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Quadratic Backward Stochastic Differential Equations with Jumps

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Dedication

To those who are not matched by anyone in the universe, to those who have made a great deal, and have given what cannot be returned, my dear mother and father, I dedicate this research to you, you have been my best supporter throughout my academic career.

To my siblings, you are my support, my strength, and my source of happiness.

To all my loving and support family, especially my uncles who gave me all the advice and encouragement.

To my friends with whom i spent enjoyable times throughout school years and we went many experiences and successes.

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Notations and Symbols

These are the different symbols and abbreviations used in this thesis.

Symbols:

- $(\Omega, \mathcal{F}, \mathbb{P})$:Probability space.
- $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$:Filtered probability space.
- $W = (W_t)_{t\geq 0}$:Brownian motion.
- N :Poisson random measure.
- $\tilde{N}$: The compensated Poisson measure.
- $E = \mathbb{R}\backslash 0$.
- $a \wedge b$: $\min(a, b)$ Minimum of two numbers.
- $\mathbf{1}_A$: The indicator function of the set A.
- $L^1(\mathbb{R})$: The space of the functions whose absolute value is integrable.
- $L^2(\Omega)$: The Banach space of $\mathbb{R}$-valued, square-integrable random variables on $(\Omega, \mathcal{F}, \mathbb{P})$.
- $L^2_T(\Omega)$: The space of all $\mathbb{R}$-valued, $\mathcal{F}_T-$measurable and square integrable random variables $X : \Omega \to \mathbb{R}$.
- $\mathcal{L}^2_\upsilon$: The set of Borelian deterministic functions $\varphi : E \to \mathbb{R}$ such that

$$\|\varphi(.)\|^2_\upsilon := \int_E |\varphi(e)|^2 \,\upsilon(de) < \infty.$$

where υ is a $\sigma-$finite measure.

- $\mathfrak{L}_{\upsilon}^{p}$: The set real-valued measurable functions u defined on $[0,T]\times E$ such that:

$$\|u(.)\|_{\upsilon,p} := \left(\int_E |u(e)|^p\, \upsilon(de)\right)^{\frac{1}{p}} < \infty.$$

- $\mathcal{C}^2(\mathbb{R})$: The space of continuous functions $\varphi : \mathbb{R} \to \mathbb{R}$ which have continuous derivatives φ' and φ''.

- $\mathbb{H}^2(\mathbb{R})$: The space of real-valued predictable processes $Z : \Omega \times [0,T] \to \mathbb{R}$ satisfying

$$\mathbb{E}\left[\int_0^T Z(t)^2 dt\right] < \infty.$$

- $\mathbb{H}_N^2(\mathbb{R})$: The space of real-valued predictable processes $U : \Omega \times [0,T] \times E \to \mathbb{R}$ satisfying

$$\mathbb{E}\left[\int_0^T \int_E |U(t,e)|^2\, \upsilon(de)dt\right] < \infty.$$

- $\mathbb{S}^2(\mathbb{R})$: The space of adapted, càdlàg processes $Y : \Omega \times [0,T] \to \mathbb{R}$ satisfying

$$\mathbb{E}\left[\sup_{t\in[0,T]} |Y(t)|^2\right] < \infty.$$

- $\mathbb{M}_S^2 = \mathbb{S}^2(\mathbb{R}) \otimes \mathbb{H}^2(\mathbb{R}) \otimes \mathbb{H}_N^2(\mathbb{R})$, which will be our solution space.

- $\mathcal{W}_1^2(\mathbb{R})$: The space of continuous functions g from $\mathbb{R}$ to $\mathbb{R}$ such that g' is continuous and g'' is integrable on $\mathbb{R}$.

- $\mathcal{W}_{p,loc}^2(\mathbb{R})$: The Sobolev space of functions g defined on $\mathbb{R}$ such that both g and its generalized derivatives g' and $g''$are locally integrable on $\mathbb{R}$.

- $\mathcal{M}^2$: The space of square integrable martingale $M = (M_t)_{t\geq 0}$, i.e. M is a martingale, and for each $t > 0$, $\mathbb{E}\left(|M_t|^2\right) < \infty$, and $M_0 = 0$

- $\bar{A}$: The closure of the set A.

- $(.) \otimes (.)$: Tensor product.

- $:=$: Equal by definition.

Notations:

BM	:	Brownien Motion.
a.s.	:	Almost surely.
a.e.	:	Almost everywhere.
$\mathbb{P}-$a.s.	:	Almost sure in probability $\mathbb{P}$.
i.e.	:	That's to say.
càdlàg	:	Right continuous with left limits.
w.r.t.	:	With respect to.
BSDEs	:	Backward stochastic differential equations.
BSDEJs	:	Backward stochastic differential equations with jumps.
QBSDEJs	:	Quadratic backward stochastic differential equations with jumps.
BDG	:	Burkholder-Davis-Gundy.
iff	:	If and only if.

Contents

Introduction

Backward stochastic differential equations, or BSDEs for short, are a class of stochastic differential equations where the solution is determined by a terminal rather than an initial condition. They are significant in various fields, such as mathematical finance, where they are used for pricing and hedging derivative securities and in stochastic control and optimal stopping problems. See for example: [2, 7, 10, 29], mathematical finance [11, 13], stochastic differential games and stochastic control [17] partial differential equations [5, 24], utility optimization and dynamic risk measures [29], Morlais [28].

BSDEs are embraced for both continuous and jump types. In the continuous setting: Linear BSDEs driven by continuous Brownian Motion initially appeared by Bismut in 1973 [9] when he studied the adjoint equation in the Pontryagin-type maximum principle for stochastic optimal control theory. The first well-posedness result for non-linear BSDEs was proved in 1990 by Pardoux-Peng [32], who proved the existence and uniqueness of solutions to BSDEs under the conditions of Lipschitz continuity on the driver and square-integrability on the terminal data, which has been recognized as the strong condition that guarantees the well-posedness of BSDEs. Many works have proposed different methods or techniques to prove that we can guarantee the existence of the solution under weaker assumptions, either on the generator or on the terminal condition.

Given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$ generated by an $\mathbb{R}$-valued Brownian motion W and a random measure N with compensator υ, the $\mathbb{R}$-valued BSDEJs given as:

$$Y_t = \xi + \int_t^T H(s, Y(s), Z(s), U(s,.))ds - \int_t^T Z(s)dW(s) - \int_t^T \int_E U(s,e)\tilde{N}(ds,de),$$

for all $0 \leq t \leq T$, where W is a Brownian motion and $\tilde{N}(ds,de) = N(ds,de) - \upsilon(de)ds$ is a compensated random measure such that $N(ds,de)$ a time random measure with compensator $\upsilon(de)ds$. We refer to the previous equation as $eq(\xi, H)$. The data ξ is called the terminal condition (the terminal value), and H is the coefficient or the generator of $eq(\xi, H)$. The key challenge in BSDEs is solving for the pair of processes (Y, Z, U) given the terminal condition ξ.

BSDEs with jumps generalize the classical BSDEs by adding a discontinuous part to the backward equation. The use of jump processes in modeling stochastic systems with jumps provides valuable insights into the behavior of these systems and helps in making predictions and decisions.

BSDEJs were first introduced and studied by Tang and Li [37] during their study of an optimal control problem, where the adjoint process is of type Linear BSDEJs. Using a fixed point approach similar to the one used in [32], they proved the existence and uniqueness of such equations in the general case (the generator is not necessarily linear). This is done under the Lipschitz continuity assumption in y, z, and u, where the jump is of the Poisson type. It is worth mentioning that the Lipschitz condition in y, z, and u is considered the strongest condition that ensures the well-posedness of BSDEJs. Many attempts have been suggested to relax the assumptions made on the generator H and the terminal datum assumptions. Barles et al. in [5] established an existence and uniqueness result of BSDEJs Under a Lipschitz condition on the generator to give a probabilistic interpretation (so-called Feynman-Kac representation) of viscosity solution of semilinear integral Partial equations. Then Pardoux [31] relaxed the Lipschitz condition of the generator on variable y by assuming a monotonicity condition, and maintain the Lipschitz condition w.r.t. (z, u). Later, Situ [35] and Mao and

Yin [39] relax the monotonicity condition of the generator to a weaker version so as to remove the Lipschitz condition on variable z. All these studies are dedicated to BSDEJs when the generator is driven by a Brownian motion and an independent Poisson random measure. In 2006, Becherer [7] studied the existence and uniqueness of bounded BSDE's driven by a Brownian motion and a random measure that is possibly inhomogeneous in time with finite jump activity solution when the generator is Lipschitz in (y, z) but not necessarily Lipschitz in the jump component (Locally Lipschitz).

Recently, researchers have also been interested in a subcategory of BSDEJs whose generator shows quadratic growth in the Brownian component. This type of equation arises from the utility maximization problem and has been extensively studied in the continuous framework by many authors. However, there are few results for the discontinuous case. Morlais improved the result of [7], by considering the exponential utility maximization problem in an incomplete market in his paper [29] and proved that the value process could be characterized as a solution of a specific quadratic BSDE with infinite activity jumps and has proved the existence and uniqueness of solution. Jeanblanc et al. [20] also studied the utility maximization problem with random horizon. El Karoui et al. [12] studied the case of unbounded BSDEs with jumps when the generator satisfies the quadratic exponential structure condition. Fujii and Takahashi [15] have studied Malliavin's differentiability of bounded solutions of BSDEs with jumps under the quadratic-exponential growth condition of [12]. Kazi-Tani et al. [21] established the existence of a solution of Lipschitz quadratic BSDE with jumps. Matoussi and Salhi [27] study quadratic BSDE with infinite activity jumps and unbounded terminal condition. Their perspective is based on the work of [12], which examined the issue in the context of finite activity jumps. Antonelli & Mancini (2016) [1] Antonelli studied quadratic BSDEs with jumps when the generator is local Lipschitz with different assumptions. Based on the stability of quadratic semimartingales, Barrieu & El Karoui (2013) [6] showed the existence of a non-necessary unique solution under the minimal assumption. Most of the above-cited references considered bounded or exponential integrable terminal data and at least continuous generators with some additional regularity properties.

This type of equation shows a quadratic growth in the Brownian component; that is, its generator is not Lipschitzian but only locally Lipschitz. Therefore, the result of Tang and Li in their seminal paper[37] is invalid and cannot be applicable in proving the existence and uniqueness of solutions. Hence, we are raising the problem of proving the existence and uniqueness of solutions to such equations when the generator is merely measurable and integrable and the terminal condition is square integrable. To solve this problem, we first use the phase space transformation (known as Zvonkin transformations) to remove the generator or the quadratic part of it. This leads to a BSDEJs with a Lipschitz generator, which enjoys the existence and uniqueness propriety. The last propriety can be transformed to the initial quadratic BSDEJ via the inverse of Zvonkin transformation. Our study covers the following cases:

$$H(y,z,u(.)) = \begin{cases} f(y)\left|z\right|^2 + \left[u\right]_f(y) =: H_f(y,z,u(.)) \\ h(y,u(.)) + cz + H_f(y,z,u(.)) \\ cz + f(y)\left|z\right|^2 + \int_E u(e)\upsilon(e) \end{cases}$$

These generators show quadratic growth in the Brownian component and non-linear functional form w.r.t. the jump term, where f is a measurable and integrable function, h enjoy some classical assumptions, $\left[u\right]_f(y)$ is a functional of the unknown processes Y, and given by

$$\left[u\right]_f(y) = \int_E \frac{F(y+u(e)) - F(y) - F'(y)u(e)}{F'(y)}\upsilon(de),$$

and the function F is defined, for every $x \in \mathbb{R}$, by

$$F(x) = \int_0^x \exp\left(2\int_0^y f(t)dt\right)dy.$$

This thesis consists of three chapters and aims to expose two results regarding the existence, uniqueness of a BSDEJ solution.

Chapter 1 (General Reminder of Stochastic Calculus):

This chapter provides an introduction to the fundamental results of probability theory utilized throughout this thesis. We cover essential topics, including random processes, continuous stochastic calculus, and various auxiliary results that support the main chapters of the dissertation.

Chapter 2 (Backward Stochastic Differential Equations with Jumps): The second chapter is divided into two parts. The first part revisits key results in the discontinuous case, with a particular focus on Itô-Krylov's formula in the context of jumps, which is initiated by Madoui et al. [26]. The second part introduces the foundational result on the existence and uniqueness of solutions for BSDEs with jumps, as established by Tang and Li (1994) [37], specifically in the case of a Lipschitzian generator.

Chapter 3 (Quadratic Backward Stochastic Differential Equations with Jumps): This chapter constitutes the core of our thesis. We establish the existence and uniqueness of solutions for certain quadratic BSDEs with jumps, where the generator is only measurable and integrable, and the terminal condition is square-integrable. The chapter is organized into the following sections: a priori estimates, existence and uniqueness of solutions, solvability of specific quadratic BSDEs with jumps, and a comparison theorem.

