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Extropy of Ordered Variates

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Dedication

"Dedicated to the guidin lights of my life :

To my parents, "Elhadi and Khaira", whose unwavering support and endless love have been my guiding light.

To my wonderful sisters "Aicha, Maroua and Bouthaina" and amazing brothers "Haider, Imad and Farouk", thank you for your contant encouragement and belief in me.

To my neices and nephews "Djawad, Takoua, Darine and Assil", whose smiles and joy light up my life.

To my sistres' husbands "Walid and Sami" and my brothers' wives "Sawsen and Yousra", whose kindness and understanding have meant so much to me

And thememory of my grandmother "Fatna" whose spirit and legacy continue to inspire me every day.

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Abstract

Shannon entropy, established by Shannon in 1948 and widely utilized in reliability and information studies, has seen a recent counterpart in extropy, proposed by Lad et al. (2015), which serves as its dual measure.

This master's thesis aims to present the concept of extropy, a measure of uncertainty. One statistical application of extropy is to score the forecasting distributions. It addresses the problem of extropy of ordered variates records and order statistics and related properties. Simplified expressions of the extropy of ordered variates are derived.

Notations and symbols

iid	Independent and identically distributed
$P(.)$	probability function
cdf	Cumulative distribution function note $F(.)$
pdf	Probability density function also density function note $f(.)$
$\bar{F}(.)$	The complementary cumulative distribution function
$F_n(.)$	The empirical (or sample) distribution function
$U(n)$	Upper record times
$L(n)$	Lower record times
U_n^X	Upper record values
L_n^X	Lower record values
$U_{n,k}^X$	nth value of upper k-records
$L_{n,k}^X$	nth value of lower k-records
$X_{k,n}$	the kth statistical order
$f_{k,n}(x)$	The pdf of the order statistic
$F_{k,n}(x)$	The cdf of the order statistic
$F_{i,j:n}(x,y)$	The joint cdf of $X_{i,n}$ and $X_{j,n}$
$f_{i,j:n}(x,y)$	The joint pdf where $(X_{i,n} \leq X_{j,n})$

$h(x)$	The hazard function
$H(x)$	The cumulative hazard function
$f_{U_n^x}(x)$	Probability density function of nth Upper record values
$f_{U_m^X,U_n^X}(x,y)$	The joint probability density function of mth and nth Upper records
$f_{U_n^X\|U_m^X}(x,y)$	The conditional probability density function of nth upper given mth upper record
$F_{U_n^X}(x)$	Cumulative distribution function of nth value of upper records
$f_{U_{n,k}^X}(x)$	Probability density function of nth value of upper k-records
$F_{U_{n,k}^X}(x)$	Cumulative distribution function of nth value of upper k-records
$f_{U_{m,k}^X,U_{n,k}^X}(x,y)$	The joint probability density function of mth and nth Upper k records
$f_{U_{n,k}^X\|U_{m,k}^X}(y\|x)$	The conditional probability density function of nth upper given mth upper k record
$f_{L_n^X}(x)$	Probability density function of nth lower record values
$f_{L_m^X,L_n^X}(x,y)$	The joint probability density function of mth and nth lower records
$f_{L_n^X\|L_m^X}(x,y)$	The conditional probability density function of nth lower given mth lower record
$f_{U_{n,k}^X}(x)$	Probability density function of nth value of upper k-records
$F_{U_{n,k}^X}(x)$	Cumulative distribution function of mth and nth Upper krecords
$f_{U_{m,k}^X,U_{n,k}^X}(x,y)$	The joint probability density function of mth and nth Upper krecords
$f_{U_{n,k}^X\|U_{m,k}^X}(y\|x)$	The conditional probability density function of nth upper given mth upper k record

$f_{L^X_{n,k}}(x)$	Probability density function of nth value of lower k-records
$F_{L^X_{n,k}}(x)$	Cumulative distribution function of mth and nth lower krecords
$f_{L^X_{m,k},L^X_{n,k}}(x,y)$	The joint probability density function of mth and nth lower krecords
$f_{L^X_{n,k}\|L^X_{m,k}}(x,y)$	The conditional probability density function of nth lower given mth lower k record
$J(X)$	The Extropy of X
$F^{-1}(u)$	The inverse function of a cumulative distribution function
X_i	Random variables with $i = 1, 2, ...$
$R.V$	Record values
$O.S$	Order statistics

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Introduction

Uncertainty is commonly understood as a result of insufficient information, whereas information is seen as the means to diminish uncertainty. When these specific interpretations of uncertainty and information risk being misunderstood with their broader meanings, it is beneficial to refer to them as information-driven uncertainty and uncertainty-driven information, respectively.

Uncertainty, often attributed to a lack of information, finds its quantitative expression through entropy, a cornerstone of information theory introduced by Shannon (1948[26]) and Wiener (1948[28]) over three decades ago. This formalism continues to thrive, applied from image restoration to socio-economic data analysis. However, as interpretations of uncertainty and information broaden, distinctions such as "information-driven uncertainty" and "uncertainty-driven information" become pivotal.

In recent years, the concept of entropy has evolved further with the proposal of extropy by Lad, Sanfilippo, and Agro (2015[14]). Extropy complements Shannon (1948[26]) entropy, offering a dual perspective on uncertainty measurement, particularly suited for ordered variables and statistical distributions beyond conventional entropy's scope. This extension finds applications in fields ranging from order statistics to advanced information measures. The extropy measure has been developed for ordered variables. Qiu (2017[20]) was the first to apply extropy for order statistics and record values and present several of their properties. After that, the researchers manifested to present extension measures of extropy.

This master's thesis is divided to two chapters, in **the first chapter**, we will present an overview of fundamental concepts related to records and order statistics. This includes an exploration of various notions and properties of order statistics and record statistics (both upper and lower) and their distributions in continuous cases.

In **the second and final chapter**, we will find into the extropy of records and order statistics. We will discuss some results and characterizations based on both records **(R.V)** and order statistics **(O.S)**. Additionally, we will perform numerical computations on the upper and lower record values, order statistics, and their extropy.

Chapter 1

Records and Order Statistics

In this chapter, we find into the record and the order statistics pertaining to continuous random variables, which comprises four sections. Section 1.1 serves as an introduction to order statistic and we explore various characterizations of order statistics derived from a sequence of continuous random variables. Section 1.2 acts as an introductory overview to order statistics, We find different characterizations of record statistics "upper and lower" obtained from a sequence of continuous random variables.

1.1 Order Statistic

In this section, we start by explaining some definitions and characteristics of order statistics. Then, we gradually move on to discussing their distributions in continuous cases.

1.1.1 Ordinary Order Statistics

Definition 1.1.1 *Let $(X_1, X_2, ..., X_n)$ be a sequence of independent and identically distributed (iid) random variable over space $(\mathbb{R}, \Omega)$, each distributed according to continuous cdf F and pdf f $F(F(x) = P(X_n \leq x)$, for $x \in \mathbb{R})$ let S_n be the set of permutation of $\{1, ..., n\}$.*

Definition 1.1.2 *We have([3]):*

- *The sample statistics $(X_1, X_2, ..., X_n)$ is the increasing arrangement $(X_1, ..., X_n)$. It is noted $(X_{1,n}, X_{2,n}, ..., X_{n,n})$. It has $X_{1,n} \leq ... \leq X_{n,n}$, and there is a random permutation $\sigma_n \in S_n$ such that*

$$(X_{1,n}, ..., X_{n,n}) = \left(X_{\sigma_n(1)}, ..., X_{\sigma_n(n)}\right).$$

  The vector $(X_{1,n}, ..., X_{n,n})$ is called the associated ordered sample $(X_1, ..., X_n)$ and $X_{k,n}$ being the k^{th} statistical order

- *The random variable minimum and maximum of the n-sample (iid) correspond better to the idea of extreme value:*

$$X_{1,n} = m(n) = \min(X_1, ..., X_n) = \min_{1 \leq i \leq n} X_i$$

$$X_{n,n} = M(n) = \max(X_1, ..., X_n) = \max_{1 \leq i \leq n} X_i$$

- $F_n(x) = \frac{1}{n}\sum\limits_{i=1}^{n} 1_{\{X_i \leq x\}}$ *denotes the empirical (or sample) distribution function*

Let mention that:

$$F_n(x) = \begin{cases} 0 & \text{if } x \prec X_{1,n} \\ \frac{i}{n} & \text{if } X_{i,n} \leq x \leq X_{i+1,n} \text{ and } 1 \leq i \leq n-1 \\ 1 & \text{if } \quad x > X_{n,n} \end{cases}$$

Remark 1.1.1 *If the law of variable X is absolutely continuous one can conclude that,*

$$P(X_{1,n} \leq X_{2,n} \leq ... \leq X_{n,n}) = 1$$

1.1.2 K-Order Probability Density Function

Proposition 1.1.1 *The cdf of the order statistic $X_{k,n}$ is given for all $x \in \mathbb{R}$ by (David [1970], Balakrishnan and Clifford Cohen[1991])*

$$F_{k,n}(x) = F_{X_{k,n}}(x) = P(X_{k,n} \leq x) = \sum_{r=k}^{n} C_n^r [F(x)]^r [1 - F(x)]^{n-r}, x \in \mathbb{R}. \tag{1.1}$$

