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Dedication

This dissertation is dedicated to all those who have played a part in shaping my academic journey. To my family, whose unwavering support has been my foundation, and to my mentors, whose guidance and wisdom have illuminated my path. To my friends, whose encouragement has been a constant source of motivation. This dissertation is a tribute to your belief in me. Thank you for being my inspiration.

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Introduction

Introduction

For many years, stochastic control has been growing and finding many uses in fields like engineering, economics, and financial mathematics. This area started in mathematics and is based on the idea that an optimal control decision stays optimal no matter when you check it. This basic idea has been used to study stochastic control for a long time. The Hamilton-Jacobi-Bellman (HJB) equations, which are central to modern stochastic control theory, were made possible by this idea and dynamic programming principles.

Traditional optimal control theory assumes that control strategies are time-consistent, see [7, 9]. This means that a decision that is optimal today will remain optimal in the future. Dynamic programming and HJB equations rely on this assumption, making it easier to solve control problems in a consistent way.

In the real world, however, time inconsistency often occurs. Time inconsistency means that preferences or decisions change over time in ways that were not planned. Examples include:

Hyperbolic Discounting [6]: People tend to choose smaller, immediate rewards over larger, delayed ones. This preference can change over time, leading to inconsistent decision-making.

Dynamic Mean-Variance Portfolio Selection Models [3]: Investors' risk preferences can change over time, affecting their investment strategies and causing time-inconsistent behavior.

To deal with time inconsistency, researchers have looked into equilibrium controls. Instead of finding one best control, equilibrium controls aim to find a balance where decisions change over time in a consistent way. This approach considers the entire time span and looks for strategies that stay balanced over time.

In the area of time-inconsistent optimal control, Ekeland and Lazrak [6] made a big breakthrough with their work. Later, Bjork and Murgoci [4], and then Bjork, Murgoci, and Zhou, [5], made important contributions to understanding time-inconsistent problems in the stochastic (random) setting. Despite these advances, the study of time-inconsistent control is still quite new.

In this dissertation, we study a general stochastic linear-quadratic (LQ) control problem, see [1] and [2], adding to the research on time-inconsistent control. Our problem includes special terms that cause time inconsistency, making it different from traditional problems. We suggest a new way to look at things using open-loop strategies, which plan the entire control path from the start, instead of feedback models that adjust controls based on current states.

To analyze the problem in detail, we use forward-backward stochastic differential equations (FBSDEs). These equations help us model the dynamic behavior of the system and the control strategies. By studying these equations carefully, we aim to understand the equilibrium conditions fully, allowing us to find the best control strategies for systems that are time-inconsistent.

The next parts of this dissertation are organized as follows:

Chapter 1: Introduces the mathematical basics needed for our study, including forward-backward stochastic differential equations (FBSDEs) and stochastic processes. We look at optimal control problems in the context of time inconsistency.

Chapter 2: Describes the specific stochastic LQ control problem we are studying. We present theoretical results on how to find equilibrium control using FBSDEs and other mathematical tools.

Chapter 3: Provides an example of a general discounting, time-inconsistent LQ model. We show a specific case to illustrate the concepts and methods developed in the previous chapters, demonstrating how the theoretical results can be applied to real-world scenarios.

Chapter 1

Introduction to Stochastic Processes

In this opening chapter, we begin by introducing fundamental terms in stochastic computation. We'll keep it straightforward, focusing on key concepts that we'll expand on later. We'll cover the basics you need before delving into more intricate topics like stochastic calculus and stochastic differential equations in the following chapters. This chapter lays the foundation for understanding the mechanics of stochastic computation and its significance across different fields. For more details about this chapter, the reader can see [10], [8].

1.1 Stochastic Processes

Definition 1.1.1 (Stochastic Process) *A stochastic process, denoted by* $X = (X(t))_{t\in[0,T]}$*, is a collection of random variables defined on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*, taking values in* $\mathbb{R}^n$*.*

Now we give some terminology:

1. The variable t is often interpreted as time.

2. If $X = (X(t))_{t\in\{0,1,\cdots,n=T\}}$, X is a discrete-time stochastic process. For a continuous index set, it is a continuous-time stochastic process.

3. The typical index set is $[0,T]$, where $T > 0$ (or $\mathbb{R}_+$).

4. Each $X(t)$ is a random variable $\omega \longrightarrow X(\omega,t)$ for $\omega \in \Omega$.

5. Fixing $\omega \in \Omega$ makes $X(\omega)$ a function $t \longrightarrow X(\omega,t)$, referred to as a path of X.

Definition 1.1.2 (Modification of Process) *A stochastic process $(X_1(t))_{t\in[0,T]}$ is considered a modification of another process $(X_2(t))_{t\in[0,T]}$ if $\mathbb{P}(X_1(t) = X_2(t)) = 1$ for all $t \in [0,T]$.*

Definition 1.1.3 (Indistinguishable Processes) *Two stochastic processes $(X_1(t))_{t\in[0,T]}$ and $(X_2(t))_{t\in[0,T]}$ are termed indistinguishable if $\mathbb{P}(X_1(t) = X_2(t), \forall t \in [0,T]) = 1$.*

Remark 1.1.1 *If two stochastic processes $(X_1(t))_{t\in[0,T]}$ and $(X_2(t))_{t\in[0,T]}$ are indistinguishable, they are modifications of each other. However, the converse is not always true.*

Definition 1.1.4 (Measurable stochastic process) *A stochastic process $X = (X(t))_{t\in[0,T]}$ is measurable if the mapping $X : \Omega\times[0,T] \to \mathbb{R}^n$ is $(\mathcal{B}([0,T])\otimes\mathcal{F},\mathcal{B})$ measurable.*

Definition 1.1.5 (Filtration) *A filtration is a family of σ-algebras $\{\mathcal{F}_t\}_{t\geq 0}$ defined on a probability space $(\Omega,\mathcal{F},\mathbb{P})$, such that $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$ for all $0 \leq s \leq t$, where $\mathcal{F}$ is the σ-algebra of all events in Ω. And we call $\left(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P}\right)$ a filtered probability space.*

- The filtration represents the increasing amount of information available to an observer as time passes, with $\mathcal{F}_t$ being the set of events that the observer can distinguish up to time t.

- A filtration $\{\mathcal{F}_t\}_{t\geq 0}$is right continuous if $\mathcal{F}_{t^+} = \cap_{\varepsilon>0}\mathcal{F}_{t+\varepsilon} = \mathcal{F}_t$ for all $t \geq 0$.

- A filtration $\{\mathcal{F}_t\}_{t\geq 0}$is complete if $\mathcal{F}_0 \subset \mathcal{F}_t$.

- A filtration that is right continuous and complete is said to satisfy the usual conditions.

Definition 1.1.6 (Adapted stochastic process) *We say that a stochastic process* $X = (X(t))_{t\in[0,T]}$ *is adapted to a filtration* $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ *if* $X(t)$ *is* $\mathcal{F}_t$*-measurable for each* t.

Definition 1.1.7 (Natural filtration) *The collection of* σ*-algebras* $\{\mathcal{G}(t)\}_{t\geq 0}$ *where*

$$\mathcal{G}(t) = \sigma\{X(s) : 0 \leq s \leq t\},$$

for all $t \geq 0$ *we call it the natural filtration of a stochastic process* $(X(t))_{t\in[0,T]}$.

We define the minimal augmented filtration generated by $(X(t))_{t\in[0,T]}$ *to be the smallest filtration that is right continuous and complete and with respect to which the process* $(X(t))_{t\in[0,T]}$ *is adapted.*

Definition 1.1.8 (Stopping time) *A stopping time* τ *with respect to* $\{\mathcal{F}_t\}_{t\geq 0}$ *is a random variable* $\tau : \Omega \to [0, +\infty]$ *such that for all* $t \in [0, T]$*, the event*

$$\{\tau \leq t\} = \{\omega \in \Omega : \tau(\omega) \leq t\} \in \mathcal{F}_t.$$

Definition 1.1.9 (σ-algebra of events prior to $[0, T]$) *If* τ *is a stopping time. The* σ*-algebra*

$$\mathcal{F}_\tau = \{A \in \mathcal{F}, A \cap \{\tau \leq t\} \in \mathcal{F}_t, \forall t \in [0, T]\},$$

is called the σ*-algebra of events prior to* $[0, T]$.

1.1.1 Brownian Motion

Definition 1.1.10 (Standard Brownian Motion) *The Standard Brownian Motion, also known as Wiener process, is a stochastic process* $(W(t))_{t\geq 0}$ *with independent and identically distributed increments such that:*

1. $W(0) = 0$ *almost surely.*

2. *For all* $0 \leq s < t$*, the increment* $W(t) - W(s)$ *is normally distributed with mean* 0 *and variance* $t - s$ *(i.e.,* $W(t) - W(s) \sim N(0, t - s)$*).*

3. *The sample paths of* $W(t)$ *are almost surely continuous.*

Definition 1.1.11 (d-dimensional Brownian Motion) *A d-dimensional Brownian motion, denoted by* $W = \left(W^{(1)}, W^{(2)}, \ldots, W^{(d)}\right)$*, is defined by considering* $W^{(i)}$ *as independent standard Brownian motions for* $i = 1, 2, \ldots, d$*.*

The filtration $\{\mathcal{F}_t\}_{t\geq 0}$ *generated by a Brownian motion* W *is defined as,*

$$\mathcal{F}_t = \sigma(W(s) : s \leq t), \quad t \geq 0,$$

and it is called the natural filtration of W *or Brownian filtration.*

1.2 Martingales

Definition 1.2.1 (Martingale) *A continuous-time martingale (resp.submartingale, supermartingale) is a stochastic process* $\{X(t), t \geq 0\}$ *satisfying the following conditions:*

1. $X(t)$ *is adapted to a filtration* $\{\mathcal{F}_t\}_{t\geq 0}$*, i.e.,* $X(t)$ *is measurable with respect to* $\mathcal{F}_t$ *for all* $t \geq 0$*.*

2. *$X(t)$ is integrable for all $t \geq 0$.*

3. *For all $0 \leq s \leq t$, $\mathbb{E}(X(t)|\mathcal{F}_s) = (\text{resp.} \leq, \geq) X(s)$ almost surely.*

Remark 1.2.1 *A process X is a martingale if it is both a submartingale and a supermartingale. If X is a martingale, then $\mathbb{E}(X(t)) = \mathbb{E}(X(0))$ for all $t \in [0,T]$.*

Example 1.2.1 *If W is a Brownian motion, then $W(t)$, $W^2(t) - t$, and $\exp\left(\sigma W(t) - \frac{\sigma^2 t}{2}\right)$ for $t \in [0,T]$ are martingales. Conversely, if X is a continuous process such that $\{X(t)\}_{t\geq 0}$ and $\{X^2(t) - t\}_{t\geq 0}$ are martingales, then X is a Brownian motion.*

Definition 1.2.2 (Local Martingale) *A stochastic process $\{M(t)\}_{t\in\mathbb{R}^+}$ adapted and caglad (right-continuous with left limits) is a local martingale if there exists an increasing sequence of stopping times (τ_n) such that $\tau_n \to +\infty$ as $n \to \infty$ and $M(t \wedge \tau_n)$ is a martingale for all n.*

Remark 1.2.2 *A positive local martingale is a supermartingale. A locally uniformly integrable martingale is a martingale.*

Definition 1.2.3 (Semimartingale) *A semimartingale is a cadlag adapted process X admitting a decomposition of the form: $X = A + M$, where M is a cadlag local martingale null at 0 and A is an adapted process of finite variation and null at 0.*

1.3 Stochastic Integration and Itô's Formula

In this section, we consider a positive real number T, and aim to define the integral

$$\mathrm{I}(\theta) = \int_0^T \theta(t) dW(t). \tag{1.1}$$

Here, $(\theta(t))_{t\geq 0}$ represents any process, and $(W(t))_{t\geq 0}$ denotes a Brownian motion. The challenge lies in giving meaning to the differential element $dW(s)$ since the function $s \to W(s)$ is not differentiable.

