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Stochastic Maximum Principle for Optimal Control Problems In Progressive Structure

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Dedication

I dedicate this dissertation to my family, whose unwavering love, encouragement, and sacrifices have been the cornerstone of my journey. Your boundless support and belief in me have fueled my determination to reach this milestone. This achievement is as much yours as it is mine, and I am forever grateful for your enduring presence and guidance.

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Introduction

The stochastic optimal control problem is very important in control theory and presents a big challenge. At the heart of this problem is the maximum principle, which is crucial for finding the best control strategies. Over time, a lot of research has been done to understand this principle better, greatly advancing the field.

A key contribution to this area is Peng's work in 1990 [2]. He established the maximum principle for forward stochastic control systems without jumps. His approach was innovative because it dealt with the complex issue of non-convex control domains and included control variables in the diffusion term using second-order variation equations.

Building on Peng's work, Situ [3] made significant progress in 1994 by deriving the maximum principle for systems with jumps. However, Situ's model did not include the control variable in the jump coefficient, indicating the need for further improvement.

Tang and Li, [4], in 1994, filled this gap by proving the maximum principle for systems where the control variable affects both diffusion and jump coefficients. Their work provided new insights and methods, enhancing our understanding of stochastic control.

This master's dissertation represents another important step in the development of stochastic control theory. It aims to optimize control for systems with random jumps. Unlike previous models, it allows the integrand of stochastic integrals with respect to the compensated Poisson point process to be progressively measurable rather than predictable, as described by Tang and Li [4].

This approach offers a new perspective on the optimal control problem by eliminating the need for the incorrect estimate found in Tang and Li's work, specifically the third one in equation (2.10). This leads to more robust and efficient control strategies.

We start by giving the difference between the two definitions, Predictable Stochastic Process and Progressive Stochastic Process.

1. Predictable Stochastic Process :

- In a predictable stochastic process, the future behavior of the process can be predicted to some extent based on past and present information.
- Formally, a stochastic process $(X(t))_{t\geq 0}$ is said to be predictable if, for each t, the value of $X(t)$ can be predicted using information available up to time t.
- Predictable stochastic processes are often encountered in situations where there is a certain level of regularity or pattern that allows for reasonable forecasting.

2. Progressive Stochastic Process :

- A progressive stochastic process is one where the randomness or uncertainty evolves over time in a progressive or continuous manner.
- Unlike predictable processes, the future behavior of a progressive stochastic process may not be entirely predictable, even with complete knowledge of past and present information.
- These processes are often used to model phenomena where randomness accumulates or changes continuously, making it difficult to precisely predict future states.

In summary, the key distinction lies in the predictability of future states : predictable stochastic processes allow for some degree of prediction based on past and present information, while progressive stochastic processes involve randomness that evolves continuously over time, making future states harder to predict accurately.

The dissertation is structured as follows :

Chapter 1 : Covers basic concepts such as stochastic processes, stochastic Itô integrals, and stochastic differential equations.

Chapter 2 : Builds on this by introducing preliminaries on stochastic integrals with respect to jumps. It also discusses stochastic differential equations with jumps and establishes the existence and uniqueness of solutions for these equations.

Chapter 3 : Is the main part of the study. Using new spike variation techniques and second-order variation equations, it establishes the maximum principle in a detailed mathematical framework.

Chapitre 1

Introduction to Stochastic Computation

Let's embark on a journey through the intriguing realm of stochastic computation. This chapter serves as our gateway, introducing foundational terms and concepts that will pave the way for deeper exploration in subsequent sections. For more details see e.g [7, 5, 1]

1.1 Stochastic Processes

Definition 1.1.1 (Stochastic Process) *A stochastic process, symbolized by* $X = (X(t))_{t\in\mathcal{T}}$*, emerges as a collection of random variables defined within a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*, with values spanning* $\mathbb{R}^n$*.*

Now, let's elucidate some key terminology,

1. The variable t typically denotes time.
2. If the index set $\mathcal{T}$ is countable, we classify X as a discrete-time stochastic process ; for a continuous index set, it's a continuous-time stochastic process.
3. Common index sets include the half-line $[0, \infty)$ or a finite interval $[0, T]$, where $T > 0$.

4. Each $X(t)$ represents a random variable, mapping $\omega \longrightarrow X(t,\omega)$ for $\omega \in \Omega$.
5. Fixing $\omega \in \Omega$ transforms $X(\omega)$ into a function $t \longrightarrow X(t,\omega)$, known as a path of X.

Definition 1.1.2 (Modification of Process) *We term a stochastic process* $(X(t))_{t\in\mathcal{T}}$ *a modification of another process* $(\bar{X}(t))_{t\in\mathcal{T}}$ *if the probability* $\mathbb{P}(X(t)=\bar{X}(t))$ *equals* 1 *for all* $t \in \mathcal{T}$*.*

Definition 1.1.3 (Indistinguishable Processes) *When* $\mathbb{P}(X(t)=\bar{X}(t), \forall t \in \mathcal{T})$ *equals* 1*, we refer to two stochastic processes* $(X(t))_{t\in\mathcal{T}}$ *and* $(\bar{X}(t))_{t\in\mathcal{T}}$ *as indistinguishable.*

Remark 1.1.1 *Indistinguishable processes are modifications of each other, but the converse isn't always true.*

Definition 1.1.4 (Measurable stochastic process) *A stochastic process* $X = (X(t))_{t\in\mathcal{T}}$ *is measurable if the mapping* $X : [0,T] \times \Omega \to \mathbb{R}^n$ *is* $(\mathcal{B}([0,T]) \otimes \mathcal{F}, \mathcal{B})$ *measurable.*

Definition 1.1.5 (Filtration) *A filtration comprises a sequence of* σ*-algebras* $\{\mathcal{F}_t\}_{t\geq 0}$ *defined over a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*, wherein* $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$ *for all* $0 \leq s \leq t$*. This structure captures the evolving information accessible to an observer over time, with* $\mathcal{F}_t$ *representing distinguishable events up to time* t*.*

- A filtration is right continuous if $\mathcal{F}_{t^+} = \cap_{\varepsilon>0}\mathcal{F}_{t+\varepsilon} = \mathcal{F}_t$ for all $t \geq 0$.
- A filtration is complete if $\mathcal{F}_0 \subset \mathcal{F}_t$, and it's termed to satisfy the usual conditions if it's both right continuous and complete.

Definition 1.1.6 (Adapted stochastic process) *We say that a stochastic process* $X = (X(t))_{t\in\mathcal{T}}$ *is adapted to a filtration* $\{\mathcal{F}_t\}_{t\geq 0}$ *if* $X(t)$ *is* $\mathcal{F}_t$*-measurable for each* t*.*

Definition 1.1.7 (Natural filtration) *The natural filtration of a stochastic process* $(X(t))_{t\in\mathcal{T}}$ *comprises the collection of* σ*-algebras* $\{\mathcal{G}(t)\}_{t\geq 0}$*, where* $\mathcal{G}(t) = \sigma\{X(s) : 0 \leq s \leq t\}$ *for all* $t \geq 0$*. It's the minimal augmented filtration generated by* $(X(t))_{t\in\mathcal{T}}$*, characterized by being both right continuous and complete.*

Definition 1.1.8 (Stopping time) *A stopping time τ with respect to $\{\mathcal{F}_t\}_{t\geq 0}$ is a random variable $\tau : \Omega \to [0, +\infty]$ such that $\{\tau \leq t\} \in \mathcal{F}_t$ for all $t \in \mathcal{T}$.*

Definition 1.1.9 (σ-algebra of events prior to $\mathbb{T}$) *For a stopping time τ, the σ-algebra $\mathcal{F}_\tau = \{A \in \mathcal{F} : A \cap \{\tau \leq t\} \in \mathcal{F}_t, \forall t \in \mathcal{T}\}$ captures events preceding $\mathcal{T}$.*

1.1.1 Brownian Motion

Definition 1.1.10 (Standard Brownian Motion) *The Standard Brownian Motion, also known as Wiener process, manifests as a stochastic process $(B(t))_{t\geq 0}$ characterized by independent and identically distributed increments,*

1. *It starts at 0 almost surely, $B(0) = 0$.*
2. *For all $0 \leq s < t$, the increment $B(t) - B(s)$ follows a normal distribution with mean 0 and variance $t - s$ (i.e., $B(t) - B(s) \sim N(0, t - s)$).*
3. *The sample paths of $B(t)$ are almost surely continuous.*

Definition 1.1.11 (d-dimensional Brownian Motion) *A d-dimensional Brownian motion, denoted by $B = \left(B^{(1)}, B^{(2)}, \ldots, B^{(d)}\right)$, is defined by considering $B^{(i)}$ as independent standard Brownian motions for $i = 1, 2, \ldots, d$. The filtration $\{\mathcal{F}_t\}_{t\geq 0}$ generated by a Brownian motion B is defined as, $\mathcal{F}_t = \sigma(B(s) : s \leq t), \quad t \geq 0$, and it is called the natural filtration of B or Brownian filtration.*

1.2 Martingales

Definition 1.2.1 (Martingale) *A continuous-time martingale (resp. submartingale, supermartingale) is a stochastic process $\{X(t), t \geq 0\}$ satisfying the following conditions,*

1. *$X(t)$ is adapted to a filtration $\{\mathcal{F}_t\}_{t\geq 0}$, i.e., $X(t)$ is measurable with respect to $\mathcal{F}_t$ for all $t \geq 0$,*

2. *$X(t)$ is integrable for all $t \geq 0$,*

3. *For all $0 \leq s \leq t$, $\mathrm{E}\left(X(t)|\mathcal{F}_s\right) = $ (resp. $\leq, \geq$)$X(s)$ almost surely.*

Remark 1.2.1 *A process X is a martingale if it is both a submartingale and a supermartingale. If X is a martingale, then $\mathrm{E}(X(t)) = \mathrm{E}(X(0))$ for all $t \in \mathcal{T}$.*

Example 1.2.1 *If B is a Brownian motion, then $B(t)$, $B^2(t) - t$, and $\exp\left(\sigma B(t) - \frac{\sigma^2 t}{2}\right)$ for $t \in \mathcal{T}$ are martingales. Conversely, if X is a continuous process such that $\{X(t)\}_{t\geq 0}$ and $\{X^2(t) - t\}_{t\geq 0}$ are martingales, then X is a Brownian motion.*

Definition 1.2.2 (Local Martingale) *A stochastic process $\{M(t)\}_{t\in\mathbb{R}^+}$ adapted and caglad (right-continuous with left limits) is a local martingale if there exists an increasing sequence of stopping times (τ_n) such that $\tau_n \to +\infty$ as $n \to \infty$ and $M(t \wedge \tau_n)$ is a martingale for all n.*

Remark 1.2.2 *A positive local martingale is a supermartingale. A locally uniformly integrable martingale is a martingale.*

Definition 1.2.3 (Semimartingale) *A semimartingale is a cadlag adapted process X admitting a decomposition of the form, $X = A + M$, where M is a cadlag local martingale null at 0 and A is an adapted process of finite variation and null at 0.*

A continuous semimartingale is a semimartingale X such that in the decomposition $X = A+M$, M and A are continuous. Such a decomposition where M and A are continuous is unique.

1.3 Stochastic Integration and Itô's Formula

In this section, we consider a positive real number T, and aim to define the integral

$$\mathbb{I}(\theta) = \int_0^T \theta(t)dB(t) \tag{1.1}$$

Here, $(\theta(t))_{t\geq 0}$ represents any process, and $(B(t))_{t\geq 0}$ denotes a Brownian motion. The challenge lies in giving meaning to the differential element $dB(s)$ since the function $s \rightarrow B(s)$ is not differentiable.

