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A second order maximum principle for singular optimal stochastic control in the classical sense

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Dedication

I dedicate this work

To the person who was their for me in every pain , my mother (Salima).

To the person who lost everything just to make me win , my father (Said).

To the special part of my familly , my sisters (Wiam and Besma) , to my brothers (Ihab , Nacer and Chawki) and to their little angels (Yehya , Rahaf , Youcef , Jebriel and Janah).

To The teachers who educate me with patience in my years of education , and every person who gave me any information.

To the younger me , for showing up, pushing through and finishing what i started.

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⟨**Dr**.**LAKHDARI** IMAD EDDINE ⟩ *and* ⟨**Pr** CHALA ADEL ⟩

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Thank you all for your help .

Contents

Symbols and Acronyms

$(\Omega, \mathcal{F}, \mathbb{F}, P)$	Complete probability space
U_{ad}	Theset of all admissible controls
$\mathbb{F} = \{\mathcal{F}_t\}_{0\leq t\leq T}$	Natural filtration
W_t	Brownian motion
$L^2_{\mathcal{F}_t}(\Omega;\mathbb{R})$	The space of $\mathbb{R}$-valued,$\mathcal{F}_t$-measurable random variables
$L^2_{\mathbb{F}}([0,T];\mathbb{R})$	the space of $\mathbb{R}$-valued,ß$([0,T]) \otimes \mathcal{F}$-measurable,$\mathbb{F}$-adapted processes
$a.e.$	Almost everywhere
$a.s.$	Almost surely
$i.e$	Abbreviation of Latin(id)
$BSDE$	Backward stochastic differential equation
$\phi_x(t,x,u)$	First order derivatives of ϕ with respect to x
$\phi_u(t,x,u)$	First order derivatives of ϕ with respect to u
$\phi_{xx}(t,x,u)$	The second order derivatives of ϕ with respectto(x,x)
$\phi_{uu}(t,x,u)$	Second order derivatives of ϕ with respectto(u,u)
$\phi_{(x,u)^2}(t,x,u)$	Second order derivatives of ϕ with respectto(x,u)
SDE	Stochastic differential equations
$\mathbb{P}-a.s$	Almost surely for the probability measure

Introduction

Examining the pointwise second order maximum pricipale for singular optimal stochastic control in the classical sense is the principal objective of this thesis .The maximum principale is one of the major approaches for discussing problems of optimisation.

One of the main issues in the theory of optimal control is determining the necessary condition for optimal control . [12] contains some early research on the first order necessary condition for stochastic optimal control where the diffusion term is unrelated to the control variable.

As for early works on the same problem but in the case of the diffusion term containing the control variable.

There may not be enough information provided by the first order necessary condition for stochastic optimal control to identify the stochastic optimal control.Since the first order necessary condition for the general situation was established in [20] , it is only reasonable to study the second order necessary condition for stochastic optimal control.Unfortunately, to the authors' best knowledge,there aren't many works are available in this respect.

A little while back, [22] derived a pointwise second-order necessary condition for stochastic optimal controls while [6] obtained an integral-type (rather than the more desired pointwise type) second-order necessary condition for stochastic optimal controls with the control variable entering into the diffusion terms, but the control region is assumed to be convex.To the best of our knowledge ,there are just two publications that deal with the second order necessary condition for stochastic optimal control [6] and [22] . In Zhang and Zhang [25] , the authors raised the pointwise second order necessary conditions for stochastic optimal controls in the general area where the control region is allowed to be non convex. Second order necessary conditions for singular optimal stochastic controls with some examples have been obtained in [11] First and second order necessary optimality conditions for local minimizers of stochastic optimal control problems with state constraints have been established in [8] .Pointwise second-order necessary conditions for stochastic optimal control with jump diffusions have been studied by Ghoul et al [1]

The structure of this thesis is as follows.

In **Chapter 1** We offer a quick reminder of stochastic calcul ,we present some concepts and definition that allow us to prove our results, such as: Probability space, Martingales, Stochastic integrals, Stochastic differential equations, Itô formula ....Also some inequality such as: Holder's inequality , Caushy schwarz's inequality ....

In **Chapter 2** we showcase Strong formulation and weak formulation , The stochastic maximum principle (SMP) (Problem formulation ,Stochastic maximum principale ,Variation equation) and the pontrygain type stochastic maximum principle methods to solve optimal control problems.

In **Chapter 3** we talk about Second order necessary condition in martingale terme ,the controlled system is described by a stochastic differential equation and the control domain is assumed to be convex. This chapter is based on the work of [24].

Chapter 1

Reminder of stochastic calculs

1.1 Probability space

Definition 1.1.1 *We called probability space , all triple* $(\Omega, \mathbb{F}, \mathcal{P})$ *or* $(\Omega, \mathbb{F})$ *with :*

- Ω : Fundamental set .
- $\mathbb{F}$: tribe defined on Ω .
- $\mathcal{P}$: probability over $(\Omega, \mathbb{F})$ .Is a measurable space.

Definition 1.1.2 *(Tribe):Let E be any set , we call tribe of any part of E , all sub set $\mathcal{A}$ of E such that:*

1. $E \in \mathcal{A}$.
2. $A \in \mathcal{A} \Rightarrow A^c \in \mathcal{A}$.
3. $A_n \subset \mathcal{A} \Rightarrow (\cup A_n) \in \mathcal{A}$.

Definition 1.1.3 *(Filtration): A filtration is a growing familly of sub tribes of $\mathbb{F}$, such that :*

$$\mathbb{F}_s \subset \mathbb{F}_t \text{ for all } s < t$$

1.2 Stochastic process

Definition 1.2.1 *The following* $(\Omega, \mathbb{F}, \mathcal{P})$ *is a complet probability space either* $T \subseteq \mathbb{R}_+$ *.All the familly* $X = (X_t)$ *of rendom variable in* $\mathbb{R}^d$ *is a stochastic process.*

Definition 1.2.2 *(Modification):We say that* $Y = (Y_t)$ *is a version or modification of the process* $X = (X_t)$ *, if:*

$$P(X_t = Y_t) = 1 \ , \ P(X_t \neq Y_t) = 0 \ \text{ for all } \ t \geqslant 0 \tag{1.1}$$

Definition 1.2.3 *(Indistinguability):We say that the process* $Y = (Y_t)$ *and* $X = (X_t)$ *are indistinguability if their trajectoirs are the same(1.1), that's mean:*

$$P(X_t = Y_t \quad t \geqslant 0) = 1$$

Proposition 1.2.1 *:If* $X = (X_t)$ *and* $Y = (Y_t)$ *are indistinguability so they are modification one to other , but the recoporad is generally false.*

Definition 1.2.4 *(Equality of processes):* X, Y *are two processes have the same law , if for all* $p \in \mathbb{N}^*$ *and* $t_1, t_2, \quad, t_p \in T$ *:*

$$(X_{t_1}, \quad, X_{t_p}) = (Y_{t_1}, \quad, Y_{t_p})$$

or

$$X \subset Y$$

Definition 1.2.5 *(Adapted processes):A process* $X = (X_t)$ *is adapted (compared with the filtration* $\mathbb{F}_t$*) if* X_t *is* $\mathbb{F}_t$ *measurable for all* t*.*

Definition 1.2.6 *(Cadlag): A process is cadlag if :*

- it's trajectoirs who's continuous to the left provided with limites to the right.

Definition 1.2.7 *(Caglad):A process is caglad if:*

- it's trajectoirs who's continuous to the right provided with limites to the left.

1.3 Brawnian motion

Definition 1.3.1 *:* $(\Omega, \mathbb{F}, P)$ *is a probability space ,* $B = (B_t)$ *is a stochastic process.*B_t *is a Brawnian Motion standard ,if:*

- $B_0 = 0$.
- for all $0 < s < t < +\infty$. $B_t - B_s$ is independent of $(B_u)_{u \in [0,s]}$.
- for all $s < t$, $B_t - B_s \sim N(0, t-s)$.
- $w \in \Omega, t \longrightarrow B_t(w)$, for all $t \in [0,T]$,all the trajectoirs are continuous.

Proposition 1.3.1 *:* $B = (B_t)$ *a stochastic process such as all the trajectoirs are continuous and* $B_0 = 0$ *, so the following properties are equivalent:*

1. process $B = (B_t)$ is standard Brawnian motion.
2. process $B = (B_t)$ is Gaussian process with

$$\begin{cases} E(B_t) & = 0 \\ cov(B_{(t,s)}) = \min(t,s) & = t \cap s \end{cases}$$

1.4 Martingal

Definition 1.4.1 :(X_t) *for all* $t \geqslant 0$ *is a process called martingal (or sub-martingal , super-martingal) ,if:*

- (X_t) is measurable (that's mean $\mathbb{F}_t - adapted$).

- (X_t) is integrable (that's mean $E(X_t) < +\infty$).
- For all $0 \leqslant s \leqslant t$:

$$\begin{cases} E(X_t/\mathcal{F}_s) & = X_s \quad \textit{is martingal} \\ E(X_t/\mathcal{F}_s) & \geqslant X_s \quad \textit{is sub}-\textit{martingal} \\ E(X_t/\mathcal{F}_s) & \leqslant X_s \quad \textit{is super}-\textit{martingal} \end{cases}$$

1.5 Stochastic integral

Definition 1.5.1 *:We called stochastic integral all the integral of the following shape:*

$$\int_a^b Xs(w)dB_s(w)$$

for all $a, b \in \mathbb{R}_+$, X_t stochastic process and $(B_s)_{s\succ 0}$ is Brawnian motion.

Proposition 1.5.1 *:*

1. $\int_a^b X_s dB_s$ is linair.
2. $\int_a^b X_s dB_s$ is continuous .
3. $\int_a^b X_s dB_s$ is $\mathbb{F}_t - adapted$ process.
4. $E(\int_a^b X_s dB_s) = 0$, $Var(\int_a^b X_s dB_s) = E(\int_a^b {X_s}^2 dB_s)$.
5. $\int_a^b X_s dB_s$ is $\mathcal{F}_t - martingale$.

Definition 1.5.2 *(Ito's process): X_t is defined as Ito's process , if it is of the following shape:*

$$X_t = x_0 + \int_a^b b(s)ds + \int_a^b \delta(s)dB_s \tag{1.2}$$

Where $\int_a^b b(s)ds < +\infty$ *and δ is a good local process.or it defined as:*

$$dX_t = b(t)dt + \delta(t)dB_t$$

Where:

$$X_0 = x$$

Where $b(t)$ is derived of the process X and $\delta(t)$ is the coefficient diffusion or volatility.