Chapter 1

General Reminder of Stochastic Calculus

This chapter's goal is to provide an overview of the primary stochastic Calculus tools that will be utilized in this thesis, especially those related to the continuous case and some useful results that particularly serve the content of chapter three, and we mention some of the references we used [3, 4, 8, 11, 14, 18, 19, 22, 23, 25, 26, 33, 36, 38].

1.1 Preliminaries

1.1.1 Preliminaries of Probability Theory

Definition 1.1.1 *($\mathbb{P}-$null set)*

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $A \subseteq \Omega$ (not necessary measurable), we say that A is $\mathbb{P}-$null set if there exists $B \in \mathcal{F}$ such that $A \subseteq B$ and $\mathbb{P}(B)$ =0. We denote by $\mathcal{N}$ the set of all $\mathbb{P}-$null set. The probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is called complete if $\mathcal{N} \subseteq \mathcal{F}$. In this thesis, the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ will always be considered completed by the collection of null probability sets. If $\mathbb{X}$ is a topological space, then $\mathcal{B}_{\mathbb{X}}$ denotes the Borel σ-algebra over $\mathbb{X}$, *that is the* $\sigma-$*algebra generated by the family of open subsets of* $\mathbb{X}$*, in particular* $\mathcal{B}_d \overset{def}{=} \mathcal{B}_{\mathbb{R}^d}$ *and* $\mathcal{B} \overset{def}{=} \mathcal{B}_1$.

Random Variable

A mapping $X : \Omega \to \mathbb{X}$ is an $\mathbb{X}-$valued random variable if for all $B \in B_{\mathbb{X}}$

$$\{X \in B\} = \{\omega \in \Omega : X(\omega) \in B\} \in \mathcal{F}.$$

- If $\mathbb{X} = \mathbb{R}$, then X will be called a real random variable (or scalar random variable).
- If $\mathbb{X} = \mathbb{R}^d$, then X will be called a $d-$dimensional random vector.

Remark 1.1.1

The random variable X *is* $(B_{\mathbb{X}}, \mathcal{F})-$*measurable.*

Definition 1.1.2

Let $X : (\Omega, \mathcal{F}, \mathbb{P}) \to \mathbb{X}$ *be a random vector. Then*

$$\sigma(X) = \left\{X^{-1}(B) : B \in \mathcal{B}_{\mathbb{X}}\right\}$$

is a σ*-algebra of subsets of* Ω *(called the* σ*-algebra generated by* X*). Since* X *is a random variable,* $\sigma(X) \subset \mathcal{F}$*, and* $\sigma(X)$ *is the smallest -algebra which makes* X *measurable. It is the class of events for which one knows whether or not they are realized once* $X(\omega)$ *is observed. In this sense,* $\sigma(X)$ *represents the information carried by* X*. We also define*

$$\mathcal{F}^X = \sigma(X) \vee \mathcal{N}$$

to be the smallest σ-algebra which contains both $\sigma(X)$ and $\mathcal{N}$. In probabilistic terms, the σ-algebra $\mathcal{F}^X$ can be interpreted as containing all relevant information about the random variable X.

1.1.2 Conditional Expectation

Definition 1.1.3 *(Conditional Expectation in Relation to a $\sigma-$algebra)*

Let X be a random variable that is either nonnegative or integrable defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathcal{G}$ and $\mathcal{H}$ will denote sub-σ-algebras of $\mathcal{F}$. The conditional expectation of X given $\mathcal{G}$, denoted $\mathbb{E}(X \mid \mathcal{G})$, is the unique random variable that satisfies:

(i) (Measurability) $\mathbb{E}(X \mid \mathcal{G})$ is $\mathcal{G}-$measurable,

(ii) (Partial averaging)

$$\int_A \mathbb{E}(X \mid \mathcal{G}) d\mathbb{P} = \int_A X d\mathbb{P}, \ \forall A \in \mathcal{G}.$$

• If $\mathcal{G}$ is the σ-algebra generated by some other random variable Y (i.e. $\mathcal{G} = \sigma(Y)$), we generally write $\mathbb{E}(X \mid Y)$ rather than $\mathbb{E}(X \mid \sigma(Y))$.

Proposition 1.1.1 *(Properties of the Conditional Expectation)*

Let X and Y be integrable random variables on $(\Omega, \mathcal{F}, \mathbb{P})$, let $\mathcal{G}$, $\mathcal{H} \subset \mathcal{F}$ be finite σ-algebras, and let α, $\beta \in \mathbb{R}$,

(i) $\mathbb{E}(\alpha X + \beta Y \mid \mathcal{G}) = \alpha \mathbb{E}(X \mid \mathcal{G}) + \beta \mathbb{E}(Y \mid \mathcal{G})$ *a.s.*

(ii) If $X \geq 0$ a.s., then $\mathbb{E}(X \mid \mathcal{G}) \geq 0$ a.s.

(iii) $\mathbb{E}(\mathbb{E}(X \mid \mathcal{G})) = \mathbb{E}(X)$.

(iv) Tower property: if $\mathcal{H} \subset \mathcal{G}$, then $\mathbb{E}(\mathbb{E}(X \mid \mathcal{G}) \mid \mathcal{H}) = \mathbb{E}(X \mid \mathcal{H})$ a.s.

(v) If X is $\mathcal{G}-$measurable, then $\mathbb{E}(X \mid \mathcal{G}) = X$.

(vi) If X is $\mathcal{G}-$measurable and $XY \in L^1(\Omega)$, then $\mathbb{E}(XY \mid \mathcal{G}) = X\mathbb{E}(Y \mid \mathcal{G})$ a.s.

(vii) If $\mathcal{H}$ and X are independent, then $\mathbb{E}(X \mid \mathcal{H}) = \mathbb{E}(X)$.

1.2 Stochastic Process

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, T an index set, and $(\mathbb{X}, \mathcal{B}_{\mathbb{X}})$ a topological space. An $(\mathbb{X}, \mathcal{B}_{\mathbb{X}})$-valued stochastic process on $(\Omega, \mathcal{F}, \mathbb{P})$ is a collection $(X_t)_{t \in T}$ of random variables $X_t : (\Omega, \mathcal{F}) \to (\mathbb{X}, \mathcal{B}_{\mathbb{X}})$, for $t \in T$. $(\Omega, \mathcal{F}, \mathbb{P})$ is called the underlying probability space of the process $(X_t)_{t \in T}$, while $(\mathbb{X}, \mathcal{B}_{\mathbb{X}})$ is the state space or phase space. An $\mathbb{R}^d$-valued stochastic process will be called a d-dimensional stochastic process. By a stochastic process, we will usually mean a one-dimensional stochastic process. For $t \in T$ fixed, the random variable X_t is the state of the process at time t. Moreover, for all $\omega \in \Omega$, the mapping

$$X(.,\omega) : t \in T \to X_t(\omega) \in \mathbb{X},$$

is called the trajectory or the sample path of the process corresponding to ω.

Definition 1.2.1

(i) A filtration is a family of sub$-\sigma-$algebra $\mathbb{F} := (\mathcal{F}_t)_{t \in \mathbb{R}^+}$ that is increasing, i.e. $\mathcal{F}_s \subseteq \mathcal{F}_t$, for each $s \leq t$. $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is called a filtred probability space (or a stochastic basis). We say that the filtration $\mathbb{F}$ is complete if and only if $\mathcal{N} \subseteq \mathcal{F}_0$. A filtred probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ is said to satisfy the usual hypothesis if and only if the filtration $\mathbb{F}$ is complete and right continuous.

(ii) The $\sigma-$field generated by the adapted left-continuous processes is called the predictable $\sigma-$field $\mathcal{P}$. A process is called predictable if it is measurable with respect to the predictable $\sigma-$field $\mathcal{P}$.

Definition 1.2.2 *(Natural Filtration)*

The natural filtration associated with a stochastic process $X : \Omega \times \mathbb{R}^+ \to \mathbb{X}$ is defined by

$$\mathcal{F}_t^X = \sigma(X_s : s \leq t) \vee \mathcal{N},$$

where $\mathcal{N}$ denotes the collection of all $\mathbb{P}$-null-sets of $\mathcal{F}$, $\mathcal{F}_t^X$ is called the history of the process X until (and including) time $t \geq 0$.

Theorem 1.2.1

- $(X_t)_{t\in\mathbb{R}^+}$ *is said to be càdlàg if it almost sure has sample paths which are right continuous with left limits.*

- $(X_t)_{t\in\mathbb{R}^+}$ *is said to be càglàd if it almost sure has sample paths which are left continuous with right limits.*

- $(X_t)_{t\in\mathbb{R}^+}$ *is said to be continuous if it almost sure has continuous sample paths.*

Theorem 1.2.2

(i) The stochastic process $X = (X_t)_{t\in\mathbb{R}^+}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be $\mathbb{F}$-adapted (or adapted with respect to $\mathbb{F}$) if

$$\forall t \geq 0,\ X_t \text{ is } \mathcal{F}_t - \text{measurable.}$$

(ii) A stochastic process X is called $(\mathcal{B}_{\mathbb{R}^+} \otimes \mathcal{F}, \mathcal{B}_{\mathbb{X}})$-measurable if $\forall A \in \mathcal{B}_{\mathbb{X}}$,

$$\{(t,\omega) : X_t(\omega) \in \mathcal{A}\} \in \mathcal{B}_{\mathbb{R}^+} \otimes \mathcal{F}.$$

In the other worlds, if the mapping $X : (\Omega\times\mathbb{R}^+, \mathcal{B}_{\mathbb{R}^+}\otimes\mathcal{F}) \to (\mathbb{X}, \mathcal{B}_{\mathbb{X}})$ is $(\mathcal{B}_{\mathbb{R}^+}\otimes\mathcal{F}, \mathcal{B}_{\mathbb{X}})-$measurable, then we shall say that X is an $\mathbb{X}$-valued measurable stochastic process.

(iii) A process $X : \Omega \times [0,T] \to \mathbb{R}$, or $X : \Omega \times [0,T] \times \zeta \to \mathbb{R}$, is called $\mathbb{F}$-predictable if it is $\mathbb{F}$-adapted and $\mathcal{P}$-measurable, or $\mathcal{P} \otimes \mathcal{B}(\zeta)$-measurable. Moreover, any measurable $\mathbb{F}$-adapted and left-continuous (with respect to t) process is predictable.

Definition 1.2.3 *(Uniformly Integrable)*

Let $\mathbf{X}$= $(X_\alpha)_{\alpha\in J}$ *be a family of random variable indexed by J, we say that the family* $\mathbf{X}$ *is* *uniformly integrable if*

$$\forall\varepsilon \geq 0, \exists M \geq 0 : \sup(\mathbb{E}(\mathbf{1}_{\{X_\alpha > M\}}\,|X_\alpha|)) < \varepsilon.$$

1.2.1 Martingale

We assume throughout that a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ is given.

Definition 1.2.4 *((Sub/super-)Martingales in continuous time)*

An $\mathbb{F}-$adapted stochastic process $(X_t)_{t\geq 0}$ such that $X_t \in L^1(\Omega)$ (i.e. $\mathbb{E}\,|X_t| < \infty$) for any $t \geq 0$, is called:

(i) A martingale if $\mathbb{E}(X_t \mid \mathcal{F}_s) = X_s$, for all $0 \leq s \leq t$,

(ii) A super-martingale if $\mathbb{E}(X_t \mid \mathcal{F}_s) \leq X_s$, for all $0 \leq s \leq t$,

(iii) A sub-martingale if $\mathbb{E}(X_t \mid \mathcal{F}_s) \geq X_s$, for all $0 \leq s \leq t$.

- *If $(X_t)_{t\geq 0}$ is a submartingale, then $(-X_t)_{t\geq 0}$ is a supermartingale.*

Proposition 1.2.1

Let $X := (X_t)_{t\geq 0}$ be an $\mathbb{F}-$adapted integrable stochastic process

(i) If X is a martingale, then $\mathbb{E}(X_t) = \mathbb{E}(X_s)$, $\forall t,\ s \geq 0$,

(ii) If X is a sub-martingale, then $\mathbb{E}(X_t) \geq \mathbb{E}(X_s)$, $\forall t,\ s \geq 0,\ s \leq t$,

(iii) If X is a super-martingale, then $\mathbb{E}(X_t) \leq \mathbb{E}(X_s)$, $\forall t,\ s \geq 0,\ s \leq t$.

A martingale $(X_t)_{t\geq 0}$ is said to be closed if there exists a random variable $Z \in L^1(\Omega)$ such that, for every $t \geq 0$,

$$X_t = \mathbb{E}(Z \mid \mathcal{F}_t).$$

Definition 1.2.5

An $\mathbb{F}-$martingale is called square integrable on $[0, T]$ if its second moments are bounded (i.e. $\in \mathcal{M}^2$).