Proposition 1.1.2 *In the case where the distribution function F is continuous and derivable almost everywhere from derivative$f$, order statistics have a law that admits a density relative to the Lebesgue measurement. The density of $X_{k,n}$ is given by*

$$f_{k,n}(x) = \frac{n!}{(k-1)!\,(n-k)!} f(x) [F(x)]^{k-1} [1 - F(x)]^{n-k}. \tag{1.2}$$

As another direct consequence of this result, it is easily shown that if $U_1, ..., U_n$ One are random variables independent of uniform law then $U_{k,n}$ follows a Beta law of k and $n-k+1$ parameters. It is recalled that the density of a Beta law of *parameters $a > 0$ and $b > 0$ is*

$$\frac{1}{\beta(a,b)} x^{a-1} (1-x)^{b-1}, x \in [0,1].$$

In particular, for all $s \in \mathbb{N}$,

$$E\left(U_{k,n}^{s}\right) = \frac{n!}{(n+k)!} \frac{(k+s-1)!}{(k-1)!}. \tag{1.3}$$

Proposition 1.1.3 *(Barry and al.(1992))If the distribution function F is continuous and derivable almost everywhere from derivative f, the density of the vector $(X_{1,n}, ..., X_{n,n})$ is*

$$f_n(x_1, ..., x_n) = n! \prod_{i=1}^{n} f(x_i), x_1 \leq x_2 \leq ... \leq x_n. \tag{1.4}$$

It can therefore be concluded from the statistics of the minimum that the cdf and the pdf are respectively

$$F_{1,n}(x) = F_{X_{1,n}}(x) = [1 - F(x)]^n$$

$$f_{1,n}(x) = f_{X_{1,n}}(x) = n[1 - F(x)]^{n-1} f(x), \tag{1.5}$$

For the maximum statistics, we have

$$F_{n,n}(x) = F_{X_{n,n}}(x) = [F(x)]^n, \tag{1.6}$$

$$f_{n,n}(x) = f_{X_{n,n}}(x) = n[F(x)]^{n-1} f(x), \tag{1.7}$$

Proof. Using the independent property of the random variables $X_1, X_2, ..., X_n$, we deduce that,

$F_{1,n}(x) = P\{X_{1,n} \leq x\} = 1 - P\{X_{1,n} \succ x\} = 1 - P\left\{\bigcap_{i=1}^{n} X_i \succ x\right\} = 1 - \prod_{i=n}^{n} P\{X_i \succ x\} = 1 - \prod_{i=1}^{n}[1 - P\{X_i \leq x\}] = [1 - F(x)]^n$, $F_{n,n}(x) = P\{X_{n,n} \leq x\} = \prod_{i=1}^{n} P\{X_i \leq x\} = [F(x)]^n$. ■

Example 1.1.1 *Let $X_1, X_2, ..., X_5$ be a simple aleatory sample and $Y_1 < Y_2 < Y_3 < Y_4 < Y_5$ are statistics of order and size $n = 5$ with density $f(x) = 3x^2$, $0 \leq x \leq 1$.*

$F(x) = \int_0^x f(t)\,dt = x^3$, $0 \leq x \leq 1$.

$F(y) = P(X \leq y) = y^3$, $0 \leq y \leq 1$.

1. $f_{1,5}(y_1) = n[1 - F(y_1)]^{n-1} f(y_1) = 5[1 - y_1^3]^{5-1} 3y_1^2 = 15y_1^2[1 - y_1^3]^4$ $F_{1,5}(y_1) = [1 - F(y_1)]^n = [1 - F(y_1)]^5 = [1 - y_1^3]^5$

2. $f_{5,5}(x) = n[F(y_1)]^{5-1} f(y_1) = 5[y_1^3]^{5-1} 3y_1^2 = 15y_1^{24}$

   $F_{5,5}(x) = [F(y_1)]^5 = [y_1^3]^5 = y_1^{15}$

3. $f_{3,n}(x) = \frac{5!}{2!2!} f(x)[F(x)]^2[1 - F(x)]^2 = \frac{5!}{2!2!} 3y_1^2[y_1^3]^2[1 - y_1^3]^2 = 30y_1^8[1 - y_1^3]^2$

   $F_{3,5}(x) = \sum_{r=3}^{5} C_5^r [F(x)]^r [1 - F(x)]^{5-r} = 5y_1^9[1 - y_1^3][2(1 - y_1^3) + y_1^3]$

Theorem 1.1.1 *The joint cdf of $X_{i,n}$ and $X_{j,n}$, where $(X_{i,n} \leq X_{j,n})$, $1 \leq i \leq j \leq n$ and $-\infty \prec x \prec y \prec +\infty$ is given by(Arnold et al.(1992) [6])*

$$F_{i,j:n}(x,y) = \sum_{s=j}^{n}\sum_{r=i}^{s} \frac{n!}{(r-1)!\,(s-r-1)!\,(n-s)!}[F(x)]^{r-1} f(x)[F(y) - F(x)]^{s-r-1} f(y)[1 - F(y)]^{n-s}. \tag{1.8}$$

For $x \geq y$ then the cdf is

$$F_{i,j:n}(x,y) = F_{(X_{i,n} \leq X_{j,n})}(x,y) = F_{X_{j,n}}(y).$$

The joint pdf where $(X_{i,n} \leq X_{j,n})$ is

$$f_{i,j:n}(x,y) = \frac{n!}{(i-1)!\,(j-i-1)!\,(n-j)!}\left[F(x)\right]^{i-1} f(x)\left[F(y)-F(x)\right]^{j-i-1}$$
$$f(y)\left[1-F(y)\right]^{n-j}. \tag{1.9}$$

Remark 1.1.2 *The joint pdf of $X_{1,n}$and $X_{n,n}$ is*

$$f_{X_{1,n},X_{n,n}}(x,y) = n(n-1)\left[F(y)-F(x)\right]^{n-2}, -\infty \prec x \prec y \prec +\infty. \tag{1.10}$$

1.2 Record Statistics

In this section, we first explain what upper and lower record statistics are and their main features. Then, we move on to how they are distributed in continuous cases.

1.2.1 Ordinary Record Statistics

Let $X_1, X_2, ...$ be a sequence of independent and identically distributed (iid) random variables and $X_{1,n} \leq ... \leq X_{n,n}$ be the corresponding order statistics.

Definition 1.2.1 *The classical upper record times $U(n)$ $(n \geq 1)$ and upper record values U_n^X as follows:*

$$U(1) = 1, U(n+1) = \min\left\{j : j \geq U(n), X_j \succ U_n^X\right\}, U_n^X = X_{U(n)}, n = 1, 2, ...$$

Definition 1.2.2 *The sequences of lower record times $L(n)$ $(n \geq 1)$ and lower record values L_n^X as follows :*

$$L(1) = 1, L(n+1) = \min\left\{j : j \geq L(n), X_j \prec L_n^X\right\}, L_n^X = X_{L(n)}, n = 1, 2, ...$$

1.2.2 Record Probability Density Function

Proposition 1.2.1 *The hazard function, also known as the failure rate or force of function, denoted as $h(x)$, represents the ratio of the probability density function $f(x)$ to the survival function $1 - F(x)$. This relationship is expressed as:*

$$h(x) = \frac{f(x)}{1 - F(x)}.$$

The cumulative hazard function

$$H(x) = -\log[1 - F(x)]. \tag{1.11}$$

Lemma 1.2.1 *Let X be a random variable having an absolutely continuous cumulative distribution function (cdf) $F(x)$and probability density function (pdf) $f(x)$, be the cumulative hazard function and the corresponding hazard rate, respectively. Then the pdf of U_n^X is obtained to be (Arnold et al. (1998)[7])*

$$f_{U_n^x}(x) = \frac{[H(x)]^{n-1}}{(n-1)!} f(x), \ x \in \mathbb{R}, n = 1, 2, ... \tag{1.12}$$

$\circ$*The joint pdf of U_m^X and U_n^X ; where $1 \leq m < n$; is given by (Arnold et* *al.(1998)[7])*

$$f_{U_m^X,U_n^X}(x,y)=\frac{[H(x)]^{m-1}}{(m-1)!}\frac{[H(y)-H(x)]^{n-m-1}}{(n-m-1)!}h(x)f(y),$$

$$-\infty\prec x\prec y\prec+\infty \tag{1.13}$$

$\circ$*From the pdf of* U_n^X *and The joint pdf of* U_m^X *and* U_n^X, *the conditional pdf of* U_n^X *given* $U_m^X(1\leq m<n)$; *can be written as*

$$f_{U_n^X|U_m^X}(x,y)=\frac{[H(y)-H(x)]^{n-m-1}}{(n-m-1)!}\frac{f(y)}{1-F(x)},\ x\prec y \tag{1.14}$$

$\circ$*The survival function of the nth record can be represented as*

$$1-F_{U_n^X}(x)=[1-F(x)]\sum_{j=0}^{n-1}\frac{[-\log(1-F(x))]^j}{j!} \tag{1.15}$$

Where

$$F_{U_n^X}(x)=\int_0^{-\log(1-F(x))}\frac{u^{n-1}}{(n-1)!}e^{-u}du.$$

Similarly, the pdf of L_n^X *is given by*

$$f_{L_n^X}(x)=\frac{[-\log F(x)]^{n-1}}{(n-1)!}f(x),\ \ -\infty\prec x\prec+\infty,n=1,2,... \tag{1.16}$$