Theorem 1.5.1 *(Taylor's theorem for one variable):Let $h \geqslant 1$ and let the real valued fonction $f : \mathbb{R} \longrightarrow \mathbb{R}$ be h times differentiable at the point $\alpha \in \mathbb{R}$ (in our case it's Brawnian motion) .Then there is a function $g_h : \mathbb{R} \longrightarrow \mathbb{R}$ such that :*

$$f(x) = f(\alpha) + f'(\alpha)(x-\alpha) + \tfrac{f''(\alpha)}{2!}(x-\alpha)^2 + \quad + \tfrac{f^h(\alpha)}{h!}(x-\alpha)^h + g_h(x)(x-\alpha)^h \tag{1.3}$$

And

$$\lim_{x \longrightarrow \alpha} g_h(x) = 0$$

Theorem 1.5.2 *(Taylor's theorem for two variables):Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is a function of two variables $f(x,y)$ whose first and second partials exist at the point (a,b) .The second* *degree Taylor's of* f *for* (x,y) *near to the point* (a,b) *is:*

$$\begin{aligned} f(x,y) &= f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \\ &+\frac{1}{2}\left[\frac{f_{xx}(a,b)}{2}(x-a)^2 + \frac{f_{yy}(a,b)}{2}(y-b)^2 + f_{xy}(a,b)(x-a)(y-b)\right] \end{aligned} \tag{1.4}$$

1.6 Stochastic differtential equation(SDE)

Definition 1.6.1 *we say that the stochastic process* $(Y_t)_{t\in[0,+\infty]}$ *is a solution of Ito's stochastic differential equation*

$$dY_t = b(Y_t,t)dt + \delta(Y_t,t)dB_t$$

If $0 \leqslant t \leqslant T$:

- $Y_t is \mathbb{F}_t - adapted$.
- $b(t,Y_t) \in L^1_{\mathcal{F}_t}([0,T])$ and $\delta(t,Y_t) \in L^2_{\mathcal{F}_t}([0,T])$.
- for b, δ are $n-$dimensional and $n\times m-$dimensional adapted processes , we got:

$$Y_t = y + \int_0^t b(Y_s,s)ds + \int_0^t \delta(Y_s,s)dB_s \ . \tag{1.5}$$

- $L^1_{\mathcal{F}_t}([0,T]) : (Y_t)_{t\in[0,+\infty]}$ adapted real-valued process space , such as:

$$E[\int_0^T \mid Y_t \mid dt] < +\infty$$

And

$$E[\int_0^T \mid Y_t \mid^2 dt] < +\infty$$

Theorem 1.6.1 *(Existence and uniqness) Let be* $b, \delta : [0,T] \times \mathbb{R} \longrightarrow \mathbb{R}$ *.borel functions satisfying*

$$| b(t,x) - b(t,y) | \leqslant b \mid x - y \mid$$

$$| \delta(t,x) - \delta(t,y) | \leqslant \delta \mid x - y \mid \quad \textit{for all} \quad x, y \in \mathbb{R} \;, \; t \in [0,T]$$

- *Liptschtiz continuity : There is a constant* $C1 \leqslant +\infty$ *, such as:*

$$| b(t,x) - b(t,y) | + | \delta(t,x) - \delta(t,y) | \leqslant C1 \mid x - y \mid$$

- *Linear growth condition: There is a constant* $C2 \leqslant +\infty$ *, such as :*

$$|| b(t,x) | + | \delta(t,x) || \leqslant C2(1 + | x |)$$

Let Y *be a random variable independent of Brownian motion* (B_t)*, such as:*

$$E \mid Y \mid ^2 < +\infty$$

Then there exists a unique solution $Y_t \in L^2_{\mathcal{F}_t}([0,T])$ *of the SDE:*

$$\begin{cases} dY_t = & b(t,Y_t)dt + \delta(t,Y_t)dB_t \\ Y_0 = & y \end{cases} \tag{1.6}$$

Definition 1.6.2 *From the past definition , we define* B_t*as* $m-$*dimensional standard Brawnian motion and* b *and* δ *are* $n-$*dimensional and* $n \times m$ $-$*dimensional adapted processes respectively.the past equation (1.6) is shorthand for (1.5)*

1.7 Ito's formula

Theorem 1.7.1 *(Ito's formula for 1 dimensional Ito process):Let* $0 = t_0 < t_1 < \dots\dots < t_n = T$ *be a partition of* $[0, T]$ *.* B_t *a Brawnian motion on* $[0, T]$ *and suppose* $f(x)$ *is* C^2 *class on* $\mathbb{R}$*.So clearly:*

$$f(B_t) = f(0) + \sum_{i=0}^{n-1} (f(B_{t_{i+1}}) - f(B_{t_i})) \tag{1.7}$$

Using Taylor's Theorem(1.3):

$$f(B_{t_{i+1}}) - f(B_{t_i}) = f'(B_{t_i})(B_{t_{i+1}} - B_{t_i}) + \tfrac{1}{2} f''(\beta_i)(B_{t_{i+1}} - B_{t_i})^2 \tag{1.8}$$

- *For all* $\beta_i \in (B_{t_{i+1}} - B_{t_i})$ *substituting (1.8) into (1.7) so:*

$$f(B_{t_i}) = f(0) + \sum_{i=0}^{n-1} f'(B_{t_i})(B_{t_{i+1}} - B_{t_i}) + \frac{1}{2} \sum_{i=0}^{n-1} f''(\beta_i)(B_{t_{i+1}} - B_{t_i})^2 \tag{1.9}$$

- If we let $\alpha = (t_{i+1} - t_i) \longrightarrow 0$, so it is clear that the terms on the right hand-side of (1.7) converge to the correspponding terms on the right hand-side of the following equation :

$$f(B_t) = f(0) + \tfrac{1}{2} \int_0^t f''(B_s) ds + \int_0^t f'(B_{s)} dB_s \tag{1.10}$$

Theorem 1.7.2 *(Ito's formula for 2 dimensional Ito process)*

X_t *is 2-dimensional Ito process satisfying the following equation :*

$$dX_t = \mu_t dt + \delta_t dB_t$$

- If $f(t,x)$ is C^2 fonction and $Y_t = f(t,X_t)$, then :

$$\begin{aligned} dY_t &= \frac{\partial f}{\partial t}(t,X_t)dt + \frac{\partial f}{\partial x}(t,X_t)dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial^2 x}(t,X_t)(dX_t)^2 \\ &= (\frac{\partial f}{\partial t}(t,X_t) + \frac{\partial f}{\partial x}(t,X_t)\mu_t + \frac{1}{2}\frac{\partial^2 f}{\partial^2 x}(t;X_t){\delta_t}^2)dt + \frac{\partial f}{\partial x}(t,X_t)\delta_t dB_t \end{aligned}$$

Remark 1.7.1 $dt \times dt = dt \times dB_t = 0$ *also* $dB_t \times dB_t = dt$.

1.7.1 Inequality

Holder's inequality

Let $\frac{1}{p} + \frac{1}{q} = 1$ with $p,q > 1$ Then Holder's inequality for integrals states that :

$$\int_a^b \mid f(x)g(x) \mid dx \leqslant [\int_a^b \mid f(x) \mid_p dx]^{\frac{1}{p}} [\int_a^b \mid g(x) \mid_q dx]^{\frac{1}{q}} \tag{1.11}$$

For

$$\mid g(x) \mid = c \mid f(x) \mid^{p-1}$$

Cauchy Schwarz's inequality

If $p = q = 2$, this is cauchy schwarz's inequality:

$$\int_a^b \mid f(x)g(x)dx \mid^2 \leqslant [\int_a^b \mid f(x) \mid^2 dx][\int_a^b \mid g(x) \mid^2 dx] \tag{1.12}$$

Burkhölder Davis-Gundy inequality

For any there $1 \leq p < \infty$ exist positive constants c_p,$C_p$such that, for all local martingales X with $X_0 = 0$ and stopping times τ

the following inequality holds.

$$c_p\mathbb{E}\left[[X]_\tau^{\frac{p}{2}}\right] \leq \mathbb{E}\left[X_\tau^*\right]^p \leq C_p\mathbb{E}\left[[X]_\tau^{\frac{p}{2}}\right]$$

Furthermore, for continuous local martingales, this statement holds for all $0 < p < \infty$. with $X_t^* = \sup_{s<t}|X_s|$ and $[X]$ denotes the quadratic variation of a process X

Grunwall's inequality

Let x, Ψ and Φ be real continuous functions defined in $[a,b]$, $\Phi \geq 0$ for $t \in [a,b]$. We suppose that on$[a,b]$ we have the inequality

$$x(t) \leq \Psi(t) + \int_a^t \Phi(s)\, x(s)\, ds$$

Then

$$x(t) \leq \Psi(t) + \int_a^t \Phi(s)\, \Psi(s) \exp\left[\int_s^t \Phi(u)\, du\right] ds$$

in $[a,b]$

Chapter 2

Stochastic optimal control problems

2.1 Strong formulation

Let $T > 0, t \in [0,T]$ and $(\Omega, \mathbb{F}, \{\mathbb{F}_t\}_{t\geq 0}, \mathcal{P})$ We consider a complete filtered probability space, which satisfies the usual conditions, and we define an $m-$dimensional standard Brawnian motion $B(\cdot)$, denote by U the separable matrix space.We denote by $U_{ad}[0,T]$ the set of all the admissible control.

The state of controlled diffusion is described by the SDE:

$$\begin{cases} dy(t) &= b(t, y(t), u(t))dt + \delta(t, y(t), u(t))dB_t \\ y(0) &= y \end{cases} \tag{2.1}$$

Where :

$b : [0,T] \times \mathbb{R}^n \times u \to \mathbb{R}^n$, $\delta : [0,T] \times u \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$ are given and $y(\cdot)$ is the variable of state , $u(\cdot)$ is called control representing the action of the decision maker(controller).At any time instant the controller knowledge about some information of what has happened up to that moment , but not able to predict what is going to happend afterward due to the uncertainty of the system (as a result , for any t the controller can't exercise his / her decision $u(t)$ before time t comes) , which can be expessed in mathematical term as

" $u(\cdot) is \{\mathbb{F}_t\}_{t>0} - adapted$ ", the control u is taken from the set

$$U[0,T] \triangleq \{u : [0,T] \times \Omega \to U\{u(\cdot) is \{\mathbb{F}_t\}_{t\in[0,T]} adapted\}$$

The cost functional has the form:

$$J(u(.)) = E[\int_0^T f(t, y(t), u(t))dt + g(y(t))]$$

Definition 2.1.1 *Let* $(\Omega, \mathbb{F}, \{\mathbb{F}_t\}_{t\in[0,T]}, \mathcal{P})$ *be given as a filtred probability space satisfying the usual conditions and let* $B(t)$ *given as* $m-$*dimensional standard* $\{\mathbb{F}_t\}_{t\in[0,T]}$ *Brawnian motion . A control* $u(\cdot)$ *called an admissible control and* $(y(\cdot), u(\cdot))$ *an admissible pair ,if:*

1. $y(\cdot)$ is the unique solution of the equation(2.1).
2. $f(., y(\cdot), u(\cdot)) \in L^1_{\mathcal{F}_t}(0, \mathbb{F}, \mathbb{R})$ and $g(y(t)) \in L^1_{\mathcal{F}_t}(\Omega, \mathbb{R})$.
3. $u(\cdot) \in U[0,T]$.

