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# Necessary conditions for relaxed control problems for nonlinear mean-field FBSDE

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# Dedicace

**I dedicate this humble work to**

**My mother Chagra Hayet and my father Abd Elhakime Benghezala**

**My systers hImen, Asmaa, Chaimaa, Wafaa, Samah, Djoumana and tasnime**

**those who are the best in my world**

**My brother Abd Eldjalile and Ayoub The mountain that when the world leans against me, Isupport myself on it in times of adversity. How could it not, when the Lord of the universe said, " We will support you with your brother."**

**I dedicace this humble work to my teacher, my beloved and my sister in God Mariem Dassa**

**And to my beloved ones, my friends, and my sisters in God, the sisters of the prayer hall (Asmaa Dat al-Nitaqin prayer hall) especialiy my friend Hanen**

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# Contents

# Introduction

The EDSRs were introduced in 1973 by J.-M. Bismut [3] in the case where the generator f is linear with respect to the variables $Y$ and $Z$. It was not until the beginning of the 90s and the work of E. Pardoux and S. Peng [6] to have the first result of existence and uniqueness of solution in the nonlinear case. Forward- backward stochastic differential equations (shortly, FBSDEs) were first studied by Antonelli in [1 ]. He showed the existence and uniqueness of the solution of a backward -forward stochastic differential equations, where the solution depends explicitly on both the past and future of its own trajectory, under a more restricture hypothesis on the Lipschitz constant. Since then they are encountered in stochastic optimal control problems and mathematical finance.

The objective of this memory is to establish the necessary conditions of optimality as maximum principle for relaxed control problem for system governed by the following FBSDE of mean-field type:

$$\begin{cases} dX_t = & \int_U b(t, X_t^\mu, \mathbb{E}[X_t^\mu], u)\mu_t(du)dt + \int_U \sigma(t, X_t^\mu, \mathbb{E}[X_t^\mu], u)\mu_t(du)dW_t \\ -dY_t = & \int_U f(t, X_t^\mu, \mathbb{E}[X_t^\mu], Y_t^\mu, \mathbb{E}[Y_t^\mu], Z_t^\mu, \mathbb{E}[Z_t^\mu], u)\mu_t(du)dt - Z_t dW_t \\ & X_0^\mu = x, \quad Y_T^\mu = g(X_T^\mu, \mathbb{E}[X_T^\mu]]), \qquad t \in [0, T]. \end{cases}$$

The functionnel cost to be minimized over the set $\mathcal{R}$ of admissible relaxed controls, is defined by the form:

$$J(\mu_.) := \mathbb{E}[\alpha(X_T^{\mu}, \mathbb{E}[X_T^{\mu}]) + \beta(Y_0^{\mu}, \mathbb{E}[Y_0^{\mu}])$$
$$+ \int_0^T \int_U l(t, X_t^{\mu}, \mathbb{E}[X_t^{\mu}], Y_t^{\mu}, \mathbb{E}[Y_t^{\mu}], Z_t^{\mu}, \mathbb{E}[Z_t^{\mu}], u)\mu_t(du)dt].$$

We say that admissible relaxed control $q_.$ is optimal control if:

$$J(q_.) = \inf_{\mu_. \in \mathcal{R}} J(\mu_.).$$

This memory is composed of two chapters:

**Chapter 1**: In this chapter, we study a type of nonlinear forward-backward stochastic differential equations this type is called mean field, where the system is dependent on the state process, as well as its distribution via the expectation of the state process. We will give the demonstration of the result of existence and uniqueness of solution for systems derived by nonlinear forward-backward stochastic differential equations of mean field type (MF-FBSDE), under Lipschitz condition and by using Picard iteration.

**Chapter 2**: Our objective in the second chapter is to establish the necessary optimality condition in the form of a stochastic maximum principle, for relaxed controls for a system governed by nonlinear forward-backward SDEs, of mean field type. To achieve this goal, we use the fact that the set of relaxed controls is convex and applying the weak (convex) perturbation method.

# Chapter 1

# The existence and uniqueness of solution for nonlinear mean-field FBSDE

## 1.1 Existence and uniqueness of solution for FBSDE:

We consider the following nonlinear forward-backward SDEs, of mean field type:

$$\begin{cases} dX_t = & b(t, X_t, \mathbb{E}[X_t])dt + \sigma(t, X_t, \mathbb{E}[X_t])dW_t \\ -dY_t = & f(t, X_t, \mathbb{E}[X_t], Y_t, \mathbb{E}[Y_t], Z_t, \mathbb{E}[Z_t])dt - Z_t dW_t \\ X_0 = x \ , & Y_T = g(X_T, \mathbb{E}[X_T]), \end{cases} \tag{1.1}$$

where the following functions must be measurable:

$$b : \Omega \times [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

$$\sigma : \Omega \times [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^{n\times d}$$

$$g : \Omega \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^m$$

$$f : \Omega \times [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times \mathbb{R}^{m\times d} \longrightarrow \mathbb{R}^m.$$

This system can be interpreted in full form as follows:

$$\begin{cases} X_t = & x + \int_0^t b(s, X_s, \mathbb{E}[X_s])ds + \int_0^t \sigma(s, X_s, \mathbb{E}[X_s])dW_s \quad ,t > 0 \\ Y_t = & g(X_T, \mathbb{E}[X_T]) + \int_t^T f(s, X_s, \mathbb{E}[X_s], Y_s, \mathbb{E}[Y_s], Z_s, \mathbb{E}[Z_s])ds - \int_t^T Z_s dW_s, \end{cases}$$

the system (1.1) is called forward and backward stochastic differential equation of mean-field type (MF-FBSDE). Such that, the coefficient $b$ is called the drift, $\sigma$ is called the diffusion of the FSDE and $f$ is called the generator of the BSDE.

-We will work with two process spaces.

-We will first denote by $S^2(\mathbb{R}^n)$ the vector space formed by the process $X_t$, progressively measurable, with values in $\mathbb{R}^n$,

such that:

$$\parallel X_t \parallel_{s^2}^2 = \mathbb{E}[\sup_{0\leq s\leq T} \mid X_t \mid^2] < \infty.$$

And $S_c^2(\mathbb{R}^n)$ the subspace formed by continuous processes.

-And then $M^2(\mathbb{R}^{m\times d})$ that formed by the process $Z_t$ progressively measurable, with values in $\mathbb{R}^{n\times d}$, such that:

$$\parallel Z_t \parallel_{M^2}^2 = \mathbb{E}[\int_0^T \parallel Z_t \parallel^2 dt] < \infty.$$

-The spaces $S^2$, $S_c^2$ and $M^2$ are Banach spaces for the defined norms previously.

-We will designate $B^2$ the Banach space

$$B^2 = S^2(\mathbb{R}^n) \times S_c^2(\mathbb{R}^n) \times M^2(\mathbb{R}^{m\times d}).$$

**Definition:**

We call solution of the system of Forward-Backward stochastic differential equations (FBSDE) of mean field type (1.1), all triple $(X, Y, Z)$ of progressively measurable processes valued in $\mathbb{R}^n\times$ $\mathbb{R}^m$ $\times\mathbb{R}^{m\times d}$ and square integrable such that:

$$\begin{cases} X_t = x + \int_0^t b(s, X_s, \mathbb{E}[X_s])ds + \int_0^t \sigma(s, X_s, \mathbb{E}[X_s])dW_s \quad ,t > 0 \\ Y_t = g(X_T, \mathbb{E}[X_T]) + \int_t^T f(s, X_s, \mathbb{E}[X_s], Y_s, \mathbb{E}[Y_s], Z_s, \mathbb{E}[Z_s])ds - \int_t^T Z_s dW_s. \end{cases}$$

### 1.1.1 Existence and uniqueness theorem:

**Theorem (Existence and uniqueness):**

Let $b$, $\sigma$, $f$ and $g$ are a Borelian functions. Assume that there is a constant $k > 0$ such that , for all $t \in [0, T]$, all $(x_1, x_1', x_2, x_2', y_1, y_1', y_2, y_2', z_1, z_1', z_2, z_2') \in \mathbb{R}^{4n+4m+4m\times d}$

**1-Lipschitz condition:**

$$\begin{aligned} &|b(t, x_1, x_1') - b(t, x_2, x_2')| + \|\sigma(t, x_1, x_1') - \sigma(t, x_2, x_2')\| \\ &\leq k(|x_1 - x_2| + |x_1' - x_2'|), \end{aligned}$$

and

$$| g(x_1, x_1') - g(x_2, x_2') | \leq k(| x_1 - x_2 | + | x_1' - x_2' |).$$

**2-Linear growth:**

$$|b(t,x,x')| + \|\sigma(t,x,x')\| \leq k(1+|x|+|x'|),$$

and

**3-**

$$\mathbb{E}\left[\int_0^T |f(s,0,0,0,0,0,0)|^2\right] ds < +\infty.$$

Then there is a unique solution $(X,Y,Z)$ of the mean-field $FBSDE$ (1.1).

**Proof:**

**1-Existence:** We construct the solution using **Picard's iteration** methode. By defining the sequence $(X^n,Y^n,Z^n)_{n\in\mathbb{N}}$ such that: $X_0 = x, Y_0 = Z_0 = 0$, $(X^{n+1},Y^{n+1},Z^{n+1})$ is the solution of the following mean-field $FBSDE$ system.

$$\begin{cases} X_t^{n+1} = & x + \int_0^t b(s,X_s^n,\mathbb{E}[X_s^n])ds + \int_0^t \sigma(s,X_s^n,\mathbb{E}[X_s^n])dW_s \\ Y_t^{n+1} = & g(X_T^n,\mathbb{E}[X_T^n]) + \int_t^T f(s,X_s^n,\mathbb{E}[X_s^n],Y_s^n,\mathbb{E}[Y_s^n],Z_s^n,\mathbb{E}[Z_s^n])ds - \int_t^T Z_s^{n+1}dW_s. \end{cases} \tag{1.2}$$

And such that the stochastic integrals are well defined because it is clear by recurrence that for each $n$, $X_t^{n+1}$ continuous and adapted, therefore the process $\sigma(s,X_s^n,\mathbb{E}[X_s^n])$ is too.

<u>**Firstly**</u>

We show the existence of solution to the SDE in (1.1), for $t\in[0,T]$, first checking by induction on $n$ that there exists a constant $C_n$ such that for all $t \in [0,T]$.

$$\mathbb{E}\left[|X_t^n|^2\right] \leq C_n.$$

Assume that $\mathbb{E}\left[|X_t^n|^2\right] \leq C_n$ and we show that

$$\mathbb{E}\left[\left|X_t^{n+1}\right|^2\right] \leq C_{n+1.} \tag{1.3}$$

We have:

$$\left|X_t^{n+1}\right|^2 = \left|x + \int_0^t b(s, X_s^n, \mathbb{E}[X_s^n])ds + \int_0^t \sigma(s, X_s^n, \mathbb{E}[X_s^n])dW_s\right|^2.$$

As we have $(a+b+c)^2 \leq 3(a^2+b^2+c^2)$, by passing to hope, we obtain:

$$\begin{aligned}\mathbb{E}\left[\left|X_t^{n+1}\right|^2\right] &\leq 3(|x|^2 + \mathbb{E}[(\textstyle\int_0^t |b(s, X_s^n, \mathbb{E}[X_s^n])|\, ds)^2] \\ &+ \mathbb{E}[(\textstyle\int_0^t \|\sigma(s, X_s^n, \mathbb{E}[x_s^n])\|\, dW_s)^2]).\end{aligned} \tag{1.4}$$

Applying isometry theorem and linear growth, we find:

$$\begin{aligned}&\mathbb{E}\left[\left(\int_0^t \sigma(s, X_s^n, \mathbb{E}[X_s^n])dW_s\right)^2\right] \\ &= \mathbb{E}\left[\int_0^t \|\sigma(s, X_s^n, \mathbb{E}[X_s^n])\|^2\, ds\right] \\ &\leq \mathbb{E}\left[\int_0^t k^2(1 + \mid X_s^n \mid^2 + \mathbb{E}[\mid X_s^n \mid^2])ds\right] \\ &\leq \int_0^t k^2(1 + \mathbb{E}[\mid X_s^n \mid^2] + \mathbb{E}[\mathbb{E}[\mid X_s^n \mid^2]])ds \\ &\leq \int_0^t k^2(1 + 2\mathbb{E}[\mid X_s^n \mid^2])ds.\end{aligned} \tag{1.5}$$

And by the Cauchy-Schwarz inequality, we have:

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_0^t b(s, X_s^n, \mathbb{E}[X_s^n])ds\right)^2\right] \qquad (1.6)\\
&\leq \mathbb{E}\left[(\int_0^t ds) \times \left(\int_0^t \mid b(s, X_s^n, \mathbb{E}[X_s^n]) \mid^2\right) ds\right]\\
&\leq T.\mathbb{E}\left[\int_0^t k^2(1+ \mid X_s^n \mid^2 + \mathbb{E}[\mid X_s^n \mid^2])ds\right]
\end{aligned}$$

replace (1.5) and (1.6) in (1.4) we find:

$$\begin{aligned}
\mathbb{E}[\mid X_t^{n+1} \mid^2] &\leq 3(\mid x \mid^2 + T.\mathbb{E}[\textstyle\int_0^t k^2(1+ \mid X_s^n \mid^2\\
&+\mathbb{E}[\mid X_s^n \mid^2])ds] + \textstyle\int_0^t k^2(1 + 2\mathbb{E}[\mid X_s^n \mid^2])ds)\\
&\leq C + C\textstyle\int_0^t \mathbb{E}[\mid X_s^n \mid^2]ds \quad , \quad \forall t \in [0, T] \ , \ C > 0.
\end{aligned}$$

Proving (1.3)

We will increase by recurrence the quantity:

$$\mathbb{E}\left[\sup_{0\leq t\leq T} \mid X_t^{n+1} - X_t^n \mid^2\right].$$

Using Doob's inequality, we obtain:

$$\begin{aligned}
\mathbb{E}[\sup\nolimits_{0\leq s\leq t} |X_s^{n+1} - X_s^n|^2] &\leq 2\mathbb{E}\left[\left|\textstyle\int_0^t(\sigma(s, X_s^n, \mathbb{E}[X_s^n]) - \sigma(s, X_s^{n-1}, \mathbb{E}[X_s^{n-1}]))dW_s\right|^2\right]\\
&+2\mathbb{E}\left[\left|\textstyle\int_0^t(b(s, X_s^n, \mathbb{E}[X_s^n]) - b(s, X_s^{n-1}, \mathbb{E}[X_s^{n-1}]))ds\right|^2\right]\\
&\leq 2\mathbb{E}\left[\textstyle\int_0^t \|\sigma(s, X_s^n, \mathbb{E}[X_s^n]) - \sigma(s, X_s^{n-1}, \mathbb{E}[X_s^{n-1}])\|^2 ds\right]\\
&+ 2T.\mathbb{E}\left[\textstyle\int_0^t |b(s, X_s^n, \mathbb{E}[X_s^n]) - b(s, X_s^{n-1}, \mathbb{E}[X_s^{n-1}])|^2 ds\right].
\end{aligned}$$

As the functions $b$ and $\sigma$ are Lipshitzian, we obtain for $t \in [0, T]$

$$\mathbb{E}\left[\sup_{0\leq s\leq t} \mid X_s^{n+1} - X_s^n \mid^2\right] \leq 4(1 + T)k^2\mathbb{E}\left[\int_0^t \mid X_s^n - X_s^{n-1} \mid^2 ds\right].$$

So:

$$\mathbb{E}\left[\sup_{0\leq s\leq t} \mid X_s^{n+1} - X_s^n \mid^2\right] \leq C\int_0^t \mathbb{E}\left[\sup_{0\leq r\leq s} \mid X_r^n - X_r^{n-1} \mid^2\right] dr. \qquad (1.7)$$

Where $C = 4(1+T)k^2$.