1.3.1 Wiener Integral

The Wiener integral is an integral of the form

$$\mathbb{I}(\theta) = \int_0^T \theta(t) dB(t) \tag{1.2}$$

with θ being a deterministic function, meaning it does not depend on the random variable ω. Define

$$L^2([0,T],\mathbb{R}) = \left\{ \theta : [0,T] \rightarrow \mathbb{R} \text{ such that } \int_0^T |\theta(s)|^2 ds < \infty \right\}.$$

Suppose θ_n is a deterministic step function defined as

$$\theta_n(t) = \sum_{i=1}^{p_n} \alpha_i 1_{[t_i^n, t_{i+1}^n]}(t),$$

where $p_n \in \mathbb{N}$, the α_i are real numbers, and $\{t_i^n\}$ is an increasing sequence in $\mathcal{T} = [0,T]$. Then, the Wiener integral is defined as

$$\mathbb{I}(\theta_n) = \int_0^T \theta_n(s) dB(s) = \sum_{i=1}^{p_n} \alpha_i \left(B\left(t_{i+1}\right) - B\left(t_i\right)\right).$$

Due to the Gaussian nature of Brownian motion and the independence of its increments, the random variable $\mathbb{I}(\theta_n)$ is a Gaussian variable with zero mean and variance

$$\begin{aligned}\operatorname{Var}(\mathbb{I}(\theta_n)) &= \sum_{i=1}^{p_n} \operatorname{Var}(\alpha_i (B(t_{i+1}) - B(t_i))) \\ &= \sum_{i=1}^{p_n} \alpha_i^2 (t_{i+1} - t_i) \\ &= \int_0^T \theta_n^2(s) ds\end{aligned}$$

Remark 1.3.1 *We observe that $\theta \to \mathbb{I}(\theta)$ is a linear function. Moreover, if b and g are two step functions, we have*

$$\mathrm{E}(\mathbb{I}(b)\mathbb{I}(g)) = \int_0^T b(s)g(s)ds.$$

We then refer to the isometry property of the Wiener integral. Now let $\theta \in L^2([0,T],\mathbb{R})$. Therefore, there exists a sequence of step functions $\{\theta_n, n \geq 0\}$ that converges in $L^2([0,T],\mathbb{R})$ to θ. According to the previous paragraph, we can construct the Wiener integrals $\mathbb{I}(\theta_n)$, which are centered Gaussians forming a Cauchy sequence by isometry. Since the space $L^2([0,T],\mathbb{R})$ is complete, this sequence converges to a Gaussian random variable denoted by $\mathbb{I}(\theta)$. It can be shown that the limit does not depend on the choice of the sequence $\theta_n, n \geq 0$. $\mathbb{I}(\theta)$ is called the Wiener integral of θ with respect to $(B(t))_{t\in\mathbb{R}}$.

1.3.2 The Itô's integral

Our objective now is to define the integral given by equation (1.2). To achieve this, we construct $\mathbb{I}(\theta)$ using discretization, similar to the approach used for the Wiener integral. Let's start by examining step processes represented by,

$$\theta_n(t) = \sum_{i=0}^{p_n} \alpha_i 1_{[t_i^n, t_{i+1}^n]}(t), \tag{1.3}$$

where $p_n \in \mathbb{N}$, (t_i^n) forms an increasing sequence in $\mathcal{T} = [0, T]$, and $\alpha_i \in L^2(\Omega, \mathcal{F}_{t_i}, \mathbb{P})$ for all $i = 0, \cdots, p_n$. We define $\mathbb{I}(\theta_n)$ as

$$\mathbb{I}(\theta_n) = \sum_{i=0}^{p_n} \alpha_i \left(B(t_{i+1}) - B(t_i)\right).$$

It can be confirmed that $\mathrm{E}(\mathbb{I}(\theta_n)) = 0$, and

$$\mathrm{Var}(\mathbb{I}(\theta_n)) = \mathrm{E}\left(\int_0^T \theta_n^2(s) ds\right).$$

Let H denote the space of caglad (left-continuous and right-limited), $\mathcal{F}_t$ adapted processes θ such that

$$\|\theta\|^2 = \mathrm{E}\left(\int_0^T |\theta(s)|^2 ds\right) < \infty.$$

We can define $\mathbb{I}(\theta)$ for any $\theta \in H$. We approximate θ using a sequence of step processes given by equation (1.3), and the limit exists in $L^2(\Omega, [0, T])$. The integral $\mathbb{I}(\theta)$ is then defined as $\lim_{n \to +\infty} \mathbb{I}(\theta_n)$, where $\mathrm{E}(\mathbb{I}(\theta)) = 0$, and

$$\mathrm{Var}(\mathbb{I}(\theta)) = \mathrm{E}\left(\int_0^T \theta^2(s) ds\right).$$

Definition 1.3.1 (Itô Process) *An Itô process is defined as a real-valued process $(X(t))_{t \in \mathcal{T}}$ satisfying the following conditions almost surely,*

$$X(t) = X(0) + \int_0^t b(s) ds + \int_0^t \sigma(s) dB(s), \quad \text{for } 0 \leq t \leq T. \tag{1.4}$$

Alternatively, it can be expressed differentially as,

$$dX(t) = b(t) dt + \sigma(t) dB(t).$$

Here, $X(0)$ is $\mathcal{F}_0$-measurable, and b and σ are two progressively measurable processes, *which satisfy almost surely,*

$$\int_0^T |b(s)|ds < \infty, \quad \text{and} \quad \int_0^T |\sigma(s)|^2 ds < \infty.$$

In other words, $b \in \mathrm{L}^1_{\mathcal{F}_t}[0,T]$ *and* $\sigma \in \mathrm{L}^2_{\mathcal{F}_t}[0,T]$. *The coefficient* b *represents the drift or derivative, and* σ *is the diffusion coefficient.*

Definition 1.3.2 (Integration by Parts Formula) *If* X *and* $\bar{X}$ *are two Itô processes, where*

$$X(t) = X(0) + \int_0^t b(s)ds + \int_0^t \sigma(s)dB(s),$$

and

$$\bar{X}(t) = \bar{X}(0) + \int_0^t \bar{b}(s)ds + \int_0^t \bar{\sigma}(s)dB(s),$$

then the integration by parts formula states that

$$X(t)\bar{X}(t) = X(0)\bar{X}(0) + \int_0^t X(s)d\bar{X}(s) + \int_0^t \bar{X}(s)dX(s) + \langle X, \bar{X}\rangle_t,$$

where

$$\langle X, \bar{X}\rangle_t = \int_0^t \sigma(s)\bar{\sigma}(s)ds.$$

1.4 Itô's Formula

Definition 1.4.1 (Itô's Formula) *Let* $b \in \mathrm{L}^1_{\mathcal{F}_t}[0,T]$, $\sigma \in \mathrm{L}^2_{\mathcal{F}_t}[0,T]$, *and let* X *be an Itô process defined as in (1.4). Define* $\langle X(t)\rangle = \int_0^t |\sigma(s)|^2 ds$. *If* $h \in C^{1,2}(\mathcal{T} \times \mathbb{R}, \mathbb{R})$, *then*

$$\begin{aligned} dh(t,X(t)) &= \partial_t h(t,X(t))dt + \partial_x h(t,X(t))dX(t) + \frac{1}{2}\partial_{xx}h(t,X(t))d\langle X\rangle_t \\ &= (\partial_t h(t,X(t)) + \partial_x h(t,X(t))b(t) + \frac{1}{2}|\sigma(t)|^2\partial_{xx}h(t,X(t)))dt \\ &+ \partial_x h(t,X(t))\sigma(t)dB(t). \end{aligned}$$

Chapitre 2

Existence and Uniqueness of Solutions for a Stochastic Differential Equation with Jumps

Consider a probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ equipped with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$, where $\mathcal{F}_t$ represents the information available up to time t. Within this space, we have a Brownian motion $\{B_t\}_{t\geq 0}$ adapted to $\{\mathcal{F}_t\}_{t\geq 0}$ and a Poisson random measure N on $\mathbb{R}_+ \times E$, where E is a standard measurable space with σ-field $\mathcal{E}$. The mean measure of N takes the form $Leb \times \lambda$, where Leb is the Lebesgue measure on $\mathbb{R}_+$ and λ is a finite measure on E.

Given $B \in \mathcal{E}$ and $t \in \mathbb{R}_+$, with $\lambda(B) < \infty$, let's define D such that $\tilde{N}(\omega, [0,t] \times B)$ is a martingale for every B. We assume that $\{\mathcal{F}_t\}_{t\geq 0}$ is generated by B and N, i.e.,

$$\mathcal{F}_t = \sigma(N([0,s], A), 0 \leq s \leq t, A \in \mathcal{E}) \vee \sigma(B_s, 0 \leq s \leq t) \vee \mathcal{N},$$

where $\mathcal{N}$ denotes the collection of P-null sets, ensuring that $\mathcal{F}_t$ satisfies the usual conditions.

Let M be a Euclidean space and $\mathcal{B}(M)$ be the Borel σ-field on M. For a given $T > 0$, we define the following,

Definition 2.0.2 (Progressive (resp. Predictable) Process) *A process $X : [0,T] \times \Omega \to M$ is termed progressive (resp. predictable) if X is $\mathcal{G}/\mathcal{B}(M)$ (resp. $\mathcal{P}/\mathcal{B}(M)$) measurable, where $\mathcal{G}$ (resp. $\mathcal{P}$) is the progressive (resp. predictable) σ-field on $[0,T] \times \Omega$.*

Definition 2.0.3 (E-Progressive (resp. E-Predictable) Process) *A process $X : [0,T] \times \Omega \times E \to M$ is labeled E-progressive (resp. E-predictable) if X is $\mathcal{G} \otimes \mathcal{E}/\mathcal{B}(M)$ (resp. $\mathcal{P} \otimes \mathcal{E}/\mathcal{B}(M)$) measurable.*

2.1 Stochastic Integral of Random Measure

This section introduces a broader definition of the stochastic integral that involves a random measure. This new definition expands on the one given in [4] and is based on the theory of stochastic integration of processes. We will use the concept of dual predictable projection, also known as the compensator, but we won't explain its definition here. For more details, you can look at [8].

Consider a process X_t that has càdlàg trajectories, meaning it is right-continuous with left limits. Here, $X_{0-} = 0$ and $\Delta X_t = X_t - X_{t-}$ for any time t. We define a measure μ on $\mathcal{F} \otimes \mathcal{B}([0,T]) \otimes \mathcal{E}$, generated by N, as follows :

$$\mu(A) = \mathrm{E} \int_0^T \int_E I_A N(ds, de).$$

For any integrable process X that is measurable with respect to $\mathcal{F} \otimes \mathcal{B}([0,T]) \otimes \mathcal{E}/\mathcal{B}(\mathbb{R})$, we denote $\mathbb{E}[X]$ as $\int X \, d\mu$. We use $\mathbb{E}[X|\mathcal{P} \otimes \mathcal{E}]$ to represent the Radon-Nikodym derivatives with respect to $\mathcal{P} \otimes \mathcal{E}$.

Remark 2.1.1 *Note that $\mathbb{E}$ here does not represent an expectation because μ is not a probability measure, although it behaves similarly.*

Consider $H = \mathbb{I}_{A\times B}$, where $A \in \mathcal{G}$ and $B \in \mathcal{E}$. We define

$$\int_0^T \int_E H\, \tilde{N}(dt, de) = \int_0^T \mathbb{I}_A\, \tilde{N}(dt, B).$$

For any simple E-progressive function H, given by :

$$H = \sum_{i=1}^n a_i \mathbb{I}_{A_i \times B_i}, \quad a_i \in \mathbb{R}, \quad A_i \in \mathcal{G}, \quad B_i \in E,$$

we extend its definition by linear extension. For an E-progressive process H such that

$$\mathbb{E}\left(\int_0^T \int_E H^2\, \tilde{N}(dt, de)\right) < \infty,$$

there exists a sequence of simple E-progressive functions H_n of the same form such that

$$\lim_{n\to\infty} \mathbb{E}\left(\int_0^T \int_E (H - H_n)^2\, \tilde{N}(dt, de)\right) = 0.$$

We can show that $\{(H_n \cdot \tilde{N})_T\}_{n\geq 1}$ is a Cauchy sequence in L^2, allowing us to define

$$\int_0^T \int_E H\, \tilde{N}(dt, de) = \lim_{n\to\infty} \int_0^T \int_E H_n\, \tilde{N}(dt, de), \text{ in } L^2.$$

Proposition 2.1.1 *Let H be a positive E-progressive process such that $\mathbb{E}\left(\int_0^T \int_E H N(dt, de)\right) < \infty$. Then, we have*

$$\left(\int_0 \int_E H N(ds, de)\right)_t^p = \int_0^t \int_E \mathbb{E}[H|\mathcal{P} \otimes \mathcal{E}]\lambda(de)ds, \tag{2.1}$$

where X^p denotes the dual predictable projection of X. If $H = \mathbb{I}_{A\times B}$ with $A \in \mathcal{G}$ and $B \in E$, then

$$\left(\int_0 \int_E H N(ds, de)\right)_t^p = \left(\int_0 \mathbb{I}_A N(ds, B)\right)_t^p = \int_0^t \mathbb{E}_B[\mathbb{I}_A|\mathcal{P}]\lambda(B)ds,$$

where $\mathbb{E}_B$ *represents the measure on* $\mathcal{B}([0,T])\otimes\mathcal{F}$ *generated by* $N([0,t]\times B)$*.*

