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Dedicace

First of all, I thank God, our creator, for giving me the strength, the will, and the courage to accomplish this modest work.

I dedicate this work to my mother, the source of tenderness and the light that guides my paths and takes me to the paths of success, for all her sacrifices and her precious advice, for all her assistance and her presence in my life.

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Introduction

The extreme values theory plays a prominent role in modeling rare events occurring across various fields such as finance, insurance losses, and environmental sciences. Regularly varying functions are the appropriate models that fit this data, appearing in numerous application fields and considered a fundamental tool to characterize the domains of attraction of distribution functions, namely Gumbel, Weibull, and Fréchet. Our focus is to explore the significance of this class of functions by considering Pareto-type models, which correspond to the Fréchet domain of attraction, exhibiting tails that behave as regularly varying functions of negative indices.

Our memory comprises four chapters:

In the first chapter, we are discussing the domains of attraction of distribution functions in relation to regularly varying functions.

The second chapter aims to present the first and second conditions for this class of functions and their corresponding uniform inequalities, known as Potter's ones. These are key tools for establishing the asymptotic behavior of estimators of the tail index.

The third chapter deals with the statistical inference of tail distribution functions, constructing the so-called Hill's estimator of the tail index and its corresponding high quantiles, then establishing the consistency and asymptotic normality of the latter.

The fourth chapter is devoted to a simulation study of Hill's estimator and the computation of the optimal sample fraction k used in the estimation of extreme values. Additionally, it presents real data applications to the asset returns of financial markets.

Chapter 1

Extreme Value Theory

Extreme Value Theory (EVT) is essential for understanding and modeling rare events in datasets. It focuses on the statistical behavior of extreme values, aiding in understanding their characteristics and facilitating the prediction and assessment of risks associated with rare events.

1.1 Asymptotic behaviour of extremes

Let us consider $\{X_i\}_{i\geq 1}$ to be a sequence of n independent and identically distributed ($i.i.d$) of random variables with cumulative distribution function (cdf) F defined by

$$F(x) = \mathbf{P}(X_i \leq x) \text{ for } i = 1, 2, ..., n.$$

We will be interested in the asymptotic behavior of the maximum of the sample $\{X_i\}_{i\geq 1}$ denoted by

$$M_n := \max(X_1, ..., X_n).$$

As the random variables are ($i.i.d$) then the cdf of M_n is given by

$$\begin{aligned} F_{M_n}(x) &= \mathbf{P}\left(M_n \leq x\right) = \mathbf{P}\left(X_1 \leq x, ..., X_n \leq x\right) \\ &= \mathbf{P}\left(X_1 \leq x\right)...\mathbf{P}(X_n \leq x) = \left[F(x)\right]^n. \end{aligned} \tag{1.1}$$

The approximate distribution of the maximum value as n tends to infinity.

$$\lim_{n\to\infty} F_{M_n}(x) = \lim_{n\to\infty} \{F(x)\}^n = \begin{cases} 1 & \text{if } F(x) = 1, \\ 0 & \text{if } F(x) < 1. \end{cases}$$

This means that the asymptotic distribution of M_n is a degenerate function. What we are interested in with extrema is the existence of a non-degenerate law for the maximum, and the Extreme Value Theorem provides an answer to this.

Theorem 1.1.1 (Fisher and Tippet (1928), Gnedenko (1943)) *Let $(X_i)_{i=1,\dots,n}$ be a sequence of i.i.d random variables (rv) with cdf F. If there are two real sequences $\{a_n\}_{n\geq 1} > 0$ and $\{b_n\}_{n\geq 1} \in \mathbb{R}$ such that for all $x \in \mathbb{R}$,*

$$\lim_{n\to\infty} \mathbf{P}\left(\frac{M_n - b_n}{a_n} \leq x\right) = \lim_{n\to\infty} F^n(a_n x + b_n) = G_\gamma(x), \tag{1.2}$$

then $G_\gamma(x)$ is of the same type as one of the following three distributions:

$$\Phi_\gamma(x) := \begin{cases} 0 & si\ x \leq 0 \\ \exp(-x^{-\frac{1}{\gamma}}) & si\ x > 0 \end{cases} \quad with\ \gamma > 0 \qquad \textit{(Fréchet)},$$

$$\Psi_\gamma(x) := \begin{cases} 1 & si\ x \geq 0 \\ \exp\left[-(-x)^{-\frac{1}{\gamma}}\right] & si\ x < 0 \end{cases} \quad with\ \gamma < 0 \qquad \textit{(Weibull)},$$

$$\Lambda_\gamma(x) := \exp\left(-\exp\left(-x\right)\right), for\ all\ x \in \mathbb{R} \qquad \textit{(Gumbel)}.$$

We refer to Φ_γ, Ψ_γ and Λ as the extreme value distributions.

A detailed proof of this theorem is given in Resnick (1987)[25].

1.2 Distribution of Generalized Extreme Values (GEV)

The behaviour of Φ_γ , Ψ_γ and Λ is completely different but they can be combined into a single distribution dependent on a single parameter that controls the tail thickness of the distribution.

Theorem 1.2.1 *If the limit* (1.2) *exists, then*

$$G_\gamma(x) := \begin{cases} \exp(-(1+\gamma x)_+^{-\frac{1}{\gamma}}) & \text{if } \gamma \neq 0, \\ \exp(-\exp(-x)) & \text{if } \gamma = 0. \end{cases}$$

with $(1+\gamma x)_+ = \max(1+\gamma x, 0) > 0$.

- $G_\gamma(x)$:Distribution G.E.V.
- $G_\gamma(x)$ is a non-degenerate function and γ called the tail index or extreme value index.

The sign of the parameter γ is an essential indicator of the shape of the tail:

1. The case $\gamma > 0$, corresponds to Fréchet's distribution with parameter $1/\gamma > 0$.
2. The case $\gamma < 0$, corresponds to the Weibull's distribution with parameter $-1/\gamma < 0$.
3. The case $\gamma = 0$, corresponds to Gumbel's distribution.

Definition 1.2.1 *The Generalised Pareto Distribution* (GPD), *with parameters* $\gamma \in \mathbb{R}$, *and* $\sigma > 0$, *is defined by its distribution function, given by:*

$$G_{\gamma,\sigma}(x) = \begin{cases} 1-(1+\frac{\gamma}{\sigma}x)^{-1/\gamma} & \text{if } \gamma \neq 0, \\ 1-\exp(-\frac{x}{\sigma}) & \text{if } \gamma = 0, \end{cases}$$

or, $x \geq 0$ *if* $\gamma \geq 0$ *and* $0 \leq x \leq \frac{\gamma}{\sigma}$ *if* $\gamma < 0$.

Remark 1.2.1 *Depending on the values of the shape parameter* γ, *the* GPD *includes the following three distributions:*

If $\gamma > 0$ *then* $G_{\gamma,\sigma}(x) \longmapsto$ *Pareto law.*

If $\gamma < 0$ *then* $G_{\gamma,\sigma}(x) \longmapsto$ *Beta law.*

If $\gamma = 0$ *then* $G_{\gamma,\sigma}(x) \longmapsto$ *exponential law.*

1.3 Domains of Attraction

Definition 1.3.1 (Domains of attraction) *A distribution is said to belong to the DA of G, denoted $F \in DA(G)$, if the distribution of the normalised maximum converges to G. In other words, if there exist real constants $a_n > 0$ and $b_n \in \mathbb{R}$ such that*

$$\lim_{n \to \infty} F^n(a_n x + b_n) = G_\gamma(x).$$

The table 1.1 presents the DA for some models.

Domaine of attraction	Models
Fréchet $(\gamma > 0)$	Fréchet, Burr,Pareto, Cauchy, Student, Chi-square.
Gumbel $(\gamma = 0)$	Gumbel, Exponential, Log-normal, Gamma, Logistic, Normal, Weibull.
Weibull $(\gamma < 0)$	Beta, Uniform.

Table 1.1: Usual models and their domaine of attractions

Characterisation of domains of attraction

Proposition 1.3.1 (Characterisation of DA(G)) *$F \in DA(G_\gamma)$ if and only if, for a certain sequence $a_n > 0$ and $b_n \in \mathbb{R}$,*

$$\lim_{n \to \infty} n\overline{F}(a_n x + b_n) = -\log G_\gamma(x), \; x \in \mathbb{R},$$

where $\overline{F} := 1 - F$ is the survival function, also known as the tail of the cdf.

Theorem 1.3.1 *We said that:*

1) F is the Fréchet domain of attraction, i.e. $F \in DA(G_\gamma)$ for $\gamma > 0$, if and only if for all $x > 0$

$$\overline{F}(x) = x^{-1/\gamma} L(x) \Leftrightarrow \lim_{t \to \infty} \frac{\overline{F}(tx)}{\overline{F}(t)} = x^{-1/\gamma}.$$

2) F is in the Weibull domain of attraction, i.e. $F \in DA(G_\gamma)$ for $\gamma < 0$, if and only if for all $x > 0$

$$\overline{F}(x^F - x^{-1}) = x^{-1/\gamma} L(x) \Leftrightarrow \lim_{t \to 0} \frac{\overline{F}(x^F - tx)}{\overline{F}(x^F - t)} = x^{-1/\gamma}.$$

3) F is in the domain of attraction of the Gumbel distribution, i.e. $F \in DA$ for $\gamma = 0$, with $x^F \leq \infty$ for all $x \in \mathbb{R}$

$$\lim_{t \to x^F} \frac{\overline{F}(t + xf(t))}{\overline{F}(t)} = \exp(-x),$$

where x^F be the end point of F.

The characterisation of domains of attraction relies heavily on the notion of functions with regular variations The latter will be discussed in Chapter 2.

Chapter 2

Regular Variation

In this section, we delve into functions with broad applications across various mathematical domains, known as regularly varying functions, and explore their fundamental properties. Our goal is to draw generalizations about these functions and their characteristics. Readers keen on further exploring this theory are encouraged to reference seminal works by authors such as Bingham *et al.* ([5]) , Embrechts *et al.* ([10]) , Beirlant *et al.* ([3]) , de Haan, Ferreira ([17]) and Resnick([24]) .

2.1 Basic concepts

Definition 2.1.1 (Regular Variation) *A measurable function* $h : \mathbb{R}_+ \longmapsto \mathbb{R}_+$, *is regularly varying at* ∞ *with index* $\rho \in \mathbb{R}$ *and we denote* $h \in RV_\rho$, *if*

$$\lim_{t\to\infty} \frac{h(tx)}{h(t)} = x^\rho, \; t > 0.$$

Example 2.1.1 *The functions* $x \mapsto x^\rho$ *and* $x \mapsto (\log(1+x))^\rho$ *are regularly varying with index* ρ.*function* $x \mapsto \sin(x)$ *is not regularly varying.*

Definition 2.1.2 *A sequence* $(h_n)_{n\in\mathbb{N}}$ *of positive numbers is called regularly varying* $(at\ \infty)$ *of index* $\rho \in \mathbb{R}$

$$\lim_{n\to\infty} \frac{h_{\lfloor nx \rfloor}}{h_n} = x^\rho, \; for\ x > 0,$$

and we write $h_n \in RV_\rho$.

Definition 2.1.3 (Extended regular variation) *A measurable function $h : \mathbb{R}_+ \to \mathbb{R}$ is said to have extended regular variation of index $1/\gamma$ which is denoted $h \in ERV_{1/\gamma}$, if there exists a positive a auxiliary function a such that, for all $x > 0$*

$$\lim_{t\to\infty} \frac{h(tx) - h(t)}{a(t)} = \frac{x^{1/\gamma} - 1}{1/\gamma}.$$

Theorem 2.1.1 *If h is regularly varying with index ρ (in the case $\rho > 0$, assuming h bounded on each interval $]0, t]$, $t > 0$, then for $0 < a \leq b < \infty$,*

$$\lim_{x\to 0} \frac{h(tx)}{h(t)} = x^{\rho},$$

uniformly in $x \in [a, b]$ if $\rho = 0$, in $x \in]0, b]$ if $\rho > 0$ and in $x \in [a, \infty[$ if $\rho < 0$.

Definition 2.1.4 *A measurable function $h : \mathbb{R}_+ \to \mathbb{R}_+$ has regular variation of index ρ in the neighbourhood of 0 denoted $(h \in RV_{\rho}^{0})$, if for all $x > 0$,*

$$\lim_{t\to 0} \frac{h(tx)}{h(t)} = x^{\rho},$$

i.e. $h(1/x)$ has a regular variation of index $-\rho$ at infinity.

Lemma 2.1.1 *Let h be a function with regular variation of index ρ, then*

$$\lim_{t\to\infty} \sup_{x\in[a,b]} \left| \frac{h(tx)}{h(t)} - x^{\rho} \right| = 0, \ \textit{for all } 0 < a < b.$$

If this

$$h(tx) = g(t)g(x) \text{ when } x \to \infty. \tag{2.1}$$

1. (2.1) is satisfied for all $t > 0$.

2. There exists $\rho \in \mathbb{R}$ such that $g(t) = t^{\rho}$, for any $t > 0$.

3. $h(x) = x^{\rho} L(x)$ where $L(x)$ is a slowly varying function (at ∞).

From this result, it is clear that to study regular variations, it is sufficient to study the properties of slowly varying functions.

Definition 2.1.5 *A measurable function $L : \mathbb{R}_+ \longmapsto \mathbb{R}_+$ has slow variation at ∞ with the index $\rho = 0$ and we denote $L \in RV_0$, if*

$$\lim_{t\to\infty} \frac{L(tx)}{L(t)} = 1, \ t > 0.$$

Example 2.1.2 *Function $x \mapsto \log x$ is slowly varying. Indeed*

$$\lim_{t\to\infty} \frac{\log(tx)}{\log t} = \lim_{t\to\infty} \frac{\log t + \log x}{\log t} = 1, \ \textit{for any } t > 0.$$

Remark 2.1.1 *A function with regular variation of index $\rho \in \mathbb{R}$ can always be written in the form*

$$h(x) = x^{\rho} L(x), \ \textit{where } L \in RV_0.$$

Proposition 2.1.1 *Let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be a regularly varying at ∞ with index ρ. Then*

i) *If h is increasing then $\rho \geq 0$.*

ii) *If h is decreasing then $\rho \leq 0$.*

iii) *If h is not decreasing and $\rho \in [0, \infty)$ then $h^{-1} \in RV_{\frac{1}{\rho}}$.*

Proposition 2.1.2 *If h, h_1 and h_2 vary regularly at infinity with indices ρ, ρ_1 and ρ_2 respectively, then*

1) *If $h \in RV_\rho$, then $h^\alpha \in RV_{\alpha\rho}$.*

2) *If $\rho \neq 0$*

$$\lim_{x\to\infty} h(x) = \begin{cases} 0 & \textit{if } \rho < 0, \\ \infty & \textit{if } \rho > 0. \end{cases}$$

3) *If h_1, $h_2 \in RV_\rho$, and $h_2(x) \to \infty$ as $x \to \infty$, then $h_1 \circ h_2 \in RV_{\rho_1\rho_2}$.*

4) *If $h_1, h_2 \in RV_\rho$, then $h_1 + h_2 \in RV_\rho$ where $\rho = \max(\rho_1, \rho_2)$.*

5) ***(Potter, 1942)*** *if $h \in RV_\rho$, then there exists t_0 such that for all $x \geq 1$, $t \geq t_0$:*

$$(1-\varepsilon)x^{\rho-\varepsilon} < \frac{h(tx)}{h(t)} < (1+\varepsilon)x^{\rho+\varepsilon}, \ \textit{for any } \varepsilon > 0.$$

6) *Suppose that h is increasing, $h(\infty) = \infty$, and $h \in RV_\rho$, $0 \leq \rho \leq \infty$, then,*

$$h^{\leftarrow} \in RV_{\rho^{-1}}.$$

7) *Suppose that* $h_1, h_2 \in RV_\rho$, $0 < \rho < \infty$, *then for* $0 \leq c \leq \infty$,

$$h_1(x) \backsim ch_2(x),\ x \to \infty,$$

is equivalent to

$$h_1^{-1}(x) \backsim c^{-\rho^{-1}} h_2^{-1}(x),\ x \to \infty.$$

8) *If* $h \in RV_\rho$, $\rho \neq 0$, *then there exists a function* h^* *which is absolutely continuous and strictly monotonic satisfying*

$$h(x) \backsim h^*(x),\ x \to \infty.$$

Proposition 2.1.3 (Drees(1998)**)** *If* $h \in RV_\rho$, *then there exists* $t_0 = t_0(\varepsilon, \delta)$ *such that, for* t, $tx \geq t_0$:

$$\left| \frac{h(tx)}{h(t)} - x^\rho \right| \leq \varepsilon \max(x^{\rho+\delta}, x^{\rho-\delta}),\ \textit{for any } \varepsilon,\ \delta > 0.$$

Theorem 2.1.2 (Karamata theorem) *Let* $h \in RV_\sigma$, *and locally bounded on* $x_0 \leq x < \infty$,

1) *If* $\sigma \geq -1$, *then,*

$$\lim_{x\to\infty} \frac{xh(x)}{\int_{x_0}^{x} h(t)dt} = \sigma + 1. \tag{2.2}$$

2) *If* $\sigma < -1$ *(If* $\sigma = -1$ *and* $\int_x^\infty h(t)dt < +\infty$*), then,*

$$\lim_{x\to\infty} \frac{xh(x)}{\int_{x}^{\infty} h(t)dt} = -(\sigma + 1). \tag{2.3}$$

Conversely, if h *verifies* (2.2) *for* $-1 < \sigma < \infty$, *then* $h \in RV_\sigma$. *And if* h *verifies* (2.3) *with* $-\infty < \sigma < -1$, *then* $h \in RV_\sigma$.

