorem 1.3.1 (see also [57, 67, 69]) shows that there exists a unique solution

$$(Y^{t,x;n}_., Z^{t,x;n}_., K^{t,x;n}_.(\cdot))_{n\geq 1} \in \mathbb{M}^2_{\mathcal{S}},$$

which solves BSDEJ (2.2). On the other hand, since f_n satisfies the property (**c**) in Lemma 2.2.1, Theorem 1.4.1 yields that, there exist three sequences of deterministic measurable functions $u^n : [0,T]\times\mathbb{R}^p \longrightarrow \mathbb{R}^q$, $v^n : [0,T]\times\mathbb{R}^p \longrightarrow \mathbb{R}^{q\times q}$ and $\theta^n : [0,T]\times\mathbb{R}^p\times E \longrightarrow \mathcal{L}^{2,q}_{\nu}$ such that

$$Y_s^{t,x;n} = u^n(s, X_s^{t,x}),\ \ Z_s^{t,x;n} = v^n(s, X_s^{t,x}) \text{ and } K_s^{t,x;n}(e) = \theta^n(s, X_{s-}^{t,x}, e).$$

Beside, we have the following deterministic expression for $n \geq 1$,

$$u^n(t,x) = \mathbb{E}\left[g(X_T^{t,x}) + \int_t^T F_n(s, X_s^{t,x})\mathrm{d}s\right],\ \forall\ (t,x) \in [0,T]\times\mathbb{R}^p, \tag{2.3}$$

where

$$F_n(t,x) = f_n(t,x,u^n(t,x),v^n(t,x),\theta^n(t,x,\cdot)).$$

Hence, keeping in mind the property **(b)**, as in Lemma 1.4.3, we can show that there exists a constant $C > 0$ (independent from n) such that for any $n \geq 1$ and $s \in [t,T]$,

$$\mathbb{E}\left[|u^n(s,X_s^{t,x})|^2 + \int_0^T \left(|Z_r^{t,x;n}|^2 + \|K_r^{t,x;n}(\cdot)\|_{q,\nu}^2\right)\mathrm{d}r\right] \leq C(1+|x|^2),$$

which imply that, since $X_t^{t,x} = x$,

$$|u^n(t,x)|^2 \leq C(1+|x|^2)\ \ \forall\ n\geq 1 \text{ and } t\in[0,T].$$

Consequently,

$$|u^n(t,x)| \leq C(1+|x|)\ \ \forall\ n\geq 1 \text{ and } t\in[0,T],$$

and thus, for any $s\in[0,T]$ and $n\geq 1$,

$$|Y_s^n| = |u^n(s,X_s^{0,x_0})| \leq C(1+|X_s^{0,x_0}|), \mathrm{d}\mathbb{P}\text{-a.s.} \tag{2.4}$$

The remainder of the proof will be broken down into the following three steps.

Step 1. We will prove, for each $(t,x)\in[0,T]\times\mathbb{R}^p$, that $(u^n(t,x))_{n\geq 1}$ has a convergent subsequence in $\mathbb{R}^q$. In one hand, since f_n satisfies the property **(b)**, the same technique as in the proof of Lemma 1.4.3, yields

$$\sup_{n\geq 1}\mathbb{E}\left[|Y_s^{0,x_0;n}|^2 + \int_0^T \left(|Z_r^{0,x_0;n}|^2 + \|K_r^{0,x_0;n}(\cdot)\|_{q,\nu}^2\right)\mathrm{d}r\right] \leq C,\ \forall\ s\leq T. \tag{2.5}$$

The property **(b)** with assumptions $(\mathbf{H}_{2.1})$, $(\mathbf{H}_{4.1})$ and the estimate (2.5), show that there exists a positive constant C such that, for any $n \geq 1$,

$$\int_0^T \int_{\mathbb{R}^p} |F_n(s,y)|^2 \,\mu(0,x_0;s,\mathrm{d}y)\mathrm{d}s = \mathbb{E}\int_0^T |F_n(s,X_s^{0,x_0})|^2\mathrm{d}s \leq C. \tag{2.6}$$

This implies that, there exists a subsequence $\{n_j\}$, (which is still labeled by $\{n\}$), and $\mathcal{B}([0,T]) \otimes \mathcal{B}(\mathbb{R}^p)$-measurable deterministic function $F(s,x)$ such that

$$F_n \longrightarrow F \text{ weakly in } L^2([0,T]\times\mathbb{R}^p;\mu(0,x_0;s,\mathrm{d}x)\mathrm{d}s). \tag{2.7}$$

So, let (t,x) be fixed, $\varepsilon > 0$, N, n, and $m \geq 1$ be integers. From (2.3), we have

$$\begin{aligned}
|u^n(t,x) - u^m(t,x)| &= \left|\mathbb{E}\left[\int_t^T (F_n(s,X_s^{t,x}) - F_m(s,X_s^{t,x}))\mathrm{d}s\right]\right| \\
&\leq \mathbb{E}\left[\int_t^{t+\varepsilon} \left|F_n(s,X_s^{t,x}) - F_m(s,X_s^{t,x})\right| \mathrm{d}s\right] \\
&\quad + \left|\mathbb{E}\left[\int_{t+\varepsilon}^T (F_n(s,X_s^{t,x}) - F_m(s,X_s^{t,x}))1\!\!1_{\{|X_s^{t,x}|\leq N\}}\mathrm{d}s\right]\right| \\
&\quad + \mathbb{E}\left[\int_{t+\varepsilon}^T \left|F_n(s,X_s^{t,x}) - F_m(s,X_s^{t,x})\right| 1\!\!1_{\{|X_s^{t,x}|> N\}}\mathrm{d}s\right] \\
&= I_1^{n,m,\varepsilon} + |I_2^{n,m,\varepsilon}| + I_3^{n,m,\varepsilon}.
\end{aligned} \tag{2.8}$$

According to Schwarz inequality and estimate (2.6), we find

$$I_1^{n,m,\varepsilon} \leq \varepsilon^{\frac{1}{2}} \left\{\mathbb{E}\left[\int_0^T \left|F_n(s,X_s^{t,x}) - F_m(s,X_s^{t,x})\right|^2 \mathrm{d}s\right]\right\}^{\frac{1}{2}} \leq C\sqrt{\varepsilon}.$$

At the same time, thanks to the L^2-domination property

$$\begin{aligned}
I_2^{n,m,\varepsilon} &= \int_{\mathbb{R}^p}\int_{t+\varepsilon}^T (F_n(s,y) - F_m(s,y))1\!\!1_{\{|y|\leq N\}}\mu(t,x;s,\mathrm{d}y)\mathrm{d}s \\
&= \int_{\mathbb{R}^p}\int_{t+\varepsilon}^T (F_n(s,y) - F_m(s,y))1\!\!1_{\{|y|\leq N\}}\phi_{t,x}(s,y)\mu(0,x_0;s,\mathrm{d}y)\mathrm{d}s.
\end{aligned}$$

Since $\phi_{t,x}(s,y) \in L^2([t+\varepsilon,T]\times[-N,N]^p;\mu(0,x_0;s,\mathrm{d}y)\mathrm{d}s)$, for $N \geq 1$, it follows from the weak convergence (2.7) that for any $t \geq 0$, $\mu(0,x_0;s,\mathrm{d}y)$-almost every $x \in \mathbb{R}^p$, we have

$$\mathbb{E}\left[\int_{t+\varepsilon}^T (F_n(s,X_s^{t,x}) - F_m(s,X_s^{t,x}))1\!\!1_{\{|X_s^{t,x}|\leq N\}}\mathrm{d}s\right] \longrightarrow 0 \text{ as } n,\ m \longrightarrow \infty.$$

Again, applying Schwarz inequality and estimate (2.6), we get

$$\begin{aligned}
I_3^{n,m,\varepsilon} &\leq \left\{\mathbb{E}\left[\int_{t+\varepsilon}^T 11_{\{|X_s^{t,x}|>N\}}\mathrm{d}s\right]\right\}^{\frac{1}{2}} \left\{\mathbb{E}\left[\int_{t+\varepsilon}^T \left|F_n(s,X_s^{t,x}) - F_m(s,X_s^{t,x})\right|^2 \mathrm{d}s\right]\right\}^{\frac{1}{2}} \\
&\leq \frac{C}{\sqrt{N}}.
\end{aligned}$$

Therefore, by letting N tend to infinity, we obtain

$$|u^n(t,x) - u^m(t,x)| \leq C\sqrt{\varepsilon}.$$

We take ε small enough so that

$$|u^n(t,x) - u^m(t,x)| \longrightarrow 0 \text{ as } n,\ m \longrightarrow \infty.$$

Thus, the sequence $(u^n(t,x))_{n\geq 1}$ has a convergent subsequence in $\mathbb{R}^q$ with limit $u(t,x)$ for any $t \geq 0$ and every $x \in \mathbb{R}^p$.

Step 2. Now, we are going to show the existence of a subsequence still denoted

$$(Y^{0,x_0;n}_{\cdot}, Z^{0,x_0;n}_{\cdot}, K^{0,x_0;n}_{\cdot})_{n\geq 1},$$

which converges in $\mathbb{M}^2$ to $(Y_{\cdot}, Z_{\cdot}, K_{\cdot}(\cdot))$ solution of the BSDEJ (2.1). From step 1, there exists a measurable function u on $[0,T]\times\mathbb{R}^p$, such that for any $t \in [0,T]$,

$$\lim_{n\to+\infty} Y^{0,x_0;n}_t = u(t, X^{0,x_0}_t), \quad \mathbb{P}\text{-a.s.}$$

Considering estimate (2.4), and using Lebesgue's dominated convergence Theorem, the sequence $(Y^{0,x_0;n}_t)_{n\geq 1}$ converges to $Y^{0,x_0}_t := u(t, X^{0,x_0}_t)$ in $\mathcal{M}^2_{\mathcal{F}}(0,T,\mathbb{R}^q)$, that is,

$$\mathbb{E}\left[\int_0^T |Y^{0,x_0;n}_t - Y^{0,x_0}_t|^2 \mathrm{d}t\right] \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{2.9}$$

Next, we will show the convergence of $(Z^{0,x_0;n}_{\cdot})_{n\geq 1}$ and $(K^{0,x_0;n}_{\cdot})_{n\geq 1}$ respectively in $\mathcal{M}^2_{\mathcal{F}}(0,T,\mathbb{R}^{q\times q})$ and $\mathcal{M}^2_{\mathcal{F}}([0,T]\times E, \mathbb{R}^q, \mathrm{d}t\nu(\mathrm{d}e))$ as $n \to +\infty$ (we omit the subscript $(0,x_0)$ for convenience). To simplify the notations, for any n, $m \geq 1$, and $s \leq T$, we set:

$$\bar{Y}^{n,m}_s := Y^n_s - Y^m_s, \quad \bar{Z}^{n,m}_s := Z^n_s - Z^m_s \text{ and } \bar{K}^{n,m}_s(\cdot) := K^n_s(\cdot) - K^m_s(\cdot),$$

and

$$\bar{f}^{n,m}(s) := f_n(s, X^{0,x_0}_s, Y^n_{s-}, Z^n_s, K^n_s(\cdot)) - f_m(s, X^{0,x_0}_s, Y^m_{s-}, Z^m_s, K^m_s(\cdot)).$$

Using Itô's formula to $|\bar{Y}^{n,m}_s|^2$, we get

$$\begin{aligned} &|\bar{Y}^{n,m}_s|^2 + \int_s^T |\bar{Z}^{n,m}_r|^2 \mathrm{d}r + \int_s^T \|\bar{K}^{n,m}_r(\cdot)\|^2_{q,\nu}\mathrm{d}r \\ &= 2\int_s^T \bar{Y}^{n,m}_r\ \bar{f}^{n,m}(r)\mathrm{d}r - (M^{n,m}_T - M^{n,m}_s) - (N^{n,m}_T - N^{n,m}_s), \end{aligned} \tag{2.10}$$

where

$$M_s^{n,m} = 2\int_0^s \bar{Y}_r^{n,m}\ \bar{Z}_r^{n,m}\ \mathrm{d}W_r - 2\int_0^s\int_E \bar{Y}_r^{n,m}\ \bar{K}_r^{n,m}(e)\widetilde{N}(\mathrm{d}r,\mathrm{d}e),$$

and

$$N_s^{n,m} = \int_0^s\int_E |\bar{K}_r^{n,m}(e)|^2\widetilde{N}(\mathrm{d}r,\mathrm{d}e)$$

are real-valued martingales. Hence, according to property **(b)**, the assumption $(\mathbf{H}_{4.1})$ and the estimate (2.5), we obtain by taking the expectation in (2.10)

$$\int_0^T \mathbb{E}\left[|\bar{Z}_r^{n,m}|^2 + \|\bar{K}_r^{n,m}(\cdot)\|_{q,\nu}^2\right]\mathrm{d}r \le C\left[\int_0^T \mathbb{E}\left[|\bar{Y}_r^{n,m}|^2\right]\mathrm{d}r\right].$$

Thanks to result (2.9), it follows that $((Z_\cdot^n)_{n\ge1},(K_\cdot^n(\cdot))_{n\ge1})$ converges to some $(Z_\cdot,K_\cdot(\cdot))$ in $\mathcal{M}_{\mathcal{F}}^2(0,T,\mathbb{R}^{q\times q})\otimes\mathcal{M}_{\mathcal{F}}^2([0,T]\times E,\mathbb{R}^q,\mathrm{d}t\nu(\mathrm{d}e))$. Finally, we have proved that for a subsequence n_j,

$$(Y_\cdot^{n_j},Z_\cdot^{n_j},K_\cdot^{n_j}(\cdot))_{j\ge1} \longrightarrow (Y_\cdot,Z_\cdot,K_\cdot(\cdot)) \text{ in } \mathbb{M}^2. \tag{2.11}$$

Step 3. We will verify that the limits of the subsequences are exactly the solutions to BSDEJ (2.1). Indeed, we need to show that $f_n(t,X_t^{0,x_0},Y_t^n,Z_t^n,K_t^n(\cdot))$ converges to $f(t,X_t^{0,x_0},Y_t,Z_t,K_t(\cdot))\ \mathrm{d}t\otimes\mathrm{d}\mathbb{P}$. For $N\ge1$, we define

$$A_N := \{(r,\omega) : |Y_r^n| + |Z_r^n| \le N\},\ \bar{A}_N := \Omega\setminus A_N. \tag{2.12}$$

We have,

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T\left|f_n(r,X_r^{0,x_0},Y_r^n,Z_r^n,K_r^n(\cdot)) - f(r,X_r^{0,x_0},Y_r,Z_r,K_r(\cdot))\right|\mathrm{d}r\right]\\
&\le \mathbb{E}\left[\int_0^T\left|f_n(r,X_r^{0,x_0},Y_r^n,Z_r^n,K_r^n(\cdot)) - f_n(r,X_r^{0,x_0},Y_r^n,Z_r^n,K_r(\cdot))\right|\mathrm{d}r\right]\\
&+\mathbb{E}\left[\int_0^T\left|(f_n-f)\,(r,X_r^{0,x_0},Y_r^n,Z_r^n,K_r(\cdot))\right|\mathbb{1}_{A_N}\mathrm{d}r\right]\\
&+\mathbb{E}\left[\int_0^T\left|(f_n-f)\,(r,X_r^{0,x_0},Y_r^n,Z_r^n,K_r(\cdot))\right|\mathbb{1}_{\bar{A}_N}\mathrm{d}r\right]\\
&+\mathbb{E}\left[\int_0^T\left|f(r,X_r^{0,x_0},Y_r^n,Z_r^n,K_r(\cdot)) - f(r,X_r^{0,x_0},Y_r,Z_r,K_r(\cdot))\right|\mathrm{d}r\right]\\
&= I_1^n + I_2^n + I_3^n + I_4^n.
\end{aligned}$$

Thanks to property (**a**) in Lemma 2.2.1 and (2.11) I_1^n converges to 0. Again according to (2.11) and the continuity of f w.r.t $(y,z,k(\cdot))$, it follows, using assumptions $(\mathbf{H}_{4.1})$, $(\mathbf{H}_{4.2})$

from chapter 1 and Lebesgue's dominated Theorem, that I_4^n converges to 0 as n tends toward infinity. Moreover, due to the facts that f_n satisfies the property **(b)**, f satisfies $(\mathbf{H}_{4.1})$, $(\mathbf{H}_{4.2})$ and the estimate (2.6), a simple computation shows that there is a positive constant C such that

$$\begin{aligned} I_3^n &\le \left\{\mathbb{E}\left[\int_0^T \mathbb{1}_{\bar{A}_N} \mathrm{d}r\right]\right\}^{\frac{1}{2}} \left\{\mathbb{E}\left[\int_0^T \left|(f_n - f)\,(r, X_r^{0,x_0}, Y_r^n, Z_r^n, K_r(\cdot))\right|^2 \mathrm{d}r\right]\right\}^{\frac{1}{2}} \\ &\le CN^{-\frac{1}{2}}. \end{aligned}$$

Now, we return to estimate the second term I_2^n. The linear growth condition **(b)** together with $(\mathbf{H}_{4.1})$ and $(\mathbf{H}_{4.2})$ imply that:

$$|\,(f_n - f)\,(r, X_r^{0,x_0}, Y_r^n, Z_r^n, K_r(\cdot))|\mathbb{1}_{A_N} \le 2C(1 + N + |X_r^{0,x_0}|).$$

On the other hand, it is easy to see that

$$\begin{aligned} &|\,(f_n - f)\,(r, X_r^{0,x_0}, Y_r^n, Z_r^n, K_r(\cdot))|\mathbb{1}_{A_N} \\ &\le \sup_{\{(y,z), |y|+|z| \le N\}} |\,(f_n - f)\,(r, X_r^{0,x_0}, y, z, K_r(\cdot))|. \end{aligned}$$

Thanks to the property **(d)** in Lemma 2.2.1, we conclude that the second term of the last inequality converges to 0. Finally, Lebesgue's dominated convergence Theorem asserts that I_2^n converges also to 0 in $L^1([0,T] \times \Omega, \mathrm{d}t \otimes \mathrm{d}\mathbb{P})$. Eventually, we find, by sending respectively n and N to infinity; the convergence of the sequence

$$(f_n(t, X_t^{0,x_0}, Y_t^n, Z_t^n, K_t^n(\cdot)))_{0 \le t \le T})_{n \ge 1}$$

to

$$(f(t, X_t^{0,x_0}, Y_t, Z_t, K_t(\cdot)))_{0 \le t \le T}$$

in $L^1([0,T] \times \Omega, \mathrm{d}t \otimes \mathrm{d}\mathbb{P})$, and then $F(t, X_t^{0,x_0}) = f(t, X_t^{0,x_0}, Y_t, Z_t, K_t(\cdot))$, $\mathrm{d}t \otimes \mathrm{d}\mathbb{P}$-a.s. Thus $(Y_., Z_., K_.(\cdot))$ solves the equation (2.1). $\square$