Proof. We claim that

$$\int_E \mathbb{E}[\mathbb{I}_{A\times B}|\mathcal{P}\otimes E]\lambda(de) = \lambda(B)\mathbb{E}_B[\mathbb{I}_A|\mathcal{P}].$$

Both sides of the equation are predictable processes. To prove this, we will check that the expectations on both sides match for any $C\in\mathcal{P}$. We start with the left-hand side :

$\mathrm{E}\Big(\int_0^T \mathbb{I}_C \int_E \mathbb{E}[\mathbb{I}_A \times B|\mathcal{P}\otimes E]\lambda(de)dt\Big) = \mathbb{E}\left(\int_0^T \int_E \mathbb{I}_C\mathbb{E}[\mathbb{I}_A \times B|\mathcal{P}\otimes E]\lambda(de)dt\right)$

(because $\mathbb{I}_C$ is independent of the integral over E)

$=\mathrm{E}\Big(\int_0^T \int_E \mathbb{E}[\mathbb{I}_C \times A\times B|\mathcal{P}\otimes E]\lambda(de)dt\Big)$

$=\mathrm{E}\Big(\int_0^T \int_E \mathbb{I}_C\cap A\times B\lambda(de)dt\Big)$, (since the indicator function $\mathbb{I}$ acts on $A\times B$)

$=\mathrm{E}\Big(\int_0^T \mathbb{I}_C\cap A\lambda(B)dt\Big)$, (because $\mathbb{I}_B$ is independent of the integral over t)

$=\mathrm{E}\Big(\int_0^T \mathbb{I}_C\cap AN(dt,B)\Big)$, (since $\lambda(B)$ is a constant and can be moved outside the integral)

$=\mathrm{E}\Big(\int_0^T \mathbb{I}_C\cap AN(dt,B)\Big)$.

Next, we consider the right-hand side

$\mathrm{E}\Big(\int_0^T \mathbb{I}_C\lambda(B)\mathbb{E}_B[\mathbb{I}_A|\mathcal{P}]dt\Big)=\lambda(B)\mathrm{E}\Big(\int_0^T \mathbb{I}_C\mathbb{E}_B[\mathbb{I}_A|\mathcal{P}]dt\Big)$,

$=\mathrm{E}\Big(\int_0^T \mathbb{I}_C\mathbb{E}_B[\mathbb{I}_A\cap B|\mathcal{P}]\lambda(de)dt\Big)$

$=\mathrm{E}\Big(\int_0^T \int_E \mathbb{I}_C\cap A\mathbb{I}_B\lambda(de)dt\Big)$, (since $\mathbb{E}_B$ is taken with respect to B)

$=\mathrm{E}\Big(\int_0^T \int_E \mathbb{I}_C\cap AN(dt,de)\Big)$, (since $\mathbb{I}_B$ is independent of the integral over t)

$=\mathrm{E}\Big(\int_0^T \mathbb{I}_C\cap AN(dt,B)\Big)$.

Now, let $\mathcal{C}=\{H=I_{A\times B}|A\in\mathcal{G},B\in\mathcal{E}\}$. We observe that $\mathcal{C}\in\mathcal{H}$, where $\mathcal{H}$ is the set of bounded and E-progressive processes satisfying (2.1). By the linear property of dual predictable projection, $\mathcal{H}$ is a linear space.

If $H^n \uparrow H$ and H is bounded, then we have,

$$\left(\int_0^{\cdot}\int_E H^n N(ds,de)\right)_t^p \to \left(\int_0^{0}\int_E HN(ds,de)\right)_t^p,$$

for each t in the L^1 sense, implying $H \in \mathcal{H}$. Hence, by the monotone class theorem, we conclude that all bounded E-progressive processes satisfy the result. For E-progressive H such that $\mathrm{E}\left[\int_0^T\int_E HN(dt,de)\right] < \infty$, we set,

$$H^n = HI_{\{|H|\leq n\}} \in \mathcal{H},$$

and take the limit to show that H satisfies (2.1). ■

Proposition 2.1.2 *Let H be an E-progressive process such that* $\mathrm{E}\left(\int_0^T\int_E H^2N(dt,de)\right) < \infty$. *Then,*

$$\int_0^T\int_E H\tilde{N}(dt,de) = \int_0^T\int_E HN(dt,de) - \left(\int_0\int_E HN(dt,de)\right)_T^p.$$

Proof. We first consider $H = \mathbb{I}_{A\times B}$, where $A \in \mathcal{G}$ and $B \in E$. By the definition of the stochastic integral, we have

$$\int_0^T\int_E H\tilde{N}(dt,de) = \int_0^T \mathbb{I}_A\tilde{N}(dt,B).$$

For the process $\mathbb{I}_A\tilde{N}(dt,B)$, the dual predictable projection can be written as

$$\left(\int_0^{\cdot} \mathbb{I}_A\tilde{N}(dt,B)\right)_T^p.$$

Therefore, we have

$$\int_0^T \mathbb{I}_A\tilde{N}(dt,B) = \int_0^T \mathbb{I}_A N(dt,B) - \left(\int_0^{\cdot} \mathbb{I}_A N(dt,B)\right)_T^p.$$

This shows that for $H = \mathbb{I}_{A\times B}$, the proposition holds. Next, we extend this result to any E-progressive simple process of the form

$$H = \sum_{i=1}^{n} a_i \mathbb{I}_{A_i \times B_i},$$

where $a_i \in \mathbb{R}$, $A_i \in \mathcal{G}$, and $B_i \in E$. By linearity of the integral and the dual predictable projection, we have

$$\int_0^T \int_E H N(dt, de) = \sum_{i=1}^{n} a_i \int_0^T \mathbb{I}_{A_i} N(dt, B_i).$$

Using the result for indicator functions, we get

$$\sum_{i=1}^{n} a_i \int_0^T \mathbb{I}_{A_i} \tilde{N}(dt, B_i) = \sum_{i=1}^{n} a_i \left(\int_0^T \mathbb{I}_{A_i} N(dt, B_i) - \left(\int_0^{\cdot} \mathbb{I}_{A_i} N(dt, B_i) \right)_T^p \right).$$

Therefore,

$$\int_0^T \int_E H \tilde{N}(dt, de) = \int_0^T \int_E H N(dt, de) - \left(\int_0 \int_E H N(dt, de) \right)_T^p.$$

Now, consider a positive E-progressive process H such that

$$\mathbb{E} \left(\int_0^T \int_E H^2 N(dt, de) \right) < \infty.$$

We can approximate H by a sequence of positive increasing simple functions H_n of the form above, such that

$$\lim_{n\to\infty} \mathbb{E} \left(\int_0^T \int_E (H - H_n)^2 N(dt, de) \right) = 0.$$

This implies that

$$\int_0^T \int_E H\tilde{N}(dt,de) = \lim_{n\to\infty} \int_0^T \int_E H_n\tilde{N}(dt,de) \quad \text{in } L^2.$$

Using the previous result for simple functions H_n, we get

$$\int_0^T \int_E H_n\tilde{N}(dt,de) = \int_0^T \int_E H_n N(dt,de) - \left(\int_0 \int_E H_n N(dt,de)\right)_T^p.$$

Taking the limit as $n \to \infty$ in L^2, we obtain

$$\int_0^T \int_E H\tilde{N}(dt,de) = \int_0^T \int_E H N(dt,de) - \left(\int_0 \int_E H N(dt,de)\right)_T^p.$$

Finally, if H is not necessarily positive, we decompose H into its positive and negative parts : $H = H^+ - H^-$. Both H^+ and H^- are positive E-progressive processes. Applying the result to H^+ and H^- separately, we obtain

$$\int_0^T \int_E H^+\tilde{N}(dt,de) = \int_0^T \int_E H^+ N(dt,de) - \left(\int_0 \int_E H^+ N(dt,de)\right)_T^p,$$

and

$$\int_0^T \int_E H^-\tilde{N}(dt,de) = \int_0^T \int_E H^- N(dt,de) - \left(\int_0 \int_E H^- N(dt,de)\right)_T^p.$$

Subtracting these two equations, we get

$$\int_0^T \int_E (H^+ - H^-)\tilde{N}(dt,de) = \int_0^T \int_E (H^+ - H^-) N(dt,de) - \left(\int_0 \int_E (H^+ - H^-) N(dt,de)\right)_T^p,$$

which simplifies to

$$\int_0^T \int_E H\tilde{N}(dt,de) = \int_0^T \int_E H N(dt,de) - \left(\int_0 \int_E H N(dt,de)\right)_T^p.$$

Thus, the result holds for any E-progressive process H. ■

Proposition 2.1.3 *Suppose H is E-progressive and* $\mathrm{E}\left(\int_0^T \int_E H^2 N(dt, de)\right) < \infty$. *Then,*

$$\int_0^T \int_E H\tilde{N}(dt, de) = \int_0^T \int_E HN(dt, de) - \int_0^T \int_E \mathbb{E}[H|\mathcal{P} \otimes \mathcal{E}]\lambda(de)dt.$$

Remark 2.1.2 *Given the conditions of the previous proposition, we find that the expected value of the stochastic integral can be expressed as*

$$\mathrm{E}\left(\int_0^T \int_E HN(dt, de)\right) = \mathrm{E}\left(\int_0^T \int_E \mathbb{E}[H|\mathcal{P} \otimes \mathcal{E}]\lambda(de)dt\right).$$

In particular, if H is E-predictable, then the integral simplifies to,

$$\mathrm{E}\left(\int_0^T \int_E HN(dt, de)\right) = \mathrm{E}\left(\int_0^T \int_E H\lambda(de)dt\right).$$

Proposition 2.1.4 *For a progressive process H with* $\mathrm{E}\left(\int_0^T \int_E H^2 N(dt, de)\right) < \infty$, *we have*

$$\Delta(H.\tilde{N})_t = \int_E HN(\{t\}, de).$$

First, consider $H = \mathbb{I}_{A\times B}$ where $A \in \mathcal{G}$ and $B \in E$. Then,

$$\Delta\left(\int_0^t \int_E HN(ds, de)\right)_t = \Delta\left(\int_0^t \mathbb{I}_A N(ds, B)\right)_t.$$

Proof. By the properties of the stochastic integral, we know that

$$\Delta\left(\int_0^t \mathbb{I}_A N(ds, B)\right)_t = \mathbb{I}_A N(\{t\}, B).$$

Since $H = \mathbb{I}_{A\times B}$, we have

$$\int_E \mathbb{I}_{A\times B} N(\{t\}, de) = \mathbb{I}_A N(\{t\}, B).$$

Thus, for $H = \mathbb{I}_{A\times B}$,

$$\Delta\left(\int_0^t \int_E HN(ds,de)\right)_t = \int_E HN(\{t\},de).$$

Next, consider a simple function H of the form

$$H = \sum_{i=1}^n a_i \mathbb{I}_{A_i\times B_i},$$

where $a_i \in \mathbb{R}$, $A_i \in \mathcal{G}$, and $B_i \in E$. For such a simple function,

$$\Delta\left(\int_0^t \int_E HN(ds,de)\right)_t = \Delta\left(\int_0^t \sum_{i=1}^n a_i \mathbb{I}_{A_i} N(ds,B_i)\right)_t.$$

By linearity of the stochastic integral and the jump operator, we get

$$\Delta\left(\int_0^t \sum_{i=1}^n a_i \mathbb{I}_{A_i} N(ds,B_i)\right)_t = \sum_{i=1}^n a_i \Delta\left(\int_0^t \mathbb{I}_{A_i} N(ds,B_i)\right)_t.$$

From our previous result for indicator functions, this equals

$$\sum_{i=1}^n a_i \mathbb{I}_{A_i} N(\{t\},B_i).$$

This is exactly

$$\int_E HN(\{t\},de).$$

Now, consider a positive E-progressive process H with $\mathbb{E}\left(\int_0^T \int_E H^2 N(dt,de)\right) < \infty$. There exists a sequence of positive increasing simple functions H_n such that

$$\lim_{n\to\infty} \mathbb{E}\left(\int_0^T \int_E (H-H_n)^2 N(dt,de)\right) = 0.$$