Definition 2.1.2 *Stochastic control problem is to find an optimal control* $\hat{u}(\cdot)$ *for all* $u(\cdot) \in U[0,T]$ *(and that's only if it exist) , such that :*

$$J(\hat{u}(\cdot)) = \inf J(u(\cdot))_{u(.)\in U[0,T]}$$

Where $\hat{u}(.)$ *is pressenting as an optimal control and the state control pair* $(\hat{y}(\cdot), \hat{u}(\cdot))$ *are an optimal state process.*

2.2 Weak formulation

Unlike in the strong formulation the filtred probability space $(\Omega, F, \{\mathbb{F}_t\}_{t\in[0,T]}, \mathcal{P})$ on which we defined $B(\cdot)$ as a Brawnian motion but it 's not the case in the weak formulation ;where we consider that as a part of the control.

Definition 2.2.1 *Let* $(\Omega, \mathbb{F}, \{\mathbb{F}_t\}_{t\in[0,T]}, \mathcal{P}, B(\cdot), u(\cdot))$ *is called* $w-$*admissible control and* $y(\cdot), u(\cdot)$ *is called* $w-admissible$ *pair ,if:*

1. $(\Omega, \mathbb{F}, \{\mathbb{F}_t\}_{t\in[0,T]}, \mathcal{P})$ is a filtered probability space satisfying the usual conditions.

2. $B(.)$ is a $m-$dimensional standard Brawnian motion defined on $(\Omega, \mathbb{F}, \{\mathbb{F}_t\}_{t\in[0,T]}, \mathcal{P})$.

3. $u(\cdot)$ is an $\{\mathbb{F}_t\}_{t\in[0,T]} - adapted$ process on $(\Omega, \mathbb{F}, P)$ taking values in U.

4. $y(\cdot)$ is the unique solution of the equation (2.1).

5. $f(., y(\cdot), u(\cdot)) \in L^1_F(0, F, \mathbb{R})$ and $g(y(t)) \in L^1_F(\Omega, \mathbb{R})$

Definition 2.2.2 *Stochastic optimal control problem under weak formulation is to find an optimal control* $\hat{u}(\cdot) \in U[0,T]$ *(and that's only if it exist) , such that :*

$$J(\hat{u}(\cdot)) =_{u(\cdot)\in U[0,T]} \inf J(u(\cdot))$$

2.3 Stochastic maximum principale

The stochastic maximum principle (SMP) is a fundamental result in stochastic optimal control. Its basic idea is to derive a set of necessary and sufficient conditions that any optimal control must satisfy. The first version of the SMP was extensively established in the 1970s by Bismut [4] , Kushner [14] , and Haussmann [12], under the condition that there is no control on the diffusion coefficient. Haussmann [11] developed a powerful form of the Stochastic Maximum Principle for the feedback class of controls using Girsanov's transformation and applied it to solve some problems in stochastic control.

2.3.1 Problem formulation

Let T be a positive real number and $(\Omega, \mathbb{F}, \{\mathbb{F}_t\}_{t\in[0,T]}, \mathcal{P})$ be a probability space that meets the standard conditions , in which a $m-$dimensional Brawnian motion such that $\mathbb{F}$

is a natural filtration ; $\{\mathbb{F}_t\} = \mathbb{F}$ for an arbitrarily fixed

arbitrarily fixed time horizon T , which we denote U by the set of all admissible control . Any element $y \in \mathbb{R}^n$ with the norm $\mid y \mid = \mid x_1 \mid + \mid x_2 \mid + ... + \mid x_n \mid$ will be identified to coloumn vector with n compositions.For function h, we denote by $h_y(resp.h_{yy})$ the Gradient or Jacobian(resp.Hessian).

Definition 2.3.1 *An admissible control is a meassurable adapted process ,* $u : [0,T] \times \Omega \to u$ *, such that :*

$$E[\int_0^T u(s)ds] < +\infty$$

Take into account the subsequent stochastic controlled system:

$$\begin{cases} dy(t) & = b(t, y(t), u(t))dt + \delta(t, y(t), u(t))dB_t \\ y(0) & = y_0 \in \mathbb{R}^n \end{cases} \tag{2.2}$$

Where $b : [0,T] \times \mathbb{R}^n \times u \to \mathbb{R}^n$; $\delta : [0,T] \times \mathbb{R}^n \times u \to \mathbb{R}^{n \times m}$ *are given.Suppose we are given a performence functional* $J(u)$ *as the follow cost :*

$$J(u) = E[\int_0^T (f(t, y(t), u(t)))dt + g(y(T))] \tag{2.3}$$

Where $f : [0,T] \times \mathbb{R}^n \times u \to \mathbb{R}$; $g : \mathbb{R}^n \to \mathbb{R}$ *.The stochastic control problem is to find an optimal control* $\hat{u} \in U$ *who can verifiy:*

$$J(\hat{u}) =_{u(\cdot)\in U} \inf J(u) \tag{2.4}$$

Let us make the following assumptions about the coefficients b, δ, f *and* g*.*

A1 The maps$b, \delta$and f are countinuously differentiable with respect to (y, u) and g is countinously differentiable in y.

A2 The derivatives $b_y, b_u, \delta_y, \delta_u, f_y, f_u$ and g_y are countinuous in (y, u) uniformly bounded.

A3 b, δ, f are bounded by $C(1+ \mid y \mid + \mid u \mid)$ and g is bounded by $C(1+ \mid y \mid)$, for all $C > 0$.

2.3.2 Optimal control and Optimal trajectory:

Definition 2.3.2 *For all* $i = \overline{1, N}$ *we defined the Hamiltonion:*

$$H(t, y, u, p, q) = f(t, y, u) + pb(t, y, u) + \sum_{i=1}^{N} q^i \delta^i(t, y, u) \tag{2.5}$$

By $H : [0, T] \times \mathbb{R}^n \times \overline{u} \times \mathbb{R}^n \times \mathbb{R}^{n \times m} \to \mathbb{R}$
Where q^i *and* δ^i *denote by coloumn of the matrix* q *and* δ*.Let* $\hat{u}$ *be an optimal control and* $\hat{y}$ *denote the corresponding optimal trajectory . Then we consider the pair* (p, q) *of square integrable adapted process associated to* $\hat{u}$ *with value in* $\mathbb{R}^n \times \mathbb{R}^{n \times m}$ *, such that :*

$$\begin{cases} dp(t) & = -H_y(t, \hat{y}(t), \hat{u}(t); p(t), q(t))dt + q(t)dB_t \\ p(T) & = h_y(\hat{y}(T)) \end{cases} \tag{2.6}$$

Theorem 2.3.1 (Necessary conditions of optimality) *Let* $\widehat{u}$ *be an optimal control minimizing the performance functional* J *over* $\mathcal{U}$*, and let* $\widehat{y}$ *be the corresponding optimal trajectory, then there exists an adapted processes* $(p, q) \in \mathbb{L}^2\left(\left([0, T]; \mathbb{R}^n\right)\right) \times \mathbb{L}^2\left(\left([0, T]; \mathbb{R}^{n \times d}\right)\right)$ *which is the unique solution of the BSDE (2.6), such that for all* $v \in U$

$$H_u\left(t, \widehat{y}\left(t\right), \widehat{u}\left(t\right), p\left(t\right), q\left(t\right)\right)\left(v_t - \widehat{u}\left(t\right)\right) \leq 0, \quad \mathbb{P} - a.s.$$

In order to give the proof this tesult , it is convenient to present the following.

2.3.3 Estimation and linearization of the solution

Definition 2.3.3 *Let* $(\hat{u}(\cdot), y(\cdot), \hat{y}(\cdot), v(\cdot))$ *an optimal solution of the problem such that* $\hat{u} + v \in U$ *, for* $\theta \in [0,1]$ *the control* $(\hat{u} + \theta v)$ *by standard arguments for stochastic cacul ; so it is easy to cheak the following convergence result:*

Lemma 2.3.1 *Under assumption* ($\boldsymbol{A}_1$) *it hold that:*

$$\lim_{\theta \to 0} E[_{t\in[0,T]}\sup \mid y^{\theta}(t) - \hat{y}(t)) \mid {}^2] = 0 \tag{2.7}$$

We define the process $z(t) = z^{\hat{u},v}(t)$

$$\begin{cases} dz(t) = & \{b_y(t, \hat{z}(t), \hat{u}(t))z(t) + b_u(t, \hat{y}(t), \hat{u}(t), v(t))\}dt \\ & +\sum_{j=1}^{m}\{\delta_y^j(t, \hat{y}(t), \hat{u}(t))w(t) + \delta_u^j(t, y(t), u(t), v(t)\}dB^j(t) \\ z(0) = & 0 \end{cases} \tag{2.8}$$

We can find a unique solution z which solves the variational equation (2.8),and the following estimation holds.

Lemma 2.3.2 *Under assumption* ($\boldsymbol{A}_1$) *we have:*

$$\lim_{\theta \to 0} E \mid \frac{y^{\theta}(t) - \hat{y}(t)}{\theta} - z(t) \mid^2 = 0 \tag{2.9}$$

Proof. To prove this lemma, you can consult [3] ■

2.3.4 Variational inequality

Let Φ be the fundamental solution of the linear matrix equation, for $0 \leq s < t \leq T$

$$\begin{cases} d\Phi_{s,t} & = b_y\left(t, \widehat{y}\left(t\right), \widehat{u}\left(t\right)\right) \Phi_{s,t} dt + \sum_{j=1}^{d} \sigma_y^j\left(t, \widehat{y}\left(t\right), \widehat{u}\left(t\right)\right) \Phi_{s,t} dB^j\left(t\right), \\ \Phi_{s,s} & = I_d, \end{cases}$$

where I_d is the $n \times n$ identity matrix, this equation is linear with bounded coefficients, then it admits a unique strong solution.