The joint pdf of L_m^X *and* L_n^X, *where* $1\leq m\prec n$, *can be written as*

$$f_{L_m^X,L_n^X}(x,y)=\frac{[-\log F(x)]^{m-1}}{(m-1)!}\frac{[\log F(x)-\log F(y)]^{n-m-1}}{(n-m-1)!}\frac{f(x)}{F(x)}f(y),$$

$$-\infty\prec x\prec y\prec+\infty. \tag{1.17}$$

The conditional pdf of L_n^X given L_m^X where $1 \leq m < n$, can be written as([4],[7])

$$f_{L_n^X|L_m^X}(x,y) = \frac{[-\log F(y) + \log F(x)]^{n-m-1}}{(n-m-1)!}\frac{f(y)}{F(x)}, x \prec y. \tag{1.18}$$

1.2.3 k-Record Statistics

Let $\{X_k; k \geq 1\}$g be a sequence of iid random variables with a continuous cdf F and pdf f. Let $X_{1:n} \leq X_{2:n} \leq ... \leq X_{n:n}$; be the order statistics of $X_1, X_2, ..., X_n$ Let

$$U_{1,k} = k,$$

and

$$U_{n,k} = \min\left\{j : j \succ U_{n-1,k}, X_j \succ X_{U_{n-1,k}-k+1:U_{n-1,k}}\right\}, \tag{1.19}$$

for a fixed integer $k \geq 1$ and $n \geq 2$. Then $U_{n,k}^X = X_{U_{n-1,k}-k+1:U_{n-1,k}}$ and $U_{n,k}(n \geq 1)$ are the sequences of upper k-record values and upper k-record times, respectively. The model of k-record values are proposed firstly by Dziubdziela and Kopocinski (1976)([7])

Theorem 1.2.1 *The pdf of $U_{n,k}^X$ is obtained to be*

$$f_{U_{n,k}^X}(x) = k^n \frac{[H(x)]^{n-1}}{(n-1)!}[1-F(x)]^{k-1} f(x), -\infty \prec x \prec +\infty. \tag{1.20}$$

Proof. By induction, the densities $f_{U_n^X}(x_1, ..., x_k)$, $n = 1, 2, ...$, satisfy the equations

$$f_{U_n^X}(x_1, ..., x_n) = \begin{cases} k! g_{U_n^X}(x_1) f(x_1) f(x_2) ... f(x_k), \; x_1 \prec x_2 \prec ... \prec x_k \\ 0, \; otherwise \end{cases}$$

where,

$$g_{U_1^X}(x) = 1$$

$$g_{U_{n+1}^X}(x) = k\int_{-\infty}^{x} g_{U_n^X}(y)\,\frac{f(y)}{1-F(y)}dy, n = 1, 2, ...$$

it is easy to verify that

$$g_{U_n^X}(x) = \frac{1}{(n-1)!}\left[-\log\left(1-F(x)\right)\right]^{n-1}, n = 1, 2, ...$$

then we have

$$\begin{aligned} f_{U_{n,k}^X}(x) &= \int_x^{\infty}\int_{x_2}^{\infty}...\int_{x_{k-1}}^{\infty}\frac{k!}{(n-1)!}f(x)\,f(x_2)\,...f(x_k)\,dx_k...dx_2 \\ &= \frac{k}{(n-1)!}\left[-k\log\left(1-F(x)\right)\right]^{n-1}\left[1-F(x)\right]^{n-1}f(x)\,. \end{aligned} \tag{1.21}$$

■

Corollary 1.2.1 *We have,*

○*The cdf of* $U_{n,k}^X$ *is obtained to be ([27])*

$$F_{U_{n,k}^X}(x) = \frac{\gamma\left(n, -k\ln F(x)\right)}{\gamma(n)} \tag{1.22}$$

where $\Gamma(a,x) = \int_0^x u^{a-1}e^{-u}du$ *is the lower incomplete gamma function.*

○*The joint pdf of* $U_{m,k}^X$ *and* $U_{n,k}^X$*, where* $1 \leq m < n$*, is given by (see, Grudzien (1982)[10]).*

$$\begin{aligned} f_{U_{m,k}^X,U_{n,k}^X}(x,y) =& \frac{k^n}{(n-1)!\,(n-m-1)!}\left[H(x)\right]^{m-1}\left[H(y)-H(x)\right]^{n-m-1}\left[1-F(y)\right]^{k-1} \\ & h(x)\,f(y)\,, -\infty \prec x \prec y \prec +\infty. \end{aligned} \tag{1.23}$$

$\circ$ *The conditional pdf of* $U_{n,k}^X$ *given* $U_{m,k}^X$ $(1 \leq m < n)$, *can be written as*

$$f_{U_{n,k}^X | U_{m,k}^X}(y \mid x) = \frac{k^{n-m}}{(n-m-1)!}\left[H(y) - H(x)\right]^{n-m-1}\left[1 - F(y)\right]^{k-1} \frac{f(y)}{\left[1 - F(x)\right]^{k}}, x \prec y. \tag{1.24}$$

Similarly, we have $L_{n,k}^X = X_{L_{n-1,k}-k+1:L_{n-1,k}}$ *is the sequences of lower k-record values and* $L_{n,k}$ $(n \geq 1)$ *is sequences of lower* $k-record\ times$,

$$L_{1,k} = k$$

with probability 1,

$$L_{n,k} =' \min\left\{j : j \succ L_{n-1,k}, X_j \prec X_{L_{n-1,k}-k+1:L_{n-1,k}}\right\}, \tag{1.25}$$

Corollary 1.2.2 *We have:*

$\circ$*The pdf of* $L_{n;k}^X$ *is given by*

$$f_{L_{n,k}^X}(x) = k^n \frac{\left[-\log F(x)\right]^{n-1}}{(n-1)!}\left[F(x)\right]^{k-1} f(x), -\infty \prec x \prec +\infty \tag{1.26}$$

$\circ$*The cdf of* $L_{n;k}^X$ *is given by([27])*

$$F_{L_{n,k}^X}(x) = \frac{\gamma\left(n, -k\ln\left[1 - F(x)\right]\right)}{\gamma(n)}, \tag{1.27}$$

Where,$\gamma(a, x) = \int_0^x u^{a-1} e^{-u} du$ *is the upper incomplete gamma function.*

$\circ$*The joint pdf of* $L_{m,k}^X$*and* $L_{n,k}^X$; *where* $1 \leq m < n$, *can be written as (see, Pawlas* *and Szynal (1998, 2000)[16] and [18])*

$$f_{L^X_{m,k},L^X_{n,k}}(x,y) = \frac{[-\log F(x)]^{m-1}}{(m-1)!}\frac{[\log F(x)-\log F(y)]^{n-m-1}}{(n-m-1)!}[F(y)]^{k-1}$$
$$\frac{f(x)}{F(x)}f(y),-\infty \prec x \prec y \prec +\infty. \tag{1.28}$$

$\circ$ *The conditional pdf of* $L^X_{n,k}$ *given* $L^X_{m,k}$ *where* $1 \leq m < n$*, can be written as*

$$f_{L^X_{n,k}|L^X_{m,k}}(x,y) = k^{n-m}\frac{[-\log F(y)+\log F(x)]^{n-m-1}}{(n-m-1)!}[F(y)]^{k-1}$$
$$\frac{f(y)}{[F(x)]^k}, x \prec y. \tag{1.29}$$

For details on generalized $k-$ *values, one can refer Arnold, Balakrishnan, and Nagaraja (1998)[7], Dziubdziela and Kopoci nski (1976)[8], and Goel, Taneja, and Kumar (2018)[11]. Now, we present a brief introduction of some univariate stochastic orders which are used to compare two random variables. For various other stochastic orders and stochastic comparison results, one can refer Shaked and Shanthikumar (2007).[25]*

Example 1.2.1 *Climate data of the Biskra area during the period (1989-2021) (Weather in Biskra, 2021).This dataset represts Relative Humidity(%) (H%) over 12 month* $(n = 12)$ *in 2021[5]*

$$\{51.32, 41.95, 43.6, 41.22, 31.05, 26.72, 24.92, 28.92, 40.1, 40.82, 47.15, 52.22\}$$

Order Statistics:

$\circ$ *We want to find the pdf and cdf of the* 3^{th} *order statistics,* $X_{(3)}$.

$$\{24.92 \leq 26.72 \leq 28.92 \leq 32.05 \leq 40.1 \leq 40.82 \leq 41.22 \leq$$
$$41.95 \leq 43.6 \leq 47.5 \leq 51.32 \leq 52.22\}$$

Given that the Relative Humidity follows a gamma distribution with shape parameter $k = 2$ and scale parameter $\theta = 3$, we can use the properties of order statistics. First, let's calculate the cdf of the gamma distribution:

$$F_X(x) = \frac{1}{\Gamma(k)} \int_0^x t^{k-1} e^{-\frac{t}{\theta}} dt$$

For $k = 2$ and $\theta = 3$, the cdf become:

$$F_X(x) = [-3xe^{-\frac{x}{3}} - 27e^{-\frac{x}{3}} + 27]$$

To find the pdf and cdf of $X_{(3)}$,we need consider the order statistics distribution formula (1.1) and (1.2), we need also the pdf of the gamma distribution, which is:

$$f_X(x) = \frac{1}{\Gamma(k)\,\theta^k} x^{k-1} e^{-\frac{x}{\theta}}$$

In our case, $k = 2$ and $\theta = 3$,

$$f_X(x) = \frac{1}{\Gamma(2)\,3^2} x^{2-1} e^{-\frac{x}{3}} = \frac{1}{9} x e^{-\frac{x}{3}}$$