Proposition 2.1.4 (Karamata's representation) *A function* $h : \mathbb{R}_+ \to \mathbb{R}$ *has regular variation of index* $1/\gamma$, *if and only if there exist two measurable functions* $c : \mathbb{R}_+ \to \mathbb{R}$ *and* $r : \mathbb{R}_+ \to \mathbb{R}$ *satisfying*

$$\lim_{t\to\infty} c(t) = c > 0 \ \textit{and} \ \lim_{t\to\infty} r(t) = 0,$$

such that

$$h(t) = t^\rho c(t) \exp\left\{ \int_{t_0}^{t} \frac{r(t)}{t} dt \right\}.$$

For some $t_0 \geq 0$.

Theorem 2.1.3 (Karamata's representation) *$L \in RV_0$, if and only if can be represented in the form*

$$L(x) = c(x) \exp\left\{\int_a^x \frac{r(t)}{t} dt\right\}, \ x \geq a,$$

where c, r measurable functions, and

$$\lim_{x\to\infty} c(x) = c_0 \in]0, +\infty[\ \text{ and } \lim_{x\to\infty} r(t) = 0.$$

2.2 First order conditions

In the contex of pareto-type models, we will give three versions of the first condition of regular variation, in terms of $\overline{F}$ the quantile function (or the generalized inverse)

$$Q(s) := F^{-1}(s) = \inf\{x, \ F(x) \geq s\}, \ 0 < s < 1,$$

and $U(t) := Q(1 - 1/t)$.

- As a function of $\overline{F}$: for all $x > 0$,

$$\lim_{t\to\infty} \frac{\overline{F}(tx)}{\overline{F}(t)} = x^{-\frac{1}{\gamma}} \Leftrightarrow \overline{F}(x) = x^{-\frac{1}{\gamma}} L(x), \ \left\{\overline{F} \in RV_{-\frac{1}{\gamma}} \Leftrightarrow F \in DA\left(\Phi_{\frac{1}{\gamma}}\right)\right\}. \tag{2.4}$$

- As a function of Q : for all $s > 0$,

$$\lim_{t\downarrow 0} \frac{Q(1 - ts)}{Q(1 - t)} = s^{-\gamma} \Leftrightarrow Q(1 - s) = s^{-\gamma} L(s), \ \left\{Q(1 - s) \in RV_{-\gamma} \Leftrightarrow F \in DA\left(\Phi_{\frac{1}{\gamma}}\right)\right\}. \tag{2.5}$$

- As a function of U :

$$\lim_{t\to\infty} \frac{U(tx)}{U(t)} = x^{\gamma}, \text{for } x > 0, \left\{U \in RV_{\gamma} \Leftrightarrow F \in DA\left(\Phi_{\frac{1}{\gamma}}\right)\right\}. \tag{2.6}$$

Proposition 2.2.1 (First-order condition of Haan and Ferreira (2006)) *The following statements are equivalent:*

- F is of Pareto-type:

$$F \in DA\left(\Phi_{1/\gamma}\right), \ \gamma > 0.$$

- U is regularly varying at ∞ with index $\gamma > 0$:

$$\lim_{t\to\infty} \frac{U(tx)}{U(t)} = x^{\gamma},\ x > 0.$$

- $\overline{F}$ is regularly varying at ∞ with index $-1/\gamma < 0$:

$$\lim_{t\to\infty} \frac{\overline{F}(tx)}{\overline{F}(t)} = x^{-1/\gamma},\ x > 0.$$

- $Q(1-s)$ is regularly varying at 0 with index $-\gamma < 0$:

$$\lim_{s\downarrow 0} \frac{Q(1-ts)}{Q(1-t)} = s^{-\gamma},\ 0 < s < 1.$$

In a semi-parametric approach, a first-order condition is generally not sufficient to study to study the properties of tail parameter estimators, in particular asymptotic normality. In this case, a second order condition is required. The most common are the following.

2.3 Second-Order conditions

Definition 2.3.1 *The tail of cdf F, $F \in DA(\Phi_\gamma)$, $\gamma > 0$, is said to satisfy the second order condition of a regular variation at ∞ if one of the following (equivalent) conditions is satisfied:*

- There exists a real constant $\rho \le 0$ and a function of constant sign $A(t) \to 0$ when $t \to \infty$, such that for all $x > 0$,

$$\lim_{t\to\infty} \frac{\frac{U(tx)}{U(t)} - x^{\gamma}}{A(t)} = x^{\gamma} \frac{x^{\rho}-1}{\rho}. \tag{2.7}$$

- There exists a real constant $\rho \le 0$ and a function of constant sign $A^*(t) \to 0$ as $t \to \infty$ such that for all $x > 0$,

$$\lim_{t\to\infty} \frac{\frac{\overline{F}(tx)}{\overline{F}(t)} - x^{-\frac{1}{\gamma}}}{A^*(t)} = x^{-\frac{1}{\gamma}} \frac{x^{\rho}-1}{\rho}. \tag{2.8}$$

- There is a real constant $\rho \le 0$ and a function $A^{**}(s) \to 0$ as $t \to 0$ such that for all $x > 0$,

$$\lim_{t\to\infty} \frac{\frac{Q(1-tx)}{Q(1-t)} - x^{-\gamma}}{A^{**}(t)} = x^{-\gamma} \frac{x^{\rho}-1}{\rho}, \tag{2.9}$$

where ρ is a second-order parameter that controls the convergence velocity in (2.6) , (2.4) and (2.5) where A check: $|A| \in RV_\rho$.

The functions A, A^* and A^{**} are regularly varying, $A^* = A\left(1/\overline{F}(t)\right)$ and $A^{**} = A(1/t)$. Their role is to control the speed of convergence in (2.7) , (2.8) and (2.9) . The relations above may be reformulated respective:

$$\lim_{t\to\infty} \frac{\log \frac{U(tx)}{U(t)} - \log x^\gamma}{A(t)} = \frac{x^\rho - 1}{\rho},$$

$$\lim_{t\to\infty} \frac{\log \frac{\overline{F}(tx)}{\overline{F}(t)} + \gamma^{-1} \log x}{A\left(1/\overline{F}(t)\right)} = \frac{x^\rho - 1}{\rho},$$

and

$$\lim_{t\to\infty} \frac{\log \frac{Q(1-tx)}{Q(1-t)} + \gamma \log x}{A(1/t)} = \frac{x^\rho - 1}{\rho}.$$

2.4 Potter's inequalities

The uniform version of the fist order condition of regular variation functions are expressed by Potter's inequalities.

Definition 2.4.1 (Potter, 1942) *Suppose $h \in RV_\alpha$. If $\delta_1, \delta_2 > 0$ are arbitrary, there exists $t_0 = t_0(\delta_1, \delta_2)$ such that for $t \geq t_0$, $tx \geq t_0$,*

$$(1 - \delta_1)\, x^\alpha \min\left(x^{\delta_2}, x^{-\delta_2}\right) < \frac{h(tx)}{h(t)} < (1 + \delta_1) x^\alpha \max\left(x^{\delta_2}, x^{-\delta_2}\right).$$

By using these inequalities we show that:

Theorem 2.4.1 *If h is regularly varying with index ρ (in the case $\rho > 0$, assuming h bounded on each interval $]0, t]$, $t > 0$, then for $0 < a \leq b < \infty$,*

$$\lim_{x\to\infty} \frac{h(tx)}{h(t)} = x^\rho, \ \textit{uniformly in } x.$$

(a) $x \in [a, b]$ if $\rho = 0$,

(b) $x \in]0, b]$ if $\rho > 0$,

(c) $x \in [a, \infty[$ if $\rho < 0$.

Theorem 2.4.2 (Local uniform convergence) *If h is regular varying function with index ρ*

then,

$$\lim_{t\to\infty}\sup_{x\in A}\left|\frac{h(tx)}{h(t)}-x^{\rho}\right|=0,$$

where

$$A=\begin{cases}[a,b] & \textit{if } \rho=0,\\ ]0,b] & \textit{assume } h \textit{ is bounded on }]0,b] \textit{ if } \rho>0,\\ [a,\infty[& \textit{if } \rho<0.\end{cases}$$

2.4.1 Uniform variation of the first order condition

Since $\overline{F}\in RV_{-1/\gamma}$ then

$$(1-\delta_1)x^{-1/\gamma}\min(x^{\delta_2},x^{-\delta_2})<\frac{\overline{F}(tx)}{\overline{F}(t)}<(1+\delta_1)x^{-1/\gamma}\max(x^{\delta_2},x^{-\delta_2}).$$

Likewise, since $U\in RV_{\gamma}$ then

$$(1-\delta_1)x^{\gamma}\min(x^{\delta_2},x^{-\delta_2})<\frac{U(tx)}{U(t)}<(1+\delta_1)x^{\gamma}\max(x^{\delta_2},x^{-\delta_2}).$$

We also have $Q(1-\cdot)\in RV_{-\gamma}$ then

$$(1-\delta_1)x^{-\gamma}\min(x^{\delta_2},x^{-\delta_2})<\frac{Q(1-tx)}{Q(1-t)}<(1+\delta_1)x^{-\gamma}\max(x^{\delta_2},x^{-\delta_2}). \tag{2.10}$$

2.4.2 Uniform variation of the second-order condition

Since $U\in RV_{\gamma}$, then

$$\left|\frac{\frac{U(tx)}{U(t)}-x^{\gamma}}{A(t)}-x^{\gamma}\frac{x^{\rho}-1}{\rho}\right|\le\varepsilon x^{\gamma+\rho/\gamma}\max(x^{\delta},x^{-\delta}),$$

likewise, we have $\overline{F}\in RV_{-1/\gamma}$, then

$$\left|\frac{\frac{\overline{F}(tx)}{\overline{F}(t)}-x^{-1/\gamma}}{A(t)}-x^{-\frac{1}{\gamma}}\frac{x^{\rho/\gamma}-1}{\gamma\rho}\right|\le\varepsilon x^{-1/\gamma+\rho/\gamma}\max(x^{\delta},x^{-\delta}).$$

Where we also have $Q(1-\cdot)\in RV_{-\gamma}$, then

$$\left|\frac{\frac{Q(1-tx)}{Q(1-t)}-x^{-\gamma}}{A(t)}-x^{-\gamma}\frac{x^{-\rho}-1}{\rho}\right|\le\varepsilon x^{-\gamma+\rho/\gamma}\max(x^{\delta},x^{-\delta}).$$

Next we only consider the Fréchet domain of attraction and its relationship of the regular variation functions.

2.5 Fréchet domain of attraction

Theorem 2.5.1 *$F \in DA\left(\Phi_{1/\gamma}\right)$ if and only if $\overline{F} \in RV_{-1/\gamma}$. In this case, $a_n = F^{-1}(1-n^{-1})$ and $b_n = 0$.*

Gnedenko's theorem, posited in 1943, provides a convenient framework for characterizing distributions $F \in DA\left(\Phi_{1/\gamma}\right)$ .In fact, we must verify

$$\lim_{t\to+\infty} \frac{\overline{F}(tx)}{\overline{F}(t)} = x^{-1/\gamma}, \text{ for any } x > 0.$$

As exemple, the strict Pareto distribution is givne by

$$F(x) = 1 - x^{-1/\gamma}, \text{ for } x \geq 1.$$

It is clear that

$$\frac{\overline{F}(tx)}{\overline{F}(t)} = \frac{(tx)^{-1/\gamma}}{(t)^{-1/\gamma}} = x^{-1/\gamma}, \text{ for any } x > 0.$$

So $F \in DA\left(\Phi_{1/\gamma}\right)$ and $\overline{F} \in RV_{-1/\gamma}$.

On the other words, the cdf F belongs to the Fréchet domain of attraction that is

$$\overline{F}(x) = x^{-\frac{1}{\gamma}} L(x), \text{ for } x > 0,$$

where $L \in RV_0$.

The following proposition gives a sufficient condition for $F \in DA\left(\Phi_{1/\gamma}\right)$.

Proposition 2.5.1 (Von Mises condition) *Let F cdf be an absolutely continuous density function f satisfying:*

$$\lim_{x\to+\infty} \frac{xf(x)}{\overline{F}(x)} = \frac{1}{\gamma} > 0,$$

then $F \in DA(\Phi_{1/\gamma})$.

Remark 2.5.1

- If F is of a Pareto model, then F belongs to the Fréchet domain of attraction.
- Along this memory, we consider only the Fréchet domain of attaraction.

2.6 Example of Pareto-type distributions

Burr model

- Probability density function:

$$f(x;\lambda,c,\sigma)=\begin{cases} c(\lambda/\sigma)\frac{(x/\sigma)^{c-1}}{(1+x^c)^{\lambda+1}} & \text{if } x>0 \\ 0 & \text{if } x\leq 0\end{cases},\ \text{for } \sigma,c,\lambda>0.$$

- Cumulative distribution function:

$$F(x;\lambda,c,\sigma)=\begin{cases} 1-\left(1+((x/\sigma)^c)^{-\lambda}\right) & \text{if } x>0, \\ 0 & \text{if } x\leq 0.\end{cases}$$

- Quantile function:

$$Q(s;\lambda,c,\sigma)=\sigma\left((1-s)^{-1/\lambda}-1\right)^{1/c},\ 0<s<1.$$

If $c=1$, Burr's model correspond to the Pareto one.