To get the convergence in $\mathbb{M}_{\mathcal{S}}^2$, we add the following assumption on the generator f:

$(\mathbf{H}_{2.4})$ $f : [0,T] \times \Omega \times \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^{q \times q} \times \mathcal{L}_\nu^{2,q} \longrightarrow \mathbb{R}^q$ is measurable and for any $(t, x, y, z, k(\cdot)) \in [0,T] \times \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^{q \times q} \times \mathcal{L}_\nu^{2,q}$, there exists a constant $C > 0$ and $0 < \beta < 1$ such that

$$|f(t, x, y, z, k(\cdot))| \le C(1 + |x| + |y| + |z| + \|k(\cdot)\|_{q,\nu})^\beta.$$

Corollary 2.2.1

Assume that $(\mathbf{H}_{4.1})$ *and* $(\mathbf{H}_{2.2})-(\mathbf{H}_{2.4})$ *are in force. Then, there exists a triple of processes* $(Y_\cdot, Z_\cdot, K_\cdot(\cdot))$ *belonging to* $\mathbb{M}^2_{\mathcal{S}}$ *that solves the BSDEJ (2.1).*

Proof. By using the inequality $|x|^\beta \leq 1+|x|$, it is easy to check that the sub-linear growth condition $(\mathbf{H}_{2.4})$ implies the linear growth condition $(\mathbf{H}_{4.2})$. Thus, the above theorem confirms that there exists a triple $(Y_\cdot, Z_\cdot, K_\cdot(\cdot))$ solution to the BSDEJ (2.1) which belongs to $\mathbb{M}^2$. It remains to prove that the sequence $(Y^n_s)_{n\geq 1} = (u^n(s, X^{(0,x_0)}_s))_{n\geq 1}$ defined in the above proof converges to $Y_\cdot$ in $\mathcal{S}^2_{\mathcal{F}}(0,T,\mathbb{R}^q)$. From equations (2.1) and (2.2), we have

$$\begin{aligned}
Y^n_r - Y_r &= \int_s^T (f_n(r, X^{0,x_0}_r, Y^n_r, Z^n_r, K^n_r(\cdot)) - f(r, X^{0,x_0}_r, Y_r, Z_r, K_r(\cdot)))\mathrm{d}r \\
&\quad - \int_s^T (Z^n_r - Z_r)\mathrm{d}W_r - \int_s^T (K^n_r(\cdot) - K_r(\cdot))\widetilde{N}(\mathrm{d}r, \mathrm{d}e).
\end{aligned}$$

Squaring both sides of $(Y^n_r - Y_r)$, taking the supremum and the conditional expectation, we get

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\leq r\leq T} |Y^n_r - Y_r|^2\right] \\
&\leq C\left[\mathbb{E}\int_0^T \left|f_n(r, X^{0,x_0}_r, Y^n_r, Z^n_r, K^n_r(\cdot)) - f(r, X^{0,x_0}_r, Y_r, Z_r, K_r(\cdot))\right|^2 \mathrm{d}r\right. \\
&\left. + \mathbb{E}\left(\sup_{0\leq s\leq T}\left|\int_s^T (Z^n_r - Z_r)\mathrm{d}W_r\right|^2\right) + \mathbb{E}\left(\sup_{0\leq s\leq T}\left|\int_s^T (K^n_r(\cdot) - K_r(\cdot))\widetilde{N}(\mathrm{d}r, \mathrm{d}e)\right|^2\right)\right].
\end{aligned}$$

Using BDG inequality, we show that there is a universal constant C such that

$$\begin{aligned}
\mathbb{E}\left[\sup_{0\leq r\leq T} |Y^n_r - Y_r|^2\right] \leq C &\left[\mathbb{E}\int_0^T \left|(f_n - f)(r, X^{0,x_0}_r, Y^n_r, Z^n_r, K_r(\cdot))\right|^2 \mathbb{1}_{A_N}\mathrm{d}r\right. \\
&+ \mathbb{E}\int_0^T \left|(f_n - f)(r, X^{0,x_0}_r, Y^n_r, Z^n_r, K_r(\cdot))\right|^2 \mathbb{1}_{\bar{A}_N}\mathrm{d}r \\
&+ \mathbb{E}\int_0^T \left|f(r, X^{0,x_0}_r, Y^n_r, Z^n_r, K_r(\cdot)) - f(r, X^{0,x_0}_r, Y_r, Z_r, K_r(\cdot))\right|^2 \mathrm{d}r \\
&\left.+ \mathbb{E}\int_0^T |Z^n_r - Z_r|^2\mathrm{d}r + \mathbb{E}\int_0^T \|(K_r - K_r)(\cdot)\|^2_{q,\nu}\mathrm{d}r\right],
\end{aligned} \tag{2.13}$$

where A_N and $\bar{A}_N$ are defined by (2.12).

Since $(Z^n_\cdot)_{n\geq 1}$ (respectively $(K^n_\cdot(\cdot))_{n\geq 1}$) converges in $\mathcal{M}^2_{\mathcal{F}}(0,T,\mathbb{R}^{q\times q})$ (respectively $\mathcal{M}^2_{\mathcal{F}}([0,T]\times E, \mathbb{R}^q, \mathrm{d}t\nu(\mathrm{d}e))$ to $Z_\cdot$ (respectively $K_\cdot(\cdot)$), the fourth and fifth terms in the right hand side of the above inequality tends to 0 as n goes towards infinity. Then, by using similar arguments to estimate I_2^n (respectively I_4^n) in the previous step, one can prove that the first (respectively the third) term also tends to 0 as n goes to infinity. Next, we proceed to estimate the second term in inequality (2.13). Since f_n satisfies the property **(b)** and f satisfies $(\mathbf{H}_{2.4})$, we get

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T \left|(f_n - f)\left(r, X_r^{0,x_0}, Y_r^n, Z_r^n, K_r(\cdot)\right)\right|^2 \mathbb{1}_{\bar{A}_N} \mathrm{d}r\right] \\
&\leq C\mathbb{E}\left[\int_0^T (1 + |X_r^{0,x_0}| + |Y_r^n| + |Z_r^n| + \|K_r(\cdot)\|_{q,\nu})^{2\beta} \mathbb{1}_{\bar{A}_N} \mathrm{d}r\right], \ \beta \in \,]0,1[\,.
\end{aligned}$$

Thanks to Hölder's inequality applied with $p = \frac{1}{\beta}$ and $q = \frac{1}{1-\beta}$, along with $(\mathbf{H}_{4.1})$ and the estimate (2.5) one can get that,

$$\begin{aligned}
&\mathbb{E}\left[\int_0^T \left|(f_n - f)\left(r, X_r^{0,x_0}, Y_r^n, Z_r^n, K_r(\cdot)\right)\right|^2 \mathbb{1}_{\bar{A}_N} \mathrm{d}r\right] \\
&\leq C\left\{\mathbb{E}\left[\int_0^T (1 + |X_r^{0,x_0}| + |Y_r^n| + |Z_r^n| + \|K_r(\cdot)\|_{q,\nu})^{2} \mathrm{d}r\right]\right\}^{\beta} \left\{\mathbb{E}\left[\int_0^T \mathbb{1}_{\bar{A}_N} \mathrm{d}r\right]\right\}^{1-\beta} \\
&\leq C\left(T\mathbb{P}(\bar{A}_N)\right)^{1-\beta}.
\end{aligned}$$

Chebyshev's inequality yields that there is a positive constant C such that:

$$\mathbb{E}\left[\int_0^T \left|(f_n - f)\left(r, X_r^{0,x_0}, Y_r^n, Z_r^n, K_r(\cdot)\right)\right|^2 \mathbb{1}_{\bar{A}_N} \mathrm{d}r\right] \leq CN^{\beta-1}.$$

Eventually, by letting N tend to infinity, the previous inequality tends to 0 as n goes to infinity. Therefore,

$$\mathbb{E}\int_0^T \left|f_n(r, X_r^{0,x_0}, Y_r^n, Z_r^n, K_r^n(\cdot)) - f(r, X_r^{0,x_0}, Y_r, Z_r, K_r(\cdot))\right|^2 \mathrm{d}r \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Consequently, $(Y_\cdot^n)_{n\geq 1}$ converges to $Y_\cdot$ in $\mathcal{S}_{\mathcal{F}}^2(0,T,\mathbb{R}^q)$. □

Remark 2.2.1

Noting that all the previous results of this section still hold true if the Markov process $X_\cdot$ is constant with value $x \in \mathbb{R}^p$ on $[0,t]$.

2.2.3 Second Case: f is Continuous in y, z and k

We claim that, from a mathematics point of view, it is hard to deal with the general case where the generator f is continuous in $(y, z, k(\cdot))$. Indeed, the difficulty comes from the fact that the process $k(\cdot)$ takes values in the functional space $\mathcal{L}_{\nu}^{2,q}$ not in $\mathbb{R}^q$, we try, in the sequel, to relax the globally Lipschitz condition on $k(\cdot)$ by considering the following special case:

$$\begin{aligned} Y_s^{t,x} &= g(X_T^{t,x}) + \int_s^T f\left(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}, \int_E K_r^{t,x}(e)\,\nu(\mathrm{d}e)\right)\mathrm{d}r \\ &\quad - \int_s^T Z_r^{t,x}\,\mathrm{d}W_r - \int_s^T\int_E K_r^{t,x}(e)\widetilde{N}(\mathrm{d}r,\mathrm{d}e), \end{aligned} \tag{2.14}$$

where $X_s^{t,x} = x$ for $s \in [0,t]$.

For a given measurable function f defined from $[0,T] \times \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^{q\times q} \times \mathbb{R}^q$ into $\mathbb{R}^q$, we consider the three following assumptions:

$(\mathbf{H}_{2.5})$ For any $(t,x,y,z,k) \in [0,T] \times \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^{q\times q} \times \mathbb{R}^q$, there exists a constant $C > 0$ such that

$$|f(t,x,y,z,k)| \le C\left(1 + |x| + |y| + |z| + |k|\right).$$

$(\mathbf{H}_{2.6})$ For any $(t,x,y,z,k) \in [0,T] \times \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^{q\times q} \times \mathbb{R}^q$, there exists a constant $C > 0$ and $0 < \beta < 1$ such that

$$|f(t,x,y,z,k)| \le C(1 + |x| + |y| + |z| + |k|)^{\beta}.$$

$(\mathbf{H}_{2.7})$ The mapping $(y,z,k) \longmapsto f(t,x,y,z,k)$ is continuous for any fixed $t \in [0,T]$ and $x \in \mathbb{R}^p$.

Theorem 2.2.2

Under $(\mathbf{H}_{4.1})$, $(\mathbf{H}_{2.5})$ and $(\mathbf{H}_{2.7})$, the BSDEJ (2.14) has at least one solution $(Y_., Z_., K_.(\cdot))$ which belongs to $\mathbb{M}^2$. Furthermore, if f satisfies $(\mathbf{H}_{4.1})$, $(\mathbf{H}_{2.6})$ and $(\mathbf{H}_{2.7})$, then, the solution is in $\mathbb{M}_{\mathcal{S}}^2$.

Proof. For a given Ψ an element of $\mathcal{C}^\infty(\mathbb{R}^q \times \mathbb{R}^{q\times q} \times \mathbb{R}^q, \mathbb{R})$ with a compact support such that

$$\int_{\mathbb{R}^{q+q\times q+q}} \Psi(\overrightarrow{u})\mathrm{d}\overrightarrow{u} = 1,$$

where $\overrightarrow{u} = (y,z,k) \in \mathbb{R}^{q+q\times q+q}$. We define

$$f(t,x,(\cdot)) * \Psi(n(\cdot))(\overrightarrow{u}) = \int_{\mathbb{R}^{q+q\times q+q}} f(t,x,\overrightarrow{v})\Psi(n(\overrightarrow{u} - \overrightarrow{v}))\mathrm{d}\overrightarrow{v}$$

and $\bar{\Psi} \in \mathcal{C}^{\infty}(\mathbb{R}^q \times \mathbb{R}^{q\times q} \times \mathbb{R}^q, \mathbb{R})$ such that

$$\bar{\Psi}(\overrightarrow{u}) = \begin{cases} 1, & |\overrightarrow{u}|^2 \leq 1, \\ 0, & |\overrightarrow{u}|^2 \geq 2. \end{cases}$$

Let f be a function satisfying $(\mathbf{H}_{2.5})$ and $(\mathbf{H}_{2.7})$. The sequence of the measurable functions $\{f_n, n \geq 1\}$, defined by

$$f_n(t,x,\overrightarrow{u}) = n^3 \bar{\Psi}(\frac{\overrightarrow{u}}{n}) \left(f(t,x,(\cdot)) * \Psi(n(\cdot))\right)(\overrightarrow{u}),$$

satisfies the following assumptions:

(i) $\sup_{t,x} |f_n(t,x,y,z,k) - f_n(t,x,y',z',k')| \leq C(|y-y'| + |z-z'| + |k-k'|)$ for some positive constant C.

(ii) $|f_n(t,x,y,z,k)| \leq C(1+|x|+|y|+|z|+|k|)$, for all (t,x,y,z,k) in the product space $[0,T] \times \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^{q\times q} \times \mathbb{R}^q$.

(iii) For all $(t,x,y,z,k) \in [0,T] \times \mathbb{R}^p \times \mathbb{R}^q \times \mathbb{R}^{q\times q} \times \mathbb{R}^q$ and $n \in \mathbb{N}$, there exists a positive constant C such that $|f_n(t,x,y,z,k)| \leq C(1+|x|)$.

(iiii) For any $(t,x) \in [0,T] \times \mathbb{R}^p$, and for any compact subset $S \subset \mathbb{R}^q \times \mathbb{R}^{q\times q} \times \mathbb{R}^q$,

$$\sup_{(y,z,k)\in S} |f_n(t,x,y,z,k) - f(t,x,y,z,k)| \longrightarrow 0 \text{ as } n \to +\infty.$$

Firstly, we define the following family of approximating BSDEJs obtained by replacing the generator f in BSDEJ (2.14) with f_n defined above.

$$\begin{aligned} Y_s^{t,x;n} &= g(X_T^{t,x}) + \int_s^T f_n\left(r, X_r^{t,x}, Y_r^{t,x;n}, Z_r^{t,x;n}, \int_E K_r^{t,x;n}(e)\nu(\mathrm{d}e)\right)\mathrm{d}r \\ &\quad - \int_s^T Z_r^{t,x;n}\,\mathrm{d}W_r - \int_s^T \int_E K_r^{t,x;n}(e)\widetilde{N}(\mathrm{d}r,\mathrm{d}e). \end{aligned} \tag{2.15}$$

The result obtained by Tang and Li (Lemma 2.4 in [69]) implies that the equation (2.15) admits a unique solution denoted

$$(Y_\cdot^{t,x;n}, Z_\cdot^{t,x;n}, K_\cdot^{t,x;n}(\cdot))_{n\geq 1}$$

belongs to $\mathbb{M}^2$.

Taking into account that f_n satisfies **(iii)**, Theorem 1.4.1 confirms the existence of three sequences of measurable and deterministic functions $u^n : [0,T] \times \mathbb{R}^p \longrightarrow \mathbb{R}^q$, $v^n : [0,T] \times \mathbb{R}^p \longrightarrow \mathbb{R}^{q\times q}$ and $\theta^n : [0,T] \times \mathbb{R}^p \times E \longrightarrow \mathcal{L}^{2,q}_{\nu}$ such that

$$Y^{t,x;n}_r = u^n(r, X^{t,x}_r),\ Z^{t,x;n}_r = v^n(r, X^{t,x}_r) \text{ and } K^{t,x;n}_r(e) = \theta^n(r, X^{t,x}_{r-}, e).$$

Besides, we have the following equality

$$u^n(t,x) = \mathbb{E}\left[g(X^{t,x}_T) + \int_t^T F_n(r, X^{t,x}_r)\mathrm{d}r\right],\ \forall\ (t,x) \in [0,T] \times \mathbb{R}^p,$$

where we have denoted by

$$F_n(t,x) = f_n\left(t, x, u^n(t,x), v^n(t,x), \int_E \theta^{(n)}(t,x,e)\nu\,(\mathrm{d}e)\right).$$

Starting from the sequence defined in (2.15) and reasoning as in the three steps of the proof of Theorem 2.2.1, we can also establish the existence of at least one solution $(Y^{t,x}_{\cdot}, Z^{t,x}_{\cdot}, K^{t,x}_{\cdot}(\cdot))$ to BSDEJ (2.14) which belongs to $\mathbb{M}^2$ provided that $(\mathbf{H}_{4.1})$, $(\mathbf{H}_{2.5})$ and $(\mathbf{H}_{2.7})$ hold true. Furthermore, using similar arguments in the proof of Corollary 2.2.1, one can prove that the solution $(Y^{t,x}_{\cdot}, Z^{t,x}_{\cdot}, K^{t,x}_{\cdot}(\cdot))$ is in fact in $\mathbb{M}^2_S$ whenever $(\mathbf{H}_{4.1})$, $(\mathbf{H}_{2.6})$ and $(\mathbf{H}_{2.7})$ are in force. □

2.3 Examples of Markov Processes Satisfying L^2-Domination Condition

Let us give some examples of Markov processes satisfying the assumption $(\mathbf{H}_{2.1})$.

1. Obviously the Brownian motion with drift, starting at x at time t: $X^{t,x}_s = B^{t,x}_s + as$, where $B^{t,x}_s = x \in \mathbb{R}^p$ for all $s \leq t$ and $B^{t,x}_{\cdot}$ is an $\mathbb{R}^p$-valued Brownian motion and $a \in \mathbb{R}^p$, satisfies the $(\mathbf{H}_{2.1})$ since it has a density.

2. Let us now consider a more general Markov process solution to the following SDE

$$X^{t,x}_s = x + \int_t^s b(r, X^{t,x}_r)\mathrm{d}r + \int_t^s \sigma(r, X^{t,x}_r)\mathrm{d}W_r$$

with $X^{t,x}_s = x$ if $s \leq t$.

The functions $b : [0,T] \times \mathbb{R}^p \longrightarrow \mathbb{R}^p$ and $\sigma : [0,T] \times \mathbb{R}^p \longrightarrow \mathbb{R}^{p\times p}$, satisfy the following conditions:

(a) σ is Lipschitz w.r.t x uniformly in t;

(b) σ is invertible and bounded and its inverse is bounded;

(c) b is Lipschitz w.r.t x uniformly in t and of linear growth.