We repeat the same method, applying Doob's inequality, to $\mid X_t^n - X_t^{n-1} \mid$ to get

$$\begin{aligned}\mid X_s^n - X_s^{n-1} \mid^2 &\leq 2\left|\int_0^s (\sigma(r, X_r^{n-1}, \mathbb{E}[X_r^{n-1}]) - \sigma(r, X_r^{n-2}, \mathbb{E}[X_r^{n-2}]))dW_r\right|^2 \\ &+2\left|\int_0^s (b(r, X_r^{n-1}, \mathbb{E}[X_r^{n-1}]) - b(r, X_r^{n-2}, \mathbb{E}[X_r^{n-2}]))dr\right|^2 \\ &\leq 2\int_0^s \| (\sigma(r, X_r^{n-1}, \mathbb{E}[X_r^{n-1}]) - \sigma(r, X_r^{n-2}, \mathbb{E}[X_r^{n-2}])) \|^2 dr \\ &+2T\left|\int_0^s (b(r, X_r^{n-1}, \mathbb{E}[X_r^{n-1}]) - b(r, X_r^{n-2}, \mathbb{E}[X_r^{n-2}]))\right|^2 dr.\end{aligned}$$

As the functions $b$ and $\sigma$ are Lipshitzian, therefore:

$$\begin{aligned}\mathbb{E}[\sup_{0\leq r\leq s} \mid X_r^n - X_r^{n-1} \mid^2] &\leq C\int_0^s \mathbb{E}[\mid X_r^{n-1} - X_r^{n-2} \mid^2]dr \\ &\leq C\int_0^s \mathbb{E}[\sup_{0\leq k\leq r} \mid X_k^{n-1} - X_k^{n-2} \mid^2]dr.\end{aligned} \qquad (1.8)$$

In the same way as (1.7) and (1.8), we can find:

$$\mathbb{E}[\sup_{0\leq k\leq r} \mid X_k^{n-1} - X_k^{n-2} \mid^2] \leq C\int_0^r \mathbb{E}[\sup_{0\leq l\leq k} \mid X_l^{n-2} - X_l^{n-3} \mid^2]dk. \qquad (1.9)$$

By replacing (1.8) and (1.9) to (1.6) we obtain:

$$\begin{aligned}\mathbb{E}\left[\sup_{0\leq s\leq t}\left|X_s^{n+1} - X_s^n\right|^2\right] &\leq C\int_0^t \mathbb{E}\left[\sup_{0\leq r\leq s}\left|X_r^n - X_r^{n-1}\right|^2\right] ds \\ &\leq C^2\int_0^t (\int_0^s \mathbb{E}\left[\sup_{0\leq k\leq r}\left|X_k^{n-1} - X_k^{n-2}\right|^2\right] dr)ds \\ &\leq C^3\int_0^t (\int_0^s (\int_0^r \mathbb{E}\left[\sup_{0\leq l\leq k}\left|X_l^{n-2} - X_l^{n-3}\right|^2\right] dk)dr)ds \\ &\leq C^3\mathbb{E}\left[\sup_{0\leq l\leq k}\left|X_l^{n-2} - X_l^{n-3}\right|^2\right]\int_0^t (\int_0^s (\int_0^r dk)dr)ds \\ &\leq C^3\mathbb{E}\left[\sup_{0\leq l\leq k}\left|X_l^{n-2} - X_l^{n-3}\right|^2\right]\int_0^t (\int_0^s rdr)ds \\ &\leq C^3\mathbb{E}\left[\sup_{0\leq l\leq k}\left|X_l^{n-2} - X_l^{n-3}\right|^2\right]\int_0^t \frac{s^2}{2}ds \\ &\leq \frac{C^3T^3}{3!}\mathbb{E}\left[\sup_{0\leq l\leq k}\left|X_l^{n-2} - X_l^{n-3}\right|^2\right].\end{aligned}$$

We repeat this method several times, we find:

$$\mathbb{E}[\sup_{0\leq s\leq T} \mid X_s^{n+1} - X_s^n \mid^2]$$
$$\leq \frac{C^nT^n}{n!}\mathbb{E}[\sup_{0\leq s\leq T} \mid X_s^1 - X_s^0 \mid^2] \leq D\frac{C^nT^n}{n!}$$

By applying Chebyshev's inequality, we have:

$$\mathbb{P}[\sup_{0\leq s\leq T} \mid X_s^{n+1} - X_s^n \mid\geq \frac{1}{2^{n+1}}]$$
$$\leq D\frac{C^nT^n}{n!}/(\frac{1}{2^{n+1}})^2 = 4D\frac{(4CT)^n}{n!}.$$

It turns out that:

$$\sum_{n=0}^{\infty} \mathbb{P}[\sup_{0\leq s\leq T} \mid X_s^{n+1} - X_s^n \mid\geq \frac{1}{2^{n+1}}] \leq \sum_{n=0}^{\infty} D\frac{(CT)^n}{n!}/(\frac{1}{2^{n+1}})^2$$
$$= 4D\sum_{n=0}^{\infty} \frac{(4TC)^n}{n!} = 4De^{4TC} < \infty.$$

What the Borel-Cantelli Lemma implies:

$$\mathbb{P}\left[\sup_{0\leq s\leq T} \mid X_s^{n+1} - X_s^n \mid> \frac{1}{2^{n+1}}, \forall n \in \mathbb{N}\right] = 0.$$

What does it mean that:

$$\mathbb{P}\left[\sup_{0\leq s\leq T} \mid X_s^{n+1} - X_s^n \mid\leq \frac{1}{2^{n+1}}, \forall n \in \mathbb{N}\right] = 1.$$

That is to say:

$$\sup_{0\leq s\leq T} \mid X_s^{n+1} - X_s^n \mid\leq \frac{1}{2^{n+1}}, \forall n \geq n_0, \text{for certain } n_0 \in \mathbb{N}.$$

With probability equal to 1. Moving on to the sum we find:

$$\sup_{0\leq s\leq T} \mid X_s^m - X_s^n \mid \leq \sum_{k=m\wedge n-1}^{m\vee n} \sup_{0\leq t\leq T} \mid X_s^{k+1} - X^k \mid$$
$$\leq \sum_{k=m\wedge n-1}^{\infty} \frac{1}{2^{k+1}} \leq \frac{1}{2^{m\wedge n}}.$$

For $m \wedge n \geq n_0(w)$; where $m \vee n = \max\{m, k\}$ . So the process $(X_n)_{n=1}^{\infty}$ is a Cauchy sequence, therefore convergent. Then there exists a continuous process $(X_t)_{t\in[0,T]}$

such as :

$$\sup_{0\leq t\leq T} \mid X_t^n - X_t \mid \longrightarrow 0,$$

when $n \longrightarrow \infty$, whith probability equal to 1.

So, $\mathbb{P} - p.s$, $X_n$ converges to a continuous process $X_t$. It is very easy to check that $X_t$ is a solution of the $SDE$ in (1.1) passing to the limit in the equation progressive in the system (1.2).

-So moving on to solve the second recurrence equation for $Y_n$.

Let us now prove that the sequence $(Y^n, Z^n)$ is a Cauchy sequence in the space of Banach $B^2$.

By applying the formula d'Itô to $e^{\alpha t} \mid Y_t^{n+1} - Y_t^n \mid^2$, we obtain:

$$\begin{aligned} d(e^{\alpha t} \mid Y_t^{n+1} - Y_t^n \mid^2) &= 2e^{\alpha t} \mid Y_t^{n+1} - Y_t^n \mid d \mid Y_t^{n+1} - Y_t^n \mid \\ &+ \alpha e^{\alpha t} \mid Y_t^{n+1} - Y_t^n \mid^2 \\ &+ e^{\alpha t} d\langle \mid Y_t^{n+1} - Y_t^n \mid, \mid Y^{n+1} - Y^n \mid\rangle_t, \end{aligned} \tag{1.10}$$

we have:

$$\begin{aligned} Y_t^{n+1} - Y_t^n = & g(X_T^n, \mathbb{E}[X_T^n]) - g(X_T^{n-1}, \mathbb{E}[X_T^{n-1}]) \\ & - \int_t^T (f(s, X_s^n, \mathbb{E}[X_s^n], Y_s^n, \mathbb{E}[Y_s^n], Z_s^n, \mathbb{E}[Z_s^n]) \\ & - f(s, X_s^{n-1}, \mathbb{E}[X_s^{n-1}], Y_s^{n-1}, \mathbb{E}[Y_s^{n-1}], Z_s^{n-1}\mathbb{E}[Z_s^{n-1}]))ds \\ & + \int_t^T (Z_s^{n+1} - Z_s^n) dW_s, \end{aligned} \tag{1.11}$$

and

$$\begin{aligned} d(Y_t^{n+1} - Y_t^n) = & -(f(s, X_s^n, \mathbb{E}[X_s^n], Y_s^n, \mathbb{E}[Y_s^n], Z_s^n, \mathbb{E}[Z_s^n]) \\ & - f(s, X_s^{n-1}, \mathbb{E}[X_s^{n-1}], Y_s^{n-1}, \mathbb{E}[Y_s^{n-1}], Z_s^{n-1}\mathbb{E}[Z_s^{n-1}])) \\ & + (Z_s^{n+1} - Z_s^n) dW_s \end{aligned} \tag{1.12}$$

By replacing (1.11) and (1.12) in (1.10) we find:

$$\begin{aligned} d(e^{\alpha t} \left|Y_t^{n+1} - Y_t^n\right|^2) = & -2e^{\alpha t} \left|Y_t^{n+1} - Y_t^n\right| (f(t, X_t^n, \mathbb{E}[X_t^n], Y_t^n, \mathbb{E}[Y_t^n], Z_t^n, \mathbb{E}[Z_t^n]) \\ & - f(t, X_t^{n-1}, \mathbb{E}[X_t^{n-1}], Y_t^{n-1}, \mathbb{E}[Y_t^{n-1}], Z_t^{n-1}\mathbb{E}[Z_t^{n-1}]))dt \\ & + 2e^{\alpha t} \left|Y_t^{n+1} - Y_t^n\right| (Z_t^{n+1} - Z_t^n) dW_t + \alpha e^{\alpha t} \left|Y_t^{n+1} - Y_t^n\right|^2 \\ & + e^{\alpha t} d\langle |Y^{n+1} - Y^n|, |Y^{n+1} - Y^n| \rangle_t. \end{aligned}$$

Passing to the integral between $t$ and $T$.

$$\begin{aligned} \textstyle\int_t^T d(e^{\alpha t} \left|Y_t^{n+1} - Y_t^n\right|^2) = & -2\langle \textstyle\int_t^T e^{\alpha s} \left|Y_s^{n+1} - Y_s^n\right|, (f(s, X_s^n, \mathbb{E}[X_s^n], Y_s^n, \mathbb{E}[Y_s^n], Z_s^n, \mathbb{E}[Z_s^n]) \\ & - f(s, X_s^{n-1}, \mathbb{E}[X_s^{n-1}], Y_s^{n-1}, \mathbb{E}[Y_s^{n-1}], Z_s^{n-1}\mathbb{E}[Z_s^{n-1}]))\rangle ds \\ & + 2 \textstyle\int_t^T \langle e^{\alpha s} \mid Y_s^{n+1} - Y_s^n \mid, (Z_s^{n+1} - Z_s^n) \rangle dW_s \\ & + \alpha \textstyle\int_t^T e^{\alpha s} \mid Y_s^{n+1} - Y_s^n \mid^2 ds + \textstyle\int_t^T e^{\alpha s} \parallel Z_s^{n+1} - Z_s^n \parallel^2 ds. \end{aligned}$$

By passing to hope and using the fact that $f$ is Lipshitzian, we have:

$$\mathbb{E}[e^{\alpha t}\left|Y_t^{n+1}-Y_t^n\right|^2]+\alpha\mathbb{E}[\int_t^T e^{\alpha s}\left|Y_s^{n+1}-Y_s^n\right|^2 ds]+\mathbb{E}[\int_t^T e^{\alpha s}\left\|Z_s^{n+1}-Z_s^n\right\|^2 ds]$$
$$\leq k\mathbb{E}[e^{\alpha T}\mid X_T^n-X_T^{n-1}\mid^2]$$
$$+2K.\mathbb{E}[\int_t^T e^{\alpha s}\left|Y_s^{n+1}-Y_s^n\right|(|X_s^n-X_s^{n-1}|+|\mathbb{E}[X_s^n-X_s^{n-1}]|+|Y_s^n-Y_s^{n-1}|$$
$$+|\mathbb{E}[Y_s^n-Y_s^{n-1}]|+\|Z_s^n-Z_s^{n-1}\|+\|\mathbb{E}[Z_s^n-Z_s^{n-1}]\|]).$$