From Itô's formula we can easily check that $d\left(\Phi_{s,t}\Psi_{s,t}\right) = 0$, and $\Phi_{s,s}\Psi_{s,s} = I_d$, where Ψ is the solution of the following equation

$$\begin{cases} d\Psi_{s,t} & = -\Psi_{s,t}\left\{ b_y\left(t, \widehat{y}\left(t\right), \widehat{u}\left(t\right)\right) - \sum_{j=1}^{d} \sigma_y^j\left(t, \widehat{y}\left(t\right), \widehat{u}\left(t\right)\right) \sigma_y^j\left(t, \widehat{y}\left(t\right), \widehat{u}\left(t\right)\right) \right\} dt \\ & \quad - \sum_{j=1}^{d} \Psi_{s,t} \sigma_y^j\left(t, \widehat{y}\left(t\right), \widehat{u}\left(t\right)\right) dB^j\left(t\right), \\ \Psi_{s,s} & = I_d, \end{cases}$$

so $\Psi = \Phi^{-1}$, if $s = 0$ we simply write $\Phi_{0,t} = \Phi_t$, and $\Psi_{0,t} = \Psi_t$. By integrating by part formula we can see that, the solution of (2.8) is given by $z\left(t\right) = \Phi_t \eta_t$, where η_t is the solution of the stochastic differential equation

$$\begin{cases} d\eta_t & = \Psi_t\left\{ b_u\left(t, \widehat{y}\left(t\right), \widehat{u}\left(t\right)\right) v\left(t\right) - \sum_{j=1}^{d} \sigma_y^j\left(t, \widehat{y}\left(t\right), \widehat{u}\left(t\right)\right) \sigma_u^j\left(t, \widehat{y}\left(t\right), \widehat{u}\left(t\right)\right) v\left(t\right) \right\} dt \\ & \quad + \sum_{j=1}^{d} \Psi_t \sigma_u^j\left(t, x_t^\star, u_t^\star\right) v\left(t\right) dB^j\left(t\right), \\ \eta_0 & = 0. \end{cases}$$

Let us introduce the following convex perturbation of the optimal control $\widehat{u}$ by

$$u^\theta = \widehat{u} + \theta v, \tag{2.10}$$

for any $v \in \mathcal{U}$, and $\theta \in (0,1)$. Since $\widehat{u}$ is an optimal control, then $\theta^{-1}\left(J\left(u^\theta\right) - J\left(\widehat{u}\right)\right) \geq 0.$

Thus a necessary condition for optimality is that

$$\lim_{\theta\to 0}\theta^{-1}\left(J\left(u^{\theta}\right)-J\left(\widehat{u}\right)\right)\geq 0. \tag{2.11}$$

The rest is devoted to the computation of the above limit. We shall see that the expression (2.11) leads to a precise description of the optimal control $\widehat{u}$ in terms of the adjoint process. First, it is easy to prove the following lemma

Lemma 2.3.3 *Under assumptions* $(H1)$, *we have*

$$\begin{aligned} I &= \lim_{\theta\to 0}\theta^{-1}\left(J\left(u^{\theta}\right)-J\left(\widehat{u}\right)\right) \\ &= \mathbb{E}\left[\int_0^T \left\{f_y\left(s,\widehat{y}\left(s\right),\widehat{u}\left(s\right)\right)z\left(s\right)+f_u\left(s,\widehat{y}\left(s\right),\widehat{u}\left(s\right)\right)v\left(s\right)\right\}ds+g_y\left(\widehat{y}\left(T\right)\right)z\left(T\right)\right]. \end{aligned} \tag{2.12}$$

Proof. We use the same notations as in the proof of (lemma 2.2.2). First, we have

$$\begin{aligned} &\theta^{-1}\left(J\left(u^{\theta}\right)-J\left(\widehat{u}\right)\right) \\ &= \mathbb{E}\left[\int_0^T\int_0^1\left\{f_y\left(s,y^{\mu,\theta}\left(s\right),u^{\mu,\theta}\left(s\right)\right)z\left(s\right)+f_u\left(s,y^{\mu,\theta}\left(s\right),u^{\mu,\theta}\left(s\right)\right)v\left(s\right)\right\}d\mu ds \right.\\ &\qquad\left. +\int_0^1 g_y\left(y^{\mu,\theta}\left(T\right)\right)z\left(T\right)d\mu\right]+\beta^{\theta}\left(t\right), \end{aligned}$$

where

$$\beta^{\theta}\left(t\right)=\mathbb{E}\left[\int_0^T\int_0^1 f_y\left(s,y^{\mu,\theta}\left(s\right),u^{\mu,\theta}\left(s\right)\right)\Gamma^{\theta}\left(s\right)d\mu ds+\int_0^1 g_y\left(y^{\mu,\theta}\left(T\right)\right)\Gamma^{\theta}\left(T\right)d\mu\right].$$

By using the (lemma 1.4.2), and since the derivatives f_y, f_u, and g_y are bounded, we have $\lim_{\theta\to 0}\beta^{\theta}\left(t\right)=0$. Then, the result follows by letting θ go to 0 in the above equality. ■

Substituting by $z\left(t\right)=\Phi_t\eta_t$ in (2.12), this leads to

$$I=\mathbb{E}\left[\int_{0}^{T}\left\{f_{y}\left(s,\widehat{y}\left(s\right),\widehat{u}\left(s\right)\right)\Phi_{s}\eta_{s}+f_{u}\left(s,\widehat{y}\left(s\right),\widehat{u}\left(s\right)\right)v\left(s\right)\right\}ds+g_{y}\left(\widehat{y}\left(T\right)\right)\Phi_{T}\eta_{T}\right].$$

Consider the right continuous version of the square integrable martingale

$$M\left(t\right):=\mathbb{E}\left[\int_{0}^{T}f_{y}\left(s,\widehat{y}\left(s\right),\widehat{u}\left(s\right)\right)\Phi_{s}ds+g_{y}\left(\widehat{y}\left(T\right)\right)\Phi_{T}\left|\mathcal{F}_{t}\right.\right].$$

By the representation theorem, there exist $Q=\left(Q^{1},..,Q^{d}\right)$ where $Q^{j}\in\mathbb{L}^{2}$, for $j=1,...,d$,

$$M\left(t\right)=\mathbb{E}\left[\int_{0}^{T}f_{y}\left(s,\widehat{y}\left(s\right),\widehat{u}\left(s\right)\right)\Phi_{s}ds+g_{y}\left(\widehat{y}\left(T\right)\right)\Phi_{T}\right]+\sum_{j=1}^{d}\int_{0}^{t}Q^{j}\left(s\right)dB^{j}\left(s\right).$$

We introduce some more notation, write $\widehat{y}\left(t\right)=M\left(t\right)-\int_{0}^{t}f_{y}\left(s,\widehat{y}\left(s\right),\widehat{u}\left(s\right)\right)\Phi_{s}ds$. The adjoint variable is the processes defined by

$$\begin{cases}p\left(t\right) & =\widehat{y}\left(t\right)\Psi_{t},\\ q^{j}\left(t\right) & =Q^{j}\left(t\right)\Psi_{t}-p\left(t\right)\sigma_{y}^{j}\left(t,\widehat{y}\left(t\right),\widehat{u}\left(t\right)\right),\ \text{for } j=1,...,d.\end{cases}\tag{2.13}$$

Theorem 2.3.2 *Under assumptions* $(H1)$, *we have*

$$I=\mathbb{E}\left[\int_{0}^{T}\left\{f_{u}\left(s,\widehat{y}\left(s\right),\widehat{u}\left(s\right)\right)+p\left(s\right)b_{u}\left(s,\widehat{y}\left(s\right),\widehat{u}\left(s\right)\right)+\sum_{j=1}^{d}q^{j}\sigma_{u}^{j}\left(s,\widehat{y}\left(s\right),\widehat{u}\left(s\right)\right)\right\}\right].$$

Proof. From the integration by part formula, and by using the definition of $p\left(t\right),q^{j}\left(t\right)$ for $j=1,..,d$, we easily check that

$$\begin{aligned}E\left[y\left(T\right)\eta\left(T\right)\right] &=\mathbb{E}\left[\int_{0}^{T}\left\{p\left(t\right)b_{u}\left(s,\widehat{y}\left(s\right),\widehat{u}\left(s\right)\right)+\sum_{j=1}^{d}q^{j}\left(s\right)\sigma_{u}^{j}\left(s,\widehat{y}\left(s\right),\widehat{u}\left(s\right)\right)\right\}v\left(t\right)dt\right.\\ &\quad-\int_{0}^{T}f_{y}\left(s,\widehat{y}\left(s\right),\widehat{u}\left(s\right)\right)\eta_{t}\Phi_{t}dt.\end{aligned}\tag{2.14}$$

Also we have

$$I=\mathbb{E}\left[y\left(T\right)\eta\left(T\right)+\int_{0}^{T}f_{y}\left(s,\widehat{y}\left(s\right),\widehat{u}\left(s\right)\right)\Phi_{t}\eta_{t}dt+\int_{0}^{T}f_{u}\left(s,\widehat{y}\left(s\right),\widehat{u}\left(s\right)\right)v\left(t\right)dt\right], \tag{2.15}$$

substituting (2.14) in (2.15), This completes the proof. ■

By analyzing the variations in the control and the corresponding variations in the state trajectory, one can derive important insights into the optimality of the control for more detail we can see [3].

Chapter 3

Second order necessary conditions for singular stochastic optimal control

3.1 Preliminaries

Let $(\Omega, \mathbb{F}, \{\mathbb{F}_t\}_{t\in[0,T]}, P)$ be filtred probability space satisfied the usal condition,and we suppose that $\mathbb{F} = \{\mathbb{F}_t\}_{t\in[0,T]}$ is the natural filtration created by $1 - dimonsional$ standard Brownian motion $B(\cdot)$ .The controlled stochastic differential equation that being considred by:

$$\begin{cases} dx(t) = & b(t, x(t), u(t))dt + \delta(t, x(t), u(t))dB_t \\ x(0) = & x \end{cases} \tag{3.1}$$

With a cost functional :

$$J(u(.)) = E[\int_0^T f(t, x(t), u(t))dt + g(x(T))] \tag{3.2}$$

Where we denote the stochastic process $u(.)$ as the control valued in the environment $U \in \mathbb{R}^m \quad (m \in \mathbb{N})$ and $x(.)$ as the state valued in $\mathbb{R}^n(n \in \mathbb{N})$ and $b, \delta : [0, T] \times \mathbb{R}^n \times U \to \mathbb{R}^n$

, $f:[0,T]\times\mathbb{R}^n\times U\rightarrow\mathbb{R}$ and we regard $h:\mathbb{R}^n\rightarrow\mathbb{R}$ as a function meeting acceptable requirements is the state valued in our trajectory to solve the stochastic optimisation of finding a control $\overline{u}(.)\in U_{ad}$, such that:

$$J(\overline{u}(.))=\inf_{\overline{u}(.)\in U_{ad}}J(u(.)) \tag{3.3}$$

The acceptable control that accomplishes the minimum $\overline{u}(.)\in U_{ad}$ is referred to as an optimal control.

Now, we define singular control in the classical sense for diffusion ,as inspired by [1] and [24] .

Definition 3.1.1 *(singular control in the classical sense): We called the admissible control* $\overline{u}(.)$ *as an singular control in the classical sense if it verified:*

$$\begin{cases} H_u\left(t,\widetilde{x}\left(t\right),\widetilde{u}\left(t\right),\widetilde{p}\left(t\right),\widetilde{q}\left(t\right)\right)=0 \quad a.s.\ a.e., \\ H_{uu}\left(t,\widetilde{x}\left(t\right),\widetilde{u}\left(t\right),\widetilde{p}\left(t\right),\widetilde{q}\left(t\right)\right)+\widetilde{P}\left(t\right)\left(\sigma_u\left(t,\widetilde{x}\left(t\right),\widetilde{u}\left(t\right)\right)\right)^2=0 \ \ a.s.\ a.e., \end{cases} \tag{3.4}$$

Where adjoint processes $(\widetilde{p}(.),\widetilde{q}(.))$ and $(\widetilde{P}(.),\widetilde{Q}(.))$ are given respectely by(3.15) and (3.16) with $(\overline{x}(.),\overline{u}(.))$ replaced by $(\widetilde{x}(.),\widetilde{u}(.))$. If $\widetilde{u}(.)$ in (3.4) is also optimal , then we call $\widetilde{u}(.)$ as a singular optimal stochastic control in the classical sense.