According to Lemma 4.3 in [40] the law $\mu(t,x;s,\mathrm{d}y)$ of $X_s^{t,x}$ satisfies ($\mathbf{H}_{2.1}$).

3. Let $(W_s^{t,x})_{t\le s\le T}$ be an $\mathbb{R}^d$-valued Brownian motion such that $W_s^{t,x}=x$ if $s\le t$ and $(A_s)_{0\le s\le T}$ an $\frac{\alpha}{2}$-stable subordinator starting at zero, $0<\alpha<2$, independent of $W_\cdot^{t,x}$ for every $\mathbb{P}_x$. Set $X_s^{t,x}=W_{A_r}^{t,x}$ a rotationally invariant α-stable process whose generator is the fractional power of order $\frac{\alpha}{2}$ of the negative Laplacian, corresponding to the Riesz potential of order α. It is well known that $X_\cdot^{t,x}$ is a Markov process and the law $\mu(t,x;s,\mathrm{d}y)$ of $X_s^{t,x}$ has a transition density $p(t,x;s,y)$ satisfying the following upper and lower estimates

$$\begin{aligned}
&c_1(s-t)^{-\frac{d}{\alpha}}\wedge\frac{s-t}{|x-y|^{d+\alpha}}\le p(t,x;s,y)\\
&=p(t-s,x-y)\le c_2(s-t)^{-\frac{d}{\alpha}}\wedge\frac{s-t}{|x-y|^{d+\alpha}},
\end{aligned}$$

for all $s\ge t$, and $x,y\in\mathbb{R}^d$. Under a simple relation between α and d, the law $\mu(t,x;s,\mathrm{d}y)$ of $X_s^{t,x}$ satisfies ($\mathbf{H}_{2.1}$).

4. Let D be an open subset of $\mathbb{R}^d$ and $\tau_D^X=\inf\{s>0: X_s\notin D\}$ be the exit time of X from D. The process X killed upon exiting D is defined by

$$X_s^D=\begin{cases} X_s & \text{if} \quad s<\tau_D^X \\ \varkappa & \text{if} \quad s\ge\tau_D^X\end{cases}=\begin{cases} W_{A_s} & \text{if} \quad s<\tau_D^X \\ \varkappa & \text{if} \quad s\ge\tau_D^X\end{cases}$$

where $\varkappa$ is an isolated point. The infinitesimal generator of the Markov process $X_\cdot^D$ is the Dirichlet fractional Laplacian, $-(-\Delta)^{\frac{\alpha}{2}}|_D$ i.e., the fractional Laplacian with zero exterior conditions. It is shown in [21, Theorem 1.1], that when D is a $\mathcal{C}^{1,1}$ open set in $\mathbb{R}^d$, $d\ge 1$ the heat kernel $p^D(t,x;s,y)$ of $-(-\Delta)^{\frac{\alpha}{2}}|_D$ which is also the transition density of $X_\cdot^D$ has the following lower and upper estimates: for every

$T > 0$ and $(s, x, y) \in (t, T] \times D \times D$,

$$\begin{aligned} & c_1 \left(1 \wedge \frac{\varrho(x)^{\frac{\alpha}{2}}}{\sqrt{s-t}}\right)\left(1 \wedge \frac{\varrho(y)^{\frac{\alpha}{2}}}{\sqrt{s-t}}\right) \\ & \leq \frac{p^D(t,x;s,y)}{p(t,x;s,y)} \\ & \leq c_2 \left(1 \wedge \frac{\varrho(x)^{\frac{\alpha}{2}}}{\sqrt{s-t}}\right)\left(1 \wedge \frac{\varrho(y)^{\frac{\alpha}{2}}}{\sqrt{s-t}}\right), \end{aligned}$$

where $\varrho(x)$ denotes the distance between x and D^c the compliment of D and $p(t, x; s, y)$ is the transition density defined in example 3. Therefore, under a simple condition on α and d, the law $\mu(t, x; s, \mathrm{d}y)$ of $X_s^{t,x}$ satisfies $(\mathbf{H}_{2.1})$.

5. For simplicity and ease of notation, we shall take $t = 0$. The Brownian motion killed upon exiting D is defined as

$$W_s^D = \begin{cases} W_s & \text{if} \quad s < \tau_D^W \\ \varkappa & \text{if} \quad s \geq \tau_D^W. \end{cases}$$

Now, we define the subordinate killed Brownian motion, $Y_s^D = W_{A_s}^D$ for all $s \geq 0$, as the process obtained by subordinating $W_\cdot^D$ via the $\frac{\alpha}{2}$-stable subordinator $A_\cdot$, that is

$$Y_s^D = \begin{cases} W_{A_s}^D & \text{if} \quad s < \tau_D^W \\ \varkappa & \text{if} \quad s \geq \tau_D^W. \end{cases}$$

Let $r^D(t, x; s, y)$ be the transition density of Y^D. It is proved in [68, Lemma 2.1] that if D is a bounded $\mathcal{C}^{1,1}$ domain in $\mathbb{R}^p$, $p \geq 1$ then for any $T > 0$, there exist two positive constants C_3 and C_4 such that for any $s \in (t, T]$ and $x, y \in D$.

$$C_3 q^D(s-t, x, y) \leq \frac{r^D(t,x;s,y)}{p(t,x;s,y)} \leq C_4 q^D(s-t, x, y),$$

where

$$q^D(s, x, y) = \left(\frac{\varrho(x)}{(s^{\frac{1}{\alpha}} + |x-y|)} \wedge 1\right)\left(\frac{\varrho(y)}{(s^{\frac{1}{\alpha}} + |x-y|)} \wedge 1\right).$$

Therefore, under a condition on α and p, the law $\mu(t, x; s, \mathrm{d}y)$ of $X_s^{t,x}$ satisfies $(\mathbf{H}_{2.1})$.

6. For $d \geq 2$, we consider the time-inhomogeneous and non-symmetric non-local operators:

$$\mathfrak{L}_t f(x) = \mathfrak{L}_t^a f(x) + b_t \cdot \nabla f(x) + \mathfrak{L}_t^\kappa f(x), \tag{2.16}$$

where

$$\mathfrak{L}_t^a f(x) = \frac{1}{2}\sum_{i,j=1}^{d} a_{i,j}(t,x)\frac{\partial^2 f}{\partial x_i x_j}(x), \ \ b_t\cdot\nabla f(x) = \sum_{i=1}^{d} b_i(t,x)\frac{\partial f}{\partial x_i}(x),$$

and

$$\mathfrak{L}_t^\kappa f(x) = \int_{\mathbb{R}^d}\Big(f(x+z)-f(x)-\mathbb{1}_{\{|z|\leq 1\}}z\cdot\nabla f(x)\Big)\frac{\kappa(t,x,z)}{|z|^{d+\alpha}}\mathrm{d}z,$$

where $a(t,x) := (a_{ij}(t,x))_{1\leq i,j\leq d}$ is a $d\times d$-symmetric matrix-valued measurable function on $\mathbb{R}_+\times\mathbb{R}^d$, $b(t,x):\mathbb{R}_+\times\mathbb{R}^d\longrightarrow\mathbb{R}^d$ and $\kappa(t,x,z):\mathbb{R}_+\times\mathbb{R}^d\times\mathbb{R}^d\longrightarrow\mathbb{R}^d$ are measurable functions, and $\alpha\in(0,2)$. We denote by $p(t,x;s,y)$ the fundamental solution of the operator $\{\mathfrak{L}_t^a, t\geq 0\}$ and $q(t,x;s,y)$ the fundamental solution of the operator $\{\mathfrak{L}_t, t\geq 0\}$. From (2.16) $\mathfrak{L}_t$ can be interpreted as a perturbation of $\mathfrak{L}_t^a$ by the operator $b_t\cdot\nabla+\mathfrak{L}_t^\kappa$, so the heat kernels p and q are related by the following Duhamel's formula:

$$\begin{aligned} q(t,x;s,y) &= p(t,x;s,y) \\ &\quad + \int_t^s\int_{\mathbb{R}^d} q(t,x;r,z)\,(b_r\cdot\nabla+\mathfrak{L}_r^\kappa)\,p(r,\cdot;s,y)(z)\mathrm{d}z\mathrm{d}r \\ &= p(t,x;s,y) \\ &\quad + \int_t^s\int_{\mathbb{R}^d} p(t,x;r,z)\,(b_r\cdot\nabla+\mathfrak{L}_r^\kappa)\,q(r,\cdot;s,y)(z)\mathrm{d}z\mathrm{d}r \end{aligned} \tag{2.17}$$

for all $0\leq t<s<\infty$ and $x,\ y\in\mathbb{R}^d$. For any $T>0$ and $\varepsilon\in[0,T)$, we denote

$$\mathbb{D}_\varepsilon^T = \Big\{(t,x;s,y) : s,\ t\geq 0 \text{ and } x,\ y\in\mathbb{R}^d \ \text{ with } \varepsilon<s-t<T\Big\}.$$

It is proved under some mild conditions of the coefficients a, b and κ (see ($\mathbf{H}^a$), and ($\mathbf{H}^\kappa$) in [20, Thoereom 1.1]) that there exists a unique heat kernel $q(t,x;s,y)$ satisfying (2.17). Moreover, $q(t,x;s,y)$ is the transition density of the Markov process X associated to the operator $\{\mathfrak{L}_t, t\geq 0\}$. The two-sided estimates below of q were established in [20, Corollary 1.5]: For any $T>0$, there exist constants C and $\lambda\geq 1$ such that on $\mathbb{D}_0^T$:

$$\begin{aligned} &bC^{-1}\left((s-t)^{-\frac{d}{\alpha}}e^{-\lambda^{-1}\frac{|x-y|^2}{s-t}} + m_\kappa(s-t)\left((s-t)^{\frac{1}{2}}+|x-y|\right)^{-d-\alpha}\right) \\ &\leq q(t,x;s,y), \end{aligned}$$

and

$$
\begin{aligned}
& q(t,x;s,y) \\
& \leq C\left((s-t)^{-\frac{d}{\alpha}} e^{-\lambda \frac{|x-y|^2}{s-t}} + \|\kappa\|_\infty (s-t)\left((s-t)^{\frac{1}{2}} + |x-y|\right)^{-d-\alpha}\right),
\end{aligned}
$$

where $m_\kappa = \inf_{(t,x)} \operatorname{essinf}_{z\in\mathbb{R}^d} \kappa(t,x,z)$. The law $\mu(t,x;s,\mathrm{d}y)$ of $X_s^{t,x}$ satisfies $(\mathbf{H}_{2.1})$.

Chapter 3

Quadratic BSDEJs and Related PIDEs

3.1 Introduction

Let $[0,T]$ be a bounded time interval, $E=\mathbb{R}\setminus\{0\}$ endowed with the Borel σ–algebra $\mathcal{B}(E)$ and $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\in[0,T]},\mathbb{P})$ a filtered probability space supporting a one–dimensional standard Brownian motion $W=\{W_t\}_{t\in[0,T]}$ and a time homogeneous Poisson random measure $N(\mathrm{d}s,\mathrm{d}e)$ with compensator $\nu(\mathrm{d}e)\mathrm{d}s$ on $([0,T]\times E,\mathcal{B}([0,T])\otimes\mathcal{B}(E))$, independent to each other. We denote by $\tilde{N}(\mathrm{d}s,\mathrm{d}e)=N(\mathrm{d}s,\mathrm{d}e)-\nu(\mathrm{d}e)\mathrm{d}s$ the compensated jump measure and we take $(\mathcal{F}_t)_{t\in[0,T]}$ as generated by W and $\tilde{N}$, completed with $\mathbb{P}$–null sets and made right continuous. Moreover, we assume that the filtration $(\mathcal{F}_t)_{t\in[0,T]}$ has the predictable representation property, that is any local martingale can be written as the sum of two stochastic integrals with predictable integrand processes, one w.r.t. W and the other w.r.t. $\tilde{N}$.

The aim of this chapter is to investigate a class of BSDEJs and related PIDEs of a quadratic type associated with a Brownian component and independent Poisson random measure. Let us point out that our work [52] is a natural continuation and extension to the jump case of the recent papers of Bahlali, Eddahbi & Ouknine (2017) [10] and Bahlali (2020) [8].

Concretely, we are concerned with $\mathbb{R}$–valued BSDEJs of quadratic type of the form

$$Y_t = \xi + \int_t^T H(Y_s, Z_s, K_s(\cdot))\mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s - \int_t^T \int_E K_s(e)\tilde{N}(\mathrm{d}s, \mathrm{d}e). \tag{3.1}$$

Herein, the terminal data ξ will be assumed to be square integrable. Our study covers the following cases:

$$H(y,z,k(\cdot)) = \begin{cases} f(y)\,|z|^2 + [k]_f(y) =: H_f(y,z,k(\cdot))) \\ h\,(y,k(\cdot)) + cz + H_f(y,z,k(\cdot))) \\ a + b\,|y| + c\,|z| + d\|k(\cdot)\|_{1,\nu} + H_f(y,z,k(\cdot)) \\ cz + f(y)\,|z|^2 - \int_E k(e)\nu(\mathrm{d}e) \\ cz + f(y)\,|z|^2 \\ h\,(y,k(\cdot)) + cz + f(y)\,|z|^2 \\ H_0\,(r,X_r) + H_f(y,z,k(\cdot)),\ (X_r)_{r\geq 0} \text{ is a Markov process,} \end{cases}$$

where f is a measurable and integrable function, $[k]_f\,(\cdot)$ is a functional of the unknown processes $Y_.$ and $K_.(\cdot)$ to be defined later and h and H_0 enjoy some classical assumptions. A long the whole chapter, we shall assume that the measurable function f is not constant. The case where f is constant has been considered under some restriction on the terminal data ξ, see [18, 19] for the continuous case and [3, 32, 44, 45] for the jump setting.
This chapter is organized as follows. Firstly, we prove Krylov's estimates, Itô-Krylov's formula and a priori estimates for eventual solutions of Eq(ξ, H_f) and then used to establish existence and uniqueness of solutions to Eq(ξ, H_f). The second result is devoted to the solvability of some QBSDEJs via several examples dealing with different generators of another quadratic form in z. These examples can not be covered by previous papers [3, 18, 19, 32, 44, 45]. Moreover, a comparison principle, for a class of QBSDEJs, is proved without any assumption on the third argument of the measurable generator H_f. The last section deals with the relationship between quadratic BSDEJs and quadratic PIDEs with measurable generators.

3.2 Existence and Uniqueness of Solutions

In this section, we will study the existence and uniqueness of solutions to one kind of quadratic BSDE with jumps and globally integrable drift H_f.

Definition 3.2.1

Given an $\mathcal{F}_T$–measurable and square integrable random variable ξ and measurable function $f : \mathbb{R} \longrightarrow \mathbb{R}$, a triple of processes (Y, Z, K), with adapted Y and predictable Z and K, is a solution of $\mathrm{Eq}(\xi, H_f)$ if (Y, Z, K) satisfies $\mathrm{Eq}(\xi, H_f)$, $\mathbb{P}$–a.s. for all $t \in [0, T]$ such that $Z \in \mathcal{M}^2_{\mathcal{F}}(0, T, \mathbb{R})$, $K \in \mathcal{M}^2_{\mathcal{F}}([0, T] \times E, \mathbb{R}, \mathrm{d}t\nu(\mathrm{d}e))$, $Y \in \mathcal{S}^2_{\mathcal{F}}(0, T; \mathbb{R})$ and $\mathbb{E}[| \int_0^T f(Y_s) |Z_s|^2 \mathrm{d}s|^2]$ is finite.

The next lemma is very useful. More precisely, it plays a crucial role in the proof of the Proposition 3.2.2 and the Theorem 3.2.2 below. In fact, the transformation F allows us to eliminate the generator H_f in the $\mathrm{Eq}(\xi, H_f)$.

Lemma 3.2.1

The function F defined by

$$F(x) = \int_0^x \exp(2 \int_0^y f(t)\mathrm{d}t)\mathrm{d}y, \tag{3.2}$$

satisfies,

$$F''(x) - 2f(x)F'(x) = 0, \ \text{for a.e. } x \in \mathbb{R}, \tag{3.3}$$

and has the following properties:

(i) F and F^{-1} are quasi–isometry, that is for any $x,\ y \in \mathbb{R}$ and $|f|_1 = \int_{\mathbb{R}} |f(x)|\, dx$

$$\begin{aligned} e^{-2|f|_1} |x-y| \le |F(x) - F(y)| \le e^{2|f|_1} |x-y| \,, \\ e^{-2|f|_1} |x-y| \le |F^{-1}(x) - F^{-1}(y)| \le e^{2|f|_1} |x-y| \,. \end{aligned} \tag{3.4}$$

(ii) F is a one-to-one function. Both F and its inverse function F^{-1} belongs to $\mathcal{W}^2_1(\mathbb{R})$.

Proof of (i). By definition the functions F and its inverse F^{-1} are continuous, one to one, strictly increasing functions, moreover $F''(x) - 2f(x)F'(x) = 0$ for a.e. $x \in \mathbb{R}$. In addition $F'(x) = \exp(2 \int_0^x f(t)\mathrm{d}t)$, hence

$$\text{for every } x \in \mathbb{R}, \quad e^{-2|f|_1} \le F'(x) \le e^{2|f|_1} \text{ and } e^{-2|f|_1} \le (F^{-1})'(x) \le e^{2|f|_1}. \tag{3.5}$$

Proof of (ii). Using the inequality (3.5), one can show that both F and F^{-1} are $\mathcal{C}^1(\mathbb{R})$. Since the second generalized derivative F'' satisfies (3.3) for almost all x, we get that F'' belongs to $L^1(\mathbb{R})$. Therefore, F belongs to the space $\mathcal{W}^2_1(\mathbb{R})$. Using again assertion (i), one can check easily that F^{-1} also belongs to $\mathcal{W}^2_1(\mathbb{R})$. □

Lemma 3.2.2

For a given real number y and a measurable function $k(\cdot)$ in $\mathcal{L}^1_\nu$,

(i) *the operator $[k]_f(y)$ given by*

$$[k]_f(y) := \int_E \frac{F(y+k(e)) - F(y) - F'(y)k(e)}{F'(y)} \nu(\mathrm{d}e) \tag{3.6}$$

is well defined. Moreover,

$$\left| [k]_f(y) \right| \leq \left(1 + e^{4|f|_1}\right) \|k(\cdot)\|_{1,\nu}. \tag{3.7}$$

(ii) *If f is non negative then $[k]_f(y) \geq 0$.*

Proof of (i). From the quasi–isometry properties of the function F defined in (3.2), for all $y \in \mathbb{R}$ and $0 \leq s \leq T$, we have

$$\begin{aligned}
\left| [k]_f(y) \right| &\leq \int_E \left| \frac{F(y+k(e)) - F(y) - F'(y)k(e)}{F'(y)} \right| \nu(\mathrm{d}e) \\
&\leq \int_E \left| \frac{F(y+k(e)) - F(y)}{F'(y)} \right| \nu(\mathrm{d}e) + \|k(\cdot)\|_{1,\nu} \\
&\leq \left(\frac{e^{2|f|_1}}{\min_y F'(y)} + 1 \right) \|k(\cdot)\|_{1,\nu}.
\end{aligned}$$

Thus,

$$\left| [k]_f(y) \right| \leq \left(1 + e^{4|f|_1}\right) \|k(\cdot)\|_{1,\nu}.$$

Hence, the operator $[k]_f(\cdot)$ is well defined.