Which implies (according to Yong's inequality $(2ab\leq\varepsilon^2a^2+\frac{1}{\varepsilon^2}b^2)$) that:

$$\mathbb{E}[e^{\alpha t}\mid Y_t^{n+1}-Y_t^n\mid^2]+\mathbb{E}\left[\int_t^T e^{\alpha s}\parallel Z_s^{n+1}-Z_s^n\parallel^2 ds\right]$$
$$\leq k.\mathbb{E}[e^{\alpha T}\left|X_T^n-X_T^{n-1}\right|^2]+(K^2\varepsilon^2-\alpha)\mathbb{E}\left[\int_t^T e^{\alpha s}\left|Y_s^{n+1}-Y_s^n\right|^2 ds\right]$$
$$+\frac{6}{\varepsilon^2}\mathbb{E}\left[\int_t^T e^{\alpha s}\left|X_s^n-X_s^{n-1}\right|^2 ds\right]+\frac{6}{\varepsilon^2}\mathbb{E}\left[\int_t^T e^{\alpha s}\left|Y_s^n-Y_s^{n-1}\right|^2 ds\right]$$
$$+\frac{6}{\varepsilon^2}\mathbb{E}\left[\int_t^T e^{\alpha s}\left\|Z_s^n-Z_s^{n-1}\right\|^2 ds\right]+\frac{6}{\varepsilon^2}\mathbb{E}\left[\int_t^T e^{\alpha s}\mathbb{E}[|X_s^n-X_s^{n-1}|^2]ds\right]$$
$$+\frac{6}{\varepsilon^2}\mathbb{E}\left[\int_t^T e^{\alpha s}\mathbb{E}[|Y_s^n-Y_s^{n-1}|^2]ds\right]+\frac{6}{\varepsilon^2}\mathbb{E}\left[\int_t^T e^{\alpha s}\mathbb{E}[\|Z_s^n-Z_s^{n-1}\|^2]\right]ds,$$

then according to Fubini's theorem:

$$\mathbb{E}[e^{\alpha t}\left|Y_t^{n+1}-Y_t^n\right|^2]+\mathbb{E}\left[\int_t^T e^{\alpha s}\parallel Z_s^{n+1}-Z_s^n\parallel^2 ds\right]$$
$$\leq k.\mathbb{E}[e^{\alpha T}\left|X_T^n-X_T^{n-1}\right|^2]+(k^2\varepsilon^2-\alpha)\mathbb{E}[\int_t^T e^{\alpha s}\left|Y_s^{n+1}-Y_s^n\right|^2 ds]$$
$$+\frac{12}{\varepsilon^2}\mathbb{E}\left[\int_t^T e^{\alpha s}\left|X_s^n-X_s^{n-1}\right|^2 ds\right]+\frac{12}{\varepsilon^2}\mathbb{E}\left[\int_t^T e^{\alpha s}\left|Y_s^n-Y_s^{n-1}\right|^2 ds\right]$$
$$+\frac{12}{\varepsilon^2}\mathbb{E}\left[\int_t^T e^{\alpha s}\mathbb{E}[\|Z_s^n-Z_s^{n-1}\|^2]ds\right].$$

We choose $\alpha$ and $\varepsilon$ such that $\frac{12}{\varepsilon^2}=\frac{1}{12}$ and $144k^2-\alpha=0$, so:

$$\mathbb{E}[e^{\alpha t}\mid Y_t^{n+1}-Y_t^n\mid^2]+\mathbb{E}[\int_t^T e^{\alpha s}\parallel Z_s^{n+1}-Z_s^n\parallel^2 ds]$$
$$\leq k\mathbb{E}[e^{\alpha T}\left|X_T^n-X_T^{n-1}\right|^2]+\frac{1}{12}\mathbb{E}[\int_t^T e^{\alpha s}(|X_s^n-X_s^{n-1}|^2$$
$$+\mid Y_s^n-Y_s^{n-1}\mid^2+\parallel Z_s^n-Z_s^{n-1}\parallel^2)ds].$$

So for $t = 0$, find:

$$\begin{aligned}
&\mathbb{E}[\sup_{0\leq t\leq T} e^{\alpha t} \mid Y_t^{n+1} - Y_t^n \mid^2] + \mathbb{E}[\int_0^T e^{\alpha s} \parallel Z_s^{n+1} - Z_s^n \parallel^2 ds] \\
&\leq k\mathbb{E}[e^{\alpha T} \mid X_T^n - X_T^{n-1} \mid^2] + \tfrac{C}{12}(\mathbb{E}[\sup_{0\leq t\leq T} e^{\alpha t}(\mid X_t^n - X_t^{n-1} \mid^2)] \\
&+\mathbb{E}[\sup_{0\leq t\leq T} e^{\alpha t} \mid Y_t^n - Y_t^{n-1} \mid^2] + \mathbb{E}[\int_0^T e^{\alpha t} \parallel Z_s^n - Z_s^{n-1} \parallel^2 ds]).
\end{aligned} \tag{1.13}$$

We repeat the same method, applying the formula d'Itô to $e^{\alpha t} \mid Y_t^n - Y_t^{n-1} \mid$, to get :

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\leq t\leq T} e^{\alpha t} \left|Y_t^n - Y_t^{n-1}\right|^2\right] + \mathbb{E}\left[\int_0^T e^{\alpha s} \parallel Z_s^n - Z_s^{n-1} \parallel^2 ds\right] \\
&\leq k'\mathbb{E}\left[e^{\alpha T} \left|X_T^{n-1} - X_T^{n-2}\right|^2\right] + \tfrac{C'}{12}(\mathbb{E}\left[\sup_{0\leq t\leq T} e^{\alpha t}(\left|X_t^{n-1} - X_t^{n-2}\right|^2\right]] \\
&+ \mathbb{E}\left[\sup_{0\leq t\leq T} e^{\alpha t} \left|Y_t^{n-1} - Y_t^{n-2}\right|^2\right] + \mathbb{E}\left[\int_0^T e^{\alpha s} \|Z_s^{n-1} - Z_s^{n-2}\|^2 ds\right]).
\end{aligned} \tag{1.14}$$

By replacing (1.14) in (1.13) we have:

$$\begin{aligned}
&\mathbb{E}[\sup_{0\leq t\leq T} e^{\alpha t} \left|Y_t^{n+1} - Y_t^n\right|^2] + \mathbb{E}[\int_0^T e^{\alpha s} \left\|Z_s^{n+1} - Z_s^n\right\|^2 ds] \\
&\leq k.\mathbb{E}[e^{\alpha T} \left|X_T^n - X_T^{n-1}\right|^2] + k'.\mathbb{E}[e^{\alpha T}(\left|X_T^{n-1} - X_T^{n-2}\right|^2)] \\
&+ \tfrac{C}{12}\mathbb{E}\left[\sup_{0\leq t\leq T} e^{\alpha t}(\left|X_t^n - X_t^{n-1}\right|^2)\right] + \tfrac{C'}{12}\mathbb{E}\left[\sup_{0\leq t\leq T} e^{\alpha t}(\left|X_t^{n-1} - X_t^{n-2}\right|^2)\right] \\
&+ \tfrac{C'}{12^2}\left(\mathbb{E}\left[\sup_{0\leq t\leq T} e^{\alpha t} \left|Y_t^{n-1} - Y_t^{n-2}\right|^2\right] + \mathbb{E}\left[\int_0^T e^{\alpha s} \parallel Z_s^{n-1} - Z_s^{n-2} \parallel^2 ds\right]\right).
\end{aligned}$$

We repeat this method several times, we find:

$$\begin{aligned}
&\mathbb{E}[\sup_{0\leq t\leq T} e^{\alpha t} \mid Y_t^{n+1} - Y_t^n \mid^2] + \mathbb{E}[\int_0^T e^{\alpha s} \parallel Z_s^{n+1} - Z_s^n \parallel^2 ds] \\
&\leq C(\mathbb{E}[e^{\alpha T} \left|X_T^n - X_T^{n-1}\right|^2] + \mathbb{E}[e^{\alpha T} \left|X_T^{n-1} - X_T^{n-2}\right|^2] + ... \\
&... + \mathbb{E}[e^{\alpha t}(|X_T^1 - X_T^0|^2)] + C'(\mathbb{E}[\sup_{0\leq t\leq T} e^{\alpha t}(\left|X_t^n - X_t^{n-1}\right|^2)]) \\
&+ \mathbb{E}[\sup_{0\leq t\leq T} e^{\alpha t}(\left|X_t^{n-1} - X_t^{n-2}\right|^2)] + ... + \mathbb{E}[\sup_{0\leq t\leq T} e^{\alpha t}(|X_t^1 - X_t^0|^2)]) \\
&+ \tfrac{C''}{12^n}(\mathbb{E}[\sup_{0\leq t\leq T} e^{\alpha t} \left|Y_t^n - Y_t^{n-1}\right|^2] + \mathbb{E}[\sup_{0\leq t\leq T} e^{\alpha t} \left|Y_t^{n-1} - Y_t^{n-2}\right|^2] \\
&+ ... + \mathbb{E}[\sup_{0\leq t\leq T} e^{\alpha t} |Y_t^1 - Y_t^0|^2] + \mathbb{E}[\int_0^T e^{\alpha s} \|Z_s^n - Z_s^{n-1}\|^2 ds] \\
&+ \mathbb{E}\left[\int_0^T e^{\alpha s} \|Z_s^{n-1} - Z_s^{n-2}\|^2 ds\right] + .... + \mathbb{E}\left[\int_0^T e^{\alpha s} \|Z_s^1 - Z_s^0\|^2 ds\right]),
\end{aligned}$$

so:

$$\mathbb{E}[\sup_{0\leq t\leq T} e^{\alpha t} \mid Y_t^{n+1} - Y_t^n \mid^2] + \mathbb{E}[\int_0^T e^{\alpha s} \parallel Z_s^{n+1} - Z_s^n \parallel^2 ds]$$
$$\leq \frac{D}{12^n} \longrightarrow 0 \text{ as } n \longrightarrow \infty.$$

Consequently, $(X^n, Y^n, Z^n)_{n\in\mathbb{N}}$ is a Cauchy sequence, therefore convergent. So, there is a triple of a stochastic process $(X_t, Y_t, Z_t) \in B^2$, such as:

$$\lim_{n\longrightarrow+\infty} \mathbb{E}\left[\sup_{0\leq t\leq T} \mid X_t^n - X_t \mid^2\right] = 0,$$
$$\lim_{n\longrightarrow+\infty} \mathbb{E}\left[\sup_{0\leq t\leq T} \mid Y_t^n - Y_t \mid^2\right] = 0$$

and

$$\lim_{n\longrightarrow+\infty} \mathbb{E}\left[\int_0^T \parallel Z_s^{n+1} - Z_s^n \parallel^2 dt\right] = 0,$$

with probability equal to 1. That's to say :

$$\lim_{n\to+\infty} X^n = X, \lim_{n\to+\infty} Y^n = Y, \lim_{n\to+\infty} Z^n = Z.$$

It is easy to verify that $(X, Y, Z)$ is a solution of FBSDE (1.1) just do a passage to the limit in the mean field type FBSDE (1.2).

**2 Uniqueness:** Suppose that $(X, Y, Z)$ and $(\hat{X}, \hat{Y}, \hat{Z})$ are two solutions of (1.1) for all $t \in [0, T]$. Since $(a + b)^2 \leq 2a^2 + 2b^2$, then

$$\mathbb{E}\left[\mid X_t - \hat{X}_t \mid^2\right] \leq 2\mathbb{E}\left[\left|\int_0^t \left(b(s, X_s, \mathbb{E}[X_s]) - b(s, \hat{X}_s, \mathbb{E}[\hat{X}_s])\right) ds\right|^2\right]$$
$$+ 2\mathbb{E}\left[\left|\int_0^t \left(\sigma(s, X_s, \mathbb{E}[X_s]) - \sigma(s, \hat{X}_s, \mathbb{E}[\hat{X}_s])\right) dW_s\right|^2\right].$$

According to the inequality of Cauchy Schwarz and by the isometry of d'itô we have:

$$\begin{aligned}\mathbb{E}\left[\left|X_t-\hat{X}_t\right|^2\right] &\leq 2T.K^2\int_0^t\mathbb{E}\left[\left|X_s-\hat{X}_s\right|^2\right]ds+2K^2\int_0^t\mathbb{E}\left[\left|X_s-\hat{X}_s\right|^2\right]ds\\ &=(2T.K^2+2K^2)\int_0^t\mathbb{E}[\mid X_s-\hat{X}_s\mid^2]ds\\ &=C\int_0^t\mathbb{E}[\mid X_s-\hat{X}_s\mid^2]ds,\end{aligned} \tag{1.15}$$

where $C=(2T.K^2+2K^2)$, either:

$$\phi(t)=\mathbb{E}[\mid X_t-\hat{X}_s\mid^2]\ ,\phi(t)\leq C\int_0^t\phi(s)ds\ ,\ \forall t\in[0,T].$$

Using Granwall's lemma, with $C_0=0$ implies $\phi=0$, we find:

$$\mathbb{E}\left[\mid X_s-\hat{X}_s\mid^2\right]=0.$$

By applying the formula d'Itô to $\mid Y_t-\hat{Y}_t\mid^2$, we find:

$$d(\mid Y_t-\hat{Y}_t\mid^2)=2\mid Y_t-\hat{Y}_t\mid d(\mid Y_t-\hat{Y}_t\mid)+d\langle Y-\hat{Y},Y-\hat{Y}\rangle_t.$$