It is important to remember that there is another kind of singularity in stochastic control problem . In this case , the control variable has two components $(u(.),\zeta(.))$, the first of which is absoulutely continuous and the second of which has bounded varaition and is non-decreasing left continuous whith the right limits and $\zeta(0)=0$.

We introduce a few different rendom and stochastic variable for any $t\in[0,T]$

- $\mathcal{L}^2_{\mathcal{F}_t}(\Omega;\mathbb{R})$ the space of $\mathbb{R}-$valued , $\mathcal{F}_t-$measurable random variable ζ such that

$$E\mid\zeta\mid^2<+\infty$$

- $\mathcal{L}^2_{\mathcal{F}_t}([0,T];\mathbb{R})$ the space of $\mathbb{R}-$valued $\beta([0,T])\otimes \mathcal{F}_t-$measurable , $F-$adapted process ψ such that

$$\| \psi \|''_{\mathcal{L}^2_{\mathcal{F}_t}([0,T];\mathbb{R})} := [E(\int_0^T (\psi(t))^2 dt)]^{\frac{1}{2}} < +\infty$$

We also suppose that

3.1.1 Assumptions

Assumptions(A1)

1. The maps b and δ are $\beta([0,T])\otimes \mathcal{F}_t-$measurable and $F-$adapted.
2. The function b and δ are continuously differetiable up to the second order with respect to (x,u) .
3. All the firdt order derivatives are continuous in (x,u) and unformly bounded.
4. There exists a constant $\alpha_1 > 0$ such that for almost all $(t,\omega) \in [0,T] \times \Omega$ and for any $x, \widetilde{x} \in \mathbb{R}$ and $u, \widetilde{u} \in U$

$$|\lambda(t,x,u)| \leq \alpha_1, \text{ for } \phi = b, \sigma,$$

$$|\lambda(t,x,u) - \lambda(t,\widetilde{x},u)| \leq \alpha_1 |x - \widetilde{x}|, \text{ for } \lambda = b, \sigma,$$

$$| \lambda_{(x,u)^2}(t,x,u) | - | \lambda_{(x,u)^2}(t,\widetilde{x},\widetilde{u}) | \leqslant \alpha_1(| x - \widetilde{x} | + | u - \widetilde{u} |), \text{for} \lambda = b, \delta$$

Assumptions(A2)

1. The process f is $\beta([0,T])\otimes \mathcal{F}_t-$measurable and $F-$adapted.
2. The rendom variable h is $\mathcal{F}_t-$measurable.
3. The process f is bounded by $\alpha_2(1+ | x |^2 + | u |^2)$ and h is bounded by $\alpha_2(1+ | x |)$.

4. The maps f and h are continuously differetiable up to the second order.

5. for any $x, \widetilde{x} \in \mathbb{R}$ and $u, \widetilde{u} \in U$

$$\begin{cases} \mid f_x(t,x,u) \mid + \mid f_u(t,x,u) \mid \leqslant \alpha_2(1+ \mid x \mid + \mid u \mid) \quad \mid h_x(x) \mid \leqslant \alpha_2(1+ \mid x \mid) \\ \mid f_{xx}(t,x,u) \mid + \mid f_{\mathrm{uu}}(t,x,u) \mid + \mid f_{xu}(t,x,u) \mid \leqslant \alpha_2 \\ \mid h_{xx}(x) \mid \leqslant \alpha_2 \quad \mid h_{xx}(x) + h_{xx}(\widetilde{x}) \mid \leqslant \alpha_2(\mid x - \widetilde{x} \mid) \\ \mid f_{(x,u)^2}(t,x,u) - f_{(x,u)^2}(t,\widetilde{x},\widetilde{u}) \mid \leqslant \alpha_2(\mid x - \widetilde{x} \mid + \mid u - \widetilde{u} \mid) \end{cases}$$

The equation (3.1) has strong and unique solution

$$x(t) = x_0 + \int_0^t b(s, x(s), u(s))ds + \int_0^t \delta(s, x(s), u(s))dWs$$

With certain assumptions (A1)and (A2) standard arguments prove that for all $C_k > 0$

$$E(\sup_{t\in[0,T]} \mid x(t) \mid^k) < C_k$$

Where C_k is a constant that depends only on α_2 . Moreover , the functional (3.2) is well defined from U_{ad} into $\mathbb{R}$.

3.2 Second order necessary condition in integral form

In this part, we prove an integral type second order necessary condition for stochastic optimal control . We assume a nonempty and bounded control region U, with a convex perturbation of the optimal control described by $u^\theta(t) = \overline{u}(t) + \theta(u(t) - \overline{u}(t))$ for $u(.) \in U_{ad}$ and $\theta \in [0,1]$. The convexity condition of the control domain guarantes that $u^\theta(.) \in U_{ad}$.

For simplicity , we shall use the following notations , denoted by $x^\theta(.)$, $\overline{x}(.)$ the trejoctery of the SDE (3.1) corresponding to $u^\theta(.)$ and $\overline{u}(.)$.

To simplify our notation , we write for $\phi = b, \sigma, f$:

$$\begin{cases} \sigma_\phi = & \phi(t, x^\theta(t), u^\theta(t)) - \phi(t, \overline{x}(t), \overline{x}(t)) \\ \phi_x = & \phi_x(t, \overline{x}(t), \overline{u}(t)), \phi_u = \phi_u(t, \overline{x}(t), \overline{u}(t)) \\ \phi_{xx} = & \phi_{xx}(t, \overline{x}(t), \overline{u}(t)), \phi_{\mathrm{uu}} = \phi_{\mathrm{uu}}(t, \overline{x}(t), \overline{u}(t)) \\ \phi_{xu} = & \phi_{xu}(t, \overline{x}(t), \overline{u}(t)) \end{cases}$$

We introduce the following variational equation:

$$\begin{cases} dy_1(t) = & (b_x(t)y_1(t) + b_u(t)v(t))dt + (\sigma_x(t)y_1(t) + \sigma_u(t)v(t))dW_t \\ y_1(0) = & 0 \end{cases} \tag{3.5}$$

And:

$$\begin{cases} dy_2(t) = & \{b_x(t)y_2(t) + b_{xx}(t)y_1(t)^2 + 2b_{xu}(t)y_1(t)v(t) + b_{\mathrm{uu}}(t)v(t)^2\}dt \\ & +\{\sigma_x(t)y_2(t) + \sigma_{xx}(t)y_1(t)^2 + 2\sigma_{xu}(t)y_1(t)v(t) + \sigma_{\mathrm{uu}}(t)v(t)^2\}dW_t \\ y_2(0) = & 0 \end{cases} \tag{3.6}$$

Remark 3.2.1 *Based on assumptions (A1)and (A2) we admit a strong unique solution* $y_1(t)$ *and* $y_2(t)$ *to the variational equation (3.5) and (3.6).*

Then , we show the proposition in obtaining a second order necessary condition .

Proposition 3.2.1 *Assumes that assumptions (A1) and (A2) varified . Then for any* $K > 0$ *we have to follow fundimontal estimates*

$$E[\sup_{t\in[0,T]} \mid x^\theta(t) - \overline{x}(t) \mid^{2k}] \leqslant C_k\theta^k \tag{3.7}$$

$$E[\sup_{t\in[0,T]} \mid y_1(t) \mid^{2k}] \leqslant C_k \tag{3.8}$$

$$E[\sup_{t\in[0,T]} \mid y_2(t) \mid^{2k}] \leqslant C_k \tag{3.9}$$

$$E[\sup_{t\in[0,T]} \mid x^\theta(t) - \overline{x}(t) - \theta y_1(t) \mid^{2k}] \leqslant C_k\theta^{2k} \tag{3.10}$$

$$E[\sup_{t\in[0,T]} \mid x^\theta(t) - \overline{x}(t) - \theta y_1(t) - \tfrac{\theta^2}{2} y_2(t) \mid^{2k}] \leqslant C_k\theta^{2k} \tag{3.11}$$

Proof. Allow $\overline{x}(.)$ and $x^\theta(.)$ be the trajectory of (3.1) corresponding to $\overline{u}(.)$ and $u^\theta(.)$ resp.Allow $y_1(.)$ and $y_2(.)$ be the answer of first and second order adjoint equation (3.5) and (3.6). Nothing that estimates (3.7) follows from standard arguments ,shall refer to equation (3.5) as the first order variational equation , aslo we call the process $y_1(.)$ as the first order variational process and we call the process $y_2(.)$ as second variational process . $x^\theta(t) - \overline{x}(t) - \theta y_1(t) - \frac{\theta^2}{2} y_2(t) = O(\theta^2)$ as $\theta \longrightarrow 0$ and that the convergence is of an appropriate order.So the estimates (3.8),(3.9) and (3.10) are obvious and standard . Now we start to prove the estimate (3.11) from (3.1) , (3.5) and (3.6).

We got:

$$\begin{aligned}
\mid x^\theta(t) - \overline{x}(t) - \theta y_1(t) - \tfrac{\theta^2}{2} y_2(t) \mid^{2k} \; =& \mid \int_0^t [\delta b(s) - [b_x(s)y_1(s) + b_u(s)v(s)] \\
& -\tfrac{\theta^2}{2}[b_x(s)y_2(s) + b_u(s)v(s) + b_{xx}(s)y_1(s)^2 \\
& +2b_{xu}(t)y_1(s)v(s) + b_{uu}(s)v(s)^2]]ds \\
& + \mid \int_0^t [\delta\sigma(s) - [\sigma_x(s)y_1(s) + \sigma_u(s)v(s)] \\
& -\tfrac{\theta^2}{2}[\sigma_x(s)y_2(s) + \sigma_u(s)v(s) + \sigma_{xx}(s)y_1(s)^2 \\
& +2\sigma_{xu}(t)y_1(s)v(s) + \sigma_{uu}(s)v(s)^2]]dWs \mid
\end{aligned}$$

Straight forward calculation by appling Caushy-Schwarz inequality(1.12) , we admit that:

$$E[\sup_{t\in[0,T]} \mid x^{\theta}(t)-\overline{x}(t)-\theta y_1(t)-\frac{\theta^2}{2}y_2(t) \mid^{2k}] \leqslant I \tag{3.12}$$

Where:

$$\begin{aligned} I \; &= E \mid 2\sup \int_0^t [\delta b(s)-[b_x(s)y_1(s)+b_u(s)v(s)] \\ &-\tfrac{\theta^2}{2}[b_x(s)y_2(s)+b_u(s)v(s)+b_{xx}(s)y_1(s)^2 \\ &+2b_{xu}(t)y_1(s)v(s)+b_{\mathrm{uu}}(s)v(s)^2]]ds \\ &+\mid \int_0^t [\delta\sigma(s)-[\sigma_x(s)y_1(s)+\sigma_u(s)v(s)] \\ &-\tfrac{\theta^2}{2}[\sigma_x(s)y_2(s)+\sigma_u(s)v(s)+\sigma_{xx}(s)y_1(s)^2 \\ &+2\sigma_{xu}(t)y_1(s)v(s)+\sigma_{\mathrm{uu}}(s)v(s)^{2k}]dWs \mid \end{aligned} \tag{3.13}$$

By applying the Caushy-schawrz inequality (1.12) and the Burkhölder Davis-Gundy inequality , with Bonnans [6] , Zhang and Zhang [24] , we have :

$$I \leqslant C_k\theta^{2k} \tag{3.14}$$

By combining (3.12) , (3.13) , the desired result (3.11) is accomplished . The proof of prposition 3.2.1 is finalized .