Local Martingales

Definition 1.2.6

A random variable $T : \Omega \to [0, \infty]$ is a stopping time of the filtration $(\mathcal{F}_t)_{t \geq 0}$ (or $(\mathcal{F}_t)_{t\geq 0}-$stopping time) if $\{T \leq t\} \in \mathcal{F}_t$, for every $t \geq 0$. The σ-field of the past before T is then defined by

$$\mathcal{F}_T = \{A \in \mathcal{F} : \forall t \geq 0,\ A \cap \{T \leq t\} \in \mathcal{F}_t\}.$$

For a process X and a stopping time T we further recall that X^T denotes the stopped process

$$X_t^T = X_{t\wedge T} = X_t \mathbf{1}_{\{t<T\}} + X_T \mathbf{1}_{\{t \geq T\}}.$$

Definition 1.2.7

An adapted, càdlàg process X is a local martingale if there exists a sequence of increasing stopping times T_n, with $\lim_{n\to\infty} T_n = \infty$ a.s., such that $X_{t\wedge T_n} \mathbf{1}_{\{T_n>0\}}$ is a uniformly integrable martingale for each n. Such a sequence (T_n) of stopping times is called a fundamental sequence.

Semi-Martingales

Definition 1.2.8 *(Finite Variation)*

1. Functions of finite variation: *If f is a function of real variable, its variation over the interval $[a, b]$ is defined as*

$$V_f([a, b]) = \sup \sum_{i=1}^{n} \left| f(t_i^n) - f(t_{i-1}^n) \right|,$$

where the supremum is taken over partitions: $a = t_0^n < t_1^n < ...t_n^n = b$, If $V_f([a, b])$ is finite then f is said to be a function of finite variation on $[a, b]$.

2. Processes of finite variation: *An adapted right-continuous process $A = (A_t : t \geq 0)$*

is called a finite variation process (or a process of finite variation) if $A_0 = 0$ *and* $t \to A_t$ *is (a function) of finite variation a.s..*

Definition 1.2.9

An $\mathbb{F}-$*semi-martingale is a càdlàg process* X *which can be written as* $X_t = X_0 + M_t + A_t$*, where* M *is an* $\mathbb{F}-$*local martingale such that* $M_0 = 0$*, and* A *is an* $\mathbb{F}-$*adapted càdlàg process with finite variation.*

1.2.2 Brownian Motion

Definition 1.2.10

A real-valued stochastic process $(W_t)_{t\geq 0}$ *is a Brownian motion (or a Wiener process) if for some real constant* σ *:*

(i) $W_0 = 0$,

(ii) *For each* $n \geq 1$ *and any times* $0 \leq t_1 \leq t_2 \leq ... \leq t_n$*, the random variables* $\left\{W_{t_r} - W_{t_{r-1}}\right\}$ *are independent,*

(iii) *For each* $s \geq 0$ *and* $t > 0$ *the random variable* $W_{t+s} - W_s$ *has the normal distribution with mean zero and variance* $\sigma^2 t$ $(W_{t+s} - W_s \sim \mathcal{N}(0, \sigma^2 t))$,

(iv) W_t *is continuous in* $t \geq 0$*.*

When $\sigma^2 = 1$*, we say we have a standard Brownian motion.*

Remark 1.2.1

Of the definition we conclude that $\forall t \geq 0$, $W_t \sim \mathcal{N}(0, t)$.

1.2.3 Quadratic Variation

Definition 1.2.11

The quadratic variation process of a càdlàg semi-martingale V*, denoted* $[V, V] = ([V, V](t))_{t\geq 0}$

is defined by

$$[V,V](t) = [V,V]_t = \lim_{n\to\infty} \sum_{i=1}^{n} \left(V_{t_{i+1}^n \wedge t} - V_{t_i^n \wedge t} \right)^2, 0 \le t \le T,$$

where $\lim\limits_{n\to\infty} \sup_{i=1,\dots,n} \left| t_{i+1}^n - t_i^n \right| = 0$*, and the convergence is uniform in probability.*

* The quadratic variation of a Brownian motion is given by $[W,W]_t = t$.

* The quadratic variation of a quadratic pure jump process J (a purely discontinuous Lévy process or a step process) is given by $[J,J]_t = \sum_{s\le t} |\Delta J(s)|^2$.

* Let M be a square-integrable martingale, that is $\sup_t \mathbb{E}(M^2(t)) < \infty$, the quadratic variation of M has the property

$$M^2(t) - [M,M](t) \text{ is a martingale.}$$

Proposition 1.2.2 *(Properties of Quadratic Variation)*

(i) $([V,V](t))_{t\in[0,T]}$ *is an increasing process.*

(ii) The jumps of $[V,V]$ *are related to the jumps of* V *by:* $\Delta [V,V](t) = |\Delta V_t|^2$*. In particular,* $[V,V]$ *has continuous sample paths if and only if* V *does.*

(iii) If V *is continuous and has paths of finite variation, then* $[V,V] = 0$*.*

(iv) If V *is a martingale and* $[V,V] = 0$*, then* $V = V_0$ *almost surely.*

1.3 Itô's Stochastic Calculus

1.3.1 Itô Integral

Definition 1.3.1 *(Itô Integral with Respect to Wiener Process)*

Let $(\Omega, \mathcal{F}, \mathbb{F} := (\mathcal{F}_t)_{t\ge 0}, \mathbb{P})$ *be a fixed filtered probability space satisfying the usual condition, let* $T > 0$ *and* θ *be an adapted process, such that* $\int_0^T \theta^2(s) ds < \infty$ *with probability one, so that*

$\int_0^t \theta(s)dW_s$ *is defined for any* $t \leq T$*. Since it is a random variable for any fixed* t*,* $\int_0^t \theta(s)dW_s$ *as a function of the upper limit* t *defines a stochastic process*

$$M(t) = \int_0^t \theta(s)dW_s.$$

- *Itô integrals are* $\mathbb{F}-$*adapted.*

Theorem 1.3.1 *(Properties of Itô Integral)*

(i) Let $V : \Omega \times [0, T] \to \mathbb{R}$ *be a predictable process satisfying*

$$\int_0^T |V(s)|^2 \, ds < \infty,$$

Then $(\int_0^t V(s)dW(s),\ 0 \leq t \leq T)$ *is a continuous local martingale with the quadratic variation process*

$$\left[\int_0^{\cdot} V(s)dW(s), \int_0^{\cdot} V(s)dW(s)\right](t) = \int_0^t |V(s)|^2 \, ds,\ 0 \leq t \leq T.$$

(ii) Let $V \in \mathbb{H}^2(\mathbb{R})$*, then* $(\int_0^t V(s)dW(s),\ 0 \leq t \leq T)$ *is a continuous, square-integrable martingale, which satisfies*

$$\mathbb{E}\left[\left|\int_0^T V(s)dW(s)\right|^2\right] = \mathbb{E}\left[\int_0^T |V(s)|^2 \, ds\right].$$

1.3.2 Itô's Formula

Theorem 1.3.2 *(Itô's Formula for Semi-martingales)*

Let X *be a semi-martingale and let* f *be a* $\mathcal{C}^2$*-real function, then* $f(X)$ *is again a semi-*

martingale, and the following formula holds:

$$f(X_t) - f(X_0) = \int_0^t f'(X_{s^-})dX_s + \frac{1}{2}\int_0^t f''(X_{s^-})d\left[X,X\right]_s^c + \sum_{0\leq s\leq t}\left(f(X_s) - f(X_{s^-}) - f'(X_{s^-})\Delta X_s\right),$$

where $[X,X]^c$ *is the continuous component of the finite variation function* $[X,X]$.

1.4 Some Auxiliary Results

Lemma 1.4.1

Let f *be a measurable function and belongs to* $L^1(\mathbb{R})$*. The function* F *defined by*

$$F(x) = \int_0^x \exp\left(2\int_0^y f(t)dt\right)dy, \tag{1.1}$$

has the following properties:

(i) F *satisfies in* $\mathbb{R}$ *the equation*

$$F''(x) - 2f(x)F'(x) = 0, \text{ for a.e. } x, \tag{1.2}$$

(ii) F *and* F^{-1} *are quasi-isometry, that is for any* x, $y \in \mathbb{R}$ *and* $\|f\|_1 = \int_{\mathbb{R}}|f(x)|\,dx$

$$\begin{aligned} e^{-2\|f\|_1}|x-y| &\leq |F(x)-F(y)| \leq e^{2\|f\|_1}|x-y|, \\ e^{-2\|f\|_1}|x-y| &\leq \left|F^{-1}(x)-F^{-1}(y)\right| \leq e^{2\|f\|_1}|x-y|. \end{aligned} \tag{1.3}$$

(iii) Both F *and its inverse function* F^{-1} *belongs to* $\mathcal{W}_1^2(\mathbb{R})$.

Proof. (i) For a.e. x, we calculate the derivative of F, we have $F(x) = \Phi(x) - \Phi(0)$, where

Φ is the primitive of $y \to \exp\left(2\int_0^y f(t)dt\right)$, then

$$F'(x) = \Phi'(x) = \exp\left(2\int_0^x f(t)dt\right),$$

we denote by Ψ the primitive of f, we get $F'(x) = \exp(2\Psi(x) - 2\Psi(0))$, that implies $F''(x) = 2\Psi'(x)\exp(2\Psi(x) - 2\Psi(0))$, then

$$F''(x) = 2f(x)\exp\left(2\int_0^x f(t)dt\right),$$

it's clear that $F''(x) - 2f(x)F'(x) = 0$, for a.e. x.

(ii) By definition the functions F and its inverse F^{-1} are continuous, one-to-one from $\mathbb{R}$ onto $\mathbb{R}$, strictly increasing functions, moreover $F'(x) = \exp\left(2\int_0^x f(t)dt\right)$, hence

$$\text{for every } x \in \mathbb{R},\ e^{-2\|f\|_1} \leq F'(x) \leq e^{2\|f\|_1} \text{ and } e^{-2\|f\|_1} \leq (F^{-1})'(x) \leq e^{2\|f\|_1}. \tag{1.4}$$

This shows that F is a quasi-isometry.

(iii) Using the inequality (1.4), we conclude that both F and F^{-1} are $\mathcal{C}^1(\mathbb{R})$. The second generalized derivative F'' satisfies $F''(x) = 2f(x)F'(x)$ for a.e. x, and since $f \in L^1(\mathbb{R})$, then F'' is an integrable function (but not necessarily continuous), thus, $F \in \mathcal{W}_1^2(\mathbb{R})$. Using again assertion (ii), we prove that F^{-1} belongs to $\mathcal{W}_1^2(\mathbb{R})$. ■

Remark 1.4.1

(i) If f is continuous, then both F and F^{-1} are of class $\mathcal{C}^2(\mathbb{R})$.

(ii) The previous lemma is very important in proving the existence and uniqueness of the solution (Theorem 3.4.1) and priori estimates (Proposition 3.3.1).

Lemma 1.4.2

For a given real number y and a measurable function $u(.) \in \mathfrak{L}_v^2$.

(i) The operator $[u]_f(y)$ *given by*

$$[u]_f(y) := \int_E \frac{F(y+u(e)) - F(y) - F'(y)u(e)}{F'(y)} \upsilon(de) \tag{1.5}$$

is well defined, where υ *is a finite Lévy measure given by (2.2). Moreover,*

$$\left|[u]_f(y)\right| \leq \left(1 + e^{4\|f\|_1}\right) \|u(.)\|_{\upsilon,1}\,, \tag{1.6}$$

such that $\|u(.)\|_{\upsilon,1} = \int_E |u(e)|\,\upsilon(de)$.

(ii) If f *is non-negative, then* $[u]_f(y) \geq 0$.

Proof. (i) From the quasi-isometry property (1.3) of the function F defined by (1.1), for all $y \in \mathbb{R}$, we have

$$\begin{aligned}
\left|[u]_f(y)\right| &\leq \int_E \left|\frac{F(y+u(e)) - F(y) - F'(y)u(e)}{F'(y)} \upsilon(de)\right| \\
&\leq \int_E \left|\frac{F(y+u(e)) - F(y)}{F'(y)}\right| \upsilon(de) + \int_E |u(e)|\,\upsilon(de) \\
&= \int_E \left|\frac{F(y+u(e)) - F(y)}{F'(y)}\right| \upsilon(de) + \|u(.)\|_{\upsilon,1}\,,
\end{aligned}$$

then

$$\begin{aligned}
\left|[u]_f(y)\right| &\leq \int_E \frac{e^{2\|f\|_1}\,|u(e)|}{F'(y)} \upsilon(de) + \|u(.)\|_{\upsilon,1} \\
&\leq \left(\frac{e^{2\|f\|_1}}{F'(y)} + 1\right) \|u(.)\|_{\upsilon,1}\,.
\end{aligned}$$

Thus

$$\left|[u]_f(y)\right| \leq \left(e^{4\|f\|_1} + 1\right) \|u(.)\|_{\upsilon,1}\,.$$

Hence the operator $[u]_f(.)$ is well defined.