1.3.1 Wiener Integral

The Wiener integral is an integral of the form (1.1) with θ being a deterministic function, meaning it does not depend on the random variable ω. Let

$$L^2([0,T],\mathbb{R}) = \left\{\theta : [0,T] \to \mathbb{R} \text{ such that, } \int_0^T |\theta(s)|^2 ds < \infty\right\}.$$

Suppose θ_n is a deterministic step function defined as,

$$\theta_n(t) = \sum_{i=1}^{p_n} \alpha_i 1_{\left[t_i^n, t_{i+1}^n\right]}(t),$$

where $p_n \in \mathbb{N}$, the α_i are real numbers, and $\left\{t_i^n\right\}$ is an increasing sequence in $[0,T]$. Then, the Wiener integral is defined as

$$\mathrm{I}(\theta_n) = \int_0^T \theta_n(s) dW(s) = \sum_{i=1}^{p_n} \alpha_i \left(W(t_{i+1}) - W(t_i)\right).$$

Due to the Gaussian nature of Brownian motion and the independence of its increments, the random variable $\mathrm{I}(\theta_n)$ is a Gaussian variable with zero mean and variance

$$\begin{aligned} \mathrm{V}(\mathrm{I}(\theta_n)) &= \mathrm{V}\left(\sum_{i=1}^{p_n} \alpha_i \left(W(t_{i+1}) - W(t_i)\right)\right) \\ &= \int_0^T \theta_n^2(s) ds. \end{aligned}$$

Remark 1.3.1 *We observe that $\theta \to \mathrm{I}(\theta)$ is a linear function. Moreover, if f and g are two step functions, we have*

$$\mathbb{E}(\mathrm{I}(f)\mathrm{I}(g)) = \int_0^T f(s)g(s)ds.$$

We then refer to the isometry property of the Wiener integral. Now let $\theta \in L^2([0,T],\mathbb{R})$. Therefore, there exists a sequence of step functions $\{\theta_n, n \geq 0\}$ that converges in $L^2([0,T],\mathbb{R})$ to θ. According to the previous paragraph, we can construct the Wiener integrals $\mathrm{I}(\theta_n)$, which are centered Gaussians forming a Cauchy sequence by isometry. Since the space $L^2([0,T],\mathbb{R})$ is complete, this sequence converges to a Gaussian random variable denoted by $\mathrm{I}(\theta)$. It can be shown that the limit does not depend on the choice of the sequence $\theta_n, n \geq 0$. $\mathrm{I}(\theta)$ is called the Wiener integral of θ with respect to $(W(t))_{t\in\mathbb{R}}$.

1.3.2 The Itô's integral

Our objective now is to define the integral given by equation (1.1). To achieve this, we construct $\mathrm{I}(\theta)$ using discretization, similar to the approach used for the Wiener integral.

Let's start by examining step processes represented by,

$$\theta_n(t) = \sum_{i=0}^{p_n} \alpha_i \mathbf{1}_{\left[t_i^n, t_{i+1}^n\right]}(t), \tag{1.2}$$

where $p_n \in \mathbb{N}$, (t_i^n) forms an increasing sequence in $[0,T]$, and $\alpha_i \in L^2\left(\Omega, \mathcal{F}_{t_i}, \mathbb{P}\right)$ for all $i = 0, \cdots, p_n$. We define $\mathrm{I}(\theta_n)$ as

$$\mathrm{I}(\theta_n) = \sum_{i=0}^{p_n} \alpha_i \left(W(t_{i+1}) - W(t_i)\right).$$

It can be confirmed that $\mathbb{E}(\mathrm{I}(\theta_n)) = 0$, and

$$\mathrm{V}(\mathrm{I}(\theta_n)) = \mathbb{E}\left(\int_0^T \theta_n^2(s)ds\right).$$

Let H denote the space of caglad (left-continuous and right-limited), $\mathcal{F}_t-$ adapted processes θ such that

$$||\theta||^2 = \mathbb{E}\left(\int_0^T |\theta(s)|^2 ds\right) < \infty.$$

We can define $\mathrm{I}(\theta)$ for any $\theta \in H$. We approximate θ using a sequence of step processes given by equation (1.2), and the limit exists in $L^2(\Omega,[0,T])$. The integral $\mathrm{I}(\theta)$ is then defined as $\lim_{n\to+\infty} \mathrm{I}(\theta_n)$, where $\mathbb{E}(\mathrm{I}(\theta)) = 0$, and

$$\mathrm{V}(\mathrm{I}(\theta)) = \mathbb{E}\left(\int_0^T \theta^2(s) ds\right).$$

Definition 1.3.1 (Itô Process) *An Itô process is defined as a real-valued process $(X(t))_{t\in[0,T]}$ satisfying the following conditions almost surely,*

$$X(t) = X(0) + \int_0^t F(s)ds + \int_0^t G(s)dW(s), \quad \textit{for } 0 \le t \le T. \tag{1.3}$$

Alternatively, it can be expressed differentially as,

$$dX(t) = F(t)dt + G(t)dW(t).$$

Here, $X(0)$ is $\mathcal{F}_0$-measurable, and F and G are two progressively measurable processes, which satisfy almost surely,

$$\int_0^T |F(s)|ds < \infty, \quad \textit{and} \quad \int_0^T |G(s)|^2 ds < \infty.$$

In other words, $F \in \mathrm{L}^1_{\mathcal{F}_t}[0,T]$ and $G \in \mathrm{L}^2_{\mathcal{F}_t}[0,T]$. The coefficient F represents the drift or derivative, and G is the diffusion coefficient.

Definition 1.3.2 (Integration by Parts Formula) *If X_1 and X_2 are two Itô processes, where*

$$X_1(t) = X_1(0) + \int_0^t F_1(s)ds + \int_0^t G_1(s)dW(s),$$

and

$$X_2(t) = X_2(0) + \int_0^t F_2(s)ds + \int_0^t G_2(s)dW(s),$$

then the integration by parts formula states that

$$X_1(t)X_2(t) = X_1(0)X_2(0) + \int_0^t X_1(s)dX_2(s) + \int_0^t X_2(s)dX_1(s) + \langle X_1, X_2\rangle_t,$$

where

$$\langle X_1, X_2\rangle_t = \int_0^t G_1(s)G_2(s)ds.$$

Definition 1.3.3 (Itô's Formula) *Let $F \in \mathrm{L}^1_{\mathcal{F}_t}[0,T]$, $G \in \mathrm{L}^2_{\mathcal{F}_t}[0,T]$, and let X be an Itô process defined as in (1.3). Define $\langle X(t)\rangle = \int_0^t |G(s)|^2 ds$. If $\kappa \in \mathcal{C}^{1,2}([0,T]\times\mathbb{R},\mathbb{R})$, then*

$$\begin{aligned} d\kappa(t,X(t)) &= \partial_t\kappa(t,X(t))dt + \partial_x\kappa(t,X(t))dX(t) + \frac{1}{2}\partial_{xx}\kappa(t,X(t))d\langle X\rangle_t, \\ &= (\partial_t\kappa(t,X(t)) + \partial_x\kappa(t,X(t))F(t) + \frac{1}{2}|G(t)|^2\partial_{xx}\kappa(t,X(t)))dt \\ &+ \partial_x\kappa(t,X(t))G(t)dW(t). \end{aligned}$$

Alternatively, in integral form

$$\begin{aligned} \kappa(t,X(t)) = \kappa(0,X(0)) &+ \int_0^t \partial_t\kappa(s,X(s))ds \\ &+ \int_0^t \left(\partial_x\kappa(s,X(s))F(s) + \frac{1}{2}\partial_{xx}\kappa(s,X(s))|G(s)|^2\right)ds \end{aligned}$$

$$+\int_0^t \partial_x \kappa(s,X(s))G(s)dW(s).$$

1.4 Stochastic Differential Equation

Let $W=(W(t))_{t\geq 0}$ denote a d-dimensional Brownian motion on a filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P})$. For $T>0$, consider two functions $F:\Omega\times[0,T]\times\mathbb{R}^n\to\mathbb{R}^n$ and $G:\Omega\times[0,T]\times\mathbb{R}^n\to\mathbb{R}^{n\times d}$. Here, $\|G\|$ denotes the trace of $GG^\top$. We aim to solve the following stochastic differential equation

$$dX(t)=F(t,X(t))\,dt+G(t,X(t))\,dW(t).$$

Definition 1.4.1 *A (strong) solution to this stochastic differential equation is a* ***continuous*** *stochastic process $X=(X(t))_{t\geq 0}$ that satisfies*

$$X(t)=X(0)+\int_0^t F(s,X(s))\,ds+\int_0^t G(s,X(s))\,dW(s),\ \mathbb{P}-a.s.$$

with the property that

$$\int_0^T \left(|F(s,X(s))|+\|G(s,X(s))\|^2\right)ds<+\infty,\ \mathbb{P}-a.s.$$

Example 1.4.1 (Black-Scholes-Merton model) *Consider the SDE*

$$dX(t)=\eta X(t)dt+\sigma X(t)dW(t),\quad X(0)=1.$$

Take $\kappa(x) = \ln x$*, then* $\kappa'(x) = 1/x$ *and* $\kappa''(x) = -1/x^2$.

$$\begin{aligned}
d(\ln X(t)) &= \frac{1}{X(t)}dX(t) + \frac{1}{2}\left(-\frac{1}{X^2(t)}\right)\sigma^2 X^2(t)dt \\
&= \frac{1}{X(t)}(\eta X(t)dt + \sigma X(t)dW(t)) - \frac{1}{2}\sigma^2 dt \\
&= \left(\eta - \frac{1}{2}\sigma^2\right)dt + \sigma dW(t).
\end{aligned}$$

So that $Z(t) = \ln X(t)$ *satisfies*

$$dZ(t) = \left(\eta - \frac{1}{2}\sigma^2\right)dt + \sigma dW(t).$$

Its integral representation gives

$$Z(t) = Z(0) + \left(\eta - \frac{1}{2}\sigma^2\right)t + \sigma W(t),$$

and $X(t) = \exp(Z(t))$

$$X(t) = X(0)e^{\left(\eta - \frac{1}{2}\sigma^2\right)t + \sigma W(t)}.$$

Existence and Uniqueness of Strong Solutions

Consider the SDE

$$dX(t) = F(t, X(t))\,dt + G(t, X(t))\,dW(t)$$

and the following conditions:

(H1) Lipschitz continuity: There exist constants $L, K > 0$ such that for all $t \geq 0$, $x, y \in \mathbb{R}^n \times \mathbb{R}^n$,

$$|F(t,x) - F(t,y)|^2 + \|G(t,x) - G(t,y)\|^2 \leq L|x-y|^2.$$

(H2) Linear growth: For $t \geq 0$,

$$|F(t,x)|^2 + \|G(t,x)\|^2 \leq K(1+|x|^2).$$

(H3) $X(0)$ is independent of W, and $\mathbb{E}\left|X(0)\right|^2 < \infty$.

The following theorem is based on Banach's fixed point theorem:

Theorem 1.4.1 *Under conditions* (H1) − (H3), *the SDE has a unique solution* X. *Additionally, for every* $t \geq 0$, *the solution satisfies*

$$\mathbb{E}\left(\sup_{t\in[0,T]} |X(t)|^2\right) \leq C\mathbb{E}\left(1+\left|X(0)\right|^2\right) < +\infty.$$

where constant C *depends only on* K *and* T.

The proof of existence is carried out by successive approximations (or contraction principle), similar to that for ordinary differential equations (Picard iterations). It can be found in [10]. It is not hard to see, by using Gronwall's lemma, that the Lipschitz condition implies uniqueness.

1.5 Backward Stochastic Differential Equation

Let $W = (W(t))_{t\geq 0}$ denote a d-dimensional Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$. We go now to the concept of Backward stochastic differential equations. We recall the martingal representation theorem. First we assume that $(\mathcal{F}_t)_{t\in[0,T]}$ is generated by a d-dimensional standard Brownian motion $(W(t))_{t\in[0,T]}$. Then, we have

$$\mathcal{F}_t = \sigma(B_s; s \leq t) \vee \mathcal{N},$$

where $\mathcal{N}$ denotes the set of $\mathbb{P}$-null sets, and $\sigma_1 \vee \sigma_2$ denotes the σ-field generated by $\sigma_1 \cup \sigma_2$.

The theorem states that

Theorem 1.5.1 *For $\mathcal{F}_T^W$-measurable $\xi : \Omega \longrightarrow \mathbb{R}^n$, such that $\mathbb{E}(|\xi|^2) < \infty$, there exists a unique progressively-measurable, $\mathbb{R}^{n\times d}$-valued processes such that $\mathbb{E}\left[\int_0^T \|\sigma(t)\|^2 dt\right] < +\infty$*

$$\xi = \mathbb{E}[\xi] + \int_0^T \sigma(t) dW(t),$$

(which is generalized later for any $\mathcal{F}^W$-martingale).

The martingale representation theorem turns out to be a special case of backward stochastic differential equations where the generator is equal to zero. The general case was studied and proven by Pardoux and Peng.

We consider now the following equation

$$-dX(t) = F(t, X(t), Z(t))dt - Z(t)dW(t), \quad X(T) = \xi, \tag{1.4}$$

or using the integral form

$$X(t) = \xi + \int_t^T F(s, X(s), Z(s))ds - \int_t^T Z(s)dW(s),$$

such that for every $(x,z) \in \mathbb{R}^n \times \mathbb{R}^{n\times d}$, the process $(F(t,x,z))_{t\geq 0}$ is progressively measurable.

