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Dedicace

I dedicate this humble work to

the princess, my mother. I inherited in her heart how to be a human being before I screamed my first scream in this world. And to my kind father, under whose care I was raised to be honest before I took my first step on the path of life.. to my sisters and brothers and my brother's wife...

to every hand I shook her hand one day by heart!

Thankfulness

First of all I would like to say all praise is due to Allah, the Lord of all the worlds. All praise is due to Allah, by whose favor good works are accomplished. Thanks to Allah for His blessings and bounties. Then I want to express my sincere gratitude for the Professor **Necir Abdelhakim** guidance and support throughout this semester. Your expertise and dedication have profoundly impacted my academic growth, and I deeply appreciate the time and effort you have invested in my education.

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Introduction

The L-statistic's robustness, versatility, and distribution-free properties make it a fundamental tool in statistical inference. Its ability to provide reliable estimates and inference under a wide range of conditions contributes to its widespread use and importance in both theoretical research and practical applications.

In essence, L-statistics provide a robust alternative to classical measures, making them particularly useful in situations where the data may be skewed or contain extreme values. They are widely employed in various fields, including economics, finance, engineering, and biology, where accurate estimation of central tendency is crucial. Common examples of L-statistics include the median (as a special case), trimmed mean, sample range, and some L-estimator. These statistics offer robustness against outliers and heavy-tailed distributions, making them invaluable tools in statistical analysis.

Our memory is composed of four chapters:

In the first chapter, we consider the order statistics and we highlighted the main order statistics distribution.

The second chapter has of object to present the main topic of our thesis, we touched on what is an L-statistics and the different written of this later as we have listed some simple application of this last one

The third chapter deals with the L-estimators ,we chose three of them which are the best linear unbiased estimator,l-moment and asymptotic efficient estimator and we have given examples for each one.

The fourth chapter is devoted to a simulation study about the performance of the previous estimators in addition to the maximum likelihood estimator where we estimated the scale and location parameters of different distributions .Finally we present real data applications Accidental Deaths in the US 1973-1978) and annual maximal levels of the River Nidd in Yorkshire.

Chapter 1

Order statistics

Order statistics involve arranging a sample of random variables in ascending or descending order. For instance, if you have a sample of n observations, the first order statistic would be the smallest observation, the second order statistic would be the second smallest, and so on. L-statistics are constructed using these ordered values, and they often play a significant role in statistical inference, particularly in nonparametric statistics and robust statistics. L-statistics often involve functions of order statistics. This connection arises because order statistics provide information about the relative positions of data points, and these positions are used in estimating central tendencies. In this chapter we will know what is an order statistic and mention its distributions, see for instance Arnold *et al.* (1992) [1].

1.1 Order statistics

Definition 1.1.1 *We say order statistics associated to the sample $X_1, \ldots, X_n$, the ordered sequence (in increasing sense) denoted by $X_{1:n} < \ldots < X_{n:n}$.*

Theorem 1.1.1 *Let $U_{1:n} < ... < U_{n:n}$ be the order statistics pertaining to sample $U_1, ..., U_n$ from uniform-$(0,1)$ random variable (rv) U and let $X_{1:n} < .... < X_{n:n}$*

be the order statistics corresponding to sample $X_1, ..., X_n$ *from a rv* X *of cumulative distribution function (cdf)* F. *The quantile (or the generalized inverse) function pertaining to the cdf* F *is defined by*

$$Q(s) := F^{-1}(s) = \sup\{x : F(x) \leq s\}, \text{ for } 0 < s < 1.$$

Then

$$\{Q(U_{1:n}), ..., Q(U_{n:n})\} \stackrel{D}{=} \{X_{1:n}, ..., X_{n:n}\},$$

where $\stackrel{D}{=}$ *denotes the equality in distribution. Moreover, if* F *is continuous, we have*

$$\{X_{1:n}, ..., X_{n:n}\} = \{Q(U_{1:n}), .., Q(U_{n:n})\}, \text{ almost surely.} \tag{1.1}$$

1.2 Distribution functions of order statistics

Let $X_1, ..., X_n$ be a sequence of independent and identically distributed (i.i.d) rv's of continuous cdf F and density function (df) f.

Let's $(F_{X_{1:n}}, F_{X_{n:n}}, F_{X_{i:n}})$ and $(f_{X_{1:n}}, f_{X_{n:n}}, f_{X_{i:n}})$ denote, respectively, the cdf's and the df's corresponding to the order statistics $X_{1:n}$, $X_{n:n}$ and $X_{i:n}$.

1.2.1 Distribution functions of smallest and largest order statistics

The cdf's of smallest and largest order statistics are respectively given by

$$F_{X_{1:n}}(x) = 1 - (1 - F(x))^n \text{ and } F_{X_{n:n}}(x) = (F(x))^n.$$

Thereby, their corresponding df's are

$$f_{X_{1:n}}(x) = nf(x)(1-F(x))^{n-1} \text{ and } f_{X_{n:n}}(x) = nf_X(x)(F_X(x))^{n-1}.$$

1.2.2 Distribution function of the *i-th* order statistic

The cdf of $X_{i:n}$ is given by

$$\begin{aligned}
F_{X_{i:n}}(x) = \mathbf{P}\left(X_{i:n} \leq x\right) &= \mathbf{P}(\bigcup_{i=1}^{n} \left\{k\left(X_k \leq x\right) \bigcap \left(n-k\right)\left(X_k > x\right)\right\} \\
&= \sum_{i=1}^{n} \mathbf{P}\left(k\left(X_k \leq x\right) \bigcap \left(n-k\right)\left(X_k > x\right)\right) \\
&= \sum_{i=1}^{n} \binom{n}{k} \left(P\left(X_k \leq x\right)\right)^k \left(P\left(X_k > x\right)\right)^{n-k} \\
&= \sum_{i=1}^{n} \binom{n}{k} \left(F_X\left(x\right)\right)^k \left(1-F_X\left(x\right)\right)^{n-k},\ i = 1, ..., n.
\end{aligned}$$

We also show that its the corresponding df is

$$f_{X_{i:n}}(x) = n\binom{n-1}{i-1} f_X(x) \left(F_X\left(x\right)\right)^{i-1} \left(1-F_X(x)\right)^{n-i},\ i = 1, ..., n.$$

1.3 Basic distributions

1.3.1 Joint distribution function of n order statistics

We define the df of the vector of order statistics $(X_{1:n}, ..., X_{n:n})$ by

$$f_{(X_{1:n},...,X_{n:n})}(x_1, .., x_n) = n! \prod_{i=1}^{n} f\left(x_i\right), \text{ for } x_1 < ... < x_n,\ i = 1, ..., n.$$

1.3.2 Joint distribution function of a couple order statistics

Let us consider the couple of order statistics $X_{i:n}$ and $X_{j:n}$ for $1 \leq i, j \leq n$. The df of $(X_{i:n}, X_{j:n})$ is given by

$$f_{(Xi:n,Xj:n)}(x_i, x_j) \tag{1.2}$$
$$= \frac{n! f(x_i) f(x_j)}{(i-1)!(j-i-1)!(n-j)!} (F(x_i))^{i-1} (F(x_j) - F(x_i))^{j-i-1} (1 - F(x_j))^{n-j},$$

for $-\infty < x_i < x_j < +\infty$.

1.4 Moments of order statistics

Let $X_{1:n} \leq ... \leq X_{n:n}$ be the order statistics corresponding to sample $X_1, ..., X_n$ from an absolutely continuous rv X of cdf F and df f. Then the k-th moments of the order statistics $X_{i:n}$ are defined by

$$\mu_{i:n}^{(k)} := \mathbf{E}\left[X_{i:n}^k\right] = \int_{-\infty}^{+\infty} x^k f_{i:n}(x) dx$$
$$= \frac{n!}{(i-1)!(n-1)!} \int_{-\infty}^{+\infty} x^k (F(x))^{i-1} (1 - F(x))^{n-i} f(x) dx,$$

for $i = 1, ..., n$, and $k \geq 1$. The expectation of the product $X_{i:n}X_{j:n}$ is given by

$$\mu_{i,j:n} := \mathbf{E}[X_{i:n}X_{j:n}] = \iint_{x_i < x_j} x_i x_j f_{i,j:n} dx_i dx_j$$
$$= \frac{n!}{(i-1)!(j-i-1)!(n-j)!}$$
$$\times \iint_{x_i < x_j} x_i x_j \{F(x_i)\}^{i-1} \{F(x_j) - F(x_i)\}^{j-i-1} \{1 - F(x_j)\}^{n-1} f(x_i) f(x_j) dx_i dx_j,$$

for $1 \leq i < j \leq n$. We deduce that the covariance of the couple $(X_{i:n}, X_{j:n})$ is

$$\sigma_{i,j:n} := \mathbf{Cov}(X_{i:n}X_{j:n}) = \mu_{i,j:n} - \mu_{i:n}\mu_{j:n},$$

where $\mu_{i:n} := \mu_{i:n}^{(1)}$, for $i = 1, ..., n$.

Next we give expectation and covariance of some probability models.

Example 1.4.1 *(**Power**) The df a cdf of the power rv are respectively defined by*

$$f_\nu(x) = \nu x^{\nu-1} \text{ and } F(x) = \int_0^x \nu t^{\nu-1}dt = x^\nu, \text{ for } 0 < x < 1, \nu > 0.$$

The corresponding quantile functions is $Q(s) = s^{1/\nu}$, for $0 < s < 1$, $\nu > 0$. The expectation of the power i-th order statistics $X_{i:n}$ is

$$\mu_{i:n} = \frac{\Gamma(n+1)}{\Gamma(n+1+1/\nu)} - \frac{\Gamma(i+1/\nu)}{\Gamma(i)}, \text{ for } i = 1, ..., n,$$

where $\Gamma(\cdot)$ denotes the complete gamma function $\Gamma(p) := \int_0^\infty e^{-t}t^{p-1}dt$. The covariance matrix corresponding to the couple of power i-th and j-th order statistics $(X_{i:n}, X_{j:n})$ is given, for $1 \leq i < j \leq n$, by

$$\sigma_{i,j:n} = \frac{\Gamma(n+1)}{\Gamma(i)}\left\{\frac{\Gamma(j+2/\nu)}{\Gamma(j+1/\nu)}\frac{\Gamma(i+1/\nu)}{\Gamma(n+1+2/\nu)} - \frac{\Gamma(i+1/\nu)}{\Gamma(n+1+1/\nu)}\mu_{i:n}\right\}.$$

Example 1.4.2 *(**Logistic**) The df a cdf of the logistic rv are respectively defined by $f(x) = e^{-x}(1+e^{-x})^{-2}$ and $F(x) = (1+e^{-x})^{-1}$, $x \in \mathbb{R}$. The corresponding quantile functions is*

$$Q(s) = F^{-1}(s) = \log\left(\frac{s}{1-s}\right), \text{ for } 0 < s < 1.$$

The expectation of the logistic i-th order statistics $X_{i:n}$ of

$$\mu_{i:n} = \psi(i) - \psi(n-i+1), \text{ for } i = 1, ..., n,$$

where $\psi(\cdot)$ *denotes the digamma (or psi) function defined by*

$$\psi(z) = d\log(\Gamma(z))/dz = \Gamma'(z)/\Gamma(z).$$

The covariance matrix corresponding to the couple $(X_{i:n}, X_{j:n})$ *is given by*

$$\sigma_{i,j:n} = \psi'(i) + \psi'(n-j+1), \ 1 \leq i < j \leq n.$$

Example 1.4.3 ***(Uniform)*** *The df and cdf of the uniform-*$(0,1)$ *rv are respectively defined by*

$$f(x) = \begin{cases} 1, & 0 \leq x \leq 1 \\ 0, & \textit{otherwise.} \end{cases} \quad \textit{and } F(x) = \begin{cases} 0 & x < 0 \\ x & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases}$$

Its corresponding quantile function is $Q(s) = s$, *for* $0 \leq s \leq 1$. *The expectation of the uniform-*$(0,1)$ *i-th order statistics* $X_{i:n}$ *is*

$$\mu_{i:n} = \frac{i}{(n+1)}, \ \textit{for } i = 1, ..., n.$$

The covariance matrix corresponding to the couple of uniform i-th and j-th order statistics $(X_{i:n}, X_{j:n})$ *is given by*

$$\sigma_{i,j:n} = \frac{i(j+1)}{(n+1)(n+2)} - \frac{ij}{(n+1)^2} = \frac{i(n+1-i)}{(n+1)(n+2)}, \ \textit{for } 1 \leq i < j \leq n.$$

Example 1.4.4 ***(Exponential)*** *The df and cdf of the standard rv are respectively defined by* $f(x) = e^{-x}$ *and* $F(x) = 1 - e^{-x}$, $x \geq 0$. *The corresponding quantile function is* $Q(s) = -\log(1-s)$, *for* $0 \leq s < 1$.*The expectation of the exponential* *i-th order statistics* $X_{i:n}$ *is*

$$\mu_{i:n} = \sum_{k=1}^{i} \frac{\mathbf{E}\left[Z_k\right]}{(n-k+1)} = \sum_{k=1}^{i} \frac{\mathbf{1}}{(n-k+1)},\ \text{for } i = 1, ..., n,$$

where $Z_k := (n-k+1)\left(X_{k:n} - X_{k-1:n}\right)$ *with* $X_{0:n} := 0$. *The covariance matrix corresponding to the couple of exponential i-th and j-th order statistics* $\left(X_{i:n}, X_{j:n}\right)$ *is given by*

$$\sigma_{i,j:n} = \sum_{k=1}^{i} \frac{1}{(n-k+1)^2},\ \text{for } 1 \le i < j \le n.$$

Example 1.4.5 ***(Weibull)*** *The df and cdf of the Weibull rv are respectively defined by* $f(x;\alpha) = \alpha x^{\alpha-1} e^{-x^{\alpha}}$ *and* $F(x;\alpha) = 1 - \exp(-x^{\alpha})$, $\alpha > 0$, $x \ge 0$. *Its corresponding quantile functions is* $Q(s) = -\log(1-s)^{1/\alpha}$, $\alpha > 0$, $0 \le s < 1$. *Let* $X_1, ..., X_n$ *be a random sample drawn from the df* F *and* $X_{1:n} \le ... \le X_{n:n}$ *be the corresponding order statistics. The expectation of the Weibull i-th order statistics* $X_{i:n}$ *of* $\mu_{i:n} = \Gamma(1 + 1/\alpha)$, $1 \le i \le n$, *where* Γ *is the gamma function in which* $\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} e^{-x} dx$. *The covariance matrix corresponding to the couple of Weibull i-th and j-th order statistics* $\left(X_{i:n}, X_{j:n}\right)$ *is given by*

$$\sigma_{i,j:n} = \frac{n!}{(i-1)!\,(j-i-1)!\,(n-1)!} \times \sum_{r=0}^{i-1} \sum_{s=0}^{j-i-1} (-1)^{j-i-1} C_r^{i-1} C_s^{j-i-1} \phi_{\alpha}(r+s+1, n-i-s),$$

for $1 \le i \le j \le n$, *where* $\phi_{\alpha}(a,b) := \alpha^2 \int_0^{\infty} \int_0^{y} e^{-ax^{\alpha} - by^{\alpha}} x^{\alpha} y^{\alpha} dx dy$ *denotes Lieblein's* ϕ*-function.*

Example 1.4.6 ***(Gumbel)*** *The df and cdf of the Gumbel rv are respectively defined by* $f(x) = e^{-\left(x+e^{-x}\right)}$ *and* $F(x) = e^{-e^{-x}}$, $x \in \mathbb{R}$. *Its quantile function is*

$$Q(s) = -\log(-\log s), 0 < s < 1.$$

The expectation of the Gumbel *i-th* order statistics $X_{i:n}$ is

$$\mu_{i:n} = \frac{n!}{(i-1)!\,(n-1)!} \sum_{r=0}^{n-i} (-1)^r\, C_r^{n-i} \int_{-\infty}^{\infty} x^k e^{-x-(i+r)e^{-x}} dx,\ \ i = 1, ..., n.$$

The covariance matrix corresponding to the couple of Gumbel *i-th* and *j-th* order statistics $(X_{i:n}, X_{j:n})$, see for instance Julius (1953) [6], is given by

$$\sigma_{i,j:n} = \frac{n!}{(i-1)!\,(j-i-1)!\,(n-j)!} \times \sum_{r=0}^{j-i-1} \sum_{s=0}^{n-j} (-1)^{r+s}\, C_r^{j-i-1} C_s^{n-j} \phi\,(i+r, j-i-r+s)\,,$$

for $1 \leq i \leq j \leq n$, where $\phi\,(a,b) = \int_{-\infty}^{\infty} \int_{-\infty}^{y} xy e^{-x-ae^{-x}} e^{-y-be^{-y}} dxdy,\ a, b > 0.$

Example 1.4.7 ***(Lognormal)*** *The df and cdf of the lognormal rv are respectively defined by*

$$f_\nu\,(x) = \frac{1}{x\sqrt{2\pi}} \exp\left(-\frac{(\log x)^2}{2\sigma}\right) \text{ and } F\,(x) = \Phi\,(\log x)\,,\ \ x > 0,$$

where Φ *denotes the cdf of the standard normal rv. Its corresponding quantiles function is* $Q\,(s) = \exp\left(\sqrt{2}\,\mathrm{erf}^{-1}\,(2s-1)\right),\ 0 < s < 1,$ *where erf is the error function given by* $\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt.$ *The expectation of the Lognormal i-th order statistics* $X_{i:n}$ *is* $\mu_{i:n} = e^{\frac{1}{2}}$*, but there is explicit formula for the covariance to couple i-th and j-th order statistics* $(X_{i:n}, X_{j:n})$ *, however one may compute the later numerically, see for insntance Gupta(1972) [9].*

Chapter 2

L-Statistics and applications

L-statistics is a linear combinations of order statistics that refers to a class of statistics used in order statistics theory. It is a branch of statistics dealing with the study of the ordered values of random variables. In this chapter we will talk about the L-statistics and some general examples by giving some asymptotic proprieties.

Definition 2.0.1 *For a given sample $X_1, \cdots, X_n$, an L-Statistics L_n is a linear combinations of order statistics $X_{1:n} \leq \cdots \leq X_{n:n}$, that is $L_n = \sum_{i=1}^{n} a_{i:n} X_{i:n}$ for some sequence of constants $a_{i,n}$. We can define the L-statistics by*

$$L_n = \frac{1}{n} \sum_{i=1}^{n} J\left(\frac{i}{n+1}\right) X_{i:n}, \tag{2.1}$$

for some measurable function J defined on $(0,1)$,

Definition 2.0.2 *The empirical quantile function pertaining to a sample $X_1, \cdots, X_n$ is defined by*

$$Q_n(t) = \sup\{x : \ F_n(x) \leq s\}, \quad 0 < s < 1,$$

where F_n is the usual empirical cdf. This many be rewritten into

$$Q_n(t) = X_{i:n}, \text{ for } \frac{i-1}{n} < t \leq \frac{i}{n}, \quad i = 1, ..., n.$$

The following proposition is useful, among others, in asymptotic properties of the L-statistics.

Proposition 2.0.1 *Let J be a measurable function defined on $(0,1)$ and set*

$$a_{i:n} = J_n(t), \text{ for } \frac{i-1}{n} < t \leq \frac{i}{n}, \text{ with } J_n(0) = a_{1,n}.$$

Then the corresponding L-statistics L_n has the following functional representation

$$L_n = n\int_0^1 J_n(t)Q_n(t)\,dt = n\int_0^1 Q_n(t)\,d\Psi_n(t),$$

where $Q_n(t) := \Psi_n(t) = \int_{\frac{1}{2}}^t J_n(s)\,ds$, for $0 \leq t \leq 1$.

Proof. It is clear that $n\int_{(i-1)/n}^{i/n} Q_n(t)\,dt = X_{i:n}$, it follows that

$$\begin{aligned} L_n &= \sum_{i=1}^n a_{i:n}X_{i:n} = \sum_{i=1}^n a_{i:n}\left\{n\int_{(i-1)/n}^{i/n} Q_n(t)\,dt\right\} \\ &= n\sum_{i=1}^n \int_{(i-1)/n}^{i/n} a_{i:n}Q_n(t)\,dt. \end{aligned}$$

Since $a_{i:n} = J_n(t)$, for $\frac{i-1}{n} < t \leq \frac{i}{n}$, with $J_n(0) = a_{1,n}$, it follows that

$$L_n = n\sum_{i=1}^n \int_{(i-1)/n}^{i/n} J_n(t)\,Q_n(t)\,dt = n\int_0^1 J_n(t)\,Q_n(t)\,dt.$$

It is obvious that $d\Psi_n(t) = d\left\{\int_{\frac{1}{2}}^t J_n(s)\,ds\right\} = J_n(t)\,dt$, therefore

$$L_n = n\int_0^1 J_n(t)\,Q_n(t)\,dt = n\int_0^1 Q_n(t)\,d\Psi_n(t),$$

as sought. ■

2.1 Application of L-statistics

Next we give some examples of the L-Statistics, see for instance Chine (2024) [5].