Where:

- *n is the sample size ,*
- *r is the order statistics we are interested in,*
- *$F_X(x)$ is the cdf of the gamma distribution,*
- *$f_X(x)$ is the pdf of the gamma distribution.*

In our case, $n = 10$ and $r = 3$, then the $X_{(3)} = 28.92$ into the order statistics formula:

1. *the pdf of* $X_{(3)}$,

$$f_{X_{(3)}}\left(x\right)=\frac{10!}{\left(3-1\right)!\left(10-3\right)!}\left[F_{X_{(3)}}\left(x\right)\right]^{3-1}\left[1-F_{X_{(3)}}\left(x\right)\right]^{10-3}f_{X_{(3)}}\left(x\right)$$

2. *The cdf of* $X_{(3)}$,

$$F_{X_{(3)}}\left(x\right)=\sum_{i=4}^{10}C_{10}^{i}\left[F_{X_{(3)}}\left(x\right)\right]^{i}\left[1-F_{X_{(3)}}\left(x\right)\right]^{10-i}$$

Record Statistics:

We have,

$$\{51.32, 41.95, 43.6, 41.22, 31.05, 26.72, 24.92, 28.92, 40.1, 40.82, 47.15, 52.22\}$$

For $n = 1$, *where* n *represents the sample size, the first lower record time is denoted as* $L(1) = 1$

$$X_{L_1} = L_1^X = X_1 = 51.32$$

When $n = 2$, *the process involves determining the lower record time and subsequently identifying the corresponding lower value. To accomplish this, we search for* j *such that it satisfies the following conditions:*

$$L\left(2\right)=\min\left\{j:j\geq L\left(1\right),X_{j}\prec L_{1}^{X}\right\}$$

we have:

$X_2 = 51.32 \succ 41.95$: *hence, the corresponding value of the second lower record should be determined* $L(2) = 2,\ X_{L(2)} = L_2^X = X_2 = 41.95.$

$X_3 = 43.6 \succ 41.95$: *does not give the required results.*

$\circ$*Alternatively, by examining the lower record values for all subsequent* n, *we observe that* $X_{L(i)} = L_i^X = X_2$ *for all* $i = 2, 3..., 12$. *This is due to* X_8 *being the smallest value in* X_j *where* $1 \leq j \leq 12$. *Our analysis concludes at* $n = 2$ *as it has reached the minimum value within the dataset.*

Thus, the lower record values within this dataset can be identified as follows: "$51.31, 41.95, 41.22, 32.05, 26.72, 24.92$".

Using a similar approach, we can determine the upper record values as follows: "$51.32, 52.22$".

We establish lower k-records from the dataset $k = 2, 3$, *as outlined below: Consider* $k = 2$ *and* $n = 1, 2, \ldots, 2$. *Let* $X_{1,12} \geq ... \geq X_{12,12}$ *be the order statistics of* $X_1, ..., X_{12}$ *as follows*

We have for $n = 1$

$$L_{1,2} = 2, L_{1,2}^X = X_{L_{1,2}-2+1:L_{1,,2}} = X_{1,2} = 51.32$$

For $n = 2$

$$L_{2,2} = \min\left\{j : j \succ L_{1,2}, X_j \prec X_{L_{1,2}-3:L_{1,2}}\right\} = \min\left\{j : j \succ 2, X_j \prec X_{1,2}\right\}$$

Now, we seek the value of j *that fulfills the following condition:*

$\min\left\{j : j \succ 2, X_j \prec X_{1,2}\right\}.$

Therefore, we find for

$$j = 3: \;\; L_{2,2} = \min\left\{j : 3 \succ 2, X_3 \prec 51.32\right\}.$$

Therefore,

$$L_{2,2} = 3, L_{2,2}^X = X_{L_{2,2}-2+1:L_{2,2}} = X_{2,3} = 43.6$$

For $n = 3$

$$L_{3,2} = \min\left\{j : j \succ L_{2,2}, X_j \prec X_{L_{2,2}-3:L_{2,2}}\right\}$$

$$= \min\left\{j : j \succ 3, X_j \prec X_{2,3}\right\}$$

Thus, $j = 4$ *satisfies the condition, and therefore we have:*

$$j = 4: \;\; L_{3,2} = \min\left\{j : 4 \succ 3, X_4 \prec 43.6\right\}$$

Therefore,

$$L_{3,2} = 4, L_{3,2}^X = X_{L_{3,2}-2+1:L_{3,2}} = X_{3,4} = 41.95$$

We proceed similarly to obtain the sequence of the lower $2 - record$*, which is presented as follows:*

$$51.32, 43.6, 41.95, 41.22, 32.05, 26.72$$

As a result, the table below displays the upper k-record for $k = 2, 3, 4, 5$

$2-record$	51.32	43.6	41.95	41.22	32.05	26.72
$3-record$	51.32	41.95	41.22			
$4-record$	51.32	43.6	41.95	41.22	32.05	
$5-record$	51.32	43.6	41.95	41.22	40.1	

Table 1.1: The Lower k-Record for $k = 2, 3, 4, 5$.

To find the pdf and cdf of $2 - record$*,we need to consider the record statistics distribution formula: (1.20) and (1.22)*

$$f_{L^X_{n,k}}(x) = k^n \frac{[-\log F(x)]^{n-1}}{(n-1)!} [F(x)]^{k-1} f(x), -\infty \prec x \prec +\infty$$

$$F_{L^X_{n,k}}(x) = \frac{\gamma(n, -k \ln [1 - F(x)])}{\gamma(n)}$$

we need also the pdf and the cdf of the gamma distribution, which are
$(In\ our\ case,\ k = 2\ and\ \theta = 3)$*:*

$$f_X(x) = \frac{1}{9} x e^{-\frac{x}{3}}$$

$$F_X(x) = [-3xe^{-\frac{x}{3}} - 27e^{-\frac{x}{3}} + 27]$$

In our case, $n = 10$ *and* $k = 2$*, then the lower 2-record statistics formula:*

$$f_{L^X_{10,2}}(x) = 2^{10} \frac{\left[-\log[-3xe^{-\frac{x}{3}} - 27e^{-\frac{x}{3}} + 27]\right]^{10-1}}{(10-1)!} [-3xe^{-\frac{x}{3}} - 27e^{-\frac{x}{3}} + 27]^{2-1} \frac{1}{9} x e^{-\frac{x}{3}},$$

$$-\infty \prec x \prec +\infty$$

$$F_{L^X_{10,2}}(x) = \frac{\gamma\left(10, -2\ln\left[1 - (-3xe^{-\frac{x}{3}} - 27e^{-\frac{x}{3}} + 27)\right]\right)}{\gamma(10)}$$

Chapter 2

Extropy of Records and Order Statistics

This chapter begins with an introduction to extropy, provided in section2.1, and some preliminaries of stochastic order in section2.2. Following that, we find into the discussion of the extropy of k-records in sections2.3 and extropy of k-order in section2.4, and the latter two are dedicated to exploring fundamental findings and characterizations of the extropy of records and order statistics. In the end, we will make Some numerical computations in 2.5.

2.1 Background

A novel measure called extropy was introduced by Lad et al. (2015) to quantify certain aspects of random variables. If we consider a random variable X with a cumulative distribution function (cdf) $F(x)$ and probability density function (pdf) $f(x)$ that are both absolutely continuous, then the extropy of X is defined as

$$J\left(x\right) = -\frac{1}{2}\int_{-\infty}^{+\infty} f^2\left(x\right)dx \tag{2.1}$$
$$= -\frac{1}{2}\int_{0}^{1} f\left(F^{-1}\left(u\right)\right)du$$

where

$$F^{-1}\left(u\right) = \inf\left\{t : F\left(t\right) \geq u\right\},\ 0 \prec u \prec 1,$$

is the quantile function of F. Lad et al. (2015) present numerous properties of the extropy measure and explore its applications. Qiu (2017)[20] further contributes to this field by establishing characterizations and lower bounds on the extropy of order and record statistics, investigating associated monotonic propreties, and exploring stochastic comparisons. This research finds into analyzing the extropy measure concerning record statistics. Lad et al. (2015) introduced an alternative measure of uncertainty, which is defined for a random variable X as

$$J\left(X\right) = -\frac{1}{2}E\left(f\left(X\right)\right). \tag{2.2}$$

Recent literature has extensively discussed the statistical applications of extropy. Qiu (2017)[20] examines various properties, characterizations, and lower bounds related to the extropy of order and record statistics. Additionally, Qiu and Jia (2018a)[21] propose estimators for extropy, which are applied in testing uniformity. Introducing the concept of residual extropy, Qiu and Jia (2018b)[22] aims to measure the residual uncertainty of a random variable. Most recently, Raqab and Qiu (2018)[23] explored the extropy of ranked set sampling, investigating numerous monotone properties and other associated bounds.