For each H_n,

$$\Delta\left(\int_0^t \int_E H_n N(ds,de)\right)_t = \int_E H_n N(\{t\},de).$$

Since H_n converges to H in L^2, we have

$$\lim_{n\to\infty} \Delta\left(\int_0^t \int_E H_n N(ds, de)\right)_t = \Delta\left(\int_0^t \int_E HN(ds, de)\right)_t.$$

Therefore,

$$\lim_{n\to\infty} \int_E H_n N(\{t\}, de) = \int_E HN(\{t\}, de).$$

Thus,

$$\Delta\left(\int_0^t \int_E HN(ds, de)\right)_t = \int_E HN(\{t\}, de).$$

Finally, if H is not necessarily positive, we decompose H into its positive and negative parts : $H = H^+ - H^-$. Both H^+ and H^- are positive E-progressive processes. Applying the result to H^+ and H^- separately, we get

$$\Delta(H \cdot N)_t = \Delta(H^+ \cdot N)_t - \Delta(H^- \cdot N)_t.$$

By the previous results, this equals

$$\int_E H^+ N(\{t\}, de) - \int_E H^- N(\{t\}, de) = \int_E (H^+ - H^-) N(\{t\}, de) = \int_E HN(\{t\}, de).$$

Thus, the proposition holds for any E-progressive process H. ■

Proposition 2.1.5 *Suppose H is E-progressive and* $\mathrm{E}\left(\int_0^T \int_E H^2 N(dt, de)\right) < \infty$*. Then, the quadratic variation process* $[H.\tilde{N}, H.\tilde{N}]$ *satisfies*

$$[H.\tilde{N}, H.\tilde{N}]_t = \int_0^t \int_E H^2 N(ds, de).$$

Proof. The proof follows similarly to the previous one. ■

2.2 Existence and Uniqueness

Consider the following stochastic differential equation with jumps,

$$X_t = x_0 + \int_0^t b\,(s, X_s)\, ds + \int_0^t \sigma\,(s, X_s)\, dB_s + \int_0^t \int_E c\,(s, X_{s-}, e)\, \tilde{N}(ds, de), \qquad (2.2)$$

where $x_0 \in \mathbb{R}^n, b : [0, T] \times \Omega \times \mathbb{R}^n \to \mathbb{R}^n, \sigma : [0, T] \times \Omega \times \mathbb{R}^n \to \mathbb{R}^{n \times d}, c : [0, T] \times \Omega \times \mathbb{R}^n \times E \to \mathbb{R}^n$, d is the dimension of the Brownian Motion, and n is the dimension of X. Let us introduce the Banach space

$$S^2[0, T] = \left\{ X : X \text{ has càdlàg paths and is adapted, and } \mathrm{E}\left[\sup_{0 \le t \le T} |X_t|^2 \right] < \infty \right\},$$

with the norm

$$\|X\|^2 = \mathrm{E}\left[\sup_{0 \le t \le T} |X_t|^2 \right].$$

We make the following assumptions,

Assumptions (H1)

- The function b is $\mathcal{G} \otimes \mathcal{B}\,(\mathbb{R}^n) / \mathcal{B}\,(\mathbb{R}^n)$ measurable, σ is $\mathcal{G} \otimes \mathcal{B}\,(\mathbb{R}^n) / \mathcal{B}\left(\mathbb{R}^{n \times d}\right)$ measurable, and c is $\mathcal{G} \otimes \mathcal{E} \otimes \mathcal{B}\,(\mathbb{R}^n) / \mathcal{B}\,(\mathbb{R}^n)$ measurable.
- The functions b, σ, and c are uniformly Lipschitz continuous with respect to x, meaning there exists a constant C_L such that, for any $t \in [0, T]$,

$$\begin{cases} |b\,(t, x) - b\,(t, y)| \le C_L\, |x - y| \\ |\sigma\,(t, x) - \sigma\,(t, y)| \le C_L\, |x - y| \\ |c\,(t, x, e) - c\,(t, y, e)| \le C_L\, |x - y| \end{cases} \qquad (2.3)$$

and

$$\mathrm{E} \int_0^T |b(t, \omega, 0)|^2 dt < \infty, \quad \mathrm{E} \int_0^T |\sigma(t, \omega, 0)|^2 dt < \infty, \quad \mathrm{E} \int_0^T \int_E |c(t, \omega, 0, e)|^2 N(ds, de) < \infty.$$

Remark 2.2.1 *See that the assumption (2.3) implies that,*

$$\begin{cases} |b(t,x)| \leq C_L |x| + |b(t,0)| \\ |\sigma(t,x)| \leq C_L |x| + |\sigma(t,0)| \\ |c(t,x,e)| \leq C_L |x| + |c(t,0)| \end{cases}$$

Theorem 2.2.1 *Assuming* (H1)*, there exists a unique solution in* $S^2[0,T]$ *for the equation defined by (2.2).*

Proof. To demonstrate the existence and uniqueness of solutions for the stochastic differential equation, we start by defining a transformation. Define the operator $\mathcal{T}$ on the space $S^2[0,T]$ by :

$$\mathcal{T} : S^2[0,T] \longrightarrow S^2[0,T],$$
$$X \longrightarrow \mathcal{T}(X),$$

such that for $X \in S^2[0,T]$

$$\mathcal{T}(X_t) = x_0 + \int_0^t b(s, X_s)\, ds + \int_0^t \sigma(s, X_s)\, dB_s + \int_0^t \int_E c(s, X_{s-}, e)\, \tilde{N}(ds, de).$$

Step 1 : Well-definedness of $\mathcal{T}$.

We need to show that $\mathcal{T}$ maps $S^2[0,T]$ into itself. That is, if $X \in S^2[0,T]$, then $\mathcal{T}(X) \in S^2[0,T]$, i.e.

$$\mathrm{E}\left[\sup_{0\leq s\leq t} |X_s|^2\right] < \infty \Longrightarrow \mathrm{E}\left[\sup_{0\leq s\leq t} |\mathcal{T}(X_s)|^2\right] < \infty.$$

For each X in $S^2[0,T]$

$$\mathcal{T}(X_t) = x_0 + \int_0^t b(s, X_s)\, ds + \int_0^t \sigma(s, X_s)\, dB_s + \int_0^t \int_E c(s, X_{s-}, e)\, \tilde{N}(ds, de).$$

Applying the inequality $(a_1 + a_2 + a_3 + a_4) \leq 4(a_1^2 + a_2^2 + a_3^2 + a_4^2)$, we have

$$\begin{aligned}
&\mathrm{E}\left[\sup_{0\leq s\leq t} |\mathcal{T}(X_s)|^2\right] \\
&= \mathrm{E}\sup_{0\leq s\leq t}\left|x_0 + \int_0^s b\left(u, X_u\right) du + \int_0^s \sigma\left(u, X_u\right) dB_u + \int_0^s \int_E c\left(u, X_{u-}, e\right) \tilde{N}(du, de)\right|^2 \\
&\leq 4\left|x_0\right|^2 + 4\mathrm{E}\sup_{0\leq s\leq t}\left|\int_0^s b\left(u, X_u\right) du\right|^2 + \mathrm{E}\sup_{0\leq s\leq t}\left|\int_0^s \sigma\left(u, X_u\right) dB_u\right|^2 \\
&+ 4\mathrm{E}\sup_{0\leq s\leq t}\left|\int_0^s \int_E c\left(u, X_{u-}, e\right) \tilde{N}(du, de)\right|^2 .
\end{aligned}$$

Using the Lipschitz condition, we have (see remark 2.2.1). Then, by applying Hölder's inequality with $p = 2, q = 2$, and $\frac{1}{p} + \frac{1}{q} = 1$, we have $\frac{p}{q} = p - 1 = 2 - 1$,

$$\begin{aligned}
\mathrm{E}\sup_{0\leq s\leq t}\left|\int_0^s b\left(u, X_u\right) du\right|^2 &\leq \mathrm{E}\left(\left[\left(\sup_{0\leq s\leq t}\int_0^t 1^2 du\right)^{\frac{1}{2}} \left(\int_0^t \left|b\left(u, X_u\right)\right|^2 du\right)^{\frac{1}{2}}\right]^2\right) \\
&\leq t\mathrm{E}\int_0^t \left|b\left(s, X_s\right)\right|^2 ds \leq tC_L\mathrm{E}\int_0^t \left(C_L\left|X_s\right| + \left|b\left(s, 0\right)\right|\right)^2 ds \\
&\leq 2t\mathrm{E}\int_0^t \left|b\left(s, 0\right)\right|^2 ds + 2tC_L^2\mathrm{E}\left(\int_0^t ds \sup_{0\leq s\leq t}\left|X_s\right|^2\right) \\
&\leq 2t\mathrm{E}\int_0^t \left|b\left(s, 0\right)\right|^2 ds + 2t^2C_L^2\mathrm{E}\left[\sup_{0\leq s\leq t}\left|X_s\right|^2\right]. \qquad (2.4)
\end{aligned}$$

Applying the Burkholder-Davis-Gundy inequalty and the assumptions (H1) :

$$\begin{aligned}
\mathrm{E}\sup_{0\leq s\leq t}\left|\int_0^s \sigma\left(u, X_u\right) dB_u\right|^2 &\leq C_2\mathrm{E}\int_0^t \left|\sigma\left(s, X_s\right)\right|^2 ds \\
&\leq C_2\mathrm{E}\int_0^t \left|C_L\left|X_s\right| + \left|\sigma\left(s, 0\right)\right|\right|^2 ds \\
&\leq 2\mathrm{E}\int_0^T \left|b\left(s, 0\right)\right|^2 ds + 2tC_L^2\mathrm{E}\left[\sup_{0\leq s\leq t}\left|X_s\right|^2\right]. \qquad (2.5)
\end{aligned}$$

With $C_2 = 1$. Since X_{s-} is left-continuous, it is progressive, and $c(s, \omega, x, e)$ is E-

progressive. Thus, $c(s, X_{s-}, e)$ is E-progressive. For any $t \in [0,T]$, we have,

$$\begin{aligned}
\mathrm{E} \sup_{0\le s\le t} \left| \int_0^t \int_E c(s, X_{s-}, e) \tilde{N}(ds, de) \right|^2 &\le \tilde{C}_2 \mathrm{E}\left[\int_0^s \int_E \left| c(u, X_{u-}, e)\right|^2 N(du, de) \right] \\
&\le \tilde{C}_2 \mathrm{E}\left[\int_0^t \int_E \left(|c(s,\omega,0,e)| + C_L \left|X_{s-}\right| \right)^2 N(ds, de) \right] \\
&\le 2\tilde{C}_2 \mathrm{E}\left[\int_0^t \int_E |c(s,\omega,0,e)|^2 N(ds, de) \right] \\
&+ 2\tilde{C}_2 C_L^2 \mathrm{E}\left[\int_0^t \int_E N(ds, de) \sup_{0\le s\le t} \left|X_{s-}\right|^2 \right] \\
&\le 2\mathrm{E}\left[\int_0^t \int_E |c(s,\omega,0,e)|^2 N(ds, de) \right] \\
&+ 2C_L^2 t\lambda(E) \mathrm{E}\left[\sup_{0\le s\le t} \left|X_s\right|^2 \right], \qquad (2.6)
\end{aligned}$$

With $\tilde{C}_2 = 1$. Then,

$$\begin{aligned}
\mathrm{E}\left[\sup_{0\le s\le t} \left|\mathcal{T}(X_s)\right|^2 \right] &\le 4\left|x_0\right|^2 + 4\left(2t^2 C_L^2 + 2tC_L^2 + 2C_L^2 t\lambda(E) \right) \mathrm{E}\left[\sup_{0\le s\le t} \left|X_s\right|^2 \right] \\
&+ 8t\mathrm{E}\int_0^t \left|b(s,0)\right|^2 ds + 8\mathrm{E}\left[\int_0^t \int_E |c(s,\omega,0,e)|^2 N(ds, de) \right] \\
&+ 8\mathrm{E}\int_0^t \left|b(s,0)\right|^2 ds.
\end{aligned}$$

Thus, $\mathrm{E}\left[\sup_{0\le s\le t} |X_s|^2\right] < \infty$ then $\mathrm{E}\left[\sup_{0\le s\le t} |\mathcal{T}(X_s)|^2\right] < \infty$.

Because by hypothesis $X \in S^2[0,T]$, and

$$8t\mathrm{E}\int_0^t \left|b(s,0)\right|^2 ds + 8\mathrm{E}\left[\int_0^t \int_E |c(s,\omega,0,e)|^2 N(ds, de) \right] + 8\mathrm{E}\int_0^t \left|b(s,0)\right|^2 ds.$$

This confirms the well-definedness of $\mathcal{T}$.