The sample minimum

The smallest value in sample,$X_{1:n}$ can be expressed as

$$X_{1:n} = \sum_{i=1}^{n} a_{i:n} X_{i:n}, \text{ where } a_{i:n} = 0, \text{ for } i \neq 1.$$

The sample maximum

The largest order statistics $X_{n:n}$ can be expressed as

$$X_{n:n} = \sum_{i=1}^{n} a_{i:n} X_{i:n}, \text{where } a_{i:n} = 0, \text{ for } i \neq n.$$

The sample mean

The sample mean $\overline{X}$ is an L-statistics:

$$\bar{X} = \sum_{i=1}^{n} a_{i:n} X_{i:n}, \text{ where } a_{i:n} = 1/n, \text{ for } i = 1, ..., n.$$

The sample median

The sample median is given by

$$\mathbf{Med}_n = \begin{cases} X_{\frac{n+1}{2}:n} & \text{if } n \text{ is odd,} \\ \frac{1}{2}\left(X_{\frac{n}{2}:n} + X_{\frac{n+1}{2}:n}\right) & \text{if } n \text{ is even.} \end{cases}$$

This many be also rewritten as an L-statistics. Indeed, if n is odd

$$\mathbf{Med}_n = \sum_{i=1}^{n} a_{i:n} X_{i:n},$$

where $a_{\frac{n+1}{2}:n} = 1$ and $a_{i:n} = 0$ otherwise. If n is even

$$\mathbf{Med}_n = \sum_{i=1}^{n} a_{i:n} X_{i:n},$$

where $a_{\frac{n}{2}:n} = 1/2 = a_{\frac{n+1}{2}:n}$ and $a_{i:n} = 0$ otherwise.

The sample range

The range $R_n := X_{n:n} - X_{1:n}$ is also an L-statistics:

$$R_n = \sum_{i=1}^{n} a_{i:n} X_{i:n},$$

where $a_{1:n} = -1$, $a_{n:n} = 1$ and $a_{i:n} = 0$ otherwise.

The trimmed mean

The trimmed mean is given by

$$S_n = \frac{1}{[np_2] - [np_1]} \sum_{i=[np_1]+1}^{[np_2]} X_{i:n},$$

which many be rewritten into $S_n = \sum_{i=1}^{n} a_{i:n} X_{i:n}$, where

$$a_{i:n} = \frac{1}{[np_2] - [np_1]} \mathbb{I}_{\{[np_1]+1 \leq i \leq [np_2]\}},$$

where $\mathbb{I}_A$ denotes the indicator function of set A.

The Gini mean difference

Gini's mean difference is given by

$$G_n = \frac{1}{n(n-1)} \sum_{i=1}^{n} \sum_{j=1}^{n} |X_i - X_j| = \sum_{i=1}^{n} a_{i:n} X_{i:n}$$

where

$$a_{i:n} = 2 \frac{n+1}{n(n-1)} \left(\frac{2i}{n+1} - 1 \right), \text{ for } i = 1, ..., n.$$

Proof. See for instance Shorack and Wellner (1986) [10] (page 676). ■

The risk measure

Definition 2.1.1 *A risk measure of an insurance losses X is defined as a mapping from a set of random variables to the real numbers. An example of the last is the well known distorted risk measure given by*

$$L_F = \int_0^\infty g(1 - F(x)) \, dx,$$

where g is an increasing function such that $g(0) = 0$ and $g(1) = 1$. We show that this may be rewritten into

$$L_F = \int_0^1 \psi(s) Q(s) \, ds, \tag{2.2}$$

where $\psi(s) = g(1 - s)$, Jones and Zitikis (2005) [3]. For a given sample $X_1, ..., X_n$ of X, we defined the empirical version of L_F by

$$L_n = \int_0^1 \psi(s) Q_n(s) \, ds = \sum_{i=1}^{n} \left(\int_{\frac{i-1}{n}}^{\frac{i}{n}} \psi(s) \, ds \right) X_{i:n},$$

where $X_{1:n} \leq \cdots \leq X_{n:n}$ *being the order statistics pertaining to* $X_1 ... X_n$*. This means that* L_n *is an L-statistics too.*

Chapter 3

Linear estimators of order statistics

Linear estimators of order statistics (L-estimators) are a class of statistical estimators used to infer population parameters based on the ordered values of a sample. These estimators are expressed as linear combinations of the order statistics (L-statistics). In this chapter we will discuss some of them.

3.1 L-moment estimators

L-moments, which are L-statistics, are used to analyze and modeling the distribution data, particularly in hydrology, meteorology, and environmental sciences. L-moments were introduced by Hosking and Wallis (1990) [7] as an alternative to traditional methods of distribution fitting, like moments, maximum likelihood,...

Definition 3.1.1 *Let $X_1, ..., X_n$ be a sample of size n of a continuous cdf F. An L-moment λ_r is defined as specific linear combination of the expectations of the order* *statistics* $X_{1:n} \leq \cdots \leq X_{n:n}$. *More precisely the* $r-th$ *L-moment is giving by*

$$\lambda_r := \frac{1}{r}\sum_{i=0}^{r-1}(-1)^i \binom{r-1}{i}\mathbf{E}\left[X_{r-i:r}\right],\ r \geq 1. \tag{3.1}$$

The first four L-moments are defined by

$$\begin{aligned}
\lambda_1 &:= \mathbf{E}\left[X_{1:1}\right], \\
\lambda_2 &:= \tfrac{1}{2}\mathbf{E}\left[X_{2:2} - X_{1:2}\right], \\
\lambda_3 &:= \tfrac{1}{3}\mathbf{E}\left[X_{3:3} - 2X_{2:3} - X_{1:3}\right], \\
\lambda_4 &:= \tfrac{1}{4}\mathbf{E}\left[X_{4:4} - 3X_{3:4} - 3X_{2:4} - X_{1:4}\right].
\end{aligned}$$

3.1.1 L-functional representation of L-moments

The r-th L-moments may be represented as follows

$$\lambda_r = \int_0^1 Q(u)\, P_{r-1}^*(u)\, du,\ r \geq 1, \tag{3.2}$$

where $P_r^*(u) = \sum_{i=0}^{r} p_{r,i}^* u^i$ is legendre system moved orthogonal polynomials, with $p_{r,i}^* := (-1)^{r+i}\binom{r}{i}\binom{r+i}{i}$. The first four expressions of this polynomial are

$$\begin{aligned}
P_0^*(u) &= 1, \\
P_1^*(u) &= (2u-1), \\
P_2^*(u) &= (6u^2 - 6u + 1), \\
P_3^*(u) &= (20u^3 - 30u^2 + 12u - 1).
\end{aligned}$$

The corresponding first four L-moments are

$$\begin{aligned}
\lambda_1 &= \textstyle\int_0^1 Q(u)\,du, \\
\lambda_2 &= \textstyle\int_0^1 (2u-1)\,Q(u)\,du, \\
\lambda_3 &= \textstyle\int_0^1 (6u^2-6u+1)\,Q(u)\,du, \\
\lambda_4 &= \textstyle\int_0^1 (20u^3-30u^2+12u-1)\,Q(u)\,du.
\end{aligned}$$

For a given sample $X_1, ..., X_n$ of cdf F, we define the empirical version for the r-th L-moments by substituting the quantile function Q with its empirical one Q_n, that is

$$\hat{\lambda}_r = \int_0^1 P_{r-1}^*(u)\,Q_n(u)\,du = \sum_{i=1}^n \left\{ \int_{\frac{i-1}{n}}^{\frac{i}{n}} P_{r-1}^*(u)\,du \right\} X_{i:n},$$

which is indeed an L-statistics.

3.1.2 L-moments based estimation

Next we give some application of the L-moments to estimating the scale and location parameters of parametric probability models.

Example 3.1.1 ***(logistic)*** *Let $X_{1:n} \leq ... \leq X_{n:n}$ be the order statistics pertaining to the random sample $X_1, ..., X_n$ from a logistic rv $X \rightsquigarrow \mathcal{LG}(\mu, \sigma)$,defined by its cdf $F(x; \mu, \sigma) = \left(1 + e^{-\frac{x-\mu}{\sigma}}\right)^{-1}$, for $x \in \mathbb{R}$ or its quantile function $Q(s; \mu, \sigma) = \mu + \sigma \log \frac{s}{1-s}$, for $0 < s < 1$. We set*

$$Z_{i,n} := \frac{X_{i:n} - \mu}{\sigma}, \; i = 1, ..., n,$$

this mean's that $X_{i:n} = \sigma Z_{i,n} + \mu$, where $Z_{1:n} \leq ... \leq Z_{n:n}$ be the order statistics pertaining the random sample $Z_1, ..., Z_n$ from a Logistics rv $Z \rightsquigarrow \mathcal{LG}(0, 1)$. The two *first L-moments are defined by*

$$\lambda_1 = \int_0^1 Q(s;\mu,\sigma)\,ds \text{ and } \lambda_2 = \int_0^1 (2s-1)\,Q(s;\mu,\sigma)\,ds.$$

In other terms $\lambda_1 = \mu + \int_0^1 \sigma \log \frac{s}{1-s} ds$ *and*

$$\lambda_2 = \int_0^1 (2s-1)\left(\mu + \sigma \log \frac{s}{1-s}\right) ds = \mu \int_0^1 (2s-1)\,ds + \sigma \int_0^1 (2s-1) \log \frac{s}{1-s} ds$$

It is easy to check that

$$\int_0^1 \log \frac{s}{1-s} ds = 0, \quad \int_0^1 (2s-1)\,ds = 0 \text{ and } \int_0^1 (2s-1) \log \frac{s}{1-s} ds = 1,$$

it follows that $\lambda_1 = \mu$ *and* $\lambda_2 = \sigma$. *It is cleat that*

$$\widehat{\mu} = \widehat{\lambda}_1 = \int_0^1 Q_n(u)\,du = \overline{X} \text{ and } \widehat{\sigma} = \widehat{\lambda}_2 = \int_0^1 (2u-1)\,Q_n(u)\,du.$$

Observe that

$$\widehat{\sigma} = \widehat{\lambda}_2 = \sum_{i=1}^{n} \int_{\frac{i-1}{n}}^{\frac{i}{n}} (2u-1)\,Q_n(u)\,du = \sum_{i=1}^{n} \left\{ \int_{\frac{i-1}{n}}^{\frac{i}{n}} (2u-1)\,du \right\} X_{i:n}.$$

Note that $\int (2u-1)\,du = u(u-1) + c,$ *then*

$$\int_{\frac{i-1}{n}}^{\frac{i}{n}} (2u-1)\,du = \frac{i}{n}\left(\frac{i}{n} - 1\right) - \frac{i-1}{n}\left(\frac{i-1}{n} - 1\right) = \frac{1}{n^2}(2i-n-1),$$

therefore

$$\widehat{\sigma} = \frac{1}{n^2} \sum_{i=1}^{n} (2i-n-1)\,X_{i:n}$$

Finally the L-moments estimators of location-scale parameters corresponding to the *logistic distribution are defined by*

$$\widehat{\mu} = \overline{X} \text{ and } \widehat{\sigma} = \frac{1}{n^2}\sum_{i=1}^{n}(2i-n-1)\,X_{i:n}.$$

Example 3.1.2 ***(Power)*** *Let $X_{1:n} \leq ... \leq X_{n:n}$ be the order statistics pertaining to the random sample $X_1, ..., X_n$ from a a Power rv $X \rightsquigarrow \mathcal{P}(\mu,\sigma)$, defined by its cdf $F(x;\mu,\sigma) = \left(\frac{x-\mu}{\sigma}\right)^{\nu}$, for $0 < x < \mu$, $\nu > 0$ or its quantile function $Q(s;\mu,\sigma) = \mu + \sigma s^{1/\nu}$, $0 < s < 1$. By using similar argument as used for the first example, we defined the two first L-moments by $\lambda_1 = \mu + \sigma \int_0^1 s^{1/\nu} ds$ and*

$$\lambda_2 = \int_0^1 (2s-1)\left(\mu + \sigma s^{1/\nu}\right) ds = \mu \int_0^1 (2s-1)\,ds + \sigma \int_0^1 (2s-1)\,s^{1/\nu} ds.$$

We have $\int_0^1 s^{1/\nu} ds = \frac{\nu}{\nu+1}$, $\int_0^1 (2s-1)\,ds = 0$, and $\int_0^1 (2s-1)\,s^{1/\nu} ds = \frac{\nu}{2\nu^2+3\nu+1}$, it follows that

$$\lambda_1 = \mu + \frac{\nu}{\nu+1}\sigma \text{ and } \lambda_2 = \frac{\nu}{2\nu^2+3\nu+1}\sigma.$$

First we deduce that

$$\sigma = \frac{2\nu^2+3\nu+1}{\nu}\lambda_2.$$

By substituting the later in the above first equation yields

$$\lambda_1 = \mu + \left(\frac{\nu}{\nu+1}\right)\left(\frac{2\nu^2+3\nu+1}{\nu}\right)\lambda_2,$$

then

$$\mu = \lambda_1 - \frac{2\nu^2+3\nu+1}{\nu+1}\lambda_2.$$

Recall that $\widehat{\lambda}_1 = \int_0^1 Q_n(u)\,du = \overline{X}$, and $\widehat{\lambda}_2 = \frac{1}{n^2}\sum_{i=1}^{n}(2i-n-1)\,X_{i:n}$, then the es-

timators of the location and scale parameters μ *and* σ *are respectively defined by*

$$\widehat{\mu} := \overline{X} - \frac{2\nu^2 + 3\nu + 1}{\nu + 1} \frac{1}{n^2} \sum_{i=1}^{n} (2i - n - 1) X_{i:n}$$

and

$$\widehat{\sigma} := \frac{2\nu^2 + 3\nu + 1}{\nu} \frac{1}{n^2} \sum_{i=1}^{n} (2i - n - 1) X_{i:n}.$$

Example 3.1.3 ***(Normal)*** *Let* $X_{1:n} \leq ... \leq X_{n:n}$ *be the order statistics pertaining to the random sample* $X_1, ..., X_n$ *from a Gaussian rv* $X \rightsquigarrow \mathcal{N}(\mu, \sigma^2)$. *We set* $Z_{i,n} := (X_{i:n} - \mu)/\sigma,\ i = 1, ..., n,$ *this mean's that* $X_{i:n} = \sigma Z_{i,n} + \mu,$ *where* $Z_{1:n} \leq ... \leq Z_{n:n}$ *be the order statistics pertaining the random sample* $Z_1, ..., Z_n$ *from a Gaussian rv* $Z \rightsquigarrow \mathcal{N}(0,1)$. *The two first L-moments are defined by* $\lambda_1 = \mathbf{E}[X_{1:1}]$ *and* $\lambda_2 = \frac{1}{2}(\mathbf{E}[X_{2,2}] - \mathbf{E}[X_{1,2}])$. *In other terms* $\lambda_1 = \mathbf{E}[\sigma Z_{1,1} + \mu] = \sigma \mathbf{E}[Z_{1,1}] + \mu,$ *and* $\lambda_2 = \frac{1}{2}\sigma(\mathbf{E}[Z_{2,2}] - \mathbf{E}[Z_{1,2}])$. *Note that* $Z_{1,1} = Z_1$ *and* $\mathbf{E}[Z_1] = 0,$ *then* $\lambda_1 = \mu$*. On the other hand* $\mathbf{E}[Z_{2,2}] = \frac{1}{\sqrt{\pi}}$ *and* $\mathbf{E}[Z_{1,2}] = -\mathbf{E}[Z_{2,2}] = -\frac{1}{\sqrt{\pi}},$ *see for instance see Balakrishnan and Chohen (1991) [**?**,] page 50. Then* $\lambda_2 = \frac{1}{\sqrt{\pi}}\sigma$*.Then*

$$\widehat{\mu} = \widehat{\lambda}_1 = \overline{X}$$

and $\widehat{\sigma} = \sqrt{\pi}\widehat{\lambda}_2 = \sqrt{\pi}\int_0^1 (2u - 1) Q_n(u)\, du,$*where* Q_n *is the empirical quantile function pertaining the random sample* $X_1, ..., X_n,$ *defined by*

$$Q_n(u) = X_{i:n}, \text{ for } \frac{i-1}{n} < u \leq \frac{i}{n},\ i = 1, ..., n.$$

Observe that

$$\widehat{\sigma} = \sqrt{\pi}\widehat{\lambda}_2 = \sqrt{\pi}\sum_{i=1}^{n}\int_{\frac{i-1}{n}}^{\frac{i}{n}} (2u - 1) Q_n(u)\, du = \sqrt{\pi}\sum_{i=1}^{n}\left\{\int_{\frac{i-1}{n}}^{\frac{i}{n}} (2u - 1)\, du\right\} X_{i:n}.$$

We have $\int (2u-1)\,du = u\,(u-1)+c,$ *then*

$$\int_{\frac{i-1}{n}}^{\frac{i}{n}} (2u-1)\,du = \frac{i}{n}\left(\frac{i}{n}-1\right) - \frac{i-1}{n}\left(\frac{i-1}{n}-1\right) = \frac{1}{n^2}\,(2i-n-1)\,,$$

therefore

$$\widehat{\sigma} = \frac{\sqrt{\pi}}{n^2}\sum_{i=1}^{n}(2i-n-1)\,X_{i:n}.$$

Example 3.1.4 *(**lognormal**) From the previous example of the Normal distribution, we deduce that the L-moment estimators of the location parameter corresponding to the lognormal mmdel is*

$$\widehat{\mu} = \frac{1}{n}\sum_{i=1}^{n}\log X_{i:n} \text{ and } \widehat{\sigma} = \frac{\sqrt{\pi}}{n^2}\sum_{i=1}^{n}(2i-n-1)\log X_{i:n}.$$

Example 3.1.5 *(**Weibull**) Let* $X_{1:n} \leq ... \leq X_{n:n}$ *be the order statistics pertaining to the random sample* $X_1, ..., X_n$ *from a Weibull rv* $X \rightsquigarrow \mathcal{W}(\alpha,\mu,\sigma)\,,$ *defined by its cdf*

$$F(x;\alpha,\mu,\sigma) = 1-\exp\left\{-\left(\frac{x-\mu}{\sigma}\right)^{\alpha}\right\},\ \alpha>0,\ \sigma>0,\ \mu\in R,\ 0<x<\mu.$$

or by its quantile function

$$Q(s;\alpha,\mu,\sigma) = \mu-\sigma\,(\log(1-s))^{1/\alpha}\,,\ 0<s<1. \tag{3.3}$$

We will show that the first two L-moments of the location and scale parameters corresponding the Weibull model are

$$\lambda_1 = \mu+\sigma\Gamma\left(1+\frac{1}{\alpha}\right) \text{ and } \lambda_2 = \sigma\left(1-2^{-1/\alpha}\right)\Gamma(1+1/\alpha)\,. \tag{3.4}$$

Indeed, for a given Weibull random variable X *of parameters* (α, μ, σ), *we have*

$$X = \mu + \sigma Z, \tag{3.5}$$

where Z *is a Weibull random variable of parameters* $(\alpha, 0, 1)$. *It is well known that* $\mathbf{E}[Z] = \Gamma(1 + 1/\alpha)$, *it follows that*

$$\lambda_1 = \mathbf{E}[X] = \mu + \sigma \mathbf{E}[Z] = \mu + \sigma \Gamma(1 + 1/\alpha).$$

Note that equation (3.3) *means that* $Q(s, \alpha, \mu, \sigma) = \mu + \sigma Q(s)$, *where* $Q(\cdot)$ *is the quantiles function corresponding to the rv* Z. *Therefore*

$$\lambda_2 = \int_0^1 (2u-1) Q(\cdot, \alpha, \mu, \sigma) \, du = \mu \int_0^1 (2u-1) \, du + \sigma \int_0^1 (2u-1) Q(u) \, du.$$