(ii) We use the properties of F' for proof it. For each $y \in \mathbb{R}$, we can write

$$
\begin{aligned}
[u]_f(y) &= \frac{1}{F'(y)} \int_E \left(F(y+u(e)) - F(y) - F'(y)u(e)\right) \upsilon(de) \\
&= \frac{1}{F'(y)} \int_E \int_y^{y+u(e)} \left(F'(x) - F'(y)\right) dx \upsilon(de) \\
&= \frac{1}{F'(y)} \int_E \int_y^{y+u(e)} \left(F'(x) - F'(y)\right) \mathbf{1}_{\{u(e)>0\}} dx \upsilon(de) \\
&+ \frac{1}{F'(y)} \int_E \int_{y+u(e)}^{y} \left(F'(y) - F'(x)\right) \mathbf{1}_{\{u(e)<0\}} dx \upsilon(de).
\end{aligned}
$$

The last two terms in the above inequality are non-negative since F' is positive and increasing.

■

Lemma 1.4.3 *(Burkholder-Davis-Gundy Inequalities)*

Let M be a martingale with càdlàg paths, and let $p \geq 1$ be fixed. Let $M_t^* = \sup_{0\leq s\leq t} |M_s|$. Then, there exist constants c_p and C_p such that

$$
c_p \mathbb{E}\left[[M,M]_t^{\frac{p}{2}}\right] \leq \mathbb{E}\left[(M_t^*)^p\right] \leq C_p \mathbb{E}\left[[M,M]_t^{\frac{p}{2}}\right], \tag{1.7}
$$

for all $0 \leq t < \infty$. The constants c_p and C_p are universal: they do not depend on the choice of M.

Lemma 1.4.4 *(Hölder Inequality for Integrals)*

Let $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. If f and g are real functions defined on $[a,b]$ and if $|f|^p$, $|g|^q$ are integrable functions on $[a,b]$ then

$$
\int_a^b |f(x)g(x)|\, dx \leq \left(\int_a^b |f(x)|^p\, dx\right)^{\frac{1}{p}} \left(\int_a^b |g(x)|^q\, dx\right)^{\frac{1}{q}}. \tag{1.8}
$$

Chapter 2

Backward Stochastic Differential Equations with Jumps

The objective of this chapter is to present some basic definitions and results about Lévy processes and introduce the notion of backward stochastic differential equations with jumps (BSDEJs for short) driven by a Brownian motion and a compensated random measure, and we consider the case of Lipschitz continuous generators. We will show the classical result of existence and uniqueness. The content of this chapter is mainly based on these references [3, 11, 16, 26, 30, 34]

2.1 Pure Jump Lévy Processes

2.1.1 Basics on Lévy Process

We recall some basic properties of Lévy processes. This is briefly what we need this thesis about Lévy processes. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space.

Definition 2.1.1 *(Lévy Process)*

A one-dimensional Lévy process is a stochastic process $\eta = \eta(t), t \geq 0$*, such that* $\eta(t) =$

$\eta(t,\omega)$, $\omega \in \Omega$, *with the following properties:*

(i) $\eta(0) = 0$, $\mathbb{P}$-*a.s.*,

(ii) η *has independent increments, that is, for all* $t > 0$ *and* $h > 0$, *the increment* $\eta(t + h) - \eta(t)$ *is independent of* $\eta(s)$ *and* $s \leq t$,

(iii) η *has stationary increments, that is, for all* $h > 0$ *the increment* $\eta(t+h) - \eta(t)$ *has the same probability law as* $\eta(h)$,

(iv) It is stochastically continuous, that is, for every $t \geq 0$ *and* $\varepsilon > 0$, *then*

$$\lim_{s \to t} \mathbb{P}\{|\eta(t) - \eta(s)| > \varepsilon\} = 0,$$

(v) η *has càdlàg paths, that is, the trajectories are right-continuous with left limits.*

Let $\eta(t)$ be a one-dimensional Lévy process on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we know that η has a càdlàg modification, and we will always use this modification. We define the jumps of η at time t by $\Delta\eta = \eta_t - \eta_{t^-}$, where

$\eta_{t^-} = \lim_{s \to t} \eta_s$, $s < t$, (η_{t^-} is the left limit at time t) and $\eta_{0^-} = \eta_0 = 0$.

Let $E := \mathbb{R} \backslash 0$, and define $\mathcal{B}(E)$ as the $\sigma-$algebra generated by the family of all Borel subsets $U \subset \mathbb{R}$, such that $\bar{U} \subset E$. If $U \in \mathcal{B}(E)$ with $\bar{U} \subset E$ and $t > 0$, we define

$$N(t, U) = \sum_{0 \leq s \leq t} \mathbf{1}_{\{\Delta\eta_s \in U\}}, \; U \in \mathcal{B}(E), \; t \geq 0, \tag{2.1}$$

that is, the number of jumps of size $\Delta\eta_s \in U$ for any s in $0 \leq s \leq t$.

Since the paths of η are càdlàg, we know that $N(t, U) < \infty$, for all $U \in \mathcal{B}(E)$, with closure not containing zero. Further, 2.1, defines in a natural way a Poisson random measure N on $\mathcal{B}(0, \infty) \times \mathcal{B}(E)$ given by

$$(a, b] \times U \to N(b, U) - N(a, U), \; 0 < a \leq b, \; U \in \mathcal{B}(E).$$

This random measure is called the jump measure of η, and we use the notation $N(dt, de)$, $t > 0$, $e \in E$, for the differential form of the jump measure. The Lévy measure υ of η is defined by

$$\upsilon(U) := \mathbb{E}[N(1, U)],\ U \in \mathcal{B}(E), \tag{2.2}$$

which is known to satisfy the condition

$$\int_E min(1, e^2)\upsilon(de) < \infty.$$

Further, we define the compensated jump measure (also compensated Poisson random measure) $\tilde{N}$ on $\mathcal{B}(\mathbb{R}) \times \mathcal{B}(E)$ by

$$\tilde{N}(dt, de) := N(dt, de) - \upsilon(de)dt,$$

it is a martingale with a mean zero.

2.1.2 Stochastic Integration with Respect to Lévy Processes

Definition 2.1.2

For any t, let $\mathcal{F}_t$ be the $\sigma-$algebra generated by the random variables $W(s)$ and $\tilde{N}(ds, de)$, $e \in E$, $s \leq t$, augmented for all $\mathbb{P}-$null sets. Let the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with the corresponding filtration $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$. For any $\mathbb{F}$-adapted process $\theta = \theta(t, e)$, $t \geq 0$, $e \in E$ such that

$$\mathbb{E}\left[\int_0^T \int_E \theta^2(s, e)\upsilon(de)ds\right] < \infty,\ \textit{for some } T > 0,$$

we can see that the process

$$M_n(t) := \int_0^t \int_{|e|\geq \frac{1}{n}} \theta(s, e)\tilde{N}(ds, de),\ 0 \leq t \leq T,$$

is a martingale and its limit

$$M_t := \lim_{n\to\infty} M_n(t) = \int_0^t \int_E \theta(s,e)\tilde{N}(ds,de),\ 0 \le t \le T,$$

is also a martingale. Moreover, we have the Itô isometry

$$\mathbb{E}\left[\left(\int_0^T \int_E \theta(s,e)\tilde{N}(ds,de)\right)^2\right] = \mathbb{E}\left(\int_0^T \int_E \theta^2(s,e)\upsilon(de)ds\right).$$

Theorem 2.1.1 *(Properties of Stochastic Integration with Respect to Lévy Processes)*

(i) Let $V : \Omega \times [0,T] \times E \to \mathbb{R}$ be a predictable process satisfying

$$\mathbb{E}\left[\int_0^T \int_E |V(s,e)|\, \upsilon(de)ds\right] < \infty,$$

where we integrate w.r.t. the compensator of a Poisson random measure N.

We may define $\int_0^t \int_E V(s,e)N(ds,de)$ and $\int_0^t \int_E V(s,e)\upsilon(de)ds$ as integrable processes of finite variation. Further,

$$\int_0^t \int_E V(s,e)\tilde{N}(ds,de) = \int_0^t \int_E V(s,e)(N(ds,de) - \upsilon(de)ds)$$

is a martingale, and we have

$$E\left(\int_0^T \int_E V(s,e)N(ds,de)\right) = E\left(\int_0^T \int_E V(s,e)\upsilon(de)ds\right).$$

(ii) Let $V : \Omega \times [0,T] \times E \to \mathbb{R}$ be a predictable process satisfying

$$\int_0^T \int_E |V(s,e)|^2\, \upsilon(de)ds < \infty.$$

Then $(\int_0^t \int_E V(s,e)\tilde{N}(ds,de),\ 0 \le t \le T)$ is a càdlàg local martingale with the quadratic *variation process, for all* $0 \leq t \leq T$

$$\left[\int_0^\cdot \int_E V(s,e)\tilde{N}(ds,de), \int_0^\cdot \int_E V(s,e)\tilde{N}(ds,de)\right](t) = \int_0^t \int_E |V(s,e)|^2 N(ds,de).$$

(iii) Let $V : \Omega \times [0,T] \times E \to \mathbb{R}$ *be a predictable process satisfying*

$$\int_0^T \int_E |V(s,e)|\, \upsilon(de)ds < \infty.$$

Then $(\int_0^t \int_E V(s,e)\tilde{N}(ds,de), 0 \leq t \leq T)$ *is a càdlàg local martingale and* $(\int_0^t \int_E V(s,e)N(ds,de),$ $0 \leq t \leq T)$ *is a càdlàg process. Let* N *be the jump measure of a càdlàg process* J*. We also have the property*

$$\int_0^t \int_E V(s,e)N(ds,de) = \sum_{s\in(0,t]} V(s,\Delta J(s))\mathbf{1}_{\Delta J(s)\neq 0}(s),\ 0 \leq t \leq T.$$

(iv) Let $V \in \mathbb{H}^2_N(\mathbb{R})$*, then is a càdlàg, square-integrable martingale which satisfies*

$$\mathbb{E}\left[\left|\int_0^T \int_E V(s,e)\tilde{N}(ds,de)\right|^2\right] = \mathbb{E}\left[\int_0^T \int_E |V(s,e)|^2\, \upsilon(de)ds\right].$$

- *We also recall that*

$$\left[\int_0^\cdot V_1(s)dW(s), \int_0^\cdot \int_E V_2(s,e)\tilde{N}(ds,de)\right](T) = 0. \tag{2.3}$$

2.1.3 Martingale Representation Theorem

Theorem 2.1.2 *(The Property of Predictable Representation)*

The predictable representation property is the key concept in the theory of BSDEJs which allows us to construct a solution to a BSDEJs. Let us consider a probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$

with a filtration $\mathbb{F} = (\mathcal{F}_t)_{0 \le t \le T}$*, any* $\mathbb{F}$*-local martingale* M *has the representation*

$$M(t) = M(0) + \int_0^t Z(s)dW(s) + \int_0^t \int_E U(s,e)\tilde{N}(ds,de), \; 0 \le t \le T, \tag{2.4}$$

where Z *and* U *are* $\mathbb{F}$*-predictable processes integrable with respect to* W *and* $\tilde{N}$*. If* M *is a locally square integrable local martingale, then the processes* Z *and* U *are locally square integrable.*

- *We can easily deduce that the representation of a square integrable martingale* M *is unique in* $\mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2_N(\mathbb{R})$*. Consequently, any square integrable* $\mathbb{F}$*-martingale* M *has the unique representation (2.4), where* $(Z,U) \in \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2_N(\mathbb{R})$*.*

- *We can also assume that any square integrable* $\mathcal{F}_T$*-measurable random variable* ξ *has the unique representation*

$$\xi = \mathbb{E}(\xi) + \int_0^T Z(s)dW(s) + \int_0^T \int_{\mathbb{R}} U(s,z)\tilde{N}(ds,dz), \tag{2.5}$$

where $(Z,U) \in \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2_N(\mathbb{R})$*. Representation (2.5) follows immediately from (2.4) by taking the martingale* $M(t) = \mathbb{E}(\xi \mid \mathcal{F}_t)$*,* $0 \le t \le T$*.*

2.2 BSDEJs with Globally Lipschitz Generators

2.2.1 Notation and Definitions

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let W be a one-dimensional Brownian motion and $N(dt,de)$ be a Poisson random measure with compensator $\upsilon(de)dt$ such that υ is a $\sigma-$finite measure on $E := \mathbb{R}\backslash 0$, equipped with its Borel field $\mathcal{B}(E)$. Let $\tilde{N}(dt,de)$ be its compensated process. Let $\mathbb{F} = (\mathcal{F}_t)_{t \ge 0}$, be the natural filtration generated both by the two mutually independant processes W and N, it is a complete right continuous filtration.