Definition 1.5.1 (BSDE Solution) *A couple $(X(t), Z(t))_{t\geq 0}$ is said to be a solution to the backward SDE (1.4) if and only if*

(i) *X and Z are progressively measurable.*

(ii) $\mathbb{E}\left[\sup_{0\leq t\leq T} |X(t)|^2\right] + \mathbb{E}\left[\int_0^T \|Z(t)\|^2 dt\right] < \infty.$

(iii) *We have* $\mathbb{P}$*-a.s*

$$X_t = \xi + \int_t^T F(s, X(s), Z(s))ds - \int_t^T Z(s)dW(s), \quad 0 \le t \le T.$$

Existence and Uniqueness Theorem

Consider the backward SDE (1.4). We suppose that the following properties hold

(A1) F is Lipschitz-continuous in (x, z) : There exists a constant L, For all $(t, x, x', z, z') \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^{n\times d} \times \mathbb{R}^{n\times d}$

$$\left|F(t,x,z) - F\left(t,x',z'\right)\right|^2 \le L\left(\left|x-x'\right|^2 + \|z-z'\|^2\right).$$

(A2) The integrability conditionn

$$\mathbb{E}\left[\left|\xi^2\right| + \int_0^T |F(r,0,0)|^2 dr\right] < +\infty.$$

(A3) (Linear growth) There exists a constant K such that

$$|F(t,x,z)|^2 \le C_1\left(1 + |x|^2 + \|z\|^2\right); \text{ for all } x \in \mathbb{R}^n.$$

Then we have the following theorem

Theorem 1.5.2 (Pardoux-Peng) *Under conditions* (A1) − (A3)*, the backward SDE* (*1.4*) *has a unique solution.*

Lemma 1.5.1 *Consider the following backward SDE*

$$X(t) = \xi + \int_t^T F(s,X(s),Z(s))ds - \int_t^T Z(s)dW(s), \quad 0 \leq t \leq T,$$

then

$$X(t) = \mathbb{E}^t\left[\xi + \int_t^T F(s,X(s),Z(s))ds\right], \quad 0 \leq t \leq T.$$

Proof. Using conditional expectation

$$\begin{aligned}
\mathbb{E}[X(t)|\mathcal{F}_t] &= \mathbb{E}^t\left[\xi + \int_t^T F(s,X(s),Z(s))ds - \int_t^T Z(s)dW(s)\right] \\
&= \mathbb{E}^t\left[\xi + \int_t^T F(s,X(s),Z(s))ds - \int_0^T Z(s)dW(s) + \int_0^t Z(s)dW(s)\right] \\
&= \mathbb{E}^t\left[\xi + \int_t^T F(s,X(s),Z(s))ds\right] - \mathbb{E}^t\left[\int_0^T Z(s)dW(s) - \int_0^t Z(s)dW(s)\right] \\
&= \mathbb{E}^t\left[\xi + \int_t^T F(s,X(s),Z(s))ds\right] - \int_0^t Z(s)dW(s) + \int_0^t Z(s)dW(s) \\
&= \mathbb{E}^t\left[\xi + \int_t^T F(s,X(s),Z(s))ds\right].
\end{aligned}$$

■

1.6 Stochastic Maximum Principle

In an attempt to understand and to control stochastic systems, many articles and works were published, proposing new methods to approach these problems. We mention among them Bellman's Dynamical programming principle [1950's] and Pontryagin's stochastic maximum principle, which will be our topic in this section.

1.6.1 Optimization Problem Formulation

We consider a d-dimensional brownian motion W over a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$, is the natural filtration of the brownian motion W.

We define a control function $u : \Omega \times [0,T] \to \Gamma$. The control function u is usually referred to as a "decision" function. The space Γ represents the control constraint, which is usually a set to determine the image of the control based on the optimization problem (The amount of money spent in a month should not overpass the monthly income). Consider now the following problem

$$\begin{cases} dX(t) = b(t,X(t),u(t))dt + \sigma(t,X(t),u(t))dW(t), \\ X(0) = x_0, \end{cases} \tag{1.5}$$

where $b : \Omega \times [0,T] \times \mathbb{R}^n \times \Gamma \to \mathbb{R}^n, \quad \sigma : \Omega \times [0,T] \times \mathbb{R}^n \times \Gamma \to \mathbb{R}^{n\times d}$. We define the cost functional J

$$J(0,X(0),u(t)) = \mathbb{E}\left[\int_0^T f(t,X(t),u(t))dt + h(X(T))\right].$$

Definition 1.6.1 (Admissible control) *A control u is called an admissible control, and ($X(\cdot), .u(\cdot)$. an admissible pair if*

(i) $u(\cdot) \in \mathcal{U}[0,T]$, $X(\cdot)$ *is the unique solution to the equation (1.5).*

(ii) $L(\cdot, X(\cdot), .u(\cdot)) \in L^1_{\mathcal{F}}([0,T], \mathbb{R}), \quad h(X(T)) \in L^1(\Omega, \mathcal{F}_T; \mathbb{R})$.

1.6.2 Stochastic Maximum Principle

We suppose a finite-horizon stochastic control problem

$$dX(t) = b(t,X(t),u(t))dt + \sigma(s,X(t),u(t))dW(t),$$

with cost functional

$$J(0,X(0),u(t)) = \mathbb{E}\left[\int_0^T L(s,X(s),u(s))ds + g(X(T))\right],$$

where $L : \Omega\times[0,T]\times\mathbb{R}^n\times\Gamma \to \mathbb{R}$ is a continuous function in (t,x) for every $u \in \mathcal{U}_{ad}$, $g : \mathbb{R}^n \to \mathbb{R}$ is a C^1-convex function, and both f,g are of quadratic growth with respect to x.

Definition 1.6.2 (Generalized Hamiltonian) *The Generalized Hamiltonian is given by* $\mathcal{H} : [0,T]\times\mathbb{R}^n\times\Gamma\times\mathbb{R}^n\times\mathbb{R}^{n\times d} \to \mathbb{R}$

$$\mathcal{H}(t,x,u,p,q) = b(t,x,u)\cdot p + \text{trace}\left(\sigma'(t,x,u)\cdot q\right) + L(t,x,u).$$

Definition 1.6.3 (Adjoint equation) *We call Adjoint Equation the following backward SDE*

$$-dp(t) = D_x H(t,X(t),u(t),p(t),q(t))dt - q(t)dW(t), \quad X_T = D_x g(X_T).$$

Theorem 1.6.1 (Verification theorem) *Let $\tilde{u} \in \mathcal{U}_{ad}$ and let $\tilde{X}$ be the controlled diffusion. Suppose that there exists a solution $(\tilde{p},\tilde{q})$ to the corresponding adjoint equation 2.5 such that a.s*

$$\mathcal{H}(t,\tilde{X}(t),\tilde{u}(t),\tilde{p}(t),\tilde{q}(t)) = \min_{u\in\mathcal{U}_{ad}} \mathcal{H}(t,\tilde{X}(t),u(t),\tilde{p}(t),\tilde{q}(t)), \quad 0 \le t \le T$$

and suppose that

$$(x,u) \longrightarrow \mathcal{H}(t,x,u,\tilde{p}(t),\tilde{q}(t)),$$

is a convex function $\forall t \in [0,T]$. Then $\tilde{u}$ is an optimal control

$$J(0,X(0),\tilde{u}(t)) = \min_{u\in\mathcal{U}_{ad}} J(0,X(0),u(t)).$$

Chapter 2

Stochastic Maximum Principle for Time-Inconsistent LQ Control Problem

Consider a filtered probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}\right)$, where: $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets, $\mathcal{F}_T = \mathcal{F}$ for a fixed finite time horizon $T > 0$, and $(\mathcal{F}_t)_{t\in[0,T]}$ satisfies the usual conditions. Assume that $(\mathcal{F}_t)_{t\in[0,T]}$ is generated by a d-dimensional standard Brownian motion $(W(t))_{t\in[0,T]}$. Then, we have $\mathcal{F}_t = \sigma(B_s; s \leq t) \vee \mathcal{N}$, where $\mathcal{N}$ denotes the set of $\mathbb{P}$-null sets, and $\sigma_1 \vee \sigma_2$ denotes the σ-field generated by $\sigma_1 \cup \sigma_2$.

2.1 Problem Statement

We consider an n-dimensional non-homogeneous linear controlled diffusion system for $t \in [0,T]$, $\xi \in \mathbb{L}^2(\Omega, \mathcal{F}_t, \mathbb{P}; \mathbb{R}^n)$, $u(\cdot) \in \mathcal{L}^2_{\mathcal{F}}(t,T;\mathbb{R}^m)$.

$$\begin{cases} dX(s) = (A(s)X(s) + B(s)u(s) + b(s))\,ds \\ \qquad + \displaystyle\sum_{j=1}^{d} \big(C_j(s)X(s) + D_j(s)u(s) + \sigma_j(s)\big)\,dW^j(s) \quad s \in [t,T], \\ X(t) = \xi. \end{cases} \tag{2.1}$$

Under certain conditions, for any initial situation (t,ξ) and any admissible control $u(\cdot)$, the state equation is uniquely solvable. We denote by $X(\cdot) = X^{t,\xi}(\cdot;u(\cdot))$ its solution for $s \in [t,T]$. Different controls $u(\cdot)$ will lead to different solutions $X(\cdot)$. Note that $\mathcal{L}^2_{\mathcal{F}}(t,T;\mathbb{R}^m)$ is the space of all admissible strategies. Our aim is to minimize the following expected discounted cost functional

$$\begin{aligned} &J(t,\xi,u(\cdot)) \\ &= \mathbb{E}^t \Bigg\{ \frac{1}{2} \int_t^T \big(\langle Q(t,s)X(s), X(s)\rangle + \big\langle \bar{Q}(t,s)\mathbb{E}^t X(s), \mathbb{E}^t X(s)\big\rangle + \langle R(t,s)u(s), u(s)\rangle \big)\,ds \\ &+ \big\langle \eta_1(t)\xi + \eta_2(t), X(T)\big\rangle + \frac{1}{2}\big(\langle G(t)X(T), X(T)\rangle + \big\langle \bar{G}(t)\mathbb{E}^t X(T), \mathbb{E}^t X(T)\big\rangle \big) \Bigg\}, \end{aligned}$$

over $u(\cdot) \in \mathcal{L}^2_{\mathcal{F}}(t,T;\mathbb{R}^m)$, where $X(\cdot) = X^{t,\xi}(\cdot;u(\cdot))$ and $\mathbb{E}^t[\cdot] = \mathbb{E}[\cdot \mid \mathcal{F}_t]$. We need to impose the following assumptions about the coefficients:

(H1) The functions

$$A(\cdot), C_j(\cdot) : [0,T] \to \mathbb{R}^{n\times n}, \quad B(\cdot), D_j(\cdot) : [0,T] \to \mathbb{R}^{n\times m}, \; b(\cdot), \sigma_j(\cdot) : [0,T] \to \mathbb{R}^n,$$

are continuous and uniformly bounded. The coefficients in the cost functional satisfy

$$Q(\cdot,\cdot),\bar{Q}(\cdot,\cdot)\in\mathcal{C}(\mathcal{D}[0,T];S^n),\qquad R(\cdot,\cdot)\in\mathcal{C}(\mathcal{D}[0,T];S^m),$$

$$G(\cdot),\bar{G}(\cdot)\in\mathcal{C}([0,T];S^n),\qquad \eta_1(\cdot)\in\mathcal{C}([0,T];\mathbb{R}^{n\times n}),$$

$$\eta_2(\cdot)\in\mathcal{C}([0,T];\mathbb{R}^n).$$

(H2) The functions $R(\cdot,\cdot)$, $Q(\cdot,\cdot)$, and $G(\cdot)$ satisfy

$$R(t,t)\geq 0,\quad G(t)\geq 0,\quad \forall t\in[0,T],\text{ and } Q(t,s)\geq 0,\quad \forall(t,s)\in\mathcal{D}[0,T].$$

Theorem 2.1.1 *Under* (H1)*, for any* $(t,\xi,u(\cdot))\in[0,T]\times\mathbb{L}^2(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}^n)\times\mathcal{L}^2_{\mathcal{F}}(t,T;\mathbb{R}^m)$*, the state equation* (2.1) *has a unique solution* $X(\cdot)\in\mathcal{S}^2_{\mathcal{F}}(t,T;\mathbb{R}^n)$*. Moreover, we have the following estimate:*

$$\mathbb{E}\left[\sup_{t\leq s\leq T}|X(s)|^2\right]\leq K\left(1+\mathbb{E}\left[|\xi|^2\right]\right),$$

for some positive constant K*.*

The optimal control problem can be formulated as follows.

Definition 2.1.1 (Stochastic linear quadratic optimal control problems (LQ)) *For any given initial pair* $(t,\xi)\in[0,T]\times\mathbb{L}^2(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}^n)$*, find a control* $\hat{u}(\cdot)\in\mathcal{L}^2_{\mathcal{F}}(t,T;\mathbb{R}^m)$ *such that*

$$J(t,\xi,\hat{u}(\cdot))=\inf_{u(\cdot)\in\mathcal{L}^2_{\mathcal{F}}(t,T;\mathbb{R}^m)}J(t,\xi,u(\cdot)).$$

An example of time-inconsistent optimal control problem

We present a simple illustration of a stochastic optimal control problem which is time-inconsistent. Our aim is to show that the classical SMP approach is not efficient in the study of this problem if viewed as time-consistent.