We have $\int_0^1 (2u-1) \, du = 0$ *then it remains calculate the second integral. By using the change of variables* $Q(u) = x$ *we write*

$$2 \int_0^1 u Q(u) \, du = 2 \int_0^\infty x F(x) \, dF(x) = 2\alpha \int_0^\infty x^\alpha \exp(-x^\alpha) (1 - \exp(-x^\alpha)) \, dx.$$

Once again by using the change of variables $x^\alpha = t$, *yields the previous integral* *becomes*

$$\begin{aligned}
&2\int_0^\infty t^{1/\alpha}\exp(-t)(1-\exp(-t))\,dt\\
&=2\int_0^\infty t^{1/\alpha}\exp(-t)\,dx-\int_0^\infty 2t^{1/\alpha}\exp(-2t)\,dt\\
&=2\int_0^\infty t^{1/\alpha+1-1}\exp(-t)\,dx-\int_0^\infty 2t^{1/\alpha}\exp(-2t)\,dt\\
&=2\Gamma(1+1/\alpha)-\int_0^\infty (s/2)^{1/\alpha}\exp(-s)\,ds\\
&=2\Gamma(1+1/\alpha)-2^{-1/\alpha}\Gamma(1+1/\alpha)\\
&=\left(2-2^{-1/\alpha}\right)\Gamma(1+1/\alpha).
\end{aligned}$$

Thus

$$\begin{aligned}
\int_0^1 (2u-1)\,Q(u)\,du&=2\int_0^1 uQ(u)\,du+\int_0^1 Q(u)\,du\\
&=\left(2-2^{-1/\alpha}\right)\Gamma(1+1/\alpha)+\mathbf{E}[Z]\\
&=\left(2-2^{-1/\alpha}\right)\Gamma(1+1/\alpha)-\Gamma(1+1/\alpha)\\
&=\left(1-2^{-1/\alpha}\right)\Gamma(1+1/\alpha),
\end{aligned}$$

thus $\lambda_2=\sigma\left(1-2^{-1/\alpha}\right)\Gamma(1+1/\alpha)$. *Observe now that the two equation* (3.4) *imply that*

$$\mu=\lambda_1-\frac{1}{(1-2^{-1/\alpha})\,\Gamma(1+1/\alpha)}\lambda_2 \text{ and } \sigma=\frac{\lambda_1}{\Gamma(1+1/\alpha)}.$$

Then for a given sample $X_1,...,X_n$ *from the rv* X, *the L-moments estimators of* (μ,σ) *are defined by*

$$\widehat{\mu}=\widehat{\lambda}_1-\frac{1}{(1-2^{-1/\alpha})\,\Gamma(1+1/\alpha)}\widehat{\lambda}_2=\overline{X}-\frac{n^{-2}\sum_{i=1}^n(2i-n-1)\,X_{i:n}}{(1-2^{-1/\alpha})\,\Gamma(1+1/\alpha)}$$

and

$$\widehat{\sigma}=\frac{\widehat{\lambda}_1}{\Gamma(1+1/\alpha)}=\frac{\overline{X}}{\Gamma(1+1/\alpha)}.$$

Example 3.1.6 ***(Gumbel)*** *Let $X_{1:n} \leq ... \leq X_{n:n}$ be the order statistics pertaining to the random sample $X_1, ..., X_n$ from a Gumbel rv $X \rightsquigarrow$ Gumbel (μ, σ), defined by its distribution function (df)*

$$F(x; \mu, \sigma) = \exp(-\exp\left(-\left(\frac{x-\mu}{\sigma}\right)\right), \ x \in R, \ \mu \in \mathbb{R}, \ \sigma > 0.$$

The quantile function corresponding to the later is given by

$$Q(s; \mu, \sigma) = \mu - \sigma \log(-\log s), \ 0 < s < 1.$$

For convenience, we set $Q(s) := Q(s; 0, 1) = -\log(-\log s)$. Let us defined $Z_{i,n} := (X_{i:n} - \mu)/\sigma, \ i = 1, ..., n$, this mean's that $X_{i:n} = \sigma Z_{i,n} + \mu$, where $Z_{1:n} \leq ... \leq Z_{n:n}$ be the order statistics pertaining the random sample $Z_1, ..., Z_n$ from a rv $Z =: (X - \mu)/\sigma \rightsquigarrow$ Gumbel (μ, σ). The first two L-moments are defined by

$$\lambda_1 = \int_0^1 Q(s; \mu, \sigma) \, ds \text{ and } \lambda_2 = \int_0^1 (2s-1) Q(s; \mu, \sigma) \, ds.$$

Since $Q(s; \mu, \sigma) = \sigma Q(s) + \mu$, then

$$\lambda_1 = \int_0^1 Q(s; \mu, \sigma) \, ds = \sigma \int_0^1 Q(s) \, ds + \mu$$

and

$$\lambda_2 = \int_0^1 (2s-1) Q(s; \mu, \sigma) \, ds = \sigma \int_0^1 (2s-1) Q(s) \, ds + \mu \int_0^1 (2s-1) \, ds.$$

Note that $\int_0^1 (2s-1) \, ds = 0$, therefore

$$\lambda_2 = \sigma \int_0^1 (2s-1) Q(s) \, ds.$$

In the Gumbel case, we have $\int_0^1 Q(s)\,ds = 0.577\,22$ *this implies that*

$$\lambda_1 = \mu + 0.577\,22\sigma. \tag{3.6}$$

On the other hand, we have $\int_0^1 (2s-1)\,Q(s)\,ds = 0.693\,15,$ *thus*

$$\lambda_2 = 0.693\,15\sigma. \tag{3.7}$$

From two equations(3.6) *and* (3.7) , *we infer that*

$$\mu = \lambda_1 - 0.832\,76\lambda_2 \text{ and } \sigma = 1.\,442\,7\lambda_2$$

From the previous system of equations, we deduce that the L-moments estimators corresponding to μ *and* σ *are*

$$\widehat{\mu} = \widehat{\lambda}_1 - 0.832\,76\widehat{\lambda}_2 \text{ and } \widehat{\sigma} = 1.\,442\,7\widehat{\lambda}_2,$$

that is

$$\widehat{\mu} := \overline{X} - \frac{0.832\,76}{n^2}\sum_{i=1}^{n}(2i-n-1)\,X_{i:n}$$

and

$$\widehat{\sigma} := \frac{1.\,442\,7}{n^2}\sum_{i=1}^{n}(2i-n-1)\,X_{i:n}.$$

3.2 Asymptotic efficient estimators

In many contexts, such as estimating the mean or variance of a population, the ideal estimator is one that is unbiased and has the smallest possible variance among all unbiased estimators. This notion is captured by the Cramer-Rao inequality, which states that the variance of any unbiased estimator is bounded below by the reciprocal of the Fisher information, which is a measure of the amount of information in the data about the parameter being estimated. Thus, an efficient estimator is one that achieves the smallest possible variance, given the amount of information in the data. In this section, we will present an asymptotically efficient estimators (AFE) both to the location and scale parameters which are proposed by Chernoff *et al.* (1967) [4].

Let $X_{1:n} \leq \cdots \leq X_{n:n}$ be the order statistics pertaining to a sample $X_1, ..., X_n$ of a rv X defined by its cdf

$$F(x; \theta_1, \theta_2) = F\left(\frac{x - \theta_1}{\theta_2}\right), \ \theta_1 \in \mathbb{R}, \ \theta_2 > 0,$$

and df

$$f(x; \theta_1, \theta_2) = \frac{1}{\theta_2} f\left(\frac{x - \theta_1}{\theta_2}\right).$$

We consider estimators to the location and scale parameters θ_1 and θ_2 as L-statistitcs, that is of the form

$$T_n = \frac{1}{n} \sum_{i=1}^{n} J\left(\frac{i}{n+1}\right) X_{i:n}. \tag{3.8}$$

The corresponding quantile function is

$$Q(u; \theta_1, \theta_2) = \theta_1 + \theta_2 Q(u), \ 0 < u < 1,$$

where $Q(u) := Q(u; 0, 1)$. The Fisher information matrix is defined by

$$I(\theta_1, \theta_2) := \left(\mathbf{E}\left\{\frac{\partial \log f(X; \theta_1, \theta_2)}{\partial \theta_i} \times \frac{\partial \log f(X; \theta_1, \theta_2)}{\partial \theta_j}\right\}\right), \ i, j = 1, ..., n.$$

Under some regularity assumptions on df f, see the aforementioned paper, by using a integration by parts we show that the matrix $I(\theta_1, \theta_2)$ may be rewritten into

$$\mathbf{E}\left[\frac{-\partial^2 \log f(X; \theta_1, \theta_2)}{\partial \theta_i \partial \theta_j}\right], \ i, j = 1, ..., n.$$

It is easy to verify that, by letting $y = (x - \theta_1)/\theta_2$, we have

$$\partial \log f(x; \theta_1, \theta_2)/\partial\theta_1 = \theta_2^{-1} L_1(y), \ \partial \log f(x; \theta_1, \theta_2)/\partial\theta_2 = \theta_2^{-1} L_2(y),$$

and $\int_{\mathbb{R}} L_1(y) f(y)\, dy = \int_{\mathbb{R}} L_2(y) f(y)\, dy = 0$, were

$$L_1(y) = -f^{'}(y)/f(y) \text{ and } L_2(y) = -(1 + yf'y/f(y)),$$

with $f(y) := f(y, 0, 1)$. Thus we may easily show that the Fisher information matrix is given by $I(\theta_1, \theta_2) = \theta_2^{-2} I$ where

$$I := \left(\int L_i(y) L_j(y) f(y)\, dy\right) = \begin{pmatrix} \int L_1^2(y) f(y)\, dy & \int L_2(y) L_1(y) f(y)\, dy \\ \int L_1(y) L_2(y) f(y)\, dy & \int L_2^2(y) f(y)\, dy \end{pmatrix}.$$

Finally, the AFE of θ_1 and θ_2, are respectively defined by

$$\widehat{\theta}_{1:n} := \frac{1}{n}\sum_{i=1}^{n} J_1\left(\frac{i}{n+1}\right) X_{i:n} \text{ and } \widehat{\theta}_{2:n} := \frac{1}{n}\sum_{i=1}^{n} J_2\left(\frac{i}{n+1}\right) X_{i:n}$$

where

$$\{J_1(u), J_2(u)\} = \{L_1'(y), L_2'(y)\} I^{-1}, \ y = Q(u).$$

Next we give examples of this estimation method corresponding to some probability models.

Example 3.2.1 ***(logistic)*** *The density function of the logistic distribution is defined by*

$$f(x; \mu, \sigma) = \frac{e^{-\frac{x-\mu}{\sigma}}}{\left(1 + e^{-\frac{x-\mu}{\sigma}}\right)^2}, \ \textit{for } x \in \mathbb{R}.$$

For convenience, we set $f(x) := f(x;0,1) = \frac{e^{-x}}{(1+e^{-x})^2}$. *We already noticed above that*

$$\frac{\partial}{\partial\mu}\log f(x;\mu,\sigma) = \frac{1}{\sigma}L_1(y),$$

where

$$L_1(y) = -\frac{f'(y)}{f(y)} = -\frac{e^{-y}-1}{e^{-y}+1}$$

with $y := (x-\mu)/\sigma$. *Likewise*

$$\frac{\partial}{\partial\sigma}\log f(x;\mu,\sigma) = \frac{1}{\sigma}L_2(y),$$

where

$$L_2(y) := -\left(1+y\frac{f'(y)}{f(y)}\right) = -\left(1+y\frac{e^{-y}-1}{e^{-y}+1}\right).$$

Recall that, the fisher information $I(\mu,\sigma)$ *is defined by* $\sigma^{-2}I$, *where*

$$I = \begin{pmatrix} \int_{-\infty}^{\infty} L_1^2(y) f(y)\,dy & \int_{-\infty}^{\infty} L_2(y) L_1(y) f(y)\,dy \\ \int_{-\infty}^{\infty} L_1(y) L_2(y) f(y)\,dy & \int_{-\infty}^{\infty} L_2^2(y) f(y)\,dy \end{pmatrix}.$$

Let us now calculate each coefficient of I. *We have* $\int_{-\infty}^{\infty} L_1^2(y) f(y)\,dy$ *equals*

$$\int_{-\infty}^{+\infty}\left(-\frac{e^{-y}-1}{e^{-y}+1}\right)^2\left(\frac{e^{-y}}{(1+e^{-y})^2}\right)dy = \int_{-\infty}^{\infty} e^{-y}\frac{(e^{-y}-1)^2}{(e^{-y}+1)^4}dy = \frac{1}{3},$$

and $\int_{-\infty}^{\infty} L_2^2(y) f(y)\,dy$ *equals*

$$\int_{-\infty}^{\infty}\left(-\left(1+y\frac{e^{-y}-1}{e^{-y}+1}\right)\right)^2\left(\frac{e^{-y}}{(1+e^{-y})^2}\right)dy$$
$$= \int_{-\infty}^{\infty}\frac{e^{-y}}{(e^{-y}+1)^4}\left(e^{-y}-y+ye^{-y}+1\right)^2 dy = 1.4300\,.$$

Likewise $\int_{-\infty}^{\infty} L_1(y) L_2(y) f(y) \, dy$ *equals*

$$= \int_{-\infty}^{\infty} \left(-\frac{e^{-y}-1}{e^{-y}+1}\right)\left(-\left(1+y\frac{e^{-y}-1}{e^{-y}+1}\right)\right)\left(\frac{e^{-y}}{(1+e^{-y})^2}\right) dy$$
$$= \int_{-\infty}^{\infty} e^{-y}\frac{e^{-y}-1}{(e^{-y}+1)^4}\left(e^{-y}-y+ye^{-y}+1\right) dy = 0$$

Therefore

$$I = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & 1.4300 \end{pmatrix} \text{ and } I^{-1} = \begin{pmatrix} 3 & 0 \\ 0 & 0.6993 \end{pmatrix}.$$

On the other hand, we show that

$$L_1' = 2\frac{e^{-y}}{(e^{-y}+1)^2}, \text{ and } L_2' = \frac{1}{(e^{-y}+1)^2}\left(-e^{2(-y)}+2ye^{-y}+1\right),$$

it follows

$$\{J_1, J_2\} = \{L_1', L_2'\} I^{-1}$$
$$= \left\{6\frac{e^{-y}}{(e^{-y}+1)^2}, \frac{0.6993}{(e^{-y}+1)^2}\left(-e^{2(-y)}+2ye^{-y}+1\right)\right\},$$

where $y = Q(s) = \log\frac{s}{1-s}$, $0 < s < 1$. *It is easy to check that*

$$\frac{e^{-\left(\log\frac{s}{1-s}\right)}}{\left(e^{-\left(\log\frac{s}{1-s}\right)}+1\right)^2} = s(1-s),$$

and

$$\frac{1}{\left(e^{-\log\frac{s}{1-s}}+1\right)^2}\left(-e^{2\left(-\log\frac{s}{1-s}\right)}+2\log\frac{s}{1-s}e^{-\log\frac{s}{1-s}}+1\right)$$
$$= 2s-1+3s(1-s)\ln\frac{s}{1-s},$$

Finally, the AFE of μ *and* σ, *are respectively defined by*

$$\widehat{u} = \frac{1}{n}\sum_{i=1}^{n} J_1\left(\frac{i}{n+1}\right) X_{i:n} \text{ and } \widehat{\sigma} = \frac{1}{n}\sum_{i=1}^{n} J_2\left(\frac{i}{n+1}\right) X_{i:n},$$

where

$$J_1(s) = 6s(1-s), \text{ for } 0 < s < 1,$$

and

$$J_2(s) = 0.699\,3\left\{2s - 1 + 2s(1-s)\log\frac{s}{1-s}\right\}, \text{ for } 0 < s < 1.$$

Note that $0.699\,32 = 9(\pi^2+3)^{-1}$, *then the function* J_2 *corresponds indeed to this stated in the paper of Chernoff et al. (1967) [4].*

Example 3.2.2 ***(Power)*** *The density function is defined by*

$$f_\nu(x;\mu,\sigma) = \frac{v}{\sigma}\left(\frac{x-\mu}{\sigma}\right)^{\nu-1}, \ \mu < x < \mu/\sigma \text{ and } \nu > 0.$$

We have $\frac{\partial}{\partial\mu}\log f_\nu(x;\mu,\sigma) = \frac{1}{\sigma}L_1(y)$, *where*

$$L_1(y) = -\frac{f_\nu'(y)}{f_\nu(y)} = -(\nu-1)\frac{1}{y}, \text{ with } y := (x-\mu)/\sigma,$$

where $f_\nu(y) := f_\nu(x;\mu,\sigma)$. *Likewise* $\frac{\partial}{\partial\sigma}\log f_\nu(x;\mu,\sigma) = \frac{1}{\sigma}L_2(y)$, *where*

$$L_2(y) := -\left(1 + y\frac{f_\nu'(y)}{f_\nu(y)}\right) = -(1+(\nu-1)) = -v.$$

Let us now calculate each coefficient of matrix I. *We have*

$$\int_0^1 L_1^2(y) f_\nu(y)\,dy = \int_0^1 \left(-(\nu-1)\frac{1}{y}\right)^2 \left(\nu y^{\nu-1}\right) dy$$
$$= \nu(\nu-1)^2 \int_0^1 y^{\nu-3}\,dy = \frac{\nu(\nu-1)^2}{\nu-2},$$

provided that $\nu > 2$. *Likewise*

$$\int_0^1 L_2^2(y) f_\nu(y)\,dy = \int_0^1 (-v)^2 \left(\nu y^{\nu-1}\right) dy = \nu^2,$$

and

$$\int_0^1 L_1(y) L_2(y) f_\nu(y)\,dy = \int_0^1 \left((\nu-1)\frac{1}{y}\right)(v)\left(\nu y^{\nu-1}\right) dy = \nu^2.$$

Therefore

$$I = \begin{pmatrix} \frac{\nu(\nu-1)^2}{\nu-2} & \nu^2 \\ \nu^2 & \nu^2 \end{pmatrix} \text{ and } I^{-1} = \begin{pmatrix} \frac{1}{\nu}(\nu-2) & -\frac{1}{\nu}(\nu-2) \\ -\frac{1}{\nu}(\nu-2) & \frac{1}{\nu^2}(\nu-1)^2 \end{pmatrix}.$$

On the other hand $L_1' = \frac{1}{y^2}(\nu-1)$, $L_2' = 0$, *this implies that*

$$\{J_1, J_2\} = \{L_1', L_2'\} I^{-1} = \left\{ \frac{1}{y^2\nu}(\nu-1)(\nu-2), \quad -\frac{1}{y^2\nu}(\nu-1)(\nu-2) \right\}.$$

Thus, the AFE of μ *and* σ, *are respectively defined by*

$$\widehat{\mu} = \frac{1}{n}\sum_{i=1}^{n} J_1\left(Q\left(\frac{i}{n+1}\right)\right) X_{i:n} = \frac{1}{n}\sum_{i=1}^{n}\frac{(\nu-1)(\nu-2)}{\nu}\left(\frac{i}{n+1}\right)^{-2/\nu} X_{i:n}$$

$$\widehat{\sigma} = -\widehat{\mu}.$$

Example 3.2.3 *(**Weibull**) The density function is defined by*

$$f(x;\mu,\sigma) = \alpha\left(\frac{x-\mu}{\sigma}\right)^{\alpha-1} e^{-\left(\frac{x-\mu}{\sigma}\right)^{\alpha}}, \ \alpha > 0, \ \sigma > 0, \ x > 0,$$

We have

$$L_1(y) = -\frac{f'(y)}{f(y)} = \frac{1}{y}\left(y^{\alpha}\alpha - \alpha + 1\right), \text{ with } y := (x-\mu)/\sigma,$$

and

$$L_2(y) := -\left(1 + y\frac{f'(y)}{f(y)}\right) = \alpha\left(y^\alpha - 1\right),$$

where $f(x) := f(x;\mu,\sigma)$. *It is easy to verify that*

$$L_1(y) = -\frac{f'(y)}{f(y)} = \frac{1}{y}\left(y^\alpha\alpha - \alpha + 1\right), \text{ with } y := (x-\mu)/\sigma;$$

and

$$L_2(y) := -\left(1 + y\frac{f'(y)}{f(y)}\right) = \alpha\left(y^\alpha - 1\right).$$