Definition 2.2.1 *(Driver, Lipschitz Driver)*

A function f is said to be a driver if

- $f : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathcal{L}_v^2 \to \mathbb{R}$

$(\omega, t, y, z, u(.)) \to f(\omega, t, y, z, u(.))$ *is* $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^2) \otimes \mathcal{B}(\mathcal{L}_v^2)$*- measurable,*

- $f(., 0, 0, 0) \in \mathbb{H}^2(\mathbb{R})$.

A driver f is called a Lipschitz driver if moreover there exists a constant $C \geq 0$ such that $d\mathbb{P} \otimes dt$-a.s., for each (y_1, z_1, u_1), (y_2, z_2, u_2)

$$|f(\omega, t, y_1, z_1, u_1) - f(\omega, t, y_2, z_2, u_2)| \leq c\left(|y_1 - y_2| + |z_1 - z_2| + \|u_1 - u_2\|_v\right).$$

Definition 2.2.2 *(BSDEs with Jumps)*

We deal with backward stochastic differential equations driven by a Brownian motion and a compensated random measure and we consider the case of Lipschitz continuous generators. The backward stochastic differential equations with jumps (BSDEJs) were first introduced and studied by Tang and Li [37] and it has the following form

$$Y_t = \xi + \int_t^T f(s, Y(s), Z(s), U(s,.))ds - \int_t^T Z(s)dW(s) - \int_t^T \int_E U(s,e)\tilde{N}(ds, de), \quad (2.6)$$

for all $0 \leq t \leq T$, where W is a Brownian motion and $\tilde{N}$ is a compensated random measure. Given a terminal condition ξ (this is why we say backward), sometimes call ξ the end point, and f a generator . (ξ, f) is called a standard parameter.

Definition 2.2.3 *(Solution of the BSDEJs)*

Solution of the BSDEJs is a triple $(Y, Z, U) \in \mathbb{M}_S^2$ such that (Y, Z, U) satisfied the dynamics

$$dY(t) = -f(t, Y(t), Z(t), U(t,.))dt + Z(t)dW(t) + \int_E U(t,e)\tilde{N}(dt, de), \quad (2.7)$$

and satisfies $Y(T) = \xi$, where ξ is a square integrable random variable ($\xi \in L_T^2(\Omega)$). The processes Z and U are called control processes. They control the process Y so that Y satisfies *the terminal condition.*

Remark 2.2.1

The predictable representation property (2.1.2) is the key concept in the theory of BSDEJs since it allows us to find an adapted solution to an equation with a random terminal condition.

2.2.2 BSDEJs with Zero Generator

Let us deal with the equation

$$\begin{cases} dY_t = Z_t dW_t + \int_E U(t,e)\tilde{N}(dt,de) \\ Y(T) = \xi. \end{cases} \tag{2.8}$$

Equation 2.8 is called a backward stochastic differential equation with zero generator. It is the simplest example of a BSDEJs. We are interested in finding a triple (Y, Z, U) which satisfies

$$Y_t = \xi - \int_t^T Z_s dW_s - \int_t^T \int_E U(s,e)\tilde{N}(ds,de),\ 0 \le t \le T. \tag{2.9}$$

Theorem 2.2.1 *(Existence and Uniqueness of Adapted Solutions)*

Assume that $\xi \in L^2_T(\mathbb{R})$. There exists a unique solution $(Y, Z, U) \in \mathbb{M}^2_S$ to the BSDEJs 2.9. The process Y has the representation

$$Y(t) = \mathbb{E}\left[\xi \mid F_t\right],\ 0 \le t \le T,$$

and the control processes (Z, U) are derived from the representation

$$\xi = \mathbb{E}(\xi) + \int_0^T Z_s dW_s + \int_0^T \int_E U(s,e)\tilde{N}(ds,de).$$

Proof. See [11] ∎

2.2.3 BSDEJs with Generator Independent to Y, Z and U

Let us deal with the equation

$$Y_t = \xi + \int_t^T f(s)ds - \int_t^T Z_s dW_s - \int_t^T \int_E U(s,e)\tilde{N}(ds,de),\ 0 \leq t \leq T, \tag{2.10}$$

which we call a BSDEJs with generator independent of (Y, Z, U). We can immediately prove the following result.

Theorem 2.2.2 *(Existence and Uniqueness of Adapted Solutions)*

Assume that $\xi \in L_T^2(\Omega)$ and $f : \Omega \times [0,T] \to \mathbb{R}$ is a predictable process satisfying $\mathbb{E}\left(\left[\int_0^T |f(s)|^2\, ds\right]\right) < \infty$. There exists a unique solution $(Y, Z, U) \in \mathbb{M}_S^2$ to the BSDEJs (2.10). The process Y has the representation

$$Y(t) = \mathbb{E}\left(\xi + \int_t^T f(s)ds \mid \mathcal{F}_t\right),\ 0 \leq t \leq T,$$

and the control processes (Z, U) are derived from the representation

$$\xi + \int_0^T f(s)ds = \mathbb{E}\left(\xi + \int_0^T f(s)ds\right) + \int_0^T Z_s dWs + \int_0^T \int_E U(s,e)\tilde{N}(ds,de).$$

Proof. See [25]. ■

2.2.4 BSDEJs in General Case

We recall that $v(de)dt$ denote the compensator of the random measure N. We assume that

(H1) The terminal value $\xi \in L_T^2(\Omega)$,

(H2) The generator $f : \Omega \times [0,T] \times \mathbb{R} \times \mathbb{R} \times \mathcal{L}_v^2(\mathbb{R}) \to \mathbb{R}$ is predictable and Lipschitz continuous in the sense that

$$|f(\omega,t,y,z,u)-f(\omega,t,y',z',u')|^2 \leq K\left(|y-y'|^2+|z-z'|^2+\int_E |u(x)-u'(x)|^2\,v(de)\right),$$

a.s., a.e. $(\omega,t)\in\Omega\times[0,T]$, for all $(y,z,u),(y',z',u')\in\mathbb{R}\times\mathbb{R}\times\mathcal{L}_v^2(\mathbb{R})$,

(H3) $\mathbb{E}\left(\int_0^T |f(t,0,0,0)|^2\,dt\right)<\infty$.

Assumptions *(H1),(H2)* and *(H3)* are called standard in the theory of BSDEJs. Let us recall that the predictability of the generator f means that

$f:\Omega\times[0,T]\times\mathbb{R}\times\mathbb{R}\times\mathcal{L}_v^2(\mathbb{R})\to\mathbb{R}$ is $\mathcal{P}\otimes\mathcal{B}(\mathbb{R})\otimes\mathcal{B}(\mathbb{R})\otimes\mathcal{B}(\mathcal{L}_v^2(\mathbb{R}))$ measurable.

We now prove the existence of a unique solution to the BSDEJs (2.6). The idea is to construct a sequence of solutions to simpler BSDEJs for which the existence of a unique solution can be established by Theorem (2.2.2) and the predictable representation property (Theorem 2.1.2). Next, we shall prove that it is a Cauchy sequence in the Banach space $\mathbb{M}_S^2$.

Theorem 2.2.3

Assume that (H1),(H2) *and* (H3) *hold. The BSDEJs (2.6) has a unique solution* $(Y,Z,U)\in\mathbb{M}_S^2$.

Proof. See for example [11, 25, 37]. ■

2.2.5 Special Case

If the BSDEJs (2.6) abandons the jumps part, we get

$$Y_t=\xi+\int_t^T f(s,Y(s),Z(s))ds-\int_t^T Z(s)dW(s),\ 0\leq t\leq T. \tag{2.11}$$

Under the integrability condition of ξ and $(f(t,0,0))_{t\in[0,T]}$ and Lipschitz condition in y and z, then, by the *Pardoux-Peng* result, the BSDEJs (2.11) has a unique solution (Y,Z).

Proof. See [32]. ■

Chapter 3

Quadratic Backward Stochastic Differential Equations with Jumps

Due to the important role played by Quadratic BSDEJs (QBSDEJs for short) in solving most problems in various fields, it is necessary to focus on their existence and uniqueness of solutions. In this chapter, we consider one-dimensional BSDEJs that show quadratic growth in the Brownian component and non-linear functionals with respect to the jump term. This type of equation does not satisfy the usual Lipschitz condition to ensure the existence and uniqueness of solutions. However, using Zvonkin transformation to eliminate the drift or a part of it, we can prove the existence and uniqueness of solutions for different types of QBSDEJs. This chapter is mainly based on [26].

3.1 Introduction

We start with a stochastic basis (a filtered probability space) $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$ with a finite time horizon $T < \infty$, $E = \mathbb{R} \backslash \{0\}$ endowed with the Borel $\sigma-$algebra $\mathcal{B}(E)$. On this stochastic basis, let $W = (W_t)_{t \in [0,T]}$ a one-dimensional standard Brownian motion, and $N(ds, de)$ a time-homogeneous Poisson random measure with compensator $\upsilon(de)ds$ on $([0,T] \times E,\ \mathcal{B}([0,T]) \otimes \mathcal{B}(E))$.

We denote by $\tilde{N}(ds, de) = N(ds, de) - \upsilon(de)ds$ the compensated jump measure, $\upsilon(de)$ is a finite Lévy measure, suppose that the Brownian motion W and the random measure $N(ds, de)$ are stochastically independent under $\mathbb{P}$. The filtation $\mathbb{F} = \{\mathcal{F}_t \mid t \in [0, T]\}$ is the natural filtration generated both by the two independant processes W and $\tilde{N}$. It is a complete with $\mathbb{P}$-null sets and right continuous, we assume that $\mathcal{F}_0$ is trivial. Moreover, we assume that the filtration $\mathbb{F}$ has the predictable representation property, that is any local martingale can be written as the sum of two stochastic integrals with predictable integrand processes, one w.r.t. W and the other w.r.t. $\tilde{N}$. Let ξ be an $\mathcal{F}_T$-measurable $\mathbb{R}$-valued random variable.

The $\mathbb{R}$-valued QBSDEJs is given by the following form

$$\begin{cases} -dY_t = H(Y_t, Z_t, U_t(.))dt - Z_t dW_t - \int_E U_t(e)\tilde{N}(dt, de), \ t \in [0, T[\,. \\ Y_T = \xi. \end{cases}$$

Or, equivalently, $\mathbb{P}-$a.s for all $t \in [0, T]$

$$Y_t = \xi + \int_t^T H(Y_s, Z_s, U_s(.))ds - \int_t^T Z_s dW_s - \int_t^T \int_E U_s(e)\tilde{N}(ds, de). \tag{3.1}$$

The previous equation will be denoted by $eq(\xi, H)$. The data ξ called the terminal condition (the terminal value), and will be assumed to be square integrable, and H called the coefficient or the generator of $eq(\xi, H)$. For the BSDEs of quadratic type associated with a Brownian component and without the jumps parts have been studied by Bahlali et al. [3], and this work is considered an extension of it.

For the special generators H of the following form:

$$H(y, z, u(.)) = \begin{cases} f(y)\,|z|^2 + [u]_f\,(y) =: H_f(y, z, u(.)) \\ h(y, u(.)) + cz + H_f(y, z, u(.)) \\ cz + f(y)\,|z|^2 - \int_E u(e)\upsilon(de) \end{cases}$$

where:

- f is a measurable and integrable function, and we assumed that it is not constant.
- $[u]_f(y)$ is a functional of the unknown processes Y, and given by

$$[u]_f(y) = \int_E \frac{F(y+u(e)) - F(y) - F'(y)u(e)}{F'(y)} \upsilon(de),$$

and the function F is defined, for every $x \in \mathbb{R}$, by

$$F(x) = \int_0^x \exp\left(2\int_0^y f(t)dt\right) dy. \tag{3.2}$$

- h enjoy some classical assumptions.

Remark 3.1.1

The function F (3.2) which has the properties mentioned in the first chapter (Lemma 1.4.1) allows us to eliminate the generator H_f in the eq(ξ, H_f).

We are interested in proving the existence and uniqueness of the solution to one type of quadratic BSDE with jumps and globally integrable generator (drift) H_f, and in the section (3.5), we will solve some quadratic BSDEJs. We racall that the $eq(\xi, H_f)$ is given by

$$Y_t = \xi + \int_t^T \left(f(Y_s)\left|Z_s\right|^2 + [U_s]_f(Y_s)\right) ds - \int_t^T Z_s dW_s - \int_t^T \int_E U_s(e)\tilde{N}(ds, de), \tag{3.3}$$

for all $t \in [0, T]$.