For $n = d = 1$, consider the following controlled SDE starting from $(t,x) \in [0,T]\times\mathbb{R}$:

$$\begin{cases} dX^{t,x}(s) = bu(s)ds + \sigma dW(s), \quad s \in [t,T], \\ X^{t,x}(t) = x, \end{cases} \tag{2.2}$$

where b and σ are real constants.

We define the cost functional by

$$J(t,x,u(\cdot)) = \frac{1}{2}\mathbb{E}\left[\int_t^T |u(s)|^2 ds + h(t)\left(X^{t,x}(T) - x\right)^2\right], \tag{2.3}$$

where $h(\cdot) : [0,T] \to (0,\infty)$, is a general deterministic non-exponential discount function satisfying $h(0) = 1$, $h(s) \geq 0$ and $\displaystyle\int_0^T h(t)dt < \infty$.

We want to address the following stochastic control problem.

Problem 2.1.1 (E) *For any given initial pair* $(t,x) \in [0,T]\times\mathbb{R}$*, find a control* $\bar{u}(\cdot) \in \mathcal{L}^2_{\mathcal{F}}(t,T;\mathbb{R})$ *such that*

$$J(t,x,\bar{u}(\cdot)) = \inf_{u(\cdot)\in\mathcal{L}^2_{\mathcal{F}}(t,T;\mathbb{R})} J(t,x,u(\cdot)).$$

At a first stage, we consider the Problem (E) as a standard time consistent stochastic linear quadratic problem. Since $J(t,x,\cdot)$ is convex and coercive, there exists then a unique optimal control for this problem for each fixed initial pair $(t,x) \in [0,T]\times\mathbb{R}$.

The usual Hamiltonian associated to this Problem (E) is $\mathbb{H} : [0,T]\times\mathbb{R}^4 \to \mathbb{R}$ such that for every $(s,y,v,p,q) \in [0,T]\times\mathbb{R}^4$ we have

$$\mathbb{H}(s,y,v,p,q) = pbv + \sigma q - \frac{1}{2}v^2.$$

Let $u^{t,x}(\cdot)$ be an admissible control for $(t,x)\in[0,T]\times\mathbb{R}$.

The corresponding first order and second order adjoint equations are given respectively by

$$\begin{cases} dp^{t,x}(s) = q^{t,x}(s)dW(s), \quad s\in[t,T], \\ p^{t,x}(T) = -h(t)\left(X^{t,x}(T)-x\right), \end{cases}$$

and

$$\begin{cases} dP^{t,x}(s) = Q^{t,x}(s)dW(s), \quad s\in[t,T], \\ P^{t,x}(T) = -h(t), \end{cases}$$

the last equation has only the solution $(P^{t,x}(s),Q^{t,x}(s)) = (-h(t),0), \forall s\in[t,T]$.

The corresponding $\mathcal{H}$-function is given by

$$\mathcal{H}(s,y,v) = \mathbb{H}\left(s,y,v,p^{t,x}(s),q^{t,x}(s)\right) = p^{t,x}(s)bv + \sigma q^{t,x}(s) - \frac{1}{2}v^2,$$

which is a concave function of v.

Then according to the sufficient condition of optimality, for any fixed initial pair $(t,x)\in[0,T]\times\mathbb{R}$, Problem (E) is uniquely solvable with an optimal control $\bar{u}^{t,x}(\cdot)$ having the representation

$$\bar{u}^{t,x}(s) = b\bar{p}^{t,x}(s), \forall s\in[t,T],$$

such that the process $\left(\bar{p}^{t,x}(\cdot),.\bar{q}^{t,x}(\cdot).\right)$ is The unique adapted solution to the backward SDE

$$\begin{cases} d\bar{p}^{t,x}(s) = \bar{q}^{t,x}(s)dW(s), \quad s \in [t,T], \\ \bar{p}^{t,x}(T) = -h(t)\left(\bar{X}^{t,x}(s) - x\right). \end{cases}$$

By standard arguments, we can show that the processes $\left(\bar{p}^{t,x}(\cdot),\vec{q}^{t,x}(\cdot).\right)$ are explicitly given by

$$\begin{cases} \bar{p}^{t,x}(s) = -M^t(s)\left(\bar{X}^{t,x}(s) - x\right), \quad s \in [t,T], \\ \bar{q}^{t,x}(s) = -\sigma M^t(s), \quad s \in [t,T], \end{cases}$$

where $\bar{X}^{t,x}(\cdot)$ is the solution of the state equation corresponding to $\bar{u}^{t,x}(\cdot)$, given by

$$\begin{cases} d\bar{X}^{t,x}(s) = b^2\bar{p}^{t,x}(s)ds + \sigma dW(s), \quad s \in [t,T], \\ \bar{X}^{t,x}(t) = x. \end{cases}$$

and

$$M^t(s) = \frac{h(t)}{b^2h(t)(T-s)+1}, \quad \forall s \in [t,T].$$

A simple computation shows that

$$\bar{u}^{t,x}(s) = -\frac{bh(t)}{b^2h(t)(T-s)+1}\left(\bar{X}^{t,x}(s) - x\right), \quad \forall s \in [t,T],$$

clearly we have

$$\bar{u}^{t,x}(s) \neq 0, \quad \forall s \in (t,T]. \tag{2.4}$$

In the next stage, we will prove that the Problem (E) is time-inconsistent, for this we first fix the initial data $(t,x) \in [0,T]\times\mathbb{R}$. Note that, if we assume that the Problem (E) is time-consistent, in the sense that for any $r \in [t,T]$ the restriction of $\bar{u}^{t,x}(\cdot)$ on $[r,T]$ is optimal for Problem (E) with initial pair $\left(r,\bar{X}^{t,x}(r)\right)$, however as Problem (E) is uniquely solvable for any initial pair, we should have then $\forall r \in (t,T]$

$$\bar{u}^{t,x}(s)=\bar{u}^{r,\bar{X}^{t,x}(r)}(s)=-\frac{bh(r)}{b^2h(r)(T-s)+1}\left(\bar{X}^{r,\bar{X}^{t,x}(r)}(s)-\bar{X}^{t,x}(r)\right),\quad \forall s\in[r,T],$$

where $\bar{X}^{r,\hat{X}^{t,x}(r)}(\cdot)$ solves the SDE

$$\begin{cases} d\bar{X}^{r,\bar{X}^{t,x}(r)}(s)=b^2\dfrac{h(r)}{b^2h(r)(T-s)+1}\left(\bar{X}^{r,\bar{X}^{t,x}(r)}(s)-\bar{X}^{t,x}(r)\right)ds+\sigma dW(s),\quad \forall s\in[r,T],\\ \bar{X}^{r,\bar{X}^{t,x}(r)}(r)=\bar{X}^{t,x}(r). \end{cases}$$

In particular by the uniqueness of solution to the state SDE we should have

$$\bar{u}^{t,x}(r)=-\frac{bh(r)}{b^2h(r)(T-r)+1}\left(\bar{X}^{r,\bar{X}^{t,x}(r)}(r)-\bar{X}^{t,x}(r)\right)=0,$$

is the only optimal solution of the Problem (E), this contradicts (2.4). Therefore, the Problem (E) is not time-consistent, and more precisely, the solution obtained by the classical SMP is wrong and the problem is rather trivial since the only optimal solution is equal to zero.

2.2 Characterization of equilibrium strategies

The purpose of this chapter is to characterize open-loop Nash equilibriums rather than focusing on optimal controls. Here is a detailed description of the game framework we will consider:

- We are examining a game where there is one player at each time point t within the interval $[0,T]$. This player embodies the controller at time t and is referred to as "player t".
- Player t has the ability to control the system exclusively at time t by adopting a strategy $u(t,\cdot):\Omega\to\mathbb{R}^m$.
- A control process $u(\cdot)$ is then viewed as a complete description of the chosen strategies of all players in the game.
- The reward to player t is given by the functional $J(t,\xi,u(\cdot))$, which depends only on the restriction of the control $u(\cdot)$ to the time interval $[t,T]$.

Noting that, for brevity, in the rest of the chapter, we suppress the subscript (s) for the coefficients $A(s),B(s),b(s),C_j(s),D_j(s)$ and $\sigma_j(s)$. In addition, sometimes we simply call $\hat{u}(\cdot)$ an equilibrium control instead of open-loop Nash equilibrium control when there is no ambiguity.

We first consider an equilibrium by local spike variation, given an admissible control $\hat{u}(\cdot)\in\mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^m)$. For any $t\in[0,T]$, $v\in\mathbb{L}^2(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}^m)$, and for any $\varepsilon>0$, define

$$u^{\varepsilon}(s)=\begin{cases}\hat{u}(s)+v, & \text{for } s\in[t,t+\varepsilon),\\ \\ \hat{u}(s), & \text{for } s\in[t+\varepsilon,T],\end{cases}\tag{2.5}$$

we have the following definition.

Definition 2.2.1 (Open-loop Nash equilibrium) *An admissible strategy* $\hat{u}(\cdot)\in\mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^m)$

is an open-loop Nash equilibrium control for Problem (LQ) if

$$\lim_{\varepsilon\downarrow 0}\frac{J(t,\hat{X}(t),u^{\varepsilon}(\cdot))-J(t,\hat{X}(t),\hat{u}(\cdot))}{\varepsilon}\geq 0, \tag{2.6}$$

for any $t\in[0,T]$ *and* $v\in\mathbb{L}^2(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}^m)$.

The corresponding equilibrium dynamics solve the following SDE

$$\begin{cases} d\hat{X}(s)=\{A\hat{X}(s)+B\hat{u}(s)+b\}ds+\sum_{j=1}^{d}\{C_j\hat{X}(s)+D_j\hat{u}(s)+\sigma_j\}dW^j(s),\forall s\in[0,T],\\ \hat{X}_0=x_0. \end{cases}$$

2.3 The flow of adjoint equations

We introduce the adjoint equations involved in the stochastic maximum principle which characterize the open-loop Nash equilibrium controls of Problem (LQ). First, define the Hamiltonian

Definition 2.3.1 *We define the Hamiltonian*

$$\mathbb{H}:\mathcal{D}[0,T]\times\mathbb{L}^1(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}^n)\times\mathbb{R}^m\times\mathbb{R}^n\times\mathbb{R}^{n\times d}\to\mathbb{R},$$

by

$$\begin{aligned}\mathbb{H}(t,s,X,u,p,q)=\langle p,AX+Bu+b\rangle+\sum_{j=1}^{d}\left\langle q_j,D_jX+C_ju+\sigma_j\right\rangle-\frac{1}{2}\langle R(t,s)u,u\rangle\\ -\frac{1}{2}\left(\langle Q(t,s)X,X\rangle+\left\langle\bar{Q}(t,s)\mathbb{E}^t[X],\mathbb{E}^t[X]\right\rangle\right).\end{aligned} \tag{2.7}$$

Let $\hat{u}(\cdot) \in \mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^m)$ and denote by $\hat{X}(\cdot)$ the corresponding controlled state process.

For each $t \in [0,T]$, we introduce the first order adjoint equation.

Definition 2.3.2 (The first order adjoint equation) *The first order adjoint equation is defined on the time interval $[t,T]$, and satisfied by the couple of processes $(p(.;t), q(.;t))$ as follows*

$$\begin{cases} dp(s;t) = -A^\top p(s;t)ds - \sum_{j=1}^{d} C_j^\top q_j(s;t)ds \\ \qquad\qquad +Q(t,s)\hat{X}(s)ds + \bar{Q}(t,s)\mathbb{E}^t[\hat{X}(s)]ds + \sum_{j=1}^{d} q_j(s;t)dW^j(s),\ \forall s \in [t,T], \\ p(T;t) = -G(t)\hat{X}(T) - \bar{G}(t)\mathbb{E}^t[\hat{X}(T)] - \eta_1(t)\hat{X}(t) - \eta_2(t), \end{cases} \tag{2.8}$$

where $q(.;t) = (q_1(.;t), \ldots, q_d(.;t))$.

For each $t \in [0,T]$, we introduce the second order adjoint equation.

Definition 2.3.3 (The second order adjoint equation) *The second order adjoint equation is defined on the time interval $[t,T]$, and satisfied by the couple of processes $(P(.;t), \Lambda(.;t))$ as follows*

$$\begin{cases} dP(s;t) = -A^\top P(s;t)ds - P(s;t)Ads - \sum_{j=1}^{d} \left(C_j^\top P(s;t)C_j + \Lambda_j(s;t)C_j + C_j^\top \Lambda_j(s;t) \right) ds \\ \qquad\qquad -Q(t,s)ds + \sum_{j=1}^{d} \Lambda_j(s;t)dW_s^j, s \in [t,T], \\ P(T;t) = -G(t), \end{cases} \tag{2.9}$$

where $\Lambda(.;t) = (\Lambda_1(.;t), \ldots, \Lambda_d(.;t))$.