Proof of (ii). Observe also that for each $y \in \mathbb{R}$, we can write

$$\begin{aligned}
[k]_f(y) &= \frac{1}{F'(y)} \int_E \left(F(y+k(e)) - F(y) - F'(y)k(e)\right) \nu(\mathrm{d}e) \\
&= \frac{1}{F'(y)} \int_E \int_y^{y+k(e)} \left(F'(x) - F'(y)\right) \mathrm{d}x \nu(\mathrm{d}e) \\
&= \frac{1}{F'(y)} \int_E \int_y^{y+k(e)} \left(F'(x) - F'(y)\right) \mathbb{1}_{\{k(e)>0\}} \mathrm{d}x \nu(\mathrm{d}e) \\
&+ \frac{1}{F'(y)} \int_E \int_{y+k(e)}^{y} \left(F'(y) - F'(x)\right) \mathbb{1}_{\{k(e)<0\}} \mathrm{d}x \nu(\mathrm{d}e).
\end{aligned}$$

The last two terms in the above inequality are non-negative since F' is positive and increasing. $\square$

Corollary 3.2.1

For a given real number y and a predictable process $k_s(e)$ on $[0,T]\times E$, such that

$$\int_0^T \|k_s(\cdot)\|_{1,\nu}\,\mathrm{d}s < +\infty \quad \mathbb{P}\text{–a.s.}$$

We have, using (3.7)

$$\int_0^T \left|[k_s]_f(y)\right|\mathrm{d}s \leq \left(1+e^{4|f|_1}\right)\int_0^T\int_E |k_s(e)|\,\nu(\mathrm{d}e)\mathrm{d}s \quad \mathbb{P}\text{–a.s. } \forall\, y\in\mathbb{R}.$$

Moreover, if $k_\cdot(\cdot)$ in $\mathcal{M}^2_{\mathcal{F}}([0,T]\times E,\mathbb{R},\mathrm{d}t\nu(\mathrm{d}e))$. Then there exists a constant $C_{f,\nu}$ (depending only on f and ν) such that

$$\int_0^T \mathbb{E}\left[\left|[k_s]_f(y)\right|^2\right]\mathrm{d}s \leq C_{f,\nu}\int_0^T \mathbb{E}\left[\|k_s(\cdot)\|_{2,\nu}\right]\mathrm{d}s. \tag{3.8}$$

3.2.1 Krylov's Estimates and Itô–Krylov's Formula for BSDEJs

Let us recall the Tanaka's formula. If $(X_t, t\geq 0)$ is a real–valued semi-martingale such that $\sum_{0\leq s\leq t}|\Delta X_s|$ is finite $\mathbb{P}$–a.s. for each $t>0$ and g is the difference between two convex functions, then, for each $x\in\mathbb{R}$, there exists an adapted process $(L_t(x), t\geq 0)$ such that, for each $t\geq 0$, with probability 1, we have

$$\begin{aligned} g(X_t) &= g(X_0) + \int_0^t g'_\ell(X_{s-})dX_s + \frac{1}{2}\int_{\mathbb{R}} g''_\ell(x)L_t(x)dx \\ &\quad + \sum_{0\leq s\leq t} g(X_s) - g(X_{s-}) - g'_\ell(X_{s-})\Delta X_s), \end{aligned}$$

where g'_ℓ stands for the left first derivatives of g and g''_ℓ is a signed measure which is the second derivative of g in the generalized function sense.

Proposition 3.2.1

Let $(Y_t, Z_t, K_t(e))_{0\leq t\leq T,\ e\in E}$ be a solution to $\mathrm{Eq}(\xi, H_f)$ in the sense of the definition 3.2.1. Put

$$\eta = 2\sup_{0\leq t\leq T}|Y_t| + \left(2+e^{4|f|_1}\right)\int_0^T \|K_s(\cdot)\|_{1,\nu}\,\mathrm{d}s,$$

then, for any measurable and integrable function ϕ on $\mathbb{R}$, we have

$$\mathbb{E}\int_0^T |\phi|\,(Y_s)|Z_s|^2\mathrm{d}s \leq 2\mathbb{E}\left[\eta\right]|\phi|_1\, e^{2|f|_1}. \tag{3.9}$$

Proof. Let x be a real number, set for notational simplicity $\psi_x(y) = (y-x)^-$. By Tanaka's formula, we have

$$\begin{aligned}
\psi_x(Y_t) &= \psi_x(Y_0) + \int_0^t \mathbb{1}_{\{Y_s<x\}} \mathrm{d}Y_s + \frac{1}{2}L_t(x) \\
&+ \int_0^t \int_E (\psi_x(Y_{s-} + K_s(e)) - \psi_x(Y_{s-}) - \mathbb{1}_{\{Y_{s-}<x\}} K_s(e))\nu(\mathrm{d}e)\mathrm{d}s \\
&= \psi_x(Y_0) + M_t + \frac{1}{2}L_t(x) \\
&- \int_0^t \mathbb{1}_{\{Y_s<x\}} \left(f(Y_s)\left|Z_s\right|^2 + [K_s]_f (Y_{s-})\right) \mathrm{d}s \\
&+ \int_0^t \int_E \left(\psi_x(Y_{s-} + K_s(e)) - \psi_x(Y_{s-}) - \mathbb{1}_{\{Y_{s-}<x\}} K_s(e)\right) \nu(\mathrm{d}e)\mathrm{d}s,
\end{aligned}$$

where

$$M_t = \int_0^t \mathbb{1}_{\{Y_s<x\}} Z_s \mathrm{d}W_s + \int_0^t \int_E \mathbb{1}_{\{Y_s<x\}} K_s(e)\tilde{N}(\mathrm{d}s, \mathrm{d}e)$$

is a martingale. Since the map $y \longmapsto \psi_x(y)$ is one–Lipschitz, it follows that:

$$\begin{aligned}
\frac{1}{2}L_t(x) &\leq |Y_t - Y_0| + \int_0^t \mathbb{1}_{\{Y_s<x\}} (f(Y_s)|Z_s|^2)\mathrm{d}s - M_t \\
&+ \int_0^T \left|[K_s]_f (Y_{s-})\right| \mathrm{d}s + 2\int_0^T \|K_s(\cdot)\|_{1,\nu} \mathrm{d}s,
\end{aligned}$$

hence

$$0 \leq L_t(x) \leq 2\eta - 2M_t + 2\int_{-\infty}^x L_t(y)\left|f(y)\right| \mathrm{d}y, \tag{3.10}$$

where we have used the estimation (3.7), the occupation time density formula and the notation (3.9). Taking the expectation in (3.10), we obtain

$$\mathbb{E}\left[L_t(x)\right] \leq 2\mathbb{E}\left[\eta\right] + 2\int_{-\infty}^x \mathbb{E}\left[L_t(y)\right]\left|f(y)\right| \mathrm{d}y.$$

Now, Gronwall's Lemma gives

$$\sup_x \mathbb{E}\left[L_t(x)\right] \leq 2\mathbb{E}\left[\eta\right] e^{2|f|_1}. \tag{3.11}$$

Let ϕ be a measurable and integrable function on $\mathbb{R}$. The time occupation formula shows that

$$\begin{aligned}
\mathbb{E}\int_0^T \left|\phi\right|(Y_s)|Z_s|^2\mathrm{d}s &= \mathbb{E}\int_0^T \left|\phi\right|(Y_s)\mathrm{d}[Y]_s^c = \mathbb{E}\int_{-\infty}^{\infty} \left|\phi\right|(x)L_T(x)dx \\
&\leq \sup_x \mathbb{E}\left[L_t(x)\right] \int_{-\infty}^{\infty} \left|\phi\right|(x)dx \leq 2\left|\phi\right|_1 e^{2|f|_1}\mathbb{E}\left[\eta\right].
\end{aligned}$$

Proposition 3.2.1 is proved since $\mathbb{E}\left[\eta\right]$ is finite thanks to Definition 3.2.1. $\square$

Now, we shall establish an Itô–Krylov's change of variable formula for the solutions of one-dimensional BSDEs with jumps.

Theorem 3.2.1

Let $(Y_t, Z_t, K_t(e))_{0\leq t\leq T,\ e\in E}$ be a solution to $\mathrm{Eq}(\xi, H)$. *Then, for any function g belonging to the space $\mathcal{W}^2_{1,loc}(\mathbb{R})$, we have with probability* 1

$$g(Y_t) = g(Y_0) + \int_0^t g'(Y_s)\mathrm{d}Y_s + \frac{1}{2}\int_0^t g''(Y_s)|Z_s|^2\mathrm{d}s \qquad (3.12)$$
$$+ \sum_{0<s\leq t} \left(g\left(Y_s\right) - g\left(Y_{s-}\right) - g'(Y_{s-})\Delta Y_s\right),$$

which can be written as

$$g(Y_t) = g(Y_0) + \int_0^t g'(Y_s)\mathrm{d}Y_s + \frac{1}{2}\int_0^t g''(Y_s)|Z_s|^2\mathrm{d}s$$
$$+ \int_0^t \int_E \left(g\left(Y_{s-} + K_s(e)\right) - g\left(Y_{s-}\right) - g'(Y_{s-})K_s(e)\right) N(\mathrm{d}s, \mathrm{d}e).$$

Proof. For $R > |Y_0|$, let $\tau_R := \inf\{t > 0 : \max(|Y_{t-}|, |Y_{t-} + K_t(e)|) \geq R\}$. Since τ_R tends to infinity as R tends to infinity, it is then enough to establish the formula (3.12) by replacing t with $t \wedge \tau_R$. The stochastic integral $\int_0^{t\wedge\tau_R} g'(Y_s)\mathrm{d}Y_s$ is well defined since g' is continuous and $(Y_s)_{0\leq s\leq T}$ is a càdlàg semi-martingale. Moreover, the jump term

$$\int_0^{t\wedge\tau_R}\int_E \left(g\left(Y_{s-} + K_s(e)\right) - g\left(Y_{s-}\right) - g'(Y_{s-})K_s(e)\right) N(\mathrm{d}e, \mathrm{d}s)$$

is also well defined since

$$\int_0^{t\wedge\tau_R}\int_E \left|\left(g\left(Y_{s-} + K_s(e)\right) - g\left(Y_{s-}\right) - g'(Y_{s-})K_s(e)\right)\right| \nu(\mathrm{d}e)\mathrm{d}s$$
$$\leq \left(M_R + \max_{0\leq t\leq T}|g'(Y_s)|\right)\int_0^{t\wedge\tau_R} \|K_s(\cdot)\|_{1,\nu}\,\mathrm{d}s,$$

and the fact that g and g' are locally Lipschitz continuous functions. Using Proposition 3.2.1, the term $\int_0^{t\wedge\tau_R} g''(Y_s)|Z_s|^2 ds$ is well defined since

$$\mathbb{E}\left|\int_0^{t\wedge\tau_R} g''(Y_s)|Z_s|^2\mathrm{d}s\right| \leq \mathbb{E}\left|\int_0^T g''(Y_s)|Z_s|^2\mathrm{d}s\right| \leq \sup_x \mathbb{E}\left[L_t(x)\right]|g''|_1.$$

Now, let g_n be a sequence of $\mathcal{C}^2$–class functions obtained via a classical regularization by convolution satisfying:

(*i*) g_n converges uniformly to g in the interval $[-R, R]$.

(*ii*) g_n' converges uniformly to g' in the interval $[-R, R]$.

(iii) g_n'' converges in $L^1([-R,R])$ to g''. Classical Itô's formula applied to $g_n(Y_{t\wedge\tau_R})$ gives

$$\begin{aligned} g_n(Y_{t\wedge\tau_R}) &= g_n(Y_0) + \int_0^{t\wedge\tau_R} g_n'(Y_s)\mathrm{d}Y_s + \frac{1}{2}\int_0^{t\wedge\tau_R} g_n''(Y_s)|Z_s|^2\mathrm{d}s \\ &+ \int_0^{t\wedge\tau_R}\int_E \left(g_n\left(Y_{s-}+K_s(e)\right) - g_n\left(Y_{s-}\right) - g_n'(Y_{s-})K_s(e)\right) N(\mathrm{d}s,\mathrm{d}e). \end{aligned} \tag{3.13}$$

Passing to the limit on n in (3.13) then use the above properties (i), (ii), (iii) and Proposition 3.2.1, to obtain

$$\begin{aligned} g(Y_{t\wedge\tau_R}) &= g(Y_0) + \int_0^{t\wedge\tau_R} g'(Y_s)\mathrm{d}Y_s + \frac{1}{2}\int_0^{t\wedge\tau_R} g''(Y_s)|Z_s|^2\mathrm{d}s \\ &+ \int_0^{t\wedge\tau_R}\int_E \left(g\left(Y_{s-}+K_s(e)\right) - g\left(Y_{s-}\right) - g'(Y_{s-})K_s(e)\right) N(\mathrm{d}s,\mathrm{d}e). \end{aligned}$$

Henceforth, the following types of Itô–Krylov's formula are used frequently throughout the remainder of the chapter. For a given generator H satisfying suitable conditions which guarantee the existence of a solution for BSDEJs

$$Y_t = \xi + \int_t^T H(Y_s,Z_s,K_s(\cdot))\mathrm{d}s - \int_t^T Z_s\mathrm{d}W_s - \int_t^T\int_E K_s(e)\tilde{N}(\mathrm{d}s,\mathrm{d}e).$$

Itô-Krylov's formula (3.12) applied to $F(Y_t)$ leads to

$$\begin{aligned} F(Y_t) = F(\xi) &+ \int_t^T \left(F'(Y_{s-})(H(Y_s,Z_s,K_s(\cdot)) - \frac{1}{2}F''(Y_s)\left|Z_s\right|^2\right)\mathrm{d}s \\ &- \int_t^T F'(Y_{s-})Z_s\mathrm{d}W_s - \int_t^T\int_E F'(Y_{s-})K_s(e)\tilde{N}(\mathrm{d}s,\mathrm{d}e) \\ &- \sum_{t<s\leq T}\left(F(Y_s) - F(Y_{s-}) - F'(Y_{s-})\Delta Y_s\right). \end{aligned}$$

Furthermore,

$$\sum_{t<s\leq T}\left(F(Y_s) - F(Y_{s-}) - F'(Y_{s-})\Delta Y_s\right) = \int_t^T\int_E \left(F(Y_s) - F(Y_{s-}) - F'(Y_{s-})K_s(e)\right) N(\mathrm{d}s,\mathrm{d}e).$$

Remember also that

$$F'(y)\left[k\right]_f(y) = \int_E \left(F\left(y+k(e)\right) - F(y) - F'(y)k(e)\right)\nu(\mathrm{d}e).$$

This implies

$$\begin{aligned} F(Y_t) = F(\xi) &+ \int_t^T \left(F'(Y_{s-})\left(H(Y_s,Z_s,K_s(\cdot)) - \left[K_s\right]_f(Y_{s-})\right) - \frac{1}{2}F''(Y_s)\left|Z_s\right|^2\right)\mathrm{d}s \\ &- \int_t^T F'(Y_{s-})Z_s\mathrm{d}W_s - \int_t^T\int_E \left(F(Y_{s-}+K_s(e)) - F(Y_{s-})\right)\tilde{N}(\mathrm{d}s,\mathrm{d}e). \end{aligned} \tag{3.14}$$

In particular, if $H(y,z,k(\cdot)) = H_f(y,z,k(\cdot)) = f(y)\,|z|^2 + [k]_f\,(y)$, we get

$$F(Y_t) = F(\xi) - \int_t^T F'(Y_{s-}) Z_s \mathrm{d}W_s - \int_t^T \int_E \left(F(Y_{s-} + K_s(e)) - F(Y_{s-})\right) \tilde{N}(\mathrm{d}s, \mathrm{d}e). \quad (3.15)$$

For each $0 \leq s \leq T$ we set $y_s := F(Y_s)$, $z_s := F'(Y_{s-})Z_s$ and $k_s\,(e) := F(Y_{s-} + K_s(e)) - F(Y_{s-})$. These notations will be used repeatedly until the remainder of this chapter. □

Before proving the existence and uniqueness of solutions to $\mathrm{Eq}(\xi, H_f)$, we establish some useful a priori estimates.