By passing to the integral from $t$ to $T$ and expectation, we have:

$$\begin{aligned}&\mathbb{E}\left[\mid Y_t-\hat{Y}_t\mid^2\right]+\mathbb{E}\left[\int_t^T\parallel Z_s-\hat{Z}_s\parallel^2ds\right]\\ &=\mathbb{E}\left[\left|g(X_T,\mathbb{E}[X_T])-g(\hat{X}_T,\mathbb{E}[\hat{X}_T])\right|^2\right]\\ &+2\mathbb{E}\left[\int_t^T\Big\langle Y_s-\hat{Y}_s,f(s,X_s,\mathbb{E}[X_s],Y_s,\mathbb{E}[Y_s],Z_s,\mathbb{E}[Z_s])\right.\\ &-\left.f(s,\hat{X}_s,\mathbb{E}[\hat{X}_s],\hat{Y}_s,\mathbb{E}[\hat{Y}_s],\hat{Z}_s,\mathbb{E}[\hat{Z}_s])\Big\rangle\right].\end{aligned}$$

So:

$$\begin{aligned}
&\mathbb{E}\left[\mid Y_t-\hat{Y}_t\mid^2\right]+\mathbb{E}\left[\int_t^T \parallel Z_s-\hat{Z}_s\parallel^2 ds\right] \\
&\quad\leq k^2\mathbb{E}\left[\mid X_T-\hat{X}_T\mid\right]+2K.\mathbb{E}\left[\int_t^T \mid Y_s-\hat{Y}_s\mid (\mid X_s-\hat{X}_s\mid \right.\\
&\qquad +\mid \mathbb{E}[X_s-\hat{X}_s]\mid + \mid Y_s-\hat{Y}_s\mid + \mid \mathbb{E}[Y_s-\hat{Y}_s]\mid \\
&\qquad \left. +\mid Z_s-\hat{Z}_s\mid + \mid \mathbb{E}[Z_s-\hat{Z}_s]\mid )\right].
\end{aligned}$$

By the Yong's inequality we have:

$$\begin{aligned}
&\mathbb{E}\left[\mid Y_t-\hat{Y}_t\mid^2\right]+\mathbb{E}\left[\int_t^T \parallel Z_s-\hat{Z}_s\parallel^2 ds\right] \\
&\leq K^2\mathbb{E}\left[\mid X_T-\hat{X}_T\mid^2\right]+2K^2\varepsilon^2.\mathbb{E}\left[\int_t^T \mid Y_s-\hat{Y}_s\mid^2 ds\right] \\
&+\tfrac{6}{\varepsilon^2}\mathbb{E}\left[\int_t^T \mid X_s-\hat{X}_s\mid^2 ds\right]+\tfrac{6}{\varepsilon^2}\mathbb{E}\left[\int_t^T \mid Y_s-\hat{Y}_s\mid^2 ds\right] \\
&\quad +\tfrac{6}{\varepsilon^2}\mathbb{E}\left[\int_t^T \parallel Z_s-\hat{Z}_s\parallel^2 ds\right].
\end{aligned}$$

We pose $\frac{6}{\varepsilon^2}=\frac{1}{2}$, then :

$$\begin{aligned}
&\mathbb{E}\left[\mid Y_t-\hat{Y}_t\mid^2\right]+\tfrac{1}{2}\mathbb{E}\left[\int_t^T \parallel Z_s-\hat{Z}_s\parallel^2 ds\right] \\
&\leq k^2\mathbb{E}\left[\mid X_T-\hat{X}_T\mid^2\right]+(24K^2+\tfrac{1}{2})\mathbb{E}\left[\int_t^T \mid Y_s-\hat{Y}_s\mid^2 ds\right] \\
&+\tfrac{1}{2}\mathbb{E}\left[\int_t^T \mid X_s-\hat{X}_s\mid^2 ds\right].
\end{aligned}$$

By inequality (1.3) we have:

$$\mathbb{E}\left[\mid X_s-\hat{X}_s\mid^2\right]=0.$$

Then $X_s=\hat{X}_s$ and $X_T=\hat{X}_T$, so:

$$\begin{aligned}
&\mathbb{E}\left[\mid Y_t-\hat{Y}_t\mid^2\right]+\tfrac{1}{2}\mathbb{E}\left[\int_t^T \parallel Z_s-\hat{Z}_s\parallel^2 ds\right] \\
&\leq C\mathbb{E}\left[\int_t^T \mid Y_t-\hat{Y}_t\mid^2 ds\right],
\end{aligned}$$

with $C=24K^2+\frac{1}{2}$.

We can extract two inequalities:

$$\mathbb{E}\left[\sup_{0\leq t\leq T} \mid Y_t - \hat{Y}_t \mid^2\right] \leq C\mathbb{E}\left[\int_t^T \mid Y_s - \hat{Y}_s \mid^2 ds\right], \tag{1.16}$$

and

$$\frac{1}{2}\mathbb{E}\left[\int_t^T \parallel Z_s - \hat{Z}_s \parallel^2 ds\right] \leq C\mathbb{E}\left[\int_t^T \mid Y_s - \hat{Y}_s \mid^2 ds\right]. \tag{1.17}$$

According to Granwall's lemma at (1.16), (we have $b = 0$), so:

$$\mathbb{E}\left[\sup_{0\leq t\leq T} \mid Y_t - \hat{Y}_t \mid^2\right] = 0. \tag{1.18}$$

Replacing (1.18) in (1.17) a gives us :

$$\mathbb{E}\left[\int_t^T \parallel Z_s - \hat{Z}_s \parallel^2 ds\right] = 0.$$

Therefore $Y_t \equiv \hat{Y}_t$, $Z_s \equiv \hat{Z}_s$. which proves uniqueness.

# Chapter 2

# Necessary optimality conditions for relaxed control problems for nonlinear MF-FBSDE

In this section, we establish necessary optimality conditions for relaxed control problems driven by systems of nonlinear MF-FBSDEs. This chapter is inspired from [2], [4] and [5].

## 2.1 Necessary optimality conditions for relaxed control problems

### 2.1.1 Relaxed control

Let $V$ be the set of Radon measure on $[0,T] \times U$ whose projection on $[0,T]$ coincides with the Lebesgue measure $dt$, equipped with the topology of the stable convergence of the measure. The space $V$ is equipped with its Borelian tribe, which is the smallest tribe such that the map $q \to \int f(s,a)q(ds,da)$ is measurable for any measurable function $f$, bounded and continuous in $a$ .

A relaxed control $q$ is a random variable $q(w, dt, da)$ with value in $V$ such that for each $t, 1_{[0,t]}q$ is $\mathcal{F}_t$-measurable ($\mathcal{F}_t = \sigma(B_s, 0 \leq s \leq t)$). Any relaxed control can be integrated into $q(w, dt, da) = dtq(w; t; da)$ where $q(t, da)$ is a progressively measurable process with values in the space of probability measures.

**Definition 2.1.1** *A relaxed control $q$ is a random variable $q(w, dt; da)$ with value in $V$ such that for each $t, 1_{[0,t]}q$ is $\mathcal{F}_t$-measurable.*

We denote by $\mathcal{R}$ all the relaxed controls.

**Remark 2.1.1** *Let $P(U)$ denote the space of probability measures on $B(U)$ equipped with the topology of weak convergence, where $U$ is a nonempty Borel compact subset of $\mathbb{R}^k$. In a relaxed control problem, the $U$-valued process $v_t$ is replaced by an $P(U)$-valued process $q_t$. Moreover, if $q_t(du) = \delta_{v_t}(du)$ is a Dirac measure charging a strict control $v_t$ for each $t$, then we get that the strict control problem is a particular case of the relaxed one.*

To establish necessary optimality conditions for relaxed control, let us consider a relaxed control problem governed by the following MF-FBSDE:

$$\begin{cases} dX_t^\mu = \int_U b(t, X_t^\mu, \mathbb{E}[X_t^\mu], u)\mu_t(du)dt + \int_U \sigma(t, X_t^\mu, \mathbb{E}[X_t^\mu], u)\mu_t(du)dW_t \\ \qquad dY_t^\mu = - \int_U f(t, X_t^\mu, \mathbb{E}[X_t^\mu], Y_t^\mu, \mathbb{E}[Y_t^\mu], Z_t^\mu, \mathbb{E}[Z_t^\mu], u)\mu_t(du)dt \\ \qquad\qquad + Z_t^\mu dW_t \\ \qquad X_0^\mu = x, \quad Y_T^\mu = h(X_T^\mu, \mathbb{E}[X_T^\mu]), \quad t \in [0, T], \end{cases} \tag{2.1}$$

and the functional cost to be minimized over the set of relaxed controls $\mathcal{R}$, is given by:

$$J(\mu_{\cdot}) := \mathbb{E}[\alpha(X_T^{\mu}, \mathbb{E}[X_T^{\mu}]) + \beta(Y_0^{\mu}, \mathbb{E}[Y_0^{\mu}]) \\ + \int_0^T \int_U l(t, X_t^{\mu}, \mathbb{E}[X_t^{\mu}], Y_t^{\mu}, \mathbb{E}[Y_t^{\mu}], Z_t^{\mu}, \mathbb{E}[Z_t^{\mu}], u)\mu_t(du)dt]. \tag{2.2}$$

We say that a relaxed control $q_{\cdot}$ is an optimal control if:

$$J(q_{\cdot}) = \inf_{\mu_{\cdot} \in \mathcal{R}} J(\mu_{\cdot}). \tag{2.3}$$

According to the fact that the set of relaxed controls is convex, then to establish necessary optimality condition we use the convex perturbation method. Let $q$ be an optimal relaxed control with associated trajectories $(X_t^q, Y_t^q, Z_t^q)$ solution of the MF-FBSDEs (2.1). Then, we can define a perturbed relaxed control by:

$$q_t^{\varepsilon} = q_t + \varepsilon(\mu_t - q_t),$$

where $\varepsilon > 0$ is sufficiently small and $\mu_{\cdot}$ is an arbitrary element of $\mathcal{R}$. Denote by $(X_t^{\varepsilon}, Y_t^{\varepsilon}, Z_t^{\varepsilon})$ the solution of the system (2.1) corresponding to the perturbed relaxed control $q_{\cdot}^{\varepsilon}$.

We shall consider in this section the following assumptions.

- (H1) (Regularity conditions)

$$\begin{cases} (i) \text{ the mappings } b, h, \sigma, \alpha \text{ are bounded and continuously differentiable with} \\ \text{respect to} (x, x^{'}), \text{ and the functions } f \text{ and } \beta \text{ are bounded and continuously} \\ \text{differentiable with respect to } (X, X^{'}, Y, Y^{'}, Z, Z^{'}) \text{ and } (X, X^{'}), \text{ respectively,} \\ (ii) \text{ the derivatives of } b, h, \sigma, f \text{ with respect to the above arguments are} \\ \text{continuous and bounded} \\ (iii) \text{ the derivatives of } l \text{ are bounded by } C(1+ \mid X \mid + \mid X' \mid + \mid Y \mid + \mid Y' \mid \\ + \mid Z \mid + \mid Z' \mid), \\ (iv) \text{ the derivatives of } \alpha \text{ and } \beta \text{ are bounded by } C(1+ \mid X \mid + \mid X' \mid) \text{ and} \\ C(1+ \mid Y \mid + \mid Y' \mid) \text{ respectively,} \end{cases}$$

for some positive constant $C$.

### 2.1.2 The variational inequality

Using the optimality of $q.$, the variational inequality will be derived from the following inequality

$$0 \leq J(q^{\varepsilon}) - J(q).$$

For this end, we need some results.

**Proposition 2.1.1** *Under assumptions **(H1)**, we have:*

$$\lim_{\varepsilon \to 0} \mathbb{E}\left[\sup_{0 \leq t \leq T} \mid X_t^{\varepsilon} - X_t^q \mid^2\right] = 0, \tag{2.4}$$

$$\lim_{\varepsilon \to 0} \mathbb{E}\left[\sup_{0 \leq t \leq T} \mid Y_t^{\varepsilon} - Y_t^q \mid^2\right] = 0, \tag{2.5}$$

$$\lim_{\varepsilon \to 0} \mathbb{E}\left[\int_0^T \parallel Z_t^{\varepsilon} - Z_t^q \parallel^2 dt\right] = 0, \tag{2.6}$$

**Proof:** We calculate $\mathbb{E}\left[\mid X_t^{\varepsilon}-X_t^{q}\mid^2\right]$ and using the definition of $q_t^{\varepsilon}$ to get:

$$\begin{aligned}\mathbb{E}[\mid X_t^{\varepsilon}-X_t^{q}\mid^2] \leq C\mathbb{E}\Big[\textstyle\int_0^t \mid \int_U b(s,X_s^{\varepsilon},\mathbb{E}[X_s^{\varepsilon}],u)q_s(du)\\ -\textstyle\int_U b(s,X_s^{q},\mathbb{E}[X_s^{q}],u)q_s(du)\mid^2 ds\Big]\\ +C.\varepsilon^2\mathbb{E}\Big[\textstyle\int_0^t \left|\int_U b(s,X_s^{\varepsilon},\mathbb{E}[X_s^{\varepsilon}],u)\left(\mu_s(du)-q_s(du)\right)\right|^2 ds\Big]\\ +C\mathbb{E}\textstyle\int_0^t \mid \int_U \sigma(s,X_s^{\varepsilon},\mathbb{E}[X_s^{\varepsilon}],u)q_s(du\\ -\textstyle\int_U \sigma(s,X_s^{q},\mathbb{E}[X_s^{q}],u)q_s(du)\mid^2 ds\Big]\\ +C.\varepsilon^2\mathbb{E}\Big[\textstyle\int_0^t \left|\int_U \sigma(s,X_s^{\varepsilon},\mathbb{E}[X_s^{\varepsilon}],u)\left(\mu_s(du)-q_s(du)\right)\right|^2 ds\Big].\end{aligned}$$

Since $b$ and $\sigma$ are uniformly Lipschitz and bounded, we can show

$$\mathbb{E}[\mid X_t^{\varepsilon}-X_t^{q}\mid^2] \leq C\mathbb{E}\left[\int_0^t \mid X_s^{\varepsilon}-X_s^{q}\mid^2 ds\right]+C\varepsilon^2.$$

Applying Granwall's lemma and Burkholder-Davis-Gundy inequality, we get (2.4).