Fréchet model

- Probability density function:

$$f(x;\gamma,\mu,\sigma)=\begin{cases} \frac{1}{\sigma\gamma}\left(\frac{x-\mu}{\sigma}\right)^{-1-1/\gamma}\exp\left(-\left(\frac{x-\mu}{\sigma}\right)^{-1/\gamma}\right) & \text{if } x>\mu \\ 0 & \text{if } x\leq \mu\end{cases},\ \text{for } \mu\geq 0,\gamma,\sigma>0.$$

- Cumulative distribution function:

$$F(x;\gamma,\mu,\sigma)=\begin{cases} \exp\left(-\left(\frac{x-\mu}{\sigma}\right)^{-1/\gamma}\right) & \text{if } x>\mu, \\ 0 & \text{if } x\leq \mu.\end{cases}$$

- Quantile function:

$$Q\left(s;\gamma,\mu,\sigma\right)=\mu-\sigma\log^{-\gamma}s,\ \ 0<s<1.$$

This is some of other Pareto-type models:

Model	$F\left(x\right)$	$f\left(x\right)$
Invers burr	$\left(1+\lambda x^{-\delta}\right)^{-\beta}$	$\beta\lambda\delta x^{-\delta-1}\left(1+\lambda x^{-\delta}\right)^{-\beta}$
Lévy	$\operatorname{erf} c\sqrt{\frac{c}{2(x-\mu)}}$	$\sqrt{\frac{c}{2\pi}}\cdot\frac{1}{(x-\mu)^{3/2}}\exp\left(-\frac{c}{2(x-\mu)}\right)$
Lomax	$1-\left[1+\frac{x}{\lambda}\right]^{-\alpha}$	$\frac{\alpha}{\lambda}\left[1+\frac{x}{\lambda}\right]^{-\alpha}$
Log-logistique	$\frac{1}{1+(x/\alpha)^{-\beta}}$	$\frac{(\beta/\alpha)(x/\alpha)^{\beta-1}}{\left(1+(x/\alpha)^{\beta}\right)^{2}}$
Pareto	$1-\left(\frac{x}{u}\right)^{\alpha}$	$\left(\frac{\alpha u^{k}}{x^{\alpha+1}}\right)$

Table 2.1: Pareto-type models

Chapter 3

Consistency and asymptotic normality of Hill's estimator

3.1 Construction of Hill's estimator

In this section, we deal with the estimation of the tail index $\gamma > 0$ whenever $F \in DA(G_\gamma)$, on the other terms $\overline{F}$ satisfies the first order condition of regular variation function:

$$\lim_{t\to\infty} \frac{\overline{F}(tx)}{\overline{F}(t)} = x^{-1/\gamma}, \text{ for } x > 0. \tag{3.1}$$

To this end, we first introduce the following result:

Proposition 3.1.1 *Assume that F satisfies the assumption, then*

$$\int_1^\infty x^{-1} \frac{\overline{F}(tx)}{\overline{F}(t)} dx \to \gamma \text{ as } t \to \infty,$$

which is equivalent to

$$\frac{1}{\overline{F}(t)} \int_t^\infty \log(x/t)\, dF(x) \to \gamma \text{ as } t \to \infty. \tag{3.2}$$

Proof. From Potter's inequalities above, for any $x \geq 1$, $\epsilon > 0$ and for large t, we have

$$(1-\epsilon)x^{-1/\gamma-\epsilon} < \frac{\overline{F}(tx)}{\overline{F}(t)} < (1+\epsilon)x^{-1/\gamma+\epsilon},$$

which implies that

$$(1-\epsilon)x^{-1/\gamma-1-\epsilon} < x^{-1}\frac{\overline{F}(tx)}{\overline{F}(t)} < (1+\epsilon)x^{-1/\gamma-1+\epsilon}.$$

It is clear that

$$(1-\epsilon)\int_1^\infty x^{-1/\gamma-1-\epsilon}dx < \int_1^\infty x^{-1}\frac{\overline{F}(tx)}{\overline{F}(t)}dx < (1+\epsilon)\int_1^\infty x^{-1/\gamma-1+\epsilon}dx.$$

We have $\int_1^\infty x^{-1/\gamma-1+\epsilon}dx = \frac{\gamma}{1-\gamma\epsilon}$ and $\int_1^\infty x^{-1/\gamma-1-\epsilon}dx = \frac{\gamma}{\gamma\epsilon+1}$, therefore

$$(1-\epsilon)\frac{\gamma}{\gamma\epsilon+1} < \int_1^\infty x^{-1}\frac{\overline{F}(tx)}{\overline{F}(t)}dx < (1+\epsilon)\frac{\gamma}{1-\gamma\epsilon}.$$

By letting $\epsilon \downarrow 0$, yields

$$\int_1^\infty x^{-1}\frac{\overline{F}(tx)}{\overline{F}(t)}dx \to \gamma, \text{ as } t \to \infty,$$

which completes the proof of the first assertion. The second one comes by using an elementary integration by part that we omit further details. ■

Let us now derive the so-called Hill's estimator of the tail index γ. Let $X_{1:n} \leq ... \leq X_{n:n}$ be the order statistics pertaining to the sample $X_1, ..., X_n$ from the cdf $F \in DA(G_\gamma)$. Recall that, for any sequence of integer $k = k_n$ such that $k \to \infty$ and $k/n \to 0$, $X_{n-k:n} \to \infty$ almost surely. Then by substituting, in (3.2), $t = X_{n-k:n}$ and F by its the empirical counterpart

$$F_n(x) = n^{-1}\sum_{i=1}^n \mathbb{I}_{\{X_{i,n} \leq x\}},$$

we derive Hill's estimator given by

$$\widehat{\gamma}_H := \frac{1}{\overline{F}_n(X_{n-k:n})}\int_{X_{n-k:n}}^\infty \log(x/X_{n-k:n})\,dF_n(x).$$

Notice that $\overline{F}_n(X_{n-k:n}) = k/n$, then

$$\begin{aligned}\widehat{\gamma}_H &= \frac{n}{k}\int_0^\infty \mathbb{I}_{\{x>X_{n-k:n}\}} \log(x/X_{n-k:n})\, dF_n(x) \\ &= \frac{1}{k}\sum_{i=n-k+1}^{n} \log(X_{i:n}/X_{n-k:n}) = \frac{1}{k}\sum_{i=1}^{k} \log(X_{n-i+1:n}/X_{n-k:n}).\end{aligned}$$

There by changing i by $n-i+1$, we end up to final formula for Hill's estimator

$$\widehat{\gamma}_H = \frac{1}{k}\sum_{i=1}^{k} \log(X_{n-i+1:n}/X_{n-k:n}).$$

3.2 Limit theorems for Pareto-type intermediate order statistics

Let $X_{1:n} \leq ... \leq X_{n:n}$ be the order statistics pertaining to the sample $X_1, ..., X_n$ for (rv) of regularly varying distribution function F with negative index $(-1/\gamma)$, notation $\overline{F} \in \mathcal{RV}_{(-1/\gamma)}$ that is:

$$\lim_{t\to\infty} \mathbf{P}(X/t > x >| X > t) = \lim_{t\to\infty} \frac{\overline{F}(xt)}{\overline{F}(t)} = x^{-1/\gamma},\ x > 0.$$

In terms of the quantile function $Q = F^{-1}$, the previous limit is equivalent to:

$$\lim_{t\downarrow 0} \frac{Q(1-st)}{Q(1-t)} = s^{-\gamma},\ s > 0. \tag{3.3}$$

Some times it is convenient to rewrite the previous limit into:

$$\lim_{t\to\infty} \frac{U(st)}{U(t)} = s^{\gamma},\ s > 0, \tag{3.4}$$

where $U(s) = Q(1-1/s),\ s > 1$.

Theorem 3.2.1 *Let $\overline{F} \in \mathcal{RV}_{(-1/\gamma)}$ and $1 < k_n < n$ be a sequence of integers such that $k_n \to \infty$ and $k_n/n \to 0$, then*

$$\frac{X_{n-k+1:n}}{Q(1-k_n/n)} \xrightarrow{P} 1,\ \text{as } n \to \infty.$$

The proof Theorem 3.2.1 requires the propositions below.

Proposition 3.2.1 *Let $U_1, ..., U_n$ be a sample from a rv U uniformly distributed on $(0,1)$ and let*

$E_1, ..., E_{n+1}$ be a sample from a rv E, independent of U, of standard exponential distribution, i.e. E follows $\exp(1)$. *Then*

$$\{U_{i:n}\}_{i=1}^{n} \overset{\mathcal{D}}{=} \left\{ \frac{S_i}{S_{n+1}} \right\}_{i=1}^{n},$$

where $S_i = \sum_{i=1}^{n} E_i,\ i = 1, ..., n+1.$

Proof. See the representation (4.24) page 44 in **Ahsanullah et al. (2013)[1]**. Indeed, observe that jointly

$$\frac{S_i}{S_{n+1}} = \frac{S_i}{S_i + S_{n-i+1}} \overset{\mathcal{D}}{=} \frac{\Gamma(i,1)}{\Gamma(i,1) + \Gamma(n-i+1,1)} \overset{\mathcal{D}}{=} beta\,(i, n-i+1),\ \ i = 1, .., n$$

On the other hand, we have jointly $U_{i:n} \overset{\mathcal{D}}{=} beta\,(i, n-i+1)$, it follows that jointly

$$U_{i:n} \overset{\mathcal{D}}{=} \frac{S_i}{S_{n+1}} \overset{\mathcal{D}}{=} beta\,(i, n-i+1).$$

■

Proposition 3.2.2 *Let $1 < k_n < n$ be a sequence of integers such that $k := k_n \to \infty$ and $k/n \to 0$, then*

$$nU_{k:n}/k \overset{p}{\to} 1 \text{ and } n(1 - U_{n-k+1:n})/k \overset{p}{\to} 1, \text{ as } n \to \infty.$$

Proof. From Proposition 3.2.1, we write

$$U_{k:n} \overset{\mathcal{D}}{=} \frac{S_k}{S_{n+1}}.$$

Observe that

$$nU_{k:n}/k \overset{\mathcal{D}}{=} \frac{n}{n+1} \frac{S_k}{k} \frac{n+1}{S_{n+1}}.$$

We have $\frac{n}{n+1} \to 1$,as $n \to \infty$ and from the law of large numbers, both $\frac{S_k}{k} \overset{p}{\to} 1$ and $\frac{S_k}{k} \overset{p}{\to} 1$ as $n \to \infty$, therefore $nU_{k:n}/k \overset{p}{\to} 1$. Since $U_{k:n} \overset{\mathcal{D}}{=} 1 - U_{n-k+1:n}$ then we also have $n(1 - U_{n-k+1:n})/k \overset{p}{\to} 1$, as $n \to \infty$ as sought. ■

We have now all the ingredients to prove Theorem 3.2.1.

Proof of Theorem 3.2.1. We first note that

$$X_{n-k+1:n} \overset{\mathcal{D}}{=} Q(U_{n-k+1:n}) = Q(1 - (1 - U_{n-k+1:n})).$$

We set $s_n := n\left(1 - U_{n-k+1:n}\right)/k$ and $t_n = k/n$, it the other terms

$$X_{n-k+1:n} \overset{\mathcal{D}}{=} Q\left(1 - t_n s_n\right).$$

By applying inequality (2.10) , we infer that

$$(1-\epsilon)s_n^{-\gamma}\min(s_n^{\epsilon}, s_n^{-\epsilon}) < \frac{Q(1-t_n s_n)}{Q(1-t_n)} < (1+\epsilon)s_n^{-\gamma}\max(s_n^{\epsilon}, s_n^{-\epsilon}).$$

From Proposition 3.2.2, we have $s_n \xrightarrow{p} 1$, therefore with probability tending to 1, we get

$$(1-\epsilon) < \frac{Q(1-t_n s_n)}{Q(1-t_n)} < (1+\epsilon),$$

this means that

$$\frac{X_{n-k+1:n}}{Q\left(1-k_n/n\right)} = \frac{Q(1-t_n s_n)}{Q(1-t_n)} \xrightarrow{p} 1, \text{ as } n \to \infty,$$

the proof is now completed.

The following Theorem establish the asymptotic normality of the ratio

$$X_{n-k+1:n}/Q\left(1-k_n/n\right).$$

To this end, we require to the second order condition corresponding to the first one (3.4) that is

$$\lim_{t\to\infty} \frac{\frac{\mathbb{U}(st)}{\mathbb{U}(t)} - s^{\gamma}}{A\left(t\right)} = s^{\gamma}\frac{s^{\rho}-1}{\rho}, \ s > 0, \tag{3.5}$$

where the $\rho < 0$ is the second order parameter and $|A| \in \mathcal{RV}_{(-1/\gamma)}$ tending to zero as $t \to \infty$. When $\overline{F}$ satisfies assumption (3.5) we write $\overline{F} \in \mathcal{R}^{(2)}\mathcal{V}_{(-1/\gamma)}$. See for instance the limit (2.3.22) in page 48 of de Haan and Ferreira (2006) [17].

Theorem 3.2.2 *Let $\overline{F} \in \mathcal{RV}_{(-1/\gamma)}$ and $1 < k_n < n$ be a sequence of integers such that $k_n \to \infty$, $k_n/n \to 0$ and $\sqrt{k}A\left(n/k\right) = O\left(1\right)$, then*

$$\sqrt{k}\left(\frac{X_{n-k+1:n}}{Q\left(1-k/n\right)} - 1\right) \xrightarrow{D} \mathcal{N}\left(0, \gamma^2\right), \ \text{as } n \to \infty$$

The proof Theorem 3.2.2 requires the propositions below.

Proposition 3.2.3 *Let $1 < k_n < n$ be a sequence of integers such that $k := k_n \to \infty$ and $k/n \to 0$,* *then*

$$\sqrt{k}\left(nU_{k:n}/k-1\right)\xrightarrow{D}\mathcal{N}\left(0,1\right) \text{ and } \sqrt{k}\left(n\left(1-U_{n-k+1:n}\right)/k-1\right)\xrightarrow{D}\mathcal{N}\left(0,1\right).$$

Proof. We already write that

$$nU_{k:n}/k\overset{\mathcal{D}}{=}\frac{n}{n+1}\frac{S_k}{k}\frac{n+1}{S_{n+1}}.$$

It easy to verify that

$$\sqrt{k}\left(nU_{k:n}/k-1\right)=T_{n1}+T_{n2}-\frac{\sqrt{k}}{n+1},$$

where

$$T_{n1}:=\frac{n}{n+1}\sqrt{k}\left(\frac{S_k}{k}-1\right)\frac{n+1}{S_{n+1}} \text{ and } T_{n2}:=\sqrt{k}\frac{n}{n+1}\left(\frac{n+1}{S_{n+1}}-1\right).$$

Using the central limit theorem, we have $\sqrt{k}\left(S_k/k-1\right)\xrightarrow{D}\mathcal{N}\left(0,1\right)$ and by the law of large numbers $S_{n+1}/\left(n+1\right)\xrightarrow{p}1$, it follows that $T_{n1}\xrightarrow{D}\mathcal{N}\left(0,1\right)$. The second term may be rewritten into

$$T_{n2}=-\sqrt{\frac{k}{n}}\frac{n}{n+1}\sqrt{n}\left(\frac{S_{n+1}}{n+1}-1\right)\frac{n+1}{S_{n+1}}.$$

Again by using the central limit theorem, we have $\sqrt{n}\left(\frac{S_{n+1}}{n+1}-1\right)\xrightarrow{D}\mathcal{N}\left(0,1\right)$ then $\sqrt{n}\left(\frac{S_{n+1}}{n+1}-1\right)$ is bounded in probability as $n\to\infty$. On the other hand $S_{n+1}/\left(n+1\right)\xrightarrow{p}1$ and $k/n\to0$, therefore $T_{n2}\xrightarrow{p}0$. It is obvious that $\frac{\sqrt{k}}{n+1}\to0$, which completes the proof. ■

Proposition 3.2.4 (Potter's inequalities) *For any $\epsilon>0$, there exists $t_0=t_0(\epsilon)>1$ such that for $t\geq t_0$, $ts\geq t_0$,*

$$\left|\frac{\frac{\mathbb{U}(st)}{\mathbb{U}(t)}-s^{\gamma}}{A_0\left(t\right)}-s^{\gamma}\frac{s^{\rho}-1}{\rho}\right|<\epsilon s^{\gamma+\epsilon}\max\left(s^{\epsilon},s^{-\epsilon}\right),$$

*for some $A_0\left(t\right)\sim A\left(t\right)$ as $t\to\infty$. See for instance **d**e Haan and Ferreira (2006) [**17**].*

All the materiels now are available to show Theorem (3.2.2).