Step 2 : Contraction Mapping on a Small Interval

Next, we need to show that $\mathcal{T}$ is a contraction mapping in $S^2[0,T]$ on a sufficiently small interval $[0,T]$.

For any $X, Y \in S^2[0,T]$, consider :

$$\mathcal{T}(X(s)) - \mathcal{T}(Y(s)) = x_0 - x_0 + \int_0^s \left(b\left(u, X_u\right) - b\left(u, Y_u\right)\right) du + \int_0^s \left(b\left(u, X_u\right) - b\left(u, Y_u\right)\right) dB_u$$
$$+ \int_0^s \int_E c\left(u, X_{u-}, e\right) - c\left(u, Y_{u-}, e\right) \tilde{N}(du, de)$$

Applying the inequality $(a_1 + a_2 + a_3) \leq 3(a_1^2 + a_2^2 + a_3^2)$,

$$\begin{aligned}\mathrm{E}\left[\sup_{0\leq s\leq t} \|\mathcal{T}(X(s)) - \mathcal{T}(Y(s))\|^2\right] &\leq 3\mathrm{E} \sup_{0\leq s\leq t} \left|\int_0^s \left(b\left(u, X_u\right) - b\left(u, Y_u\right)\right) du\right|^2 \\ &+ 3\mathrm{E} \sup_{0\leq s\leq t} \left|\int_0^s \left(b\left(u, X_u\right) - b\left(u, Y_u\right)\right) dB_u\right|^2 \\ &+ 3\mathrm{E} \sup_{0\leq s\leq t} \left|\int_0^s \int_E \left[c\left(u, X_{u-}, e\right) - c\left(u, Y_{u-}, e\right)\right] \tilde{N}(du, de)\right|^2\end{aligned}$$

Applying the same principles and inequalities as previously stated :

Using the same method as in equation (2.4), we obtain :

$$\mathrm{E} \sup_{0\leq s\leq t} \left|\int_0^s \left(b\left(u, X_u\right) - b\left(u, Y_u\right)\right) du\right|^2 \leq tC_L^2 \mathrm{E}\left[\sup_{0\leq s\leq t} |X_s - X_s|^2\right]$$

Employing the same approach as in equation (2.5), we find :

$$\mathrm{E} \sup_{0\leq s\leq t} \left|\int_0^s \left(b\left(u, X_u\right) - b\left(u, Y_u\right)\right) dB_u\right|^2 \leq tC_L^2 \mathrm{E}\left[\sup_{0\leq s\leq t} |X_s - X_s|^2\right]$$

Utilizing the same technique as in equation (2.6), we get :

$$\mathrm{E} \sup_{0\leq s\leq t} \left|\int_0^s \int_E c\left(u, X_{u-}, e\right) - c\left(u, Y_{u-}, e\right) \tilde{N}(du, de)\right|^2 \leq C_L^2 t\lambda(E) \mathrm{E}\left[\sup_{0\leq s\leq t} |X_s|^2\right]$$

Then,

$$\mathrm{E}\left[\sup_{0\leq s\leq t} \|\mathcal{T}(X(s)) - \mathcal{T}(Y(s))\|^2\right] \leq 3C_L^2\left(t + t^2 C_L^2 + t\lambda(E)\right) \mathrm{E}\left[\sup_{0\leq s\leq t} |X_s - X_s|^2\right]$$

Thus,

$$\begin{aligned}\sqrt{\mathrm{E}\left[\sup_{0\leq s\leq t}\|\mathcal{T}(X(s))-\mathcal{T}(Y(s))\|^2\right]} &\leq C_L\sqrt{3}\sqrt{(t+t^2+t\lambda(E))}\sqrt{\mathrm{E}\left[\sup_{0\leq s\leq t}|X_s-X_s|^2\right]}\\ &\leq C_L\sqrt{3}\sqrt{(T(1+\lambda(E))+T^2)}\sqrt{\mathrm{E}\left[\sup_{0\leq s\leq t}|X_s-X_s|^2\right]}\\ &\leq C_L\sqrt{3}\sqrt{T}\left(\sqrt{(1+\lambda(E))}+\sqrt{T}\right)\sqrt{\mathrm{E}\left[\sup_{0\leq s\leq t}|X_s-X_s|^2\right]}\end{aligned}$$

For T sufficiently small, the constant $\sqrt{3}C_L\sqrt{T}\left(\sqrt{(1+\lambda(E))}+\sqrt{T}\right)$ can be made

$$\sqrt{3}C_L\sqrt{T}\left(\sqrt{(1+\lambda(E))}+\sqrt{T}\right)<1,$$

making $\mathcal{T}$ a contraction mapping. By the Banach fixed-point theorem, there exists a unique fixed point $X\in S^2[0,T]$ in the interval $[0,T]$, i.e. there exists a point $X\in S^2[0,T]$ such that

$$\mathcal{T}(X)=X\in S^2[0,T].$$

Step 3 : Extending to the Whole Interval [0,T]

By partitioning the interval $[0,T]$ into smaller intervals $[0,\tilde{T}]$, $[\tilde{T},2\tilde{T}]$, ..., we can apply the contraction mapping argument on each subinterval with an initiale value in 0, $\tilde{T}$, ..., and extend the solution to the whole interval $[0,T]$. Thus, we have established the existence and uniqueness of the solution to the stochastic differential equation over the interval $[\mathbf{0},T]$. ∎

In a similar vein, we find a minor distinction. Presented below is the L^p estimate theorem,

Theorem 2.2.2 *For $p\geq 2$, let $X^i, i=1,2$, be solutions of the following equations,*

$$X_t^i=x_0^i+\int_0^t b^i\left(s,X_s^i\right)ds+\int_0^t\sigma^i\left(s,X_s^i\right)dB_s+\int_0^t\int_E c^i\left(s,X_{s-}^i,e\right)\tilde{N}(ds,de),$$

satisfying assumption H1. Then we obtain,

$$
\begin{aligned}
\mathrm{E}\left[\sup_{0\le t\le T}\left|X_t^1-X_t^2\right|^p\right] &\le C\left|x_0^1-x_0^2\right|^p + C\mathrm{E}\left[\int_0^T \left|b^1\left(t,X_t^2\right)-b^2\left(t,X_t^1\right)\right|^p dt\right] \\
&\quad + C\mathrm{E}\left[\left(\int_0^T \left|\sigma^1\left(t,X_t^2\right)-\sigma^2\left(t,X_t^1\right)\right|^2 dt\right)^{\frac{p}{2}}\right] \\
&\quad + C\mathrm{E}\left[\left(\int_0^T\int_E \left|c^1\left(t,X_{t-}^2,e\right)-c^2\left(t,X_{t-}^1,e\right)\right|^2 N(dt,de)\right)^{\frac{p}{2}}\right],
\end{aligned}
$$

where M is a positive real number dependent on p, T, and the Lipschitz constant C_L.

Proof. By simple calculation, we observe that

$$
\begin{aligned}
X_t^1-X_t^2 &= x_0^1+\int_0^t b^1\left(s,X_s^1\right)ds+\int_0^t \sigma^1\left(s,X_s^1\right)dB_s+\int_0^t\int_E c^1\left(s,X_{s-}^1,e\right)\tilde{N}(ds,de) \\
&\quad - x_0^2-\int_0^t b^2\left(s,X_s^2\right)ds-\int_0^t \sigma^2\left(s,X_s^2\right)dB_s-\int_0^t\int_E c^2\left(s,X_{s-}^2,e\right)\tilde{N}(ds,de)
\end{aligned}
$$

Utilizing the inequality

$$\left(\sum_{i=1}^{8} a_i{}^p\right)^p \le 8^{p-1}\left(\sum_{i=1}^{8} a_i^p\right),$$

we find that

$$\mathrm{E}\left[\sup_{0\le t\le T}\left|X_t^1-X_t^2\right|^p\right] \le 8^{p-1}\sum_{i=1}^{8} I_i^p$$

With $I_1^p = \left|x_0^1-x_0^2\right|^p$. Then, by Hölder's inequality with $p,q>1$, and $\frac{1}{p}+\frac{1}{q}=1$, we have

$\frac{p}{q} = p - 1$. Applying the Lipschitz condition, we obtain :

$$
\begin{aligned}
I_2^p &= \mathrm{E} \sup_{0 \leq t \leq T} \left| \int_0^t (b^1\left(s, X_s^1\right) - b^1\left(s, X_s^2\right)) ds \right|^p \\
&\leq \mathrm{E}\left[\left(\left(\int_0^T 1 ds \right)^{\frac{1}{q}} \left(\int_0^T \left|(b^1\left(s, X_s^1\right) - b^1\left(s, X_s^2\right))\right|^p ds \right)^{\frac{1}{p}} \right)^p \right] \\
&= T^{p-1} \mathrm{E} \int_0^T \left|(b^1\left(s, X_s^1\right) - b^1\left(s, X_s^2\right))\right|^p ds \\
&\leq T^{p-1} C_L^p \int_0^T \mathrm{E}\left|X_s^1 - X_s^2\right|^p ds \\
&\leq T^{p-1} C_L^p T \mathrm{E}\left[\sup_{0 \leq t \leq T} \left|X_t^1 - X_t^2\right|^p \right]. \\
&= T^p C_L^p \mathrm{E}\left[\sup_{0 \leq t \leq T} \left|X_t^1 - X_t^2\right|^p \right].
\end{aligned}
$$

Similarly, we have,

$$
I_3^p = \mathrm{E} \sup_{0 \leq t \leq T} \left| \int_0^t (b^2\left(s, X_s^1\right) - b^2\left(s, X_s^2\right)) ds \right|^p \leq T^p C_L^p \mathrm{E}\left[\sup_{0 \leq t \leq T} \left|X_t^1 - X_t^2\right|^p \right].
$$

And

$$
I_4^p = \mathrm{E} \sup_{0 \leq t \leq T} \left| \int_0^t (b^1\left(s, X_s^2\right) - b^2\left(s, X_s^1\right)) ds \right|^p \leq T^{p-1} \mathrm{E} \int_0^T \left|b^1\left(s, X_s^2\right) - b^2\left(s, X_s^1\right))\right|^p ds.
$$

Applying the Burkholder-Davis-Gundy inequality **??**, along with Hölder's inequality with $p', q' > 1$, and $\frac{1}{p'} + \frac{1}{q'} = 1$, where $p' = \frac{p}{2}$, we have $\frac{p'}{q'} = p' - 1 = \frac{p}{2} - 1$. Utilizing the

Lipschitz condition, we arrive at

$$
\begin{aligned}
I_5^p &= \mathrm{E}\sup_{0\le t\le T}\left|\int_0^t (\sigma^1\left(s, X_s^1\right) - \sigma^1\left(s, X_s^2\right))dB_s\right|^p \\
&\le C_p\mathrm{E}\left(\left(\int_0^T \left|(b^1\left(s, X_s^1\right) - b^1\left(s, X_s^2\right))\right|^2 ds\right)^{\frac{p}{2}}\right) \\
&\le C_p\mathrm{E}\left[\left(\left(\int_0^T 1ds\right)^{\frac{1}{q'}}\left(\int_0^T \left|(\sigma^1\left(s, X_s^1\right) - \sigma^1\left(s, X_s^2\right))\right|^{2p'} ds\right)^{\frac{1}{p'}}\right)^{\frac{p}{2}}\right] \\
&= C_pT^{p'-1}\mathrm{E}\left[\left(\int_0^T \left|(\sigma^1\left(s, X_s^1\right) - \sigma^1\left(s, X_s^2\right))\right|^{2p'} ds\right)^{\frac{p}{2}\frac{1}{p'}}\right] \\
&= C_pT^{\frac{p}{2}-1}\mathrm{E}\left(\int_0^T \left|(\sigma^1\left(s, X_s^1\right) - \sigma^1\left(s, X_s^2\right))\right|^p ds\right) \\
&\le C_pT^{\frac{p}{2}-1}C_L^pT\mathrm{E}\left[\sup_{0\le t\le T}\left|X_t^1 - X_t^2\right|^p\right] \\
&= C_pC_L^pT^{\frac{p}{2}}\mathrm{E}\left[\sup_{0\le t\le T}\left|X_t^1 - X_t^2\right|^p\right]
\end{aligned}
$$