2.2 Preliminaries

Consider two random variables, X and Y, each characterized by cumulative distribution functions (cdf's) denoted by F and G, and survival functions represented by $\bar{F} = 1 - F$ and $\bar{G} = 1 - G$, respectively. The probability density functions (pdf's) corresponding to X and Y are f and g, respectively. Let $F^{-1}(x) = inf\{t : F(t) \geq x\}$ and $G^{-1}(x) = inf\{t : G(t) \geq x\}$ represent the left continuous inverses of F and G, where $0 < x < 1$.

Definition 2.2.1 (Shaked and Shanthikumar, 2007[25]) *It is stated that X is less than Y*

1. *Usual stochastic order $(X \leq_{st} Y)$ if $\bar{F}(t) \leq \bar{G}(t)$ for all $t \in [0, +\infty[$.*
2. *Likelihood ratio order $(X \leq_{lr} Y)$ if the ratio $g(t)/f(t)$ is increasing in $t \in [0, +\infty[$.*
3. *In the hazard rate order $(X \leq_{hr} Y)$ if $h_X(t) \geq h_Y(t)$ for all $t \in [0, +\infty[$, $h_X = f/\bar{F}$ $\left(h_Y = g/\bar{G}\right)$ is the hazard rate function of X (Y).*
4. *In the convex transform order $(X \leq_c Y)$ if $G^{-1}F(t)$ is convex in $t \in [0, +\infty[$.*
5. *In the star order $(X \leq_* Y)$ if $G^{-1}F(t)$ is star-shaped (that is, if $G^{-1}F(t)/t$ increase in $x > 0$) .*
6. *In the supper-additive order $(X \leq_{su} Y)$ if $G^{-1}F(t)$ is super-additive, equivalently, $G^{-1}F(t+s) \geq G^{-1}F(t) + G^{-1}F(s)$ for all $t, s \geq 0$.*
7. *In the increasing concave (convex) order $(X \leq_{icv} Y \ (X \leq_{icx} Y))$ if $E\Phi(X) \leq E\Phi(Y)$ for all increasing concave (convex) functions $\Phi : \mathbb{R} \longrightarrow \mathbb{R}$, provided the expectations exist;*

8. *In the dispersive order, denoted by* $(X \leq_{disp} Y)$ *if* $G^{-1}F(t) - t$ *is increasing in* $t \geq 0$. *The connections between the earlier mentioned stochastic orders are described in the following diagram (see Shaked and Shanthikumar, 2007[25])*

$$X \leq_{lr} Y \Rightarrow X \leq_{hr} Y \Rightarrow X \leq_{st} Y \Rightarrow X \leq_{icx} Y,$$

$$X \leq_{c} Y \Rightarrow X \leq_{*} Y \Rightarrow X \leq_{su} Y.$$

Lemma 2.2.1 *Suppose* X *and* Y *are two non-negative random variables with probability density functions* f *and* g*, respectively, where* $f(0) \geq g(0) > 0$.
If $X \leq_{su} Y$ $(X \leq_{c} Y$ *or* $X \leq_{*} Y)$, *then* $X \leq_{disp} Y$. *([1])*

Lemma 2.2.2 *If* $X \leq_{disp} Y$, *then* $J(X) \leq J(Y)$.

Proof. It is evident that

$$\begin{aligned} J(X) &= -\frac{1}{2}\int_{s} f^{2}(x)\,dx \\ &= -\frac{1}{2}\int_{s} f(x)\,dF(x) \\ &= -\frac{1}{2}\int_{0}^{1} f\left(F^{-1}(u)\right)du \end{aligned}$$

If $X \leq_{disp} Y$, $G^{-1}F(x) - x$ is increasing in $x \geq 0$ defintion. Hence, we obtain $f(F^{-1}(v)) \geq g(G^{-1}(v))$ for all $v \in [0,1]$.
Thus,

$$J(X) = -\frac{1}{2}\int_{0}^{1} f\left(F^{-1}(v)\right)dv \leq -\frac{1}{2}\int_{0}^{1} g\left(G^{-1}(v)\right)dv = J(Y)$$

Suppose we have independent and identically distributed (iid) observations $X_1, ..., X_n$

with cumulative distribution function F and probability density function f. The order statistics (OS) of the sample are denoted by $X_{1:n} \leq X_{2:n} \leq ... \leq X_{n:n}$. Similarly, denote the OS of Y by $Y_{i:n}$, where $i = 1, 2, ..., n$. If X is less than or equal to Y in distribution $X \leq_{disp} Y$, in Shaked and Shanthikumar (2007)[25] asserts that $X_{i:n} \leq_{disp} Y_{i:n}$ for $i = 1, 2, ..., n$. ■

2.3 Extropy of Record Statistics

This section will initially provide the expressions for the extropy of the nth value of upper k-records and the nth value of lower k-records. Drawing from these expressions, we will then outline the symmetric properties existing between the extropy of nth upper k-record values and nth lower k-record values.

2.3.1 The Extropy of nth Value of k-Records

In this subsection, we begin by presenting the expression for the extropy of $U_{n,k}^X$ and $L_{n,k}^X$. To achieve this, we substitute 2.1 into 1.26.

Theorem 2.3.1 *The extropy of nth value of lower k-records is given by*

$$J\left(L_{n,k}^X\right) = -\frac{k^{2n}}{2\left(2k-1\right)^{2n-1}} \begin{pmatrix} 2n-1 \\ n-1 \end{pmatrix} Ef\left(F^{-1}\left(e^{-V_{2n-1,2k-1}}\right)\right). \qquad (2.3)$$

Proof.

we have,

$$J\left(L_{n,k}^X\right) = -\frac{1}{2}\int_{-\infty}^{+\infty} k^{2n} \frac{\left[-\log F\left(x\right)\right]^{2(n-1)}}{\left[\left(n-1\right)!\right]^2} \left[F\left(x\right)\right]^{2(k-1)} f^2\left(x\right) dx$$

We use the transformation $v = -log(F(x))$ thus $exp(-v) = F(x)$, we immediately have

$$J\left(L_{n,k}^{X}\right) = -\frac{k^{2n}}{2}\int_{0}^{+\infty}\frac{v^{2(n-1)}}{[(n-1)!]^{2}}e^{-2v(k-1)}f\left(F^{-1}\left(e^{-v}\right)\right)dv$$

$$= -\frac{k^{2n}}{2\left((n-1)!\right)^{2}}\int_{0}^{+\infty}v^{2(n-1)}e^{-2v(k-1)}f\left(F^{-1}\left(e^{-v}\right)\right)\frac{\Gamma\left(2n-1\right)\left(2k-1\right)^{2n-1}}{\Gamma\left(2n-1\right)\left(2k-1\right)^{2n-1}}dv$$

$$= -\frac{k^{2n}\Gamma\left(2n-1\right)}{2\left((n-1)!\right)^{2}\left(2k-1\right)^{2n-1}}\int_{0}^{+\infty}\frac{v^{2(n-1)}e^{-2v(k-1)}\left(2k-1\right)^{2n-1}}{\Gamma\left(2n-1\right)}f\left(F^{-1}\left(e^{-v}\right)\right)dv$$

$$= -\frac{k^{2n}}{2\left(2k-1\right)^{2n-1}}\frac{(2n-2)!}{(n-1)!\,(n-1)!}\int_{0}^{+\infty}\frac{v^{2(n-1)}e^{-2v(k-1)}\left(2k-1\right)^{2n-1}}{\Gamma\left(2n-1\right)}f\left(F^{-1}\left(e^{-v}\right)\right)dv$$

$$= -\frac{k^{2n}}{2\left(2k-1\right)^{2n-1}}\begin{pmatrix}2n-2\\ n-1\end{pmatrix}\int_{0}^{+\infty}\frac{v^{2(n-1)}e^{-2v(k-1)}\left(2k-1\right)^{2n-1}}{\Gamma\left(2n-1\right)}f\left(F^{-1}\left(e^{-v}\right)\right)dv$$

As a result of this, the extropy of the nth value of lower k-records can be expressed as

$$J\left(L_{n,k}^{X}\right) = -\frac{k^{2n}}{2\left(2k-1\right)^{2n-1}}\begin{pmatrix}2n-2\\ n-1\end{pmatrix}Ef\left(F^{-1}\left(e^{-V_{2n-1,2k-1}}\right)\right)$$

where $V_{2n-1,2k-1}$ is $G(2n-1, 1/(2k-1))$ variable. For convenience, let's express the probability density function (pdf) of $G(n,k)$ as:

$$g(v) = \frac{v^{n-1} e^{-kv}}{\Gamma(n)}$$

■

Remark 2.3.1 *Proceeding similarly, the extropy of nth value of upper k-records is given by*

$$J\left(U_{n,k}^{X}\right) = -\frac{k^{2n}}{2} \int_{0}^{+\infty} \frac{v^{2(n-1)}}{\left[(n-1)!\right]^{2}} e^{-2v(k-1)} f\left(\Psi\left(v\right)\right) dv$$

$$= -\frac{k^{2n}}{2\left(2k-1\right)^{2n-1}} \begin{pmatrix} 2n-2 \\ n-1 \end{pmatrix} Ef\left(\Psi\left(V_{2n-1,2k-1}\right)\right). \tag{2.4}$$