3.2.2 A Priori Estimates

Proposition 3.2.2

Let $\xi \in L^2(\Omega)$ and $f \in L^1(\mathbb{R})$. If (Y, Z, K) satisfies the $\mathrm{Eq}(\xi, H_f)$, then we have:

(i) $(z_r)_{0\leq r\leq T}$, $(Z_r)_{0\leq r\leq T} \in \mathcal{M}^2_{\mathcal{F}}(0, T, \mathbb{R})$ *and* $(k_r(e))_{0\leq r\leq T, e\in E}$,
$(K_r(e))_{0\leq r\leq T, e\in E} \in \mathcal{M}^2_{\mathcal{F}}([0, T] \times E, \mathbb{R}, \mathrm{d}t\nu(\mathrm{d}e))$,

(ii) $(y_r)_{0\leq r\leq T}$, $(Y_r)_{0\leq r\leq T} \in \mathcal{S}^2_{\mathcal{F}}\,(0, T; \mathbb{R})$,

(iii) $\mathbb{E}\left|\int_0^T f(Y_r)\,|Z_r|^2\,\mathrm{d}r\right|^2$ *is finite.*

Proof of (i). From Itô–Krylov's formula (3.15), we have

$$F(Y_t) = F(\xi) - \int_t^T F'(Y_{s-}) Z_s \mathrm{d}W_s - \int_t^T \int_E \left(F(Y_{s-} + K_s(e)) - F(Y_{s-})\right) \tilde{N}(\mathrm{d}s, \mathrm{d}e), \quad (3.16)$$

since F satisfies (3.3). For $t = 0$ we get

$$\int_0^T F'(Y_{r-}) Z_r \mathrm{d}W_r + \int_0^T \int_E \left(F(Y_{r-} + K_r(e)) - F(Y_{r-})\right) \widetilde{N}(\mathrm{d}r, \mathrm{d}e) = F(\xi) - F(Y_0). \quad (3.17)$$

Take the square of the $L^2(\Omega)$ norm in (3.17), thanks to the orthogonality of the martingales $W_{\cdot}$ and $\int_0^{\cdot} \int_E \widetilde{N}(\mathrm{d}r, \mathrm{d}e)$ together with the inequalities (3.4) and (3.5), we get

$$\begin{aligned}
&e^{-4|f|_1}\left(\int_0^T \mathbb{E}\left[|Z_r|^2\right]\mathrm{d}r+\int_0^T \mathbb{E}\left[\|K_r(\cdot)\|^2_{2,\nu}\right]\mathrm{d}r\right)\\
&\leq \mathbb{E}\left|\int_0^T F'(Y_{r-})Z_r\mathrm{d}W_r\right|^2+\mathbb{E}\int_0^T\int_E |F(Y_{r-}+K_r(e))-F(Y_{r-})|^2\,\nu(\mathrm{d}e)\mathrm{d}r\\
&=\mathbb{E}\left|\int_0^T z_r\mathrm{d}W_r\right|^2+\mathbb{E}\left|\int_0^T\int_E k_r(e)\widetilde{N}(\mathrm{d}r,\mathrm{d}e)\right|^2\\
&=\int_0^T \mathbb{E}\left[|z|_r^2\right]\mathrm{d}r+\int_0^T \mathbb{E}\left[\|k_r(\cdot)\|^2_{2,\nu}\right]\mathrm{d}r\\
&\leq F^2(Y_0)+\mathbb{E}\left[F^2(\xi)\right]\leq F^2(Y_0)+e^{4|f|_1}\mathbb{E}\left[\xi^2\right]<\infty.
\end{aligned}$$

This implies that $z,\ Z\in\mathcal{M}^2_{\mathcal{F}}(0,T,\mathbb{R})$ and $k,\ K\in\mathcal{M}^2_{\mathcal{F}}([0,T]\times E,\mathbb{R},\mathrm{d}t\nu(\mathrm{d}e))$.

Proof of (ii). From Itô–Krylov's formula (3.15), we have

$$F(Y_t)=F(\xi)-\int_t^T F'(Y_{s-})Z_s\mathrm{d}W_s-\int_t^T\int_E\left(F(Y_{s-}+K_s(e))-F(Y_{s-})\right)\tilde{N}(\mathrm{d}s,\mathrm{d}e). \quad (3.18)$$

Thanks to (3.4) and $F(0)=0$,

$$\begin{aligned}
e^{-2|f|_1}|Y_t|\leq |F(Y_t)|&\\
&\leq |F(\xi)|+\left|\int_t^T F'(Y_{s-})Z_s\mathrm{d}W_s\right|\\
&+\left|\int_t^T\int_E\left(F(Y_{s-}+K_s(e))-F(Y_{s-})\right)\tilde{N}(\mathrm{d}s,\mathrm{d}e)\right|\\
&\leq |F(\xi)|+\sup_{0\leq t\leq T}\left|\int_0^t z_s\mathrm{d}W_s\right|+\sup_{0\leq t\leq T}\left|\int_0^t\int_E k_s(e)\tilde{N}(\mathrm{d}s,\mathrm{d}e)\right|.
\end{aligned}$$

Using convex inequality and taking the supremum over $[0,T]$ lead to

$$\begin{aligned}
&e^{-4|f|_1}\sup_{0\leq t\leq T}|Y_t|^2\leq\sup_{0\leq t\leq T}|y_t|^2\\
&\leq 2^2\left(e^{4|f|_1}|\xi|^2+\sup_{0\leq t\leq T}\left|\int_0^t z_s\mathrm{d}W_s\right|^2+\sup_{0\leq t\leq T}\left|\int_0^t\int_E k_s(e)\widetilde{N}(\mathrm{d}s,\mathrm{d}e)\right|^2\right).
\end{aligned}$$

Now, by taking the expectation and using BDG inequality, we get

$$\begin{aligned}
&e^{-4|f|_1}\mathbb{E}\left[\sup_{0\leq t\leq T}|Y_t|^2\right]\leq\mathbb{E}\left[\sup_{0\leq t\leq T}|y_t|^2\right]\\
&\leq 4\left(e^{4|f|_1}\mathbb{E}\left[|\xi|^2\right]+\int_0^T\mathbb{E}\left[|z_s|^2\right]\mathrm{d}s+\int_0^T\mathbb{E}\left[\|k_s(\cdot)\|^2_{2,\nu}\right]\mathrm{d}s\right).
\end{aligned}$$

The right-hand side of the above inequality is finite by (i).

Proof of (iii). Since $(Y_\cdot, Z_\cdot, K_\cdot(\cdot))$ satisfies $\mathrm{Eq}(\xi, H_f)$, thus

$$\int_0^T \left(f(Y_r)\left|Z_r\right|^2\right)\mathrm{d}r = \int_0^T Z_r \mathrm{d}W_r + \int_0^T\int_E K_r(e)\widetilde{N}(\mathrm{d}r, \mathrm{de}) + Y_0 - \xi - \int_0^T \left[K_r\right]_f \left(Y_{r-}\right)\ \mathrm{d}r.$$

Using convex inequality and taking the expectation, we obtain

$$\begin{aligned}\mathbb{E}\left|\int_0^T f(Y_r)\left|Z_r\right|^2\mathrm{d}r\right|^2 &\leq 2^4\left(\mathbb{E}\left|\int_0^T Z_r\mathrm{d}W_r\right|^2 + \mathbb{E}\left|\int_0^T\int_E K_r(e)\widetilde{N}(\mathrm{d}r,\mathrm{de})\right|^2\right)\\ &\quad + 2^4\left(\left|Y_0\right|^2 + \mathbb{E}\left|\xi\right|^2 + T\int_0^T \mathbb{E}\left|\left[K_r\right]_f\left(Y_{r-}\right)\right|^2\mathrm{d}r\right)\\ &\leq 2^4\left(\mathbb{E}\int_0^T \left|Z_r\right|^2\mathrm{d}r + \mathbb{E}\int_0^T \left\|K_r(\cdot)\right\|_{2,\nu}^2\mathrm{d}r\right)\\ &\quad + 2^4\left(\left|Y_0\right|^2 + \mathbb{E}\left|\xi\right|^2 + TC_f\mathbb{E}\int_0^T\left\|K_r(\cdot)\right\|_{2,\nu}^2\mathrm{d}r\right).\end{aligned}$$

Finally $\mathbb{E}\left|\int_0^T f(Y_r)\left|Z_r\right|^2\mathrm{d}r\right|^2$ is finite thanks to (i). □

We now state and prove the main result, which is given by Theorem **3.2.2** below. We shall refer to the following simple equation

$$y_t = F(\xi) - \int_t^T z_s \mathrm{d}W_s - \int_t^T\int_E k_s(e)\tilde{N}(\mathrm{d}s, \mathrm{de})$$

as $\mathrm{Eq}(F(\xi), 0)$.

Theorem 3.2.2

Let ξ be an $\mathcal{F}_T$–measurable and square integrable random variable. If f is an integrable function, then $(Y_t, Z_t, K_t(e))_{0\leq t\leq T,\ e\in E}$ is a solution to $\mathrm{Eq}(\xi, H_f)$ if and only if $(y_t, z_t, k_t(e))_{0\leq t\leq T,\ e\in E}$ is a solution to $\mathrm{Eq}(F(\xi), 0)$.

Proof. If (Y, Z, K) is a solution to $\mathrm{Eq}(\xi, H_f)$, then (3.16) shows that $(y_t, z_t, k_t(e))_{0\leq t\leq T,\ e\in E}$ satisfies $\mathrm{Eq}(F(\xi), 0)$. Moreover, thanks to the Proposition 3.2.2, $(y_t, z_t, k_t(e))_{0\leq t\leq T,\ e\in E}$ is a solution to $\mathrm{Eq}(F(\xi), 0)$ in the sense of the Definition 3.2.1.

Conversely: Let $(y_t, z_t, k_t(e))_{0\leq t\leq T,\ e\in E}$ be a solution to $\mathrm{Eq}(F(\xi), 0)$, then Itô-Krylov's formula (3.12) applied to $Y_t = F^{-1}(y_t)$ (since F^{-1} belongs to $\mathcal{W}_1^2(\mathbb{R})$) shows that

$$\begin{aligned}F^{-1}(y_t) = \xi &- \int_t^T (F^{-1})'(y_{s-})z_s\mathrm{d}W_s - \int_t^T\int_E (F^{-1})'(y_{s-})k_s(e)\tilde{N}(\mathrm{d}s, \mathrm{de})\\ &- \frac{1}{2}\int_t^T (F^{-1})''(y_s)\left|z_s\right|^2\mathrm{d}s - \sum_{t<s\leq T}\left(F^{-1}(y_s) - F^{-1}(y_{s-}) - (F^{-1})'(y_{s-})\Delta y_s\right),\end{aligned}$$

then,

$$
\begin{aligned}
Y_t &= \xi - \int_t^T (F^{-1})'(y_{s-}) z_s \mathrm{d}W_s - \int_t^T \int_E (F^{-1})'(y_{s-}) k_s(e) \tilde{N}(\mathrm{d}s, \mathrm{d}e) \\
&- \frac{1}{2} \int_t^T (F^{-1})''(y_s) \left| z_s \right|^2 \mathrm{d}s \\
&- \int_t^T \int_E \Big(F^{-1}(y_{s-} + k_s(e)) - F^{-1}(y_{s-}) - (F^{-1})'(y_{s-}) k_s(e) \Big) N(\mathrm{d}s, \mathrm{d}e) \\
&= \xi - \int_t^T (F^{-1})'(y_{s-}) z_s \mathrm{d}W_s - \int_t^T \int_E (F^{-1})'(y_{s-}) k_s(e) \tilde{N}(\mathrm{d}s, \mathrm{d}e) \\
&- \frac{1}{2} \int_t^T (F^{-1})''(y_s) \left| z_s \right|^2 \mathrm{d}s \\
&+ \int_t^T \int_E \Big(F^{-1}(y_{s-} + k_s(e)) - F^{-1}(y_{s-}) - (F^{-1})'(y_{s-}) k_s(e) \Big) \nu(\mathrm{d}e) \mathrm{d}s \\
&- \int_t^T \int_E \Big(F^{-1}(y_{s-} + k_s(e)) - F^{-1}(y_{s-}) - (F^{-1})'(y_{s-}) k_s(e) \Big) \widetilde{N}(\mathrm{d}s, \mathrm{d}e).
\end{aligned} \tag{3.19}
$$

Notice that:

$$
(F^{-1})'(x) = \frac{1}{F'(F^{-1}(x))} \text{ and } (F^{-1})''(x) = -\frac{F''(F^{-1}(x))}{(F'(F^{-1}(x)))^2} (F^{-1})'(x) = -\frac{F''(F^{-1}(x))}{(F'(F^{-1}(x)))^3}.
$$

Set $Z_s = (F^{-1})'(y_{s-}) z_s$ and $K_s(e) = F^{-1}(y_{s-} + k_s(e)) - F^{-1}(y_{s-})$ this implies

$$
\begin{aligned}
\frac{1}{2} (F^{-1})''(y_s) \left| z_s \right|^2 &= -\frac{1}{2} \frac{F''(Y_s)}{(F'(Y_s))^3} \frac{\left| Z_s \right|^2}{((F^{-1})'(y_{s-}))^2} \overset{\mathrm{d}s \text{ a.e.}}{=} -\frac{1}{2} \frac{F''(Y_s)}{F'(Y_s)} \left| Z_s \right|^2 \\
&= -f(Y_s) \left| Z_s \right|^2
\end{aligned} \tag{3.20}
$$

and

$$
\begin{aligned}
&\int_E \Big(F^{-1}(y_{s-} + k_s(e)) - F^{-1}(y_{s-}) - (F^{-1})'(y_{s-}) k_s(e) \Big) \nu(\mathrm{d}e) \\
&= \int_E \left(K_s(e) - \frac{1}{F'(Y_s)} \left(F(Y_s) - F(Y_{s-}) \right) \right) \nu(\mathrm{d}e) \\
&= -\int_E \left(\frac{F(Y_s) - F(Y_{s-}) - F'(Y_s) K_s(e)}{F'(Y_s)} \right) \nu(\mathrm{d}e) = -\left[K_s \right]_f (Y_{s-}).
\end{aligned} \tag{3.21}
$$

Substituting (3.20) and (3.21) in (3.19), we end up with

$$
Y_t = \xi + \int_t^T \Big(f(Y_s) \left| Z_s \right|^2 + \left[K_s \right]_f (Y_{s-}) \Big) \mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s - \int_t^T \int_E K_s(e) \tilde{N}(\mathrm{d}s, \mathrm{d}e).
$$

Observe that as in the proof of the Proposition 3.2.2 and thanks to the properties (3.4) and (3.5) we show easily that

$$
\left| Y_s \right| \leq e^{2|f|_1} \left| y_s \right|, \ \left| Z_s \right| \leq e^{2|f|_1} \left| z_s \right| \text{ and } \left| K_s(e) \right| \leq e^{2|f|_1} \left| k_s(e) \right|.
$$

Consequently,

$$(Y_s, Z_s, K_s(e))_{0\le s\le T,\ e\in E} := (F^{-1}(y_s), (F^{-1})'(y_{s-})z_s, F^{-1}(y_{s-}+k_s(e))-F^{-1}(y_{s-}))_{0\le s\le T,\ e\in E}$$

is a solution to $\mathrm{Eq}(\xi, H_f)$ in the sense of Definition 3.2.1.

Comment: Taking the conditional expectation on both sides in $\mathrm{Eq}(F(\xi), 0)$, we get

$$y_t = \mathbb{E}\left[F(\xi) \mid \mathcal{F}_t\right],$$

and thus

$$Y_t = F^{-1}\left(\mathbb{E}\left[F(\xi) \mid \mathcal{F}_t\right]\right).$$

Since F and F^{-1} are globally Lipschitz then $F(\xi)$ belongs to $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$ if and only if ξ belongs to $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$, then the martingale representation theorem shows that there exists a unique predictable process $(z, k) \in \mathcal{M}^2_{\mathcal{F}}(0, T, \mathbb{R}) \otimes \mathcal{M}^2_{\mathcal{F}}([0, T] \times E, \mathbb{R}, \mathrm{d}t\nu(\mathrm{d}e))$ satisfying $\mathrm{Eq}(F(\xi), 0)$. Now, comparing with $\mathrm{Eq}(F(\xi), 0)$, one can easily arrive at

$$Y_s = F^{-1}\left(y_s\right), \quad Z_s = \frac{z_s}{F'(F^{-1}\left(y_{s-}\right))} \quad \text{and} \quad K_s\left(e\right) = F^{-1}\left(y_{s-} + k_s\left(e\right)\right) - F^{-1}\left(y_{s-}\right).$$

We deduce finally that $\mathrm{Eq}(\xi, H_f)$ admits a unique solution if and only if $\mathrm{Eq}(F(\xi), 0)$ admits a unique solution. □

Corollary 3.2.2

Let f be a bounded and integrable function on $\mathbb{R}$. The $\mathrm{Eq}(\xi, H_f)$ admits a unique solution for any $\mathcal{F}_T$–measurable square integrable random variable ξ.

Remark 3.2.1

If the function f is bounded and integrable on $\mathbb{R}$, then

$$|H_f(y, z, k(\cdot))| \le \sup_{x\in\mathbb{R}}(|f(x)||z|^2) + (1 + e^{4|f|_1})\,\|k(\cdot)\|_{1,\nu} =: H^*(z, k(\cdot)).$$

Therefore, the existence and uniqueness of the solution to $\mathrm{Eq}(\xi, H^*)$ requires the existence of exponential moments or the boundedness of the terminal value ξ. However, $\mathrm{Eq}(\xi, H_f)$ has a unique solution for ξ in $L^2(\Omega, \mathcal{F}_T, \mathbb{P})$.

3.3 Solvability of some Quadratic BSDEJs

In the present section, we shall use the results of the previous theorem to solve some QBSDEJs in the following examples. Let $h : \mathbb{R} \times \mathcal{L}^2_\nu \longrightarrow \mathbb{R}$ be a measurable function which satisfies:

($\mathbf{A}_{3.1}$) For all $k(\cdot)$, $\acute{k}(\cdot)$ in $\mathcal{L}^2_\nu$ and y, $\acute{y}$ in $\mathbb{R}$

$$|h(y,k(\cdot)) - h(\acute{y},\acute{k}(\cdot))| \leq L(|y-\acute{y}| + \|k(\cdot)-\acute{k}(\cdot)\|_{2,\nu}),$$

and there exists a constant $C>0$ such that

$$|h(y,k(\cdot))| \leq C.$$

($\mathbf{A}_{3.2}$) f is bounded and integrable.

($\mathbf{A}_{3.3}$) The function $h\,(y,k(\cdot))$ is of the form $h(y, \int_E g(k(e))\nu(\mathrm{d}e))$ and the mapping $(y,k) \longmapsto h(y,k)$ is continuous

$$\left|h(y, \int_E g(k(e))\nu(\mathrm{d}e))\right| \leq L(|y| + \|k(\cdot)\|_{1,\nu}).$$

Example 1: $\mathrm{Eq}(\xi, h\,(y,k(\cdot)) + cz + H_f(y,z,k(\cdot)))$

The formula (3.14) corresponding to

$$H_{h,f}(y,z,k(\cdot)) := h\,(y,k(\cdot)) + cz + H_f(y,z,k(\cdot)),\ c \in \mathbb{R},$$

shows that $\mathrm{Eq}(\xi, H_{h,f})$ is equivalent to $\mathrm{Eq}(F(\xi), H_1 + cz)$ where

$$H_1(y,k(\cdot)) := F'(F^{-1}\,(y))(h(F^{-1}(y), F^{-1}(y+k(\cdot)) - F^{-1}(y))).$$

The equation $\mathrm{Eq}(F(\xi), H_1+cz)$ has a unique solution if and only if H_1 is Lipschitz and $F(\xi)$ is square integrable. Which is the case under assumptions $(\mathbf{A}_{3.1}) - (\mathbf{A}_{3.2})$. Therefore, $\mathrm{Eq}(F(\xi), H_1+cz)$ has a unique solution, thus our original $\mathrm{Eq}(\xi, H_{h,f})$ admits a unique solution.