Similarly, we get,

$$
\begin{aligned}
I_6^p &= \mathrm{E}\sup_{0\le t\le T}\left|\int_0^t (\sigma^2\left(s, X_s^1\right) - \sigma^2\left(s, X_s^2\right))dB_s\right|^p \\
&\le C_pC_L^pT^{\frac{p}{2}}\mathrm{E}\left[\sup_{0\le t\le T}\left|X_t^1 - X_t^2\right|^p\right]
\end{aligned}
$$

also we get

$$
\begin{aligned}
I_7^p &= \mathrm{E}\sup_{0\le t\le T}\left|\int_0^t \left[\sigma^1\left(s, X_s^2\right) - \sigma^2\left(s, X_s^1\right)\right] dB_s\right|^p \\
&\le C_pT^{\frac{p}{2}-1}\mathrm{E}\left[\int_0^T \left|(\sigma^1\left(s, X_s^1\right) - \sigma^2\left(s, X_s^1\right))\right|^p ds\right],
\end{aligned}
$$

and

$$\begin{aligned} I_8^p &= \mathrm{E} \sup_{0 \leq t \leq T} \left| \int_0^t \int_E \left[c^1 \left(s, X_{s-}^1, e \right) - c^1 \left(s, X_{s-}^2, e \right) \right] \tilde{N}(ds, de) \right|^p \\ &\leq \tilde{C}_p \mathrm{E} \left[\left(\int_0^T \int_E \left| c^1 \left(s, X_{s-}^1, e \right) - c^1 \left(s, X_{s-}^2, e \right) \right|^2 N(ds, E) \right)^{\frac{p}{2}} \right] \\ &\leq \tilde{C}_p C_L^p \mathrm{E} \left[\left(\int_0^T \int_E \left| X - X_{s-}^2 \right|^2 N(ds, E) \right)^{\frac{p}{2}} \right]. \end{aligned}$$

Applying the modified Burkholder-Davis-Gundy inequality we have

$$\begin{aligned} I_9^p &= \mathrm{E} \sup_{0 \leq t \leq T} \left| \int_0^t \int_E \left[c^2 \left(s, X_{s-}^1, e \right) - c^2 \left(s, X_{s-}^2, e \right) \right] \tilde{N}(ds, de) \right|^p \\ &\leq \tilde{C}_p C_L^p \mathrm{E} \left[\left(\int_0^T \int_E \left| X^1 - X_{s-}^2 \right|^2 N(ds, E) \right)^{\frac{p}{2}} \right], \end{aligned}$$

where $\tilde{C}_p$ is a positive real number dependent on p.

Similarly, we have,

$$\begin{aligned} I_{10}^p &= \mathrm{E} \sup_{0 \leq t \leq T} \left| \int_0^t \int_E \left[c^1 \left(s, X_{s-}^2, e \right) - c^2 \left(s, X_{s-}^1, e \right) \right] \tilde{N}(ds, de) \right|^p \\ &\leq \tilde{C}_p \mathrm{E} \left[\left(\int_0^t \int_E \left| c^1 \left(s, X_{s-}^2, e \right) - c^2 \left(s, X_{s-}^1, e \right) \right|^2 N(ds, de) \right)^{\frac{p}{2}} \right]. \end{aligned}$$

Then

$$\begin{aligned}
&\mathrm{E}\left[\sup_{0\le t\le T}\left|X_t^1-X_t^2\right|^p\right]\\
&\le C\left|x_0^1-x_0^2\right|^p+MT^p\mathrm{E}\left[\sup_{0\le t\le T}\left|X_t^1-X_t^2\right|^p\right]\\
&+CT^{\frac{p}{2}}\mathrm{E}\left[\sup_{0\le t\le T}\left|X_t^1-X_t^2\right|^p\right]+C\mathrm{E}\left[\left(\int_0^T\left|X_{t-}^1-X_{t-}^2\right|^2N(dt,E)\right)^{\frac{p}{2}}\right]\\
&+C\mathrm{E}\left[\int_0^T\left|b^1\left(t,X_t^2\right)-b^2\left(t,X_t^1\right)\right|^p dt\right]+C\mathrm{E}\left[\left(\int_0^T\left|\sigma^1\left(t,X_t^2\right)-\sigma^2\left(t,X_t^1\right)\right|^2 dt\right)^{\frac{p}{2}}\right]\\
&+C\mathrm{E}\left[\left(\int_0^T\int_E\left|c^1\left(t,X_{t-}^2,e\right)-c^2\left(t,X_{t-}^1,e\right)\right|^2N(dt,de)\right)^{\frac{p}{2}}\right],
\end{aligned}\tag{2.7}$$

where C is a positive real number dependent on p,T,C_L.

Now, we define $H_t=\left|X_{t-}^1-X_{t-}^2\right|^2$ and $A_t=\int_0^t H_sN(ds,E)$. Since A_t is a pure jump process, we have,

$$\begin{aligned}
A_T^{\frac{p}{2}}&=\sum_{s\le T}\left(\left|A_{s-}+H_s\right|^{\frac{p}{2}}-A_{s-}^{\frac{p}{2}}\right)I_{\{N(\{s\},E)\neq0\}}\\
&=\int_0^T\left|A_{s-}+H_s\right|^{\frac{p}{2}}-A_{s-}^{\frac{p}{2}}N(ds,E)\\
&\le C\int_0^T\left(A_{s-}^{\frac{p}{2}}+H_s^{\frac{p}{2}}\right)N(ds,E).
\end{aligned}$$

Given that A_- and H are predictable, we find,

$$\mathrm{E}\left[A_T^{\frac{p}{2}}\right]\le C\mathrm{E}\left[\int_0^T\left(A_s^{\frac{p}{2}}+H_s^{\frac{p}{2}}\right)ds\right]\le CT\mathrm{E}\left[A_T^{\frac{p}{2}}\right]+CT\mathrm{E}\left[\sup_{0\le t\le T}\left|X_t^1-X_t^2\right|^p\right].$$

By selecting T sufficiently small such that $CT<1$, we obtain,

$$\mathrm{E}\left[A_T^{\frac{p}{2}}\right]\le\frac{CT}{1-CT}\mathrm{E}\left[\sup_{0\le t\le T}\left|X_t^1-X_t^2\right|^p\right].$$

Substituting (2.7) and subtracting $\left(T^p+T^{\frac{p}{2}}+\frac{CT}{1-CT}\right)\mathrm{E}\left[\sup_{0\le t\le T}\left|X_t^1-X_t^2\right|^p\right]$ on both sides yields the estimate in small time duration. For any T, we can partition T into smaller intervals to reach the desired conclusion. ■

Remark 2.2.2 *We can assume, without loss of generality, that*

$$\mathrm{E}\left[\sup_{0\leq t\leq T}\left|X_t^1 - X_t^2\right|^p\right] < \infty, \tag{2.8}$$

in the earlier proof. If this condition doesn't hold, we can introduce a sequence of stopping times to ensure (2.8) and subsequently obtain the L^p estimate using these stopping times. Finally, we can take limits, allowing us to subtract that term from both sides of (2.7).

Chapitre 3

The Maximum Principle for Progressive Optimal Stochastic Control Problems with Random Jumps

3.1 Statement of the Problem

Given a time duration $T > 0$, let $\{T_n\}_{n\geq 1}$ be the sequence of jump times defined by $T_n = \inf\{t : N([0,t] \times E) \geq n\}$, where $N([0,t] \times E)$ denotes the number of jumps up to time t in the space E. The sequence $\{T_n\}_{n\geq 1}$ is strictly increasing. We also consider a nonempty subset U of $\mathbb{R}$.

Definition 3.1.1 *We define the admissible control set U_{ad} as the set of all controls u*

satisfying the following conditions,

$$U_{ad} = \Big\{ u : u \text{ is progressive, taking values in } U, \sup_{0 \le t \le T} \mathrm{E}\left[|u_t|^p\right] < \infty \text{ for } p > 1,$$
$$\text{and } \mathrm{E} \int_0^T |u_t|^2 \, N(dt, E) < \infty \Big\}.$$

Definition 3.1.2 *For any admissible control $u \in U_{ad}$ and initial state $x_0 \in \mathbb{R}$, we consider the following progressive stochastic system with jumps,*

$$X_t = x_0 + \int_0^t b\left(s, X_s, u_s\right) ds + \int_0^t \sigma\left(s, X_s, u_s\right) dB_s + \int_0^t \int_E c\left(s, X_{s-}, u_s, e\right) \tilde{N}(ds, de), \quad (3.1)$$

along with the cost functional,

$$J(u) = \mathrm{E}\left(\int_0^T f\left(t, X_t, u_t\right) dt + g\left(X_T\right) \right), \quad (3.2)$$

where

$$b : [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}, \quad \sigma : [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R},$$
$$c : [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R} \times E \to \mathbb{R}, \quad f : [0,T] \times \Omega \times \mathbb{R} \times \mathbb{R} \to \mathbb{R},$$
$$g : \Omega \times \mathbb{R} \to \mathbb{R}.$$

Definition 3.1.3 (Optimal control) *The optimal control is to find an element $u \in U_{ad}$ such that*

$$J(u) = \inf_{v \in U_{ad}} J(v).$$

We aim to find necessary conditions for an optimal control in U_{ad}. To do so, we introduce the following assumption.

Assumption H

1. The functions b, σ, f are $\mathcal{G} \otimes \mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}) / \mathcal{B}(\mathbb{R})$ measurable, and c is $\mathcal{G} \otimes \mathcal{E} \otimes \mathcal{B}(\mathbb{R}) \otimes$

$\mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$ measurable. The function g is $\mathcal{F}_T \otimes \mathcal{B}(\mathbb{R})/\mathcal{B}(\mathbb{R})$ measurable.

2. The functions b, σ, c are twice continuously differentiable with respect to x, have bounded first and second order derivatives, and there exists a constant C such that

$$|(b, \sigma, c)(t, x, u)| \leq C(1 + |x| + |u|).$$

3. f and g are twice continuously differentiable with respect to x, with bounded second-order derivatives. There exists a constant C such that

$$|f_x(t, x, u)| \leq C(1 + |x| + |u|), \quad |f(t, x, u)| \leq C\left(1 + |x|^2 + |u|^2\right),$$

and

$$|g_x(x)| \leq C(1 + |x|), \quad |g(x)| \leq C\left(1 + |x|^2\right),$$

while satisfying

$$\mathrm{E}\int_0^T |b(t, \omega, 0, 0)|^2 dt < \infty, \quad \mathrm{E}\int_0^T |\sigma(t, \omega, 0, 0)|^2 dt < \infty,$$

and

$$\mathrm{E}\int_0^T \int_E |c(t, \omega, e, 0, 0)|^2 N(ds, de) < \infty.$$

Remark 3.1.1 *Assuming H, we establish the existence of a unique solution to (3.1) for any admissible control, as proven in Theorem 2.2.2.*

3.2 Spike Variation

Given that U may not be convex, we resort to spike variations. Let $u \in U_{ad}$ denote the optimal control. For any $\bar{t} \in [0, T]$, the spike variation of u is defined as follows,

$$u^{\varepsilon} = \begin{cases} v, & \text{if } (s, \omega) \in \mathcal{O} = [[\bar{t}, \bar{t} + \varepsilon]] \setminus \bigcup_{n=1}^{\infty} [[T_n]] \, , \\ u, & \text{otherwise,} \end{cases} \tag{3.3}$$

where

$$[[T_n]] = \{(t, \omega) \in [0, T] \times \Omega | T_n(\omega) = t\} \, ,$$

represents the graph of T_n, and v is a bounded $\mathcal{F}_{\bar{t}}$ measurable function taking values in U. As T_n is a stopping time, $[[T_n]]$ is a progressive set. Hence, u^{ε} is progressive, and it can be shown that it belongs to U_{ad}.