Proof of Theorem 3.2.2. Observe that

$$X_{n-k+1:n}\overset{\mathcal{D}}{=}Q\left(1-\frac{1}{t_ns_n}\right)=\mathbb{U}\left(t_ns_n\right),$$

where

$$t_n:=n/k \text{ and } s_n:=\frac{1}{n\left(1-U_{n-k+1:n}\right)/k}.$$

It is clear that

$$\sqrt{k}\left(\frac{X_{n-k+1:n}}{Q\left(1-k/n\right)}-1\right)=S_{n1}+S_{n2},$$

where

$$S_{n1}:=\sqrt{k}\left(\frac{\mathbb{U}\left(t_n s_n\right)}{\mathbb{U}\left(t_n\right)}-s_n^{\gamma}\right) \text{ and } S_{n2}:=\sqrt{k}\left(s_n^{\gamma}-1\right).$$

Let write

$$S_{n1}:=\sqrt{k}A_0\left(t_n\right)\left\{\frac{\frac{\mathbb{U}(t_n s_n)}{\mathbb{U}(t_n)}-s_n^{\gamma}}{A_0\left(t_n\right)}-s_n^{\gamma}\frac{s_n^{\rho}-1}{\rho}\right\}+\sqrt{k}A_0\left(t_n\right)s_n^{\gamma}\frac{s_n^{\rho}-1}{\rho}.$$

From Proposition 3.2.4, we have

$$\left|\frac{\frac{\mathbb{U}(t_n s_n)}{\mathbb{U}(t_n)}-s_n^{\gamma}}{A_0\left(t_n\right)}-s_n^{\gamma}\frac{s_n^{\rho}-1}{\rho}\right|<\epsilon s_n^{\gamma+\epsilon}\max\left(s_n^{\epsilon},s_n^{-\epsilon}\right),$$

and by Proposition 3.2.2 $s_n \xrightarrow{p} 1$, it follows that

$$\frac{\frac{\mathbb{U}(t_n s_n)}{\mathbb{U}(t_n)}-s_n^{\gamma}}{A_0\left(t_n\right)}-s_n^{\gamma}\frac{s_n^{\rho}-1}{\rho}\xrightarrow{p}0.$$

First note that $|A|\in\mathcal{RV}_{(\rho)}$ then $|A_0|\in\mathcal{RV}_{(\rho)}$ too, then since $\sqrt{k}A\left(t_n\right)=O\left(1\right)$ its implies that $\sqrt{k}A_0\left(t_n\right)$ too. Since $s_n^{\gamma}\frac{s_n^{\rho}-1}{\rho}\xrightarrow{p}0$ then $S_{n1}\xrightarrow{p}0$ as well. Let now focus on the second term S_{n2}. Making use of the mean theorem we write

$$S_{n2}:=\sqrt{k}\left(s_n^{\gamma}-1\right)=\gamma\sqrt{k}\left(s_n-1\right)c_n^{\gamma-1},$$

where c_n is between 1 and s_n. From Proposition 3.2.2 we have $s_n\xrightarrow{p}1$, it follows that $c_n\xrightarrow{p}1$ as well. On the other hand,

$$\begin{aligned}\sqrt{k}\left(s_n-1\right)&=\sqrt{k}\left(\frac{1}{n\left(1-U_{n-k+1:n}\right)/k}-1\right)\\&=-\gamma\frac{1}{n\left(1-U_{n-k+1:n}\right)/k}\sqrt{k}\left(n\left(1-U_{n-k+1:n}\right)/k-1\right).\end{aligned}$$

Again from Proposition 3.2.2 $n\left(1-U_{n-k+1:n}\right)/k\xrightarrow{p}1$ and by Proposition 3.2.3 we have

$$\sqrt{k}\left(n\left(1-U_{n-k+1:n}\right)/k-1\right)\xrightarrow{D}\mathcal{N}\left(0,1\right),$$

therefore $S_{n2}\xrightarrow{D}\mathcal{N}\left(0,\gamma^2\right)$, which completes the proof.

3.3 Empirical tail processes and applications

The uniform tail empirical process pertaining to a sample $U_1, ..., U_n$ from an uniform $(0,1)$ (rv) is defined by

$$\alpha_n(s) := \sqrt{k}\left(\frac{n}{k}G_n(ks/n) - s\right),\ 0 \le s \le 1,$$

where uniform

$$G_n(s) := \frac{1}{k}\sum_{i=1}^{n}\mathbb{I}_{(U_i \le s)},\ \text{for } 0 \le s \le 1.$$

Denote the corresponding uniform distribution function, where $1 < k_n < n$ be a sequence of integers such that $k_n \to \infty$, $k_n/n \to 0$.

Theorem 3.3.1 *There exists a standard Wiener process* $\{W(s);\ s \ge 0\}$ *such that for every* $0 < \epsilon < 1/2:$

$$\sup_{0 \le s \le 1} s^{-\epsilon}\left|\alpha_n(s) - W(s)\right| \xrightarrow{p} 0,\ \text{as } n \to \infty.$$

Let $X_{1:n} \le ... \le X_{n:n}$ be the order statistics pertaining to the sample $X_1, ..., X_n$ for random variable (rv) of regularly varying distribution function F with negative index $(-1/\gamma)$, notation $\overline{F} \in \mathcal{RV}_{(-1/\gamma)}$. The empirical df is defined pertaining to the sample $X_1, ..., X_n$ is defined by

$$F_n(x) = \frac{1}{n}\sum_{i=1}^{n}\mathbb{I}_{(X_i \le x)},$$

where $\mathbb{I}_A(\cdot)$ denotes the indicator function. The corresponding tail empirical process is defined by

$$D_n(x) = \sqrt{k}\left\{\frac{n}{k}\overline{F}_n(xX_{n-k+1:n}) - x^{-1/\gamma}\right\},\ x > 0.$$

Theorem 3.3.2 *Assume that* $\overline{F} \in \mathcal{RV}_{(-1/\gamma)}$ *and let* $1 < k_n < n$ *be a sequence of integers such that* $k_n \to \infty$, $k_n/n \to 0$. *Then for every,* $x_0 > 0$, $0 < \epsilon < 1/2:$

$$\sup_{x \ge x_0}\left|k^{-1/2}D_n(x)\right| \xrightarrow{p} 0,\ \text{as } n \to \infty.$$

In addition if $\overline{F} \in \mathcal{RV}^{(2)}_{(-1/\gamma)}$, *then*

$$\sup_{x \ge x_0} x^{(1/2-\epsilon)/\gamma}\left|D_n(x) - \left\{W\left(x^{-1/\gamma}\right) - x^{-1/\gamma}W(1)\right\} - x^{-1/\gamma}\frac{x^{\rho/\gamma}-1}{\rho\gamma}\sqrt{k}A_0(n/k)\right| \xrightarrow{p} 0, \quad (3.6)$$

as $n \to \infty$, *where* $\{W(s); s \ge 0\}$ *is the same standard Wiener process given in Theorem 3.3.1. See*

***d**e Haan and Ferreira (2006) [17] (pages 52 and 161).*

Theorem 3.3.3 *Let $\overline{F} \in \mathcal{RV}^{(2)}_{(-1/\gamma)}$ and $1 < k_n < n$ be a sequence of integers such that $k_n \to \infty$, $k_n/n \to 0$ and $\sqrt{k}A(n/k) = O(1)$, then*

$$\sqrt{k}\left(\frac{X_{n-k+1:n}}{Q(1-k/n)} - 1\right) = \gamma W(1) + o_p\left(\sqrt{k}A(n/k)\right), \text{ as } n \to \infty.$$

We deduce that

$$\sqrt{k}\left(\frac{X_{n-k+1:n}}{Q(1-k/n)} - 1\right) \xrightarrow{\mathcal{D}} \mathcal{N}\left(0, \gamma^2\right), \text{ as } n \to \infty.$$

Proof. Let us make the following decomposition

$$\frac{X_{n-k+1:n}}{Q(1-k/n)} - 1 = T_{n1} + T_{n2},$$

where

$$T_{n1} := \frac{X_{n-k+1:n}}{Q(1-k/n)} - (n(1-U_{n-k+1:n})/k)^{-\gamma},$$

and

$$T_{n2} := (n(1-U_{n-k+1:n})/k)^{-\gamma} - 1.$$

From representation of the proof of 3.2.2, we have

$$T_{n1} = A_0(t_n)\frac{\frac{\mathbb{U}(t_n s_n)}{\mathbb{U}(t_n)} - s_n^{\gamma}}{A_0(t_n)},$$

and also we showed that $T_{n1} = o_p(A_0(t_n))$, as $n \to \infty$. Let us now treat the second term, to this end let us first recall that we write $1 - U_{n-k+1:n} \overset{\mathcal{D}}{=} U_{k:n}$, its follows that

$$T_{n2} \overset{\mathcal{D}}{=} (nU_{k:n}/k)^{-\gamma} - 1.$$

By using the mean value theorem (finite increments theorem), there exsits a constant c betewen x and b such that

$$f(b) - f(x) = f'(c)(b - x).$$

If we set $f(x) = x^{-\gamma}$, $f'(x) = -\gamma x^{-\gamma-1}$, $b = nU_{k:n}/k$ and $x = 1$, then

$$T_{n2} = (nU_{k:n}/k)^{-\gamma} - 1 = (nU_{k:n}/k)^{-\gamma} - 1^{-\gamma}$$
$$= \gamma \eta_n^{-\gamma-1} (1 - nU_{k:n}/k),$$

were η_n is between 1 and $nU_{k:n}/k$. We already showed that $nU_{k:n}/k = 1 + o_p(1)$, this means $\eta_n^{-\gamma-1} = 1 + o_p(1)$ too, therefore

$$T_{n2} = (nU_{k:n}/k)^{-\gamma} - 1 = (1 + o_p(1)) \gamma (1 - nU_{k:n}/k).$$

On the other hand, it easy to verify that

$$\alpha_n(nU_{k:n}/k) = \sqrt{k} \left(\frac{n}{k} G_n(k(nU_{k:n}/k)/n) - nU_{k:n}/k \right).$$

Let us denote $nU_{k:n}/k = s$, yields

$$\alpha_n(s) = \sqrt{k} \left(\frac{n}{k} G_n(ks/n) - s \right).$$

We have $G_n(U_{k:n}) = k/n$, then $\frac{n}{k} G_n(U_{k:n}) = 1$, therefore

$$\alpha_n(nU_{k:n}/k) = \sqrt{k}(1 - nU_{k:n}/k),$$

it follows that

$$\sqrt{k} T_{n2} = (1 + o_p(1)) \gamma \alpha_n(nU_{k:n}/k),$$

where $\alpha_n(\cdot)$ being the uniform tail empirical process defined above. Making use of Theorem 3.3.1, we get

$$\alpha_n(nU_{k:n}/k) = \gamma W(nU_{k:n}/k) + o_p(1).$$

Using similar argument as used in Benchaira et al. (2016) [**4**], we infer that

$$|W(nU_{k:n}/k) - W(1)| \leq 2\epsilon, \text{ almost surely,}$$

therefore $W(nU_{k:n}/k) = W(1) + o_p(1)$, this means that

$$\sqrt{k} T_{n2} = (1 + o_p(1)) \gamma (W(1) + o_p(1)).$$

Since is stochastically bounded, $W(1) = O_p(1)$, then $\sqrt{k}T_{n2} = \gamma W(1) + o_p(1)$. Finally we showed that

$$\sqrt{k}\left(\frac{X_{n-k+1:n}}{Q(1-k/n)} - 1\right) = \gamma W(1) + o_p(1).$$

Since $W(1) \rightsquigarrow \mathcal{N}(0,1)$, then

$$\sqrt{k}\left(\frac{X_{n-k+1:n}}{Q(1-k/n)} - 1\right) \to \mathcal{N}\left(0,\gamma^2\right).$$

■

Theorem 3.3.4 *Let $\overline{F} \in \mathcal{RV}_{(-1/\gamma)}$ and $1 < k_n < n$ be a sequence of integers such that $k_n \to \infty$, $k_n/n \to 0$, then*

$$\widehat{\gamma}_H \xrightarrow{p} \gamma, \text{ as } n \to \infty.$$

In addition if $\overline{F} \in \mathcal{RV}^{(2)}_{(-1/\gamma)}$ then

$$\sqrt{k}\left(\widehat{\gamma}_H - \gamma\right) = \gamma \int_0^1 s^{-1}\left(W(s) - W(1)\right) ds + \frac{1}{1-\rho}\sqrt{k}A(n/k) + o_p(1),$$

provided that $\sqrt{k}A(n/k) = O_p(1)$. Moreover, if $\sqrt{k}A(n/k) \to \lambda < \infty$, then

$$\sqrt{k}\left(\widehat{\gamma}_H - \gamma\right) \xrightarrow{D} \mathcal{N}\left(\frac{\lambda}{1-\rho}, \gamma^2\right), \text{ as } n \to \infty.$$

Proof. We have

$$\widehat{\gamma}_H = \int_1^\infty x^{-1}\frac{\overline{F}_n(xX_{n-k:n})}{\overline{F}_n(X_{n-k:n})}dx = \int_1^\infty x^{-1}\left\{\frac{n}{k}\overline{F}_n(xX_{n-k:n})\right\}dx.$$

Observe that

$$\widehat{\gamma}_H - \gamma = \int_1^\infty x^{-1}\left(\frac{n}{k}\overline{F}_n(xX_{n-k:n}) - x^{-1/\gamma}\right)dx = \int_1^\infty x^{-1}\left\{k^{-1/2}D_n(x)\right\}dx.$$

Let us fix $0 < \epsilon < 1/2$ and write

$$\int_1^\infty x^{-1}k^{-1/2}D_n(x)\,dx = \int_1^\infty x^{-1-(1/2-\epsilon)/\gamma}\left\{x^{(1/2-\epsilon)/\gamma}k^{-1/2}D_n(x)\right\}dx.$$

It is obvious that

$$\left|\int_1^\infty x^{-1}k^{-1/2}D_n(x)\,dx\right| \leq \left\{\sup_{x\geq 1}\left|x^{(1/2-\epsilon)/\gamma}k^{-1/2}D_n(x)\right|\right\}\left\{\int_1^\infty x^{-1-(1/2-\epsilon)/\gamma}dx\right\}.$$

From the first part of Theorem (3.3.2) , the first factor converges in probability to zero as $n \to \infty$. For the second factor, we have

$$\int_1^\infty x^{-1-(1/2-\epsilon)/\gamma}dx = \frac{1}{-(1/2-\epsilon)/\gamma}\left[x^{-(1/2-\epsilon)/\gamma}\right]_1^\infty = \frac{\gamma}{1/2-\epsilon}.$$

Since $0 < \epsilon < 1/2$, the previous integral is finite and equal to $\frac{\gamma}{1/2-\epsilon}$, and therefore $\widehat{\gamma}_H \xrightarrow{p} \gamma$ as $n \to \infty$. Let us now establish the asymptotic normality of $\widehat{\gamma}_H$. On again by using representation,

$$\sqrt{k}\left(\widehat{\gamma}_H - \gamma\right) = \int_1^\infty x^{-1}D_n(x)\,dx.$$

Making use of the second part of Theorem (3.3.2) , we write

$$\sqrt{k}\left(\widehat{\gamma}_H - \gamma\right) = T_{n1} + T_{n2} + T_{n3},$$

where

$$T_{n1} := \int_1^\infty x^{-1}\left\{D_n(x) - \left\{W\left(x^{-1/\gamma}\right) - x^{-1/\gamma}W(1)\right\} - x^{-1/\gamma}\frac{x^{\rho/\gamma}-1}{\rho\gamma}\sqrt{k}A_0(n/k)\right\}dx,$$

$$T_{n2} := \int_1^\infty x^{-1}\left\{x^{-1/\gamma}\frac{x^{-\rho/\gamma}-1}{\rho\gamma}\sqrt{k}A_0(n/k)\right\}dx,$$

and

$$T_{n3} := \int_1^\infty x^{-1}\left(W\left(x^{-1/\gamma}\right) - x^{-1/\gamma}W(1)\right)dx.$$

By using weak approximation (3.6) , we obtain

$$T_{n1} = o_p(1)\int_1^\infty x^{-1+(1/2-\epsilon)/\gamma}dx = o_p(1)\,,\ \text{as } n \to \infty.$$

For the second term, we have

$$T_{n2} = \sqrt{k}A_0(n/k)\int_1^\infty x^{-1}\left\{x^{-1/\gamma}\frac{x^{\rho/\gamma}-1}{\rho\gamma}\right\}dx = \frac{\sqrt{k}A_0(n/k)}{1-\rho}.$$