Definition 3.1.1 *(Solution of QBSDEJs)*

Let ξ is an $\mathcal{F}_T$-measurable and square integrable random variable and $f : \mathbb{R} \to \mathbb{R}$ is a measurable function. A triple of processes (Y, Z, U), with adapted Y and predictable Z and U, is a solution of eq(ξ, H_f) if:

- $\mathbb{E}\left[\left|\int_0^T f(Y_s)\left|Z_s\right|^2 ds\right|^2\right] < \infty,$

- (Y, Z, U) *satisfies*

$$Y_t = \xi + \int_t^T H_f(Y_s, Z_s, U_s(.))ds - \int_t^T Z_s dW_s - \int_t^T \int_E U_s(e)\tilde{N}(ds, de),$$

$\mathbb{P}$*-a.s. for all* $t \in [0, T]$, *such that* $Z \in \mathbb{H}^2(\mathbb{R})$, $U \in \mathbb{H}^2_N(\mathbb{R})$, $Y \in \mathbb{S}^2(\mathbb{R})$.

Corollary 3.1.1

For a given real number y *and a predictable process* $u_s(e)$ *on* $[0,T]\times E$, *such that* $\int_0^T \|u_s(.)\|_{\upsilon,1} ds < \infty$, $\mathbb{P}-a.s.$ *We have, using (1.6)*

$$\int_0^T \left|[u_s]_f(y)\right| ds \leq \left(1 + e^{4\|f\|_1}\right) \int_0^T \int_E |u_s(e)|\, \upsilon(de)ds, \ \mathbb{P}-a.s., \ \forall y \in \mathbb{R}.$$

Moreover, if $u \in \mathbb{H}^2_N(\mathbb{R})$, *then there exists a constant* $C_{f,\upsilon}$ *(depending only on* f *and* υ*) such that*

$$\int_0^T \mathbb{E}\left(\left|[u_s]_f(y)\right|^2\right) ds \leq C_{f,\upsilon} \int_0^T \mathbb{E}\,\|u_s(.)\|^2_{\upsilon,2}\, ds. \tag{3.4}$$

3.2 Krylov's Estimates and Itô-Krylov's Formula for BSDEJs

We establish a Krylov-type estimate and an Itô-Krylov change of variable formula for the solutions of one-dimensional quadratic backward stochastic differential equations with jump (QBSDEJs) defined by (3.1). This allows us to prove various existence and uniqueness results for some classes of QBSDEJs.

3.2.1 Krylov's Estimates in BSDEJs

Proposition 3.2.1

Let $(Y, Z, U) := (Y_t, Z_t, U_t(e))_{0\leq t\leq T, e\in E}$ *be a solution to* $eq(\xi, H_f)$ *in the sense of the Defi-*

nition (3.1.1). Put

$$\theta = 2\sup_{0\leq t\leq T} |Y_t| + \left(2 + e^{4\|f\|_1}\right) \int_0^T \|U_s(.)\|_{v,1}\, ds,$$

then, for any measurable and integrable function ϕ on $\mathbb{R}$, we have

$$\mathbb{E}\left(\int_0^T |\phi|\,(Y_s)\,|Z_s|^2\, ds\right) \leq 2\mathbb{E}(\theta)\,\|\phi\|_1\, e^{2\|f\|_1}. \tag{3.5}$$

3.2.2 Itô–Krylov Change of Variable Formula in BSDEJs

Theorem 3.2.1

Let $(Y_t, Z_t, U_t(e))_{0\leq t\leq T, e\in E}$ be a solution to $eq(\xi, H)$. Then, for any function g belonging to the space $\mathcal{W}^2_{1,loc}(\mathbb{R})$, we have with probability 1

$$\begin{aligned} g(Y_t) &= g(Y_0) + \int_0^t g'(Y_{s^-})dY_s + \frac{1}{2}\int_0^t g''(Y_s)d\,[Y,Y]^c_s \\ &\quad + \int_0^t \int_E \left[g(Y_{s^-} + U_s(e)) - g(Y_{s^-}) - g'(Y_{s^-})U_s(e)\right] N(ds, de), \end{aligned} \tag{3.6}$$

where $d\,[Y,Y]^c_s = |Z_s|^2\, ds$.

Proof. See [26] for full proof. ■

The new Equation

If (Y, Z, U) is a solution to $eq(\xi, H_f)$, then we apply Itô–Krylov's formula (3.6) to $F(Y_t)$ such that F is given by (1.1) to obtain an equation with zero generator, and we get

$$\begin{aligned} F(Y_t) &= F(\xi) + \int_t^T \left(F'(Y_{s^-})\left[f(Y_s)\,|Z_s|^2 + [U_s]_f\,(Y_s)\right] - \frac{1}{2}F''(Y_s)\,|Z_s|^2\right) ds \\ &\quad - \int_t^T F'(Y_{s^-})Z_s dW_s - \int_t^T \int_E F'(Y_{s^-})U_s(e)\tilde{N}(ds, de) \\ &\quad - \int_t^T \int_E \left[F(Y_{s^-} + U_s(e)) - F(Y_{s^-}) - F'(Y_{s^-})U_s(e)\right] N(ds, de), \end{aligned}$$

remember that

$$F'(y)\left[u\right]_f(y) = \int_E \left[F(y+u(e)) - F(y) - F'(y)u(e)\right] v(de), \tag{3.7}$$

and we use the property (1.2) of F and (3.7), this implies

$$\begin{aligned} F(Y_t) = F(\xi) &+ \int_t^T \int_E \left(F(Y_{s^-}+U_s(e)) - F(Y_{s^-}) - F'(Y_{s^-})U_s(e)\right) v(de)ds \\ &- \int_t^T F'(Y_{s^-})Z_s dW_s + \int_t^T \int_E F'(Y_{s^-})U_s(e)v(de)ds \\ &- \int_t^T \int_E \left(F(Y_{s^-}+U_s(e)) - F(Y_{s^-})\right) N(ds,de), \end{aligned}$$

then

$$F(Y_t) = F(\xi) - \int_t^T F'(Y_{s^-})Z_s dW_s - \int_t^T \int_E \left(F(Y_{s^-}+U_s(e)) - F(Y_{s^-})\right) \tilde{N}(ds,de). \tag{3.8}$$

For each $0 \le s \le T$, we set $y_s := F(Y_s)$, $z_s := F'(Y_{s-})Z_s$ and $u_s(e) := F(Y_{s^-}+U_s(e)) - F(Y_{s^-})$.

Before proving existence and uniqueness of solutions to $eq(\xi, H_f)$, we establish some useful a priori estimates.

3.3 A Priori Estimates

Proposition 3.3.1

Let $\xi \in L^2_T(\Omega)$ and f is an integrable function ($\in L^1(\mathbb{R})$). If (Y, Z, U) satisfies the $eq(\xi, H_f)$, then we have:

(i) $(z_s)_{0\le s\le T}$, $(Z_s)_{0\le s\le T} \in \mathbb{H}^2(\mathbb{R})$ *and* $(u_s(e))_{0\le s\le T, e\in E}$, $(U_s(e))_{0\le s\le T, e\in E} \in \mathbb{H}^2_N(\mathbb{R})$,

(ii) $(y_s)_{0\le s\le T}$, $(Y_s)_{0\le s\le T} \in \mathbb{S}^2(\mathbb{R})$,

(*iii*) $\mathbb{E}\left[\left|\int_0^T f(Y_s)\left|Z_s\right|^2 ds\right|^2\right] < \infty.$

Proof. *In this proof we will repeatedly use Properties in subsections (1.3.1, 2.1.2) of stochastic integrals. We recall that the convex inequality is given by*

$$(a+b)^2 \leq 2(a^2+b^2). \tag{3.9}$$

(*i*) *We take* $t=0$ *and the square of the* $L^2(\Omega)$ *norm in Itô–Krylov's formula (3.8), we get*

$$\mathbb{E}\left[\left|\int_0^T F'(Y_{s^-})Z_s dW_s + \int_0^T \int_E \left[F(Y_{s^-}+U_s(e)) - F(Y_{s^-})\right]\tilde{N}(ds,de)\right|^2\right]$$
$$= \mathbb{E}\left|F(\xi) - F(Y_0)\right|^2.$$

Thanks to the orthogonality of the martingales $W_.$ *and* $\int_0^. \int_E \tilde{N}(ds,de)$ *(the Property 2.3) together with the inequalities (1.3) , (1.4) and (3.9), we get*

$$\begin{aligned}
&e^{-4\|f\|_1}\left[\mathbb{E}\int_0^T |Z_s|^2 ds + \mathbb{E}\int_0^T \int_E |U_s(e)|^2 \upsilon(de)ds\right]\\
&\leq \mathbb{E}\left|\int_0^T F'(Y_{s^-})Z_s dW_s\right|^2 + \mathbb{E}\left|\int_0^T \int_E \left[F(Y_{s^-}+U_s(e)) - F(Y_{s^-})\right]\tilde{N}(ds,de)\right|^2\\
&= \mathbb{E}\left|\int_0^T z_s dW_s\right|^2 + \mathbb{E}\left|\int_0^T \int_E u_s(e)\tilde{N}(ds,de)\right|^2\\
&= \mathbb{E}\int_0^T |z_s|^2 ds + \mathbb{E}\int_0^T \int_E |U_s(e)|^2 \upsilon(de)ds\\
&\leq \mathbb{E}\left[e^{4\|f\|_1}|\xi - Y_0|^2\right] \leq 2e^{4\|f\|_1}\left[Y_0^2 + \mathbb{E}|\xi|^2\right] < \infty.
\end{aligned}$$

This implies that z, $Z \in \mathbb{H}^2(\mathbb{R})$ *and* u, $U \in \mathbb{H}_N^2(\mathbb{R})$.

(ii) From Itô–Krylov's formula (3.8), quasi-isometry property (1.3) and $F(0) = 0$, *we have*

$$\begin{aligned}
e^{-2\|f\|_1} |Y_t| \leq |F(Y_t)| \\
&\leq |F(\xi)| + \left| \int_t^T F'(Y_{s^-}) Z_s dW_s \right| \\
&+ \left| \int_t^T \int_E \left[F(Y_{s^-} + U_s(e)) - F(Y_{s^-}) \right] \tilde{N}(ds, de) \right| \\
&\leq |F(\xi)| + \sup_{t \in [0,T]} \left| \int_0^t z_s dW_s \right| + \sup_{t \in [0,T]} \left| \int_0^t \int_E u_s(e) \tilde{N}(ds, de) \right|,
\end{aligned}$$

we use convex inequality (3.9) and take the squar and the supremum over $[0, T]$

$$\begin{aligned}
e^{-4\|f\|_1} \sup_{t \in [0,T]} |Y_t|^2 \leq \sup_{t \in [0,T]} |F(Y_t)|^2 \\
&\leq 2^2 \left(e^{4\|f\|_1} |\xi|^2 + \sup_{t \in [0,T]} \left| \int_0^t z_s dW_s \right|^2 + \sup_{t \in [0,T]} \left| \int_0^t \int_E u_s(e) \tilde{N}(ds, de) \right|^2 \right)
\end{aligned}$$

using BDG inequality (1.7), we have

$$\begin{aligned}
&\mathbb{E} \left(\sup_{t \in [0,T]} \left| \int_0^t z_s dW_s \right| \right)^2 \leq C_2 \mathbb{E} \left(\left[\int_0^{\cdot} z_s dW_s, \int_0^{\cdot} z_s dW_s \right]_T \right) = C_2 \mathbb{E} \left(\int_0^T |z_s|^2 ds \right), \\
&\mathbb{E} \left(\sup_{t \in [0,T]} \left| \int_0^t \int_E u_s(e) \tilde{N}(ds, de) \right| \right)^2 \leq \acute{C}_2 \mathbb{E} \left(\int_0^T \int_E |u_s(e)|^2 \upsilon(de) ds \right).
\end{aligned}$$

Now, by taking the expectation and previous inequalities, we get

$$\begin{aligned}
e^{-4\|f\|_1} \mathbb{E} \left(\sup_{t \in [0,T]} |Y_t|^2 \right) \leq \mathbb{E} \left[\sup_{t \in [0,T]} |y_t|^2 \right] \\
&\leq 4 \left(e^{4\|f\|_1} \mathbb{E} |\xi|^2 + C_2 \mathbb{E} \int_0^T |z_s|^2 ds + \acute{C}_2 \mathbb{E} \int_0^T \|u_s(.)\|_{\upsilon,2}^2 ds \right).
\end{aligned}$$

The right-hand side of the above inequality is finite by (i).