By using elementary calculations, yields

$$\begin{aligned} a &:= \int_0^\infty L_1^2(y) f(y)\,dy = \int_0^\infty \left(y^{-1}\left(\alpha y^\alpha - \alpha + 1\right)\right)^2 \left(\alpha y^{\alpha-1} e^{-y^\alpha}\right) dy \\ &= \alpha \int_0^\infty y^{\alpha-3} e^{-y^\alpha} \left(\alpha y^\alpha - \alpha + 1\right)^2 dy \\ &= \alpha^2 \Gamma(3 - 2/\alpha) + 2\alpha(1-\alpha)\Gamma(2 - 2/\alpha) + (\alpha-1)^2 \Gamma(1 - 2/\alpha), \end{aligned}$$

provided that $\alpha > 2$. *Likewise*

$$\begin{aligned} b &:= \int_0^\infty L_2^2(y) f(y)\,dy = \int_0^\infty \left(\alpha y^\alpha - \alpha\right)^2 \left(\alpha y^{\alpha-1} e^{-y^\alpha}\right) dy \\ &= \int_0^\infty \left(\frac{1}{y} y^\alpha \frac{\alpha^3}{e^{y^\alpha}} - \frac{2}{y} y^{2\alpha} \frac{\alpha^3}{e^{y^\alpha}} + \frac{1}{y} y^\alpha y^{2\alpha} \frac{\alpha^3}{e^{y^\alpha}}\right) dy \\ &= \alpha^3 \left\{\int_0^\infty y^{\alpha-1} e^{-y^\alpha} dy\right\} - 2\left\{\int_0^\infty y^{2\alpha-1} e^{-y^\alpha} dy\right\} + \left\{\int_0^\infty y^{3\alpha-1} e^{-y^\alpha} dy\right\} \\ &= \frac{1}{\alpha}\alpha^3 \left\{\int_0^\infty t^{1/\alpha-1} \left(t^{1/\alpha}\right)^{\alpha-1} e^{-t} dt\right\} = \alpha^2, \end{aligned}$$

and

$$c := \int_{-\infty}^{\infty} L_1(y) L_2(y) f(y) dy = \int_0^{+\infty} y^{-1}(\alpha y^{\alpha} - \alpha + 1)(\alpha y^{\alpha} - \alpha) \alpha y^{\alpha-1} e^{-y^{\alpha}} dy$$
$$= \alpha^2 \left\{ \int_0^{\infty} y^{2\alpha-2} e^{-y^{\alpha}} dy \right\} - 2\alpha^3 \left\{ \int_0^{\infty} y^{2\alpha-2} e^{-y^{\alpha}} dy \right\}$$
$$- \alpha^2 \left\{ \int_0^{\infty} y^{\alpha-2} e^{-y^{\alpha}} dy \right\} + \alpha^3 \left\{ \int_0^{\infty} y^{\alpha-2} e^{-y^{\alpha}} dy \right\}$$
$$+ \alpha^3 \left\{ \int_0^{\infty} y^{3\alpha-2} e^{-y^{\alpha}} dy \right\}$$
$$= \alpha^2 (1 - 2\alpha) \left\{ \int_0^{\infty} y^{2\alpha-2} e^{-y^{\alpha}} dy \right\} + \alpha^2 (\alpha - 1) \left\{ \int_0^{\infty} y^{\alpha-2} e^{-y^{\alpha}} dy \right\}$$
$$+ \alpha^3 \left\{ \int_0^{\infty} y^{3\alpha-2} e^{-y^{\alpha}} dy \right\}$$
$$= \alpha (1 - 2\alpha) \Gamma(2 - 1/\alpha) + \alpha(\alpha - 1) \Gamma(1 - 1/\alpha) + \alpha^2 \Gamma(3 - 1/\alpha).$$

provided that $\alpha > 1$. *We set*

$$I = \begin{pmatrix} a & c \\ c & b \end{pmatrix},$$

Let us write

$$(J_1, J_2) = \begin{pmatrix} L_1'(y), & L_2'(y) \end{pmatrix} \begin{pmatrix} a & c \\ c & b \end{pmatrix}^{-1}$$
$$= \begin{pmatrix} L_1'(y), & L_2'(y) \end{pmatrix} \begin{pmatrix} \frac{b}{ab-c^2} & -\frac{c}{ab-c^2} \\ -\frac{c}{ab-c^2} & \frac{a}{ab-c^2} \end{pmatrix}$$
$$= \frac{1}{ab - c^2} \begin{pmatrix} aL_1'(y) - cL_2'(y), & bL_2'(y) - cL_1'(y) \end{pmatrix}$$

Thereby, the AFE of μ *and* σ, *are respectively defined by*

$$\widehat{\mu} = \frac{1}{n} \sum_{i=1}^{n} \left[k_1 L_1' \left(F^{-1} \left(\frac{i}{n+1} \right) \right) - k_3 L_2' \left(F^{-1} \left(\frac{i}{n+1} \right) \right) \right] X_{i:n},$$

and

$$\widehat{\sigma} = \frac{1}{n} \sum_{i=1}^{n} \left[k_2 L_2' \left(F^{-1} \left(\frac{i}{n+1} \right) \right) - k_3 L_1' \left(F^{-1} \left(\frac{i}{n+1} \right) \right) \right] X_{i:n},$$

where

$$k_1 := \frac{a}{ab - c^2},\ k_2 := \frac{b}{ab - c^2} \text{ and } k_3 = \frac{c}{ab - c^2}$$

where $Q(u) = (-\log(1-u))^{1/\alpha}$, *provided that* $\alpha > 3$. *Let us now give explicit formulas of the weights functions* J_1 *and* J_2. *We have*

$$L_1'(y) = \frac{d}{dy} \left(y^{-1} (\alpha y^{\alpha} - \alpha + 1) \right) = y^{-2} (\alpha - 1) (\alpha y^{\alpha} + 1)$$

and

$$L_2'(y) = \frac{d}{dy} (\alpha y^{\alpha} - \alpha) = y^{\alpha - 1} \alpha^2$$

where $y = Q(u)$, *therefore*

$$\begin{aligned} L_1'(y) &= \left((-\log(1-u))^{1/\alpha} \right)^{-2} (\alpha - 1) \left(\alpha \left((-\log(1-u))^{1/\alpha} \right)^{\alpha} + 1 \right) \\ & \left((-\log(1-u))^{1/\alpha} \right)^{-2} (\alpha - 1) (1 - \alpha \log(1-u)) \\ &= \frac{(\alpha - 1)(1 - \alpha \log(1-u))}{(-\log(1-u))^{2/\alpha}} := \varphi_1(u) \end{aligned}$$

and

$$L_2'(y) = \alpha^2 (-\log(1-u))^{1-1/\alpha} := \varphi_2(u).$$

Then the weigh functions corresponding to AFE of μ *and* σ *are given by:*

$$J_1(u) := k_2 \varphi_1(u) - k_3 \varphi_2(u)$$

and

$$J_2(u) := k_1 \varphi_2(u) - k_3 \varphi_1(u),$$

so that

$$\widehat{\mu} = \frac{1}{n}\sum_{i=1}^{n} J_1\left(\frac{i}{n+1}\right) X_{i:n} \text{ and } \widehat{\sigma} = \frac{1}{n}\sum_{i=1}^{n} J_2\left(\frac{i}{n+1}\right) X_{i:n}.$$

provide that $\alpha > 1$.

Example 3.2.4 *(**Normal**) The density function is defined on* $\mathbb{R}$ *by*

$$f(x;\mu,\sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}, \ \mu \in \mathbb{R} \text{ and } \sigma > 0.$$

We have $\frac{\partial}{\partial\mu}\log f(x;\mu,\sigma) = \frac{1}{\sigma}L_1(y)$, *where*

$$L_1(y) = -\frac{f'(y)}{f(y)} = y, \text{ with } y := (x-\mu)/\sigma;$$

and $\frac{\partial}{\partial\sigma}\log f(x;\mu,\sigma) = \frac{1}{\sigma}L_2(y)$, *where*

$$L_2(y) := -\left(1 + y\frac{f'(y)}{f(y)}\right) = -\left(1-y^2\right).$$

Let us now calculate each coefficient of I. *We have*

$$\int_{-\infty}^{\infty} L_1^2(y) f(y)\, dy = \int_{-\infty}^{+\infty} y^2 \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\right) dy = 1,$$

$$\int_{-\infty}^{\infty} L_2^2(y) f(y)\, dy = \int_{-\infty}^{+\infty} \left(-\left(1-y^2\right)\right)^2 \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\right) dy = 2$$

and

$$\int_{-\infty}^{\infty} L_1(y) L_2(y) f(y)\, dy = \int_{-\infty}^{+\infty} y\left(-\left(1-y^2\right)\right) \left(\frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\right) dy = 0.$$

Therefore

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}, \text{ then } I^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}.$$

On the other hand we have $L_1' = 1$, $L_2' = 2y$, *therefore*

$$\{J_1, J_2\} = \{L_1', L_2'\} I^{-1} = \begin{pmatrix} 1 & y \end{pmatrix}.$$

Hence, the AFE of μ *and* σ, *are respectively defined by*

$$\widehat{\mu} = \frac{1}{n}\sum_{i=1}^{n} J_1 \left(Q \left(\frac{i}{n+1} \right) \right) X_{i:n} = \frac{1}{n}\sum_{i=1}^{n} X_{i,n} = \overline{X}$$
$$\widehat{\sigma} = \frac{1}{n}\sum_{i=1}^{n} J_2 \left(Q \left(\frac{i}{n+1} \right) \right) X_{i:n} = \frac{1}{n}\sum_{i=1}^{n} \psi^{-1} \left(\frac{i}{n+1} \right) X_{i:n},$$

where ψ^{-1} *is the quantile function corresponding the standard rv.*

Example 3.2.5 ***(lognormal)****. The density function is defined by*

$$f(x; \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{\log x - \mu}{\sigma}\right)^2}, \ x > 0, \ \mu \in \mathbb{R} \text{ and } \sigma > 0.$$

From the previous example of the Normal distribution, we infer that the AFE estimators of the location an scale parameters are

$$\widehat{\mu} = \frac{1}{n}\sum_{i=1}^{n} \log X_{i:n} \text{ and } \widehat{\sigma} = \frac{1}{n}\sum_{i=1}^{n} \psi^{-1} \left(\frac{i}{n+1} \right) \log X_{i:n}.$$

Example 3.2.6 ***(Gumbel)*** *The density function is defined by*

$$f(x; \mu, \sigma) = \exp\left(-\frac{x-\mu}{\sigma} - e^{-\frac{x-\mu}{\sigma}} \right), \ x \in \mathbb{R}, \ \mu \in \mathbb{R} \text{ and } \sigma > 0,$$

We have $\frac{\partial}{\partial \mu} \log f(x; \mu, \sigma) = \frac{1}{\sigma} L_1(y)$, *where*

$$L_1(y) = -\frac{f'(y)}{f(y)} = 1 - e^{-y}, \text{ with } y := (x-\mu)/\sigma,$$

and $\frac{\partial}{\partial\sigma}\log f(x;\mu,\sigma)=\frac{1}{\sigma}L_2(y)$, *where*

$$L_2(y):=-\left(1+y\frac{f'(y)}{f(y)}\right)=-y\left(e^{-y}-1\right)-1$$
$$=y-ye^{-y}-1. \tag{3.9}$$

Let us now calculate each coefficient of I. *We have*

$$\int_{-\infty}^{\infty}L_1^2(y)\,f(y)\,dy=\int_{-\infty}^{+\infty}\left(1-e^{-y}\right)^2\left(e^{-y}e^{-e^{-y}}\right)dy=1,$$

$$\int_{-\infty}^{\infty}L_2^2(y)\,f(y)\,dy=\int_{-\infty}^{+\infty}\left(y-ye^{-y}-1\right)^2\left(e^{-y}e^{-e^{-y}}\right)dy=1.8237,$$

and

$$\int_{-\infty}^{\infty}L_1(y)\,L_2(y)\,f(y)\,dy=\int_{-\infty}^{\infty}\left(1-e^{-y}\right)\left(y-ye^{-y}-1\right)\left(e^{-y}e^{-e^{-y}}\right)dy$$
$$=-0.42278,$$

Then

$$I=\begin{pmatrix}1 & -0.42278\\ -0.42278 & 1.8237\end{pmatrix},\ \textit{thereby}\ I^{-1}=\begin{pmatrix}1.1087 & 0.25702\\ 0.25702 & 0.60792\end{pmatrix}.$$

It is clear that

$$\begin{pmatrix}u & v\end{pmatrix}\begin{pmatrix}1.1087 & 0.25702\\ 0.25702 & 0.60792\end{pmatrix}$$
$$=\begin{pmatrix}1.1087u+0.25702v, & 0.25702u+0.60792v\end{pmatrix},$$

therefore

$$\begin{aligned}\{J_1, J_2\} &= \{L'_1, L'_2\}\, I^{-1} \\ &= \left\{ \begin{array}{cc} 1.108\,7L'_1 + 0.257\,02L'_2 & 0.257\,02L'_2 + 0.607\,92L'_1 \end{array} \right\},\end{aligned}$$

where $L'_1 = e^{-y}$ *and* $L'_2 = ye^{-y} - e^{-y} + 1$. *Let us now give explicit formulas of the weights functions* J_1 *and* J_2. *By substituting*

$$y = Q(u) = -\log(-\log u)\,,\ \ 0 < s < 1,$$

yields

$$L'_1(y) = -\log u \ \text{and}\ L'_2(y) = \log u + \log u \log(-\log u) + 1.$$

Then the weigh functions corresponding to the AFE of μ *and* σ *are given by*

$$\begin{aligned}\{J_1, J_2\} &= \{L'_1, L'_2\}\, I^{-1} \\ &= \left\{ \begin{array}{cc} 1.108\,7L'_1 + 0.257\,02L'_2 & 0.257\,02L'_1 + 0.607\,92L'_2 \end{array} \right\}.\end{aligned}$$

$$\textit{More explicitly}\left\{ \begin{array}{l} J_1(u) = -1.108\,7\log u + 0.257\,02\left(\log u + \log u \log(-\log u) + 1\right) \\ J_2(u) = -0.257\,02\log u + 0.607\,92\left(\log u + \log u \log(-\log u) + 1\right) \end{array} \right.$$

which may be simplified into

$$\left\{ \begin{array}{l} J_1(u) = 0.257\,02\log(u)\log(-\log u) - 0.851\,68\log u + 0.257\,02 \\ J_2(u) = 0.350\,9\log u + 0.607\,92\log u \log(-\log u) + 0.607\,92. \end{array} \right.$$

Hence, the AFE of μ and σ, are respectively defined by

$$\widehat{\mu} = \frac{1}{n}\sum_{i=1}^{n} J_1\left(\frac{i}{n+1}\right) X_{i:n} \text{ and } \widehat{\sigma} = \frac{1}{n}\sum_{i=1}^{n} J_2\left(\frac{i}{n+1}\right) X_{i:n}.$$

3.3 Best linear unbiased estimators

In this section we present the best linear unbiased estimation method of scale and location parameters of probability model. The obtained estimators have minimum variances among the linear unbiased ones. We will see that this method produce estimators belong to the class of L-statistics, see Balakrishnan and Cohen (1991) [2].

Definition 3.3.1 *Let* $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ *be the order statistics pertaining to a sample* $X_1, ..., X_n$ *from a rv* X *of location-scale parameter distribution. Let's denote* $Z_{i:n} = (X_{i:n} - \mu) / \sigma$, $\mathbf{E}[Z_{i:n}] = \alpha_{i,n}$ *and* $\mathbf{Cov}(Z_{i:n}, Z_{j:n}) = \beta_{ij:n}$ *for* $1 \leq i \leq n$. *For convenience, we set*

$$X := (X_{1:n}, X_{2:n}, ..., X_{n:n})^t, \ \alpha := (\alpha_{1:n}, \alpha_{2:n}, ..., \alpha_{n:n})^t,$$

and

$$\beta := \begin{pmatrix} \beta_{1,1:n} & \beta_{1,2:n} & ... & \beta_{1,n:n} \\ \beta_{1,2:n} & \beta_{2,2:n} & ... & \beta_{2,n:n} \\ .... & .... & .... & ... \\ \beta_{1,n:n} & \beta_{2,n:n} & .... & \beta_{n,n:n} \end{pmatrix}.$$

It is clear that

$$\mathbf{E}[X] = \mu\mathbf{1} + \sigma\alpha \text{ and } \mathbf{Var}(X) = \sigma^2\beta,$$

where $\mathbf{1} := (1, ..., 1)^t$ denotes the vector of n ones. The generalized variance of rv X is given by

$$(X - \mu\mathbf{1} - \sigma\alpha)^t \beta^{-1} (X - \mu\mathbf{1} - \sigma\alpha),$$

where β^{-1} denotes the inverse matrix of β. It is easy to verify that this variance equals to

$$\begin{aligned}
& X^t\beta^{-1}X - \mu\mathbf{1}^t\beta^{-1}X - \sigma\alpha^t\beta^{-1}X - \mu X^t\beta^{-1}\mathbf{1} \\
&\qquad - \mu^2\mathbf{1}^t\beta^{-1} + \mu\sigma\alpha^t\beta^{-1}\mathbf{1} - \sigma X^t\beta^{-1}\alpha + \mu\sigma\mathbf{1}^t\beta^{-1}\alpha + \sigma^2\alpha^t\beta^{-1}\alpha \\
&= X^\top\beta^{-1}X + \mu^2\mathbf{1}^t\beta^{-1}\mathbf{1} + \sigma^2\alpha^t\beta^{-1}\alpha - 2\mu\mathbf{1}^t\beta^{-1}X - 2\sigma\alpha^t\beta^{-1}X + 2\mu\sigma\alpha^t\beta^{-1}\mathbf{1}.
\end{aligned}$$

By minimizing the previous expression with respect to μ and σ, yields

$$\mu\mathbf{1}^t\beta^{-1}\mathbf{1} + \sigma\alpha^t\beta^{-1}\mathbf{1} = \mathbf{1}^t\beta^{-1}X \tag{3.10}$$

and

$$\mu\alpha^t\beta^{-1}\mathbf{1} + \sigma\alpha^t\beta^{-1}\alpha = \alpha^t\beta^{-1}X. \tag{3.11}$$

Upon solving Eqs (3.10) and (3.11) , we derive, respectively, expressions of the BLUEs of μ and σ as follows:

$$\begin{aligned}
\widehat{\mu}^* &= \left\{ \frac{\alpha^t\beta^{-1}\alpha\mathbf{1}^t\beta^{-1} - \alpha^t\beta^{-1}\mathbf{1}\alpha^t\beta^{-1}}{(\alpha^t\beta^{-1}\alpha)(\mathbf{1}^t\beta^{-1}\mathbf{1}) - (\alpha^t\beta^{-1}\mathbf{1})^2} \right\} X \\
&= -\alpha^t\Delta X = \sum_{i=1}^n a_{i:n}X_{i:n} \text{ (L-statistics)}
\end{aligned} \tag{3.12}$$

and

$$\begin{aligned}
\widehat{\sigma}^* &= \left\{ \frac{\mathbf{1}^t\beta^{-1}\mathbf{1}\alpha^t\beta^{-1} - \mathbf{1}^t\beta^{-1}\alpha\mathbf{1}^t\beta^{-1}}{(\alpha^t\beta^{-1}\alpha)(\mathbf{1}^t\beta^{-1}\mathbf{1}) - (\alpha^t\beta^{-1}\mathbf{1})^2} \right\} X \\
&= \mathbf{1}^t\Delta X = \sum_{i=1}^n b_{i:n}X_{i:n}, \text{ (L-statistics)}
\end{aligned} \tag{3.13}$$

where

$$\Delta := \frac{\beta^{-1}(\mathbf{1}\alpha^t - \alpha\mathbf{1}^t)\beta^{-1}}{(\alpha^t\beta^{-1}\alpha)(\mathbf{1}^t\beta^{-1}\mathbf{1}) - (\alpha^t\beta^{-1}\mathbf{1})^2}.$$

The coefficients $(a_{i,n})$ and $(b_{i,n})$ are respectively the elements of two vectors $-\alpha^t\Delta$ and $\mathbf{1}^t\Delta$. By using expression (3.12) and (3.13) , we end up with

$$\mathbf{Var}\left(\widehat{\mu}^*\right) = \sigma^2 \left\{ \frac{\left(\alpha^t\beta^{-1}\alpha\right)}{\left(\alpha^t\beta^{-1}\alpha\right)\left(\mathbf{1}^t\beta^{-1}\mathbf{1}\right) - \left(\alpha^t\beta^{-1}\mathbf{1}\right)^2} \right\},$$

$$\mathbf{Var}\left(\widehat{\sigma}^*\right) = \sigma^2 \left\{ \frac{\left(\mathbf{1}^t\beta^{-1}\mathbf{1}\right)}{\left(\alpha^t\beta^{-1}\alpha\right)\left(\mathbf{1}^t\beta^{-1}\mathbf{1}\right) - \left(\alpha^t\beta^{-1}\mathbf{1}\right)^2} \right\},$$

and

$$\mathbf{Cov}(\mu^*, \sigma^*) = -\sigma^2 \left\{ \frac{\alpha^t\beta^{-1}\mathbf{1}}{\left(\alpha^t\beta^{-1}\alpha\right)\left(\mathbf{1}^t\beta^{-1}\mathbf{1}\right) - \left(\alpha^t\beta^{-1}\mathbf{1}\right)^2} \right\}.$$

Chapter 4

Simulation study and real data applications

4.1 Simulation study

In terms of the absolute bias (ABIAS) and the mean square error (MSE), we will compare the performance of the L-moments, asymptotically efficient, best linear unbiased and maximum likelihood estimators $\theta := (\theta_1, \theta_2) = (\mu, \sigma)$. These two standards are defined by

$$\mathbf{ABIAS}\,(r) = \frac{1}{M}\left|\sum_{j=1}^{M}\left(\widehat{\theta}_{n,j}^{(r)} - \theta\right)\right|,\ \mathbf{MSE}\,(r) := \frac{1}{M}\sum_{j=1}^{M}\left(\widehat{\theta}_{n,j}^{(r)} - \theta\right)^2,\ r = 1, 2,$$

where $\widehat{\theta}_{n,j}^{(r)}$ denotes the j-th estimated value of the parameter $\widehat{\theta}_r$ corresponding to the j-th replication among the M one. To this end, we chose five probability models, namely Normal, the Logistic, Weibull, Gumbel and Lognormal. For each model we generate $M = 2000$ random sample of lengths $N = 50, 100$, then compute respectively the **ABIAS** and the **MSE** corresponding to each estimator.