It follows that

$$\sqrt{k}\left(\widehat{\gamma}_H-\gamma\right)=\int_1^{\infty}x^{-1}\left(W\left(x^{-1/\gamma}\right)-x^{-1/\gamma}W\left(1\right)\right)dx+\frac{\sqrt{k}A_0\left(n/k\right)}{1-\rho}+o_p\left(1\right).$$

Next we show that

$$\sigma^2:=\mathbf{var}\left[\int_1^{\infty}x^{-1}\left(W\left(x^{-1/\gamma}\right)-x^{-1/\gamma}W\left(1\right)\right)dx\right]=\gamma^2.$$

Indeed, it is clear that

$$\mathbf{E}\left[\int_1^{\infty}x^{-1}\left(W\left(x^{-1/\gamma}\right)-x^{-1/\gamma}W\left(1\right)\right)dx\right]=\int_1^{\infty}\mathbf{E}\left[x^{-1}W\left(x^{-1/\gamma}\right)-x^{-1/\gamma-1}W\left(1\right)\right]dx,$$

because $\mathbf{E}\left[W\left(s\right)\right]=0$, for any s so we have

$$\mathbf{E}\left[\int_1^{\infty}x^{-1}\left(W\left(x^{-1/\gamma}\right)-x^{-1/\gamma}W\left(1\right)\right)dx\right]=0.$$

■

Then

$$\sigma^2:=\mathbf{E}\left(\int_1^{\infty}x^{-1}\left(W\left(x^{-1/\gamma}\right)-x^{-1/\gamma}W\left(1\right)\right)dx\right)^2,$$

which equals

$$\begin{aligned}&\mathbf{E}\left(\int_1^{\infty}x^{-1}\left(W\left(x^{-1/\gamma}\right)-x^{-1/\gamma}W\left(1\right)\right)dx\right)\left(\int_1^{\infty}y^{-1}\left(W\left(y^{-1/\gamma}\right)-y^{-1/\gamma}W\left(1\right)\right)dy\right)\\&=\mathbf{E}\int_1^{\infty}\int_1^{\infty}\left(x^{-1}\left(W\left(x^{-1/\gamma}\right)-x^{-1/\gamma}W\left(1\right)\right)\right)\left(y^{-1}\left(W\left(y^{-1/\gamma}\right)-y^{-1/\gamma}W\left(1\right)\right)\right)dxdy\\&=\int_1^{\infty}\int_1^{\infty}\mathbf{E}\left[\left(x^{-1}\left(W\left(x^{-1/\gamma}\right)-x^{-1/\gamma}W\left(1\right)\right)\right)\left(y^{-1}\left(W\left(y^{-1/\gamma}\right)-y^{-1/\gamma}W\left(1\right)\right)\right)\right]dxdy\\&=\int_1^{\infty}\int_1^{\infty}x^{-1}y^{-1}\mathbf{E}\left[\left(W\left(x^{-1/\gamma}\right)-x^{-1/\gamma}W\left(1\right)\right)\left(W\left(y^{-1/\gamma}\right)-y^{-1/\gamma}W\left(1\right)\right)\right]dxdy.\end{aligned}$$

Using the obvious formula $\left(a-b\right)\left(c-d\right)=ac-ad-bc+bd$, we decompose the previous integral into the sum of

$$I_1:=\int_1^{\infty}\int_1^{\infty}x^{-1}y^{-1}\mathbf{E}\left[W\left(x^{-1/\gamma}\right)W\left(y^{-1/\gamma}\right)\right]dxdy,$$

$$I_2:=-\int_1^{\infty}\int_1^{\infty}x^{-1}y^{-1}\mathbf{E}\left[W\left(x^{-1/\gamma}\right)y^{-1/\gamma}W\left(1\right)\right]dxdy$$

$$I_3 := -\int_1^{\infty}\int_1^{\infty} x^{-1}y^{-1}\mathbf{E}\left[x^{-1/\gamma}W(1)\,W\left(y^{-1/\gamma}\right)\right]dxdy,$$

and

$$I_4 := \int_1^{\infty}\int_1^{\infty} x^{-1}y^{-1}\mathbf{E}\left[x^{-1/\gamma}W(1)\,y^{-1/\gamma}W(1)\right]dxdy.$$

Let us begin by I_1. Observe that

$$\begin{aligned} I_1 &= \int_1^{\infty}\int_1^{\infty} x^{-1}y^{-1}\mathbf{E}\left[W\left(x^{-1/\gamma}\right)W\left(y^{-1/\gamma}\right)\right]dxdy \\ &= \int_1^{\infty}\int_1^{\infty} x^{-1}y^{-1}\min\left(W\left(x^{-1/\gamma}\right),W\left(y^{-1/\gamma}\right)\right)dxdy. \end{aligned}$$

Let us use the change of variables $s = x^{-1/\gamma}$ and $t = y^{-1/\gamma}$, which is equivalente to $x = s^{-\gamma}$ and $y = t^{-\gamma}$, it follows that

$$I_1 = \int_1^{0}\int_1^{0} \left(s^{-\gamma}\right)^{-1}\left(t^{-\gamma}\right)^{-1}\min(s,t)\,ds^{-\gamma}ds^{-\gamma}.$$

Note that $ds^{-\gamma} = -\gamma s^{-\gamma-1}ds$ and $dt^{-\gamma} = -\gamma t^{-\gamma-1}dt$, then

$$\begin{aligned} I_1 &= \int_0^{1}\int_0^{1} \left(s^{-\gamma}\right)^{-1}\left(t^{-\gamma}\right)^{-1}\min(s,t)\left(-\gamma s^{-\gamma-1}ds\right)\left(-\gamma t^{-\gamma-1}dt\right) \\ &= \gamma^2\int_0^1\left(\int_0^1 s^{-1}t^{-1}\min(s,t)\,ds\right)dt. \end{aligned}$$

Observe now that

$$I_1 = \gamma^2\int_0^1\int_0^t s^{-1}t^{-1}\min(s,t)\,dsdt + \gamma^2\int_0^1\int_t^1 s^{-1}t^{-1}\min(s,t)\,dsdt,$$

which equals

$$I_1 = \gamma^2\int_0^1\int_0^t s^{-1}t^{-1}sdsdt + \gamma^2\int_0^1\int_t^1 s^{-1}t^{-1}tdsdt,$$

then

$$\begin{aligned} I_1 &= \gamma^2\left(\int_0^1 t^{-1}\int_0^t ds\right)dt + \gamma^2\left(\int_0^1\int_t^1 s^{-1}ds\right)dt \\ &= \gamma^2 - \gamma^2\int_0^1\log t\,dt = 2\gamma^2, \end{aligned}$$

since $-\int_0^1 \log t dt = 1$. For I_2 we have

$$I_2 = -\int_1^\infty \int_1^\infty x^{-1} y^{-1} y^{-1/\gamma} \min\left(x^{-1/\gamma}, 1\right) dxdy$$
$$= -\int_1^\infty \int_1^\infty x^{-1} y^{-1} y^{-1/\gamma} x^{-1/\gamma} dxdy = -\left(\int_1^\infty x^{-1/\gamma - 1} dx\right)^2 = -\gamma^2.$$

By similar arguments we also show that $I_3 = -\gamma^2$. The fourth integral equals

$$I_4 = \int_1^\infty \int_1^\infty x^{-1/\gamma-1} y^{-1/\gamma-1} \mathbf{E}\left[W^2(1)\right] dxdy$$
$$= \int_1^\infty \int_1^\infty x^{-1/\gamma-1} y^{-1/\gamma-1} dxdy = \left(\int_1^\infty x^{-1/\gamma-1} dx\right)^2 = \gamma^2.$$

Finally we have $I = 2\gamma^2 - 2\gamma^2 + \gamma^2 = \gamma^2$.

3.4 High quantile

The high quantile estimation of heavy tailed distributions has many important applications. There are theoretical difficulties in studying heavy tailed distributions since they often have infinite moments.

Definition 3.4.1 *The quantile of order p is the number x_p denoted by*

$$x_p = \inf\{x, F(x) \geq p\}, \ \textit{for all small } p.$$

Definition 3.4.2 *We call extreme quantile the quantile of order* $(1-p)$, *defined by*

$$x_p = \inf\{x \in \mathbb{R}, F(x) \geq p\} = F^{-1}(1-p). \ \textit{for all small } p.$$

Weissman (1978) proposed the following semi parametric estimator of a high quantile.

Weissman estimator

Weissman's estimator, Weissman (1978) [28], of the Pareto-type tail distribution function $\overline{F} := 1 - F$ is defined by

$$\widehat{\overline{F}}(x) = \frac{k}{n}\left(\frac{x}{X_{n-k:n}}\right)^{-1/\widehat{\gamma}_H}, \text{ for all large } x,$$

where $\widehat{\gamma}_H$ is the so called Hill estimator of the tail index γ.The corresponding high quantile estimator is

$$F^{-1}(1-p) = X_{n-k:n}(\frac{k}{np})^{-\widehat{\gamma}_H}, \text{ for all small } p.$$

Chapter 4

Simulation study and real data applications

4.1 Simulation study

In this section, we check the finite sample behavior of Hill's estimator $\widehat{\gamma}_H$ in terms of consistency and asymptotic normality. To this end, let us consider sets of data drawn from three parameters Burr (λ, c, σ) and three parameters Fréchet (γ, μ, σ) models with respective cdf's

$$F(x;\lambda,c,\sigma)=\begin{cases} 1-\left(1+\left((x/\sigma)^{c}\right)^{-\lambda}\right) & \text{if } x>0 \\ 0 & \text{if } x\leq 0 \end{cases}, \quad \sigma,c,\lambda>0,$$

and

$$F(x;\gamma,\mu,\sigma)=\begin{cases} \exp\left(-\left(\frac{x-\mu}{\sigma}\right)^{-1/\gamma}\right) & \text{if } x>\mu \\ 0 & \text{if } x\leq \mu \end{cases}, \quad \mu,\sigma,\gamma>0.$$

First, we fix the values $(0,1)$ for (μ,σ), $\{0.5, 1, 1.5, 2\}$ for γ, 0.25 for λ, then by using the notation $c = 1/(\gamma\lambda)$ we get the corresponding values $\{8, 2, 1.333, 1\}$ for c. Then we vary the common size $n = 1000$, 5000 of sample $(X_1, ..., X_n)$, then for each size, we generate 1000 independent replicates. For the selection of the optimal numbers of upper order statistics, denoted $\widehat{k}$, used in the computation of $\widehat{\gamma}_H$, we apply the algorithm of Reiss and Thomas:

$$\widehat{k} := \arg\min_{1<k<n} \frac{1}{k}\sum_{i=1}^{k} i^{1/3}\left|\widehat{\gamma}_H(i) - \text{median}\left\{\widehat{\gamma}_H(1), ..., \widehat{\gamma}_H(k)\right\}\right|. \tag{4.1}$$

The results are summarized in two Tables 4.1 and 4.2. As illustration of the consistency of $\widehat{\gamma}_H$, we plot this one as function of sample fraction k, see Figures 4.1 and 4.2. For the asymptotic normality we fit $Z_k = \sqrt{k}\left(\widehat{\gamma}_H - \gamma\right)/\gamma$ by the normal distribution and then we plot together the histogram and the corresponding density of the fitting normal model, see Figures 4.3, 4.4, 4.5 and 4.6.

	$\gamma = 0.5$		$\gamma = 1$		$\gamma = 1.5$		$\gamma = 2$	
n	$\widehat{\gamma}$	$\widehat{k}$	$\widehat{\gamma}$	$\widehat{k}$	$\widehat{\gamma}$	$\widehat{k}$	$\widehat{\gamma}$	$\widehat{k}$
1000	0.695	21	0.867	95	1.603	100	1.946	100
5000	0.668	100	1.043	158	1.513	172	2.002	109

Table 4.1: Computation of optimal sample franction k and its corresponding Hill's estimator for Burr's model.

	$\gamma = 0.5$		$\gamma = 1$		$\gamma = 1.5$		$\gamma = 2$	
n	$\widehat{\gamma}$	$\widehat{k}$	$\widehat{\gamma}$	$\widehat{k}$	$\widehat{\gamma}$	$\widehat{k}$	$\widehat{\gamma}$	$\widehat{k}$
1000	0.443	19	0.992	100	1.421	100	2.281	35
5000	0.5473	500	1.180	500	1.688	496	2.021	499

Table 4.2: Computation of optimal sample franction k and its corresponding Hill's estimator for Fréchet's model.

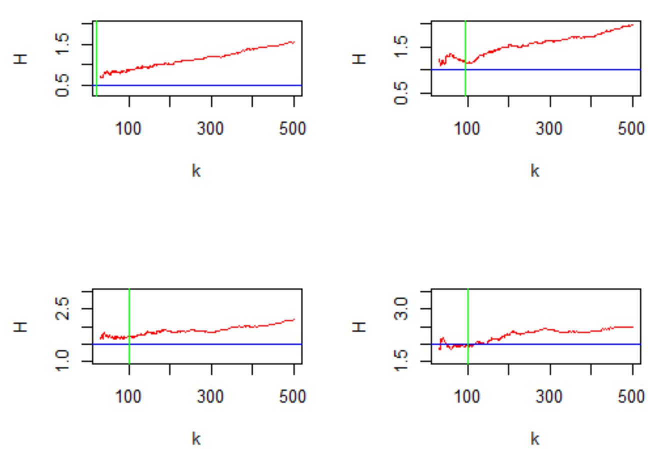


Figure 4.1: Performance of Hill's estimator for a sample of size $N = 1000$ (Burr's model) with $\gamma = \{0.5, 1\}$ top panel and $\gamma = \{1.5, 2\}$ (bottom panel).

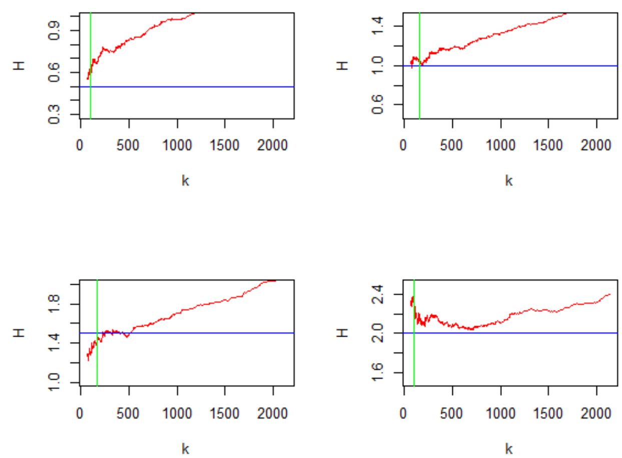


Figure 4.2: Performance of Hill's estimator for a sample of size $N = 5000$ (Burr's model) with $\gamma = \{0.5, 1\}$ top panel and $\gamma = \{1.5, 2\}$ (bottom panel)

- **Asymptotic normality of Hill's estimator for $\gamma = 0.5$ (Burr's model)**:

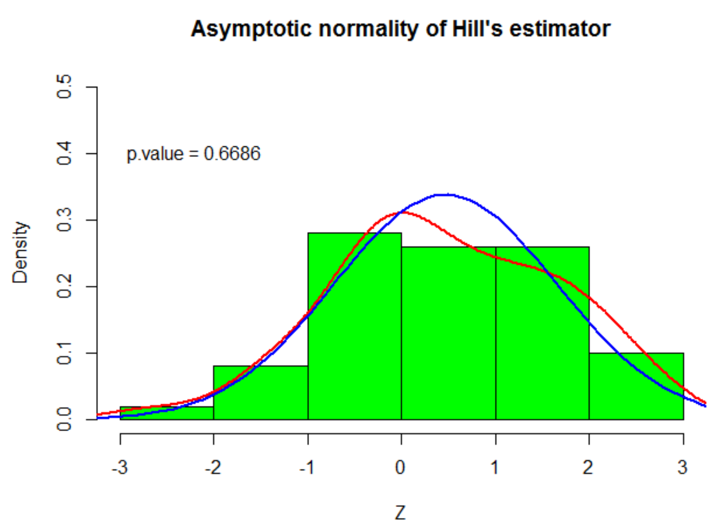


Figure 4.3: Asymptotic normality of Hill's estimator for $\gamma = 0.5$ (Burr's model).