Theorem 2.3.1 *Under* (H1) *the backward SDE* (2.8) *is uniquely solvable in* $\mathcal{S}^2_{\mathcal{F}}(t,T;\mathbb{R}^n) \times$

$\mathcal{L}_{\mathcal{F}}^2\left(t,T;\mathbb{R}^{n\times d}\right)$. *Moreover there exists a constant $K>0$ such that*

$$\mathbb{E}\left[\sup_{t\leq s\leq T}|p(s;t)|_{\mathbb{R}^n}^2\right]+\mathbb{E}\left[\int_t^T|q(s;t)|_{\mathbb{R}^{n\times d}}^2ds\right]\leq K\left(1+|x_0|^2\right). \tag{2.10}$$

Remark 2.3.1 *Noting that the final data of the equation (2.9) is deterministic, it is straightforward to look at a deterministic solution. In addition we have the following representation*

$$\begin{cases} dP(s;t)=-A^\top P(s;t)ds-P(s;t)Ads-\sum_{j=1}^d C_j^\top P(s;t)C_jds \\ \qquad +Q(t,s)ds, \forall s\in[t,T], \\ P(T;t)=-G(t), \end{cases} \tag{2.11}$$

which is a uniquely solvable matrix-valued ordinary differential equation.

Next, for each $t\in[0,T]$, associated with the 5 -tuple $(\hat{u}(\cdot),\hat{X}(\cdot),.p(.;t),q(.,t),P(.;t))$.

Definition 2.3.4 *We define the $\mathcal{H}_t$-function as follows*

$$\mathcal{H}_t(s,X,u)=\mathbb{H}(t,s,X,\hat{u}(s)+u,p(s;t),q(s;t))+\frac{1}{2}u^\top\left\{\sum_{j=1}^d D_j^\top P(s;t)D_j\right\}u, \tag{2.12}$$

where $(s,X,u)\in[t,T]\times\mathbb{L}^1(\Omega,\mathcal{F},\mathbb{P};\mathbb{R}^n)\times\mathbb{R}^m$.

Remark 2.3.2 *In the rest of the chapter, we will keep the following notation, for* $(s,t)\in\mathcal{D}[0,T]$

$$\delta\mathbb{H}(t;s)=\mathbb{H}(t,s,\hat{X}(s),\hat{u}(s)+u,p(s;t),q(s;t))-\mathbb{H}(t,s,\hat{X}(s),\hat{u}(s),p(s;t),q(s;t)).$$

A stochastic maximum principle for equilibrium controls

In this section, we present a detailed version of Pontryagin's stochastic maximum principle, which characterizes the equilibrium controls for the Problem (LQ). To derive this result, we use the second-order Taylor expansion in a specific form known as spike variation (2.5).

The following theorem represents a comprehensive and precise condition that is both necessary and sufficient for characterizing the open-loop Nash equilibrium controls in the context of the time-inconsistent Problem (LQ).

Theorem 2.3.2 (Stochastic Maximum Principle For Equilibriums) *Let* (H1) *holds. Then an admissible control* $\hat{u}(\cdot) \in \mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^m)$ *is an open-loop Nash equilibrium, if and only if, for any* $t \in [0,T]$*, there exist a unique couple of adapted processes* $(p(.;t), q(.;t))$ *which satisfy the backward SDE* (2.8) *and a deterministic matrix-valued function* $P(.;t)$ *which satisfies the ODE* (2.11)*, such that the following condition holds, for all* $u \in \mathbb{R}^m$

$$\delta\mathbb{H}(t;t) + \frac{1}{2}u^\top \left\{ \sum_{j=1}^{d} D_j^\top P(t;t) D_j \right\} u \leq 0, \ \mathbb{P}-a.s. \tag{2.13}$$

Or equivalently, we have the following two conditions. The first order equilibrium condition

$$R(t,t)\hat{u}(t) - B^\top p(t;t) - \sum_{j=1}^{d} D_j^\top q_j(t;t) = 0, \ \mathbb{P}-a.s, \tag{2.14}$$

and the second order equilibrium condition

$$R(t,t) - \sum_{j=1}^{d} D_j^\top P(t;t) D_j \geq 0. \tag{2.15}$$

Remark 2.3.3 *Note that for each $t \in [0,T]$, (2.8) and (2.9) are backward stochastic differential equations. So, as we consider all t in $[0,T]$, all their corresponding adjoint equations form essentially a "flow" of backward SDEs. Moreover, there is an additional constraint (2.13) which is equivalent to the conditions (2.14) and (2.15) that acts on the flow only when $s = t$, while the Pontryagin's stochastic maximum principle for optimal control involves only one system of forward-backward stochastic differential equation.*

2.3.1 Proof of Theorem 2.3.2

Our goal now is to give a proof of Theorem 2.3.2. The main idea is still based on the variational techniques in the same spirit of proving the stochastic Pontryagin's maximum principle.

Let $\hat{u}(\cdot) \in \mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^m)$ be an admissible control and $\hat{X}(\cdot)$ the corresponding controlled process solution to the state equation. Consider the perturbed control $u^{\varepsilon}(\cdot)$ defined by the spike variation (2.5) for some fixed arbitrary $t \in [0,T]$, $v \in \mathbb{L}^2(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}^m)$, and $\varepsilon \in [0,T-t]$. Denote by $X^{\varepsilon}(\cdot)$ the solution of the state equation corresponding to $u^{\varepsilon}(\cdot)$. Since the coefficients of the controlled state equation are linear, then by the standard perturbation approach, we have

$$\hat{X}^{\varepsilon}(s) = \hat{X}(s) + y^{\varepsilon}(s) + z^{\varepsilon}(s), \quad s \in [t,T], \tag{2.16}$$

where $y^{\varepsilon}(\cdot)$ and $z^{\varepsilon}(\cdot)$ solve the following linear stochastic differential equations, respectively

$$\begin{cases} dy^{\varepsilon}(s) = Ay^{\varepsilon}(s)ds + \sum_{j=1}^{d}\left\{C_j y^{\varepsilon}(s) + D_j v 1_{[t,t+\varepsilon)}(s)\right\}dW^j(s) \quad \forall s \in [t,T], \\ y^{\varepsilon}(t) = 0. \end{cases} \tag{2.17}$$

and

$$\begin{cases} dz^{\varepsilon}(s) = \left(Az^{\varepsilon}(s) + Bv1_{[t,t+\varepsilon)}(s)\right)ds + \sum_{j=1}^{d} C_j z^{\varepsilon}(s)dW^j(s) \quad \forall s \in [t,T], \\ z^{\varepsilon}(t) = 0. \end{cases} \tag{2.18}$$

First, we present the following technical lemma needed later in this study,

Lemma 2.3.1 *Under assumption* (H1)*, the following estimates hold*

$$\mathbb{E}^t\left[y^{\varepsilon}(s)\right] = 0, \ a.e. \ s \in [t,T] \text{ and } \sup_{s\in[t,T]} \left|\mathbb{E}^t\left[z^{\varepsilon}(s)\right]\right|^2 = O\left(\varepsilon^2\right), \tag{2.19}$$

$$\mathbb{E}^t \sup_{s\in[t,T]} \left|y^{\varepsilon}(s)\right|^2 = O(\varepsilon) \text{ and } \mathbb{E}^t \sup_{s\in[t,T]} \left|z^{\varepsilon}(s)\right|^2 = O\left(\varepsilon^2\right). \tag{2.20}$$

Moreover, we have the equality

$$\begin{aligned} &J\left(t,\hat{X}(t),u^{\varepsilon}(\cdot)\right) - J(t,\hat{X}(t),\hat{u}(\cdot)) \\ &= -\mathbb{E}^t\left[\int_t^T \left(\delta\mathbb{H}(t;s) + \frac{1}{2}\sum_{j=1}^{d} v^{\top}D_j^{\top}P(s;t)D_j v\right)1_{[t,t+\varepsilon)}(s)ds\right] + o(\varepsilon). \end{aligned} \tag{2.21}$$

Proof. Let $t \in [0,T], v \in \mathbb{L}^2(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}^m)$ and $\varepsilon \in [0,T-t]$. Since $\mathbb{E}^t\left[y^{\varepsilon}(\cdot).\right]$ and $\mathbb{E}^t\left[z^{\varepsilon}(\cdot).\right]$ solve the following ODEs, respectively

$$\begin{cases} d\mathbb{E}^t\left[y^{\varepsilon}(s)\right] = A\mathbb{E}^t\left[y^{\varepsilon}(s)\right]ds, s \in [t,T], \\ \mathbb{E}^t\left[y^{\varepsilon}(t)\right] = 0, \end{cases}$$

and

$$\begin{cases} d\mathbb{E}^t[z^\varepsilon(s)] = \left\{A\mathbb{E}^t[z^\varepsilon(s)] + B\mathbb{E}^t[v]1_{[t,t+\varepsilon)}(s)\right\}ds, s \in [t,T], \\ \mathbb{E}^t[z^\varepsilon(t)] = 0. \end{cases}$$

Thus, it is clear that $\mathbb{E}^t[y^\varepsilon(s)] = 0$, a.e. $s \in [t,T]$. According to Gronwall's inequality there exists a positive constant K such that

$$\sup_{s\in[t,T]} \left|\mathbb{E}^t[z^\varepsilon(s)]\right|^2 \leq K\varepsilon^2.$$

Moreover, by Lemma 2.1. in [11], we obtain (2.20). By these estimates, we can calculate the difference, we consider

$$\begin{aligned} &J(t,\hat{X}(t),u^\varepsilon(t)) - J(t,\hat{X}(t),\hat{u}(t)) \\ &= \mathbb{E}^t\Bigg\{\frac{1}{2}\int_t^T (\langle Q(t,s)X(s),X(s)\rangle - \langle Q(t,s)\hat{X}(s),\hat{X}(s)\rangle)ds \\ &+ \frac{1}{2}\int_t^T (\left\langle \bar{Q}(t,s)\mathbb{E}^t X^\varepsilon(s),\mathbb{E}^t X^\varepsilon(s)\right\rangle - \left\langle \bar{Q}(t,s)\mathbb{E}^t\hat{X}(s),\mathbb{E}^t\hat{X}(s)\right\rangle)ds \\ &+ \frac{1}{2}\int_t^T (\langle R(t,s)u^\varepsilon(s),u^\varepsilon(s)\rangle - \langle R(t,s)\hat{u}(s),\hat{u}(s)\rangle)ds \\ &+ \left\langle \eta_1(t)\hat{X}(t)+\eta_2(t),X^\varepsilon(T)\right\rangle - \left\langle \eta_1(t)\hat{X}(t)+\eta_2(t),\hat{X}(T)\right\rangle \\ &+ \frac{1}{2}(\langle G(t)X^\varepsilon(T),X^\varepsilon(T)\rangle - \langle G(t)\hat{X}(T),\hat{X}(T)\rangle) \\ &+ \frac{1}{2}(\left\langle \bar{G}(t)\mathbb{E}^t X^\varepsilon(T),\mathbb{E}^t X^\varepsilon(T)\right\rangle - \left\langle \bar{G}(t)\mathbb{E}^t\hat{X}(T),\mathbb{E}^t\hat{X}(T)\right\rangle)\Bigg\}. \end{aligned}$$

Because, $X^{\varepsilon}(s)-\hat{X}(s)=y^{\varepsilon}(s)+z^{\varepsilon}(s)$, and

$$u^{\varepsilon}(s)-\hat{u}(s)=\begin{cases} v, & \text{for } s\in[t,t+\varepsilon) \\ \\ 0, & \text{for } s\in[t+\varepsilon,T] \end{cases} = v1_{[t,t+\varepsilon)}(s),$$