(iii) Since (Y, Z, U) satisfies $eq(\xi, H_f)$ and for $t = 0$, thus

$$\int_0^T f(Y_s)\left|Z_s\right|^2 ds = Y_0 - \xi - \int_0^T \left[U_s\right]_f (Y_{s^-})ds + \int_0^T Z_s dW_s + \int_0^T \int_E U_s(e)\tilde{N}(ds, de).$$

Now, using the inequalities: convex (3.9) , Hölder's (1.8) and (3.4) in Corollary (3.1.1), and taking the expectation, we obtain

$$\begin{aligned}
\mathbb{E}\left|\int_0^T f(Y_s)\left|Z_s\right|^2 ds\right|^2 &\leq 2^4\left(\mathbb{E}\left|\int_0^T Z_s dW_s\right|^2 + \mathbb{E}\left|\int_0^T \int_E U_s(e)\tilde{N}(ds, de)\right|^2\right) \\
&+ 2^4\left(\left|Y_0\right|^2 + \mathbb{E}\left|\xi\right|^2 + T\int_0^T \mathbb{E}\left|\left[U_s\right]_f (Y_{s^-})\right|^2 ds\right) \\
&\leq 16\left(\int_0^T \mathbb{E}\left|Z_s\right|^2 ds + \int_0^T \mathbb{E}\left\|U_s(.)\right\|_{\upsilon,2}^2 ds\right) \\
&+ 16\left(\left|Y_0\right|^2 + \mathbb{E}\left|\xi\right|^2 + TC_{f,\upsilon}\int_0^T \mathbb{E}\left\|U_s(.)\right\|_{\upsilon,2}^2 ds\right).
\end{aligned}$$

Finally $\mathbb{E}\left|\int_0^T f(Y_s)\left|Z_s\right|^2 ds\right|^2$ *is finite thanks to (i).* ■

3.4 Existence and Uniqueness of the Solution

We denote by $eq(F(\xi), 0)$ for the following simple equation

$$y_t = F(\xi) - \int_t^T z_s dW_s - \int_t^T \int_E u_s(e)\tilde{N}(ds, de).$$

Theorem 3.4.1

Let ξ be an $\mathcal{F}_T$-measurable and square integrable random variable. If f is an integrable function, then $(Y_t, Z_t, U_t(e))_{0\leq t\leq T, e\in E}$ is a unique solution to $eq(\xi, H_f)$ if and only if $(y_t, z_t, u_t(e))_{0\leq t\leq T, e\in E}$ is a unique solution to $eq(F(\xi), 0)$.

Proof. • *If $(Y_t, Z_t, U_t(e))_{0\leq t\leq T, e\in E}$ is a solution to $eq(\xi, H_f)$, then (3.8) shows that $(y_t, z_t, u_t(e))_{0\leq t\leq T, e\in E}$ satisfies $eq(F(\xi), 0)$.*

Since F and F^{-1} are globally Lipschitz, it follows that $F(\xi)$ is square integrable if and only if ξ is square integrable. Moreover, thank to the Proposition (3.3.1), we have $(y_t, z_t, u_t(e))_{0\le t\le T, e\in E} \in \mathbb{M}^2_S$. Then, by the Theorem (2.2.1) in the chapter 2,

$(y_t, z_t, u_t(e))_{0\le t\le T, e\in E}$ is a unique solution to $eq(F(\xi), 0)$.

- *If $(y_t, z_t, u_t(e))_{0\le t\le T, e\in E}$ be a solution to $eq(F(\xi), 0)$, then Itô–Krylov's formula (3.6) applied to $Y_t = F^{-1}(y_t)$ (Because according to Lemma (1.4.1), F^{-1} belongs to $\mathcal{W}^2_1(\mathbb{R})$) shows that*

$$
\begin{aligned}
F^{-1}(y_t) = F^{-1}(F(\xi)) &- \int_t^T (F^{-1})'(y_{s^-}) z_s dW_s \\
&- \int_t^T \int_E (F^{-1})'(y_{s^-}) u_s(e) \tilde{N}(ds, de) - \frac{1}{2}\int_t^T (F^{-1})''(y_s) \left|z_s\right|^2 ds \\
&- \int_t^T \int_E \left(F^{-1}(y_{s^-} + u_s(e)) - F^{-1}(y_{s^-}) - (F^{-1})'(y_{s^-}) u_s(e)\right) N(ds, de),
\end{aligned}
$$

then, by adding and subtracting the same term

$$
\int_t^T \int_E \left[F^{-1}(y_{s^-} + u_s(e)) - F^{-1}(y_{s^-}) - (F^{-1})'(y_{s^-}) u_s(e)\right] \upsilon(de) ds,
$$

we get

$$
\begin{aligned}
Y_t = \xi &- \int_t^T (F^{-1})'(y_{s^-}) z_s dW_s - \frac{1}{2}\int_t^T (F^{-1})''(y_s) \left|z_s\right|^2 ds \\
&- \int_t^T \int_E \left(F^{-1}(y_{s^-} + u_s(e)) - F^{-1}(y_{s^-})\right) \tilde{N}(ds, de) \\
&- \int_t^T \int_E \left(F^{-1}(y_{s^-} + u_s(e)) - F^{-1}(y_{s^-}) - (F^{-1})'(y_{s^-}) u_s(e)\right) \upsilon(de) ds.
\end{aligned}
\tag{3.10}
$$

We remember the derivatives of inverse function is given by

$$
(F^{-1})'(x) = \frac{1}{F'\left(F^{-1}(x)\right)} \qquad \textit{and} \qquad (F^{-1})''(x) = -\frac{F''(F^{-1}(x))}{[F'\left(F^{-1}(x)\right)]^3}.
$$

We set $Z_s = (F^{-1})'(y_{s^-})z_s$ *and* $U_s(e) = F^{-1}(y_{s^-} + u_s(e)) - F^{-1}(y_{s^-})$, *this implies*

$$\frac{1}{2}(F^{-1})''(y_s)\left|z_s\right|^2 = -\frac{F''\left(Y_s\right)}{2\left(F'\left(Y_s\right)\right)^3}\frac{\left|Z_s\right|^2}{\left[(F^{-1})'(y_{s^-})\right]^2} \overset{ds\ a.e}{=} -\frac{F''\left(Y_s\right)}{2F'\left(Y_s\right)}\left|Z_s\right|^2 = -f(Y_s)\left|Z_s\right|^2, \quad (3.11)$$

and

$$\begin{aligned}
&\int_E \left(F^{-1}(y_{s^-} + u_s(e)) - F^{-1}(y_{s^-}) - (F^{-1})'(y_{s^-})u_s(e)\right)\upsilon(de) \qquad (3.12)\\
&= \int_E \left(U_s(e) - \frac{1}{F'\left(Y_{s^-}\right)}\left(F(Y_{s^-} + U_s(e)) - F(Y_{s^-})\right)\right)\upsilon(de)\\
&= -\int_E \frac{F(Y_{s^-} + U_s(e)) - F(Y_{s^-}) - F'\left(Y_{s^-}\right)U_s(e)}{F'\left(Y_{s^-}\right)}\upsilon(de)\\
&= -\left[U_s\right]_f\left(Y_{s^-}\right).
\end{aligned}$$

Substituting (3.11) *and* (3.12) *in* (3.10), *we end up with*

$$Y_t = \xi + \int_t^T \left(f(Y_s)\left|Z_s\right|^2 + \left[U_s\right]_f\left(Y_{s^-}\right)\right)ds - \int_t^T Z_s dW_s - \int_t^T\int_E U_s(e)\tilde{N}(ds, de).$$

Thanks to the properties (1.3) *and* (1.4) *of* F, *we show easily that*

$$\left|Y_s\right| \leq e^{2\|f\|_1}\left|y_s\right|, \ \left|Z_s\right| \leq e^{2\|f\|_1}\left|z_s\right| \ \text{ and } \ \left|U_s(e)\right| \leq e^{2\|f\|_1}\left|u_s(e)\right|.$$

This means that Y_s, Z_s *and* U_s *belong respectively to* $\mathbb{S}^2(\mathbb{R})$, $\mathbb{H}^2(\mathbb{R})$ *and* $\mathbb{H}^2_N(\mathbb{R})$.

Consequently

$$\begin{aligned}
&(Y_s, Z_s, U_s(e))_{0\leq t\leq T, e\in E}\\
&:= \left(F^{-1}(y_s), (F^{-1})'(y_{s^-})z_s, F^{-1}(y_{s^-} + u_s(e)) - F^{-1}(y_{s^-})\right)_{0\leq t\leq T, e\in E}
\end{aligned}$$

is a solution to $eq(\xi, H_f)$ *in the sense of Definition* (3.1.1).

Summary: *According to Theorem* (2.2.1), *the BSDEJs* $eq(F(\xi), 0)$ *has a unique solution* *such that the process* $(y_t)_{t\in[0,T]}$ *is given by*

$$y_t = \mathbb{E}\left[F(\xi) \mid \mathcal{F}_t\right],$$

and thus

$$Y_t = F^{-1}\left(\mathbb{E}\left[F(\xi) \mid \mathcal{F}_t\right]\right).$$

Then the martingale representation Theorem (2.1.2) shows that there exists a unique predictable process $(z, u) \in \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2_N(\mathbb{R})$ *satisfying* $eq(F(\xi), 0)$*. Now, comparing with* $eq(F(\xi), 0)$ *(3.8), one can easily arrive at*

$$Y_t = F^{-1}(y_t), \quad Z_t = \frac{z_t}{F'(F^{-1}(y_{t^-}))} \text{ and } U_t(e) = F^{-1}(y_{t^-} + u_t(e)) - F^{-1}(y_{t^-}).$$

We deduce finally that the equation $eq(\xi, H_f)$ *admits a unique solution if and only if* $eq(F(\xi), 0)$ *admits a unique solution.* ■

Corollary 3.4.1

Let f *be a bounded and integrable function on* $\mathbb{R}$*. The* $eq(\xi, H_f)$ *admits a unique solution for any* $\mathcal{F}_T$*-measurable square integrable random variable* ξ*.*

3.5 Solvability of Some Quadratic BSDEJs

In this section we shall use the results of the previous section to solve some QBSDEJs. First of all, we apply Itô-Krylov's formula to $eq(\xi, H)$ given by (3.1), where the generator H satisfying suitable conditions which guarantee the existence and uniqeness of a solution of

BSDEJs, leads to

$$
\begin{aligned}
F(Y_t) = F(\xi) + \int_t^T \left[F'(Y_{s^-}) H\left(Y_s, Z_s, U_s(.)\right) - \frac{1}{2} F''(Y_s) \left|Z_s\right|^2 \right] ds \\
- \int_t^T F'(Y_{s^-}) Z_s dW_s - \int_t^T \int_E F'(Y_{s^-}) U_s(e) \tilde{N}(ds, de) \\
- \int_t^T \int_E \left[F(Y_{s^-} + U_s(e)) - F(Y_{s^-}) - F'(Y_{s^-}) U_s(e) \right] N(ds, de).
\end{aligned}
$$

Moreover, by adding and subtracting the same term

$$
\int_t^T \int_E \left(F(Y_{s^-} + U_s(e)) - F(Y_{s^-}) \right) \upsilon(de) ds,
$$

and use (3.7), we get

$$
\begin{aligned}
F(Y_t) = F(\xi) + \int_t^T \left[F'(Y_{s^-}) \Big(H(Y_s, Z_s, U_s(.)) - \left[U_s\right]_f (Y_{s^-}) \Big) - \frac{1}{2} F''(Y_s) \left|Z_s\right|^2 \right] ds \qquad (3.13) \\
- \int_t^T F'(Y_{s^-}) Z_s dW_s - \int_t^T \int_E \left[F(Y_{s^-} + U_s(e)) - F(Y_{s^-}) \right] \tilde{N}(ds, de)
\end{aligned}
$$

Secondly, we use the result of the section (2.2.4) to show the existence and uniqueness of the solutions to two types of QBSDEJs with different quadratic generators.

Let $h : \mathbb{R} \times \mathfrak{L}^p_\upsilon \to \mathbb{R}$ be a measurable function which satisfying:

(**H1**) **Lipschitz and bounded**: For all $u(.)$, $\acute{u}(.)$ in $\mathfrak{L}^p_\upsilon$, and y, $\acute{y}$ in $\mathbb{R}$

$$
\left| h(y, u(.)) - h(\acute{y}, \acute{u}(.)) \right| \leq L \Big(|y - \acute{y}| + \|u(.) - \acute{u}(.)\|_{\upsilon,2} \Big),
$$

and there exists a constant $c > 0$, such that $|h(y, u(.))| \leq c$.

(**H2**) f is bounded and integrable.