- **Asymptotic normality of Hill's estimator for $\gamma = 1$ (Burr's model)**:

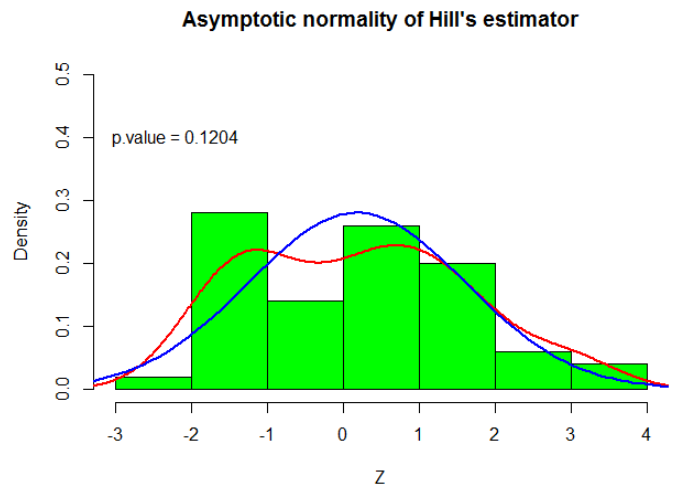


Figure 4.4: Asymptotic normality of Hill's estimator for $\gamma = 1$ (Burr's model).

- **Asymptotic normality of Hill's estimator for $\gamma = 1.5$ (Burr's model)**:

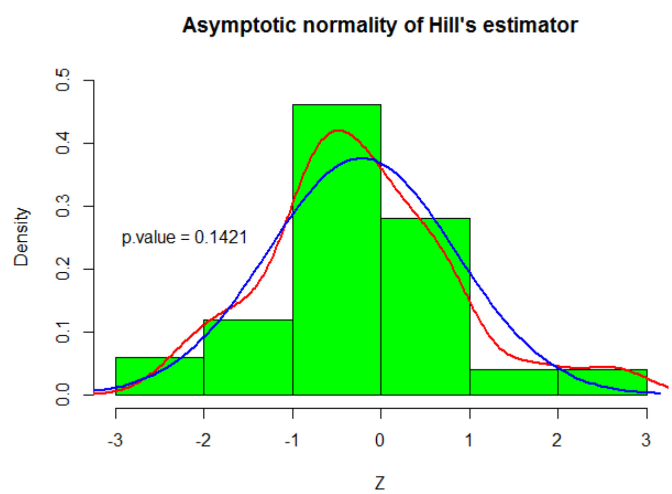


Figure 4.5: Asymptotic normality of Hill's estimator for $\gamma = 1.5$ (Burr's model).

- **Asymptotic normality of Hill's estimator for $\gamma = 2$ (Burr's model):**

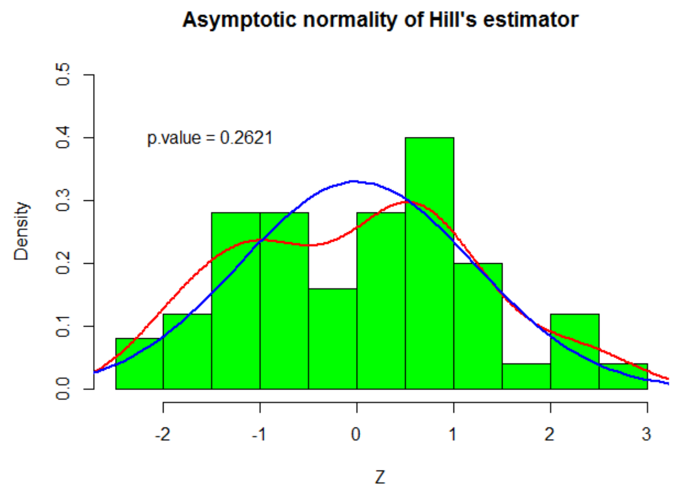


Figure 4.6: Asymptotic normality of Hill's estimator for $\gamma = 2$ (Burr's model).

4.2 Real data applications

In this section, we deal with dataset that belong to the Fréchet domain of attraction; then we choose among others the Burr (λ, c, σ) distribution as the fitting model. The selected data are given in Table 4.3 in which the MLE parameters estimation and the p-values of Cramer-Von mises (cvm) and Kolmogorov-Smirnov (ks) goodnes-of-fit tests (goft) are given. The fitting some of the data are illustrated in Figures 4.9, 4.10, 4.11 and 4.12. We note that all these data are of "small sample size". The obtained p-vlaue are all greater than the level 0.05, this means among other that are of pareto-type data. The EVT based estimation requires a large sample size to get meaningful results, then the selected data do not be considered in our study. To this end, we selected other data which meet our need, namely the financial data, the Dow Jones Industrial Average (Dow Jones), the S&P 500, the Russell 2000, and individual stocks such as Apple (APPL), experience daily fluctuations in sales volumes and returns. These fluctuations are vital indicators for understanding market dynamics and investor sentiment, crucial for effective risk management. The NASDAQ Composite Index, which heavily represents technology stocks, significantly contributes to this daily volatility. By incorporating the concept of Value at Risk (VaR) into our analysis, we can quantitatively assess the potential downside risk associated with these fluctuations. Therefore, by examining daily sales volumes, returns, and VaR for these markets, we gain a comprehensive understanding of their behavior and the associated risk levels.

Next will deal with the asset returns of the four financial maktes, namely S&P 500, APPL, Russell 2000 and NAS daq and the exchange rates JPY/USD and EUR/USD. In the first step we test the belonging of these data to the Fréchet domain of atrraction this by using the bloc maxima method. The results are summarized in Table 4.4. The fitting of the S&P 500 stock market is illustrated is Figure 4.8. In the second step, we fit our data by Burr (λ, c, σ) model. The results are summarized in Table 4.5. The given fitting of the S&P 500 stock market is illustrated in Figure 4.7.

	MLE parameters estimation			p-value goft	
data-name (package)	λ	c	σ	cvm p-value	ks p-value
cancer1	2.103	1.427	12.232	0.996	0.858
lung.cancer (SMPracticals)	7.490	1.353	10718.88	0.959	0.835
nidd.thresh (evir)	0.072	44.917	68.123	0.618	0.642
nidd.annual (evir)	0.192	10.752	79.011	0.520	0.978
Cars93 (MASS)	0.809	4.127	15.994	0.873	0.774
accdeaths (MASS)	0.137	41.59	7677.694	0.222	0.888
Insurance (MASS)	1.168	1.122	166.003	0.991	0.990
WorldPhones (datasets)	0.323	2.032	1376.847	0.463	0.383
Nile (datasets)	1.050	8.731	904.092	0.730	0.906
euro (datasets)	0.204	1.906	1.775	0.958	0.479
oldage (evd)	308.597	71.066	112.940	0.381	0.102
oxford (evd)	1.497	31.149	86.881	0.871	0.650
lossalae (evd)	1.740	1.090	10717.44	0.744	0.835
lisbon (evd)	1.037	12.728	100.769	0.964	0.796
fox (evd)	325.57	2.839	34.166	0.805	0.845
failure (evd)	53.810	1.553	1304.352	0.310	0.449
annMax (extremeStat)	1.479	4.867	69.446	0.961	0.690

Table 4.3: Fiting data with Burr's model

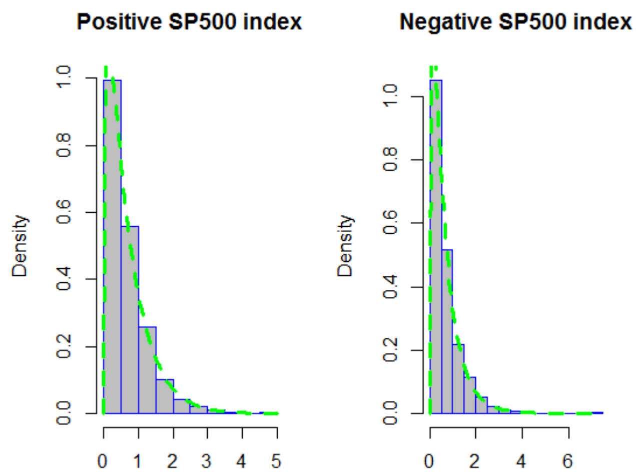


Figure 4.7: Histogram the Positive and Negative SP500 index by fitted Burr's density

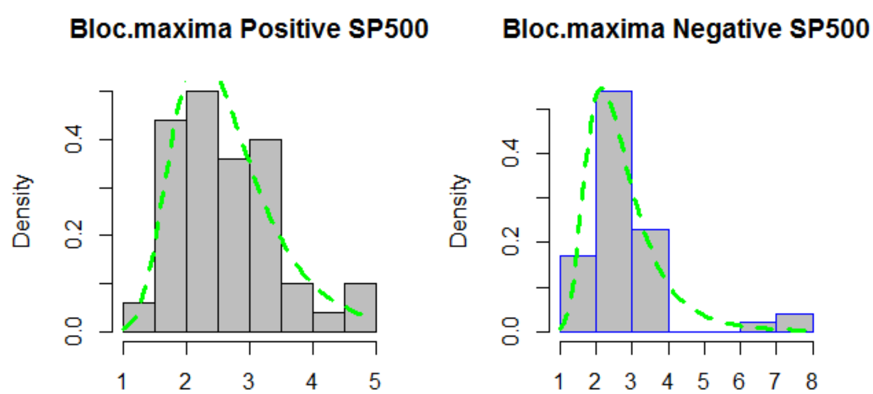


Figure 4.8: Histogram the Bloc maxima corresponding to the Positive and Negative SP500 index and the fitting by Burr's density

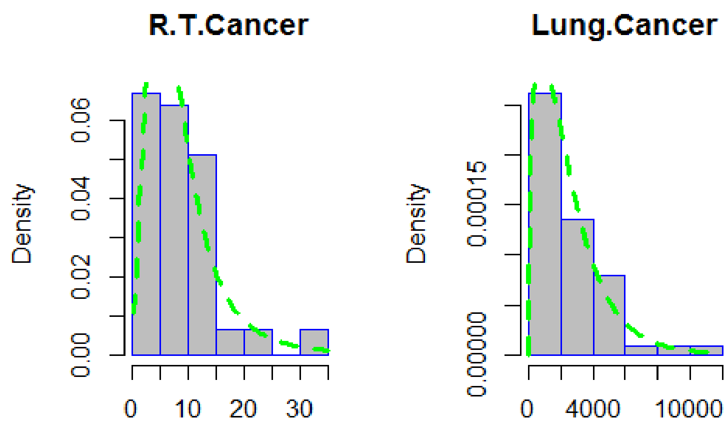


Figure 4.9: Histogram of R.T. Cancer and Lung Cancer and the fitting by Burr's density.

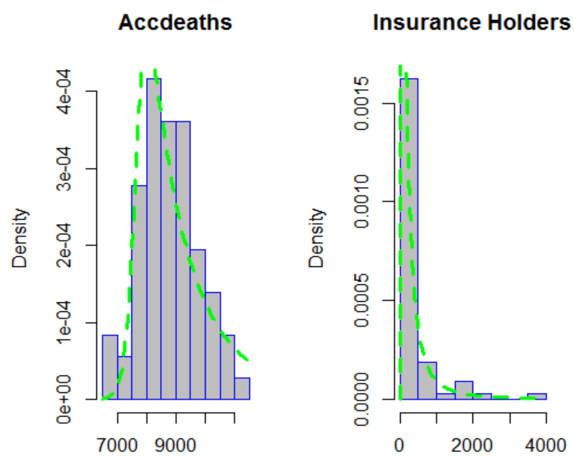


Figure 4.10: Histogram of Accdeaths and Insurance Holders and the fitting by Burr's density.

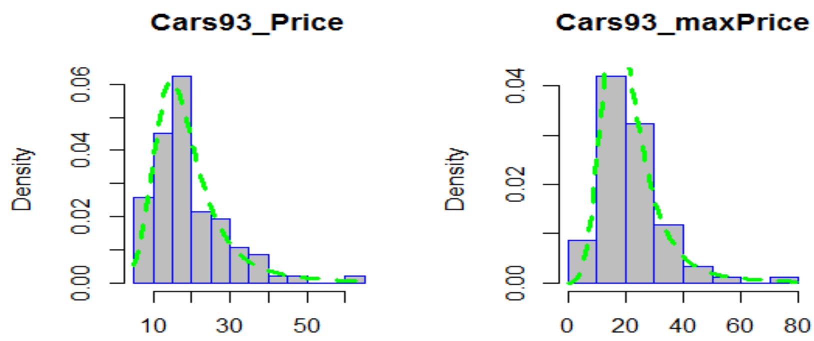


Figure 4.11: Histogram of Cars93 and the fitting by Burr's density

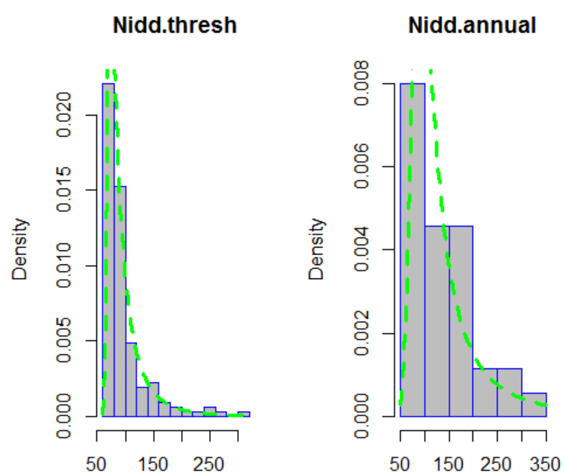


Figure 4.12: Histogram of nidd.thresh and nidd.annual data and the fitting by Burr's density.

	σ	γ	p-value cvm goft
S&P 500	2.660	1.506	0.444
APPL	2.411	1.483	0.579
Dowjones	2.961	1.623	0.368
Russell 2000	3.675	3.012	0.907
NAS daq	3.148	3.378	0.886
JPY to USD	0.844	0.339	0.992
EUR to USD	2.997	0.694	0.183

Table 4.4: Fitting data by Fréchet's model using the block maxima method.

	σ	c	λ	p-value cvm goft
S&P 500	8.560	1.103	4.380	0.950
APPL	3.732	1.265	4.771	0.836
Dowjones	6.558	1.143	3.322	0.325
Russell 2000	6.969	1.162	0.651	0.810
NAS daq	4.838	1.212	3.141	0.152
JPY to USD	4156.716	1.043	1182.62	0.981
EUR to USD	34.464	1.204	6.120	0.988

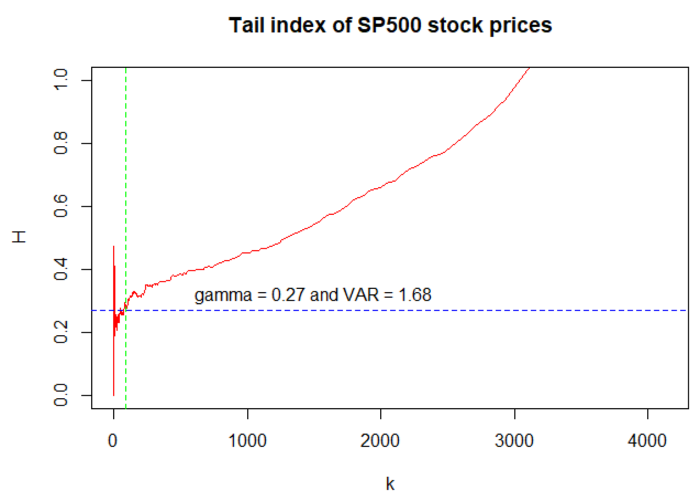
Table 4.5: Fiting data by Burr's model.