By the fact that $\langle ab,b\rangle-\langle ac,c\rangle=\langle a(b-c),b-c\rangle+2\langle ac,b-c\rangle$ then

$$\begin{aligned}
&J(t,\hat{X}(t),u^{\varepsilon}(t))-J(t,\hat{X}(t),\hat{u}(t))\\
&=\mathbb{E}^{t}\Big\{\frac{1}{2}\int_{t}^{T}\langle Q(t,s)\left[y^{\varepsilon}(s)+z^{\varepsilon}(s)+2\hat{X}(s)\right],[y^{\varepsilon}(s)+z^{\varepsilon}(s)]\rangle ds\\
&+\frac{1}{2}\int_{t}^{T}\langle\bar{Q}(t,s)\mathbb{E}^{t}\left[y^{\varepsilon}(s)+z^{\varepsilon}(s)+2\hat{X}(s)\right],\mathbb{E}^{t}\left[y^{\varepsilon}(s)+z^{\varepsilon}(s)\right]\rangle ds\\
&+\frac{1}{2}\int_{t}^{T}\langle R(t,s)\left[v+2\hat{u}(s)\right],v\rangle 1_{[t,t+\varepsilon)}(s)ds\\
&+\left\langle\eta_{1}(t)\hat{X}(t)+\eta_{2}(t),(y^{\varepsilon}(T)+z^{\varepsilon}(T))\right\rangle\\
&+\frac{1}{2}\left\langle G(t)\left[y^{\varepsilon}(T)+z^{\varepsilon}(T)+2\hat{X}(t)\right],[y^{\varepsilon}(T)+z^{\varepsilon}(T)]\right\rangle\\
&+\frac{1}{2}\left\langle\bar{G}(t)\mathbb{E}^{t}\left[y^{\varepsilon}(T)+z^{\varepsilon}(T))+2\hat{X}(t)\right],\mathbb{E}^{t}\left[y^{\varepsilon}(T)+z^{\varepsilon}(T)\right]\right\rangle\Big\}.
\end{aligned}$$

In the other hand, from (H1) and (2.19) - (2.20) the following estimate holds

$$\begin{aligned}
&\mathbb{E}^{t}\left[\frac{1}{2}\int_{t}^{T}\left(\langle\bar{Q}(t,s)\mathbb{E}^{t}(y^{\varepsilon}(s)+z^{\varepsilon}(s)),\mathbb{E}^{t}(y^{\varepsilon}(s)+z^{\varepsilon}(s))\rangle\right)ds\right.\\
&\left.+\frac{1}{2}\left(\langle\bar{G}(t)\mathbb{E}^{t}(y^{\varepsilon}(T)+z^{\varepsilon}(T)),\mathbb{E}^{t}(y^{\varepsilon}(T)+z^{\varepsilon}(T))\rangle\right)\right]\\
&=o(\varepsilon).
\end{aligned}$$

Then

$$
\begin{aligned}
&J(t,\hat{X}(t),u^{\varepsilon}(t))-J(t,\hat{X}(t),\hat{u}(t))\\
&=\mathbb{E}^{t}\Bigg\{\frac{1}{2}\int_{t}^{T}\langle Q(t,s)[y^{\varepsilon}(s)+z^{\varepsilon}(s)],[y^{\varepsilon}(s)+z^{\varepsilon}(s)]\rangle ds\\
&+\frac{1}{2}\int_{t}^{T}\langle Q(t,s)\big[2\hat{X}(s)\big]+\bar{Q}(t,s)\mathbb{E}^{t}\big[2\hat{X}(s)\big],[y^{\varepsilon}(s)+z^{\varepsilon}(s)]\rangle ds\\
&+\frac{1}{2}\int_{t}^{T}\langle R(t,s)[v+2\hat{u}(s)],v\rangle 1_{[t,t+\varepsilon)}(s)ds\\
&+\big\langle\eta_{1}(t)\hat{X}(t)+\eta_{2}(t)+G(t)\hat{X}(T)+\bar{G}(t)\mathbb{E}^{t}[\hat{X}(T)],y^{\varepsilon}(T)+z^{\varepsilon}(T)\big\rangle\\
&+\frac{1}{2}\langle G(t)[y^{\varepsilon}(T)+z^{\varepsilon}(T)],[y^{\varepsilon}(T)+z^{\varepsilon}(T)]\rangle\Bigg\}\\
&+o(\varepsilon).
\end{aligned}
$$

Then, from the terminal conditions (2.8) and (2.9) (in the adjoint equations), it follows that

$$
\begin{aligned}
&J\big(t,\hat{X}(t),u^{\varepsilon}(t)\big)-J(t,\hat{X}(t),\hat{u}(t))\\
&=\mathbb{E}^{t}\Bigg\{\frac{1}{2}\int_{t}^{T}\langle Q(t,s)[y^{\varepsilon}(s)+z^{\varepsilon}(s)],[y^{\varepsilon}(s)+z^{\varepsilon}(s)]\rangle ds\\
&+\frac{1}{2}\int_{t}^{T}\langle Q(t,s)\big[2\hat{X}(s)\big]+\bar{Q}(t,s)\mathbb{E}^{t}\big[2\hat{X}(s)\big],[y^{\varepsilon}(s)+z^{\varepsilon}(s)]\rangle ds\\
&+\frac{1}{2}\int_{t}^{T}\langle R(t,s)[v+2\hat{u}(s)],v\rangle 1_{[t,t+\varepsilon)}(s)ds\\
&-\langle p(T;t),y^{\varepsilon}(T)+z^{\varepsilon}(T)\rangle\\
&-\frac{1}{2}\langle P(T;t)(y^{\varepsilon}(T)+z^{\varepsilon}(T)),y^{\varepsilon}(T)+z^{\varepsilon}(T)\rangle\Bigg\}\\
&+o(\varepsilon).
\end{aligned}
\tag{2.22}
$$

Recalling that $(p(\cdot;t),Q(\cdot,t),\bar{Q}(\cdot,t))$ solve (2.8). Now, by applying Ito's formula on $[t,T]$ to $\langle p(s;t),y^{\varepsilon}(s)+z^{\varepsilon}(s)\rangle$ and by taking the conditional expectation, we get

$$
\begin{aligned}
&\mathbb{E}^t\left[\langle p(T;t),y^{\varepsilon}(T)+z^{\varepsilon}(T)\rangle\right]\\
&=\mathbb{E}^t\left\{\int_t^T\left\{v^{\top}B^{\top}p(s;t)1_{[t,t+\varepsilon)}(s)+(y^{\varepsilon}(s)+z^{\varepsilon}(s))^{\top}\left(Q(t,s)\hat{X}(s)+\bar{Q}(t,s)\mathbb{E}^t[\hat{X}(s)]\right)\right.\right.\\
&\quad\left.\left.+\sum_{j=1}^{d}v^{\top}D_j^{\top}q_j(s;t)1_{[t,t+\varepsilon)}(s)\right\}ds\right\}.
\end{aligned}
\tag{2.23}
$$

Again, $(p(\cdot;t),D(\cdot,t),\bar{Q}(\cdot,t))$ solve (2.11), by applying Ito's formula to

$$
s\mapsto\langle P(s;t)(y^{\varepsilon}(s)+z^{\varepsilon}(s)),y^{\varepsilon}(s)+z^{\varepsilon}(s)\rangle
$$

on $[t,T]$, we get by taking the conditional expectation

$$
\begin{aligned}
&\mathbb{E}^t\left[\langle P(T;t)(y^{\varepsilon}(T)+z^{\varepsilon}(T)),y^{\varepsilon}(T)+z^{\varepsilon}(T)\rangle\right]\\
&=\mathbb{E}^t\left\{\int_t^T\left\{2(y^{\varepsilon}(s)+z^{\varepsilon}(s))^{\top}P(s;t)Bv1_{[t,t+\varepsilon)}(s)\right.\right.\\
&+(y^{\varepsilon}(s)+z^{\varepsilon}(s))^{\top}Q(t,s)(y^{\varepsilon}(s)+z^{\varepsilon}(s))\\
&\quad\left.+\sum_{j=1}^{d}\left\{2(y^{\varepsilon}(s)+z^{\varepsilon}(s))^{\top}C_j^{\top}+v^{\top}D_j^{\top}\right\}P(s;t)D_jv1_{[t,t+\varepsilon)}(s)ds\right\}.
\end{aligned}
\tag{2.24}
$$

Moreover, we conclude from (H1) together with (2.8) and (2.10) that

$$
\mathbb{E}^t\left[\int_t^T(y^{\varepsilon}(s)+z^{\varepsilon}(s))^{\top}P(s;t)Bv1_{[t,t+\varepsilon)}(s)ds\right]=o(\varepsilon),
$$

$$\mathbb{E}^t\left[\int_t^T\left(y^\varepsilon(s)+z^\varepsilon(s)\right)^\top C_j^\top P(s;t)D_j v 1_{[t,t+\varepsilon)}(s)ds\right]=o(\varepsilon),. \tag{2.25}$$

Then by invoking (2.25) it holds

$$\begin{aligned}
&\frac{1}{2}\mathbb{E}^t\left[\langle P(T;t)(y^\varepsilon(T)+z^\varepsilon(T)),y^\varepsilon(T)+z^\varepsilon(T)\rangle\right]\\
&=\frac{1}{2}\mathbb{E}^t\left[\int_t^T\left\{\left(y^\varepsilon(s)+z^\varepsilon(s)\right)^\top Q(t,s)\left(y^\varepsilon(s)+z^\varepsilon(s)\right)\right.\right.\\
&\left.\left.+\sum_{j=1}^d v^\top D_j^\top P(s;t)D_j v 1_{[t,t+\varepsilon)}(s)\right\}ds\right]+o(\varepsilon).
\end{aligned} \tag{2.26}$$

By taking (2.23) and (2.26) in (2.22), it follows that

$$\begin{aligned}
&J\left(t,\hat{X}(t),u^\varepsilon(\cdot)\right)-J(t,\hat{X}(t),\hat{u}(\cdot))\\
&=-\mathbb{E}^t\left[\int_t^T\left\{v^\top B^\top p(s;t)+\sum_{j=1}^d v^\top D_j^\top q_j(s,t)\right.\right.\\
&\left.\left.+\frac{1}{2}\sum_{j=1}^d v^\top D_j^\top P(s;t)D_j v-v^\top R(t,s)\hat{u}(s)-\frac{1}{2}v^\top R(t,s)v\right\}1_{[t,t+\varepsilon)}(s)ds\right]+o(\varepsilon),
\end{aligned}$$

which is equivalent to (2.21) ■

Now, we are ready to give the proof of the Theorem 2.3.2.

Proof of Theorem 2.3.2 . Given an open-loop Nash equilibrium $\hat{u}(\cdot)$, then for any $t\in[0,T]$ and $v\in\mathbb{L}^2(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R}^m)$, we have clearly

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\left(J(t,\hat{X}(t),\hat{u}(\cdot))-J\left(t,\hat{X}(t),u^\varepsilon(\cdot)\right)\right)\leq 0,$$

which leads from (2.21) to

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\mathbb{E}^t\int_t^T\left(\delta\mathbb{H}(t;s)+\left(\frac{1}{2}\sum_{j=1}^d v^\top D_j^\top P(s;t)D_j v\right)1_{[t,t+\varepsilon)}(s)ds\right)\leq 0.$$

From which we deduce

$$\delta\mathbb{H}(t;t)+\frac{1}{2}v^\top\left(\sum_{j=1}^d D_j^\top P(t;t)D_j\right)v\leq 0,\mathbb{P}-a.s.$$

Therefore, the inequality (2.13) is ensured by stetting $v\equiv u$ for an arbitrarily $u\in\mathbb{R}^m$. Conversely, given an admissible control $\hat{u}(\cdot)\in\mathcal{L}^2_{\mathcal{F}}(0,T;\mathbb{R}^m)$. Suppose that for any $t\in[0,T]$, the variational inequality (2.13) holds. Then for any $v\in\mathbb{L}^2(\Omega,\mathcal{F}(t),\mathbb{P};\mathbb{R}^m)$ it yields

$$\delta\mathbb{H}(t;t)+\frac{1}{2}v^\top\left(\sum_{j=1}^d D_j^\top P(t;t)D_j\right)v\leq 0,\quad \mathbb{P}-a.s,$$

consequently

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\mathbb{E}^t\left[\int_t^{t+\varepsilon}\left(\delta\mathbb{H}(t;s)+\frac{1}{2}v^\top\left(\sum_{j=1}^d D_j^\top P(s;t)D_j\right)v\right)ds\right]\leq 0.$$

Hence

$$\lim_{\varepsilon\downarrow 0}\frac{1}{\varepsilon}\left(J(t,\hat{X}(t),\hat{u}(\cdot))-J\left(t,\hat{X}(t),u^\varepsilon(\cdot)\right)\right)\leq 0.$$

Thus $\hat{u}(\cdot)$ is an equilibrium control. Easy manipulations show that the variational inequality (2.13) is equivalent to

$$\mathcal{H}_t(t,\hat{X}(t),0)=\max_{u\in\mathbb{R}^m}\mathcal{H}_t(t,\hat{X}(t),u),$$

then (2.14) and (2.15) follow respectively from the following first order and second order conditions at the maximum point $u=0$ for the quadratic function $\mathcal{H}_t(t,\hat{X}(t),u)$

$$\mathcal{D}_u\mathcal{H}_t(t,\hat{X}(t),0)=0 \text{ and } \mathcal{D}_u^2\mathcal{H}_t(t,\hat{X}(t),u)\leq 0,$$

where we denote by $\mathcal{D}_u\mathcal{H}_t$ (resp. $\mathcal{D}_u^2\mathcal{H}_t$) the gradient (resp. the Hessian) of $\mathcal{H}_t$ with respect to the variable u. Then, the required result is directly follows. ■