On the other hand, applying Itô's formula to $(Y_t^{\varepsilon}-Y_t^{q})^2$, taking expectation and applying Young's inequality, to obtain

$$\begin{aligned}&\mathbb{E}\left[|Y_t^{\varepsilon}-Y_t^{q}|^2\right]+\mathbb{E}\left[\textstyle\int_t^T \parallel Z_s^{\varepsilon}-Z_s^{q}\parallel^2 ds\right]\\ &\leq \mathbb{E}\left[|h(X_T^{\varepsilon},\mathbb{E}[X_T^{\varepsilon}])-h(X_T^{q},\mathbb{E}[X_T^{q}])|^2\right]+\tfrac{1}{\theta}\mathbb{E}\left[\textstyle\int_t^T |Y_s^{\varepsilon}-Y_s^{q}|^2 ds\right]\\ &+\theta\mathbb{E}\Big[\textstyle\int_t^T \Big|\int_U f(s,X_s^{\varepsilon},\mathbb{E}[X_s^{\varepsilon}],Y_s^{\varepsilon},\mathbb{E}[Y_s^{\varepsilon}],Z_s^{\varepsilon},\mathbb{E}[Z_s^{\varepsilon}],u)q_s^{\varepsilon}(du)\\ &-\textstyle\int_U f(s,X_s^{q},\mathbb{E}[X_s^{q}],Y_s^{q},\mathbb{E}[Y_s^{q}],Z_s^{q},\mathbb{E}[Z_s^{q}],u)q_s(du)\Big|^2 ds\Big].\end{aligned}$$

Then,

$$
\begin{aligned}
&\mathbb{E}[| Y_t^\varepsilon - Y_t^q |^2] + \mathbb{E}[\textstyle\int_t^T \| Z_s^\varepsilon - Z_s^q \|^2 \, ds] \\
\leq{}& \mathbb{E}\left[|h(X_T^\varepsilon, \mathbb{E}[X_T^\varepsilon]) - h(X_T^q, \mathbb{E}[X_T^q])|^2\right] + \tfrac{1}{\theta}\mathbb{E}\left[\textstyle\int_t^T |Y_s^\varepsilon - Y_s^q|^2 \, ds\right] \\
&+ C\theta\mathbb{E}\Big[\textstyle\int_t^T \Big|\int_U f(s, X_s^\varepsilon, \mathbb{E}[X_s^\varepsilon], Y_s^\varepsilon, \mathbb{E}[Y_s^\varepsilon], Z_s^\varepsilon, \mathbb{E}[Z_s^\varepsilon], u) q_s^\varepsilon(du) \\
&- \textstyle\int_U f(s, X_s^\varepsilon, \mathbb{E}[X_s^\varepsilon], Y_s^\varepsilon, \mathbb{E}[Y_s^\varepsilon], Z_s^\varepsilon, \mathbb{E}[Z_s^\varepsilon], u) q_s(du)\Big|^2 ds\Big] \\
&+ C\theta\mathbb{E}\Big[\textstyle\int_t^T \Big|\int_U f(s, X_s^\varepsilon, \mathbb{E}[X_s^\varepsilon], Y_s^\varepsilon, \mathbb{E}[Y_s^\varepsilon], Z_s^\varepsilon, \mathbb{E}[Z_s^\varepsilon], u) q_s(du) \\
&- \textstyle\int_U f(s, X_s^q, \mathbb{E}[X_s^q], Y_s^q, \mathbb{E}[Y_s^q], Z_s^q, \mathbb{E}[Z_s^q], u) q_s(du)\Big|^2 ds\Big].
\end{aligned}
$$

Using the definition of $q_t^\varepsilon$, we obtain:

$$
\begin{aligned}
&\mathbb{E}[| Y_t^\varepsilon - Y_t^q |^2] + \mathbb{E}\left[\textstyle\int_t^T \| Z_s^\varepsilon - Z_s^q \|^2 \, ds\right] \\
\leq{}& \mathbb{E}[|h(X_T^\varepsilon, \mathbb{E}[X_T^\varepsilon]) - h(X_T^q, \mathbb{E}[X_T^q])|^2] \\
&+ \tfrac{1}{\theta}\mathbb{E}\left[\textstyle\int_t^T |Y_s^\varepsilon - Y_s^q |^2 \, ds\right] \\
&+ C\theta\varepsilon^2\mathbb{E}\Big[\textstyle\int_t^T \Big|\int_U f(s, X_s^\varepsilon, \mathbb{E}[X_s^\varepsilon], Y_s^\varepsilon, \mathbb{E}[Y_s^\varepsilon], Z_s^\varepsilon, \mathbb{E}[Z_s^\varepsilon], u) \mu_s(du) \\
&- \textstyle\int_U f(s, X_s^\varepsilon, \mathbb{E}[X_s^\varepsilon], Y_s^\varepsilon, \mathbb{E}[Y_s^\varepsilon], Z_s^\varepsilon, \mathbb{E}[Z_s^\varepsilon], u) q_s(du)\Big|^2 ds\Big] \\
&+ C\theta\mathbb{E}\Big[\textstyle\int_t^T \Big|\int_U f(s, X_s^\varepsilon, \mathbb{E}[X_s^\varepsilon], Y_s^\varepsilon, \mathbb{E}[Y_s^\varepsilon], Z_s^\varepsilon, \mathbb{E}[Z_s^\varepsilon], u) q_s(du) \\
&- \textstyle\int_U f(s, X_s^q, \mathbb{E}[X_s^q], Y_s^q, \mathbb{E}[Y_s^q], Z_s^q, \mathbb{E}[Z_s^q], u) q_s(du)\Big|^2 ds\Big].
\end{aligned}
$$

Since $f$ and $h$ are uniformly Lipschitz with respect to their arguments, we have:

$$
\begin{aligned}
&\mathbb{E}[| Y_t^\varepsilon - Y_t^q |^2] + \mathbb{E}\left[\textstyle\int_t^T \| Z_s^\varepsilon - Z_s^q \|^2 \, ds\right] \\
\leq{}& (\tfrac{1}{\theta} + C\theta)\mathbb{E}\left[\textstyle\int_t^T | Y_s^\varepsilon - Y_s^q |^2 \, ds\right] \\
&+ C\theta\mathbb{E}\left[\textstyle\int_t^T \| Z_s^\varepsilon - Z_s^q \|^2 \, ds\right] + \phi_t^\varepsilon,
\end{aligned}
\tag{2.7}
$$

where:

$$
\phi_t^\varepsilon = \mathbb{E}[|X_T^\varepsilon - X_T^q |^2] + C\theta\mathbb{E}\left[\int_t^T |X_s^\varepsilon - X_s^q|^2 \, ds\right] + C\theta\varepsilon^2.
$$

From (2.4) we can show that

$$\lim_{\varepsilon \longrightarrow 0} \phi_t^\varepsilon = 0. \tag{2.8}$$

Choose $\theta = \frac{1}{2C} > 0$, then the inequality (2.7) becomes

$$\begin{aligned} &\mathbb{E}[\mid Y_t^\varepsilon - Y_t^q \mid^2] + \tfrac{1}{2}\mathbb{E}\left[\int_t^T \parallel Z_s^\varepsilon - Z_s^q \parallel^2 ds\right] \\ &\leq (2C + \tfrac{1}{2})\mathbb{E}\left[\int_t^T \mid Y_s^\varepsilon - Y_s^q \mid^2 ds\right] + \phi_t^\varepsilon, \end{aligned}$$

we derive from this inequality, two inequalities

$$\mathbb{E}[\mid Y_t^\varepsilon - Y_t^q \mid^2] \leq (2C + \frac{1}{2})\mathbb{E}\left[\int_t^T \mid Y_s^\varepsilon - Y_s^q \mid^2 ds\right] + \phi_t^\varepsilon, \tag{2.9}$$

and

$$\mathbb{E}\left[\int_t^T \parallel Z_s^\varepsilon - Z_s^q \parallel^2 ds\right] \leq (4C + 1)\mathbb{E}\left[\int_t^T \mid Y_s^\varepsilon - Y_s^q \mid^2 ds\right] + 2\phi_t^\varepsilon. \tag{2.10}$$

Applying Granwall's lemma and Burkholder-Davis-Gundy inequality in (2.9) and using (2.4) and (2.8) to get (2.5). Finally (2.6) derived from (2.5), (2.8) and (2.10).

### 2.1.3 Variational equations

**Proposition 2.1.2** *Let $(\hat{X}_t, \hat{Y}_t, \hat{Z}_t)$, be the solution of the following variational equations of MF-FBSDE* (2.1)

$$\begin{cases} d\hat{X}_t = \int_U b_X(t, X_t^q, \mathbb{E}[X_t^q], u)q_t(du)\hat{X}_t dt + \mathbb{E}[\int_U b_{X'}(t, X_t^q, \mathbb{E}[X_t^q], u)q_t(du)\mathbb{E}[\hat{X}_t]]dt \\ \quad + \int_U(\sigma_X(t, X_t^q, \mathbb{E}[X_t^q], u)\hat{X}_t + \mathbb{E}[\sigma_{X'}(t, X_t^q, \mathbb{E}[X_t^q], u)\mathbb{E}[\hat{X}_t]])q_t(du)dW_t \\ \qquad +(\int_U b(t, X_t^q, \mathbb{E}[X_t^q], u)q_t(du) - \int_U b(t, X_t^q, \mathbb{E}[X_t^q], u)\mu_t(du))dt \\ \qquad +(\int_U \sigma(t, X_t^q, \mathbb{E}[X_t^q], u)q_t(du) - \int_U \sigma(t, X_t^q, \mathbb{E}[X_t^q], u)\mu_t(du))dW_t \\ d\hat{Y}_t = -(\int_U f_X(t, \pi_t^q, u)q_t(du)\hat{X}_t + \mathbb{E}[\int_U f_{X'}(t, \pi_t^q, u)q_t(du)\mathbb{E}[\hat{X}_t]] \\ \qquad + \int_U f_Y(t, \pi_t^q, u)q_t(du)\hat{Y}_t + \mathbb{E}[\int_U f_{Y'}(t, \pi_t^q, u)q_t(du)]\mathbb{E}[\hat{Y}_t] \\ \qquad + \int_U f_Z(t, \pi_t^q, u)q_t(du)\hat{Z}_t + \mathbb{E}[\int_U f_{Z'}(t, \pi_t^q, u)q_t(du)\mathbb{E}[\hat{Z}_t]] \\ \qquad +(\int_U f(t, \pi_t^q, u)q_t(du) - \int_U f(t, \pi_t^q, u)\mu_t(du)))dt + \hat{Z}_t dW_t, \\ \hat{X}_0 = 0, \; \hat{Y}_T = h_X(X_T^q, \mathbb{E}[X_T^q])\hat{X}_T + \mathbb{E}[h_{X'}(X_T^q, \mathbb{E}[X_T^q])\mathbb{E}[\hat{X}_T]], \end{cases} \tag{2.11}$$

where $(t, \pi_t^q, u) := (t, X_t^q, \mathbb{E}[X_t^q], Y_t^q, \mathbb{E}[Y_t^q], Z_t^q, \mathbb{E}[Z_t^q], u)$.

We have the following estimates:

$$\lim_{\varepsilon\to 0} \mathbb{E}\left[\sup_{0\le t\le T} \mid \frac{1}{\varepsilon}(X_t^\varepsilon - X_t^q) - \hat{X}_t \mid^2\right] = 0, \tag{2.12}$$

$$\lim_{\varepsilon\to 0} \mathbb{E}\left[\sup_{0\le t\le T} \mid \frac{1}{\varepsilon}(Y_t^\varepsilon - Y_t^q) - \hat{Y}_t \mid^2\right] = 0, \tag{2.13}$$

$$\lim_{\varepsilon\to 0} \mathbb{E}\left[\int_0^T \parallel \frac{1}{\varepsilon}(Z_t^\varepsilon - Z_t^q) - \hat{Z}_t \parallel^2 dt\right] = 0. \tag{2.14}$$

**Proof.** For simplicity, denote by:

$$\mathbb{X}_t^\varepsilon = \frac{1}{\varepsilon}(X_t^\varepsilon - X_t^q) - \hat{X}_t, \; \mathbb{Y}_t^\varepsilon = \frac{1}{\varepsilon}(Y_t^\varepsilon - Y_t^q) - \hat{Y}_t, \; \mathbb{Z}_t^\varepsilon = \frac{1}{\varepsilon}(Z_t^\varepsilon - Z_t^q) - \hat{Z}_t. \tag{2.15}$$

i) Let us prove (2.12). From (2.1), (2.11) and notations (2.15), we have:

$$
\begin{aligned}
\mathbb{X}_t^\varepsilon &= \frac{1}{\varepsilon}\int_0^t \left[\int_U b(s,X_s^\varepsilon,\mathbb{E}[X_s^\varepsilon],u)q_s^\varepsilon(du) - \int_U b(s,X_s^q,\mathbb{E}[X_s^q],u)q_s^\varepsilon(du)\right] ds \\
&+ \frac{1}{\varepsilon}\int_0^t \left[\int_U b(s,X_s^q,\mathbb{E}[X_s^q],u)q_s^\varepsilon(du) - \int_U b(s,X_s^q,\mathbb{E}[X_s^q],u)q_s(du)\right] ds \\
&+ \frac{1}{\varepsilon}\int_0^t \left[\int_U \sigma(s,X_s^\varepsilon,\mathbb{E}[X_s^\varepsilon],u)q_s^\varepsilon(du) - \int_U \sigma(s,X_s^q,\mathbb{E}[X_s^q],u)q_s^\varepsilon(du)\right] dW_s \\
&+ \frac{1}{\varepsilon}\int_0^t \left[\int_U \sigma(s,X_s^q,\mathbb{E}[X_s^q],u)q_s^\varepsilon(du) - \int_U \sigma(s,X_s^q,\mathbb{E}[X_s^q],u)q_s(du)\right] dW_s \\
&- \int_0^t \int_U b_X(s,X_s^q,\mathbb{E}[X_s^q],u)q_s(du)\hat{X}_s ds \\
&- \int_0^t \mathbb{E}\left[\int_U b_{X'}(s,X_s^q,\mathbb{E}[X_s^q],u)q_s(du)\mathbb{E}[\hat{X}_s]\right] ds \\
&- \int_0^t \int_U \sigma_X(s,X_s^q,\mathbb{E}[X_s^q],u)q_s(du)\hat{X}_s dW_s \\
&- \int_0^t \mathbb{E}\left[\int_U \sigma_{X'}(s,X_s^q,\mathbb{E}[X_s^q],u)q_s(du)\mathbb{E}[\hat{X}_s]\right] dW_s \\
&- \int_0^t \left(\int_U b(s,X_s^q,\mathbb{E}[X_s^q],u)q_s(du) - \int_U b(s,X_s^q,\mathbb{E}[X_s^q],u)\mu_s(du)\right) ds \\
&- \int_0^t \left(\int_U \sigma(s,X_s^q,\mathbb{E}[X_s^q],u)q_s(du) - \int_U \sigma(s,X_s^q,\mathbb{E}[X_s^q],u)\mu_s(du)\right) dW_s.
\end{aligned} \tag{2.16}
$$