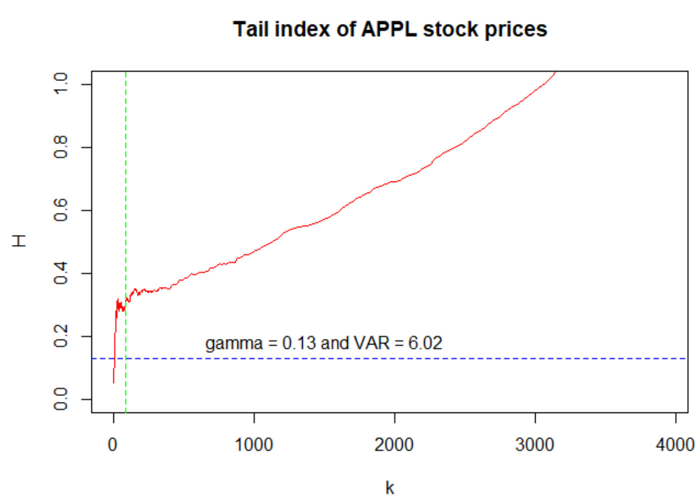
4.2.1 High quantile (var)

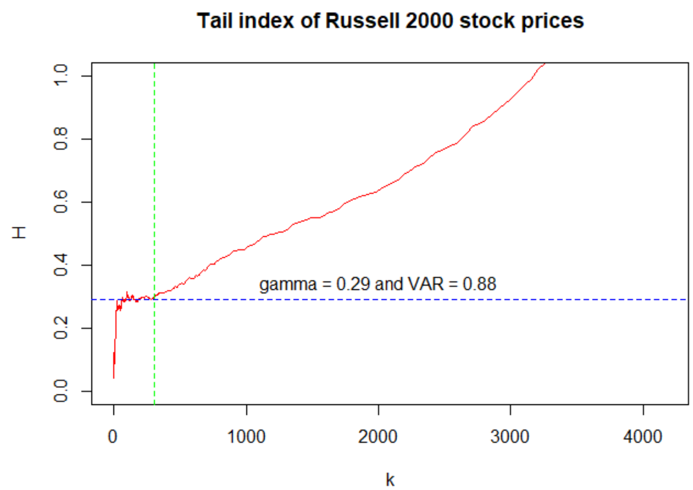
In this section, we deal with the estimation of the high quantile, some times called the value-at-risk (var), for the daily logarithmic returns corresponding the stock prices of S&P 500, APPL, Dowjones, Russell 2000, NAS daq, JPY/USD and EUR/USD. To this end we use he logarithmic return $R(t)$ at the finest time resolution is calculated by

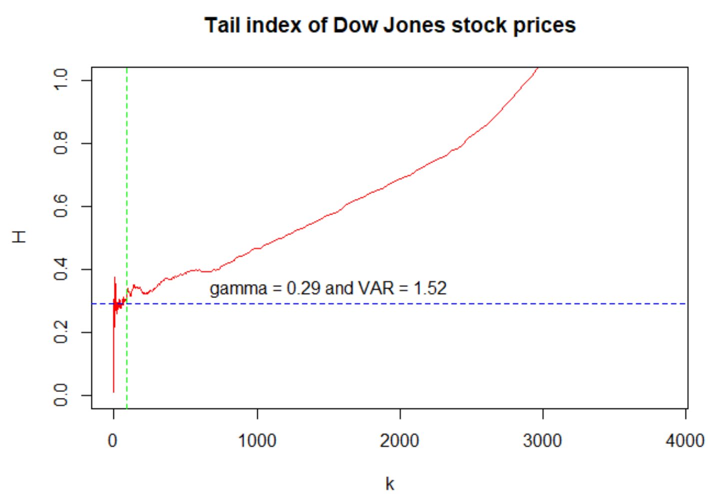
$$R_t := \log \frac{X_t}{X_{t-1}},\ t = 1, 2, ..., n,$$

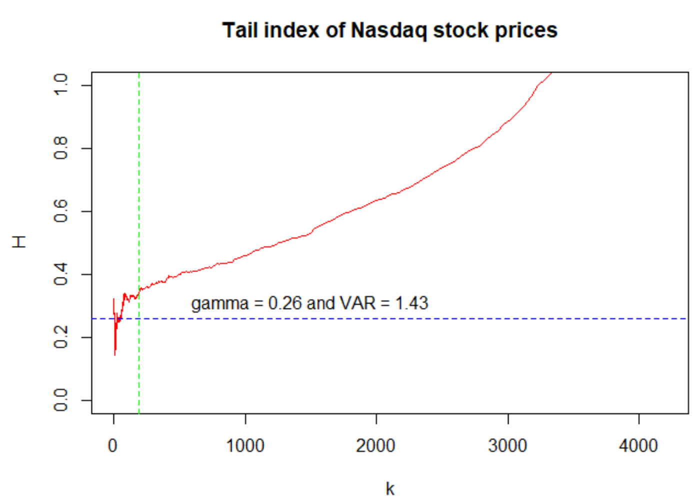
where X_t is the price of the asset at day t. It is worth mentioning that the distributions of financial returns are known to be non-normal and tend to be heavy-tailed. Indeed, the financial models with heavy-tailed distributions have been introduced to overcome problems with the realism of classical financial models. These classical models of financial time series typically assume homoskedasticity and normality cannot explain stylized phenomena such as skewness, heavy tails, and volatility clustering of the empirical asset returns in finance. In 1963, Benoit Mandelbrot first used the stable (or α-stable) distribution to model the empirical distributions which have the skewness and heavy-tail property. Since α-stable distributions have infinite p-th moments for all $p > \alpha$, the tempered stable processes have been proposed for overcoming this limitation of the stable distribution. The Figures illustrate the computation of the tail indices and the var's corresponding to the aforementioned stock prices.

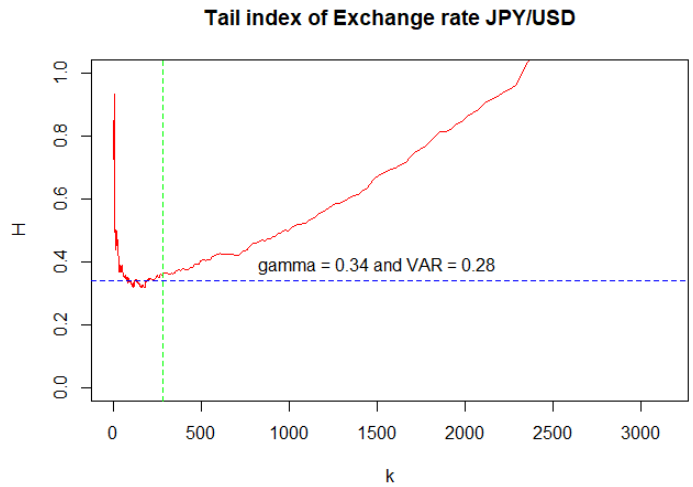


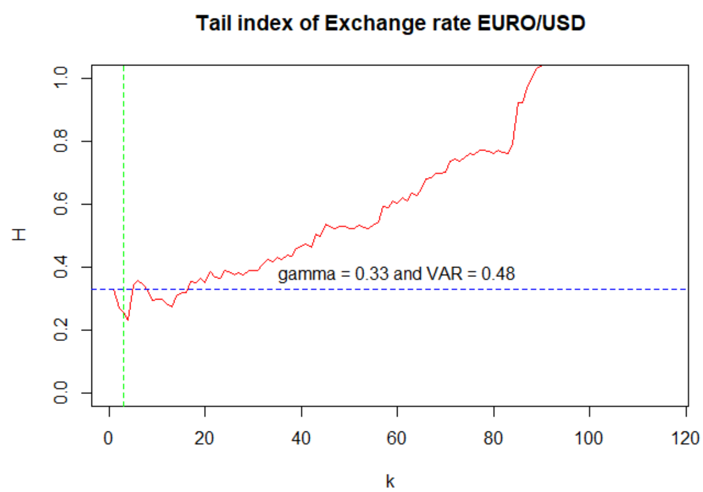



Tail index of Russell 2000 stock prices
gamma = 0.29 and VAR = 0.88
H
k
0.0
0.2
0.4
0.6
0.8
1.0
0
1000
2000
3000
4000


Tail index of Dow Jones stock prices
gamma = 0.29 and VAR = 1.52
H
k
0.0
0.2
0.4
0.6
0.8
1.0
0
1000
2000
3000
4000


Tail index of Nasdaq stock prices
gamma = 0.26 and VAR = 1.43
H
k
0.0
0.2
0.4
0.6
0.8
1.0
0
1000
2000
3000
4000


Tail index of Exchange rate JPY/USD
H
k
gamma = 0.34 and VAR = 0.28
0.0
0.2
0.4
0.6
0.8
1.0
0
500
1000
1500
2000
2500
3000


Tail index of Exchange rate EURO/USD
H
k
gamma = 0.33 and VAR = 0.48
0.0
0.2
0.4
0.6
0.8
1.0
0
20
40
60
80
100
120

Conclusion

The regularly varying functions of negative index play a prominent role is extreme values analysis and modelling heavy-tailed data pertaining to rare events. The flexibility of this class of functions allow us to derive estimators for the distribution tails and its corresponding high quantile. For example, in financial data analysis the regularly varying functions can be used to estimate the Value at Risk (VaR) helping to assess market risks and provide financial decisions. Therefore, these functions play a significant tool in managing risks in various fields, namely the Insurance, Finance, Economy, Environment and others.

Bibliography

[1] Ahsanullah, M., Nevzorov, V. B., Shakil, M. (2013). An introduction to order statistics (Vol. 8). Paris: Atlantis Press.

[2] Bailly, P. Carrère, C. (2015). Statistiques descriptives : Théorie et applications, PUG, coll.

[3] Beirlant, J., Goegebeur, Y., Segers, J., Teugels, J. L. (2006). Statistics of extremes: theory and applications. John Wiley & Sons.

[4] BENAMEUR, S. (2018). Modeling of rare events for risk management (Doctoral dissertation, University Mohamed khider Biskra).

[5] Benchaira, S., Meraghni, D., Necir, A. (2016). Tail product-limit process for truncated data with application to extreme value index estimation. Extremes, 19(2), 219-251.

[6] Bingham, N. H., Goldie, C. M., Teugels, J. L. (1987). Regular Variation Cambridge Univ. Press, Cambridge.

[7] Ciuperca, G., Mercadier, C. (2010). Semi-parametric estimation for heavy tailed distributions. Extremes, 13, 55-87.

[8] de Haan, L. F. M. (1970). On regular variation and its application to the weak convergence of sample extremes (Vol. 32). Mathematisch Centrum.

[9] De Haan, L., Stadtmüller, U. (1996). Generalized regular variation of second order. Journal of the Australian Mathematical Society, 61(3), 381-395.

[10] Deme, E. H. (2013). Quelques contributions à la Théorie univariée des Valeurs Extrêmes et Estimation des mesures de risque actuariel pour des pertes à queues lourdes. Université Gaston Berger.

[11] Embrechts, P., Klüppelberg, C., Mikosch, T. (2013). Modelling extremal events: for insurance and finance (Vol. 33). Springer Science & Business Media.

[12] Fréchet, M. (1927). Sur la loi de probabilité de l'écart maximum. Ann. de la Soc. Polonaise de Math.

[13] Fisher, R. A., Tippett, L. H. C. (1928, April). Limiting forms of the frequency distribution of the largest or smallest member of a sample. In Mathematical proceedings of the Cambridge philosophical society (Vol. 24, No. 2, pp. 180-190). Cambridge University Press.

[14] Financiers, M., des Matieres, M. (2002). Théorie des valeurs extrêmes.

[15] Gardes, L. (2020). Théorie des valeurs extrêmes. Université de Strasbourg.

[16] Gomes, M. I. (1984). Penultimate limiting forms in extreme value theory. Annals of the Institute of Statistical Mathematics, 36(1), 71-85.

[17] Gnedenko, B. (1943). Sur la distribution limite du terme maximum d'une serie aleatoire. Annals of mathematics, 423-453.

[18] Haan, L., Ferreira, A. (2006). Extreme value theory: an introduction (Vol. 3). New York: springer.

[19] Johnson, N. L., Kotz, S., Balakrishnan, N. (1995). Continuous univariate distributions, volume 2 (Vol. 289). John wiley & sons.

[20] Klüppelberg, C. (2004). Risk management with extreme value theory. Extreme Values in Finance, Telecommunication and the Environment, 101-168.

[21] Meraghni, D. (2008). Modeling Distribution Tails. Doctorat thesis, Med Khider University, Biskra-Algeria.

[22] Mousavinasr, S. (2019). Domain of attraction for extremes and mallows distance convergence.

[23] Necir, A. (2024) Rapport technique . Laboratory of applied mathematics. Biskra University.

[24] Pickands III, J. (1975). Statistical inference using extreme order statistics. the Annals of Statistics, 119-131.

[25] Resnick, S. I. (2007). Heavy-tail phenomena: probabilistic and statistical modeling. Springer Science & Business Media.

[26] Resnick, S. I. (2008). Extreme values, regular variation, and point processes (Vol. 4). Springer Science & Business Media.

[27] Reiss, R. D., Thomas, M., Reiss, R. D. (1997). Statistical analysis of extreme values (Vol. 2). Basel: Birkhäuser.

[28] Weissman, I. (1978). Estimation of parameters and large quantiles based on the k largest observations. J. Am. Statist. Assoc. 73: 812-815.

Annex A: R Software

What is the R language?

• The R language is a programming language and mathematical environment used for data processing. It allows for both simple and complex statistical analyses such as linear or non-linear models, hypothesis testing, time series modeling, classification, etc. It also features many highly useful and professional-quality graphical functions.

• R was created by Ross Ihaka and Robert Gentleman in 1993 at the University of Auckland, New Zealand, and is now developed by the R Development Core Team.

• The name of the language originates, on one hand, from the initials of the first names of the two authors (Ross Ihaka and Robert Gentleman), and on the other hand, from a play on words with the name of the language S to which it is related.