Example 2: $\mathrm{Eq}(\xi, a + b\,|y| + c\,|z| + d\,\|k(\cdot)\|_{1,\nu} + H_f(y,z,k(\cdot)))$

Similarly to the Example 1, $\mathrm{Eq}(\xi, a + b\,|y| + c\,|z| + d\,\|k(\cdot)\|_{1,\nu} + H_f(y,z,k(\cdot)))$ is equivalent to $\mathrm{Eq}(F(\xi), H_2)$ where

$$H_2(y,z,k) := F'(F^{-1}\,(y))(a + b|F^{-1}\,(y)\,| + c\,|z| + d\|F^{-1}(y+k(\cdot)) - F^{-1}(y)\|_{1,\nu}).$$

The generator H_2 is continuous in (y,z,k) and is of linear growth in $|y|$, $|z|$ and $\|k(\cdot)\|_{1,\nu}$, therefore, thanks to the result of Qin and Xia [65] the $\mathrm{Eq}(F(\xi),H_2)$ has at least one solution whenever ξ is square integrable and f is integrable.

Example 3: $\mathrm{Eq}(\xi, cz+f(y)\,|z|^2-\int_E k(e)\nu(de))$

Consider the following BSDEJ

$$\begin{aligned}Y_t=\xi+\int_t^T\left(cZ_s+f(Y_s)\,|Z_s|^2-\int_E K_s(e)\nu(\mathrm{d}e)\right)\mathrm{d}s\\ -\int_t^T Z_s\mathrm{d}W_s-\int_t^T\int_E K_s(e)\tilde{N}(\mathrm{d}s,\mathrm{d}e).\end{aligned}$$

By taking $H(y,z,k(\cdot))=cz+f(y)\,|z|^2-\int_E k(e)\nu(\mathrm{d}e)$ in (3.14), we get

$$\begin{aligned}F(Y_t)=F(\xi)-\int_t^T\int_E\left(F(Y_{s-}+K_s(e))-F(Y_{s-})\right)\nu\left(\mathrm{d}e\right)\mathrm{d}s+c\int_t^T F'(Y_{s-})Z_s\mathrm{d}s\\ -\int_t^T F'(Y_{s-})Z_s\mathrm{d}W_s-\int_t^T\int_E\left(F(Y_{s-}+K_s(e))-F(Y_{s-})\right)\tilde{N}(\mathrm{d}s,\mathrm{d}e),\end{aligned}$$

or equivalently as

$$y_t=F(\xi)+\int_t^T\left(cz_s-\int_E k_s\left(e\right)\nu\left(\mathrm{d}e\right)\right)\mathrm{d}s-\int_t^T z_s\mathrm{d}W_s-\int_t^T\int_E k_s\left(e\right)\tilde{N}(\mathrm{d}s,\mathrm{d}e).$$

Since the generator $H_3(z,k\left(\cdot\right))=cz-\int_E k\left(e\right)\nu\left(\mathrm{d}e\right)$ of the above equation is linear in z and k and $F(\xi)$ is square integrable for square integrable ξ, then it has a unique solution. Thus, $\mathrm{Eq}(F(\xi),H_3)$ has a unique solution.

Example 4: $\mathrm{Eq}(\xi, cz+f(y)\,|z|^2)$

Consider the following BSDEJ

$$Y_t=\xi+\int_t^T\left(cZ_s+f(Y_s)\,|Z_s|^2\right)\mathrm{d}s-\int_t^T Z_s\mathrm{d}W_s-\int_t^T\int_E K_s(e)\tilde{N}(\mathrm{d}s,\mathrm{d}e).$$

Then, by taking $H(y,z,k(\cdot))=cz+f(y)\,|z|^2$ in (3.14), we arrive at

$$y_t=F(\xi)+\int_t^T\left(cz_s+\int_E H(y_{s-},k_s(e))\nu\left(\mathrm{d}e\right)\right)\mathrm{d}s-\int_t^T z_s\mathrm{d}W_s-\int_t^T\int_E k_s\left(e\right)\tilde{N}(\mathrm{d}s,\mathrm{d}e),$$

where $H(y,k(e)):=F'(F^{-1}(y))\left(F^{-1}(y+k(e))-F^{-1}(y)\right)-k(e)$.

Simple computations show that

$$|H(y,k(e))-H(\acute{y},\acute{k}(e))|\leq L(1+|k(e)|)\,|y-\acute{y}|+(1+e^{4|f|})|k(e)-\acute{k}(e)|$$

and

$$|H(y,k(e))|\leq\min((1+e^{4|f|})\,|k(e)|\,;|k(e)|+2e^{4|f|}\left(|k(e)|+2\,|y|\right)).$$

Consequently, the generator

$$H_4(y,z,k(\cdot)) = cz + \int_E H(y,k(e))\nu\,(\mathrm{d}e)$$

is Lipschitz in z and $k(\cdot)$, continuous in y and is of linear growth in all arguments, this implies that $\mathrm{Eq}(F(\xi),H_4)$ admits at least one solution whenever $F(\xi)$ is square integrable (see e.g. [65]).

Consequently $\mathrm{Eq}(\xi, cz + f(y)\,|z|^2)$ has a solution.

Example 5: $\mathrm{Eq}(\xi, h\,(y,k(\cdot)) + cz + f(y)\,|z|^2)$

Let h satisfy $(\mathbf{A}_{3.3})$ and f be integrable. Consider the BSDEJ

$$Y_t = \xi + \int_t^T \left(h\,(Y_{s-},K_s(\cdot)) + cZ_s + f(Y_s)\,|Z_s|^2\right)\mathrm{d}s - \int_t^T Z_s\mathrm{d}W_s - \int_t^T\int_E K_s(e)\tilde{N}(\mathrm{d}s,\mathrm{d}e).$$

By using Itô–Krylov's formula (3.14) and taking $H(y,z,k(\cdot)) = h\,(y,k(\cdot)) + cz + f(y)\,|z|^2$, we obtain

$$\begin{aligned} F(Y_t) &= F(\xi) + \int_t^T F'(Y_{s-})\,(h\,(Y_{s-},K_s(\cdot)) + cZ_s)\,\mathrm{d}s \\ &\quad - \int_t^T\int_E (F(Y_s) - F(Y_{s-}) - F'(Y_{s-})K_s(e))\,\nu(\mathrm{d}e)\mathrm{d}s \\ &\quad - \int_t^T F'(Y_{s-})Z_s\mathrm{d}W_s - \int_t^T\int_E (F(Y_{s-}+K_s(e)) - F(Y_{s-}))\,\tilde{N}(\mathrm{d}s,\mathrm{d}e), \end{aligned}$$

which can be written as

$$y_t = F(\xi) + \int_t^T H_5(y_s,z_s,k_s(\cdot))\mathrm{d}s - \int_t^T z_s\mathrm{d}W_s - \int_t^T\int_E k_s\,(e)\,\tilde{N}(\mathrm{d}s,\mathrm{d}e),$$

where

$$H_5(y,z,k(\cdot)) := H_1(y,k(\cdot)) + H_4(y,z,k(\cdot)).$$

But H_1 of the Example 1 is continuous under $(\mathbf{A}_{3.3})$ and H_4 of the Example 4 is continuous in y, Lipschitz in z and $k(\cdot)$ and is of linear growth on y and $k(\cdot)$.

Therefore, by the result of [65], the equation $\mathrm{Eq}(F(\xi),H_5)$ has a solution. Finally $\mathrm{Eq}(\xi, h\,(y,k(\cdot)) + cz + f(y)\,|z|^2)$ has at least one solution.

Example 6: $\mathrm{Eq}(\xi, H_0(r,x) + cz + H_f(y,z,k(\cdot)))$

For a given adapted stochastic process $(X_s)_{0\leq s\leq T}$ and a measurable bounded function $H_0 : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ such that $\int_0^T \mathbb{E}[|H_0(s,X_s)|^2]\mathrm{d}s$ is finite, we consider the BSDEJ

$$Y_t = \xi + \int_t^T (H_0(s,X_s) + cZ_s + H_f(Y_s,Z_s,K_s(\cdot)))\,\mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s - \int_t^T \int_E K_s(e)\tilde{N}(\mathrm{d}s,\mathrm{d}e),$$

where the terminal value ξ is a square integrable random variable and f is bounded and integrable.

With the same transformation as before the equation $\mathrm{Eq}(\xi, H_0(r,x)+cz+H_f(y,z,k(\cdot)))$ is equivalent to the new equation below

$$y_t = F(\xi) + \int_t^T H_6(\omega,s,y_s,z_s)\mathrm{d}s - \int_t^T z_s \mathrm{d}W_s - \int_t^T \int_E k_s\,(e)\,\tilde{N}(\mathrm{d}s,\mathrm{d}e),$$

with stochastic Lipschitz coefficient of the form $H_6(\omega,s,y,z) = F'(F^{-1}\,(y))H_0(s,X_s) + cz$ which is Lipschitz in y due to the boundedness of f and linear in z. Hence $\mathrm{Eq}(F(\xi), H_6)$ has a unique solution.

Remark 3.3.1

If we replace cz by $c|z|$ in all the previous examples the results still hold true.

Remark 3.3.2

We learned from the above particular QBSDEJs that the choice of $H_f(y,z,k(\cdot))$ as the drift of our principle QBSDEJs $\mathrm{Eq}(\xi, H_f(y,z,k(\cdot)))$ is very important since it allows us to eliminate the both parts ($f(y)\,|z|^2$ and $[k]_f\,(y)$) of the drift and get a BSDEJ without drift. Any other quadratic drift in z leads to simple BSDEJs but sometimes we get non Lipschitz generators, therefore, uniqueness of solution is not possible in general. Nevertheless, under some additional assumption, one can get the existence and uniqueness of solutions still under square integrability of the terminal condition and non-continuous f.

3.4 Comparison and Strict Comparison Theorems

The comparison principle below was proved for the first time by Eddahbi and Ouknine [28] when f is a measure in the Brownian setting. In Theorem 3.4.1 below, we shall compare solutions associated with comparable data (ξ_1, f_1) and (ξ_2, f_2). Notice that technically the proofs of comparison and strict comparison principles for this class of QBSDEJ are completely different from the classical proofs for ordinary BSDEJs.

Theorem 3.4.1

(Comparison principle) Let ξ_1, ξ_2 be $\mathcal{F}_T$–measurable and square integrable random variables. Let f_1, f_2 be elements of $L^1(\mathbb{R})$. Let (Y^1, Z^1, K^1), (Y^2, Z^2, K^2) be respectively the solution of $\mathrm{Eq}(\xi_1, H_{f_1})$ *and* $\mathrm{Eq}(\xi_2, H_{f_2})$.

(i) *If $\xi_1 \leq \xi_2$ $\mathbb{P}$–a.s. and $f_1 \leq f_2$–a.e. Then $Y_t^1 \leq Y_t^2$ $\mathbb{P}$–a.s.*

(ii) *If, in addition to (i), $Y_0^1 = Y_0^2$, then $\xi_1 = \xi_2$ and $H_{f_1}(y, z, k(\cdot)) = H_{f_2}(y, z, k(\cdot))$ a.e.*

(iii) *(Strict comparison) In addition to (i), if $\mathbb{P}(\xi_2 > \xi_1) > 0$ then $\mathbb{P}(Y_t^2 > Y_t^1$ for all $t \in [0, T]) > 0$, in particular $Y_0^2 > Y_0^1$.*

Proof (i). Notice that the solutions (Y^1, Z^1, K^1) and (Y^2, Z^2, K^2) belong to $\mathbb{M}^2_{\mathcal{S}}$. For a given integrable function f_i, remember that F_i associated to f_i is defined by (3.2) and satisfies (3.3). We first apply Itô's formula to $F_1(Y_t^2)$, to obtain

$$\begin{aligned}
F_1(Y_T^2) &= F_1(Y_t^2) + \int_t^T F_1'(Y_{s-}^2)\mathrm{d}Y_s^2 + \frac{1}{2}\int_t^T F_1''(Y_{s-}^2)\mathrm{d}[Y_\cdot^2]_s^c \\
&+ \sum_{t<s\leq T}\Big(F_1(Y_s^2) - F_1(Y_{s-}^2) - F_1'(Y_{s-}^2)\triangle Y_s^2\Big) \\
&= F_1(Y_t^2) - \int_t^T F_1'(Y_{s-}^2) f_2(Y_s^2)|Z_s^2|^2\mathrm{d}s \\
&+ \int_t^T F_1'(Y_{s-}^2)Z_s^2\mathrm{d}W_s + \int_t^T\int_E F_1'(Y_{s-}^2)K_s^2(e)\tilde{N}(\mathrm{d}s, \mathrm{d}e) \\
&+ \frac{1}{2}\int_t^T F_1''(Y_{s-}^2)|Z_s^2|^2\mathrm{d}s - \int_t^T F_1'(Y_{s-}^2)[K_s^2]_{f_2}(Y_{s-}^2)\mathrm{d}s \\
&+ \int_t^T\int_E \Big(F_1(Y_s^2) - F_1(Y_{s-}^2) - F_1'(Y_{s-}^2)K_s^2(e)\Big)\, N(\mathrm{d}s, \mathrm{d}e).
\end{aligned}$$

Arranging terms and use the definition of the operator $[K_s^2]_{f_2}(\cdot)$, we get

$$\begin{aligned}F_1(Y_T^2) &= F_1(Y_t^2) + (M_T - M_t) \\ &\quad + \int_t^T \left(\frac{1}{2}F_1''(Y_{s-}^2) - F_1'(Y_{s-}^2)f_2(Y_s^2)\right)|Z_s^2|^2 \mathrm{d}s,\end{aligned}$$

where

$$M_t = \int_0^t F_1'(Y_{s-}^2)Z_s^2 \mathrm{d}W_s + \int_0^t \int_E \left(F_1(Y_s^2) - F_1(Y_{s-}^2)\right)\tilde{N}(\mathrm{d}s, \mathrm{d}e)$$

is an $\mathcal{F}_t$–martingale. Remember F_1 satisfies (3.3), thus according to Lemma 3.2.1, we obtain

$$F_1(Y_T^2) = F_1(Y_t^2) + (M_T - M_t) - \int_t^T \left(F_1'(Y_{s-}^2)\left(f_2(Y_s^2) - f_1(Y_s^2)\right)|Z_s^2|^2\right)\mathrm{d}s. \tag{3.22}$$

Since the term

$$\int_t^T F_1'(Y_{s-}^2)(f_2(Y_s^2) - f_1(Y_s^2))|Z_s^2|^2 \mathrm{d}s$$

is positive for all $t \in [0, T]$, then

$$F_1(Y_t^2) \geq F_1(Y_T^2) - (M_T - M_t).$$

Passing to conditional expectation and using the fact that F_1 is an increasing function and $\xi_2 \geq \xi_1$, we get

$$\begin{aligned}F_1(Y_t^2) &\geq \mathbb{E}\left[F_1(Y_T^2) \mid \mathcal{F}_t\right] = \mathbb{E}\left[F_1(\xi_2) \mid \mathcal{F}_t\right] \\ &\geq \mathbb{E}\left[F_1(\xi_1) \mid \mathcal{F}_t\right] = F_1(Y_t^1).\end{aligned}$$

Taking F_1^{-1} in both sides, we conclude $Y_t^2 \geq Y_t^1$ for all $t \in [0, T]$.

Proof (ii). From (3.22) for $t = 0$, we get

$$F_1(\xi_2) = F_1(Y_0^2) + M_T - \int_0^T \left(F_1'(Y_{s-}^2)\left(f_2(Y_s^2) - f_1(Y_s^2)\right)|Z_s^2|^2\right)\mathrm{d}s, \tag{3.23}$$

and from (3.18) for $t = 0$ we have

$$F_1(Y_0^1) = F_1(\xi_1) - N_T, \tag{3.24}$$

where

$$N_t = \int_0^t F_1'(Y_{s-}^1)Z_s^1 \mathrm{d}W_s - \int_0^t \int_E \left(F_1(Y_{s-}^1 + K_s^1(e)) - F_1(Y_{s-}^1)\right)\tilde{N}(\mathrm{d}s, \mathrm{d}e).$$

If $Y_0^1 = Y_0^2$, then we get, by substituting (3.24) in (3.23)

$$F_1(\xi_2) = F_1(\xi_1) + M_T - N_T - \int_0^T \left(F_1'(Y_{s-}^2)\left(f_2(Y_s^2) - f_1(Y_s^2)\right)|Z_s^2|^2\right) \mathrm{d}s,$$

taking the expectation we obtain

$$\mathbb{E}\left[F_1(\xi_2) - F_1(\xi_1)\right] + \mathbb{E}\int_0^T \left(F_1'(Y_{s-}^2)\left(f_2(Y_s^2) - f_1(Y_s^2)\right)|Z_s^2|^2\right) \mathrm{d}s = 0.$$

Since the quantities inside the expectation are positive, we conclude that

$$F_1(\xi_2) = F_1(\xi_1) \;\; \mathbb{P}\text{–a.s. and thus } \xi_2 = \xi_1 \; \mathbb{P}\text{–a.s.}$$

In addition to that, we have $f_2 = f_1$ dx–a.e. Finally

$$H_{f_1}(y, z, k(\cdot)) = H_{f_2}(y, z, k(\cdot)) \text{ a.e.}$$

Proof (iii). We have from (3.22)

$$F_1(Y_t^2) = F_1(\xi_2) - (M_T - M_t) + \int_t^T \left(F_1'(Y_{s-}^2)\left(f_2(Y_s^2) - f_1(Y_s^2)\right)|Z_s^2|^2\right) \mathrm{d}s.$$

Moreover, thanks to (3.18)

$$F_1(Y_t^1) = F_1(\xi_1) - (N_T - N_t)\,.$$

Therefore, subtracting both sides of the above inequalities leads to

$$\begin{aligned} F_1(Y_t^2) - F_1(Y_t^1) &= F_1(\xi_2) - F_1(\xi_1) + (N_T - N_t) - (M_T - M_t) \\ &\quad + \int_t^T \left(F_1'(Y_{s-}^2)\left(f_2(Y_s^2) - f_1(Y_s^2)\right)|Z_s^2|^2\right) \mathrm{d}s. \end{aligned}$$

Conditioning on $\mathcal{F}_t$, we get

$$F_1(Y_t^2) - F_1(Y_t^1) \geq \mathbb{E}\left[F_1(\xi_2) - F_1(\xi_1) \mid \mathcal{F}_t\right].$$

Consequently, $F_1(Y_t^2) - F_1(Y_t^1) > 0$ on the set $\{\xi_1 < \xi_2\}$, finally taking into account the fact that the function F_1 is one to one, $\mathbb{P}\left(Y_t^2 > Y_t^1\right) > 0$ for all $t \in [0, T]$. This achieves the proof. $\square$

Remark 3.4.1

The comparison theorem 3.4.1 does not require any additional condition on the generator H_f in particular w.r.t its third argument $k(\cdot)$. No monotonicity assumption is needed.