From our simulation study, we first noticed that maximum likelihood estimator

$N = 50$	Normal Distribution			
Estimators	ABIAS.mu	ABIAS.sigma	MSE.mu	MSE.sigma
MLE	0.00007	0.0163	0.0199	0.0103
AFE	0.00007	0.0929	0.0199	0.0172
BLUE	0.0087	0.9677	0.0262	1.0117
LM	0.00007	0.0407	0.0199	0.0755

Table 4.1: Absolute biases and mse of MLE, AFE, BLUE and LM estimators corresponding to the normal model for sample sizes: $N = 50$, with M=2000 replications.

$N = 100$	Normal Distribution			
Estimators	ABIAS.mu	ABIAS.sigma	MSE.mu	MSE.sigma
MLE	0.0032	0.0112	0.0098	0.0053
AFE	0.0032	0.0563	0.0098	0.0079
BLUE	0.0015	0.9833	0.0127	1.0022
LM	0.0032	0.0291	0.0098	0.0370

Table 4.2: Absolute biases and mse of MLE, AFE, BLUE and LM estimators corresponding to the normal model for sample sizes: $N = 100$, with M=2000 replications.

(MLE) does not work in the case of weibull(α) whenever $\alpha > 3$ with (very) small sample size. However, in overall when the sample size exceed 100 the MLE, LME, AFE estimators performed better compared to the BLUE one. The results of our simulation are recupulates tables 4.1 – 4.12.

4.1.1 The most packages used in our simulation

maxLik: Maximum likelihood estimation.

$N = 50$	Logistic Distribution			
Estimators	ABIAS.mu	ABIAS.sigma	MSE.mu	MSE.sigma
ML	0.0075	0.0126	0.0623	0.0139
AFE	0.0076	0.0123	0.0648	0.0140
BLUE	0.5284	0.5960	0.3469	0.4098
LME	0.0037	0.0441	0.0674	0.0831

Table 4.3: Absolute biases and mse of MLE, AFE, BLUE and LM estimators corresponding to the logistic model for sample sizes: $N = 50$, with M=2000 replications.

$N = 100$	**Logistic Distribution**			
Estimators	ABIAS.mu	ABIAS.sigma	MSE.mu	MSE.sigma
ML	0.0001	0.0059	0.0320	0.0075
AFE	0.0001	0.0051	0.0326	0.0075
BLUE	0.5556	0.5274	0.3437	0.3045
LME	0.0001	0.0198	0.0345	0.0422

Table 4.4: Absolute biases and mse of MLE, AFE, BLUE and LM estimators corresponding to the logistic model for sample sizes: $N = 1000$, with M=2000 replications.

$N = 50; \alpha = 2.5$	**Weibull Distribution**			
Estimators	ABIAS.mu	ABIAS.sigma	MSE.mu	MSE.sigma
MLE	0.0327	0.0424	0.0474	0.0593
AFE	0.0026	0.0263	0.0013	0.0048
BLUE	0.0038	0.0038	0.0302	0.4240
LM	0.0195	0.0223	0.0082	0.0099

Table 4.5: Absolute biases and mse of MLE, AFE, BLUE and LM estimators corresponding to the Weibull (2.5) model for sample sizes: $N = 50$, with M=2000 replications.

$N = 100, \alpha = 2.5$	**Weibull Distribution**			
Estimators	ABIAS.mu	ABIAS.sigma	MSE.mu	MSE.sigma
ML	0.0854	0.1023	0.0120	0.0169
AFE	0.0235	0.0387	0.0008	0.0023
BLUE	0.1190	0.6363	0.0221	0.6382
LME	0.0235	0.0557	0.0041	0.0048

Table 4.6: Absolute biases and mse of MLE, AFE, BLUE and LM estimators corresponding to the Weibull (2.5) model for sample sizes: $N = 100$, with M=2000 replications.

$N = 50, \alpha = 3$	**Weibull Distribution**			
Estimators	ABIAS.mu	ABIAS.sigma	MSE.mu	MSE.sigma
ML	$failure$	$failure$	$failure$	$failure$
AFE	0.0474	0.0673	0.0035	0.0068
BLUE	0.2184	0.7169	0.0747	0.8115
LME	0.0474	0.0673	0.0088	0.0095

Table 4.7: Absolute biases and mse of MLE, AFE, BLUE and LM estimators corresponding to the Weibull (3) model for sample sizes: $N = 50$, with M=2000 replications.

$N = 100, \alpha = 3$	**Weibull Distribution**			
Estimators	ABIAS.mu	ABIAS.sigma	MSE.mu	MSE.sigma
ML	0.1170	0.1293	0.0238	0.0288
AFE	0.0351	0.0450	0.0019	0.0031
BLUE	0.1597	0.6449	0.0398	0.6563
LME	0.0351	0.0450	0.0044	0.0047

Table 4.8: Absolute biases and mse of MLE, AFE, BLUE and LM estimators corresponding to the Weibull (3) model for sample sizes: $N = 100$, with M=2000 replications.

$N = 50$	**Gumbel Distribution**			
Estimators	ABIAS.mu	ABIAS.sigma	MSE.mu	MSE.sigma
ML	0.0089	0.0127	0.0217	0.0132
AFE	0.0343	0.1101	0.0233	0.0227
BLUE	0.1369	0.0001	0.0292	0.0155
LME	0.0147	0.1841	0.0226	0.0447

Table 4.9: Absolute biases and mse of MLE, AFE, BLUE and LM estimators corresponding to the gumbel model for sample sizes: $N = 50$, with M=2000 replications.

$N = 100$	**Gumbel Distribution**			
Estimators	ABIAS.mu	ABIAS.sigma	MSE.mu	MSE.sigma
ML	0.0032	0.0093	0.0109	0.0061
AFE	0.0191	0.0684	0.0114	0.0104
BLUE	0.1013	0.0072	0.0161	0.0731
LME	0.0082	0.1748	0.0112	0.0363

Table 4.10: Absolute biases and mse of MLE, AFE, BLUE and LM estimators corresponding to the gumbel model for sample sizes: $N = 100$, with M=2000 replications.

$N = 50$	**Lognormal Distribution**			
Estimators	ABIAS.mu	ABIAS.sigma	MSE.mu	MSE.sigma
ML	0.0007	0.0163	0.0199	0.0103
AFE	0.0007	0.2929	0.0199	0.1498
BLUE	0.0029	0.0120	0.0142	0.6241
LME	0.0007	0.0407	0.0199	0.0755

Table 4.11: Absolute biases and mse of MLE, AFE, BLUE and LM estimators corresponding to the Lognormal model for sample sizes: $N = 50$, with M=2000 replications.

$N = 100$	**Lognormal Distribution**			
Estimators	ABIAS.mu	ABIAS.sigma	MSE.mu	MSE.sigma
ML	0.0032	0.0112	0.0098	0.0053
AFE	0.0032	0.0563	0.0098	0.0079
BLUE	0.0017	0.0224	0.0082	0.6859
LME	0.0032	0.0291	0.0098	0.0370

Table 4.12: Absolute biases and mse of MLE, AFE, BLUE and LM estimators corresponding to the Lognormal model for sample sizes: $N = 100$, with M=2000 replications.

extraDistr: Additional Univariate and Multivariate Distributions.

pracma: Practical Numerical Math Functions.

univariateML: Maximum likelihood estimation for Univariate Densities.

ForestFit: Statistical Modelling for Plant Size Distribution.

FAdist: Distribution that are Sometimes Used in Hydrology.

4.2 Real data applications

In this part we present applications to two real data set:

- The "USAccDeaths" (stated in "datasets" package) presents the Accidental Deaths data in the US 1973-1978. The source: P. J. Brockwell and R. A. Davis (1991) Time Series: Theory and Methods. Springer, New York.
- The "nidd.annual" (stated in "evir" package), are which presents the annual maximal levels of the River Nidd in Yorkshire. The River Nidd is a tributary of the River Ouse in the English county of North Yorkshire.

The lengths of the two data sets are respectively 72 and 35.

Our task is to choose among the four probability models, logistic, normal,lognormal and gumbel, which better fits our data. In the first step, we choose the asymptotic

Distributions	**location**	**scale**	**cvm test p-value**
logistic	8871.9	546.56	0.3292
normal	8788.7	896.25	0.7933
gumbel	8294.1	1223.2	0.0004
lognormal	9.0754	0.1017	0.9105

Table 4.13: Fitting the (location-scale) normal, lognormal, logistic and Gumbel models to USAccDeathes data using the Efficient estimation method.

efficient method to estimate the location-scale parameters of each models. In the second step, we apply the Cramer-von mise goodness of fit (goft) test to the selected model. The best model adjusting the data is the one that corresponds to the largest "p-value". The results are summarized in two Tables 4.13 – 4.14. From Table 4.13, although the logistic and normal models provide p-values greater than 0.05, but the lognomal model is the best one to fit the USAccDeaths data. However, for a p-vlaue 0.0004, the gumbel it is far from being an adequate model for this data set. Table 4.13 indicates that all the models well fit the nidd.annual data, but the gumbel one is the best.

It worth mentioning that the choice of the asymptotic efficient estimation method is subjective, that is one may apply the other estimation methods and compare which provide greater p-values. The Weibull (α) is not considered in our study, since the Asymptotic efficient estimation method, presented above, focus only on the location-scale parameters that is the other parameter as the shape one α is out of our task. We also avoided the maximum estimation method because this one presents bugs when one deals with the small sample sizes.

4.2.1 The packages used in our real data program

fitdistrplus: Help to Fit of a Parametric Distribution to Non-Censored or Censored Data

Distributions	**location**	**scale**	**cvm test p-value**
logistic	133.12	33.338	0.3646
normal	136.66	51.854	0.2171
gumbel	110.36	47.481	0.5607
lognormal	4.8297	0.3706	0.3463

Table 4.14: Fitting the (location-scale) normal, lognormal, logistic and Gumbel models to nidd.annual data using the Efficient estimation method.

extremeStat: Extreme Value Statistics and Quantile Estimation

goftest: Classical Goodness of Fit Tests for Univarite Distributions

weibullness: Goodness of Fit Test for weibull Distribution(weibullness)

survival: survival Analysis

ismev: An Introduction to statical Modeling of Extreme Values

FeedbackTS: Analysis of Feedback in Time Series

Remark 4.2.1 *All programs and codes for simulation and real data they are gathered in the Appendix below.*

Conclusion

L-statistics have a wide range applications in statistical inference. They are a class of statistics used to estimate population parameters or make inferences about populations based on sample data. These statistics are versatile and applicable in various areas of statistical analysis, including hypothesis testing, confidence interval estimation and model fitting. One of the key advantages of L-statistics is their robustness against outliers and non-normality in the data, making them suitable for analyzing real-world datasets where these issues are common. They also offer flexibility in their formulation, allowing statisticians to tailor them to specific research questions or data characteristics. In hypothesis testing, L-statistics are often used to construct test statistics that help determine whether observed differences between groups or variables are statistically significant. Similarly, in confidence interval estimation, L-statistics provide a framework for estimating the range within which a population parameter is likely to fall. Moreover, in regression analysis and model fitting, L-statistics are utilized to assess the goodness-of-fit of models, identify influential observations, and evaluate the overall performance of the model.

Overall, the broad applicability and robustness of L-statistics make them indispensable tools in inferential statistics, enabling researchers and analysts to draw reliable conclusions from sample data about the populations they represent.

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Chapter 5

Appendix

5.1 Appendix A: R Software

5.1.1 What is the R language?

• The R language is a programming language and mathematical environment used for data processing. It allows you to perform both simple and complex statistical analyzes such as linear or non-linear models, hypothesis testing, time series modeling, classification, etc. It also has many very useful, professional-quality graphics functions.

• R was created by Ross Ihaka and Robert Gentleman in 1993 at the University of Auckland, New Zealand, and is now developed by the R Development Core Team.

The origin of the name of the language comes, on the one hand, from the initials of the first names of the two authors (Ross Ihaka and Robert Gentleman) and, on the other hand, from a play on words on the name of the language S to which it is related.

5.2 Appendix B: Abbreviations and Notations

The various abbreviations and notations used throughout this brief are explained below:

μ	:	Expactation
σ	:	Covariance
ψ	:	Digamma function
ϕ	:	Lieblein′s ϕ-function
Γ	:	Gamma function
ML	:	Maximum likelihood
LME	:	L-moment
AFE	:	Efficient
$BLUE$	:	Best linear unbaised etimator
$ABIAS$	:	Absolute bias
MSE	:	Mean square error

5.3 Appendix C

5.3.1 Asymptotic proprities

In the context of statistical theory and econometrics, understanding the asymptotic properties of linear functions is crucial for analyzing the behavior of estimators and statistical methods as sample sizes become large. Let's discuss some general concepts related to the asymptotic properties of linear functions.

The asymptotic normality of L_n has been established either by putting conditions on the weights or the weight function, or by assuming F to be close to an $Exp(1)$ cdf. See Serfling(1980) [8] and Shorack and Wellner (1986) [10] discuss the asymptotic normality of L_n under various regularity conditions. We will present one result which assumes J to be sufficiently smooth with very little restriction on F. It is a simplified version of a result due to Stegler (1974) [11]. First, let us define

$$\mu(J,F) = \int_{-\infty}^{+\infty} xJ(F(x))dF(x) = \int_{-\infty}^{+\infty} J(u)Q(u)du \tag{5.1}$$

and

$$\begin{aligned}\sigma^2(J,F) &= 2 \iint\limits_{-\infty<x<y<+\infty} J(F(x))J(F(y))F(x)\left(1-F(y)\right)dxdy \\ &= 2 \iint\limits_{0<u_1<u_2<1} J(u_1)J(u_2)u_1\left(1-u_2\right)dQ\left(u_1\right)dQ\left(u_2\right).\end{aligned} \tag{5.2}$$

Theorem 5.3.1 *Assume* $\mathbf{E}\left|X\right|^3$ *is finite where* X *represents the population random variable with cdf* F*. Let the weight function* J *be bounded and be continuous at every discontinuity point of* F^{-1}*. Further, suppose that*

$$\left|J(u)-J(v)\right| \leq K\left|u-v\right|^{\delta+\frac{1}{2}}$$

for some constant K *and* $\delta>0$, $0<u<v<1$, *except perhaps for a finite number of values of* u *and* v*. Then, the following results hold*

$$\begin{gathered}\sqrt{n}\left(\mathbf{E}\left[L_n\right]-\mu\left(J,F\right)\right)\to 0, \text{ as } n\to\infty, \\ n\mathbf{Var}\left(L_n\right)\to\sigma^2\left(J,F\right), \text{ as } n\to\infty, \\ \sqrt{n}\left(L_n-\mu\left(J,F\right)\right)\xrightarrow{D}\mathcal{N}\left(0,\sigma^2\left(J,F\right)\right), \text{ as } n\to\infty,\end{gathered}$$

where $\mu\left(J,F\right)$ *and* $\sigma^2\left(J,F\right)$ *are givin in* (5.1) *and* (5.2) *respectively.*

5.3.2 Simulation codes

ABIAS and MSE of Normal distribution for N=50

```
rm(list=ls())
set.seed(50.1);
N=50
M=2000 # replications number
F=function(x){pnorm(x)} # the distribution function
f=function(x){dnorm(x)} # the density function
beta=read.table("cov-normale-1-50")
colnames(beta)<-NULL
rownames(beta)<-NULL
beta=as.matrix(beta)
sbeta=solve(beta) # Inverse of the covariance matrix
e=rep(NA,N)
for(i in 1:N){
H<-function (x){
c1=factorial(N)
c2=factorial(i-1)
c3=factorial(N-i)
c=c1/(c2*c3)
c*x*(F(x)^i)*((1-F(x))^(N-i))*f(x)}
e[i]=integrate(H,-3,3)$value}
m=0; s=1 # Theoretical value of location-scale parameters
M1=rep(NA,M);M2=rep(NA,M);M3=rep(NA,M);M4=rep(NA,M)
S1=rep(NA,M);S2=rep(NA,M);S3=rep(NA,M);S4=rep(NA,M)
for (j in 1:M ){
```

```
x=rnorm(N,mean = m,sd=s)

###MLE estimators:

library(maxLik)

L=function(param) {

mu1=param[1]

sigma1=param[2]

sum(dnorm(x, mean =mu1, sd=sigma1, log = TRUE))

}

mle <- maxLik(logLik = L, start = c(mu1 = 0, sigma1= 1))

M1[j]=mle$estimate[1]; S1[j]=mle$estimate[2]

##Asymptotically efficient:

X=sort(x)

W1=rep(NA,N)

W2=rep(NA,N)

J1=function(u){1}

J2=function(u){qnorm(u)}

for (i in 1:N){

W1[i]=J1(i/(N+1))*X[i]

W2[i]=J2(i/(N+1))*X[i]

}

M2[j]=mean(W1);S2[j]=mean(W2)

#### BLUE:

I=rep(1,N)

R1=sbeta%*%(I%*%t(e)-e%*%t(I))%*%sbeta
```

```
R2=(t(e)%*%sbeta%*%e)%*%(t(I)%*%sbeta%*%I)-
(t(e)%*%sbeta%*%I)%*%(t(e)%*%sbeta%*%I)
M3[j]=-t(e)%*%R1%*%X/R2
S3[j]=t(I)%*%R1%*%X/R2
#### L-moments:
L1=function(u){qnorm(u)}
L2=function(u){(2*u-1)*qnorm(u)}
a1=integrate(L1,0,1)$value
a2=integrate(L2,0,1)$value
K1=rep(NA,N)
K2=rep(NA,N)
for (i in 1:N){
K1[i]=X[i]
K2[i]=(2*i/(N+1)-1)*X[i]}
M4[j]=(mean(K2)*a1-mean(K1)*a2)/(a1-a2)
S4[j]=(mean(K1)-mean(K2))/(a1-a2)
}
####### Biases of the fours estimators
ABIAS.mu.1=abs(mean(M1-m)); MSE.mu.1=mean((M1-m)^2)
ABIAS.mu.2=abs(mean(M2-m)); MSE.mu.2=mean((M2-m)^2)
ABIAS.mu.3=abs(mean(M3-m)); MSE.mu.3=mean((M3-m)^2)
ABIAS.mu.4=abs(mean(M4-m));MSE.mu.4=mean((M4-m)^2)
####### Mean-squared-errors of the fours estimators
ABIAS.sigma.1=abs(mean(S1-s)); MSE.sigma.1=mean((S1-s)^2)
```

```
ABIAS.sigma.2=abs(mean(S2-s)); MSE.sigma.2=mean((S2-s)^2)

ABIAS.sigma.3=abs(mean(S3-s)); MSE.sigma.3=mean((S3-s)^2)

ABIAS.sigma.4=abs(mean(S4-s));MSE.sigma.4=mean((S4-s)^2)

####### results for the BIAS and RMSE for mu

res=data.frame(

Estimators=c("ML","AFE","BLUE","LME"),

ABIAS.mu=c(ABIAS.mu.1,ABIAS.mu.2,ABIAS.mu.3,ABIAS.mu.4),

ABIAS.sigma=c(ABIAS.sigma.1,ABIAS.sigma.2,ABIAS.sigma.3,ABIAS.sigma.4),

MSE.mu=c(MSE.mu.1,MSE.mu.2,MSE.mu.3,MSE.mu.4),

MSE.sigma=c(MSE.sigma.1,MSE.sigma.2,MSE.sigma.3,MSE.sigma.4)

)

print(res)
```