Define the Humiltonian function $H:[0,T]\times\mathbb{R}\times U\times\mathbb{R}\times\mathbb{R}$ by

$$H(t,x,u,p,q)=b(t,x,u)p+\sigma(t,x,u)q-f(t,x,u)$$

Now , we present the first adjoint equation

$$\begin{cases} dP(t)= & -\{b_x(t)p(t)+\sigma_x(t)q(t)-f_x(t)\}dt+q(t)dW(t) \\ p(T)= & -h_{xx}(\overline{x}(T)) \end{cases} \tag{3.15}$$

And the second adjoint equation :

$$\begin{cases} dP(t) = & -2b_x(t)P(t) + \sigma_x(t)^2 Q(t) + 2\sigma_x(t)Q(t) + H_{xx}(t)dt + Q(t)dW(t) \\ P(T) = & -h_{xx}(\overline{x}(t)) \end{cases} \tag{3.16}$$

Where:

$$H_{xx}(t) = b_{xx}(t)p + \sigma_{xx}(t)q - f_{xx}(t)$$

It is easy to show that for assumption (A1) and (A2) , equation (3.15) and (3.16) are classical linear backward stochastic differential equation who have a strong and unique solution , such as:

$$(p(t), q(t)) \in L^2_{\mathcal{F}_t}([0,T];\mathbb{R}) \times L^2_{\mathcal{F}_t}([0,T];\mathbb{R})$$
$$(P(t), Q(t)) \in L^2_{\mathcal{F}_t}([0,T];\mathbb{R}) \times L^2_{\mathcal{F}_t}([0,T];\mathbb{R})$$

Also , we present the functional $\mathcal{H} : [0,T] \times \mathbb{R} \times U \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ as

$$\mathcal{H}(t,x,u,p,q,P,Q) = H_{xx}(t,x,u,p,q) + b_u(t,x,u)P(t) + \sigma_u(t,x,u)Q(t) + \sigma_u(t,x,u)P(t)\sigma_x(t,x,u) \tag{3.17}$$

To make our notation easier , we set :

$$\mathcal{H}(t) = \mathcal{H}(t, \overline{x}(t), \overline{u}(t), p(t), q(t), P(t), Q(t) \quad , \quad t \in [0,T]$$

■

Lemma 3.2.1 *Let* $(p(t), q(t))$ *be the solution of the adjoint equation (3.15) ,* $(P(t), Q(t))$ *be the solution of the adjoint equation (3.16) and* $y_1(t), y_2(t)$ *be the solution of the first and the second variational equation (3.5),(3.6) resp. Then the following duality relation hold:*

$$-E[p(T)y_1(T)] = -E[\int_0^T \{p(t)(b_u(t)v(t)) + q(t)(\sigma_u(t)v(t))\}dt] - E[\int_0^T f_x(t)y_1(t)dt] \quad (3.18)$$

$$\begin{cases} -E[p(T)y_2(T)] & = -E[\int_0^T p(t)\{(b_{xx}(t)y_1(t)^2 + (2b_{xu}(t)y_1(t)v(t) + (b_{\mathrm{uu}}(t)v(t)^2\}dt] \\ & +E[\int_0^T q(t)\{\sigma_{xx}(t)y_1(t)^2 + 2\sigma_{xu}(t)y_1(t)v(t) + \sigma_{xu}(t)v(t)^2\}dt] \\ & -E[\int_0^T f_x(t)y_2(t)dt] \end{cases} \quad (3.19)$$

And

$$\begin{cases} E[P(T)y_1(T)^2] & = -2E[\int_0^T \{P(t)y_1(t)(b_u(t)v(t)) + P(t)\sigma_x(t)y_1(t)(\sigma_u(t)v(t))\}dt] \\ & -2E[\int_0^T \{Q(t)\sigma_u(t)y_1(t)v(t)\}dt] - E[\int_0^T P(t)(\sigma_u(t)v(t))^2dt] \\ & +E[\int_0^T H_{xx}(t)y_1(t)^2dt] \end{cases} \quad (3.20)$$

Proof. This lemma 's proof proceeds directly from ito's formula to $p(t)y_1(t)$ and takes expectation where $y_1(0) = 0$, we have

$$E[P(T)y_1(T)] = -E\int_0^T p(t)dy_1(t) - E\int_0^T y_1(t)dp(t) \\ E\int_0^T q(t)\{\sigma_x(t)y_1(t) + \sigma_u(t)v(t)\}dt \tag{3.21}$$

Where

$$-E\int_0^T p(t)dy_1(t) = -E\int_0^T p(t)[b_x(t)y_1(t) + b_u(t)v(t)]dt \tag{3.22}$$

Consequently

$$E\int_0^T y_1(t)dp(t) \\ = E\int_0^T y_1(t)[b_x(t)p(t) + \sigma_x(t)q(t) - f_x(t)]dt \tag{3.23}$$

Substituting (3.22) , (3.23) into (3.21) then the desired result (3.18) is satisfied . Now , by using Ito's formula in $p(t)y_2(t)$ and assuming that $y_2(0) = 0$, we have

$$-E[p(T)y_2(T)] = -E\int_0^T p(t)dy_2(t) - E\int_0^T y_2(t)dp(t) \\ -E[\int_0^T q(t)\{\sigma_x(t)y_2(t) + \sigma_{xx}(t)y_1(t)^2 \\ +2\sigma_{xu}(t)y_1(t)v(t) + \sigma_{\mathrm{uu}}(t)v(t)^2\}dt] \tag{3.24}$$

Where

$$-E\int_0^T p(t)dy_2(t) = -E\int_0^T p(t)\{b_x(t)y_2(t) + b_{xx}(t)y_1(t)^2 \\ +2b_{xu}(t)y_1(t)v(t) + b_{\mathrm{uu}}(t)v(t)^2\}dt \tag{3.25}$$

And

$$
\begin{aligned}
&-E\int_0^T y_2(t)dp(t) \\
&= E\int_0^T y_2(t)[b_x(t)p(t)+\sigma_x(t)q(t)-f_x(t)]dt
\end{aligned}
\tag{3.26}
$$

Substituting (3.25) , (3.26) into (3.24) we obtain the desired result (3.19) .

Next applying Ito's formula to $P(t)y_1(t)$, where $y_1(0)=0$, we have

$$
\begin{aligned}
[P(T)y_1(T)] &= \int_0^t P(t)dy_1(t) + \int_0^T y_1(t)dP(t) \\
&+\int_0^T Q(t)\{\sigma_x(t)y_1(t)+\sigma_u(t)v(t)\}dt \\
&= I_1+I_2+I_3
\end{aligned}
\tag{3.27}
$$

Where

$$
\begin{aligned}
I_1 &= \int_0^T P(t)dy_1(t) \\
&= \int_0^T P(t)\{b_x(t)y_1(t)+b_u(t)v(t)\}dt \\
&+\int_0^T P(t)\{\sigma_x(t)y_1(t)+\sigma_u(t)v(t)\}dW(t)
\end{aligned}
$$

By easy calculation , we can demonstrate

$$
\begin{aligned}
I_2 &= \int_0^T y_1(t)dP(t) \\
&= -\int_0^T y_1(t)\{2b_x(t)P(t)+\sigma_x(t)^2P(t)+2\sigma_x(t)Q(t) \\
&+H_{xx}(t)dt + \int_0^T y_1(t)Q(t)dW(t)
\end{aligned}
$$

$$I_3 = \int_0^T \{Q(t)\sigma_x(t)y_1(t) + Q(t)\sigma_u(t)v(t)\}dt$$

Then , we can write (3.27) as follows

$$\begin{aligned}[P(T)y_1(T)] \;&= \int_0^T [P(t)b_u(t)v(t)dt + Q(t)\sigma_u(t)v(t) \\ &-y_1(t)b_x(t)P(t) - y_1(t)\sigma_x(t)^2P(t) - y_1(t)Q(t)\sigma_x(t) \\ &-y_1(t)H_{xx}(t)]dt \\ &+\int_0^T [P(t)\sigma_x(t)y_1(t) + P(t)\sigma_u(t)v(t) + y_1(t)Q(t)]dW(t)\end{aligned}$$

Now , we applying Ito's formula to $(P(t)y_1(t))y_1(t)$ and taking expectation , we obtain

$$\begin{aligned}&-E[P(T)y_1(T)^2 \\ &= -E\int_0^T P(t)y_1(t)dy_1(t) - E\int_0^T y_1(t)d(P(t)y_1(t)) \\ &-E[\int_0^T \{\sigma_x(t)y_1(t) + P(t)\sigma_u(t)v(t)\}\{P(t)\sigma_x(t)y_1(t) + P(t)\sigma_u(t)v(t) + y_1(t)Q(t)\}dt] \\ &= J_1 + J_2 + J_3\end{aligned} \tag{3.28}$$

Where

$$J_1 = -E\int_0^T P(t)y_1(t)dy_1(t) \tag{3.29}$$

$$= -E\int_0^T P(t)y_1(t)\{b_x(t)y_1(t) + b_u(t)v(t)\}dt \tag{3.30}$$

$$\begin{aligned} J_2 &= -E\int_0^T y_1(t)d(P(t)y_1(t)) \\ &= -E\int_0^T y_1(t)[P(t)b_u(t)v(t)dt + Q(t)\sigma_u(t)v(t) \\ &-y_1(t)b_x(t)P(t) - y_1(t)\sigma_x(t)^2P(t) - y_1(t)\sigma_x(t)Q(t) \\ &- - y_1(t)H_{xx}(t)]dt \end{aligned} \tag{3.31}$$

And it's simple to show that

$$\begin{aligned} J_3 &= -E[\int_0^T \{y_1(t)\sigma_x(t) + \sigma_u(t)v(t)\}\{P(t)\sigma_x(t)y_1(t) + y_1(t)Q(t)\}dt] \\ &= -E[\int_0^T \{P(t)(y_1(t)\sigma_x(t))^2 + 2P(t)y_1(t)\sigma_x(t)v(t)\sigma_u(t) + y_1(t)^2\sigma_x(t)Q(t) \\ &+P(t)(v(t)\sigma_u(t))^2 + Q(t)y_1(t)v(t)\sigma_u(t)\}dt] \end{aligned} \tag{3.32}$$

Likewise , we have at last substituted (3.29) , (3.31) , (3.32) into (3.28) and then (3.20) is satisfied .