3.5 Application to Quadratic PIDEs

3.5.1 BSDEs and PIDEs: Existence and Uniqueness

In this subsection, we show, under mild conditions, that the BSDEJ has a unique solution. We then put ourselves in a Markovian framework, in which case a certain function defined through the solution of the BSDE is the unique viscosity solution of a system of parabolic PIDEs. These results were obtained by Barles, Buckdahn, and Pardoux [12]. We consider the following SDE:

$$X_s = \zeta + \int_t^s b'(r, X_r)\mathrm{d}r + \int_t^s \sigma'(r, X_r)\mathrm{d}W_r + \int_t^s \int_E \varphi'(r, X_{r-}, e)\tilde{N}(\mathrm{d}r, \mathrm{d}e), \tag{3.25}$$

where $\zeta \in L^2\left(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^d\right)$,

The functions $b' : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^d$ and $\sigma' : [0,T] \times \mathbb{R}^d \longrightarrow \mathbb{R}^{d\times d}$ be globally Lipschitz and $\varphi' : [0,T] \times \mathbb{R}^d \times E \longrightarrow \mathbb{R}^d$ be measurable and such that for some real L , and for all $e \in E$,

$$\begin{aligned}
|\varphi'(t,x,e)| &\leq L(1 \wedge e), \quad x \in \mathbb{R}^d, t \in [0;T], \\
|\varphi'(t,x,e) - \varphi'(t,x',e)| &\leq L\,|x - x'|\,(1 \wedge e), \quad x, x' \in \mathbb{R}^d.
\end{aligned}$$

We denote by $(X_s^{t,\zeta})_{s\in[t,T]}$ the unique solution of equation (3.25) starting from ζ at time $s = t$. We introduce the following generator:

$$f : [0,T] \times \Omega \times \mathbb{R}^q \times \mathbb{R}^{q\times d} \times \mathcal{L}_\nu^{2,q} \longrightarrow \mathbb{R}^q,$$

that satisfies the following assumptions:

- $\mathbb{E}\displaystyle\int_0^T |f(t,0,0,0)|^2 dt < \infty$;
- For any $t \in [0,T], y, y' \in \mathbb{R}^q, z, z' \in \mathbb{R}^{q\times d}$ and $k, k' \in \mathcal{L}_\nu^{2,q}$, there exists $L > 0$ such that

$$|f(t,y,z,k(\cdot)) - f(t,y,z,k'(\cdot))| \leq L\left(|y - y'| + |z - z'| + \|(k - k')(\cdot)\|_{q,\nu}\right).$$

According to Theorem 1.3.1 in chapter 1, there exists a unique triple $(Y, Z, K) \in \mathbb{M}^2_{\mathcal{S}}$ which solves the following BSDEJ:

$$Y_t = \zeta + \int_t^T f(r, Y_r, Z_r, K_r(\cdot))\mathrm{d}r - \int_t^T Z_r \mathrm{d}W_r - \int_t^T \int_E K_r(e)\widetilde{N}(\mathrm{d}r, \mathrm{d}e), 0 \leq t \leq T.$$

In the sequel, we are concerned by a specific class of BSDEJs where both ζ and for each t, y, z, k, the process $(f(s, y, z, k))_{t \leq s \leq T}$ are given functions of the process $X_s^{t,\zeta}$. More precisely, we are given two continuous functions

$$g : \mathbb{R}^d \longrightarrow \mathbb{R}^d, \quad f : [0, T] \times \mathbb{R}^d \times \mathbb{R}^q \times \mathbb{R}^{q \times d} \times \mathbb{R}^q \longrightarrow \mathbb{R}^q,$$

and a measurable function $\gamma : \mathbb{R}^d \times E \longrightarrow \mathbb{R}^q$ such that, for each $1 \leq i \leq q$, $f_i(t, x, y, z, k)$ depends on the matrix z only through its $i-$th column z_i, and on the vector k only through its $i-$th coordinate k_i. We assume specifically that:

($\mathbf{A}_{5.1}$) $f(t, x, 0, 0, 0) \leq C(1 + |x|^p)$, $|g(x)| \leq C(1 + |x|^p)$, for some $C, p > 0$;

($\mathbf{A}_{5.2}$) $f = f(t, x, y, z, k)$ is globally Lipschitz in (y, z, k), uniformly in (t, x);

($\mathbf{A}_{5.3}$) for each $(t, x, y, z) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^q \times \mathbb{R}^{q \times d}, 1 \leq i \leq q$, the function

$$p \longmapsto f_i(t, x, y, z, p) \text{ is non-decreasing;}$$

($\mathbf{A}_{5.4}$) there is some real C , such that for all $1 \leq i \leq q$,

$$0 \leq \gamma_i(t, x, e) \leq C(1 \wedge e), \quad x \in \mathbb{R}^d, e \in E,$$
$$|\gamma_i(t, x, e) - \gamma_i(t, x', e)| \leq C\, |x - x'|\, (1 \wedge e), \quad x, x' \in \mathbb{R}^d, e \in E.$$

Under the assumptions $(\mathbf{A}_{5.1}) - (\mathbf{A}_{5.4})$, for each $t \in [0, T]$ and $x \in \mathbb{R}^d$, we consider the BSDEJ

$$\begin{aligned} Y_{s,i}^{t,x} &= g_i(X_T^{t,x}) + \int_s^T f_i(r, X_r^{t,x}, Y_r^{t,x}, Z_{r,i}^{t,x}, \int_E K_{r,i}^{t,x}(e)\gamma_i(r, X_r^{t,x}, e)\nu(\mathrm{d}e))\mathrm{d}r \\ &- \int_s^T Z_{r,i}^{t,x}\mathrm{d}W_r - \int_s^T \int_E K_{r,i}^{t,x}(e)\widetilde{N}(\mathrm{d}r, \mathrm{d}e), 1 \leq i \leq q, t \leq s \leq T. \end{aligned} \tag{3.26}$$

The proof of the next result can be found in Barles, Buckdahn and Pardoux [12].

Corollary 3.5.1

For each $t \in [0,T], x \in \mathbb{R}^d$, the BSDEJ (3.26) has a unique solution

$$(Y_s^{t,x}, Z_s^{t,x}, K_s^{t,x}(\cdot))_{s \le T} \in \mathbb{M}_{\mathcal{S}}^2$$

and $(t,x) \longmapsto Y_t^{t,x}$ defines a deterministic mapping from $[0,T] \times \mathbb{R}^d$ into $\mathbb{R}^q$.

We consider the system of PIDEs of parabolic type

$$\begin{cases} \frac{-\partial}{\partial t} u_i(t,x) - L_i u_i(t,x) - f_i(t,x,u(t,x),(\nabla u_i \sigma')(t,x), B_i u_i(t,x)) = 0, \\ u_i(T,x) = g_i(x), \end{cases} \tag{3.27}$$

where the integral operators L_i and B_i were defined in [12]. We now show that BSDEJ (3.26) provides a viscosity solution of (3.27), where the definition of a viscosity solution will be presented in Subsection 3 below. The following theorem is proved in [12].

Theorem 3.5.1

The function given by $u(t,x) = Y_t^{t,x}, (t,x) \in [0,T] \times \mathbb{R}^d$ is a viscosity solution of PIDE (3.27).

Now we give a uniqueness result for (3.27). This result is obtained under more restrictive conditions than the existing one: namely we need the two following additional assumptions:

- $|f_i(t,x,r,p,k) - f_i(t,y,r,p,k)| \le m_R^i(|x-y|(1+|p|))$, for $1 \le i \le q$, where $m_R^i(s)$ goes to 0 when s goes to 0^+, for all $t \in [0,T], |x|, |y| \le R, |r| \le R, p \in \mathbb{R}^d, k \in \mathbb{R}$ $(\forall R < \infty)$.

- For γ, we assume in addition

$$|\gamma_i(t,x,e) - \gamma_i(t,y,e)| \le C_1 |x-y| (1 \wedge |e|^2), 1 \le i \le q,$$

for some constant $C_1 > 0$ and for any $x, y \in \mathbb{R}^d, e \in E$.

Theorem 3.5.2

Assume that f, g, and γ satisfy the previous assumptions. Then, there exists at most one viscosity solution u of (3.27) such that

$$\lim_{|x|\to+\infty} |u(t,x)|\, e^{-\widetilde{A}[log(|x|)]^2} = 0, \tag{3.28}$$

uniformly for $t \in [0,T]$, for some $\widetilde{A} > 0$. In particular, the function $u(t,x) = Y_t^{t,x}$ is the unique viscosity solution of (3.27) in the class of solutions which satisfy (3.28) for some $\widetilde{A} > 0$.

Proof: See [12] p. 74. ■

We give now a mutual correspondence between solutions of quadratic BSDEJs associated with two independent stochastic processes a Brownian component and Poisson process and solution of a class of partial integral differential equation with a non-linear functional.

3.5.2 Solution of QBSDEs with Jumps via Solution of QPIDEs

We introduce the forward SDEJ that will generate a Markov process to be used to solve some QPIDEs. For a given t as the initial time and $\zeta \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R})$ as the initial state, let $(X_s^{t,\zeta})_{s\in[t,T]}$ be the solution of the following SDE with jumps:

$$X_s^{t,\zeta} = \zeta + \int_t^s b(r, X_r^{t,\zeta})\mathrm{d}r + \int_t^s \sigma(r, X_r^{t,\zeta})\mathrm{d}W_r + \int_t^s \int_E \varphi(r, X_{r-}^{t,\zeta}, e)\tilde{N}(\mathrm{d}r, \mathrm{d}e) \tag{3.29}$$

where $X_s^{t,\zeta} = \zeta$ for all $0 \le s \le t$ and the mappings

$$b : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}, \quad \sigma : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R} \text{ and } \varphi : [0,T] \times \mathbb{R} \times E \longrightarrow \mathbb{R}$$

satisfy the following assumptions:

($\mathbf{A}_{5.5}$) For every fixed $(x,e) \in \mathbb{R} \times E$, the mappings $r \longmapsto b(r,x)$, $r \longmapsto \sigma(r,x)$ and $r \longmapsto \varphi(r,x,e)$ are continuous.

($\mathbf{A}_{5.6}$) There exists a $L > 0$ such that, for all $r \in [0,T]$, $x,\ \acute{x} \in \mathbb{R}$

$$|b(r,x) - b(r,\acute{x})| + |\sigma(r,x) - \sigma(r,\acute{x})| \le L\,|x - \acute{x}|\,.$$

($\mathbf{A}_{5.7}$) There exists a function $\rho : E \longrightarrow \mathbb{R}^{+}$ with $\int_E \rho^2(e)\nu(\mathrm{d}e) < +\infty$, such that, for any $r \in [0,T]$, $x,\ \acute{x} \in \mathbb{R}$ and $e \in E$

$$|\varphi(r,x,e) - \varphi(r,\acute{x},e)| \leq \rho(e)\,|x-\acute{x}| \quad \text{and} \quad |\varphi(r,0,e)| \leq \rho(e).$$

It is clear the above conditions imply that b, σ and φ satisfy the global linear growth conditions: that is there exists some $C>0$ such that, for all $0 \leq r \leq T$, $x \in \mathbb{R}$

$$|b(r,x)| + |\sigma(r,x)| \leq C(1+|x|) \quad \text{and} \quad |\varphi(r,x,e)| \leq \rho(e)(1+|x|).$$

It is well known that under ($\mathbf{A}_{5.5}$) $-$ ($\mathbf{A}_{5.7}$) the SDEJ (3.29) has a unique strong solution. Moreover, there exists $C>0$ such that, for any $t \in [0,T]$ and $\zeta,\ \zeta' \in L^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R})$,

$$\mathbb{E}\left[\sup_{t\leq s\leq T} |X_s^{t.\zeta} - X_s^{t.\zeta'}|^2 \mid \mathcal{F}_t\right] \leq |\zeta - \zeta'|^2 \ \mathbb{P}\text{–a.s.} \tag{3.30}$$

and

$$\mathbb{E}\left[\sup_{t\leq s\leq T} |X_s^{t.\zeta}|^2 \mid \mathcal{F}_t\right] \leq C\left(1+|\zeta|^2\right) \ \mathbb{P}\text{–a.s.} \tag{3.31}$$

Now, we shall introduce the generators of our Markovian BSDEJ. Given

$$g : \mathbb{R} \longrightarrow \mathbb{R}, \quad H_0 : [0,T]\times\mathbb{R} \longrightarrow \mathbb{R},$$

that satisfy the following conditions:

($\mathbf{A}_{6.4}$) There exists $L, C > 0$, such that for all $r \in [0,T]$ and $x,\ \acute{x} \in \mathbb{R}$

$$|g(x) - g(\acute{x})| + |H_0(r,x) - H_0(r,\acute{x})| \leq L\,|x-\acute{x}| \quad \text{and} \quad |H_0(r,x)| \leq C.$$

For any $h : \mathbb{R}\times\mathcal{L}_\nu^{2,1} \longrightarrow \mathbb{R}$ and f satisfying ($\mathbf{A}_{3.1}$) $-$ ($\mathbf{A}_{3.2}$), we set

$$\begin{aligned}\mathcal{G} = \{&H_f(y,z,k(\cdot)), \quad h(y,k(\cdot)) + cz + H_f(y,z,k(\cdot)),\ a + b\,|y| + c\,|z| + d\,\|k(\cdot)\|_{1,\nu} \\ &+ H_f(y,z,k(\cdot)), cz + f(y)\,|z|^2 - \int_E k(e)\nu(\mathrm{d}e), \quad cz + f(y)\,|z|^2, \\ &h(y,k(\cdot)) + cz + f(y)\,|z|^2\}.\end{aligned}$$

It is clear from the results in the section 3 that the BSDEJ $\mathrm{Eq}(g(X_T^{t;x}), H_0 + H)$ has at least one solution for H in $\mathcal{G}$.

In what follows we show that this solution can be represented by as a deterministic function of the Markov process X. which is the solution of a PIDE which will be specified below. For any smooth function ϑ we define for any $t \in [0,T]$ and $x \in \mathbb{R}$, the following operators:

$$\mathcal{L}(\vartheta)(t,x) := \frac{\partial \vartheta}{\partial t}(t,x) + \frac{\sigma^2(t,x)}{2}\frac{\partial^2 \vartheta}{\partial x^2}(t,x),$$

and for $\Delta\vartheta_\varphi(\cdot)(t,x) := \vartheta(t, x+\varphi(t,x,\cdot)) - \vartheta(t,x)$, we put

$$\mathcal{I}(\vartheta,\varphi)(t,x) := \int_E \left(\Delta\vartheta_\varphi(e)(t,x) - \varphi(t,x,e)\frac{\partial \vartheta}{\partial x}(t,x)\right)\nu(\mathrm{d}e).$$

Remember also that form the definition of $[k]_f(\cdot)$ we can write

$$[\Delta\vartheta_\varphi]_f(\vartheta(t,x)) := \int_E \frac{F(\vartheta(t,x) + \Delta\vartheta_\varphi(e)(t,x)) - F(\vartheta(t,x))}{F'(\vartheta(t,x))} - \Delta\vartheta_\varphi(e)(t,x)\nu(\mathrm{d}e).$$

Let θ be the $\mathcal{C}^{1,2}$ classical solution of the following PIDE

$$\begin{cases} (\mathcal{L}(\theta) + \mathcal{I}(\theta,\varphi) + H_0 + H(\theta, \sigma\frac{\partial\theta}{\partial x}, \Delta\theta_\varphi(\cdot)))(t,x) = 0, \\ \theta(T,x) = g(x), \end{cases} \tag{3.32}$$

where H is one of the elements of $\mathcal{G}$. Then we have the following result.