We have: $\mathbb{X}_t^\varepsilon = \frac{1}{\varepsilon}(X_t^\varepsilon - X_t^q) - \hat{X}_t$ , so: $\mathbb{X}_t^\varepsilon + \hat{X}_t = \frac{1}{\varepsilon}(X_t^\varepsilon - X_t^q)$, and using the definition of $q_s^\varepsilon$:

$$
\begin{aligned}
&\frac{1}{\varepsilon}\int_0^t \left[\int_U b(s,X_s^\varepsilon,\mathbb{E}[X_s^\varepsilon],u)q_s^\varepsilon(du) - \int_U b(s,X_s^q,\mathbb{E}[X_s^q],u)q_s^\varepsilon(du)\right] ds \\
&+ \frac{1}{\varepsilon}\int_0^t [\int_U \sigma(s,X_s^\varepsilon,\mathbb{E}[X_s^\varepsilon],u)q_s^\varepsilon(du) - \int_U \sigma(s,X_s^q,\mathbb{E}[X_s^q],u)q_s^\varepsilon(du)] dW_s \\
&= \frac{1}{\varepsilon}\int_0^t [\int_U (b(s,X_s^\varepsilon,\mathbb{E}[X_s^\varepsilon],u) - b(s,X_s^q,\mathbb{E}[X_s^q],u))(q_s + \varepsilon(\mu_s - q_s))(du)] ds \\
&+ \frac{1}{\varepsilon}\int_0^t [\int_U (\sigma(s,X_s^\varepsilon,\mathbb{E}[X_s^\varepsilon],u) - \sigma(s,X_s^q,\mathbb{E}[X_s^q],u))(q_s + \varepsilon(\mu_s - q_s))(du)] dW_s.
\end{aligned}
$$

According to the development, we have:

$$\gamma = \frac{1}{\varepsilon}\int_0^t \left( \int_U (b(s, X_s^\varepsilon, \mathbb{E}[X_s^\varepsilon], u) - b(s, X_s^q, \mathbb{E}[X_s^q], u))(q_s + \varepsilon(\mu_s - q_s))(du) \right) ds \tag{2.17}$$

$$\begin{aligned}
&+ \frac{1}{\varepsilon}\int_0^t [\int_U (\sigma(s, X_s^\varepsilon, \mathbb{E}[X_s^\varepsilon], u) - \sigma(s, X_s^q, \mathbb{E}[X_s^q], u))(q_s + \varepsilon(\mu_s - q_s))(du)] dW_s \\
&= \frac{1}{\varepsilon}\int_0^t \int_0^1 \left( \int_U b_X(X^\varepsilon + \lambda(X^\varepsilon - X^q))(X^\varepsilon - X^q)(q_s + \varepsilon(\mu_s - q_s))(du) \right) ds \\
&+ \frac{1}{\varepsilon}\int_0^t \int_0^1 \left( \int_U \mathbb{E}\left[b_{X'}(X^\varepsilon + \lambda(X^\varepsilon - X^q))\mathbb{E}\left[X^\varepsilon - X^q\right](q_s + \varepsilon(\mu_s - q_s))(du)\right] \right) ds \\
&+ \frac{1}{\varepsilon}\int_0^t \int_0^1 \left( \int_U \sigma_X(X^\varepsilon + \lambda(X^\varepsilon - X^q))(X^\varepsilon - X^q)(q_s + \varepsilon(\mu_s - q_s))(du) \right) dW_s \\
&+ \frac{1}{\varepsilon}\int_0^t \int_0^1 \left( \int_U \mathbb{E}\left[\sigma_{X'}(X^\varepsilon + \lambda(X^\varepsilon - X^q))\mathbb{E}\left[X^\varepsilon - X^q\right](q_s + \varepsilon(\mu_s - q_s))(du)\right] \right) dW_s \\
&= \frac{1}{\varepsilon}\int_0^t \int_0^1 \left( \int_U b_X(X^\varepsilon + \lambda\varepsilon(\mathbb{X}_t^\varepsilon + \hat{X}_t))\varepsilon(\mathbb{X}_t^\varepsilon + \hat{X}_t)(q_s + \varepsilon(\mu_s - q_s))(du) \right) ds \\
&+ \frac{1}{\varepsilon}\int_0^t \int_0^1 \int_U \mathbb{E}\left[b_{X'}(X^\varepsilon + \lambda\varepsilon(\mathbb{X}_t^\varepsilon + \hat{X}_t))\varepsilon\mathbb{E}\left[(\mathbb{X}_t^\varepsilon + \hat{X}_t)\right] (q_s + \varepsilon(\mu_s - q_s))(du)ds\right) \\
&\frac{1}{\varepsilon}\int_0^t \int_0^1 \left( \int_U \sigma_X(X^\varepsilon + \lambda\varepsilon(\mathbb{X}_t^\varepsilon + \hat{X}_t))\varepsilon(\mathbb{X}_t^\varepsilon + \hat{X}_t)(q_s + \varepsilon(\mu_s - q_s))(du) \right) dW_s \\
&\frac{1}{\varepsilon}\int_0^t \int_0^1 \left( \int_U \mathbb{E}\left[\sigma_{X'}(X^\varepsilon + \lambda\varepsilon(\mathbb{X}_t^\varepsilon + \hat{X}_t))\varepsilon\mathbb{E}\left[(\mathbb{X}_t^\varepsilon + \hat{X}_t)\right](q_s + \varepsilon(\mu_s - q_s))(du)\right] \right) dW_s
\end{aligned}$$

Using publishing and compensation (2.17) in (2.16) and taking expectation, we obtain:

$$\begin{aligned}
\mathbb{E}[\mathbb{X}_t^\varepsilon] \leq C\mathbb{E}&\left[\int_0^t \int_0^1 \int_U \mid b_X(s, \Lambda_s^\varepsilon, u)\mathbb{X}_t^\varepsilon \mid^2 q_s(du)d\lambda ds\right] \\
&+ C\mathbb{E}\left[\int_0^t \int_0^1 \int_U \left|\mathbb{E}[b_{X'}(s, \Lambda_s^\varepsilon, u)\mathbb{E}[\mathbb{X}_t^\varepsilon]]\right|^2 q_s(du)d\lambda ds\right] \\
&+ C\mathbb{E}\left[\int_0^t \int_0^1 \int_U |\sigma_X(s, \Lambda_s^\varepsilon, u)\mathbb{X}_t^\varepsilon|^2 q_s(du)d\lambda dW_s\right] \\
&+ C\mathbb{E}\left[\int_0^t \int_0^1 \int_U \left|\mathbb{E}[\sigma_{X'}(s, \Lambda_s^\varepsilon, u)\mathbb{E}[\mathbb{X}_t^\varepsilon]]\right|^2 q_s(du)d\lambda dW_s\right] + C\mathbb{E}[|\Gamma_t^\varepsilon|^2],
\end{aligned}$$

where $(s, \Lambda_s^\varepsilon, u) := (s, X_s^q + \lambda\varepsilon(\mathbb{X}_s^\varepsilon + \hat{X}_s), \mathbb{E}[X_s^q + \lambda\varepsilon(\mathbb{X}_s^\varepsilon + \hat{X}_s)], u)$,

and

$$
\begin{aligned}
\Gamma_t^{\varepsilon} = & \int_0^t \int_0^1 \int_U b_X(s,\Lambda_s^{\varepsilon},u)(X_s^{\varepsilon}-X_s^q)\mu_s(du)d\lambda ds \\
& + \int_0^t \int_0^1 \int_U \mathbb{E}[b_{X'}(s,\Lambda_s^{\varepsilon},u)\mathbb{E}[X_s^{\varepsilon}-X_s^q]]\mu_s(du)d\lambda ds \\
& - \int_0^t \int_0^1 \int_U b_X(s,\Lambda_s^{\varepsilon},u)(X_s^{\varepsilon}-X_s^q)q_s(du)d\lambda ds \\
& - \int_0^t \int_0^1 \int_U \mathbb{E}[b_{X'}(s,\Lambda_s^{\varepsilon},u)\mathbb{E}[X_s^{\varepsilon}-X_s^q]]q_s(du)d\lambda ds \\
& + \int_0^t \int_0^1 \int_U (b_X(s,\Lambda_s^{\varepsilon},u)\hat{X}_s + \mathbb{E}[b_{X'}(s,\Lambda_s^{\varepsilon},u)\mathbb{E}[\hat{X}_s]])q_s(du)d\lambda ds \\
& + \int_0^t \int_0^1 \int_U \sigma_X(s,\Lambda_s^{\varepsilon},u)(X_s^{\varepsilon}-X_s^q)\mu_s(du)d\lambda dW_s \\
& + \int_0^t \int_0^1 \int_U \mathbb{E}[\sigma_{X'}(s,\Lambda_s^{\varepsilon},u)\mathbb{E}[X_s^{\varepsilon}-X_s^q]]\mu_s(du)d\lambda dW_s \\
& - \int_0^t \int_0^1 \int_U \sigma_X(s,\Lambda_s^{\varepsilon},u)(X_s^{\varepsilon}-X_s^q)q_s(du)d\lambda dW_s \\
& - \int_0^t \int_0^1 \int_U \mathbb{E}[\sigma_{X'}(s,\Lambda_s^{\varepsilon},u)\mathbb{E}[X_s^{\varepsilon}-X_s^q]]q_s(du)d\lambda dW_s \\
& + \int_0^t \int_0^1 \int_U (\sigma_X(s,\Lambda_s^{\varepsilon},u)\hat{X}_s + \mathbb{E}[\sigma_{X'}(s,\Lambda_s^{\varepsilon},u)\mathbb{E}[\hat{X}_s]])q_s(du)d\lambda dW_s \\
& - \int_0^t \int_U b_X(s,X_s^q,\mathbb{E}[X_s^q],u)\hat{X}_s q_s(du)ds \\
& - \int_0^t \int_U \mathbb{E}[b_{X'}(s,X_s^q,\mathbb{E}[X_s^q],u)\mathbb{E}[\hat{X}_s]]q_s(du)ds \\
& - \int_0^t \int_U \sigma_X(s,X_s^q,\mathbb{E}[X_s^q],u)\hat{X}_s q_s(du)dW_s \\
& - \int_0^t \int_U \mathbb{E}[\sigma_{X'}(s,X_s^q,\mathbb{E}[X_s^q],u)\mathbb{E}[\hat{X}_s]]q_s(du)dW_s
\end{aligned}
$$

since $b_X, b_{X'}, \sigma_X, \sigma_{X'}$ are continuous and bounded we have:

$$
\mathbb{E}[\mid \mathbb{X}_t^{\varepsilon} \mid^2] = C\mathbb{E}\left[\int_0^t |\mathbb{X}_t^{\varepsilon}|^2\, ds\right] + C\mathbb{E}[|\Gamma_t^{\varepsilon}|^2], \tag{2.18}
$$

and

$$\lim_{\varepsilon\to 0}\mathbb{E}[|\Gamma_t^{\varepsilon}|^2]=0. \tag{2.19}$$

By using (2.19), Granwall's lemma and Burkholder-Davis-Gundy inequality in (2.18), one can show (2.12).

**ii)** By the same method as in (2.12), we can prove (2.13) and (2.14).

**Proposition 2.1.3** *(Variational inequality). Let (**H1**), holds. Let $q.$ be an optimal relaxed control with associated trajectories $(X_t^{q\cdot}, Y_t^{q\cdot}, Z_t^{q\cdot})$. Then, for any element $\mu.$ of $\mathcal{R}$, we have:*

$$\begin{aligned}
0 \leq\ & \mathbb{E}[\alpha_X(X_T^q,\mathbb{E}[X_T^q])\hat{X}_T+\mathbb{E}[\alpha_{X'}(X_T^q,\mathbb{E}[X_T^q])\mathbb{E}[\hat{X}_T]]] \\
&+\mathbb{E}[\beta_Y(Y_0^q,\mathbb{E}[Y_0^q])\hat{Y}_0+\mathbb{E}[\beta_{Y'}(Y_0^q,\mathbb{E}[Y_0^q])\mathbb{E}[\hat{Y}_0]] \\
&+\mathbb{E}[\int_0^T\int_U(l_X(t,\pi_t^q,u)\hat{X}_t+\mathbb{E}[l_{X'}(t,\pi_t^q,u)\mathbb{E}[\hat{X}_t]] \\
&+l_Y(t,\pi_t^q,u)\hat{Y}_t+\mathbb{E}[l_{Y'}(t,\pi_t^q,u)\mathbb{E}[\hat{Y}_t]] \\
&+l_Z(t,\pi_t^q,u)\hat{Z}_t+\mathbb{E}[l_{Z'}(t,\pi_t^q,u)\mathbb{E}[\hat{Z}_t]])q_t(du)dt] \\
&+\mathbb{E}[\int_0^T(\int_U l(t,X_t^q,\mathbb{E}[X_t^q],Y_t^q,\mathbb{E}[Y_t^q],Z_t^q,\mathbb{E}[Z_t^q],u)\mu_t(du) \\
&-\int_U l(t,X_t^q,\mathbb{E}[X_t^q],Y_t^q,\mathbb{E}[Y_t^q],Z_t^q,\mathbb{E}[Z_t^q],u)q_t(du))dt].
\end{aligned} \tag{2.20}$$