This complet the proof of lemma 3.2.1. ■

The following technical result is required to demonstrate the main theorem

Proposition 3.2.2 *Let (A1)-(A2) hold . Then , for any* $u(.) \in U_{ad}$ *we have*

$$\begin{aligned} &J(u^\theta(.)) - J(\overline{u}(.)) \\ &= -E\int_0^T [\theta\{H_u(t)v(t)\} + \tfrac{\theta^2}{2}\{H_{uu}(t)v(t)^2\} \\ &+\tfrac{\theta^2}{2}\{P(t)(v(t)\sigma_u(t))^2 + \theta^2\{H(t)y_1(t)v(t)\}dt] + o(\theta^2), (\theta \longrightarrow 0^+) \end{aligned}$$

Where

$$H_u(t) = H_u(t, \overline{x}(t), \overline{u}(t), p(t), q(t))$$

$$H_{uu}(t) = H_{uu}(t, \overline{x}(t), \overline{u}(t), p(t), q(t))$$

Proof. By applying Taylor's formula , we get

$$\begin{aligned}
&J(u^\theta(.)) - J(\overline{u}(.)) \\
&= E[\int_0^T \{\varphi f(t)\}dt] + E[h(x^\theta(T) - h(\overline{x}(T)] \\
&= E[\int_0^T \{f_x(t)(x^\theta(t) - \overline{x}(t)) + f_u(t)(u^\theta(t) - \overline{u}(t)) + \tfrac{1}{2} f_{xx}(t)(x^\theta(t) - \overline{x}(t))^2 \\
&+ f_{xu}(t)(x^\theta(t) - \overline{x}(t))(u^\theta(t) - \overline{u}(t)) + \tfrac{1}{2} f_{uu}(t)(u^\theta(t) - \overline{u}(t))^2\}dt] \\
&+ E[h_x(\overline{x}(T))(x^\theta(T) - \overline{x}(T)) + \tfrac{1}{2} h_{xx}(\overline{x}(T))(x^\theta(T) - \overline{x}(T))^2] + o(\theta^2)
\end{aligned} \tag{3.33}$$

Using proposition 3.2.1 , we have

$$\begin{aligned}
&J(u^\theta(.)) - J(\overline{u}(.)) \\
&= E[\int_0^T \{\theta f_x(t) y_1(t) + \tfrac{\theta^2}{2} f_x(t) y_2(t) + \theta f_u(t) v(t) \\
&+ \tfrac{\theta^2}{2}(f_{xx}(t) y_1(t)^2 + 2 f_{xu}(t) y_1(t) v(t) + f_{uu}(t) v(t)^2)\}dt] \\
&+ E[\theta h_x(\overline{x}(T)) y_1(T) + \tfrac{\theta^2}{2} h_x(\overline{x}(T)) y_2(T) + \tfrac{\theta^2}{2} h_{xx}(x(T)) y_1(T)^2] + o(\theta^2), \theta \longrightarrow 0^+
\end{aligned} \tag{3.34}$$

■

Now , by proposition 3.2.2 , we can prove the following second order necessary condition in integral form for stochastic optimal control (3.1) ,(3.2)

Theorem 3.2.1 *Let (A1)-(A2) hold . If* $u(.)$ *is a singular optimal control in the classical sense for the control problem (3.2)-(3.3) , Then we have*

$$E\int_0^T H(t) y_1(t)(u(t) - \overline{u}(t))dt \leqslant 0 \tag{3.35}$$

For any $u(.) \in U_{ad}$ *, where the Hamiltonian* H *is defined by*(3.17) *and* $y_1(t)$ *is a solution of the first order adjoint equation given by*

$$y_1(t) = \int_0^T \{b_x(s)y_1(s) + b_u(s)v(s)\}ds + \int_0^T \{\sigma_x(s)y_1(s) + \sigma_u(s)v(s)\}dW(s)$$

Proof. The desired result (3.35) and proposition (3.2.2) follows directly from (3.1). This completes the proof of Theorem. ■

3.3 Martingale terms of second order maximum principle

In this part , by applying the martingale representation theorem and the property of Ito's integrals , we prove the second order necessary condition for singular optimal control which is pointwise maximum principale in terms of the martingale with respect to the time variable t . The folowing lemma play an important role to prove our result.

Lemma 3.3.1 $y_1(.)$ *is a unique strong solution of the first variational equation* (3.1) *which is represented by the following:*

$$\begin{aligned} y_1(t) &= \phi(t)[\int_0^t \psi(s)(b_u(s) - \sigma_x(s)\sigma_u(s))v(s)ds \\ &+ \int_0^t \psi(s)\sigma_u(s)v(s)dW(s)] \end{aligned} \tag{3.36}$$

Where $\phi(t)$ *is a defined by the following linear stochastic differential equation :*

$$\begin{cases} d\phi(t) = b_x(t)\phi(t)dt + \sigma_x(t)\phi(t)dW(t) \\ \phi(0) = 1 \end{cases} \tag{3.37}$$

and $\psi(t)$ *is inverse*

Proof. (3.5) is a linear equation with bounded coefficients , then it admitsa strong unique solution . Moreover , this solution is inversible and its inverse $\psi(t) = \phi^{-1}(t)$ given by:

$$\begin{cases} d\psi(t) = [\sigma_x^2(t)\psi(t) - b_x(t)\psi(t)]dt - [\sigma_x(t)\psi(t)]dW(t) \\ \psi(0) = 1 \end{cases} \tag{3.38}$$

Applying Ito's formula to $\psi(t)y_1(t)$ we have

$$\begin{aligned} d[\psi(t)y_1(t)] &= y_1(t)d\psi(t) + \psi(t)dy_1(t) \\ &-[\sigma_x(t)\psi(t)][\sigma_x(t)y_1(t) + \sigma_u(t)v(t)]dt \\ &= I_1 + I_2 + I_3 \end{aligned} \tag{3.39}$$

Where

$$\begin{aligned} I_1 &= y_1(t)d\psi(t) \\ &= [y_1(t)\sigma_x^2(t)\psi(t) - y_1(t)b_x(t)\psi(t)]dt \\ &-y_1(t)\psi(t)\sigma_x(t)dW(t) \\ &-y_1(t) \end{aligned} \tag{3.40}$$

With an easy computations ,we get

$$\begin{aligned} I_2 &= y_1(t)\psi(t)dy_1(t) \\ &= [y_1(t)b_x(t)\psi(t) + v(t)b_u(t)\psi(t)] \\ &+[y_1(t)\sigma_x(t)\psi(t) + v(t)\sigma_u(t)\psi(t)]dW(t) \end{aligned} \tag{3.41}$$

And

$$I_3 = -[\sigma_x(t)\psi(t)][y_1(t)\sigma_x(t) + v(t)\sigma_u(t)]dt \tag{3.42}$$

Substituting (3.37) , (3.38) and (3.39) into (3.37) , we get

$$\begin{aligned}&\psi(t)y_1(t)-\psi(0)y_1(0)\\&=[\int_0^t\psi(s)[b_u(s)-\sigma_x(s)\sigma_u(s)]v(s)ds\\&+\int_0^t\psi(s)\sigma_u(s)v(s)ds\end{aligned} \tag{3.43}$$

Since $y_1(0)=0$ and $\psi^{-1}(t)=\phi(t)$, then from (3.43) the desired result (3.36) is satisfied .

This complets the proof of lemma (3.3.1). ■

To show the main theorem we need to use the follwing technical lemma.

Lemma 3.3.2 *Let (A1)-(A2) hold . then we have*

1. $H(.)\in L^2_{\mathbb{F}}([0,T],\mathbb{R})$
2. $\forall v\in U$, $\exists\phi_v(.,t)\in L^2_{\mathbb{F}}([0,T],\mathbb{R})$ *such that*

$$H(t)(v-\overline{u}(t))=E[H(t)(v-\overline{u}(t))]+\int_0^t\phi_v(s,t)dW(s) \tag{3.44}$$

$$a.e.t\in[0,T],P-a.s$$

Proof. : (1) the proof is directly in [24].

(2) the proof of (3.44) follows from Tang and Li in [21] ■

Now , we return to integral type of second order necessary condition and substituting the explicit representation (3.36) of $y_1(.)$ into (3.35) we notice that there is a "bad" term in the form

$$E\int_0^T[H(t)\phi(t)\int_0^t\psi(s)\sigma_u(s)v(s)dW(s)]v(t)dt \tag{3.45}$$

Now , in order to derive a pointwise second order necessarycondition from the integral form in (3.36) , for the optimal control , we must use the following needle variation . $\overline{u}(.)$:

$$u(t) = \begin{cases} v, t \in A_\theta \\ \overline{u}(t), t \in [0,T] \mid A_\theta \end{cases} \tag{3.46}$$

Where $\tau \in [0,T]$, $v \in U$, and $A_\theta = [\tau, \tau+\theta]$ in order that $\theta > 0$ and $\tau + \theta \leqslant T$. Stand for $\mathcal{X}_{A_\theta}(.)$ the characteristic function of the set A_θ.

Then we have $v(.) = u(.) = \overline{u}(t) = (v - \overline{u}(.))\mathcal{X}_{A_\theta}(.)$.

The following theorem constitutes the main contribution of the result.