Theorem 3.5.3

The solution of $\mathrm{Eq}(g(X_T^{t,x}), H_0 + H)$ *can be represented by*

$$Y_s^{t,x} = \theta(s, X_s^{t,x}), \quad Z_s^{t,x} = \sigma(s, X_{s-}^{t,x})\frac{\partial\theta}{\partial x}(s, X_{s-}^{t,x}),$$

and

$$K_s^{t,x}(e) = \theta(s, X_{s-}^{t,x} + \varphi(s, X_{s-}^{t,x}, e)) - \theta(s, X_{s-}^{t,x}) = \Delta\theta_\varphi(e)(s, X_{s-}^{t,x}),$$

for $t \leq s \leq T$ and $e \in E$. Moreover, we have the representation

$$Y_t^{t,x} = \mathbb{E}\left[g(X_T^{t,x})\right] + \mathbb{E}[\int_t^T (H_0(r, X_r^{t,x}) + H(Y_r^{t,x}, Z_r^{t,x}, K_r^{t,x}(\cdot)))\mathrm{d}r] := \theta(t,x).$$

Proof. Applying Itô's formula to $\theta(s, X_s)$, we obtain

$$\theta(T, X_T) - \theta(t, X_t) = \int_t^T \mathcal{L}(\theta)(s, X_s)\mathrm{d}s + M_T - M_t$$
$$+ \sum_{0<s\leq T}\left(\theta(s, X_s) - \theta(s, X_{s-}) - \frac{\partial\theta}{\partial x}(s, X_{s-})\Delta X_s\right),$$

where

$$M_t = \int_0^t \frac{\partial\theta}{\partial x}(s, X_s)\sigma(s, X_s)\mathrm{d}W_s + \int_0^t \int_E \frac{\partial\theta}{\partial x}(s, X_{s-})\varphi(s, X_{s-}^{t,x}, e)\tilde{N}(\mathrm{d}s, \mathrm{d}e)$$

is a martingale. Writing the above equation in terms of $\tilde{N}(\mathrm{d}r, \mathrm{d}e)$ and use the identity $X_s = X_{s-} + \varphi(s, X^{t,x}_{s-}, e)$ lead to

$$\begin{aligned}
&\sum_{t<s\leq T}\left(\theta(s,X_s)-\theta(s,X_{s-})-\frac{\partial\theta}{\partial x}(s,X_{s-})\Delta X_s\right)\\
&=\int_t^T\int_E\left(\theta(s,X_s)-\theta(s,X_{s-})-\frac{\partial\theta}{\partial x}(s,X_{s-})\varphi(s,X^{t,x}_{s-},e)\right)N(\mathrm{d}s,\mathrm{d}e)\\
&=\int_t^T\int_E\left(\theta(s,X_s)-\theta(s,X_{s-})-\frac{\partial\theta}{\partial x}(s,X_{s-})\varphi(s,X^{t,x}_{s-},e)\right)\tilde{N}(\mathrm{d}s,\mathrm{d}e)\\
&+\int_t^T\int_E\left(\theta(s,X_s)-\theta(s,X_{s-})-\frac{\partial\theta}{\partial x}(s,X_{s-})\varphi(s,X^{t,x}_{s-},e)\right)\nu(\mathrm{d}e)\mathrm{d}s,
\end{aligned}$$

hence,

$$\theta(T,X_T)-\theta(t,X_t)=\int_t^T\left(\mathcal{L}\left(\theta\right)(s,X_s)+\mathcal{I}(\theta,\varphi)(s,X_s)\right)\mathrm{d}s+N_T-N_t,$$

where

$$N_t=\int_0^t\sigma(s,X_s)\frac{\partial\theta}{\partial x}(s,X_s)\mathrm{d}W_s+\int_0^t\int_E\left(\theta(s,X_s)-\theta(s,X_{s-})\right)\tilde{N}(\mathrm{d}s,\mathrm{d}e)$$

is also a martingale. Now, taking account that $Y_t = \theta(t, X_t)$ and $\theta(T, X_T) = g(X_T)$, we obtain

$$\begin{aligned}
Y_t &= g(X_T)-\int_t^T\left(\mathcal{L}\left(\theta\right)+\mathcal{I}(\theta,\varphi)\right)(s,X_{s-})\mathrm{d}s-(N_T-N_t)\\
&= g(X_T)+\int_t^T\left(H_0+H(\theta,\frac{\partial\theta}{\partial x},\Delta\theta_\varphi(\cdot))\right)(s,X_{s-})\mathrm{d}s-\left(N_T-N_t\right),
\end{aligned}$$

from which we get the desired result by replacing X_s by $X_{s-}+\varphi\left(s,X_{s-},e\right)$ in the discontinuous martingale part of $N_.$ and the fact that θ satisfies (3.32). □

3.5.3 Probabilistic Representation of Solution to QPIDE

We consider the following quadratic PIDE with a non-linear functional term

$$\begin{cases}\left(\mathcal{L}\left(\theta\right)+\mathcal{I}(\theta,\varphi)+H_0+H_f(\theta,\sigma\frac{\partial\theta}{\partial x},\Delta\theta_\varphi(\cdot))\right)(t,x)=0,\\ \theta(T,x)=g(x).\end{cases}\tag{3.33}$$

We want to give a probabilistic representation of the equation (3.33) by means of the solution of the QBSDEJ $\mathrm{Eq}(g(X^{t,x}_T), H_0+H_f)$. Let $(Y^{t,x}_s)_{0\leq s\leq T}$ be the unique solution of $\mathrm{Eq}(g(X^{t,x}_T), H_0+H_f)$ where $Y^{t,x}_s = Y^{t,x}_t$ for all $0\leq s\leq t$ and $t\leq T$.

Proposition 3.5.1

The mapping $(t,x) \longmapsto Y_t^{t.x}$ *is continuous. Moreover, there exist two constants* $C,\ L>0$ *such that, for all* $0 \le t \le T,\ x,\ \acute{x} \in \mathbb{R}$

$$|Y_t^{t,x} - Y_t^{t,\acute{x}}| \le L|x-\acute{x}| \text{ and } |Y_t^{t,x}| \le C\,(1+|x|)\,.$$

Proof. For t in $[0,T]$, we set $y_t^{t,x} := F(Y_t^{t,x})$. Remember that $(y_s^{t,x}, z_s^{t,x}, k_s^{t,x}(\cdot))_{0\le s\le T}$ is the unique solution of the BSDEJ Eq$(F(g(X_T^{t,x})), H)$, where $H(t,x,y) = F'(F^{-1}(y))H_0(t,x)$. For any $0 \le t \le T$ and $x \in \mathbb{R}$, we set $\alpha(t,x) = y_t^{t,x}$.
The mapping $(t,x) \longmapsto \alpha(t,x)$ is continuous on $[0,T]\times\mathbb{R}^n$ since $|H(t,x,y)| \le C\,(1+|x|)$ and $|H(t,x,y) - H(t,\acute{x},\acute{y})| \le C\,(|x-\acute{x}| + |y-\acute{y}|)$. Finally, due to the continuity of F^{-1} the mapping $(t,x) \longmapsto \theta(t,x) = F^{-1}(y_t^{t,x}) = Y_t^{t,x}$ is continuous on $[0,T]\times\mathbb{R}$. $\square$

Viscosity solution:

Set $\mathcal{C}_b^{1,2}([0,T]\times\mathbb{R})$: the space of all real functions, which has a bounded continuous first derivative w.r.t t, and up to the second derivatives w.r.t x.

Definition 3.5.1

A continuous mapping $[0,T]\times\mathbb{R} \ni (t,x) \longmapsto \theta(t,x)$ *is called a*

(i) *viscosity sub-solution of (3.33) if* $\theta(T,x) \le g(x)$ *and for all* $\vartheta \in \mathcal{C}_b^{1,2}([0,T]\times\mathbb{R})$ *at all maximum point* (t,x) *of function* $\theta - \vartheta$ *such that* $\theta(t,x) = \vartheta(t,x)$*, one has*

$$(\mathcal{L}\,(\vartheta) + \mathcal{I}^\delta(\vartheta,\theta,\varphi) + H_0 + H_f(\theta, \sigma\frac{\partial\theta}{\partial x}, \Delta\theta_\varphi(\cdot)))(t,x) \ge 0,$$

where for any $\delta>0$, $E_\delta = \{e \in E : |e| \le \delta\}$ *and*

$$\begin{aligned}\mathcal{I}^\delta(\vartheta,\theta,\varphi)(t,x) &= \int_{E_\delta}\left(\vartheta(t,x+\varphi\,(x,e)) - \vartheta(t,x) - \varphi\,(x,e)\,\frac{\partial\vartheta}{\partial x}(t,x)\right)\nu(\mathrm{de})\\ &\quad + \int_{E_\delta^c}\left(\theta(t,x+\varphi\,(x,e)) - \theta(t,x) - \varphi\,(x,e)\,\frac{\partial\vartheta}{\partial x}(t,x)\right)\nu(\mathrm{de}),\end{aligned}$$

(ii) *viscosity super-solution of (3.33) if* $\theta(T,x) \ge g(x)$ *and for all* $\vartheta \in \mathcal{C}_b^{1,2}([0,T]\times\mathbb{R})$ *at all minimum point* (t,x) *of function* $\theta - \vartheta$ *one has*

$$(\mathcal{L}\,(\vartheta) + \mathcal{I}^\delta(\vartheta,\theta,\varphi) + H_0 + H_f(\theta, \sigma\frac{\partial\theta}{\partial x}, \Delta\theta_\varphi(\cdot)))(t,x) \le 0,$$

(iii) *viscosity solution if it is a viscosity super-solution and sub-solution.*

The proof of the following lemma can be performed as in Lemma 3.3 in [12] p. 66.

Lemma 3.5.1

A continuous mapping $[0,T]\times\mathbb{R}\ni(t,x)\longmapsto\theta(t,x)$ *is called a*

(i) *viscosity sub-solution of* (3.33) *if* $\theta(T,x)\le g(x)$ *and for all* $\vartheta\in\mathcal{C}_b^{1,2}([0,T]\times\mathbb{R})$ *at all maximum point* (t,x) *of function* $\theta-\vartheta$ *one has*

$$(\mathcal{L}(\vartheta)+\mathcal{I}(\vartheta,\theta,\varphi)+H_0+H_f(\theta,\sigma\frac{\partial\theta}{\partial x},\Delta\theta_\varphi(\cdot)))(t,x)\ge 0,$$

(ii) *viscosity super-solution of* (3.33) *if* $\theta(T,x)\ge g(x)$ *and for all* $\vartheta\in\mathcal{C}_b^{1,2}([0,T]\times\mathbb{R})$ *at all minimum point* (t,x) *of function* $\theta-\vartheta$ *one has*

$$(\mathcal{L}(\vartheta)+\mathcal{I}(\vartheta,\theta,\varphi)+H_0+H_f(\theta,\sigma\frac{\partial\theta}{\partial x},\Delta\theta_\varphi(\cdot)))(t,x)\le 0.$$

Consider PIDE

$$\begin{cases}\left(\mathcal{L}(\alpha)+\mathcal{I}(\alpha,\varphi)+F'(F^{-1}(\alpha))H_0\right)(t,x)=0\\ \alpha(T,x)=F(g(x)).\end{cases}\tag{3.34}$$

Theorem 3.5.4

The continuous function $\theta(t,x)=Y_t^{t,x}$ *is a viscosity solution of* (3.33) *if and only if the continuous function* $\alpha(t,x)=F(Y_t^{t,x})$ *is a viscosity solution of* (3.34).

Proof. Let $\theta(t,x)$ is a viscosity sub-solution of (3.33) so $\theta(T,x)\le g(x)$ implies that $\alpha(T,x)=F(\theta(T,x))\le F(g(x))$ since F is increasing. Let $\vartheta\in\mathcal{C}_b^{1,2}([0,T]\times\mathbb{R})$. If (t,x) is a maximum point of $\theta-\vartheta$ such that $\theta(t,x)=\vartheta(t,x)$, its is also a maximum point of $F(\theta)-F(\vartheta)$. Applying the operators $\mathcal{L}$ and $\mathcal{I}$ to $\overline{\vartheta}(t,x)=F(\vartheta(t,x))$, we get

$$\begin{aligned}\mathcal{L}(\overline{\vartheta})(t,x)&=\left(\left(\frac{\partial\vartheta}{\partial t}+\frac{\sigma^2}{2}\frac{\partial^2\vartheta}{\partial x^2}\right)F'(\vartheta)+\frac{\sigma^2}{2}\left(\frac{\partial\vartheta}{\partial x}\right)^2F''(\vartheta)\right)(t,x)\\&=\left(\mathcal{L}(\vartheta)+f(\vartheta)\left(\sigma\frac{\partial\vartheta}{\partial x}\right)^2\right)F'(\vartheta)(t,x)\end{aligned}$$

and

$$\begin{aligned}
\mathcal{I}(\overline{\vartheta},\varphi)(t,x) &= \int_E \left(\Delta\overline{\vartheta}_\varphi\left(e\right)\left(t,x\right) - \varphi\left(t,x,e\right)\frac{\partial\overline{\vartheta}}{\partial x}(t,x)\right)\nu(\mathrm{de}) \\
&= \int_E \left(\Delta\overline{\vartheta}_\varphi\left(e\right)\left(t,x\right) - \varphi\left(t,x,e\right)\frac{\partial\vartheta}{\partial x}(t,x)F'(\vartheta(t,x))\right)\nu(\mathrm{de}) \\
&= F'(\vartheta(t,x))\int_E \left(\frac{\Delta\overline{\vartheta}_\varphi\left(e\right)\left(t,x\right)}{F'(\vartheta(t,x))} - \varphi\left(t,x,e\right)\frac{\partial\vartheta}{\partial x}(t,x)\right)\nu(\mathrm{de}) \\
&= F'(\vartheta(t,x))\int_E \left(\frac{F\left(\vartheta(t,x+\varphi\left(t,x,e\right))\right) - F\left(\vartheta(t,x)\right)}{F'(\vartheta(t,x))} - \Delta\overline{\vartheta}_\varphi\left(e\right)\left(t,x\right)\right)\nu(\mathrm{de}) \\
&+ F'(\vartheta(t,x))\int_E \left(\Delta\overline{\vartheta}_\varphi\left(e\right)\left(t,x\right) - \varphi\left(t,x,e\right)\frac{\partial\vartheta}{\partial x}(t,x)\right)\nu(\mathrm{de}) \\
&= \left(F'(\vartheta)\left(\left[\Delta\vartheta_\varphi\right]_f(\vartheta) + \mathcal{I}(\vartheta,\varphi)\right)\right)(t,x),
\end{aligned}$$

thus

$$\begin{aligned}
&\left(\mathcal{L}\left(\overline{\vartheta}\right) + \mathcal{I}(\overline{\vartheta},\varphi) + F'(F^{-1}(\overline{\vartheta}))H_0\right)(t,x) \\
&= \left(F'(\vartheta)\left(\mathcal{L}\left(\vartheta\right) + \mathcal{I}(\vartheta,\varphi) + H_0 + H_f(\vartheta,\sigma\frac{\partial\vartheta}{\partial x},\Delta\vartheta_\varphi(\cdot))\right)\right)(t,x),
\end{aligned}$$

but $\Delta\vartheta_\varphi(\cdot)(t,x) = \Delta\theta_\varphi(\cdot)(t,x)$ which implies that the right-hand side of the above equality is non-negative as soon as θ is a viscosity sub-solution of (3.33) since $F'(x) > 0$ for all $x \in \mathbb{R}$. $\square$

Theorem 3.5.5

The function θ given by $\theta(t,x) := Y_t^{t,x}$, for all $(t,x) \in [0,T] \times \mathbb{R}$ is a viscosity solution to the QPIDE (3.33).

Proof. From Theorem 3.5.4, $\theta(t,x) = Y_t^{t,x}$ is a viscosity solution of (3.33) if and only if $\alpha(t,x) = F(Y_t^{t,x})$ is a viscosity solution of (3.34). But $\alpha(t,x)$ is a viscosity solution of (3.34) thanks to the result in subsection 1. Observe that we do not need the monotonicity type condition on $k(\cdot)$ because our generator in the (3.34) is independent from $k(\cdot)$.
Let $H(y,z,k(\cdot)) = cz + f(y)\left|z\right|^2 - \int_E k(e)\nu(\mathrm{de})$ and consider the following PIDE

$$\begin{cases} (\mathcal{L}\left(\theta\right) + \mathcal{I}(\theta,\varphi) + H(\theta,\sigma\frac{\partial\theta}{\partial x},\Delta\theta_\varphi(\cdot)))(t,x) = 0, \\ \theta(T,x) = g(x). \end{cases} \tag{3.35}$$

We want to give a probabilistic representation of the equation (3.35) via the solution of the associated QBSDEJ.

Let $(Y_s^{t,x})_{0\leq s\leq T}$ be the unique solution of $\mathrm{Eq}(g(X_T^{t,x}),H)$ where $Y_s^{t,x}=Y_t^{t,x}$ for all $0\leq s\leq t$ and $t\leq T$.

One can check as in the Proposition 3.5.1 that the mapping $(t,x)\longmapsto Y_t^{t.x}$ is continuous. Applying the transformation F, the $\mathrm{Eq}(g(X_T^{t,x}),H)$ is equivalent to $\mathrm{Eq}(F(g(X_T^{t,x})),\tilde{H})$ where $\tilde{H}(z,k(\cdot))=cz-\int_E k(e)\nu(\mathrm{d}e)$ is a Lipschitz function, without the quadratic term. Now, observe that the generator $\tilde{H}$ satisfy the required assumptions of f_i, thus the mapping $(t,x)\longmapsto y_t^{t,x}$ is a viscosity to the PIDE

$$\begin{cases}(\mathcal{L}(\alpha)+\mathcal{I}(\alpha,\varphi)+\tilde{H}(\sigma\frac{\partial\alpha}{\partial x},\Delta\alpha_\varphi(\cdot)))(t,x)=0\\ \alpha(T,x)=F(g(x)).\end{cases}$$

Finally, thanks to Theorem 3.5.4, we conclude that the mapping $(t,x)\longmapsto Y_t^{t,x}$ is a viscosity solution of (3.35). □

Corollary 3.5.2

For $H(y,z,k(\cdot))=cz+f(y)\left|z\right|^2-\int_E k(e)\nu(\mathrm{d}e)$, we consider the QPIDE

$$\begin{cases}(\mathcal{L}(\theta)+\mathcal{I}(\theta,\varphi)+H_0+H(\theta,\sigma\dfrac{\partial\theta}{\partial x},\Delta\theta_\varphi(\cdot)))(t,x)=0\\ \theta(T,x)=g(x).\end{cases}\tag{3.36}$$

The mapping $(t,x)\longmapsto Y_t^{t,x}$ is a viscosity solution of (3.35).

Proof. This is a consequence of Theorem 3.5.4 and the fact that $(t,x)\longmapsto F(Y_t^{t,x})$ is a viscosity solution to

$$\begin{cases}\left(\mathcal{L}(\alpha)+\mathcal{I}(\alpha,\varphi)+F'(F^{-1}(\alpha))H_0+\tilde{H}(\sigma\frac{\partial\alpha}{\partial x},\Delta\alpha_\varphi(\cdot))\right)(t,x)=0\\ \alpha(T,x)=F(g(x)).\end{cases}$$

This leads to the result. □

Conclusion

Our greatest contribution to this Ph.D. dissertation is that we successfully discussed new results for different categories of BSDEs with jumps driven by a Poisson random measure and independent Brownian motion. The main results of this thesis are summarized as follows.

The first main topic is about the global existence of the solutions for a class of continuous multidimensional Markovian BSDEJs. We first, generalized the representation obtained by El Karoui *et al.* [30] to the jump case which claims that the solution of Markovian BSDEJ with Lipschitz generator can be represented in terms of the Markov process and some deterministic functions.

This result with the help of the so-called L^2-domination condition, on the law of the underlying Markov process, played a crucial role in proving the main results. More precisely, we proved that BSDEJ (0.1) in the case where its generator is continuous w.r.t y and z and globally Lipschitz in $k(\cdot)$ has at least a solution. Then, we have extended the later result by studying a particular form of BSDEJ (0.1) whose generator is continuous in y, z and k.

We hope to treat in future research the more general case where the generator of BSDEJ (0.1) is totally continuous w.r.t all its state variables to fill the gaps and solve this open problem.

The second main topic suggests establishing the existence and uniqueness of solutions to a class of BSDEJs of quadratic type Eq(ξ, H_f) when the coefficient is measurable and integrable and the terminal condition is square integrable. Before this, we proved Krylov's estimates, Itô-Krylov's formula and priori estimates for eventual solutions of

Eq(ξ, H_f) and then used to prove Theorem 3.2.2. On the other hand, a comparison and strict comparison theorems are proved without any hypothesis on the third argument of the generator H_f. Then, we investigated to standard relationship between QBSDEJs and QPIDEs in the Markovian setting. In particular, a probabilistic representation of viscosity solutions of some quadratic PIDEs is proved by utilizing the Feynman-Kac formula.

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