ABIAS and MSE of Normal distribution for N=100

```
rm(list=ls())

set.seed(100.1);

N=100

M=2000 # replications number

F=function(x){pnorm(x)} # the distribution function

f=function(x){dnorm(x)} # the density function

beta=read.table("cov-normale-1-100")

colnames(beta)<-NULL

rownames(beta)<-NULL

beta=as.matrix(beta)

sbeta=solve(beta) # Inverse of the covariance matrix
```

```
e=rep(NA,N)
for(i in 1:N){
H<-function (x){c1=factorial(N)
c2=factorial(i-1)
c3=factorial(N-i)c=c1/(c2*c3)
c*x*(F(x)^i)*((1-F(x))^(N-i))*f(x)}
e[i]=integrate(H,-3,3)$value}
m=0; s=1 # Theoretical value of location-scale parameters
M1=rep(NA,M);M2=rep(NA,M);M3=rep(NA,M);M4=rep(NA,M)
S1=rep(NA,M);S2=rep(NA,M);S3=rep(NA,M);S4=rep(NA,M)
for (j in 1:M ){
x=rnorm(N,mean = m,sd=s)
###MLE estimators:
library(maxLik)
L=function(param) {
mu1=param[1]
sigma1=param[2]
sum(dnorm(x, mean =mu1, sd=sigma1, log = TRUE))
}
mle <- maxLik(logLik = L, start = c(mu1 = 0, sigma1= 1))
M1[j]=mle$estimate[1]; S1[j]=mle$estimate[2]
##Asymptotically efficient:
X=sort(x)
W1=rep(NA,N)
```

```
W2=rep(NA,N)
J1=function(u){1}
J2=function(u){qnorm(u)}
for (i in 1:N){
W1[i]=J1(i/(N+1))*X[i]
W2[i]=J2(i/(N+1))*X[i]
}
M2[j]=mean(W1);S2[j]=mean(W2)
#### BLUE:
I=rep(1,N)
R1=sbeta%*%(I%*%t(e)-e%*%t(I))%*%sbeta
R2=(t(e)%*%sbeta%*%e)%*%(t(I)%*%sbeta%*%I)-
(t(e)%*%sbeta%*%I)%*%(t(e)%*%sbeta%*%I)
M3[j]=-t(e)%*%R1%*%X/R2
S3[j]=t(I)%*%R1%*%X/R2
#### L-moments:
L1=function(u){qnorm(u)}
L2=function(u){(2*u-1)*qnorm(u)}
a1=integrate(L1,0,1)$value
a2=integrate(L2,0,1)$value
K1=rep(NA,N)
K2=rep(NA,N)
for (i in 1:N){
K1[i]=X[i]
```

```
K2[i]=(2*i/(N+1)-1)*X[i]}
M4[j]=(mean(K2)*a1-mean(K1)*a2)/(a1-a2)
S4[j]=(mean(K1)-mean(K2))/(a1-a2)
}
####### Biases of the fours estimators
ABIAS.mu.1=abs(mean(M1-m)); MSE.mu.1=mean((M1-m)^2)
ABIAS.mu.2=abs(mean(M2-m)); MSE.mu.2=mean((M2-m)^2)
ABIAS.mu.3=abs(mean(M3-m)); MSE.mu.3=mean((M3-m)^2)
ABIAS.mu.4=abs(mean(M4-m));MSE.mu.4=mean((M4-m)^2)
####### Mean-squared-errors of the fours estimators
ABIAS.sigma.1=abs(mean(S1-s)); MSE.sigma.1=mean((S1-s)^2)
ABIAS.sigma.2=abs(mean(S2-s)); MSE.sigma.2=mean((S2-s)^2)
ABIAS.sigma.3=abs(mean(S3-s)); MSE.sigma.3=mean((S3-s)^2)
ABIAS.sigma.4=abs(mean(S4-s));MSE.sigma.4=mean((S4-s)^2)
####### results for the BIAS and RMSE for mu
res=data.frame(
Estimators=c("ML","AFE","BLUE","LME"),
ABIAS.mu=c(ABIAS.mu.1,ABIAS.mu.2,ABIAS.mu.3,ABIAS.mu.4),
ABIAS.sigma=c(ABIAS.sigma.1,ABIAS.sigma.2,ABIAS.sigma.3,ABIAS.sigma.4),
MSE.mu=c(MSE.mu.1,MSE.mu.2,MSE.mu.3,MSE.mu.4),
MSE.sigma=c(MSE.sigma.1,MSE.sigma.2,MSE.sigma.3,MSE.sigma.4)
)
print(res)
```

A**BIAS and MSE of Logistic distribution for N=50**

```
rm(list=ls())
set.seed(50.2);
N=50
library(extraDistr)
F=function(x){plogis(x)} # the distribution function
f=function(x){dlogis(x)} # the density function
beta=read.table("cov-logistic-50")
colnames(beta)<-NULL
rownames(beta)<-NULL
beta=as.matrix(beta)
sbeta=solve(beta) # Inverse of the covariance matrix
e=rep(NA,N)
for(i in 1:N){
H<-function (x){
c1=factorial(N)
c2=factorial(i-1)
c3=factorial(N-i)
c=c1/(c2*c3)
c*x*(F(x)^i)*((1-F(x))^(N-i))*f(x)}
e[i]=integrate(H,-3,3)$value}
m=0; s=1 # Theoretical value of location-scale parameters
M=2000 # replications number
M1=rep(NA,M);M2=rep(NA,M);M3=rep(NA,M);M4=rep(NA,M)
S1=rep(NA,M);S2=rep(NA,M);S3=rep(NA,M);S4=rep(NA,M)
```

```
for (j in 1:M ){
x=rlogis(N,location = m,scale=s)
###MLE estimators:
library(maxLik)
L=function(param) {
mu1=param[1]
sigma1=param[2]
sum(dlogis(x, location =mu1, scale=sigma1, log = TRUE))
}
mle <- maxLik(logLik = L, start = c(mu1 = 0, sigma1= 1))
M1[j]=mle$estimate[1]; S1[j]=mle$estimate[2]
##Asymptotically efficient:
X=sort(x)
W1=rep(NA,N)
W2=rep(NA,N)
J1=function(u){6*u*(1-u)}
J2=function(u){(9*(pi^2+3)^(-1))*(2*u-1+2*u*(1-u)*(log(u)-log(1-u)))}
for (i in 1:N){
W1[i]=J1(i/(N+1))*X[i]
W2[i]=J2(i/(N+1))*X[i]
}
M2[j]=mean(W1);S2[j]=mean(W2)
#### BLUE:
I=rep(1,N)
```

```
R1=sbeta%*%(I%*%t(e)-e%*%t(I))%*%sbeta
R2=(t(e)%*%sbeta%*%e)%*%(t(I)%*%sbeta%*%I)-
(t(e)%*%sbeta%*%I)%*%(t(e)%*%sbeta%*%I)M3[j]=-t(e)%*%R1%*%X/R2
S3[j]=t(I)%*%R1%*%X/R2
#### L-moments:
L1=function(u){qlogis(u)}
L2=function(u){(2*u-1)*qlogis(u)}
a1=integrate(L1,0,1)$value
a2=integrate(L2,0,1)$value
K=rep(NA,N)
K1=rep(NA,N)
K2=rep(NA,N)
for (i in 1:N){
K1[i]=X[i]
K2[i]=(2*i/(N+1)-1)*X[i]}
M4[j]=(mean(K2)*a1-mean(K1)*a2)/(a1-a2)
S4[j]=(mean(K1)-mean(K2))/(a1-a2)
}
####### Biases of the fours estimators
ABIAS.mu.1=abs(mean(M1-m)); MSE.mu.1=mean((M1-m)^2)
ABIAS.mu.2=abs(mean(M2-m)); MSE.mu.2=mean((M2-m)^2)
ABIAS.mu.3=abs(mean(M3-m)); MSE.mu.3=mean((M3-m)^2)
ABIAS.mu.4=abs(mean(M4-m));MSE.mu.4=mean((M4-m)^2)
####### Mean-squared-errors of the fours estimators
```

```
ABIAS.sigma.1=abs(mean(S1-s)); MSE.sigma.1=mean((S1-s)^2)
ABIAS.sigma.2=abs(mean(S2-s)); MSE.sigma.2=mean((S2-s)^2)
ABIAS.sigma.3=abs(mean(S3-s)); MSE.sigma.3=mean((S3-s)^2)
ABIAS.sigma.4=abs(mean(S4-s));MSE.sigma.4=mean((S4-s)^2)
####### results for the BIAS and RMSE for mu
res=data.frame(
Estimators=c("ML","AFE","BLUE","LME"),
ABIAS.mu=c(ABIAS.mu.1,ABIAS.mu.2,ABIAS.mu.3,ABIAS.mu.4),
ABIAS.sigma=c(ABIAS.sigma.1,ABIAS.sigma.2,ABIAS.sigma.3,ABIAS.sigma.4),
MSE.mu=c(MSE.mu.1,MSE.mu.2,MSE.mu.3,MSE.mu.4),
MSE.sigma=c(MSE.sigma.1,MSE.sigma.2,MSE.sigma.3,MSE.sigma.4)
)
print(res)
```

ABIAS and MSE of Logistic distribution for N=100

```
rm(list=ls())
set.seed(100.2);
N=100
library(extraDistr)
F=function(x){plogis(x)} # the distribution function
f=function(x){dlogis(x)} # the density functionbeta=read.table("cov-logistic-100")
colnames(beta)<-NULL
rownames(beta)<-NULL
beta=as.matrix(beta)
sbeta=solve(beta) # Inverse of the covariance matrix
```

```
e=rep(NA,N)
for(i in 1:N){
H<-function (x){
c1=factorial(N)
c2=factorial(i-1)
c3=factorial(N-i)
c=c1/(c2*c3)
c*x*(F(x)^i)*((1-F(x))^(N-i))*f(x)}
e[i]=integrate(H,-3,3)$value}
m=0; s=1 # Theoretical value of location-scale parameters
M=2000 # replications number
M1=rep(NA,M);M2=rep(NA,M);M3=rep(NA,M);M4=rep(NA,M)
S1=rep(NA,M);S2=rep(NA,M);S3=rep(NA,M);S4=rep(NA,M)
for (j in 1:M ){
x=rlogis(N,location = m,scale=s)
###MLE estimators:
library(maxLik)
L=function(param) {
mu1=param[1]
sigma1=param[2]
sum(dlogis(x, location =mu1, scale=sigma1, log = TRUE))
}
mle <- maxLik(logLik = L, start = c(mu1 = 0, sigma1= 1))
M1[j]=mle$estimate[1]; S1[j]=mle$estimate[2]
```

```
##Asymptotically efficient:
X=sort(x)
W1=rep(NA,N)
W2=rep(NA,N)
J1=function(u){6*u*(1-u)}
J2=function(u){(9*(pi^2+3)^(-1))*(2*u-1+2*u*(1-u)*(log(u)-log(1-u)))}
for (i in 1:N){
W1[i]=J1(i/(N+1))*X[i]
W2[i]=J2(i/(N+1))*X[i]
}
M2[j]=mean(W1);S2[j]=mean(W2)
#### BLUE:
I=rep(1,N)R1=sbeta%*%(I%*%t(e)-e%*%t(I))%*%sbeta
R2=(t(e)%*%sbeta%*%e)%*%(t(I)%*%sbeta%*%I)-
(t(e)%*%sbeta%*%I)%*%
(t(e)%*%sbeta%*%I)
M3[j]=-t(e)%*%R1%*%X/R2
S3[j]=t(I)%*%R1%*%X/R2
#### L-moments:
L1=function(u){qlogis(u)}
L2=function(u){(2*u-1)*qlogis(u)}
a1=integrate(L1,0,1)$value
a2=integrate(L2,0,1)$value
K=rep(NA,N)
```

```
K1=rep(NA,N)
K2=rep(NA,N)
for (i in 1:N){
K1[i]=X[i]
K2[i]=(2*i/(N+1)-1)*X[i]}
M4[j]=(mean(K2)*a1-mean(K1)*a2)/(a1-a2)
S4[j]=(mean(K1)-mean(K2))/(a1-a2)
}
####### Biases of the fours estimators
ABIAS.mu.1=abs(mean(M1-m)); MSE.mu.1=mean((M1-m)^2)
ABIAS.mu.2=abs(mean(M2-m)); MSE.mu.2=mean((M2-m)^2)
ABIAS.mu.3=abs(mean(M3-m)); MSE.mu.3=mean((M3-m)^2)
ABIAS.mu.4=abs(mean(M4-m));MSE.mu.4=mean((M4-m)^2)
####### Mean-squared-errors of the fours estimators
ABIAS.sigma.1=abs(mean(S1-s)); MSE.sigma.1=mean((S1-s)^2)
ABIAS.sigma.2=abs(mean(S2-s)); MSE.sigma.2=mean((S2-s)^2)
ABIAS.sigma.3=abs(mean(S3-s)); MSE.sigma.3=mean((S3-s)^2)
ABIAS.sigma.4=abs(mean(S4-s));MSE.sigma.4=mean((S4-s)^2)
####### results for the BIAS and RMSE for mu
res=data.frame(
Estimators=c("ML","AFE","BLUE","LME"),
ABIAS.mu=c(ABIAS.mu.1,ABIAS.mu.2,ABIAS.mu.3,ABIAS.mu.4),
ABIAS.sigma=c(ABIAS.sigma.1,ABIAS.sigma.2,ABIAS.sigma.3,ABIAS.sigma.4),
MSE.mu=c(MSE.mu.1,MSE.mu.2,MSE.mu.3,MSE.mu.4),
```

```
MSE.sigma=c(MSE.sigma.1,MSE.sigma.2,MSE.sigma.3,MSE.sigma.4)
)
print(res)
```

ABIAS and MSE of Gumbel distribution for N=50

```
rm(list=ls())
set.seed(50.3)
library(extraDistr)
library(pracma)
N=50; M=2000 ####
m=0; s=1 # Theoretical value of location-scale parameters
F=function(x){pgumbel(x)} # the distribution function
f=function(x){dgumbel(x)} # the density function
######### the expected value of the i-th order statistics#####
e=rep(NA,N)
for(i in 1:N){
H<-function (x){
c1=factorial(N)
c2=factorial(i-1)
c3=factorial(N-i)c=c1/(c2*c3)
c*x*(F(x)^(i-1))*((1-F(x))^(N-i))*f(x)}
e[i]=quadinf(H,-Inf,Inf)$Q}
###### calling the covariance matrix
beta=read.table("covgumbel_50M100")
rownames(beta)<-NULL
```

```
colnames(beta)<-NULL
beta=as.matrix(beta)
sbeta=solve(beta,tol=1e-100) #### inverse matrix
####################MLE estimators
M1=rep(NA,M);M2=rep(NA,M);M3=rep(NA,M);M4=rep(NA,M)
S1=rep(NA,M);S2=rep(NA,M);S3=rep(NA,M);S4=rep(NA,M)
for (j in 1:M){
x=rgumbel(N,mu=m,sigma=s)
###MLE estimators:
library(univariateML)
mle=mlgumbel(x)
M1[j]=mle[[1]]
S1[j]=mle[[2]]
###########Asymptotically efficient estimators
x=rgumbel(N)
X=sort(x)
J1=function(u){-0.85168*log(u)+0.25702*log(u)*log(-log (u))+0.25702}
J2=function(u){0.3509*log(u)+0.60792*log(u)*log(-log (u))+0.60792}
Z1=rep(NA,N);Z2=rep(NA,N)
for (i in 1:N){
Z1[i]=J1(i/(N+1))*X[i] ; Z2[i]=J2(i/(N+1))*X[i]
}
M2[j]=mean(Z1);S2[j]=mean(Z2)
#########BLU-estimators
```

```
I=rep(1,N)

R1=sbeta%*%(I%*%t(e)-e%*%t(I))%*%sbeta

R2=(t(e)%*%sbeta%*%e)%*%(t(I)%*%sbeta%*%I)-

(t(e)%*%sbeta%*%I)%*%(t(e)%*%sbeta%*%I)

M3[j]=-t(e)%*%R1%*%X/R2

S3[j]=t(I)%*%R1%*%X/R2

######################L-moments estimators

u=rep(NA,N);v=rep(NA,N)

K1=rep(NA,N);K2=rep(NA,N)

for (i in 1:N){

u[i]=X[i]

v[i]=0.83276*(2*i-N-1)*u[i]/N

K1[i]=u[i]-v[i]

K2[i]=1.4427*v[i]}

M4[j]=mean(K1)

S4[j]=mean(K2)}

####### Absolute Biases of the fours estimators

BIAS.mu.1=abs(mean(M1-m)); MSE.mu.1=mean((M1-m)^2)

BIAS.mu.2=abs(mean(M2-m)); MSE.mu.2=mean((M2-m)^2)

BIAS.mu.3=mean(abs(M3-m)); MSE.mu.3=mean((M3-m)^2)

BIAS.mu.4=abs(mean(M4-m));MSE.mu.4=mean((M4-m)^2)

####### Mean-squared-errors of the fours estimators

BIAS.sigma.1=abs(mean(S1-s)); MSE.sigma.1=mean((S1-s)^2)

BIAS.sigma.2=abs(mean(S2-s)); MSE.sigma.2=mean((S2-s)^2)
```

```
BIAS.sigma.3=abs(mean(S3-s)); MSE.sigma.3=mean((S3-s)^2)
BIAS.sigma.4=abs(mean(S4-s));MSE.sigma.4=mean((S4-s)^2)
####### results for the BIAS and RMSE for mu
res=data.frame(
Estimators=c("ML","AFE","BLUE","LME"),
MAE.mu=c(BIAS.mu.1,BIAS.mu.2,BIAS.mu.3,BIAS.mu.4),
MAE.sigma=c(BIAS.sigma.1,BIAS.sigma.2,BIAS.sigma.3,BIAS.sigma.4),
MSE.mu=c(MSE.mu.1,MSE.mu.2,MSE.mu.3,MSE.mu.4),
MSE.sigma=c(MSE.sigma.1,MSE.sigma.2,MSE.sigma.3,MSE.sigma.4)
)
print(res)
```