**Proof.** From the optimality of $q$. we have:

$$
\begin{aligned}
0 &\leq \mathbb{E}[\alpha(X_T^\varepsilon, \mathbb{E}[X_T^\varepsilon]) - \alpha(X_T^q, \mathbb{E}[X_T^q])] \\
&+ \mathbb{E}[\beta(Y_0^\varepsilon, \mathbb{E}[Y_0^\varepsilon]) - \beta(Y_0^q, \mathbb{E}[Y_0^q])] \\
&+ \mathbb{E}[\int_0^T (\int_U l(t, X_t^\varepsilon, \mathbb{E}[X_t^\varepsilon], Y_t^\varepsilon, \mathbb{E}[Y_t^\varepsilon], Z_t^\varepsilon, \mathbb{E}[Z_t^\varepsilon], u) q_t^\varepsilon(du) \\
&- \int_U l(t, X_t^q, \mathbb{E}[X_t^q], Y_t^q, \mathbb{E}[Y_t^q], Z_t^q, \mathbb{E}[Z_t^q], u) q_t^\varepsilon(du)) dt] \\
&+ \mathbb{E}[\int_0^T (\int_U l(t, X_t^q, \mathbb{E}[X_t^q], Y_t^q, \mathbb{E}[Y_t^q], Z_t^q, \mathbb{E}[Z_t^q], u) q_t^\varepsilon(du) \\
&- \int_U l(t, X_t^q, \mathbb{E}[X_t^q], Y_t^q, \mathbb{E}[Y_t^q], Z_t^q, \mathbb{E}[Z_t^q], u) q_t(du)) dt].
\end{aligned}
$$

Let us divide this inequality by $\varepsilon$ and using the definition of $q_t^\varepsilon$ and from the notation (2.15), we have:

$$
\begin{aligned}
0 &\leq \mathbb{E}[\int_0^1 (\alpha_X(\Lambda_T^\varepsilon)\hat{X}_T + \mathbb{E}[\alpha_{X'}(\Lambda_T^\varepsilon)\mathbb{E}[\hat{X}_T]]) d\lambda] \\
&+ \mathbb{E}[\int_0^1 (\beta_Y(Y_0^q + \lambda\varepsilon(\mathbb{Y}_0^\varepsilon + \hat{Y}_0), \mathbb{E}[Y_0^q + \lambda\varepsilon(\mathbb{Y}_0^\varepsilon + \hat{Y}_s)])\hat{Y}_0 \\
&+ \mathbb{E}[\beta_{Y'}(Y_0^q + \lambda\varepsilon(\mathbb{Y}_0^\varepsilon + \hat{Y}_0), \mathbb{E}[Y_0^q + \lambda\varepsilon(\mathbb{Y}_0^\varepsilon + \hat{Y}_s)])\mathbb{E}[\hat{Y}_0]]) d\lambda] \\
&+ \mathbb{E}[\int_0^T \int_0^1 \int_U (l_X(t, \Delta_t^\varepsilon, u)\hat{X}_t + \mathbb{E}[l_{X'}(t, \Delta_t^\varepsilon, u)\mathbb{E}[\hat{X}_t]] \\
&+ l_Y(t, \Delta_t^\varepsilon, u)\hat{Y}_t + \mathbb{E}[l_{Y'}(t, \Delta_t^\varepsilon, u)\mathbb{E}[\hat{Y}_t]] \\
&+ l_Z(t, \Delta_t^\varepsilon, u)\hat{Z}_t + \mathbb{E}[l_{Z'}(t, \Delta_t^\varepsilon, u)\mathbb{E}[\hat{Z}_t]]) q_t(du) d\lambda dt] \\
&+ \mathbb{E}[\int_0^T (\int_U l(t, X_t^q, \mathbb{E}[X_t^q], Y_t^q, \mathbb{E}[Y_t^q], Z_t^q, \mathbb{E}[Z_t^q], u)\mu_t(du) \\
&- \int_U l(t, X_t^q, \mathbb{E}[X_t^q], Y_t^q, \mathbb{E}[Y_t^q], Z_t^q, \mathbb{E}[Z_t^q], u) q_t(du)) dt] + \nabla_t^\varepsilon,
\end{aligned} \tag{2.21}
$$

where $\nabla_t^\varepsilon$ is given by:

$$
\begin{aligned}
\nabla_t^\varepsilon &= \mathbb{E}[\int_0^1 (\alpha_X(\Lambda_T^\varepsilon)\mathbb{X}_T^\varepsilon + \mathbb{E}[\alpha_{X'}(\Lambda_T^\varepsilon)\mathbb{E}[\mathbb{X}_T^\varepsilon]])d\lambda] \\
&+ \mathbb{E}[\int_0^1 (\beta_Y(Y_0^q + \lambda\varepsilon(\mathbb{Y}_0^\varepsilon + \hat{Y}_0), \mathbb{E}[Y_0^q + \lambda\varepsilon(\mathbb{Y}_0^\varepsilon + \hat{Y}_s)])\mathbb{Y}_0^\varepsilon \\
&+ \mathbb{E}[\beta_{Y'}(Y_0^q + \lambda\varepsilon(\mathbb{Y}_0^\varepsilon + \hat{Y}_0), \mathbb{E}[Y_0^q + \lambda\varepsilon(\mathbb{Y}_0^\varepsilon + \hat{Y}_s)])\mathbb{E}[\hat{Y}_0^\varepsilon]])d\lambda] \\
&+ \mathbb{E}[\int_0^T \int_0^1 \int_U (l_X(t, \Delta_t^\varepsilon, u)(X_t^\varepsilon - X_t^q) + \mathbb{E}[l_{X'}(t, \Delta_t^\varepsilon, u)\mathbb{E}[X_t^\varepsilon - X_t^q]] \\
&+ l_Y(t, \Delta_t^\varepsilon, u)(Y_t^\varepsilon - Y_t^q) + \mathbb{E}[l_{Y'}(t, \Delta_t^\varepsilon, u)\mathbb{E}[Y_t^\varepsilon - Y_t^q]] \\
&+ l_Z(t, \Delta_t^\varepsilon, u)(Z_t^\varepsilon - Z_t^q) + \mathbb{E}[l_{Z'}(t, \Delta_t^\varepsilon, u)\mathbb{E}[Z_t^\varepsilon - Z_t^q]])\mu_t(du)d\lambda dt] \\
&- \mathbb{E}[\int_0^T \int_0^1 \int_U (l_X(t, \Delta_t^\varepsilon, u)(X_t^\varepsilon - X_t^q) + \mathbb{E}[l_{X'}(t, \Delta_t^\varepsilon, u)\mathbb{E}[X_t^\varepsilon - X_t^q]] \\
&+ l_Y(t, \Delta_t^\varepsilon, u)(Y_t^\varepsilon - Y_t^q) + \mathbb{E}[l_{Y'}(t, \Delta_t^\varepsilon, u)\mathbb{E}[Y_t^\varepsilon - Y_t^q]] \\
&+ l_Z(t, \Delta_t^\varepsilon, u)(Z_t^\varepsilon - Z_t^q) \ + \mathbb{E}[l_{Z'}(t, \Delta_t^\varepsilon, u)\mathbb{E}[Z_t^\varepsilon - Z_t^q]])q_t(du)d\lambda dt] \\
&+ \mathbb{E}[\int_0^T \int_0^1 \int_U (l_X(t, \Delta_t^\varepsilon, u)\mathbb{X}_t^\varepsilon + \mathbb{E}[l_{X'}(t, \Delta_t^\varepsilon, u)\mathbb{E}[\mathbb{X}_t^\varepsilon]] \\
&+ l_Y(t, \Delta_t^\varepsilon, u)\mathbb{Y}_t^\varepsilon + \mathbb{E}[l_{Y'}(t, \Delta_t^\varepsilon, u)\mathbb{E}[\mathbb{Y}_t^\varepsilon]] \\
&+ l_Z(t, \Delta_t^\varepsilon, u)\mathbb{Z}_t^\varepsilon + \mathbb{E}[l_{Z'}(t, \Delta_t^\varepsilon, u)\mathbb{E}[\mathbb{Z}_t^\varepsilon]])q_t(du)d\lambda dt]
\end{aligned}
$$

Since the derivatives $\alpha_X, \alpha_{X'}, \beta_Y, \beta_{Y'}, l_X, l_{X'}, l_Y, l_{Y'}, l_Z, l_{Z'}$ are continuous and bounded, then by using, (2.4), (2.5), (2.6), (2.12), (2.13), (2.14) and the Cauchy-Schwartz inequality we show that

$$\lim_{\varepsilon \to 0} \mathbb{E}[\mid \nabla_t^\varepsilon \mid^2] = 0.$$

Then let $\varepsilon$ go to 0 in (2.21), we get the variational inequality.

### 2.1.4 *Necessary optimality conditions for relaxed control.*

Let us introduce the adjoint equations of the MF-FBSDE (2.1) and then gives the maximum principle.

Define the Hamiltonian $H$ from:

$$[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\times\mathbb{R}^{m\times d}\times U\times\mathbb{R}^m\times\mathbb{R}^n,$$

to $\mathbb{R}$ by

$$\begin{aligned}
&H(t,X,X^{'},Y,Y^{'},Z,Z^{'},\mu,\Phi,\Psi,\Sigma) \\
&:=\Phi\int_U b(t,X,X^{'},u)\mu(du)+\Sigma\int_U \sigma(t,X,X^{'},u)\mu(du) \\
&+\Psi\int_U f(t,X,X^{'},Y,Y^{'},Z,Z^{'},u)\mu(du) \\
&+\int_U l(t,t,X,X^{'},Y,Y^{'},Z,Z^{'},u)\mu(du).
\end{aligned} \tag{2.22}$$

**Theorem 4.4. (Necessary optimality conditions for relaxed control)**

Assume that (**H1**), holds. Let $q.\in\mathcal{R}$ an optimal relaxed control. Let $(X^q,Y^q,Z^q)$ be the associated solution of MF-FBSDE (2.1). Then there exists a unique solution $(\Phi^q,\Psi^q,\Sigma^q)$ of the following adjoint equations of MF-FBSDE (2.1):

$$\left\{\begin{aligned}
d\Phi_t^q&=-(H_X(t,\zeta_t^q,q_t,\chi_t^q)+\mathbb{E}[H_{X'}(t,\zeta_t^q,q_t,\chi_t^q)])dt+\Sigma_t^q dW_t,\\
d\Psi_t^q&=(H_Y(t,\zeta_t^q,q_t,\chi_t^q)+\mathbb{E}[H_{Y'}(t,\zeta_t^q,q_t,\chi_t^q)])dt\\
&\quad+(H_Z(t,\zeta_t^q,q_t,\chi_t^q)+\mathbb{E}[H_{Z'}(t,\zeta_t^q,q_t,\chi_t^q)])dW_t,\\
\Psi_0^q&=\beta_Y(Y_0^q,\mathbb{E}[Y_0^q])+\mathbb{E}[k_{X'}(Y_0^q,\mathbb{E}[Y_0^q])],\\
\Phi_T^q&=\alpha_X(X_T^q,\mathbb{E}[X_T^q])+\mathbb{E}[\alpha_{X'}(X_T^q,\mathbb{E}[X_T^q])]+h_X(X_T^q,\mathbb{E}[X_T^q])\Psi_T^q\\
&\quad+\mathbb{E}[h_{X'}(X_T^q,\mathbb{E}[X_T^q])\mathbb{E}[\Psi_T^q]],
\end{aligned}\right. \tag{2.23}$$

such that:

$$H(t, X_t^q, \mathbb{E}[X_t^q], Y_t^q, \mathbb{E}[Y_t^q], Z_t^q, \mathbb{E}[Z_t^q], q_t, \Phi_t^q, \Psi_t^q, \Sigma_t^q) \tag{2.24}$$
$$\leq H(t, X_t^q, \mathbb{E}[X_t^q], Y_t^q, \mathbb{E}[Y_t^q], Z_t^q, \mathbb{E}[Z_t^q], \mu_t, \Phi_t^q, \Psi_t^q, \Sigma_t^q),$$

a.e.t, $\mathbb{P} - a.s$, $\forall \mu \in P(U)$, where

$$(t, \zeta_t^q, q_t, \chi_t^q) := (t, X_t^q, \mathbb{E}[X_t^q], Y_t^q, \mathbb{E}[Y_t^q], Z_t^q, \mathbb{E}[Z_t^q], q_t, \Phi_t^q, \Psi_t^q, \Sigma_t^q).$$

***Proof.*** From (2.23), the inequality variational (2.20) becomes

$$0 \leq \mathbb{E}[\langle \Phi_T^q, \hat{X}_T \rangle] - [\mathbb{E}[h_X(X_T^q, \mathbb{E}[X_T^q])\Psi_T^q \tag{2.25}$$
$$+ \mathbb{E}[h_{X'}(X_T^q, \mathbb{E}[X_T^q])\mathbb{E}[\Psi_T^q]]] + \mathbb{E}[\langle \Psi_0^q, \hat{Y}_0 \rangle]$$
$$+ \mathbb{E}[\int_0^T \int_U (l_X(t, \pi_t^q, u)\hat{X}_t + \mathbb{E}[l_{X'}(t, \pi_t^q, u)\mathbb{E}[\hat{X}_t]]$$
$$+ l_Y(t, \pi_t^q, u)\hat{Y}_t + \mathbb{E}[l_{Y'}(t, \pi_t^q, u)\mathbb{E}[\hat{Y}_t]]$$
$$+ l_Z(t, \pi_t^q, u)\hat{Z}_t + \mathbb{E}[l_{Z'}(t, \pi_t^q, u)\mathbb{E}[\hat{Z}_t]])q_t(du)dt]$$
$$+ \mathbb{E}[\int_0^T \int_U (l(t, X_t^q, \mathbb{E}[X_t^q], Y_t^q, \mathbb{E}[Y_t^q], Z_t^q, \mathbb{E}[Z_t^q], u)\mu_t(du)$$
$$- \int_U (l(t, X_t^q, \mathbb{E}[X_t^q], Y_t^q, \mathbb{E}[Y_t^q], Z_t^q, \mathbb{E}[Z_t^q], u)q_t(du))dt].$$