Remark 3.2.1 *As known, T_n is not a predictable time, so $[[T_n]]$ is unpredictable, meaning u^{ε} is not predictable. This underscores the necessity of the integrand of the stochastic integral to be progressive. Indeed, T_n represents totally unpredictable times.*

Let X denote the trajectory of u, and X^{ε} the trajectory of u^{ε}. Through the SDE estimate and noticing $(Leb \times P)([[T_n]]) = 0$, we derive,

$$u^{\varepsilon} - u = \begin{cases} v - u, & \text{if } (s, \omega) \in \mathcal{O} = [[\bar{t}, \bar{t} + \varepsilon]] \setminus \bigcup_{n=1}^{\infty} [[T_n]] \, , \\ 0, & \text{otherwise,} \end{cases} \tag{3.4}$$

and applying the Burkholder-Davis-Gundy inequality, along with Hölder's inequality, and assumption H. A similar way to proof [?], we get

$$\begin{aligned}
&\mathrm{E}\left(\sup_{0\leq t\leq T}|X_t^{\varepsilon}-X_t|^p\right)\\
&\leq C\mathrm{E}\left\{\left(\int_0^T|b\left(t,X_t,u_t^{\varepsilon}\right)-b\left(t,X_t,u_t\right)|\,dt\right)^p\right.\\
&+\left(\int_0^T|\sigma\left(t,X_t,u_t^{\varepsilon}\right)-\sigma\left(t,X_t,u_t\right)|^2\,dt\right)^{\frac{p}{2}}\\
&\left.+\left(\int_0^T\int_E|c\left(t,X_{t-},u_t^{\varepsilon},e\right)-c\left(t,X_{t-},u_t,e\right)|^2\,N(dt,de)\right)^{\frac{p}{2}}\right\}\\
&\leq C\mathrm{E}\left(\left(\int_t^{t+\varepsilon}|u-v|dt\right)^p+\left(\int_t^{t+\varepsilon}|u-v|^2dt\right)^{\frac{p}{2}}+\left(\int_0^T I_{\mathcal{O}}|u-v|^2N(dt,E)\right)^{\frac{p}{2}}\right).
\end{aligned}$$

As there are no jumps on $\mathcal{O}$, we obtain,

$$\mathrm{E}\left(\sup_{0\leq t\leq T}|X_t^{\varepsilon}-X_t|^p\right)=O\left(\varepsilon^p\right)+O(\varepsilon^{\frac{p}{2}}). \tag{3.5}$$

This indicates that the jump term does not affect the order of variation.. We then introduce the variation equations,

$$\begin{aligned}
\hat{X}_t=&\int_0^t(b_x\left(s,X_s,u_s\right)\hat{X}_s+\delta b)ds+\int_0^t(\sigma_x\left(s,X_s,u_s\right)\hat{X}_s+\delta\sigma)dB_s\\
&+\int_0^t\int_E c_x\left(s,X_{s-},u_s,e\right)\hat{X}_{s-}\tilde{N}(ds,de)
\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}
\hat{Y}_t=&\int_0^t(b_x\left(s,X_s,u_s\right)\hat{Y}_s+\frac{1}{2}b_{xx}\left(s,X_s,u_s\right)\hat{X}_s^2)ds\\
&+\int_0^t(\sigma_x\left(s,X_s,u_s\right)\hat{Y}_s+\frac{1}{2}\sigma_{xx}\left(s,X_s,u_s\right)\hat{X}_s^2+\delta\sigma_x\hat{X}_s)dB_s\\
&+\int_0^t\int_E(c_x\left(s,X_{s-},u_s,e\right)\hat{Y}_{s-}+\frac{1}{2}c_{xx}\left(s,X_{s-},u_s,e\right)\hat{X}_{s-}^2)\tilde{N}(ds,de),
\end{aligned} \tag{3.7}$$

where

$$\delta\phi = \phi\left(s, X_s, u_s^\varepsilon\right) - \phi\left(s, X_s, u_s\right), \delta\phi_x = \phi_x\left(s, X_s, u_s^\varepsilon\right) - \phi\left(s, X_s, u_s\right), \phi = b, \sigma.$$

It can be demonstrated that (3.6) and (3.7) have unique solutions. We establish some basic estimates about $\hat{X}$ and $\hat{Y}$.

For $p \geq 2$, we can establish the following estimates,

Lemma 3.2.1 *For $p \geq 2$, we have,*

$$\mathrm{E}\left(\sup_{0\leq t\leq T} |\hat{X}_t|^p\right) \leq C\varepsilon^{\frac{p}{2}}, \qquad \text{and} \qquad \mathrm{E}\left(\sup_{0\leq t\leq T} \left|\hat{Y}_t\right|^p\right) \leq C\varepsilon^p.$$

Proof. Using the Burkholder-Davis-Gundy inequality, in conjunction with Hölder's inequality and assumption H for $\hat{X}$, we obtain :

$$\mathrm{E}\left(\sup_{0\leq t\leq T} |\hat{X}_t|^p\right) \leq C\mathrm{E}\left(\left(\int_0^T |\delta b| dt\right)^p\right) + C\mathrm{E}\left(\left(\int_0^T |\delta\sigma|^2 dt\right)^{\frac{p}{2}}\right) = O\left(\varepsilon^p\right) + O(\varepsilon^{\frac{p}{2}}).$$

For $\hat{Y}$, given the boundedness of $b_{xx}, \sigma_{xx}, c_{xx}$,

$$\begin{aligned}
\mathrm{E}\left(\sup_{0\leq t\leq T} \left|\hat{Y}_t\right|^p\right) &\leq C\mathrm{E}\left(\left(\int_0^T |\frac{1}{2} b_{xx}\left(s, X_s, u_s\right) \hat{X}_s^2| dt\right)^p\right) \\
&+ C\mathrm{E}\left(\left(\int_0^T |\frac{1}{2}\sigma_{xx}\left(s, X_s, u_s\right) \hat{X}_s^2 + \delta\sigma_x \hat{X}_s|^2 dt\right)^{\frac{p}{2}}\right) \\
&+ C\mathrm{E}\left(\left(\int_0^T \int_E |\frac{1}{2} c_{xx}\left(s, X_{s-}, u_s, e\right) \hat{X}_{s-}^2|^2 N(dt, de)\right)^{\frac{p}{2}}\right) \\
&= O\left(\varepsilon^p\right).
\end{aligned}$$

■

Lemma 3.2.2 *We have*

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon^2}\mathrm{E}\left(\sup_{0\le t\le T}|X_t^\varepsilon - X_t - \hat{X}_t - \hat{Y}_t|^2\right) = 0$$

Proof. See [9] for the proof. ■

Consider the stochastic control problem with the cost functional

$$J(u) = \mathbb{E}\left(\int_0^T f(t, X_t, u_t)\, dt + g(X_T)\right),$$

where u is the control, X_t is the state of the system at time t, and f and g are given functions defining the cost.

Define the variation in the cost functional as :

$$\begin{aligned}\hat{J} = \mathrm{E}&\left(\int_0^T \left(f_x\,(t, X_t, u_t)\,(\hat{X}_t + \hat{Y}_t) + \frac{1}{2}f_{xx}\,(t, X_t, u_t)\,\hat{X}_t^2 + \delta f\right) dt\right)\\ &+ \mathrm{E}\left[g_x\,(X_T)\,(\hat{X}_t + \hat{Y}_t) + \frac{1}{2}g_{xx}\,(X_T)\,(\hat{X}_T)^2\right].\end{aligned} \tag{3.8}$$

where Y_t represents a small variation in the state trajectory caused by a small perturbation in the control.

Then we have the following lemma.

Lemma 3.2.3 *We have*

$$\lim_{\varepsilon\to 0}\frac{J\,(u^\varepsilon) - J(u) - \hat{J}}{\varepsilon} = 0.$$

where u^ϵ is a perturbation of the control u.

Express $J(u) + J$ in terms of u^ϵ, X_t, and Y_t. This involves expanding the terms in the cost functional and rearranging them appropriately. Estimate the difference $J(u^\epsilon) - J(u) - J$ using suitable inequalities and bounds. This step involves careful manipulation of expectations, integrals, and the properties of f and g. Show that the estimated difference tends to zero as ϵ approaches zero. Utilize properties of f and g, such as boundedness and differentiability, to simplify expressions and establish convergence. Conclude that the limit of the expression as ϵ tends to zero is zero, proving the lemma.

Proof. The proof involves rigorous mathematical arguments, including : Expressing the cost functional in terms of variations in the control and state trajectories. Applying the definition of expectations and integrals to obtain a suitable expression for $J(u) + J$. Estimating the difference $J(u^\epsilon) - J(u) - J$ using appropriate inequalities and bounds. Utilizing the properties of the functions f and g, such as boundedness and differentiability, to simplify expressions and establish convergence. This concludes the detailed derivation and proof of the variation equation for the cost functional in stochastic control theory.

$$\begin{aligned}
J(u) + \hat{J} &= \mathrm{E} \int_0^T \left(\left(f\left(t, X_t, u_t\right) + f_x\left(t, X_t, u_t\right) \left(\hat{X}_t + \hat{Y}_t\right) + \frac{1}{2} f_{xx}\left(t, X_t, u_t\right) \hat{X}_t^2 + \delta f \right) \right) dt \\
&\quad + \mathrm{E}(g\left(X_T\right) + g_x\left(X_T\right) \left(\hat{X}_T + \hat{Y}_T\right) + \frac{1}{2} g_{xx}\left(X_T\right) \left(\hat{X}_T\right)^2) \\
&= \mathrm{E} \int_0^T \left(f(t, X_t + \hat{X}_t + \hat{Y}_t, u_t^\varepsilon) + H \right) dt + \mathrm{E}(g(X_T + \hat{X}_T + \hat{Y}_T) + I),
\end{aligned}$$

where

$$\begin{aligned}
H &= \frac{1}{2} f_{xx}\left(s, X_s, u_s\right) \hat{X}_s^2 - \delta f_x \left(\hat{X}_s + \hat{Y}_s \right) - A_f (\hat{X}_s + \hat{Y}_s)^2 \\
I &= - \int_0^1 \int_0^1 \alpha g(X_T + \alpha\beta(\hat{X}_T + \hat{Y}_T)) d\alpha d\beta (\hat{X}_T + \hat{Y}_T)^2 + \frac{1}{2} g_{xx}\left(X_T\right) (\hat{X}_T)^2.
\end{aligned}$$

Then

$$\begin{aligned}
|J\left(u^\varepsilon\right) - J(u) - \hat{J}|^2 &\leq C \mathrm{E} \left(\int_0^T |f(t, X_t + \hat{X}_t + \hat{Y}_t, u_t^\varepsilon) - f\left(t, X_t^\varepsilon, u_t^\varepsilon\right)|^2 dt + \left(\int_0^T H dt \right)^2 \right) \\
&\quad + \mathrm{E} \left(|g(X_T + \hat{X}_T + \hat{Y}_T) - g(X_T^\varepsilon)|^2 + I^2 \right) \\
&\leq C \mathrm{E} \left(\sup_{0 \leq t \leq T} |X_t^\varepsilon - X_t - \hat{X}_t - \hat{Y}_t|^2 \right) + \mathrm{E} \left(\left(\int_0^T H dt \right)^2 + I^2 \right) \\
&= o\left(\varepsilon^2\right).
\end{aligned}$$

By the same method we can show that

$$\mathrm{E}\left(\left(\int_0^T H dt\right)^2 + I^2\right) = o\left(\varepsilon^2\right),$$

which proves the result. ■

3.3 Adjoint Equations and the Maximum Principle.

We introduce the first order and second order adjoint equation.

1. First order,

$$\begin{aligned} p_t = g_x\left(X_T\right) + \int_t^T \left(b_x p_s + \sigma_x q_s + f_x + \int_E \mathbb{E}\left[c_x|\mathcal{P}\otimes\mathcal{E}\right] k_s \lambda(de)\right) ds \\ - \int_t^T q_s dB_s - \int_t^T \int_E k_s \tilde{N}(ds, de). \end{aligned} \tag{3.9}$$

2. Second order,

$$\begin{aligned} P_t = g_{xx}\left(X_T\right) + \int_t^T \Big(2b_x P_s + 2\sigma_x Q_s + f_{xx} + b_{xx} p_s + \sigma_{xx} q_s + P_s \sigma_x^2 \\ + \int_E \mathbb{E}\left[\left(c_x^2 + 2c_x\right)|\mathcal{P}\otimes\mathcal{E}\right] K_s + \mathbb{E}\left[c_{xx}|\mathcal{P}\otimes\mathcal{E}\right] k_s + \mathbb{E}\left[c_x^2|\mathcal{P}\otimes\mathcal{E}\right] P_s \lambda(de)\Big) ds \\ - \int_t^T Q_s dB_s - \int_t^T \int_E K_s \tilde{N}(ds, de). \end{aligned} \tag{3.10}$$

where $\phi_x = \phi_x\left(t, X_t, u_t\right), \phi_{xx} = \phi_{xx}\left(t, X_t, u_t\right)$. To achieve the existence and uniqueness of the two backward equations mentioned, we refer to Lemma 2.4 in [4]. Since ϕ_x, ϕ_{xx} are bounded, there exists a unique solution to equation (3.9) $(p, q, k) \in S^2[0,T] \times M^2[0,T] \times F^2[0,T]$ and a unique solution to equation (3.10) $(P, Q, K) \in S^2[0,T] \times M^2[0,T] \times F^2[0,T]$.