Proposition 2.3.1 *The extropy of the nth value of upper $k-records$, denoted by $U_{n,k}^{X}$, and the $n-th$ value of lower $k-records$, denoted by $L_{n,k}^{X}$, can be expressed as follows[11]:*

$$J\left(U_{n,k}^{X}\right) = -\frac{k^{2n}}{2\left(2k-1\right)^{2n-1}} \begin{pmatrix} 2n-2 \\ n-1 \end{pmatrix} Ef\left(\Psi\left(V_{2n-1,2k-1}\right)\right)$$

and

$$J\left(L_{n,k}^{X}\right) = -\frac{k^{2n}}{2\left(2k-1\right)^{2n-1}} \begin{pmatrix} 2n-2 \\ n-1 \end{pmatrix} Ef\left(F^{-1}\left(e^{-V_{2n-1,2k-1}}\right)\right)$$

The random variable $V_{2n-1,2k-1}$ *follows a gamma distribution with parameters* $(2n-1) > 0$ *and* $1\ /(2k-1) > 0.$

Remark 2.3.2 *Setting* $k = 1$ *allows us to readily derive the classical records from the sequence of* $k - records$, *the extropy of nth value of upper and lower* $k - records:$

$$J\left(L_{n,1}^{X}\right) = -\frac{1}{2}\begin{pmatrix} 2n-2 \\ n-1 \end{pmatrix} Ef\left(F^{-1}\left(e^{-V_{2n-1}}\right)\right),$$

and

$$J\left(U_{n,1}^{X}\right) = -\frac{1}{2}\begin{pmatrix} 2n-2 \\ n-1 \end{pmatrix} Ef\left(\Psi\left(V_{2n-1}\right)\right).$$

2.3.2 Some Results and characterizations based on record statistics

In this subsection, we present certain findings concerning the extropy of k-record statistics, while also outlining symmetric properties of the extropy of the nth value from upper k-records and the nth value from lower k-records.

Lemma 2.3.1 *If* X *is a random variable with cumulative distribution function* F, *probability density function* f, *and finite mean* μ, *then* $f\left(\mu + x\right) = f\left(\mu - x\right)$ *for all* $x \succ 0$ *if and only if* $f\left(F^{-1}\left(v\right)\right) = f\left(F^{-1}\left(1-v\right)\right)$ *for all* $v \in [0,1]$. *(Fashandi and Ahmadi, (2012)[9]).*

Lemma 2.3.2 *The sequence of Laguerre functions* $\Phi_n\left(x\right) = e^{-\frac{x}{2}}L_n\left(x\right)/n!$, *for* $n \geq 0$ *forms a complete orthonormal system for the space* $L^2\left(0,\infty\right)$, *where* $L_n\left(x\right)$

represents the Laguerre polynomial. These polynomials are defined as the coefficients of e^{-x} in the nth derivative of $x^n e^{-x}$.

Property 2.3.1 *If the probability density function of X exhibits symmetry around its finite mean μ, then*

$$J\left(U_{n,k}^{X}\right) = J\left(L_{n,k}^{X}\right)$$

Proof. we have

$$J\left(U_{n,k}^{X}\right) = -\frac{k^{2n}}{2\left(2k-1\right)^{2n-1}} \begin{pmatrix} 2n-2 \\ n-1 \end{pmatrix} Ef\left(\Psi\left(U_{2n-1,2k-1}\right)\right)$$

It is straightforward to verify that by the lemma 2.3.1 and proposition 2.3.1

$$\begin{aligned} J\left(U_{n,k}^{X}\right) &= -\frac{k^{2n}}{2\left(2k-1\right)^{2n-1}} \begin{pmatrix} 2n-2 \\ n-1 \end{pmatrix} Ef\left(F^{-1}\left(e^{-U_{2n-1,2k-1}}\right)\right) \\ &= J\left(L_{n,k}^{X}\right). \end{aligned}$$

■

Proposition 2.3.2 *Suppose X is a random variable with cdf F and pdf f, then*

$$J\left(U_{n,k}^{X}\right) - J\left(L_{n,k}^{X}\right) \geq -\frac{k^{2n}}{2(2k-1)^{2n-1}} \begin{pmatrix} 2n-2 \\ n-1 \end{pmatrix} \left\|\delta\left(v\right)\right\|_2 \left(\frac{(2k-1)\begin{pmatrix} 4n-4 \\ 2n-2 \end{pmatrix}}{2^{4n-3}}\right)^{\frac{1}{2}}$$

where

$\delta\left(v\right) = f\left(F^{-1}\left(1-e^{-v}\right)\right) - f\left(F^{-1}\left(e^{-v}\right)\right),$

and

$||\delta(v)||_2 = \left(\int_0^\infty \delta^2(v)\,dv\right).^{\frac{1}{2}}$

The equality is true if and only if

$$F(x) = 1 - exp\left(-g^{-1}_{2n-1,2k-1}\frac{\left[(2k-1)\begin{pmatrix}4n-4\\2n-2\end{pmatrix}\right]^{\frac{1}{2}}}{2^{4n-3}||\delta(v)||_2}\left[f(x) - f\left(F^{-1}\left(\bar{F}(x)\right)\right)\right]\right).$$

Proposition 2.3.3 *Let X and Y be two rv's with cdf's F and G, pdf's f and g, respectively, if $E\left[log^2 f(X)\right] \leq +\infty, E\left[log^2 g(X)\right] \leq +\infty$, then*
$X \overset{d}{=} Y \Leftrightarrow J\left(U_n^X\right) = J\left(U_n^Y\right), \forall n \geq 1$

Remark 2.3.3 *The same conclusion holds true for lower RV's.*

Proposition 2.3.4 *Let X denote a random variable with cdf F and pdf, then*

$$\frac{J\left(U^X_{n,k}\right) - J\left(L^X_{n,k}\right)}{\left|\left|\delta(v)\,e^{-\frac{u}{2}}\right|\right|_2} \geq -\frac{k^{2n}\begin{pmatrix}2n-2\\n-1\end{pmatrix}}{2^{2n}(k+1)^{2n-1}}\left[\frac{(2k+2)^{4n-2}\begin{pmatrix}4n-4\\2n-2\end{pmatrix}}{(4k+5)^{4n-3}} - \left(\frac{k+1}{2k+3}\right)^{2n-1} + 1\right]^{\frac{1}{2}}$$

.

Proposition 2.3.5 *Let X be a random variable with the cdf F and the pdf f. If $f(F^{-1}(x))$ is non-decreasing in x, then for $n \geq k$, $J(U^X_{n,k})$ decreases as n increases.[12]*

2.4 Extropy of order Statistics

This section begins by presenting the formulas for the extropy of the mth order statistic. Using these formulas as a basis, we will subsequently highlight the symmetrical properties that characterize the extropy of the mth order statistic.

2.4.1 The Extropy of mth order statistics

In this subsection, we first provide expressions for the extropy of mth order statistics $X_{m:n}$. To this end, substituting 1.2 into 2.1 .

Theorem 2.4.1 *if η is a continuous function on the interval$[0,1]$, such that $\int_0^1 x^n \eta\left(x\right) dx = 0$ for $n \geq 0$, the $\eta\left(x\right) = 0$ for any $x \in \left[1,0\right].$*
Furthermore, consider the probability density function (pdf) of the mth order statistic $X_{m:n}$ in a sample of size n with an underlying distribution X.

$$f_{m:n}\left(x\right) = \frac{F^{m-1}\left(x\right)\bar{F}^{n-m}\left(x\right) f\left(x\right)}{\beta\left(m, n-m+1\right)} \tag{2.5}$$

The term $\beta\left(m, n-m+1\right) = \frac{\Gamma(m)\Gamma(n-m+1)}{\Gamma(n+1)}$ denotes the beta function with parameters m and $(n-m+1)$. t's noteworthy that the mth order statistic in a sample of size n signifies the lifespan of an $(n-m+1)$. In the ensuing theorem, we demonstrate that the characteristics of the extropy information of $X_{m:n}$ can be indicative of the parent distribution.

Property 2.4.1 *Let X random variable with cdf's F and pdf's f, the extropy of mth order statistic is*

$J\left(X_{m:n}\right)=-\frac{1}{2}\left(\binom{n}{m-1}\right)^{2}\int_{-\infty}^{+\infty}[F(x)]^{2(m-1)}[1-F(x)]^{2(n-m)}f^{2}(x)\ dx$

Proof. We have

$J\left(X_{m:n}\right)=-\frac{1}{2}\int_{-\infty}^{+\infty}\left[\frac{n!}{(m-1)!(n-m)!}f(x)[F(x)]^{m-1}[1-F(x)]^{n-m}\right]^{2}\ dx$

then,

$J\left(X_{m:n}\right)=-\frac{1}{2}\int_{-\infty}^{\infty}\left(\frac{m!}{(m-1)!(n-m)!}\right)^{2}f^{2}\left(x\right)\left(F\left(x\right)\right)^{2(m-1)}\left(1-F\left(x\right)\right)^{2(n-m)}dx$

In the last,

$J\left(X_{m:n}\right)=-\frac{1}{2}\left(\binom{n}{m-1}\right)^{2}\int_{-\infty}^{+\infty}[F(x)]^{2(m-1)}[1-F(x)]^{2(n-m)}f^{2}(x)\ dx$ ■

2.4.2 Some Results and characterizations based on order statistics

In this subsection, we explore the extropy associated with mth order statistics, focusing on its implications and properties.(The upcoming outcome, stemming as a corollary from the Stone–Weierstrass Theorem (Aliprantis and Burkinshaw, 1981[4]), will be instrumental in demonstrating the principal findings within this subsection.