2.4 Linear Feedback Stochastic Equilibrium Control

This section focuses on the case where the Brownian motion is one-dimensional ($d=1$) for ease of presentation, although the results generalize to multidimensional Brownian motions. Our aim is to derive a state feedback representation of an equilibrium control for Problem (LQ) using a class of ordinary differential equations. We begin by examining the following system of coupled generalized Riccati equations for $(t,s)\in\mathcal{D}[0,T]$

$$\begin{cases} \dfrac{\partial M}{\partial s}(t,s)+M(t,s)A+A^{\top}M(t,s)+C^{\top}M(t,s)C \\ -\left[M(t,s)B+C^{\top}M(t,s)D\right]\mathcal{Q}(s)+Q(t,s)=0, \\ \dfrac{\partial \bar{M}}{\partial s}(t,s)+\bar{M}(t,s)A+A^{\top}\bar{M}(t,s)-\bar{M}(t,s)B\mathcal{Q}(s)+\bar{Q}(t,s)=0, \\ \dfrac{\partial \Upsilon}{\partial s}(t,s)+A^{\top}\Upsilon(t,s)=0, \\ \dfrac{\partial \kappa}{\partial s}(t,s)+(M(t,s)+\bar{M}(t,s))(b-B\psi(s))+A^{\top}\kappa(t,s)+C^{\top}M(t,s)(\sigma-D\psi(s))=0, \\ M(t,T)=G(t),\bar{M}(t,T)=\bar{G}(t),\Upsilon(t,T)=\eta_1(t),\kappa(t,T)=\eta_2(t),t\in[0,T], \end{cases} \tag{2.27}$$

where

$$\det\left(R(t,t)+D^{\top}M(t,t)D\right)\neq 0,\forall t\in[0,T].$$

The functions $\Xi(\cdot),.\mathcal{Q}(\cdot)$ and $\psi(\cdot)$ are given for $t\in[0,T]$ by

$$\begin{cases}\Xi(t)=\left\{R(t,t)+D^{\top}M(t,t)D\right\}^{-1},\\ \mathcal{Q}(t)=\Xi(t)\left(B^{\top}(M(t,t)+\bar{M}(t,t)+\Upsilon(t,t))+D^{\top}M(t,t)C\right)\\ \psi(t)=\Xi(t)\left(B^{\top}\kappa(t,t)+D^{\top}M(t,t)\sigma\right).\end{cases}\tag{2.28}$$

Theorem 2.4.1 *Assuming that conditions* (H1) − (H2) *are satisfied, if a solution exists for the system* (*2.27*)*, then the stochastic control problem* (*2.6*) *subject to the stochastic differential equation* (*2.1*) *possesses a feedback Nash equilibrium solution given by*

$$\hat{u}(t)=-\mathcal{Q}(t)\hat{X}(t)-\psi(t),\quad\forall t\in[0,T].\tag{2.29}$$

Proof. Suppose $\hat{u}(t)$ represents an equilibrium control, and let $\hat{X}(t)$ denote the corresponding controlled process. According to Theorem 2.3.2, there exists an adapted process that solves the following forward-backward stochastic differential equation parameterized by

$t \in [0,T]$:

$$
\begin{cases}
d\hat{X}(s) = \{A\hat{X}(s) + B\hat{u}(s) + b\}ds + \{C\hat{X}(s) + D\hat{u}(s) + \sigma\}dW(s) \quad \forall s \in [0,T], \\
dp(s;t) = -\{A^\top p(s;t) + C^\top q(s;t) - Q(t,s)\hat{X}(s) - \bar{Q}(t,s)\mathbb{E}^t[\hat{X}(s)]\}ds \\
\qquad\qquad + q(s;t)dW(s), \quad \forall s \in [t,T], \\
\hat{X}_0 = x_0, p(T;t) = -G(t)\hat{X}(T) - \bar{G}(t)\mathbb{E}^t[\hat{X}(T)] - \eta_1(t)\hat{X}(t) - \eta_2(t), t \in [0,T],
\end{cases}
\tag{2.30}
$$

such that the ensuing condition is satisfied

$$
R(t,t)\hat{u}(t) - B^\top p(t;t) - D^\top q(t;t) = 0, \mathbb{P} - a.s, \forall t \in [0,T]. \tag{2.31}
$$

Now, to solve the above stochastic system, we propose a conjecture regarding the relationship between $\hat{X}(t)$ and $p(s;t)$ for t in the interval $[0,T)$. For any pair of times (t,s) within this range, we suggest that $p(s;t)$

$$
p(s,t) = -M(t,s)\hat{X}(s) - \bar{M}(t,s)\mathbb{E}^t\left[\hat{X}(s)\right] - \Upsilon(t,s)\hat{X}(t) - \kappa(t,s), \tag{2.32}
$$

can be expressed as a combination of terms involving $\hat{X}(s)$, along with specific deterministic functions of t and s. These functions belong to the class $\mathcal{C}^{0,1}(\mathcal{D}[0,T],\mathbb{R}^{n\times n})$ for $M(.,)$, $\bar{M}(.,)$, $\Upsilon(\cdot,\cdot)$, and to $\mathcal{C}^{0,1}(\mathcal{D}[0,T],\mathbb{R}^n)$ for $\kappa(\cdot,\cdot)$. They satisfy certain conditions at the final time T, namely

$$
M(t,T) = G(t), \bar{M}(t,T) = \bar{G}(t), \Upsilon(t,T) = \eta_1(t), \kappa(t,T) = \eta_2(t). \tag{2.33}
$$

By applying Itô's formula to equation (2.32) and utilizing equation (2.30), we can derive

$$
\begin{aligned}
dp(s;t) &= -\frac{\partial M}{\partial s}(t,s)\hat{X}(s)ds - \frac{\partial \bar{M}}{\partial s}(t,s)\mathbb{E}^t[\hat{X}(s)]ds - \frac{\partial \Upsilon}{\partial s}(t,s)\hat{X}(t)ds - \frac{\partial \kappa}{\partial s}(t,s)ds \\
&- M(t,s)(A\hat{X}(s) + B\hat{u}(s) + b)ds - \bar{M}(t,s)\left(A\mathbb{E}^t[\hat{X}(s)] + B\mathbb{E}^t[\hat{u}(s)] + b\right)ds \\
&- M(t,s)(C\hat{X}(s) + D\hat{u}(s) + \sigma)dW(s) \\
&= -\left[A^\top p(s;t) + C^\top q(s;t) - Q(t,s)\hat{X}(s) - \bar{Q}(t,s)\mathbb{E}^t[\hat{X}(s)]\right]ds \\
&+ q(s;t)dW(s). \qquad (2.34)
\end{aligned}
$$

From the equation above, we can derive the expression for $q(s;t)$ as follows:

$$
q(s;t) = -M(t,s)(C\hat{X}(s) + D\hat{u}(s) + \sigma). \qquad (2.35)
$$

Substituting the expression for $q(s;t)$ into equation (2.31), we get:

$$
\begin{aligned}
& R(t,t)\hat{u}(t) + B^\top((M(t,t) + \bar{M}(t,t) + \Upsilon(t,t))\hat{X}(t) + \kappa(t,t)) \\
&+ D^\top M(t,t)(C\hat{X}(t) + D\hat{u}(t) + \sigma) \\
&= R(t,t)\hat{u}(t) + B^\top M(t,t)\hat{X}(t) + B^\top \bar{M}(t,t)\hat{X}(t) \\
&+ B^\top \Upsilon(t,t)\hat{X}(t) + B^\top \kappa(t,t) + D^\top M(t,t)C\hat{X}(t) \\
&+ D^\top M(t,t)D\hat{u}(t) + D^\top M(t,t)\sigma \\
&= \left(R(t,t) + D^\top M(t,t)D\right)\hat{u}(t) \\
&+ \left(B^\top((M(t,t) + \bar{M}(t,t) + \Upsilon(t,t)) + D^\top M(t,t)C\right)\hat{X}(t) \\
&+ B^\top \kappa(t,t) + D^\top M(t,t)\sigma
\end{aligned}
$$

$$= \Xi(t)^{-1}\hat{u}(t) + \Xi(t)^{-1}\mathcal{Q}(t)\hat{X}(t) + \Xi(t)^{-1}\psi(t).$$

With the above notations, we obtain:

$$\Xi(t)^{-1}(\hat{u}(t) + \mathcal{Q}(t)\hat{X}(t) + \psi(t)) = 0, \quad \forall t \in [0,T].$$

Thus, equation (2.29) holds, and for any $(t,s) \in \mathcal{D}[0,T]$, we have

$$\mathbb{E}^t[\hat{u}(s)] = -\mathcal{Q}(s)\mathbb{E}^t[\hat{X}(s)] - \psi(s). \tag{2.36}$$

Next, by comparing the ds term in equation (2.34) with the one in the backward SDE from equation (2.30), and using the expressions in equations (2.29) and (2.36), we obtain:

$$\begin{aligned}
&\Big\{\frac{\partial M}{\partial s}(t,s) + M(t,s)A + A^\top M(t,s) + C^\top M(t,s)C\Big\}\hat{X}(s)\\
&-\big\{\big(M(t,s)B + C^\top M(t,s)D\big)\mathcal{Q}(s) + Q(t,s)\big\}\hat{X}(s)\\
&+\Big\{\frac{\partial \bar{M}}{\partial s}(t,s) + \bar{M}(t,s)A + A^\top \bar{M}(t,s) - \bar{M}(t,s)B\mathcal{Q}(s) + \bar{Q}(t,s)\Big\}\mathbb{E}^t[\hat{X}(s)]\\
&+\Big\{\frac{\partial \Upsilon}{\partial s}(t,s) + A^\top \Upsilon(t,s)\Big\}\hat{X}(t) + \frac{\partial \kappa}{\partial s}(t,s) + (M(t,s) + \bar{M}(t,s))(b - B\psi(s))\\
&+ A^\top \kappa(t,s) + C^\top M(t,s)(\sigma - D\psi(s)) = 0.
\end{aligned}$$

This suggests that the functions $M(\cdot)$, $\bar{M}(\cdot)$, $\Upsilon(\cdot,\cdot)$, and $\kappa(\cdot,\cdot)$ solve the system (2.27). Note that we can verify that $\mathcal{Q}(\cdot)$ and $\psi(\cdot)$ in equation (2.28) are both uniformly bounded. Then, the following linear SDE for $s \in [0,T]$ can be derived from equation (2.29):

$$\begin{cases} d\hat{X}(s) = \{(A - BQ(s))\hat{X}(s) + b - B\psi(s)\}ds + \{(C - DQ(s))\hat{X}(s) + \sigma - D\psi(s)\}dW(s), \\ \hat{X}(0) = x_0, \end{cases}$$

which is uniquely solvable, and the following estimate holds

$$\mathbb{E}\left[\sup_{s\in[0,T]} |\hat{X}(s)|^2\right] \le K\left(1 + x_0^2\right).$$

Therefore, the control $\hat{u}$ defined by equation (2.29) is admissible. ■

Chapter 3

Application: General Discounting Linear Quadratic Regulator

In this chapter, we delve into an example of a general discounting, time-inconsistent LQ model. The objective is to minimize the expected cost functional earned during a finite time horizon:

$$J(t,\xi,u(\cdot)) = \frac{1}{2}\mathbb{E}^t\left[\int_t^T |u(s)|^2 ds + h(t)|X(T)-\xi|^2\right]. \tag{3.1}$$

Here, $h(\cdot):[0,T]\to(0,\infty)$ is a general deterministic non-exponential discount function satisfying $h(0)=1$, $h(s)\geq 0$, and $\int_0^T h(t)dt<\infty$. This optimization is subject to a controlled one-dimensional stochastic differential equation (SDE) parameterized by $(t,\xi)\in[0,T]\times\mathbb{L}^2(\Omega,\mathcal{F}_t,\mathbb{P};\mathbb{R})$:

$$\begin{cases} dX(s) = \{aX(s)+bu(s)\}ds+\sigma dW(s), \quad s\in[0,T], \\ X(t)=\xi, \end{cases} \tag{3.2}$$

with, a and b are real constants. This scenario presents a time-inconsistent version of the classical linear quadratic regulator problem. The objective is to control the system so that the final state $X(T)$ is close to ξ, while simultaneously minimizing the control energy (formalized by the running cost). It's noteworthy that in this case, time-inconsistency arises due to the terminal cost depending explicitly on both the current state ξ and the current time t. Consequently, there are two distinct sources of time-inconsistency. For this example, the system (2.27) reduces, for all $(t,s) \in \mathcal{D}[0,T]$, to:

$$\begin{cases} \frac{\partial M}{\partial s}(t,s) + 2aM(t,s) - b^2 M(t,s)\{M(s,s) + \bar{M}(s,s) + \Upsilon(s,s)\} = 0, \\ \frac{\partial \bar{M}}{\partial s}(t,s) + 2a\bar{M}(t,s) - b^2 \bar{M}(t,s)\{M(s,s) + \bar{M}(s,s) + \Upsilon(s,s)\} = 0, \\ \frac{\partial \Upsilon}{\partial s}(t,s) + a\Upsilon(t,s) = 0, \\ \frac{\partial \kappa}{\partial s}(t,s) + a\kappa(t,s) - b^2\{M(t,s) + \bar{M}(t,s)\}\kappa(s,s) = 0, \\ M(t,T) = h(t),\ \bar{M}(t,T) = 0,\ \Upsilon(t,T) = h(t),\ \kappa(t,T) = 0, \forall t \in [0,T]. \end{cases} \tag{3.3}$$