- The simulation program.

```
#Graphic of Hill estimator of the tail index.
≫dburr=function(x, a , c,L ) {(c*a/L)*(x/L)^(c-1)/(1+(x/L)^c)^(a+1)}
≫pburr=function(x, a , c,L ){1-(1+(x/L)^c)^(-a)}
≫qburr=function(s,a,c,L){L*((1-s)^(-1/a)-1)^(1/c)}
# consistency of Hill's estimator.
≫gamma=1.5
≫L=0.4
≫c=0.6
≫a=1/(gamma*c)
```

```
≫N=500
≫library(extremefit)
≫library(fitdistrplus)
≫library(extraDistr)
≫library(goftest)
≫library(SMPracticals)
≫library(goftest)
≫library(SMPracticals)
≫library(evir)
≫library(MASS)
≫library(fitdistrplus)
≫M=500
≫r=seq(0.5,0.9,length=M)
≫k=floor(N^r)
≫H=rep(NA,M)
≫U=runif(N)
≫x=qburr(U,a,c,L)
≫X=sort(x)
≫for(j in 1:M){
m=k[j]
d=N-m+1
H[j]=sum(log(X[d:N]))/m-log(X[N-m])}
≫plot(k,H,type = "l",col="red")
≫abline(h=gamma,col="blue",ylim=c(0,1))
# the optimale sample fraction k.
≫n1=floor(N/50) ; n2=floor(N/10) ; alpha=numeric(n2)
≫for(i in 1:n2){
alpha[i]=(1/i)*sum(log(X[(N-i+1):N]),na.rm = FALSE)-log(X[(N-i)])}
```

```
≫s=rep(NA,n2)

≫for(k in 1:n2){

u=numeric(k)

for(w in 1:k){

u[w]=(w^0.3)*abs(alpha[w]-median(alpha[1:k]))}

s[k]=(1/k)*sum(u[1:k])}

ss=numeric(n2-n1+1)

≫for(h in 1:(n2-n1+1)){

ss[h]=s[n1+h-1]}

≫krep=(n1-1+which.min(ss))

≫kopt=floor(krep)

≫print(kopt)

≫abline(v=kopt,col="green")

≫gamma1=H[kopt]

≫print(gamma1)

≫Tail=function(t){(kopt/N)*(t/X[N-kopt])^(-1/gamma1)*(t>X[N-kopt])}

≫Hq=function(s){X[N-kopt]*(kopt/(N*s))^(-gamma1)*(s<kopt/N)}

≫p=0.01 #### so that be < kopt/N

≫VAR=Hq(p); print(VAR)

# Asymptotic normality of Hill's estimator.

≫N=1000

≫M=50

≫Z=rep(NA,M)

≫for (j in 1:M){

U=runif(N)

x=qburr(U,a,c,L)

X=sort(x)

n1=floor(N/50) ; n2=floor(N/10) ; alpha=numeric(n2)
```

```
for(i in 1:n2){
alpha[i]=(1/i)*sum(log(X[(N-i+1):N]),na.rm = FALSE)-log(X[(N-i)])}
for(k in 1:n2){
u=numeric(k)
for(w in 1:k){
u[w]=(w^0.3)*abs(alpha[w]-median(alpha[1:k]))}
s[k]=(1/k)*sum(u[1:k])}
ss=numeric(n2-n1+1)
for(h in 1:(n2-n1+1)){
ss[h]=s[n1+h-1]}
krep=(n1-1+which.min(ss))
kopt=floor(krep)
h=(1/kopt)*sum(log(X[(N-kopt+1):N]),na.rm = FALSE)-log(X[(N-kopt)])
Z[j]=sqrt(kopt)*(h-gamma)/gamma}
≫Z=na.omit(Z)
≫p.value=round(shapiro.test(Z)[[2]],4)
≫print(p.value)
≫hist(Z,freq = FALSE,ylim=c(0,0.3),main="Asymptotic normality of Hill's estimator" ,col="green")
≫lines(density(Z),col="red",lw=2)
≫myfit=fitdist(Z,"norm")
≫mean1=myfit$estimate[[1]]
≫sd1=myfit$estimate[[2]]
≫curve(dnorm(x,mean1,sd1),-3,4,add=TRUE,col="blue",lw=2)
≫text(-2.5, 0.25,paste("p.value","=",p.value))
≫legend(2,0.3,legend = c("Historgram", "Kernel desity","Curve density"), lw= c(1,2,2), col = c("green", "red","blue"),bty = "n")
```

- Fitting small sample data with Burr's distribution.

```
≫cancer_data=c(13.80,5.85,7.09,5.32,4.33,2.83,8.37,14.77,8.53,11.98,1.76,4.40,34.26,2.07,17.12,12.63,7.66,4.18,
```

```
13.29,23.63,3.25,7.63,3.31,2.26,2.69,11.79,5.34,6.93,10.75,13.11,7.39,13.80,5.85,7.09,5.32,4.33,2.83,8.37,14.77,
8.53,11.98,1.76,4.40,34.26,2.07,17.12,12.63,7.66,4.18,13.29,23.63,3.25,7.63,2.87,3.31,2.26,2.69,11.79,5.34,6.93,
10.75,13.11,7.39)
≫library(fitdistrplus)
≫library(goftest)
≫dburr=function(x, k , c,L ) {(c*k/L)*(x/L)^(c-1)/(1+(x/L)^c)^(k+1)}
≫pburr=function(q, k , c,L ){1-(1+(q/L)^c)^(-k)}
≫qburr=function(s,k,c,L){L*((1-s)^(-1/k)-1)^(1/c)}
≫myfit=mledist(cancer_data,"burr",start=list(k=1,c=1,L=1))
≫k1=myfit$estimate[[1]]
≫c1=myfit$estimate[[2]]
≫L1=myfit$estimate[[3]]
≫mytest=cvm.test(cancer_data,"burr",k1,c1,L1)
≫p_value=mytest$p.value
≫print(p_value)
≫U=runif(length(cancer_data))
≫y=qburr(U,k1,c1,L1)
≫ks.test(y,cancer_data)
```

- Fitting large-sample data to Burr's distribution.

```
≫burr1=function(x, k , c,L ) {(c*k/L)*(x/L)^(c-1)/(1+(x/L)^c)^(k+1)}
≫pburr1=function(q, k , c,L ){1-(1+(q/L)^c)^(-k)}
≫qburr1=function(s,k,c,L){L*((1-s)^(-1/k)-1)^(1/c)}
≫options("getSymbols.warning4.0"=FALSE)
≫options("getSymbols.yahoo.warning"=FALSE)
≫dowJones <- new.env()
≫getSymbols("^DJI", env = dowJones, src = "yahoo",from = as.Date("1990-01-01"), to = as.Date("2023-
01-01"))
≫mydata=dowJones$DJI$DJI.Open
```

```
≫N=length(mydata)
≫X=as.numeric(mydata)
≫index=diff(log(X))
≫Ip=as.numeric(na.omit(index[index>0])*100); N=print(length(Ip))
≫#In=abs(as.numeric(na.omit(index[index<0])*1000)); N=print(length(In))
≫myfit=fitdist(Ip,"burr1",start=list(k=2,c=1,L=1))
≫k1=myfit$estimate[[1]]
≫c1=myfit$estimate[[2]]
≫L1=myfit$estimate[[3]]
≫mytest=cvm.test(Ip,"burr1",k1,c1,L1)
≫p_value=mytest$p.value
≫print(p_value)
≫M=500
≫r=seq(0.01,0.99,length=M)
≫k=floor(N^r)
≫H=rep(NA,M)
≫X=sort(Ip)
≫for(j in 1:M){
m=k[j]
d=N-m+1
H[j]=sum(log(X[d:N]))/m-log(X[N-m])}
≫plot(k,H,type = "l",ylim=c(0,1),main="Tail index of Dow Jones stock prices",col="red")
# the optimal sample fraction k.
≫n1=floor(N/50) ; n2=floor(N/5) ; alpha=numeric(n2)
≫for(i in 1:n2){
alpha[i]=(1/i)*sum(log(X[(N-i+1):N]),na.rm = FALSE)-log(X[(N-i)])}
≫s=rep(NA,n2)
≫for(k in 1:n2){
```

```
u=numeric(k)
for(w in 1:k){
u[w]=(w^0.3)*abs(alpha[w]-median(alpha[1:k]))}
s[k]=(1/k)*sum(u[1:k])}
≫ss=numeric(n2-n1+1)
≫for(h in 1:(n2-n1+1)){
ss[h]=s[n1+h-1]}
≫krep=(n1-1+which.min(ss))
≫kopt=floor(krep)
≫print(kopt)
≫gamma1=round(H[kopt],2)
≫print(gamma1)
# High quantile estimation
≫q=function(s){X[N-kopt]*(kopt/(N*s))^(-gamma1)*(s<kopt/N)}
≫p=1/365
≫VAR=round(q(p),2); print(VAR)
≫abline(v=kopt,col="green",lty=2)
≫abline(h=gamma1,col="blue",lty=2)
≫text(1500,gamma1+0.05,paste("gamma","=",gamma1,"and","VAR","=",VAR))
```

- Fitting large-sample data to Frechet's distribution using the block maxima method.

```
≫library(extremefit)
≫library(fitdistrplus)
≫library(extraDistr)
≫library(goftest)
≫library(SMPracticals)
≫library(goftest)
≫library(SMPracticals)
≫library(evir)
```

```
≫library(MASS)
≫library(fitdistrplus)
# these data are fitted ny Burr model
≫data("SP500")
≫mydata=SP500[SP500>0]
≫mydata=abs(SP500[SP500<0])
≫m=rep(NA,100)
≫for (i in 1:100){
z=sample(mydata,30)m[i]=max(z)}
# Fitting by cramer-Von-Mises test
≫myfit=fitdist(m,"frechet",method="mle",start = list(lambda = 1, mu = 1, sigma = 1))
≫print(myfit)
≫lambda1=myfit$estimate[[1]]
≫mu1=myfit$estimate[[2]]
≫sigma1=myfit$estimate[[3]]
≫test=cvm.test(m,"frechet",lambda1,mu1,sigma1)
≫p_value=test$p.value
≫print(p_value)
# Fitting by Kolmogorov-Smirnov test
≫U=runif(length(mydata))
≫y=qfrechet(U,lambda1,mu1,sigma1)
≫ks.test(y,m,exact=TRUE)
≫hist(m,freq=FALSE,col = "gray", border = "blue")
≫curve(dfrechet(x,lambda1,mu1,sigma1),add=TRUE,col="green",lty = 2, lwd = 3)
```

- Programme for calculating tail indicators and corresponding variables for the SP500.

```
≫dburr1=function(x, k , c,L ) {(c*k/L)*(x/L)^(c-1)/(1+(x/L)^c)^(k+1)}
≫pburr1=function(q, k , c,L ){1-(1+(q/L)^c)^(-k)}
≫qburr1=function(s,k,c,L){L*((1-s)^(-1/k)-1)^(1/c)}
```

```
# downloading financial data from its own website
≫options("getSymbols.warning4.0"=FALSE)
≫options("getSymbols.yahoo.warning"=FALSE)
≫sp500 <- new.env()
≫getSymbols("^GSPC", env = sp500, src = "yahoo",from = as.Date("1990-01-01"), to = as.Date("2023-01-01"))
≫mydata=sp500$GSPC$GSPC.Open
≫N=length(mydata)
≫X=as.numeric(mydata)
≫index=rep(NA,N)
≫for (i in 1:N){
index[i]=log(X[i+1]/X[i])}
≫Ip=as.numeric(na.omit(index[index>0])*100); N=print(length(Ip))
≫#In=abs(as.numeric(na.omit(index[index<0])*1000)); N=print(length(In))
≫myfit=fitdist(Ip,"burr1",start=list(k=2,c=1,L=1))
≫k1=myfit$estimate[[1]]
≫c1=myfit$estimate[[2]]
≫L1=myfit$estimate[[3]]
≫mytest=cvm.test(Ip,"burr1",k1,c1,L1)
≫p_value=mytest$p.value
≫print(p_value)
≫M=500
≫r=seq(0.01,0.99,length=M)
≫k=floor(N^r)
≫H=rep(NA,M)
≫X=sort(Ip)
≫for(j in 1:M){
m=k[j]
d=N-m+1
```

```
H[j]=sum(log(X[d:N]))/m-log(X[N-m])}

≫plot(k,H,type = "l",ylim=c(0,1),main="Tail index of SP500 stock prices",col="red")

# the optimal sample fraction k.

≫n1=floor(N/50) ; n2=floor(N/5) ; alpha=numeric(n2)

≫for(i in 1:n2){

≫alpha[i]=(1/i)*sum(log(X[(N-i+1):N]),na.rm = FALSE)-log(X[(N-i)])}

≫s=rep(NA,n2)

≫for(k in 1:n2){

u=numeric(k)

for(w in 1:k){

u[w]=(w^0.3)*abs(alpha[w]-median(alpha[1:k]))}

s[k]=(1/k)*sum(u[1:k])}

≫ss=numeric(n2-n1+1)

≫for(h in 1:(n2-n1+1)){

ss[h]=s[n1+h-1]}

≫krep=(n1-1+which.min(ss))

≫kopt=floor(krep)

≫print(kopt)

≫gamma1=round(H[kopt],2)

≫print(gamma1)

# High quantile estimation.

≫q=function(s){X[N-kopt]*(kopt/(N*s))^(-gamma1)*(s<kopt/N)}

≫p=1/365

≫VAR=round(q(p),2); print(VAR)

≫abline(v=kopt,col="green",lty=2)

≫abline(h=gamma1,col="blue",lty=2)

≫text(1500,gamma1+0.05,paste("gamma","=",gamma1,"and","VAR","=",VAR))

# m is frechet (k) <=> 1/m is Weibull(k).
```

```
≫R=100
≫M=10
≫m=rep(NA,R)
≫for (i in 1:R){
m[i]=max(sample(Ip,M))}
≫T=1/m
≫myfit1=fitdist(T,"weibull")
≫k=myfit1$estimate[[1]]
≫a=myfit1$estimate[[2]]
≫mytest1=cvm.test(T,"pweibull",k,a)
≫p_value1=mytest1$p.value
≫print(p_value1)
```

Annex B: Abreviations and Notations

The following is an explanation of the various abbreviations and notations which are in use throughout this report:

cdf	:	cumulative distribution function.
rv	:	random variable.
DA	:	Domains of attraction.
$i.i.d$	:	Independent and identically distributed.
x^F	:	End point.
$F(.)$	:	distribution function.
$\overline{F}(.)$	:	tail function.
$F_n(.)$	:	empirical distribution function.
EVT	:	extreme values theory.
evi	:	extreme values index.
RV_γ	:	regularly varying functions with index γ.
$RV_\gamma^{(2)}$	:	regular second order variation.
$L(.)$	:	slowly varying function.
GPD	:	Generalised Pareto Distribution.
$\mathbb{I}_A(\cdot)$	:	the indicator function.
$i.e.$	:	in other words.

MLE	:	Maximum Likelihood Estimator.
$\alpha_n(\cdot)$	:	the uniform tail empirical process.
$\lfloor nx \rfloor$	:	Integer part of np.
$\Gamma(i,1)$	:	gamma law with parameters i and 1.
$\overset{\mathcal{D}}{=}$	:	of the same law.
$\overset{p}{\rightarrow}$	:	convergence in probability.
$o_p(1)$	:	Asymptotically negligible in probability.
$O(1)$	:	stochastically bounded.
$\backsim$	:	$A \backsim B$: A has distribution B.

ABSTRACT

The memorandum addresses the application of regularly varying functions in Extreme Value Theory (EVT) and their importance in modeling rare events and extreme risks. We also discuss the domains of attraction of distribution functions related to regularly varying functions, focusing on the Fréchet domain of attraction. We cover the first and second conditions for these functions and their regular variations, then delve into the statistical inference of tail distribution functions, constructing a Hill estimator for tail indices and high quantiles, and proving consistency and asymptotic normality. We conclude our work with a simulation study of the Hill estimator and a real-life application in financial markets, particularly in analyzing asset returns and risk management using concepts of tail distribution functions.

Keywords: regularly varying, Fréchet domain of attraction, Hill estimator, tail distribution, high quantiles.

ملخــــــص

تتناول المذكرة تطبيق الدوال المتغيرة بانتظام في نظرية القيم القصوى (EVT) وأهميتها في نمذجة الأحداث النادرة، المخاطر الخطرة. نتناول ايضا مجالات جذب دوال التوزيع المتعلقة بالدوال المتغيرة بانتظام ركزنا على مجال جذب فريشي. كما تناولنا الشرطين الأول والثاني لهذه الدوال ومتبايناتها المنتظمة، ثم تعمقنا في الاستدلال الإحصائي لدوال توزيع الذيل، بناء مقدر هيل لمؤشر الذيل وكمياته العالية، وإثبات الاتساق والطبيعية التقاربية. أنهينا عملنا بدراسة محاكاة لمقدّر هيل وتطبيق حقيقي على الأسواق المالية، لا سيما تحليل عوائد الأصول وإدارة المخاطر باستخدام مفاهيم دالة التوزيع الذيل.

الكلمات المفتاحية: التباين المنتظم، مجال الجذب فريشي، مقدر هيل، توزيع الذيل، الكميات العالية.

RÉSUMÉ

Le mémoire aborde l'application de fonctions régulièrement variables dans la Théorie des Valeurs Extrêmes (EVT) et leur importance dans la modélisation des événements rares et des risques extrêmes. Nous discutons également des domaines d'attraction des fonctions de distribution liées aux fonctions régulièrement variables, en mettant l'accent sur le domaine d'attraction de Fréchet. Nous couvrons les premières et deuxièmes conditions pour ces fonctions et leurs variations régulières, puis nous nous plongeons dans l'inférence statistique des fonctions de distribution de la queue, en construisant un estimateur de Hill pour les indices de queue et les quantiles élevés, et en prouvant la cohérence et la normalité asymptotique. Nous concluons notre travail par une étude de simulation de l'estimateur de Hill et une application concrète dans les marchés financiers, notamment dans l'analyse des rendements des actifs et la gestion des risques en utilisant des concepts de fonctions de distribution de la queue.

Mots clés : variation régulière, domaine d'attraction de Fréchet, estimateur de Hill, queue de distribution, .quantiles élevés