Theorem 3.3.1 *Let (A1) , (A2) hold. If the singular optimal control in the classical sense* $u(.)$ *is for the stochastic control (3.2)-(3.3) then for any* $v \in U$ *it holds that*

$$E(H(\tau)b_u(\tau)(v-\overline{u}(\tau))^2) + \partial_\tau^+(H(\tau)(v-\overline{u}(\tau))^2\sigma_u(\tau)) \leqslant 0 \quad a.e.\tau \in [0,T] \tag{3.47}$$

Where

$$\begin{aligned} &\partial_\tau^+(H(\tau)(v-\overline{u}(\tau))^2\sigma_u(\tau)) \\ &:= 2 \lim_{o \longrightarrow 0^+} \sup \tfrac{1}{\theta^2} E \int_\tau^{\tau+\theta} \int_\tau^t [\phi_v(s,t)\phi(\tau)\psi(s)\sigma_u(s)(v(-\overline{u}(s))]dsdt \end{aligned} \tag{3.48}$$

$\phi_v(.,t)$ *is determined by(3.44) ,* $\phi(.)$ *is given by the following process*

$$\phi(t) = \phi(0) + \int_0^t b_x(s)\phi(s)ds + \int_0^t \sigma_x(s)\phi(s)dW(s)$$

And $\psi(.)$ *is given by*

$$\begin{aligned}\psi(t) &= \psi(0) + \int_0^t [-b_x(s)\psi(s) + \sigma_x(s)^2\psi(s)]ds \\ &- \int_0^t [\sigma_x(s)\psi(s)dW(s)\end{aligned}$$

Proof. From (3.43) we have $v(.) = u(.) = -\overline{u}(t) = (v - \overline{u}(.))\mathcal{X}_{A_\theta}(.)$ and the corresponding solution $y_1(.)$ to (3.5) is given by

$$\begin{aligned}y_1(t) &= \phi(t)\int_0^t \psi(s)(b_u(s) - \sigma_x(s)\sigma_u(s))(v - \overline{u}(s))\mathcal{X}_{A_\theta}(s)ds \\ &+\phi(t)\int_0^t \psi(s)\sigma_u(s)(v - \overline{u}(s))\mathcal{X}_{A_\theta}(s)dW(s)\end{aligned} \tag{3.49}$$

Substituting $v(.) = u(.) = -\overline{u}(t) = (v - \overline{u}(.))\mathcal{X}_{A_\theta}(.)$ and (3.49) into (3.35) , we have

$$\begin{aligned}0 &\geqslant \tfrac{1}{\theta^2}E\int_\tau^{\tau+\theta} [H(t)y_1(t)(v - \overline{u}(t))]dt \\ &= \tfrac{1}{\theta^2}E\int_\tau^{\tau+\theta} [H(t)\phi(t)\int_\tau^t \psi(s)(b_u(s) - \sigma_x(s)\sigma_u(s))(v - \overline{u}(s))ds(v - \overline{u}(t))]dt \\ &+\tfrac{1}{\theta^2}E\int_\tau^{\tau+\theta} [H(t)\phi(t)\int_\tau^t \psi(s)\sigma_u(s)(v - \overline{u}(s))dW(s)(v - \overline{u}(t))dt \\ &= J_1^\theta + J_2^\theta\end{aligned} \tag{3.50}$$

From [24] **Lemma 4.1**,we got

$$\begin{aligned}\lim_{o\longrightarrow 0^+} J_1^\theta &= \lim_{o\longrightarrow 0^+} \tfrac{1}{\theta^2}E\int_\tau^{\tau+\theta} [H(t)\phi(t)\int_\tau^t \psi(s)(b_u(s) - \sigma_x(s)\sigma_u(s))(v - \overline{u}(s))ds(v - \overline{u}(t))]dt \\ &= \tfrac{1}{2}E(H(\tau)(b_u(s) - \sigma_x(s)\sigma_u(s))(v - \overline{u}(s))^2)\end{aligned} \tag{3.51}$$

On the other hand , by (3.36) , it follows that

$$
\begin{aligned}
J_2^\theta &= \frac{1}{\theta^2}\int_\tau^{\tau+\theta} E[H(t)\phi(t)\int_\tau^t \psi(s)\sigma_u(s)(v-\overline{u}(s))dW(s)(v-\overline{u}(t))]dt \\
&= \frac{1}{\theta^2}\int_\tau^{\tau+\theta} E[H(t)\phi(t)\int_\tau^t \psi(s)\sigma_u(s)(v-\overline{u}(s))dW(s)(v-\overline{u}(t))]dt \\
&+\frac{1}{\theta^2}\int_\tau^{\tau+\theta} E[H(t)\int_\tau^t (b_x(s)\phi(s)ds \\
&\times\int_\tau^t \psi(s)\sigma_u(s)(v-\overline{u}(s))dW(s)(v-\overline{u}(t))]dt \\
&+\frac{1}{\theta^2}E\int_\tau^{\tau+\theta} [H(t)\int_\tau^t (\sigma_x(s)\phi(s)dW(s) \\
&\times\int_\tau^t \psi(s)\sigma_u(s)(v-\overline{u}(s))dW(s)(v-\overline{u}(t))]dt \\
&= J_{2,1}^\theta + J_{2,2}^\theta + J_{2,3}^\theta + J_{2,4}^\theta
\end{aligned}
\tag{3.52}
$$

By lemma [?] , we have

$$
\begin{aligned}
&\limsup_{o\longrightarrow 0^+} J_{2,1}^\theta \\
&= \limsup_{o\longrightarrow 0^+}\frac{1}{\theta^2}\int_\tau^{\tau+\theta} E[H(t)\phi(\tau)\int_\tau^t \psi(s)\sigma_u(s)(v-\overline{u}(s))dW(s)(v-\overline{u}(t))]dt \\
&= \limsup_{o\longrightarrow 0^+}\frac{1}{\theta^2}\int_\tau^{\tau+\theta} E[\int_\tau^t \phi(\tau)\psi(s)\sigma_u(s)(v-\overline{u}(s))dW(s)E[H(t)(v-\overline{u}(t))]]dt \\
&+\limsup_{o\longrightarrow 0^+}\frac{1}{\theta^2}\int_\tau^{\tau+\theta} E[\int_\tau^t \phi(\tau)\psi(s)\sigma_u(s)(v-\overline{u}(s))dW(s)\int_0^t \phi_v(s,t)]dt \\
&= \limsup_{o\longrightarrow 0^+}\frac{1}{\theta^2}\int_\tau^{\tau+\theta}\int_\tau^t E\{\phi(\tau)\psi(s)\sigma_u(s)(v-\overline{u}(s))\phi_v(s,t)\}dsdt \\
&= \frac{1}{2}\partial_\tau^+(H(\tau)(v-\overline{u}(\tau))^2\sigma_u(\tau)), \forall\tau\in[0,T]
\end{aligned}
\tag{3.53}
$$

Because of the martingale representation theorem in Lemma [?] , it is imerative that we only know that $\phi_v(.,t)\in L^2_{\mathbb{F}}([0,T],\mathbb{R})$ for every $v\in U$ and consequently that the function

$$\varphi_t(s) = E[\phi(\tau)\psi(s)\sigma_u(s)(v - \overline{u}(s))\phi_v(s,t)], s \in [0,T], t \in [0,T]$$

For each $\tau \in [0,T]$ is in $L^1_{\mathbb{F}}([0,T],\mathbb{R})$. For more details for the following superior limit see [24]

$$\lim_{o \longrightarrow 0^+} \frac{1}{\theta^2} \int_\tau^{\tau+\theta} \int_\tau^t \varphi_t(s) ds dt$$

By simple computations , the last term in (3.52) is in fact a process with zero expectation Now by using the similair method in [24], we have

$$\begin{aligned} \lim J^\theta_{2,2} &= \lim_{o \longrightarrow 0^+} \tfrac{1}{\theta^2} \int_\tau^{\tau+\theta} E\{H(t) \int_\tau^t b_x(s)\phi(s) ds \\ &\times \int_\tau^t \psi(s)\sigma_u(s)(v - \overline{u}(s)) dW(s)(v - \overline{u}(s))\} dt \\ &= 0 \end{aligned} \tag{3.54}$$

$$\begin{aligned} \lim J^\theta_{2,3} &= \lim_{o \longrightarrow 0^+} \tfrac{1}{\theta^2} \int_\tau^{\tau+\theta} E\{H(t) \int_\tau^t {}_x\sigma(s)\phi(s) dW(s) \\ &\times \int_\tau^t \psi(s)\sigma_u(s)(v - \overline{u}(s)) dW(s)(v - \overline{u}(s))\} dt \\ &= \tfrac{1}{2} E(H(\tau)(\sigma_x(\tau)\sigma_u(\tau))(v - \overline{u}(s))^2) \end{aligned} \tag{3.55}$$

And finally , substituting (3.50) , (3.52) , (3.53) , (3.54) , (3.55) in(3.49) we obtain

$$E(H(\tau)b_u(\tau)(v - \overline{u}(\tau))^2) + \partial^+_\tau(H(\tau)b_u(\tau)(v - \overline{u}(\tau))^2\sigma_u(\tau)) \leqslant 0 \quad a.e. \tau \in [0,T]$$

This complet the proof of theorem 3.3.1. ■

Conclusion

The thesis focuses on second order necessary condition for stochastic optimal control problems in two different classes of singular optimal controls. It employs convex perturbation techniques to derive Taylor's expansion of the cost functional and provescnecessary condition for stochastic singular optimals controls in integral form . Assuming a convex control region and degeneration of the first order condition , the discussion on secind order necessary condition becomes crucial .The integral equation derived are vital for solving optimal control problem , providing a foundation for finding optimal solutions and gaining important insights and results . Overall , the derivation of these necessary conditions in integral form plays a significant role in addressing stochastic optimal control problems , offering essential support in dolving them effectively .

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Abstract

In this work, we establish second-order necessary conditions for singular optimal controls in the classical sense, we consider the convex case, i.e., the control region is allowed to be convex, and the control variable enters into both the drift and the diffusion terms of the control systems. By introducing two vatariational equations and two adjoint equations, we obtain the desired necessary conditions for stochastic singular optimal controls in integral form and in martingale forms

Keywords : , Stochastic optimal control , SDE (Stochastic Differential Equation). needle variation, variational equation, adjoint equation.

Résumé

Dans ce travail, nous établissons des conditions nécessaires du second ordre pour les contrôles optimaux singulier aux sens classique. Nous considérions le cas convexe, c'est-à-dire que la région de contrôle doit être convexe, et la variable de contrôle intervient à la fois dans les termes de diffusion des systèmes de contrôle. En introduisant deux équations variationnelles et deux equations adjointes, nous obtenons les conditions nécessaires souhaitées pour les contrôles optimaux stochastiques singuliers sous forme intégrale et sous forme de martingales.

Mots cles: contrôles optimaux singulier aux sens classique , équations variationnelles , équations adjointes , contrôles optimaux stochastiques singuliers sous forme intégrale . et sous forme de martingales

الملخص

في هذا العمل، سوف ندرس الشروط الضرورية من الدرجة الثانية للتحكم الامثل المنفردة بالمعنى الكلاسيكي. نعتبر الحالة المحدبة، أي يجب أن تكون منطقة التحكم محدبة، وتدخل متغير التحكم في كل من مصطلحات الانتشار لأنظمة التحكم. من خلال إدخال معادلتين تفاضليتين ومعادلتين مترافقين، نحصل على الشروط الضرورية المطلوبة للتحكمات المثلى العشوائية المنفردة على شكل تكامل وعلى شكل (مارتينجال).

كلمات مفتاحية

التحكم الأمثل العشوائي بالمعنى التقليدي , الحركة البراونية للتحكم الأمثل العشوائي, المعادلة التفاضلية العشوائية التحكمات المثلى المنفردة بالمعنى الكلاسيكي، المعادلات التفاضلية، المعادلات المترافقة، التحكمات المثلى العشوائية المنفردة في شكل تكاملي وعلى شكل (مارتينجال).