Certainly, the formulation for the solution to the homogeneous equation is given by:

$$\begin{cases} M(t,s) = c_1 e^{\int_s^T [2a - -b^2\{M(\tau,\tau)+\bar{M}(\tau,\tau)+\Upsilon(\tau,\tau)\}]d\tau}, \\ \bar{M}(t,s) = c_2 e^{\int_s^T [2a - -b^2\{M(\tau,\tau)+\bar{M}(\tau,\tau)+\Upsilon(\tau,\tau)\}]d\tau},, \\ \Upsilon(t,s) = c_3 e^{\int_s^T a d\tau}, \\ \kappa(t,s) = c_3 e^{\int_s^T a d\tau}, \\ M(t,T) = h(t), \bar{M}(t,T) = 0, \Upsilon(t,T) = h(t), \kappa(t,T) = 0, \forall t \in [0,T]. \end{cases} \tag{3.4}$$

By employing the method of "variation of constants, $\forall (t,s) \in \mathcal{D}[0,T]$, $c_1'(t,s) = 0$,, and thus $\forall (t,s) \in \mathcal{D}[0,T]$, $c_1(t,s) = c_1(t,t)$. By utilizing the terminal condition, we conclude that $\forall (t,s) \in \mathcal{D}[0,T]$, $c_1(t,s) = c_1(t,T) = M(t,T) = h(t)$. Furthermore, we can infer by analogy that

$$\begin{cases} M(t,s) = M(t,T)e^{\int_s^T \{2a-b^2(M(\tau,\tau)+\bar{M}(\tau,\tau)+\Upsilon(\tau,\tau))\}d\tau}, \\ \bar{M}(t,s) = \bar{M}(t,T)e^{\int_s^T \{2a-b^2(M(\tau,\tau)+\bar{M}(\tau,\tau)+\Upsilon(\tau,\tau))\}d\tau}, \\ \varkappa(t,s) = \varkappa(t,T)e^{a(T-s)} - b^2\int_s^T e^{a(\tau-s)}\{M(t,\tau)+\bar{M}(t,\tau)\}\varkappa(\tau,\tau)d\tau. \end{cases} \tag{3.5}$$

Alternatively, given $\bar{M}(t,T) = \varkappa(t,T) = 0$, equation (3.5), simplifies $\forall(t,s) \in \mathcal{D}[0,T]$ to

$$\begin{cases} M(t,s) = M(t,T)e^{\int_s^T \{2a-b^2(M(r,r)+\Upsilon(r,r))\}dr}, \\ \bar{M}(t,s) = 0, \\ \varkappa(t,s) = -b^2\int_s^T e^{a(\tau-s)}M(t,\tau)\varkappa(\tau,\tau)d\tau. \end{cases} \tag{3.6}$$

It's evident that if $M(t,s)$ is the solution to the initial equation (3.6), then:

$$\varkappa(s,s) = -b^2\int_s^T e^{a(\tau-s)}M(s,\tau)\varkappa(\tau,\tau)d\tau, \quad \forall s \in [0,T].$$

Thus, there exists some constant $L > 0$ such that $|\varkappa(s,s)| \le L\int_s^T |\varkappa(\tau,\tau)|d\tau$. By Gronwall's Lemma, we conclude that $\varkappa(s,s) = 0, \forall s \in [0,T]$. Therefore $\varkappa(t,s) = 0, \forall(t,s) \in \mathcal{D}[0,T]$, is the unique solution to the last equation in the system (3.6). Now, it remains to solve the first equation in the system (3.6). Upon examination, the first equation in the system (3.6) can be rewritten as:

$$\begin{cases} \frac{\partial M}{\partial s}(t,s) + 2aM(t,s) - b^2M(t,s)\{M(s,s)+\Upsilon(s,s)\} = 0, \quad \forall(t,s) \in \mathcal{D}[0,T], \\ M(t,T) = h(t). \end{cases} \tag{3.7}$$

We propose a solution of the form $M(t,s) = h(t)N(s)$, yielding the following ordinary differential equation (ODE) for $N(s)$,

$$\begin{cases} \dfrac{dN}{ds}(s) + \left(2a + b^2\Upsilon(s,s)\right)N(s) - b^2 h(s)N(s)^2 = 0, \quad s \in [0,T], \\ N(T) = 1. \end{cases} \tag{3.8}$$

Substituting $N(s) = \dfrac{1}{y(s)}$, we obtain an explicitly solvable equation,

$$\begin{cases} \dfrac{dy}{ds}(s) - \left(2a + b^2\Upsilon(s,s)\right)y(s) + b^2 h(s) = 0, \quad s \in [0,T], \\ y(T) = 1. \end{cases}$$

Its solution is given by,

$$y(s) = e^{-\int_s^T \{2a+b^2\Upsilon(\tau,\tau)\}d\tau}\left(1 + b^2\int_s^T e^{\int_\tau^T \{2a+b^2\Upsilon(l,l)\}dl} h(\tau)d\tau\right), \quad s \in [0,T].$$

In view of Theorem 2.4.1, the representation (2.29) of the Nash equilibrium control gives,

$$\hat{u}(s) = -b\{\Upsilon(s,s) + M(s,s)\}\hat{X}(s), \quad \forall s \in [0,T]. \tag{3.9}$$

The corresponding equilibrium dynamics solve the following stochastic differential equation,

$$\begin{cases} d\hat{X}(s) = \{a - b^2(\Upsilon(s,s) + M(s,s))\}\hat{X}(s)ds + \sigma dW(s), \quad s \in [0,T], \\ \hat{X}(0) = x_0. \end{cases} \tag{3.10}$$

Conclusion

A game-theoretic method was used in this dissertation, which was about dynamic decision problems with time inconsistency. We were able to get around the problems caused by time not being consistent by using open-loop Nash equilibrium controls instead of optimal controls. The method was based on the stochastic maximum principle, which meant that we looked at forward-backward stochastic differential equations in a planned way when they were at their highest levels. We used real-life examples to show how well and how widely our suggested study could be used, reaffirming its importance in solving problem in the real world.

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Abbreviations and Notations

The various abbreviations and notations used throughout this thesis are explained below: For any Euclidean space $H = \mathbb{R}^n, \mathbb{R}^{n\times m}$ or S^n with Frobenius norm $|\cdot|$, we define,

FBSDE	:	Forward-Backward-Stochastic-Differential-Equation.
Problem (LQ)	:	Problem Linear-Quadratic.
$\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$	:	A filtered probability space.
$\mathbb{E}(\cdot)$	:	Mathematical expectation.
$\mathbb{E}(\cdot\vert\mathcal{F}_t) = \mathbb{E}^t(\cdot)$	:	Mathematical conditional expectation.
$\mathrm{COV}(\cdot,\cdot)$	:	Mathematical covariance of the pair $(\cdot,\cdot)$ (V$(\cdot)$,Variance).
$\mathcal{C}([0,T];H)$:	:	the set of $f:[0,T]\to H$ such that f is continuous.
$\mathcal{D}[0,T]$	:	the set of $(t,s)\in[0,T]\times[0,T]$, such that $s\geq t$.
$\mathcal{C}(\mathcal{D}[0,T];H)$	:	the set of $f(.,):\mathcal{D}[0,T]\to H(t,s)$ such that $f(.,)$ is continuous.
$\mathcal{C}^{0,1}(\mathcal{D}[0,T];H)$:	:	the set of $f(\cdot):\mathcal{D}[0,T]\to H$ such that $f(.,)$ $and \dfrac{\partial f}{\partial s}(.,)$ are continuous.
$\mathcal{C}^{1,2}([0,T]\times\mathbb{R},\mathbb{R})$	:	$\phi:[0,T]\times\mathbb{R}\to\mathbb{R}\,\vert\,\phi$ is continuous (in t), $\dfrac{\partial\phi}{\partial x}, \dfrac{\partial^2\phi}{\partial x^2}$, exists and are continuous (in x).
$\mathbb{L}^p(\Omega,\mathcal{F}_t,\mathbb{P};H)$	:	the set of $\xi:\Omega\to H$ such that ξ is $\mathcal{F}_t$ measurable and $\mathbb{E}[\vert\xi\vert^p]<\infty,\ p\geq 1$
$.\mathcal{S}^2_{\mathcal{F}}(t,T;H)$	:	the set of $X(\cdot):[t,T]\times\Omega\to H$ such that $X(\cdot)$ is $(\mathcal{F}_s)_{s\in[t,T]}-$adapted, $s\mapsto X(s)$ is RCLL (right continuous with left limits), and $\mathbb{E}\sup\limits_{s\in[t,T]}\vert X(s)\vert^2<\infty$.
$\mathcal{L}^2_{\mathcal{F}}(t,T;H)$	:	the set of $X(\cdot):[t,T]\times\Omega\to H$ such that $X(\cdot)$ is $(\mathcal{F}_s)_{s\in[t,T]}-$adapted, and $\mathbb{E}\displaystyle\int_t^T\vert X(s)\vert^2 ds<\infty$.
S^n	:	the set of $n\times n$ symmetric real matrices.
$C^\top$	:	the transpose of the vector (or matrix) C.
$\langle .,. \rangle$	:	the inner product in some Euclidean space.

الملخص

هذه المذكرة تبحث في نوع معين من مشاكل اتخاذ القرارات الديناميكية اللينة الرباعية الاحتمالية، والتي تختلف عن المبدأ الاعتيادي للأمثلية بيلمان لأنها تظهر عدم التناسق مع مرور الوقت. يرجع ذلك إلى استخدام معاملات تخفيض عامة وتكاليف تربيعية في التكاليف الحالية والنهائية، مما يجعل الوضع غير متناسق مع مرور الوقت. نحن نستخدم مبدأ الحد الأقصى الاحتمالي لاختيار الضوابط للتوازن في نظام ناش مفتوح بدلاً من الضوابط الأمثل القياسية. لتخطيط التوازن، نستخدم تدفقاً من المعادلات التفاضلية الاحتمالية التقدمية والرجوعية مع شرط الحد الأقصى. نحصل على تمثيلات واضحة لاستراتيجيات التوازن على شكل ردود فعل عكسية من خلال فصل التدفق عن العملية المرافقة. هذه الطريقة مهمة لأنها توفر شروطاً ضرورية وكافية لوصف استراتيجيات التوازن، وهذا يختلف عن الأعمال المبنية على البرمجة الديناميكية ومعادلات هاملتونجاكوبيبيلمان الموسعة، التي عادةً ما تقدم شروطاً كافية فقط.

*A*bstract

This dissertation looks into a certain type of stochastic linear quadratic dynamic decision problems that are different from the usual Bellman optimality principle because they are time inconsistent. Since general discounting coefficients and quadratic terms are used in both the running and terminal costs, the situation is time inconsistent. We use the stochastic maximum principle to choose open-loop Nash equilibrium controls over standard optimal controls. We make plans for equilibrium by using a flow of forward-backward stochastic differential equations with a maximum condition. We get an explicit representations of equilibrium strategies in feedback form by separating the flow of the adjoint process. This method is important because it gives both necessary and sufficient conditions for describing equilibrium strategies. This is different from works that was based on dynamic programming and extended Hamilton-Jacobi-Bellman equations, which mostly only gave sufficient conditions.

Résumé

Cette thèse examine un certain type de problèmes de décision dynamique linéaire quadratique stochastique qui diffèrent du principe d'optimalité de Bellman habituel car ils sont inconsistants dans le temps. En raison de l'utilisation de coefficients d'actualisation généraux et de termes quadratiques dans les coûts courants et terminaux, la situation devient inconsistante dans le temps. Nous utilisons le principe du maximum stochastique pour choisir des contrôles d'équilibre de Nash en boucle ouverte plutôt que des contrôles optimaux standards.

Pour planifier l'équilibre, nous utilisons un flux d'équations différentielles stochastiques progressive rétrograde avec une condition de maximum. Nous obtenons des représentations explicites des stratégies d'équilibre sous forme de rétroaction en séparant le flux du processus adjoint. Cette méthode est importante car elle fournit des conditions à la fois nécessaires et suffisantes pour décrire les stratégies d'équilibre. Cela diffère des travaux basés sur la programmation dynamique et les équations étendues de Hamilton-Jacobi-Bellman, qui ne fournissaient généralement que des conditions suffisantes.