ABIAS and MSE of Gumbel distribution for N=100

```
rm(list=ls())
set.seed(100.3)
library(extraDistr)
library(pracma)
N=100; M=2000 ####
m=0; s=1 # Theoretical value of location-scale parameters
F=function(x){pgumbel(x)} # the distribution function
f=function(x){dgumbel(x)} # the density function
######### the expected value of the i-th order statistics#######
e=rep(NA,N)
for(i in 1:N){
H<-function (x){
```

```
c1=factorial(N)
c2=factorial(i-1)
c3=factorial(N-i)
c=c1/(c2*c3)
c*x*(F(x)^(i-1))*((1-F(x))^(N-i))*f(x)}
e[i]=quadinf(H,-Inf,Inf)$Q}
###### calling the covariance matrixbeta=read.table("covgumbel_100M100")
rownames(beta)<-NULL
colnames(beta)<-NULL
beta=as.matrix(beta)
sbeta=solve(beta,tol=1e-100) #### inverse matrix
###################MLE estimators
M1=rep(NA,M);M2=rep(NA,M);M3=rep(NA,M);M4=rep(NA,M)
S1=rep(NA,M);S2=rep(NA,M);S3=rep(NA,M);S4=rep(NA,M)
for (j in 1:M){
x=rgumbel(N,mu=m,sigma=s)
###MLE estimators:
library(univariateML)
mle=mlgumbel(x)
M1[j]=mle[[1]]
S1[j]=mle[[2]]
##########Asymptotically efficient estimators
x=rgumbel(N)
X=sort(x)
```

```
J1=function(u){-0.85168*log(u)+0.25702*log(u)*log(-log (u))+0.25702}

J2=function(u){0.3509*log(u)+0.60792*log(u)*log(-log (u))+0.60792}

Z1=rep(NA,N);Z2=rep(NA,N)

for (i in 1:N){

Z1[i]=J1(i/(N+1))*X[i] ; Z2[i]=J2(i/(N+1))*X[i]

}

M2[j]=mean(Z1);S2[j]=mean(Z2)

########BLU-estimators

I=rep(1,N)

R1=sbeta%*%(I%*%t(e)-e%*%t(I))%*%sbetaR2=(t(e)%*%sbeta%*%e)%*%(t(I)%*%sbeta%*%

(t(e)%*%sbeta%*%I)%*%(t(e)%*%sbeta%*%I)

M3[j]=-t(e)%*%R1%*%X/R2

S3[j]=t(I)%*%R1%*%X/R2

######################L-moments estimators

u=rep(NA,N);v=rep(NA,N)

K1=rep(NA,N);K2=rep(NA,N)

for (i in 1:N){

u[i]=X[i]

v[i]=0.83276*(2*i-N-1)*u[i]/N

K1[i]=u[i]-v[i]

K2[i]=1.4427*v[i]}

M4[j]=mean(K1)

S4[j]=mean(K2)
```

```
}
####### Biases of the fours estimators
BIAS.mu.1=abs(mean(M1-m)); MSE.mu.1=mean((M1-m)^2)
BIAS.mu.2=abs(mean(M2-m)); MSE.mu.2=mean((M2-m)^2)
BIAS.mu.3=mean(abs(M3-m)); MSE.mu.3=mean((M3-m)^2)
BIAS.mu.4=abs(mean(M4-m));MSE.mu.4=mean((M4-m)^2)
####### Mean-squared-errors of the fours estimators
BIAS.sigma.1=abs(mean(S1-s)); MSE.sigma.1=mean((S1-s)^2)
BIAS.sigma.2=abs(mean(S2-s)); MSE.sigma.2=mean((S2-s)^2)
BIAS.sigma.3=abs(mean(S3-s)); MSE.sigma.3=mean((S3-s)^2)
BIAS.sigma.4=abs(mean(S4-s));MSE.sigma.4=mean((S4-s)^2)
####### results for the BIAS and RMSE for mu
res=data.frame(
Estimators=c("ML","AFE","BLUE","LME"),
MAE.mu=c(BIAS.mu.1,BIAS.mu.2,BIAS.mu.3,BIAS.mu.4),
MAE.sigma=c(BIAS.sigma.1,BIAS.sigma.2,BIAS.sigma.3,BIAS.sigma.4),
MSE.mu=c(MSE.mu.1,MSE.mu.2,MSE.mu.3,MSE.mu.4),
MSE.sigma=c(MSE.sigma.1,MSE.sigma.2,MSE.sigma.3,MSE.sigma.4)
)
print(res)
```

ABIAS and MSE of Lognormal distribution for N=50

```
rm(list=ls())
set.seed(50.4)
library(extraDistr)
```

```
library(pracma)

N=50; M=2000 ####

m=0; s=1 # Theoretical value of location-scale parameters

F=function(x){plnorm(x)} # the distribution function

f=function(x){dlnorm(x)} # the density function

######### the expected value of the i-th order statistics#######

e=rep(NA,N)

for(i in 1:N){

H<-function (x){

c1=factorial(N)

c2=factorial(i-1)

c3=factorial(N-i)c=c1/(c2*c3)

c*x*(F(x)^(i-1))*((1-F(x))^(N-i))*f(x)}

e[i]=quadinf(H,-Inf,Inf)$Q}

###### calling the covariance matrix

beta=read.table("covlognormal_50M100")

rownames(beta)<-NULL

colnames(beta)<-NULL

beta=as.matrix(beta)

sbeta=solve(beta,tol=1e-100) #### inverse matrix

##################MLE estimators

M1=rep(NA,M);M2=rep(NA,M);M3=rep(NA,M);M4=rep(NA,M)

S1=rep(NA,M);S2=rep(NA,M);S3=rep(NA,M);S4=rep(NA,M)

for (j in 1:M){
```

```
x=rlnorm(N)
###MLE estimators:
library(univariateML)
mle=mllnorm(x)
M1[j]=mle[[1]]
S1[j]=mle[[2]]
##########Asymptotically efficient estimatorsX=sort(x)
W1=rep(NA,N)
W2=rep(NA,N)
J1=function(u){1}
J2=function(u){qlnorm(u)}
for (i in 1:N){
W1[i]=J1(i/(N+1))*log(X[i])
W2[i]=J2(i/(N+1))*log(X[i])
}
M2[j]=mean(W1);S2[j]=mean(W2)
########BLU-estimators
I=rep(1,N)
R1=sbeta%*%(I%*%t(e)-e%*%t(I))%*%sbeta
R2=(t(e)%*%sbeta%*%e)%*%(t(I)%*%sbeta%*%I)-
(t(e)%*%sbeta%*%I)%*%(t(e)%*%sbeta%*%I)
M3[j]=t(e)%*%R1%*%X/R2
S3[j]=t(I)%*%R1%*%X/R2
#### L-moments:
```

```
L1=function(u){qnorm(u)}
L2=function(u){(2*u-1)*qnorm(u)}
a1=integrate(L1,0,1)$value
a2=integrate(L2,0,1)$value
K1=rep(NA,N)
K2=rep(NA,N)
for (i in 1:N){
K1[i]=log(X[i])
K2[i]=(2*i/(N+1)-1)*log(X[i])}
M4[j]=(mean(K2)*a1-mean(K1)*a2)/(a1-a2)
S4[j]=(mean(K1)-mean(K2))/(a1-a2)
}
####### Absolute Biases of the fours estimators
ABIAS.mu.1=abs(mean(M1-m)); MSE.mu.1=mean((M1-m)^2)
ABIAS.mu.2=abs(mean(M2-m)); MSE.mu.2=mean((M2-m)^2)
ABIAS.mu.3=abs(mean(M3-m)); MSE.mu.3=mean((M3-m)^2)
ABIAS.mu.4=abs(mean(M4-m));MSE.mu.4=mean((M4-m)^2)
####### Mean-squared-errors of the fours estimators
ABIAS.sigma.1=abs(mean(S1-s)); MSE.sigma.1=mean((S1-s)^2)
ABIAS.sigma.2=abs(mean(S2-s)); MSE.sigma.2=mean((S2-s)^2)
ABIAS.sigma.3=abs(mean(S3-s)); MSE.sigma.3=mean((S3-s)^2)
ABIAS.sigma.4=abs(mean(S4-s));MSE.sigma.4=mean((S4-s)^2)
####### results for the BIAS and RMSE for mu
res=data.frame(
```

```
Estimators=c("ML","AFE","BLUE","LME"),
ABIAS.mu=c(ABIAS.mu.1,ABIAS.mu.2,ABIAS.mu.3,ABIAS.mu.4),
ABIAS.sigma=c(ABIAS.sigma.1,ABIAS.sigma.2,ABIAS.sigma.3,ABIAS.sigma.4),
MSE.mu=c(MSE.mu.1,MSE.mu.2,MSE.mu.3,MSE.mu.4),
MSE.sigma=c(MSE.sigma.1,MSE.sigma.2,MSE.sigma.3,MSE.sigma.4)
)
print(res)
```

ABIAS and MSE of Lognormal distribution for N=100

```
rm(list=ls())
set.seed(100.4)
library(extraDistr)
library(pracma)
N=100; M=2000 ####
m=0; s=1 # Theoretical value of location-scale parameters
F=function(x){plnorm(x)} # the distribution function
f=function(x){dlnorm(x)} # the density function
######### the expected value of the i-th order statistics#######
e=rep(NA,N)
for(i in 1:N){
H<-function (x){
c1=factorial(N)
c2=factorial(i-1)
c3=factorial(N-i)
c=c1/(c2*c3)
```

```
c*x*(F(x)^(i-1))*((1-F(x))^(N-i))*f(x)}
e[i]=quadinf(H,-Inf,Inf)$Q}
###### calling the covariance matrixbeta=read.table("covlognormal_100M100")
rownames(beta)<-NULL
colnames(beta)<-NULL
beta=as.matrix(beta)
sbeta=solve(beta,tol=1e-100) #### inverse matrix
###################MLE estimators
M1=rep(NA,M);M2=rep(NA,M);M3=rep(NA,M);M4=rep(NA,M)
S1=rep(NA,M);S2=rep(NA,M);S3=rep(NA,M);S4=rep(NA,M)
for (j in 1:M){
x=rlnorm(N)
###MLE estimators:
library(univariateML)
mle=mllnorm(x)
M1[j]=mle[[1]]
S1[j]=mle[[2]]
##########Asymptotically efficient estimatorsX=sort(x)
W1=rep(NA,N)W2=rep(NA,N)
J1=function(u){1}
J2=function(u){qlnorm(u)}
for (i in 1:N){
W1[i]=J1(i/(N+1))*log(X[i])
W2[i]=J2(i/(N+1))*log(X[i])
```

```
}
M2[j]=mean(W1);S2[j]=mean(W2)
########BLU-estimators
I=rep(1,N)
R1=sbeta%*%(I%*%t(e)-e%*%t(I))%*%sbeta
R2=(t(e)%*%sbeta%*%e)%*%(t(I)%*%sbeta%*%I)-
(t(e)%*%sbeta%*%I)%*%(t(e)%*%sbeta%*%I)
M3[j]=t(e)%*%R1%*%X/R2
S3[j]=t(I)%*%R1%*%X/R2
#### L-moments:
L1=function(u){qnorm(u)}
L2=function(u){(2*u-1)*qnorm(u)}
a1=integrate(L1,0,1)$value
a2=integrate(L2,0,1)$value
K1=rep(NA,N)
K2=rep(NA,N)
for (i in 1:N){
K1[i]=log(X[i])
K2[i]=(2*i/(N+1)-1)*log(X[i])}
M4[j]=(mean(K2)*a1-mean(K1)*a2)/(a1-a2)
S4[j]=(mean(K1)-mean(K2))/(a1-a2)
}
####### Absolute Biases of the fours estimators
ABIAS.mu.1=abs(mean(M1-m)); MSE.mu.1=mean((M1-m)^2)
```

```
ABIAS.mu.2=abs(mean(M2-m)); MSE.mu.2=mean((M2-m)^2)
ABIAS.mu.3=abs(mean(M3-m)); MSE.mu.3=mean((M3-m)^2)
ABIAS.mu.4=abs(mean(M4-m));MSE.mu.4=mean((M4-m)^2)
####### Mean-squared-errors of the fours estimators
ABIAS.sigma.1=abs(mean(S1-s)); MSE.sigma.1=mean((S1-s)^2)
ABIAS.sigma.2=abs(mean(S2-s)); MSE.sigma.2=mean((S2-s)^2)
ABIAS.sigma.3=abs(mean(S3-s)); MSE.sigma.3=mean((S3-s)^2)
ABIAS.sigma.4=abs(mean(S4-s));MSE.sigma.4=mean((S4-s)^2)
####### results for the BIAS and RMSE for mu
res=data.frame(
Estimators=c("ML","AFE","BLUE","LME"),
ABIAS.mu=c(ABIAS.mu.1,ABIAS.mu.2,ABIAS.mu.3,ABIAS.mu.4),
ABIAS.sigma=c(ABIAS.sigma.1,ABIAS.sigma.2,ABIAS.sigma.3,ABIAS.sigma.4),
MSE.mu=c(MSE.mu.1,MSE.mu.2,MSE.mu.3,MSE.mu.4),
MSE.sigma=c(MSE.sigma.1,MSE.sigma.2,MSE.sigma.3,MSE.sigma.4)
)
print(res)
```

ABIAS and MSE of Weibull distribution for $\alpha = 2.5$ for N=50

```
rm(list=ls())
set.seed(50.5)
library(extraDistr)
library(pracma)
library(ForestFit)
library(FAdist)
```

```
r=2.5;N=50; M=2000 #### shape parameter, sample size and replications number
m=0; s=1 # Theoretical value of location-scale parameters
F=function(x){pweibull(x,shape = r)} # the distribution function
f=function(x){dweibull(x,shape=r)} # the density function
######### the expected value of the i-th order statistics#############
e=rep(NA,N)
for(i in 1:N){
H<-function (x){
c1=factorial(N)
c2=factorial(i-1)
c3=factorial(N-i)
c=c1/(c2*c3)
c*x*(F(x)^(i-1))*((1-F(x))^(N-i))*f(x)}
e[i]=quadinf(H,0,Inf)$Q}
###### calling the covariance matrix
beta=read.table("cov-weibull-2v5-50")
rownames(beta)<-NULL
colnames(beta)<-NULLbeta=as.matrix(beta)
sbeta=solve(beta,tol=1e-200) #### inverse matrix
###################MLE estimators
M1=rep(NA,M);M2=rep(NA,M);M3=rep(NA,M);M4=rep(NA,M)
S1=rep(NA,M);S2=rep(NA,M);S3=rep(NA,M);S4=rep(NA,M)
library(univariateML)
for (j in 1:M){
```

```
x=rweibull3(N,r,s,m)

fit=fitWeibull(x, TRUE, "ml", starts<-c(3,1,0))$estimate

M1[j]=fit[[3]]

S1[j]=fit[[2]]

##########Asymptotically efficient estimators

x=rweibull(N,r)

X=sort(x)

a=r^2*gamma(3-2/r)+2*r*(1-r)*gamma(2-2/r)+(r-1)^2*gamma(1-2/r)

b=r^2

c=r*(1-2*r)*gamma(2-1/r)+r*(r-1)*gamma(1-1/r)+r^2*gamma(3-1/r)

d=a*b-c^2

k1=a/d;k2=b/d;k3=c/d

f1=function(u){(r-1)*(1-r*log(1-u))/(-log(1-u))^(2/r)}

f2=function(u){r^2*(-log(1-u))^(1-1/r)}

J1=function(u){k2*f1(u)-k3*f2(u)}

J2=function(u){k1*f2(u)-k3*f1(u)}

Z1=rep(NA,N);Z2=rep(NA,N)

for (i in 1:N){

Z1[i]=J1(i/(N+1))*X[i] ; Z2[i]=J2(i/(N+1))*X[i]

}

M2[j]=mean(Z1);S2[j]=mean(Z2)

########BLUE-estimators

I=rep(1,N)

R1=sbeta%*%(I%*%t(e)-e%*%t(I))%*%sbeta
```

```
R2=(t(e)%*%sbeta%*%e)%*%(t(I)%*%sbeta%*%I)-
(t(e)%*%sbeta%*%I)%*%(t(e)%*%sbeta%*%I)
M3[j]=t(e)%*%R1%*%X/R2
S3[j]=t(I)%*%R1%*%X/R2
######################L-moments estimators
u=rep(NA,N);v=rep(NA,N)
K1=rep(NA,N);K2=rep(NA,N)
y=gamma(1+1/r)
z=1-2^(-1/r)
for (i in 1:N){
u[i]=X[i]
v[i]=(2*i-N-1)*u[i]/N
K1[i]=u[i]-v[i]/z
K2[i]=v[i]/(z*y)}
M4[j]=mean(K1)
S4[j]=mean(K2)}
#M1=na.omit(M1);M2=na.omit(M2);M3=na.omit(M3);M4=na.omit(M4)
#S1=na.omit(S1);S2=na.omit(S2);S3=na.omit(S3);S4=na.omit(S4)
####### Biases of the fours estimators
ABIAS.mu.1=abs(mean(M1-m)); MSE.mu.1=mean((M1-m)^2)
ABIAS.mu.2=abs(mean(M2-m)); MSE.mu.2=mean((M2-m)^2)
ABIAS.mu.3=abs(mean(M3-m)); MSE.mu.3=mean((M3-m)^2)
ABIAS.mu.4=abs(mean(M4-m));MSE.mu.4=mean((M4-m)^2)
####### Mean-squared-errors of the fours estimators
```

ABIAS.sigma.1=abs(mean(S1-s)); MSE.sigma.1=mean((S1-s)^2)

ABIAS.sigma.2=abs(mean(S2-s)); MSE.sigma.2=mean((S2-s)^2)

ABIAS.sigma.3=abs(mean(S3-s)); MSE.sigma.3=mean((S3-s)^2)

ABIAS.sigma.4=abs(mean(S4-s));MSE.sigma.4=mean((S4-s)^2)

####### results for the BIAS and RMSE for mu

res=data.frame(

Estimators=c("ML","AFE","BLUE","LME"),

ABIAS.mu=c(ABIAS.mu.1,ABIAS.mu.2,ABIAS.mu.3,ABIAS.mu.4),

ABIAS.sigma=c(ABIAS.sigma.1,ABIAS.sigma.2,ABIAS.sigma.3,ABIAS.sigma.4),

MSE.mu=c(MSE.mu.1,MSE.mu.2,MSE.mu.3,MSE.mu.4),

MSE.sigma=c(MSE.sigma.1,MSE.sigma.2,MSE.sigma.3,MSE.sigma.4)

)

print(res)

ABIAS and MSE of Weibull distribution for $\alpha = 2.5$ for N=100

rm(list=ls())

set.seed(100.5)

library(extraDistr)

library(pracma)

library(ForestFit)

library(FAdist)

r=2.5;N=100; M=2000 #### shape parameter, sample size and replications number

m=0; s=1 # Theoretical value of location-scale parameters

F=function(x){pweibull(x,shape = r)} # the distribution function

```
f=function(x){dweibull(x,shape=r)} # the density function
######### the expected value of the i-th order statistics#############
e=rep(NA,N)
for(i in 1:N){
H<-function (x){
c1=factorial(N)
c2=factorial(i-1)
c3=factorial(N-i)
c=c1/(c2*c3)
c*x*(F(x)^(i-1))*((1-F(x))^(N-i))*f(x)}
e[i]=quadinf(H,0,Inf)$Q}
###### calling the covariance matrix
beta=read.table("cov-weibull-2v5-100")
rownames(beta)<-NULL
colnames(beta)<-NULL
beta=as.matrix(beta)
sbeta=solve(beta,tol=1e-200) #### inverse matrix
##################MLE estimatorsM1=rep(NA,M);M2=rep(NA,M);M3=rep(NA,M
S1=rep(NA,M);S2=rep(NA,M);S3=rep(NA,M);S4=rep(NA,M)
library(univariateML)
for (j in 1:M){
x=rweibull3(N,r,s,m)
fit=fitWeibull(x, TRUE, "ml", starts<-c(3,1,0))$estimate
M1[j]=fit[[3]]
```

```
S1[j]=fit[[2]]
##########Asymptotically efficient estimators
x=rweibull(N,r)X=sort(x)
a=r^2*gamma(3-2/r)+2*r*(1-r)*gamma(2-2/r)+(r-1)^2*gamma(1-2/r)
b=r^2
c=r*(1-2*r)*gamma(2-1/r)+r*(r-1)*gamma(1-1/r)+r^2*gamma(3-1/r)
d=a*b-c^2
k1=a/d;k2=b/d;k3=c/d
f1=function(u){(r-1)*(1-r*log(1-u))/(-log(1-u))^(2/r)}
f2=function(u){r^2*(-log(1-u))^(1-1/r)}
J1=function(u){k2*f1(u)-k3*f2(u)}
J2=function(u){k1*f2(u)-k3*f1(u)}
Z1=rep(NA,N);Z2=rep(NA,N)
for (i in 1:N){
Z1[i]=J1(i/(N+1))*X[i] ; Z2[i]=J2(i/(N+1))*X[i]
}
M2[j]=mean(Z1);S2[j]=mean(Z2)
########BLU-estimators
I=rep(1,N)
R1=sbeta%*%(I%*%t(e)-e%*%t(I))%*%sbeta
R2=(t(e)%*%sbeta%*%e)%*%(t(I)%*%sbeta%*%I)-
(t(e)%*%sbeta%*%I)%*%(t(e)%*%sbeta%*%I)
M3[j]=t(e)%*%R1%*%X/R2
S3[j]=t(I)%*%R1%*%X/R2
```

```
######################L-moments estimators
u=rep(NA,N);v=rep(NA,N)
K1=rep(NA,N);K2=rep(NA,N)
y=gamma(1+1/r)
z=1-2^(-1/r)
for (i in 1:N){
u[i]=X[i]
v[i]=(2*i-N-1)*u[i]/N
K1[i]=u[i]-v[i]/z
K2[i]=v[i]/(z*y)}
M4[j]=mean(K1)
S4[j]=mean(K2)}
#M1=na.omit(M1);M2=na.omit(M2);M3=na.omit(M3);M4=na.omit(M4)
#S1=na.omit(S1);S2=na.omit(S2);S3=na.omit(S3);S4=na.omit(S4)
####### Biases of the fours estimators
BIAS.mu.1=mean(abs(M1-m)); MSE.mu.1=mean((M1-m)^2)
BIAS.mu.2=mean(abs(M2-m)); MSE.mu.2=mean((M2-m)^2)
BIAS.mu.3=mean(abs(M3-m)); MSE.mu.3=mean((M3-m)^2)
BIAS.mu.4=mean(abs(M2-m));MSE.mu.4=mean((M4-m)^2)
####### Mean-squared-errors of the fours estimators
BIAS.sigma.1=mean(abs(S1-s)); MSE.sigma.1=mean((S1-s)^2)
BIAS.sigma.2=mean(abs(S2-s)); MSE.sigma.2=mean((S2-s)^2)
BIAS.sigma.3=mean(abs(S3-s)); MSE.sigma.3=mean((S3-s)^2)
BIAS.sigma.4=mean(abs(S4-s));MSE.sigma.4=mean((S4-s)^2)
```

```
####### results for the BIAS and RMSE for mu
res=data.frame(
Estimators=c("ML","AFE","BLUE","LME"),
MAE.mu=c(BIAS.mu.1,BIAS.mu.2,BIAS.mu.3,BIAS.mu.4),
MAE.sigma=c(BIAS.sigma.1,BIAS.sigma.2,BIAS.sigma.3,BIAS.sigma.4),
MSE.mu=c(MSE.mu.1,MSE.mu.2,MSE.mu.3,MSE.mu.4),
MSE.sigma=c(MSE.sigma.1,MSE.sigma.2,MSE.sigma.3,MSE.sigma.4)
)
print(res)
```