Now applying Itô's formula to compute $\langle \Phi_t^q, \hat{X}_t \rangle$ and $\langle \Psi_t^q, \hat{Y}_t \rangle$ and taking the expectations we derive:

$$\begin{aligned}
d\langle \Phi_t^q, \hat{X}_t\rangle &= \Phi_t^q d\hat{X}_t + \hat{X}_t d\Phi_t^q + d\langle \Phi, \hat{X}\rangle_t \\
&= \Phi_t^q \Bigg( \int_U b_X(t, X_t^q, \mathbb{E}[X_t^q], u) q_t(du)\hat{X}_t dt \\
&+\mathbb{E}\left[\int_U b_{X'}(t, X_t^q, \mathbb{E}[X_t^q], u) q_t(du)\mathbb{E}[\hat{X}_t]\right] dt \\
&+ \int_U \sigma_X(t, X_t^q, \mathbb{E}[X_t^q], u) q_t(du)\hat{X}_t dW_t \\
&+\mathbb{E}\left[\int_U \sigma_{X'}(t, X_t^q, \mathbb{E}[X_t^q], u) q_t(du)\mathbb{E}[\hat{X}_t]\right] dW_t \\
&+ \int_U b(t, X_t^q, \mathbb{E}[X_t^q], u)\left(q_t(du) - \mu_t(du)\right) dt \\
&+ \int_U \sigma(t, X_t^q, \mathbb{E}[X_t^q], u)\left(q_t(du) - \mu_t(du)\right) dW_t \;) \\
&- \hat{X}_t\left((H_X(t, \zeta_t^q, q_t, \chi_t^q) + \mathbb{E}[H_{X'}(t, \zeta_t^q, q_t, \chi_t^q)]\right)dt \\
&+\Sigma_t^q dW_t) + \Sigma_t^q \Bigg( \int_U (\sigma_X(t, X_t^q, \mathbb{E}[X_t^q], u)\hat{X}_t \\
&+\mathbb{E}[\sigma_{X'}(t, X_t^q, \mathbb{E}[X_t^q], u)\mathbb{E}[\hat{X}_t]]) q_t(du) dW_t \\
&+ \int_U \sigma(t, X_t^q, \mathbb{E}[X_t^q], u)\left(q_t(du) - \mu_t(du)\right) dW_t \Bigg).
\end{aligned}$$

Integring between 0 to $T$, taking expectation and using the fact that $\hat{X}_0 = 0$, we find:

$$
\begin{aligned}
\mathbb{E}[\langle \Phi_T^q, \hat{X}_T\rangle] = \mathbb{E}\Bigg[\int_0^T \Phi_t^q \bigg( &\int_U b_X(t, X_t^q, \mathbb{E}[X_t^q], u) q_t(du) \hat{X}_t \\
&+\mathbb{E}\left[\int_U b_{X'}(t, X_t^q, \mathbb{E}[X_t^q], u) q_t(du) \mathbb{E}[\hat{X}_t]\right] \\
&+ \int_U b(t, X_t^q, \mathbb{E}[X_t^q], u) \left(q_t(du) - \mu_t(du)\right)\bigg) \, dt\Bigg] \\
&- \mathbb{E}\left[\int_0^T \hat{X}_t \left(H_X(t, \zeta_t^q, q_t, \chi_t^q) + \mathbb{E}[H_{X'}(t, \zeta_t^q, q_t, \chi_t^q)]\right)\right) dt\Bigg] \\
&\mathbb{E}\Bigg[\int_0^T \Sigma_t^q \bigg(\int_U \sigma_X(t, X_t^q, \mathbb{E}[X_t^q], u) q_t(du) \hat{X}_t \\
&+\mathbb{E}\left[\int_U \sigma_{X'}(t, X_t^q, \mathbb{E}[X_t^q], u) q_t(du) \mathbb{E}[\hat{X}_t]\right] \\
&+ \int_U \sigma(t, X_t^q, \mathbb{E}[X_t^q], u) \left(q_t(du) - \mu_t(du)\right)\bigg) \, dW_t\Bigg],
\end{aligned}
$$

by the definition of the Hamiltonian $H$, we get

$$
\begin{aligned}
\mathbb{E}[\langle \Phi_T^q, \hat{X}_T\rangle] = -\mathbb{E}\Bigg[\int_0^T \Psi_t^q \bigg( &\int_U f_X(t, \pi_t^q, u) q_t(du) \hat{X}_t \\
&+\mathbb{E}\left[\int_U f_{X'}(t, \pi_t^q, u) q_t(du) \mathbb{E}[\hat{X}_t]\right]\bigg) \, dt \\
&+ \mathbb{E}\left[\int_0^T \int_U (l_X(t, \pi_t^q, u) \hat{X}_t + \mathbb{E}[l_{X'}(t, \pi_t^q, u) \mathbb{E}[\hat{X}_t]) dt\right] \\
&+ \mathbb{E}\left[\int_0^T \Phi_t^q \int_U b(t, X_t^q, \mathbb{E}[X_t^q], u) \left(q_t(du) - \mu_t(du)\right) dt\right] \\
&+ \mathbb{E}\left[\int_0^T \Sigma_t^q \int_U \sigma(t, X_t^q, \mathbb{E}[X_t^q], u) \left(q_t(du) - \mu_t(du)\right) dW_t\right],
\end{aligned}
$$

and

$$
\begin{aligned}
d\langle \Psi_t^q, \hat{Y}_t\rangle &= \Psi_t^q d\hat{Y}_t + \hat{Y}_t d\Psi_t^q + d\langle \Psi, \hat{Y}\rangle_t \\
&= -\Psi_t^q \left( \int_U f_X(t, \pi_t^q, u) q_t(du) \hat{X}_t \right. \\
&+ \mathbb{E}\left[ \int_U f_{X'}(t, \pi_t^q, u) q_t(du) \mathbb{E}[\hat{X}_t] \right] \\
&+ \int_U f_Y(t, \pi_t^q, u) q_t(du) \hat{Y}_t \\
&+ \mathbb{E}\left[ \int_U f_{Y'}(t, \pi_t^q, u) q_t(du) \mathbb{E}[\hat{Y}_t] \right] \\
&+ \int_U f_Z(t, \pi_t^q, u) q_t(du) \hat{Z}_t \\
&+ \mathbb{E}\left[ \int_U f_{Z'}(t, \pi_t^q, u) q_t(du) \mathbb{E}[\hat{Z}_t] \right] \\
&+ \left. \int_U f(t, \pi_t^q, u) q_t(du) - \mu_t(du) \right) dt + \Psi_t^q \hat{Z}_t dW_t \\
&+ \hat{Y}_t \left( \left( H_Y(t, \zeta_t^q, q_t, \chi_t^q) + \mathbb{E}[H_{Y'}(t, \zeta_t^q, q_t, \chi_t^q)] \right) dt \right. \\
&+ \left. \left( H_Z(t, \zeta_t^q, q_t, \chi_t^q) + \mathbb{E}[H_{Z'}(t, \zeta_t^q, q_t, \chi_t^q)] \right) dW_t \right) \\
&+ \left( H_Z(t, \zeta_t^q, q_t, \chi_t^q) + \mathbb{E}[H_{Z'}(t, \zeta_t^q, q_t, \chi_t^q)] \right) \hat{Z}_t dt
\end{aligned}
$$

Integring between 0 to $T$, taking expectation and using the definition of the Hamiltonian $H$, we find:

$$\begin{aligned}
\mathbb{E}[\langle\Psi_0^q,\hat{Y}_0\rangle] &= \mathbb{E}[\langle\Psi_T^q,\hat{Y}_T\rangle] \\
&+\mathbb{E}\left[\int_0^T\left\langle\Psi_t^q,\int_U f_X(t,\pi_t^q,u)q_t(du)\hat{X}_t\right.\right. \\
&+\int_U\mathbb{E}\left[f_{X'}(t,\pi_t^q,u)\mathbb{E}[\hat{X}_t]\right]q_t(du)\rangle dt] \\
&-\mathbb{E}\left[\int_0^T\left(\int_U l_Y(t,\pi_t^q,u)q_t(du)\hat{Y}_t\right.\right. \\
&\left.\left.+\mathbb{E}[\int_U l_{Y'}(t,\pi_t^q,u)])q_t(du)\mathbb{E}\left[\hat{Y}_t\right]\right)dt\right] \\
&-\mathbb{E}\left[\int_0^T\left(\int_U l_Z(t,\pi_t^q,u)q_t(du)\hat{Z}_t\right.\right. \\
&\left.\left.+\mathbb{E}[\int_U l_{Z'}(t,\pi_t^q,u)])q_t(du)\mathbb{E}\left[\hat{Z}_t\right]\right)dt\right] \\
&+\mathbb{E}[\int_0^T\Psi_t^q\int_U f(t,\pi_t^q,u)\left(q_t(du)-\mu_t(du)\right)dt.
\end{aligned}$$

Substitute the above equalities in inequality (2.23) to get, for every $\mu\in\mathcal{R}$, we get

$$\begin{aligned}
&\mathbb{E}[\int_0^T(H(t,X_t^q,\mathbb{E}[X_t^q],Y_t^q,\mathbb{E}[Y_t^q],Z_t^q,\mathbb{E}[Z_t^q],q_t,\Phi_t^q,\Psi_t^q,\Sigma_t^q))dt] \\
&\le\mathbb{E}[\int_0^T(H(t,X_t^q,\mathbb{E}[X_t^q],Y_t^q,\mathbb{E}[Y_t^q],Z_t^q,\mathbb{E}[Z_t^q],\mu_t,\Phi_t^q,\Psi_t^q,\Sigma_t^q))dt],
\end{aligned}$$

a.e.t, $\mathbb{P}-a.s$, $\forall\mu\in P(U)$. Therefore inequality (2.22) follows by a standard arguments.

# Conclusion

We have established in this memory the necessary conditions of optimality for a relaxed control problem, for systems governed by nonlinear forward-backward stochastic differential equations of mean field type (MF-FBSDE). Here the coefficients of the system depend on the state processes as well as their distribution via the expectation of state processes. In addition, the cost functional is also of mean field type.

The relaxed control problem is a generalization of the strict control problem. Indeed, if $q_t(da) = \delta_{u_t}(da)$ is a Dirac measure concentrated at a single point which is the strict control $u_t \in U$, then we obtain that the strict control problem is a particular case of the relaxed control problem. If the coefficients of the system depend on the state processes as well as their distribution, in this case the system is called forward-backward stochastic differential equations of Mckean–Vlasov type (MV-FBSDE).

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# Annexe A:

**Lemma 2.1.1 (Granwall's Lemma):**

Let $g : [0, T] \to \mathbb{R}$ be a continuous function verifying:

$$\forall t \in [0, T] \ , 0 \leq g(t) \leq \alpha(t) + \beta \int_0^t g(s)ds.$$

For a constant $\beta \geq 0$ and for a function $\alpha : [0, T] \to \mathbb{R}$ integrable with respect to the Lebesgue measure, we then have:

$$\forall t \in [0, T] \ , g(T) \leq \alpha e^{\beta t},$$

and if $\alpha = 0$, we have $g = 0$.

**•Chebyshev's inequality**

$$\left\{ P(\mid X - E[X] \mid \geq K) \leq \frac{Var(X)}{K^2} \right\}.$$

**•Doop's inequality:**

$$\left\{ \begin{array}{c} P(\sup_{0 \leq t \leq T} B_t \geq C) = P(\sup_{0 \leq t \leq T} \exp(\lambda B_t) \geq \exp(\lambda C)) \\ \leq \frac{E[\exp(\lambda B_T)]}{\exp(\lambda C)} \\ = \exp(\frac{1}{2}\lambda^2 T - \lambda C) \end{array} \right\}$$

.

**•Picard's inequality:**

$$\| T(x)(t) - T(y)(t) \| = \| \int_a^t f(s, x(s))ds - \int_a^t f(s, y(s))ds \|$$
$$\leq \int_a^t \| f(s, x(s)) - \int_a^t f(s, y(s)) \| ds$$
$$\leq \int_a^t L \| x(s) - y(s) \| ds$$
$$\leq (t-a)L \| x - y \|_\infty .$$

# Annexe B: Abreviations et Notations

Les différentes abréviations et notations utilisées tout au long de ce mémoire sont expliquées ci-dessous:

| | | |
|---|---|---|
| $\mathbb{E}[.]$ | : | Expectation. |
| $FBSDE$ | : | Forward and backward stochastic differential equation. |
| $MF-FBSDE$ | : | Forward and backward stochastic differential equation of mean-field type |
| $\mu$ | : | relaxed Control |
| $q$ | : | optimal relaxed control |
| $\mathbb{P}-p.s$ | : | Almost certainly for the probability measure $\mathbb{P}$ |
| lim | : | limit |
| $J(.)$ | : | The cost function to be minimized |
| $H$ | : | The Hamiltonian |

## RÉSUMÉ

Dans ce travail, nous étudions les conditions nécessaires d'optimalité des contrôles relaxé pour les équations différentielles stochastique progressives et rétrogrades (EDSPR) non linéaires de type champ moyen. Dans le premier chapitre, nous démontrons un théorème d'existence et d'unicité de solution des EDSPR non linéaires de type champ moyen. Dans le deuxième chapitre nous établirons les conditions nécessaires d'optimalités sous forme d'un principe de maximum stochastique pour le contrôle relaxé des systèmes des EDSPR non linéaires de type champ moyen.



## ABSTRACT

In this work we study the necessary optimality conditions of relaxed controls for nonlinear forward-backward stochastic differential equations (FBSDEs) of mean-field type. In the first chapter, we demonstrate a theorem of existence and uniqueness of the solutions of nonlinear FBSDEs of mean-field type. In the second chapter, we establish the necessary optimality conditions in the form of a stochastic maximum principle for relaxed controls for system of nonlinear FBSDEs of mean-field-type.



## الملخص

في هذا العمل نقوم بدراسة الشروط اللازمة للتحكم المرخي للمعادلات التفاضلية العشوائية الزمنية التقدمية التراجعية غير الخطية من نوع المجال المتوسط، وفي الفصل الأول نقوم ببرهان نظرية وجود ووحدانية الحل للمعادلات التفاضلية العشوائية الزمنية التقدمية التراجعية غير الخطية من نوع المجال المتوسط. الفصل الثاني ندرس الشروط اللازمة لنظام متوسط المجال في شكل مبدأ أقصى عشوائي لعناصر التحكم المرخي.