Proposition 2.4.1 *Consider two random variables X and Y with cumulative distribution functions F and G, probability density functions f and g respectively, where the supports S_1 and S_2 have a common lower boundary b. For a fixed value of m where $(1\leq m\leq n)$,[20]*

$$X\overset{d}{=}Y\Leftrightarrow J\left(X_{m:n}\right)=J\left(Y_{m:n}\right),\forall n\geq m.$$

Remark 2.4.1 *By considering $m=1$*

we have,

$$X\overset{d}{=}Y\Leftrightarrow J\left(X_{1:n}\right)=J\left(Y_{1:n}\right),\forall n\geq 1$$

The exponential distribution holds significant relevance within reliability theory. Hereafter, we present a novel characterization for this distribution.

Proposition 2.4.2 *An exponential distribution is defined by the formula* $\bar{F}(x) = e^{-ux}$, $\mu > 0$ *can be characterized by ([20])*

$$J(X_{1:n}) = nJ(X), \forall n \geq 1. \tag{2.6}$$

Proof. If $\bar{F}(x) = e^{-\mu x}$, then $J(X_{1:n}) = -n\mu/4 = nJ(X)$ for all $n \geq 1$.

We can rewrite the condition $J(X_{1:n}) = nJ(X)$ for all $n \geq 1$ using equations 2.1 and 1.2,

$$-\frac{1}{2}\int_{-\infty}^{\infty} \bar{F}^{2n-2}(x) f^2(x)\, dx = \frac{1}{n} J(X)$$

By employing the transformation $v = \bar{F}^2(x)$, it can be deduced that

$$\int_0^1 \left[\frac{1}{4}\lambda_X\left(F^{-1}\left(1-\sqrt{v}\right) + J(X)\right)\right] v^{n-1} dv = 0$$

We have $\lambda_X(F^{-1}(1-\sqrt{v})) = -4J(X)$ for all $u \in [0,1]$. In exponential distributions, the hazard rate functionμ is constant and relates to $J(X)$ through the equation: $\mu = -4J(X)$. ■

Remark 2.4.2 *The distribution* $F(x) = e^{\mu x}$, *where* $x < 0$ *and* $\mu > 0$, *can be defined by the condition that* $J(X_{n:n}) = nJ(X)$ *holds true for all* $n \geq 1$.

2.5 Some numerical computation

Next, we conduct some numerical computations for the extropy of various statistical measures. Specifically, we will focus on the upper and lower record values, the extropy of upper record values, and the extropy of the first-order statistics. These computations is applied on R program, and it will provide insights into the behavior and properties of extropy in the context of record values and order statistics, enhancing our understanding of these important concepts in statistical analysis.

2.5.1 Upper and Lower record values

R implementation:

- Generate a sample from the exponential distribution:

  We use the exponential distribution, parameterized by rate ($\lambda = 2$) as provided in the code below (See the figure 2.1).

```
set.seed(10)
z<-rexp(100,2)
```

```
> z
  [1] 0.007478203 0.460110602 0.376079469 0.787520925 0.115829308 0.543336502
  [7] 1.163811436 0.364561911 0.644155052 0.336134143 0.213264896 0.557710973
 [13] 0.658273534 0.206646914 0.338287665 0.816491940 0.035597038 1.284425807
 [19] 0.872359347 0.146475033 0.322652580 0.173461201 0.397541315 0.698149591
 [25] 0.699053713 0.555229720 0.085210798 0.959390665 0.083221970 0.485144995
 [31] 0.005285596 1.396504857 1.177305890 0.333619194 0.261121743 0.073356449
 [37] 0.375617153 1.144283106 0.034426784 0.141356577 0.025739318 1.514796570
 [43] 0.045859840 0.592099785 0.072070561 0.077239967 0.123549602 1.007737983
 [49] 1.862417187 0.297993006 0.395211384 0.859583597 0.873754053 0.525806506
 [55] 0.609828962 0.334778330 0.703548497 1.514173018 0.165383281 0.193508233
 [61] 0.618345001 0.927035767 0.028539601 0.202947550 0.089822835 0.114469551
 [67] 0.200620528 1.152056226 0.065425756 1.005859946 0.815321479 0.307768269
 [73] 1.636034790 0.269040514 0.791112994 0.213195651 0.036848924 0.148369514
 [79] 0.719589780 0.782013951 0.322390375 0.673610375 0.112520610 1.084684784
 [85] 0.068315502 1.095398457 0.822505474 0.513487150 0.913394275 0.779077928
 [91] 0.427724467 0.035510093 0.016401494 0.670372547 0.673104150 0.262044408
 [97] 0.364926685 0.263189780 0.195155187 0.261796804
```

Figure 2.1: The sample of exponential distribution

- Extract Upper k-Record Values:

  Derive the upper record of the previous sample, we extract the k-upper records when $k = 1, 2, 6, 10$ as shown in the following table 2.1.

$1-record$	0.007478203 0.460110602 0.787520925 1.163811436 1.284425807 1.396504857 1.514796570 1.862417187
$2-record$	0.007478203 0.376079469 0.460110602 0.543336502 0.787520925 0.816491940 1.163811436 1.284425807 1.396504857 1.514796570 1.636034790
$6-record$	0.007478203 0.115829308 0.364561911 0.376079469 0.460110602 0.543336502 0.557710973 0.644155052 0.658273534 0.698149591 0.699053713 0.787520925 0.816491940 0.872359347 0.959390665 1.144283106 1.163811436 1.177305890 1.284425807

$10-record$	0.007478203 0.115829308 0.213264896 0.336134143 0.338287665 0.364561911 0.376079469 0.460110602 0.543336502 0.557710973 0.644155052 0.658273534 0.698149591 0.699053713 0.787520925 0.816491940 0.872359347 0.873754053 0.959390665 1.007737983 1.144283106

Table 2.1: The Upper k-Record values for $n = 100$ when $k = 1, 2, 6, 10$.

- Extract Lower k-Record Values:

  Derive the lower record of the previous sample, we extract the k-lower records when $k = 1, 2, 6, 10$ as shown in the following table 2.2.

$1-record$	0.007478203 0.005285596
$2-record$	0.460110602 0.376079469 0.115829308 0.035597038 0.007478203
$6-record$	0.78752093 0.54333650 0.46011060 0.37607947 0.36456191 0.33828767 0.33613414 0.21326490 0.20664691 0.17346120 0.14647503 0.11582931 0.08521080 0.08322197 0.07335645 0.04585984 0.03559704 0.03551009 0.03442678
$10-record$	1.16381144 0.78752093 0.64415505 0.55771097 0.54333650 0.46011060 0.37607947 0.36456191 0.33828767 0.33613414 0.32265258 0.21326490 0.20664691 0.17346120 0.14647503 0.14135658 0.11582931 0.08521080 0.08322197 0.07723997 0.07335645 0.07207056 0.06831550 0.06542576 0.04585984

Table 2.2: The Lower k-Record values for $n = 100$ when $k = 1, 2, 6, 10$.

2.5.2 Extropy of upper record

Plotting the extropy of upper k-record value in R:

- The Extropy of Upper k-Record Values in R by applying the density of the upper k-record value of a exponential distribution (with parameters $\lambda = 2$):

$$J(U^{Z}_{n,k}) = -\left(\frac{\lambda k}{2^{2n}}\right)\binom{2n-1}{n-1}$$

impliying,

$$J(U^{Z}_{n,k}) = -\left(\frac{2k}{2^{2n}}\right)\binom{2n-1}{n-1}$$

Now, let's shaw the folloing code:

R code:

```
# Function of extropy for exponential distribution
p <- function(n, k, l) {
  g <- (-1 * k / 2^(2 * n)) * (factorial(2 * n - 2) / (
      factorial(n - 1)^2))
  return(list(n = n, k = k, d = g))
}
p(20, 4, 2)
```

- by utilizing the code below, we can visualize the extropy of upper k-records statistics where $k = 2, 5, 9$ showing at the the graph 2.2

  R code:

```
n <- c(1:10)
plot(x1, type = 'l')
plot(n, x1, type = "o", col = "blue", pch = "o", lty = 1,
    ylim = c(-8, 0), xlab = "n-value", ylab = "J(n,k)")
points(n, x2, col = "red", pch = "*")
lines(n, x2, col = "red", lty = 2)
points(n, x3, col = "dark red", pch = "+")
lines(n, x3, col = "dark red", lty = 3)
legend(2, -4, legend = c("J(n,2)", "J(n,5)", "J(n,9)"), col =
     c("blue", "red", "black"), pch = c("o", "*", "+"), lty =
    c(1, 2, 3), ncol = 1)
```

  The plot:

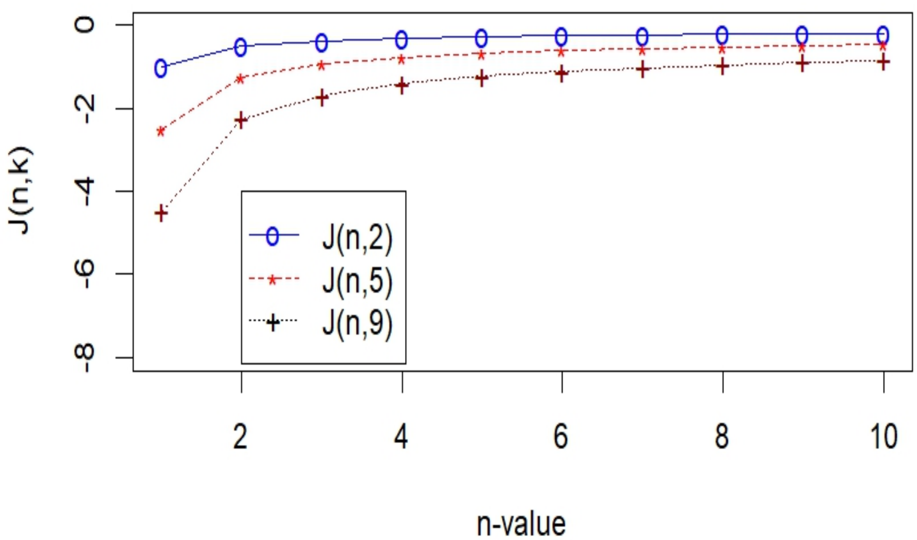

Figure 2.2: Extropy of the n-th Value of Upper k-Records from Exp(2)

Conclusion:

The plot effectively demonstrates the behavior of the function $J(n, k)$ under the condition that $f(F^{-1}(x))$ is non-decreasing x 2.3.5. For $n \geq k, J(n, k)$ in is shown to decrease with increasing n, which aligns with the theoretical condition. The visualization clearly captures this decreasing trend across different values of k $(2, 5, 9)$, supporting the mathematical relationship between the distribution functions and the extropy function J.

2.5.3 Extropy of the first order statistic

According to proposition (2.4.2), if f follows an exponential distribution, the extropy of the first-order statistic is given by $J(X_{1:n}) = nJ(X)$ for all $n \geq 1$.

Given an exponential distribution with $\lambda = 2$, this implies:

$$f(z) = 2e^{-2z}$$

then,

$$\begin{aligned} J(Z) &= -\frac{1}{2}\int_0^{\infty} (2e^{-2z})^2 \, dz \\ &= -\frac{1}{2} \end{aligned}$$

when we Apply the equation(2.6),

$$J\left(Z_{1:n}\right) = -\frac{n}{2}$$

we Apply it with a random varibale with n-value is 100 the we have

n	10	20	50	100
$J_{m:n}\left(Z\right)$	-5	-10	-25	-50

Table 2.3: Extropy of the first order statistic for $n = 100$

Conclusion

This master's thesis explores the concept of extropy in the context of records and order statistics, particularly focusing on its application introduced by Lad et al. in 2015. Extropy serves a crucial role in understanding historical trends, patterns, and outcomes within datasets, providing benchmarks for performance evaluation and goal setting across various fields.

Extropy enables the comparison of uncertainty levels among record statistics across different datasets and conditions. Unlike traditional methods tied to specific data distributions, extropy's versatility allows it to be applied to various types of record statistics regardless of their underlying distribution. This flexibility makes it a valuable tool for researchers seeking deeper insights into ordered data behavior.

However, the master's thesis also identifies some difficulties in the application of extropy of record Statistics which needs more time to explore. The interpretation and application of extropy can vary based on its specific definition, influencing its effectiveness in different scenarios. Moreover, extropy's approach frequires deriving separate equations tailored to each order within the dataset (such as 1st order, 2nd order, etc.), which contrasts with statistics that have universal formulas applicable across all orders.

Despite these challenges, the master's thesis remains optimistic about extropy's potential in analyzing record and order statistics. As research progresses, extropy is expected to play an increasingly significant role in quantifying uncertainty within this domain. By acknowledging both its strengths and limitations, researchers can effectively use extropy to enhance their understanding of record-breaking phenomena and the behavior of ordered data.

Overall, the master's thesis underscores extropy as a promising avenue for advancing the analysis of record and order statistics, contributing to informed decision-making and continuous improvement in diverse fields.

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Annexe A: Software R

2.6 What is the R language?

• The R language is a programming language and mathematical environment used for data processing. It allows you to perform both simple and complex statistical analyzes such as linear or non-linear models, hypothesis testing, time series modeling, classification, etc. It also has many very useful, professional-quality graphics functions.

• R was created by Ross Ihaka and Robert Gentleman in 1993 at the University of Auckland, New Zealand, and is now developed by the R Development Core Team.

The origin of the name of the language comes, on the one hand, from the initials of the first names of the two authors (Ross Ihaka and Robert Gentleman) and, on the other hand, from a play on words on the name of the language S to which it is related.

2.7 R code used at chapter 2

R code:

```r
# Upper
upper.record.values <- function(sqnc, k) {
  if (sum(is.na(sqnc)) > 0) {
    stop("There are missing values in data. The function is not designated for such a data!")
  } else {
    if (length(k) == 1 && k >= 1 && k <= length(sqnc) && is.numeric(k) && floor(k) == k) {
      sqnc <- unique(sqnc)
      dl <- length(sqnc)
      wndw <- sort(sqnc[1:k])
      records <- wndw[1]
      if (k < dl) {
        for (i in (k+1):dl) {
          if (sqnc[i] > wndw[1]) {
            if (i <= dl - k + 2) {
              w <- which(sqnc[i] > wndw)
            } else {
              w <- which(sqnc[i] > wndw[1:(dl-i+2)])
            }
            where <- w[length(w)]
            if (where > 1) {
              wndw[1:(where-1)] <- wndw[2:where]
            }
            wndw[where] <- sqnc[i]
            records <- c(records, wndw[1])
          }    }  }
      return(records)
```

```
  } else {
    stop("k must be an integer between 1 and length(sqnc)")
  }}}
```

```
# Lower
lower.record.values <- function(sqnc, k) {
  if (sum(is.na(sqnc)) > 0) {
    stop("There are missing values in data. The function is not
       designated for such a data!")
  } else {
    if (length(k) == 1 && k >= 1 && k <= length(sqnc) && is.
       numeric(k) && floor(k) == k) {
      sqnc <- -unique(sqnc)
      dl <- length(sqnc)
      wndw <- sort(sqnc[1:k])
      records <- wndw[1]
      if (k < dl) {
        for (i in (k+1):dl) {
          if (sqnc[i] > wndw[1]) {
            if (i <= dl - k + 2) {
              w <- which(sqnc[i] > wndw)
            } else {
              w <- which(sqnc[i] > wndw[1:(dl-i+2)])
            }
            where <- w[length(w)]
            if (where > 1) {
              wndw[1:(where-1)] <- wndw[2:where]
            }
            wndw[where] <- sqnc[i]
            records <- c(records, wndw[1])
          }  }  }
      return(-records)
```

```
    } else {
      stop("k must be an integer between 1 and length(sqnc)")
    }  }}
```

Abstract

Shannon entropy, established by Shannon in 1948 and widely utilized in reliability and information studies, has seen a recent counterpart in extropy, proposed by Lad et al. (2015), which serves as its dual measure.

This master's thesis aims to present the concept of extropy, a measure of uncertainty. One statistical application of extropy is to score the forecasting distributions. It addresses the problem of extropy of ordered variates records and order statistics and related properties. Simplified expressions of the extropy of ordered variates are derived.

Key words: Extropy, order statistic, record statistic, measure of uncertainty

Résumé

L'entropie de Shannon, établie par Shannon en 1948 et largement utilisée dans les études de fiabilité et d'information, a récemment trouvé une contrepartie dans l'extropie, proposée par Lad et al. (2015), qui sert de mesure duale.

Cette thèse de master vise à présenter le concept d'extropie, une mesure de l'incertitude. Une application statistique de l'extropie est de noter les distributions de prévision. Elle aborde le problème de l’extropies des variés ordonnées, des records et des statistiques d'ordre et les propriétés associées. Des expressions simplifiées de l'extropie des variés ordonnées sont dérivées.

Mots clé : Extropy, Statistique d'ordre, Record statistique, mesure de l'incertitude

الملخص

أنتروبيا شانون، التي أسسها شانون في عام 1948 والمستخدمة على نطاق واسع في دراسات الموثوقية والمعلومات، وجدت مؤخرًا نظيرًا لها في الإكستروبيا، التي اقترحها لاد وآخرون (2015)، والتي تعمل كمقياس مزدوج

تهدف هذه الأطروحة إلى تقديم مفهوم الإكستروبيا، وهو مقياس لعدم اليقين. إحدى التطبيقات الإحصائية للإكستروبيا هي تقييم توزيعات التنبؤ. تتناول الأطروحة مشكلة الإكستروبيا للمتغيرات المرتبة، والسجلات، وإحصاءات الترتيب والخصائص ذات الصلة. يتم اشتقاق تعبيرات مبسطة للإكستروبيا للمتغيرات المرتبة

كلمات مفتاحية: اكستروبيا, الإحصاء الترتيبي, الإحصاء السجلي, مقياس عدم اليقين