(**H3**) The function $h(y, u(.))$ is of the form $h(y, \int_E g(u(e)) \upsilon(de))$ and the mapping $(y, u) \to$

$h(y,u)$ is continuous

$$\left| h(y, \int_E g(u(e))\upsilon(de)) \right| \leq L \left(|y| + \|u(.)\|_{\upsilon,1} \right).$$

(1) $eq(\xi, h(y,u(.)) + cz + H_f(y,z,u(.)))$:

The formula (3.13) corresponding to $H_{h,f} := h(y,u(.)) + cz + H_f(y,z,u(.))$, $c \in \mathbb{R}$ give

$$F(Y_t) = F(\xi) + \int_t^T \left(F'(Y_{s^-}) \left[h(Y_s, U_s(e)) + cZ_s + f(Y_s) |Z_s|^2 \right] - \frac{1}{2} F''(Y_s) |Z_s|^2 \right) ds$$
$$- \int_t^T F'(Y_{s^-}) Z_s dW_s - \int_t^T \int_E \left[F(Y_{s^-} + U_s(e)) - F(Y_{s^-}) \right] \tilde{N}(ds, de),$$

by using (1.2), we get

$$F(Y_t) = F(\xi) + \int_t^T F'(Y_{s^-}) \left[h(Y_s, U_s(e)) + cZ_s \right] ds$$
$$- \int_t^T F'(Y_{s^-}) Z_s dW_s - \int_t^T \int_E \left[F(Y_{s^-} + U_s(e)) - F(Y_{s^-}) \right] \tilde{N}(ds, de),$$

is equivalent to

$$y_t = F(\xi) + \int_t^T \left[H_1(y_s, u_s(.)) + cz_s \right] ds - \int_t^T z_s dW_s - \int_t^T \int_E u_s(e) \tilde{N}(ds, de),$$

(using the same notation in 3.2.2), where

$$H_1(y, u(.)) := F'(F^{-1}(y)) h \left(F^{-1}(y), F^{-1}(y + u(.)) - F^{-1}(y) \right).$$

The equation $eq(F(\xi), H_1 + cz)$ has a unique solution if and only if H_1 is Lipschitz and $F(\xi)$ is square integrable. Under the evious prassumptions, we deduce H_1 is Lipschitz. Therefore, $eq(F(\xi), H_1 + cz)$ has a unique solution, thus our original $eq(\xi, H_{h,f})$ admits a unique solution.

(2) $eq(\xi, cz + f(y) |z|^2 - \int_E u(e)\upsilon(de))$:

Consider the following BSDEJs

$$Y_t = \xi + \int_t^T \left[cZ_s + f(Y_s)\left|Z_s\right|^2 - \int_E U_s(e)\upsilon(de) \right] ds - \int_t^T Z_s dW_s$$
$$- \int_t^T \int_E U_s(e)\tilde{N}(ds, de),$$

by applying formula (3.13) to this equation, we get

$$F(Y_t) = F(\xi) + \int_t^T F'(Y_{s^-}) \left(cZ_s + f(Y_s)\left|Z_s\right|^2 - \int_E U_s(e)\upsilon(de) - \left[U_s\right]_f (Y_{s^-}) \right) ds$$
$$- \frac{1}{2}\int_t^T F''(Y_s)\left|Z_s\right|^2 ds - \int_t^T F'(Y_{s^-})Z_s dW_s$$
$$- \int_t^T \int_E \left[F(Y_{s^-} + U_s(e)) - F(Y_{s^-})\right] \tilde{N}(ds, de).$$

We simplify the integrals with respect to the Lebesgue measure, we obtain

$$F(Y_t) = F(\xi) + \int_t^T \left[cF'(Y_{s^-})Z_s - \int_E \left[F(Y_{s^-} + U_s(e)) - F(Y_{s^-})\right] \upsilon(de) \right] ds$$
$$- \int_t^T F'(Y_{s^-})Z_s dW_s - \int_t^T \int_E \left[F(Y_{s^-} + U_s(e)) - F(Y_{s^-})\right] \tilde{N}(ds, de).$$

Or equivalently as

$$y_t = F(\xi) + \int_t^T \left[cz_s - \int_E u_s(e)\upsilon(de) \right] ds - \int_t^T z_s dW_s - \int_t^T \int_E u_s(e)\tilde{N}(ds, de)$$

Since the generator $H_2(z, u(.)) := cz - \int_E u(e)\upsilon(de)$ of the above equation is differentiable w.r.t. z and u and its derivatives are bounded, and $F(\xi)$ is square integrable for square integrable ξ, then it has a unique solution. Thus, $eq(F(\xi), H_2)$ has a unique solution.

3.6 Comparision Theorem

This comparison theorem of the solutions for QBSDEJs is novel in jump setting in that it does so even in the absence of the Lipschitz, continuous and convex condition of both generator.

Theorem 3.6.1 *(Comparison principle)*

Let ξ_1, ξ_2 be $\mathcal{F}_T$-measurable and square integrable random variables. Let $f \in L^1(\mathbb{R})$. Let (Y^1, Z^1, U^1), (Y^2, Z^2, U^2) be respectively the solution of eq(ξ_1, H_f) and eq(ξ_2, H_f).

(i) If $\xi_1 \leq \xi_2$ $\mathbb{P}$-a.s., then $Y_t^1 \leq Y_t^2$ $\mathbb{P}$-a.s.

(ii) If in addition to (i), $Y_0^1 = Y_0^2$, then $\xi_1 = \xi_2$.

(iii) (strict comparison) In addition to (i) if $\mathbb{P}(\xi_2 > \xi_1) > 0$, then $\mathbb{P}\left(Y_t^2 > Y_t^1,\ \forall t \in [0,T]\right) > 0)$, in particular $Y_0^2 > Y_0^1$.

Proof. (i) Notice that the solutions (Y^1, Z^1, U^1) and (Y^2, Z^2, U^2) belong to $\mathbb{S}^2(\mathbb{R}) \times \mathbb{H}^2(\mathbb{R}) \times \mathbb{H}^2_N(\mathbb{R})$. For a given integrable function f, remember that F associated to f is defined by (3.2) and satisfies (1.2). We first apply Itô–Krylov's formula to $F(Y_t^2)$, to obtain

$$\begin{aligned}
F(Y_T^2) = F(Y_t^2) &- \int_t^T F'(Y_{s^-}^2) f(Y_s^2) \left|Z_s^2\right|^2 ds \\
&+ \int_t^T F'(Y_{s^-}^2) Z_s^2 dW_s + \int_t^T \int_E F'(Y_{s^-}^2) U_s^2(e) \tilde{N}(ds, de) \\
&+ \int_t^T \frac{1}{2} F''(Y_{s^-}^2) \left|Z_s^2\right|^2 ds - \int_t^T F'(Y_{s^-}^2) \left[U_s^2\right]_f (Y_{s^-}^2) ds \\
&+ \int_t^T \int_E \left[F(Y_{s^-}^2 + U_s^2(e)) - F(Y_{s^-}^2) - F'(Y_{s^-}^2) U_s^2(e)\right] N(ds, de),
\end{aligned}$$

using the definition of the operator $[U_s^2]_f(.)$ and the fact F satisfies (1.2), we obtain

$$F(Y_T^2) = F(Y_t^2) + (M_T - M_t), \tag{3.14}$$

where

$$M_t = \int_0^t F'(Y^2_{s^-})Z^2_s dW_s + \int_0^t \int_E \left[F(Y^2_{s^-} + U^2_s(e)) - F(Y^2_{s^-})\right] \tilde{N}(ds, de),$$

is an $\mathcal{F}_t$-martingale. Passing to conditional expectation and using the fact that F is an increasing function and $\xi_2 \geq \xi_1$, we get

$$\begin{aligned} F(Y^2_t) &= \mathbb{E}(F(Y^2_T) \mid \mathcal{F}_t) = \mathbb{E}(F(\xi_2) \mid \mathcal{F}_t) \\ &\geq \mathbb{E}(F(\xi_1) \mid \mathcal{F}_t) = F(Y^1_t). \end{aligned}$$

Taking F^{-1} in both sides, we conclude $Y^2_t \geq Y^1_t$, for all $t \in [0, T]$.

(ii) From (3.14) for $t = 0$, we get

$$F(\xi_2) = F(Y^2_0) + M_T, \tag{3.15}$$

and from (3.8) for $t = 0$ we have

$$F(Y^1_0) = F(\xi_1) - N_T, \tag{3.16}$$

where

$$N_t = \int_0^t F'(Y^1_{s^-})Z^1_s dW_s - \int_0^t \int_E \left[F(Y^1_{s^-} + U^1_s(e)) - F(Y^1_{s^-})\right] \tilde{N}(ds, de).$$

If $Y^1_0 = Y^2_0$, then we get, by substituting (3.16) in (3.15)

$$F(\xi_2) = F(\xi_1) - N_T + M_T,$$

taking the expectation we obtain $\mathbb{E}\left[F(\xi_2) - F(\xi_1)\right] = 0$.

Since the quantity inside the expectation is positive, we conclude that

$$F(\xi_2) = F(\xi_1) \ \mathbb{P}\text{−a.s. and thus } \xi_2 = \xi_1 \ \mathbb{P}\text{−a.s.}$$

(iii) We have from (3.14)

$$F(Y_T^2) = F(Y_t^2) + (M_T - M_t).$$

Moreover, thanks to (3.8)

$$F(Y_t^1) = F(\xi_1) - (N_T - N_t).$$

Therefore, subtracting both sides of the above equalities leads to

$$F(Y_t^2) - F(Y_t^1) = F(\xi_2) - F(\xi_1) + (N_T - N_t) - (M_T - M_t),$$

taking the conditional expectation we get

$$F(Y_t^2) - F(Y_t^1) = \mathbb{E}[F(\xi_2) - F(\xi_1) \mid \mathcal{F}_t].$$

Consequently $F(Y_t^2) - F(Y_t^1) > 0$ on the set $\{\xi_2 > \xi_1\}$, finally taking into account the fact that the function F is one to one $\mathbb{P}\,(Y_t^2 > Y_t^1) > 0$ for all $t \in [0, T]$. This achieves the proof.

■

Conclusion

The main objective of this thesis is to study the existence and uniqueness of the square-integrable solution for two types of backward stochastic differential equations with jumps (2.6) driven by a Poisson random measure and independent Brownian motion. The first fundamental result is initiated and studied by Tang & Li [37] in 1994. This paper cover the case where the generator f *is globally Lipschitz with respect to the variables* y*,* z *and* u*, and the terminal condition* ξ *and the processe* $(f(t,0,0,0))_{t\in[0,T]}$ *are square-integrable. As for the second result, we have extended Itô-Krylov formula to the BSDEJs with jumps framework. Building on this, the third significant result is the proof of an existence and uniqueness theorem for quadratic BSDEJs. This proof leverages Itô-Krylov formula and the phase space transformation* F *(Zvonkin transformations), which simplifies the quadratic BSDEJ by eliminating its drift or parts of it, resulting in an equation with a zero or Lipschitz generator. If the transformed BSDEJ has a unique solution, then our original quadratic* $eq(\xi, H_f)$ *also admits a unique solution. Finaly, we gave a comparison and strict comparison theorems for the solutions of BSDEJs with non-Lipschitz coefficients.*

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ملخص

تقدم هذه المذكرة دراسة بحثية تتعلق بمشكلة وجود و تفرد الحل لنوعين من المعادلات التفاضلية العشوائية التراجعية المولدة بواسطة قياس بواسون العشوائي و الحركة البراونية المستقلة، الأولى ذات مولد ليبشيتزي والثانية ذات مولد يتميز بنمو تربيعي في المكون البراوني و القيمة النهائية هي متغير عشوائي مربعه قابل للتكامل.

الكلمات المفتاحية : المعادلات التفاضلية العشوائية التراجعية؛ عملية القفز؛ قياس بواسون العشوائي؛ الحركة البراونية.

Abstract

In this master’s thesis, we study a BSDE with Jumps (BSDEJs in short), and prove the existence and uniquenes of the solutions, when the driver have quadratic growth in the Brownian companente and non-linear functional form with respect to the jump part, and the terminal condition is square integrable random variable. After proving the result of the existence and uniqueness of solutions under the strongest condition (Lipschitz condition).

Key words: Backward stochastic differential equations; Jump process; Poisson random measure; Brownian motion.

Résumé

L'objectif de ce mémoire est d'analyser la question de l'existence et de l'unicité des solutions de EDSRs quadratiques avec sauts (EDSRQSs en abrégé), où les générateurs acceptent une croissance quadratique par rapport au terme de la composante Brownienne et une forme fonctionnelle non linéaire par rapport au terme de saut. Nous présentons également le résultat de l'existence et de l'unicité des solutions dans le cas du générateur Lipschitzienne, et la condition finale est de carrée intégrable.

Mots-clés: Équations différentielles stochastiques rétrogrades; Processus de saut; Mesure aléatoire de Poisson; Mouvement Brownien.