ABias and mse of Weibull distribution for $\alpha = 3$ for N=50

```
rm(list=ls())
set.seed(50.6)
library(extraDistr)
library(pracma)
library(ForestFit)
library(FAdist)
r=3;N=50; M=2000 #### shape parameter, sample size and replications number
m=0; s=1 # Theoretical value of location-scale parameters
F=function(x){pweibull(x,shape = r)} # the distribution function
f=function(x){dweibull(x,shape=r)} # the density function
######### the expected value of the i-th order statistics#############
e=rep(NA,N)
for(i in 1:N){
H<-function (x){
```

```
c1=factorial(N)
c2=factorial(i-1)
c3=factorial(N-i)
c=c1/(c2*c3)
c*x*(F(x)^(i-1))*((1-F(x))^(N-i))*f(x)}
e[i]=quadinf(H,0,Inf)$Q}
###### calling the covariance matrix
beta=read.table("cov-weibull-3-50")
rownames(beta)<-NULL
colnames(beta)<-NULL
beta=as.matrix(beta)
sbeta=solve(beta,tol=1e-200) #### inverse matrix
###################MLE estimators
M1=rep(NA,M);M2=rep(NA,M);M3=rep(NA,M);M4=rep(NA,M)
S1=rep(NA,M);S2=rep(NA,M);S3=rep(NA,M);S4=rep(NA,M)
library(univariateML)
for (j in 1:M){
x=rweibull3(N,r,s,m)
fit=fitWeibull(x, TRUE, "ml", starts<-c(3,1,0))$estimate
M1[j]=fit[[3]]
S1[j]=fit[[2]]
##########Asymptotically efficient estimators
x=rweibull(N,r)
X=sort(x)
```

```
a=r^2*gamma(3-2/r)+2*r*(1-r)*gamma(2-2/r)+(r-1)^2*gamma(1-2/r)
b=r^2
c=r*(1-2*r)*gamma(2-1/r)+r*(r-1)*gamma(1-1/r)+r^2*gamma(3-1/r)
d=a*b-c^2
k1=a/d;k2=b/d;k3=c/d
f1=function(u){(r-1)*(1-r*log(1-u))/(-log(1-u))^(2/r)}
f2=function(u){r^2*(-log(1-u))^(1-1/r)}
J1=function(u){k2*f1(u)-k3*f2(u)}
J2=function(u){k1*f2(u)-k3*f1(u)}
Z1=rep(NA,N);Z2=rep(NA,N)
for (i in 1:N){
Z1[i]=J1(i/(N+1))*X[i] ; Z2[i]=J2(i/(N+1))*X[i]
}M2[j]=mean(Z1);S2[j]=mean(Z2)
########BLU-estimators
I=rep(1,N)
R1=sbeta%*%(I%*%t(e)-e%*%t(I))%*%sbeta
R2=(t(e)%*%sbeta%*%e)%*%(t(I)%*%sbeta%*%I)-
(t(e)%*%sbeta%*%I)%*%(t(e)%*%sbeta%*%I)
M3[j]=t(e)%*%R1%*%X/R2
S3[j]=t(I)%*%R1%*%X/R2
######################L-moments estimators
u=rep(NA,N);v=rep(NA,N)
K1=rep(NA,N);K2=rep(NA,N)y=gamma(1+1/r)
z=1-2^(-1/r)
```

```
for (i in 1:N){
u[i]=X[i]
v[i]=(2*i-N-1)*u[i]/N
K1[i]=u[i]-v[i]/z
K2[i]=v[i]/(z*y)}
M4[j]=mean(K1)
S4[j]=mean(K2)}
#M1=na.omit(M1);M2=na.omit(M2);M3=na.omit(M3);M4=na.omit(M4)
#S1=na.omit(S1);S2=na.omit(S2);S3=na.omit(S3);S4=na.omit(S4)
####### Biases of the fours estimators
BIAS.mu.1=mean(abs(M1-m)); MSE.mu.1=mean((M1-m)^2)
BIAS.mu.2=mean(abs(M2-m)); MSE.mu.2=mean((M2-m)^2)
BIAS.mu.3=mean(abs(M3-m)); MSE.mu.3=mean((M3-m)^2)
BIAS.mu.4=mean(abs(M2-m));MSE.mu.4=mean((M4-m)^2)
####### Mean-squared-errors of the fours estimators
BIAS.sigma.1=mean(abs(S1-s)); MSE.sigma.1=mean((S1-s)^2)
BIAS.sigma.2=mean(abs(S2-s)); MSE.sigma.2=mean((S2-s)^2)
BIAS.sigma.3=mean(abs(S3-s)); MSE.sigma.3=mean((S3-s)^2)
BIAS.sigma.4=mean(abs(S2-s));MSE.sigma.4=mean((S4-s)^2)
####### results for the BIAS and RMSE for mu
res=data.frame(
Estimators=c("ML","AFE","BLUE","LME"),
MAE.mu=c(BIAS.mu.1,BIAS.mu.2,BIAS.mu.3,BIAS.mu.4),
MAE.sigma=c(BIAS.sigma.1,BIAS.sigma.2,BIAS.sigma.3,BIAS.sigma.4),
```

```
MSE.mu=c(MSE.mu.1,MSE.mu.2,MSE.mu.3,MSE.mu.4),
MSE.sigma=c(MSE.sigma.1,MSE.sigma.2,MSE.sigma.3,MSE.sigma.4)
)
print(res)
```

ABIAS and MSE of Weibull distribution for $\alpha = 3$ for N=100

```
rm(list=ls())
set.seed(100.6)
library(extraDistr)
library(pracma)
library(ForestFit)
library(FAdist)
r=3;N=100; M=2000 #### shape parameter, sample size and replications number
m=0; s=1 # Theoretical value of location-scale parameters
F=function(x){pweibull(x,shape = r)} # the distribution function
f=function(x){dweibull(x,shape=r)} # the density function
######### the expected value of the i-th order statistics#############
e=rep(NA,N)
for(i in 1:N){
H<-function (x){
c1=factorial(N)
c2=factorial(i-1)
c3=factorial(N-i)
c=c1/(c2*c3)
c*x*(F(x)^(i-1))*((1-F(x))^(N-i))*f(x)}
```

```
e[i]=quadinf(H,0,Inf)$Q}
###### calling the covariance matrix
beta=read.table("cov-weibull-3-100")
rownames(beta)<-NULL
colnames(beta)<-NULL
beta=as.matrix(beta)
sbeta=solve(beta,tol=1e-200) #### inverse matrix
###################MLE estimators
M1=rep(NA,M);M2=rep(NA,M);M3=rep(NA,M);M4=rep(NA,M)
S1=rep(NA,M);S2=rep(NA,M);S3=rep(NA,M);S4=rep(NA,M)
library(univariateML)
for (j in 1:M){
x=rweibull3(N,r,s,m)
fit=fitWeibull(x, TRUE, "ml", starts<-c(3,1,0))$estimate
M1[j]=fit[[3]]
S1[j]=fit[[2]]
##########Asymptotically efficient estimators
x=rweibull(N,r)
X=sort(x)
a=r^2*gamma(3-2/r)+2*r*(1-r)*gamma(2-2/r)+(r-1)^2*gamma(1-2/r)
b=r^2
c=r*(1-2*r)*gamma(2-1/r)+r*(r-1)*gamma(1-1/r)+r^2*gamma(3-1/r)
d=a*b-c^2
k1=a/d;k2=b/d;k3=c/d
```

```
f1=function(u){(r-1)*(1-r*log(1-u))/(-log(1-u))^(2/r)}

f2=function(u){r^2*(-log(1-u))^(1-1/r)}

J1=function(u){k2*f1(u)-k3*f2(u)}

J2=function(u){k1*f2(u)-k3*f1(u)}

Z1=rep(NA,N);Z2=rep(NA,N)

for (i in 1:N){

Z1[i]=J1(i/(N+1))*X[i] ; Z2[i]=J2(i/(N+1))*X[i]

}M2[j]=mean(Z1);S2[j]=mean(Z2)

########BLU-estimators

I=rep(1,N)

R1=sbeta%*%(I%*%t(e)-e%*%t(I))%*%sbeta

R2=(t(e)%*%sbeta%*%e)%*%(t(I)%*%sbeta%*%I)-

(t(e)%*%sbeta%*%I)%*%(t(e)%*%sbeta%*%I)

M3[j]=t(e)%*%R1%*%X/R2

S3[j]=t(I)%*%R1%*%X/R2

#####################L-moments estimators

u=rep(NA,N);v=rep(NA,N)

K1=rep(NA,N);K2=rep(NA,N)

y=gamma(1+1/r)

z=1-2^(-1/r)

for (i in 1:N){

u[i]=X[i]

v[i]=(2*i-N-1)*u[i]/N

K1[i]=u[i]-v[i]/z
```

```
K2[i]=v[i]/(z*y)}
M4[j]=mean(K1)
S4[j]=mean(K2)}
#M1=na.omit(M1);M2=na.omit(M2);M3=na.omit(M3);M4=na.omit(M4)
#S1=na.omit(S1);S2=na.omit(S2);S3=na.omit(S3);S4=na.omit(S4)
####### Biases of the fours estimators
BIAS.mu.1=mean(abs(M1-m)); MSE.mu.1=mean((M1-m)^2)
BIAS.mu.2=mean(abs(M2-m)); MSE.mu.2=mean((M2-m)^2)
BIAS.mu.3=mean(abs(M3-m)); MSE.mu.3=mean((M3-m)^2)
BIAS.mu.4=mean(abs(M2-m));MSE.mu.4=mean((M4-m)^2)
####### Mean-squared-errors of the fours estimators
BIAS.sigma.1=mean(abs(S1-s)); MSE.sigma.1=mean((S1-s)^2)
BIAS.sigma.2=mean(abs(S2-s)); MSE.sigma.2=mean((S2-s)^2)
BIAS.sigma.3=mean(abs(S3-s)); MSE.sigma.3=mean((S3-s)^2)
BIAS.sigma.4=mean(abs(S2-s));MSE.sigma.4=mean((S4-s)^2)
####### results for the BIAS and RMSE for mu
res=data.frame(
Estimators=c("ML","AFE","BLUE","LME"),
MAE.mu=c(BIAS.mu.1,BIAS.mu.2,BIAS.mu.3,BIAS.mu.4),
MAE.sigma=c(BIAS.sigma.1,BIAS.sigma.2,BIAS.sigma.3,BIAS.sigma.4),
MSE.mu=c(MSE.mu.1,MSE.mu.2,MSE.mu.3,MSE.mu.4),
MSE.sigma=c(MSE.sigma.1,MSE.sigma.2,MSE.sigma.3,MSE.sigma.4)
)
print(res)
```

5.3.3 Real data codes

Real data 'USAccDeaths' code

```
####################### logistic #############
library(goftest)
library(datasets)
##############################
data("USAccDeaths") #### requires datasets package
Y=USAccDeaths
Y=as.numeric(Y)
N=length(Y)
X=sort(Y)
##Asymptotically efficient:
W1=rep(NA,N)
W2=rep(NA,N)
J1=function(u){6*u*(1-u)}
J2=function(u){(9*(pi^2+3)^(-1))*(2*u-1+2*u*(1-u)*(log(u)-log(1-u)))}
for (i in 1:N){
W1[i]=J1(i/(N+1))*X[i]
W2[i]=J2(i/(N+1))*X[i]
}
mu=mean(W1);sigma=mean(W2)
print(c(mu,sigma))
########################################
pF=function(x){plogis(x,location = mu, scale = sigma)} # the distribution function
```

```
dF=function(x){dlogis(x,location = mu, scale = sigma)} # the density function
#############################################
mytest=cvm.test(X,"F") ###requires goftest package
pvlogis=mytest$p.value
############################ normal ##########"
##Asymptotically efficient:
W1=rep(NA,N)
W2=rep(NA,N)
J1=function(u){1}
J2=function(u){qnorm(u)}
for (i in 1:N){
W1[i]=J1(i/(N+1))*X[i]
W2[i]=J2(i/(N+1))*X[i]
}
mu=mean(W1);sigma=mean(W2)
print(c(mu,sigma))
#######################################
pF=function(x){pnorm(x,mean = mu, sd = sigma)} # the distribution function
dF=function(x){dnorm(x,mean = mu, sd = sigma)} # the density function
#############################################
mytest=cvm.test(X,"F") ###requires goftest package
pvnorm=mytest$p.value
########################## gumbel ##############
##########Asymptotically efficient estimators
```

```
J1=function(u){-0.85168*log(u)+0.25702*log(u)*log(-log (u))+0.25702}

J2=function(u){0.3509*log(u)+0.60792*log(u)*log(-log (u))+0.60792}

Z1=rep(NA,N);Z2=rep(NA,N)

for (i in 1:N){

Z1[i]=J1(i/(N+1))*X[i] ; Z2[i]=J2(i/(N+1))*X[i]

}

mu=mean(Z1);sigma=mean(Z2)

print(c(mu,sigma))

##########################################

PF=function(x,mu,sigma){exp(-exp((mu-x)/sigma))} # the distribution function

dF=function(x){dgumbel(x,mu,sigma)} # the density function

#################################################

mytest=cvm.test(X,"F") ###requires goftest package

pvgum=mytest$p.value

########################## lognormal #################"

###AFE estimators:

W1=rep(NA,N)

W2=rep(NA,N)

J1=function(u){1}

J2=function(u){qnorm(u)}

for (i in 1:N){

W1[i]=J1(i/(N+1))*log(X[i])

W2[i]=J2(i/(N+1))*log(X[i])

}
```

```
mu=mean(W1);sigma=mean(W2)
print(c(mu,sigma))
##########################################
pF=function(x){plnorm(x,meanlog = mu, sdlog = sigma)} # the distribution func-
tion
dF=function(x){dlnorm(x,meanlog = mu, sdlog = sigma)} # the density function
##################################################
mytest=cvm.test(Y,"F") ###requires goftest package
pvlogn=mytest$p.value
print(c(pvlogis,pvnorm,pvgum,pvlogn))
Z=log(Y)
shapiro.test(Z)
```

Real dat nidd.annual code

```
########################################## gumbel
library(goftest)
library(evir)
###############################
data("nidd.annual") #### requires datasets package
Y=nidd.annual
Y=as.numeric(Y)
N=length(Y)
X=sort(Y)
##########Asymptotically efficient estimators
J1=function(u){-0.85168*log(u)+0.25702*log(u)*log(-log (u))+0.25702}
```

```
J2=function(u){0.3509*log(u)+0.60792*log(u)*log(-log (u))+0.60792}
Z1=rep(NA,N);Z2=rep(NA,N)
for (i in 1:N){
Z1[i]=J1(i/(N+1))*X[i] ; Z2[i]=J2(i/(N+1))*X[i]
}
mu=mean(Z1);sigma=mean(Z2)
print(c(mu,sigma))
##########################################
pF=function(q){exp(-exp((mu-q)/sigma))} # the distribution function
dF=function(x){dgumbel(x,mu,sigma)} # the density function
##################################################
mytest=cvm.test(Y,"F") ###requires goftest package
pvgum=mytest$p.value
######################################## logistic
##Asymptotically efficient:
W1=rep(NA,N)
W2=rep(NA,N)
J1=function(u){6*u*(1-u)}
J2=function(u){(9*(pi^2+3)^(-1))*(2*u-1+2*u*(1-u)*(log(u)-log(1-u)))}
for (i in 1:N){
W1[i]=J1(i/(N+1))*X[i]
W2[i]=J2(i/(N+1))*X[i]
}
mu=mean(W1);sigma=mean(W2)
```

```
print(c(mu,sigma))
########################################
pF=function(x){plogis(x,location = mu, scale = sigma)} # the distribution function
dF=function(x){dlogis(x,location = mu, scale = sigma)} # the density function
###############################################
mytest=cvm.test(X,"F") ###requires goftest package
pvlogis=mytest$p.value
#################################### lognormal
###AFE estimators:
W1=rep(NA,N)
W2=rep(NA,N)
J1=function(u){1}
J2=function(u){qnorm(u)}

for (i in 1:N){
W1[i]=J1(i/(N+1))*log(X[i])
W2[i]=J2(i/(N+1))*log(X[i])
}
mu=mean(W1);sigma=mean(W2)
print(c(mu,sigma))
########################################
pF=function(x){plnorm(x,meanlog = mu, sdlog = sigma)} # the distribution function
dF=function(x){dlnorm(x,meanlog = mu, sdlog = sigma)} # the density function
###############################################
```

```
mytest=cvm.test(Y,"F") ###requires goftest package
pvlogn=mytest$p.value
################################# normal
##Asymptotically efficient:
W1=rep(NA,N)
W2=rep(NA,N)
J1=function(u){1}
J2=function(u){qnorm(u)}
for (i in 1:N){
W1[i]=J1(i/(N+1))*X[i]
W2[i]=J2(i/(N+1))*X[i]
}
mu=mean(W1);sigma=mean(W2)
print(c(mu,sigma))
###########################################
pF=function(x){pnorm(x,mean = mu, sd = sigma)} # the distribution function
dF=function(x){dnorm(x,mean = mu, sd = sigma)} # the density function
###################################################
mytest=cvm.test(X,"F") ###requires goftest package
pvnorm=mytest$p.value
print(c(pvnorm,pvlogn,pvlogis,pvgum))
```

Abstract

L-statistics offer significant advantages in statistic analysis due to its robustness, simplicity, and wide applicability. Their ability to summarize data using linear combinations of order statistics makes them particularly resistant to outliers and non-normal data distributions. This robustness is essential for producing reliable results in real-world scenarios where data often deviate from ideal conditions. The simplicity of L-statistics lies in their straightforward computational nature. Unlike more complex methods that require intricate algorithms and intensive calculations.

Résumé

Les L-statistiques offrent des avantages significatifs dans l'analyse statistique en raison de leur robustesse, de leur simplicité et de leur large applicabilité. Leur capacité à résumer des données en utilisant des combinaisons linéaires de statistiques d'ordre les rendent particulièrement résistants aux distributions de données aberrantes et non normales. Cette robustesse est essentielle pour produire des résultats fiables dans des scénarios réels où les données s'écartent souvent des conditions idéales.La simplicité des L-statistiques réside dans leur nature informatique directe. Contrairement aux méthodes plus complexes qui nécessitent des algorithmes complexes et des calculs intensifs.

ملخص

توفر الإحصاءات L مزايا كبيرة في التحليل الإحصائي نظرًا لقوتها وبساطتها وقابليتها للتطبيق على نطاق واسع. إن قدرتها على تلخيص البيانات باستخدام مجموعات خطية من إحصاءات الطلبات تجعلها مقاومة بشكل خاص للقيم المتطرفة وتوزيع البيانات غير العادية. هذه القوة ضرورية لتحقيق نتائج موثوقة في سيناريوهات العالم الحقيقي حيث غالبًا ما تنحرف البيانات عن الظروف المثالية. تكمن بساطة إحصائيات L في طبيعتها الحسابية المباشرة. على عكس الطرق الأكثر تعقيدًا التي تتطلب خوارزميات معقدة وحسابات مكثفة.