Next, we need an Itô's formula for processes with jumps, referring to Theorem 32 and Theorem 33 from [4].

Lemma 3.3.1 *Let $X^1, X^2, \ldots, X^d$ be semimartingales, and let F be a C^2 function on $\mathbb{R}^d$. Set $X = \left(X^1, X^2, \ldots, X^d\right)$, then*

$$F\left(X_t\right) - F\left(X_0\right) = \sum_{i=1}^{d} \int_0^t \frac{\partial F}{\partial x_i}\left(X_{s-}\right) dX_s^i + \frac{1}{2} \sum_{i=1,j=1}^{d} \int_0^t \frac{\partial^2 F}{\partial x_i \partial x_j}\left(X_{s-}\right) d\left[X^i, X^j\right]_s + \sum_{s \leq t} \eta_s(F),$$

where

$$\eta_s(F) = F\left(X_s\right) - F\left(X_{s-}\right) - \sum_{i=1}^{d} \frac{\partial F}{\partial x_i}\left(X_{s-}\right) \Delta X_s^i - \frac{1}{2} \sum_{i=1,j=1}^{d} \frac{\partial^2 F}{\partial x_i \partial x_j}\left(X_{s-}\right) \Delta X_s^i \Delta X_s^j.$$

and

$$\Delta X_s^i = X_s^i - X_{s-}^i.$$

Applying Itô's formula to $p_t \hat{X}_t$, $p_t \hat{Y}_t$, and $P_t |\hat{X}_t|^2$ yields the following expressions,

1.

$$\begin{aligned} \mathrm{E}[p_T \hat{X}_T] &= \mathrm{E} \int_0^T p_{t-} d\hat{X}_t + \mathrm{E} \int_0^T \hat{X}_{t-} dp_t + \mathrm{E}[p, \hat{X}]_T \\ &= \mathrm{E} \int_0^T \left(p_t \delta b + q_t \delta \sigma - \hat{X}_t f_x\right) dt. \end{aligned} \tag{3.11}$$

2.

$$\begin{aligned} \mathrm{E}[p_T \hat{Y}_T] &= \mathrm{E} \int_0^T p_{t-} d\hat{Y}_t + \mathrm{E} \int_0^T \hat{Y}_{t-} dp_t + \mathrm{E}[p, \hat{Y}]_T \\ &= \mathrm{E} \int_0^T \left(\frac{1}{2} b_{xx} p_t |\hat{X}_t|^2 + \frac{1}{2} \sigma_{xx} q_t |\hat{X}_t|^2 - \hat{Y}_t f_x + \delta \sigma_x \hat{X}_t q_t \right. \\ &\quad \left. + \int_E \frac{1}{2} \mathbb{E}[c_{xx} | \mathcal{P} \otimes \mathcal{E}] k_t \hat{X}_t^2 \right) dt. \end{aligned} \tag{3.12}$$

3.

$$\begin{aligned}
\mathrm{E}[P_T|\hat{X}_T|^2] &= \mathrm{E}\left[\int_0^T |\hat{X}_t|^2 dP_t + \int_0^T 2P_{t-}\hat{X}_{t-}d\hat{X}_t\right] \\
&\quad + \mathrm{E}\left[\int_0^T P_{t-}d[\hat{X},\hat{X}]_t + \int_0^T 2\hat{X}_{t-}d[\hat{X},P]_t + \sum_{t\le T}\Delta P_t(\Delta\hat{X}_t)^2\right] \\
&= \mathrm{E}\int_0^T \Bigg(P_t(\delta\sigma)^2 - \hat{X}_t^2\left(f_{xx} + p_t b_{xx} + q_t\sigma_{xx} + \int_E k_t\mathbb{E}[c_{xx}|\mathcal{P}\otimes\mathcal{E}]\lambda(de)\right) \\
&\quad + 2P_t\hat{X}_t\delta b + 2Q_t\hat{X}_t\delta\sigma + 2P_t\sigma_x\hat{X}_t\delta\sigma\Bigg)\, dt. \qquad (3.13)
\end{aligned}$$

Remark 3.3.1 *In equation (3.13), we utilize the following identity,*

$$\begin{aligned}
\sum_{t\le T}\Delta P_t(\Delta\hat{X}_t)^2 &= \sum_{t\le T}\int_E K_t N(\{t\},de)\left(\int_E c_x\hat{X}_{t-}N(\{t\},de)\right)^2 \\
&= \sum_{t\le T}\int_E K_t c_x^2\hat{X}_{t-}^2 N(\{t\},de) \\
&= \int_0^T\int_E K_t c_x^2\hat{X}_{t-}^2 N(dt,de).
\end{aligned}$$

The second equality follows from the property that for any $A\in\mathcal{E}$, $N(\{t\},A)=1$ or 0.

From equations (3.11)-(3.13), we can deduce the expressions for $g_x(X_T)(X_T+Y_T)$ and $g_{xx}(X_T)X_T^2$. Thus, we arrive at

$$\hat{J} = \mathrm{E}\left[\int_0^T \left(p_t\delta b + q_t\delta\sigma + \delta f + \frac{1}{2}P_t(\delta\sigma)^2\right)dt\right] + o(\varepsilon) \qquad (3.14)$$

where $o(\varepsilon)$ denotes

$$\mathrm{E}\left[\int_0^T \left(\delta\sigma_x\hat{X}_t q_t + P_t\sigma_x\hat{X}_t\delta\sigma + P_t\hat{X}_t\delta b + \hat{X}_t\delta\sigma Q_t\right)dt\right].$$

We introduce the function

$$H(t,x,u,p,q) = pb(t,x,u) + q\sigma(t,x,u) + f(t,x,u).$$

Then, we establish the following theorem,

Theorem 3.3.1 *Under the assumption that* (E) *satisfies and given that* u *represents the optimal control and* X *denotes the trajectory of* u*, with* (p, q) *satisfying* (3.9) *and* P *satisfying* (3.10)*, we can conclude almost everywhere and almost surely that for any* $v \in U$*,*

$$H(t, X_t, v, p_t, q_t) - H(t, X_t, u_t, p_t, q_t) + \frac{1}{2} P_t(\sigma(t, X_t, v) - \sigma(t, X_t, u_t))^2 \geq 0.$$

Proof. Observe that $\bigcup_{n=1}^{\infty} [[T_n]]$ is negligible under $P \times \text{Leb}$. From Equation (3.14), it follows that,

$$\begin{aligned}\hat{J} = \mathrm{E} \int_0^T 1_{(\bar{t}, \bar{t}+\varepsilon]} \{&(p_t(b(t, X_t, v) - b(t, X_t, u)) + q_t(\sigma(t, X_t, v) - \sigma(t, X_t, u)) \\ &+ (f(t, X_t, v) - f(t, X_t, u)) + \frac{1}{2} P_t(\sigma(t, X_t, v) - \sigma(t, X_t, u))^2)\} dt + o(\varepsilon).\end{aligned}$$

Dividing both sides by ε and letting $\varepsilon \to 0$, we obtain for almost every $\bar{t}$,

$$\mathrm{E} \left(H(\bar{t}, X_{\bar{t}}, v, p_{\bar{t}}, q_{\bar{t}}) - H(\bar{t}, X_{\bar{t}}, u, p_{\bar{t}}, q_{\bar{t}}) + \frac{1}{2} P_{\bar{t}} (\sigma(\bar{t}, X_{\bar{t}}, v) - \sigma(\bar{t}, X_{\bar{t}}, u))^2 \right) \geq 0.$$

Then, for any $A \in \mathcal{F}_{\bar{t}}$ and $w \in U$, setting $v = w1_A + u1_{A^c}$, we have,

$$\mathrm{E}1_A \left(H(\bar{t}, X_{\bar{t}}, w, p_{\bar{t}}, q_{\bar{t}}) - H(\bar{t}, X_{\bar{t}}, u, p_{\bar{t}}, q_{\bar{t}}) + \frac{1}{2} P_{\bar{t}} (\sigma(\bar{t}, X_{\bar{t}}, w) - \sigma(\bar{t}, X_{\bar{t}}, u))^2 \right) \geq 0.$$

This inequality holds almost everywhere, which implies,

$$H(\bar{t}, X_{\bar{t}}, w, p_{\bar{t}}, q_{\bar{t}}) - H(\bar{t}, X_{\bar{t}}, u, p_{\bar{t}}, q_{\bar{t}}) + \frac{1}{2} P_{\bar{t}} (\sigma(\bar{t}, X_{\bar{t}}, w) - \sigma(\bar{t}, X_{\bar{t}}, u))^2 \geq 0,$$

for almost every $\bar{t}$. ■

Conclusion

In this study, we dive into the world of stochastic processes, focusing on a new variation method aimed at overcoming a significant hurdle : jumps. With our innovative variation technique, we delve into the complexities of estimating Lp-norms within these processes. The beauty of our approach lies in its scalability with the growth of pp, making our variation equations more effective.

Surprisingly, despite the presence of jumps, our maximum principle retains a familiar structure observed in systems without jumps. This curious similarity arises from the fact that both principles hold almost everywhere, almost surely. The minimal influence of jumps on our results can be attributed to their negligible measure under specific probability measures.

Moreover, we take pride in the rigor and clarity of our derived maximum principle, laying a robust foundation for further theoretical and practical exploration. Looking ahead, our future research aspirations include delving deeper into optimal control strategies during jump instances and exploring the myriad real-world applications of our findings.

Bibliographie

[1] Friedman, A. (1975). Stochastic differential equations and applications : Volume 1. Academic Press.

[2] Peng, S. (1990). A general stochastic maximum principle for optimal control problems. SIAM Journal on Control and Optimization, 28(4), 966-979.

[3] Situ, R. (1991). A maximum principle for optimal controls of stochastic systems with random jumps. In Proceedings of the National Conference on Control Theory and Applications.

[4] Tang, S., & Li, X. (1994). Necessary conditions for optimal control of stochastic systems with random jumps. SIAM Journal on Control and Optimization, 32(5), 1447-1475.

[5] Øksendal, B. (2003). Stochastic differential equations. In Stochastic differential equations : An introduction with applications (pp. 65-84). Springer Berlin Heidelberg. https ://doi.org/10.1007/978-3-642-14394-6_5

[6] Protter, P. E. (2004). Stochastic integration and differential equations (pp. 195-197).

[7] Jeanblanc, M. (2006). Cours de Calcul stochastique Master 2IF EVRY. Lecture Notes, University of Évry. Retrieved from http ://www.maths.univ-evry.fr/pages_perso/jeanblanc

[8] He, S., Wang, J., & Yan, J. (2019). Semimartingale theory and stochastic calculus. Routledge.

[9] Song, Y., Tang, S., & Wu, Z. (2020). The maximum principle for progressive optimal stochastic control problems with random jumps. SIAM Journal on Control and Optimization, 58(4), 2171-2187.

Abbreviations and Notations

The various abbreviations and notations used throughout this dissertation are explained below :

SDE	:	Stochastic differential equation.
$(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$		A complete probability space with filtration $\{\mathcal{F}_t\}_{t\geq 0}$.
$\mathcal{N}$		The collection of $\mathbb{P}$-null sets.
$C^{1,2}([0,T]\times\mathbb{R},\mathbb{R})$	:	$f:[0,T]\times\mathbb{R}\to\mathbb{R}\,:\,\phi$ is continuous (in t), and f_t, f_x, f_{xx}, exists
$S^2[0,T]$	:	$X:\otimes\times\mathbb{R}\to\mathbb{R}:X$ has càdlàg paths and adapted
		and $\mathrm{E}\left[\sup_{0\leq t\leq T}\|X_t\|^2\right]<\infty$.
$M^2[0,T]$	:	X is predictable, and $\mathrm{E}\left(\int_0^T \|Z_s\|^2\,ds\right)<\infty$
$F^2[0,T]$	:	$K:[0,T]\times\Omega\times E\to M:\;K$ is $E-$ predictable,
		and $\|K\|^2=\mathrm{E}\left(\int_0^T\int_E \|K_s\|^2\,\lambda(de)dt\right)$
M	:	A Euclidean space and $\mathcal{B}(M)$ the Borel σ-field on M.
U_{ad}		Admissible control set.

Abstract :

This master's dissertation introduces a straightforward method for dealing with jumps in stochastic processes. Here's an explanation of the key points : The approach we present is simple and easy to apply, which makes it accessible for those working with Stochastic processes can experience sudden changes or "jumps." This method specifically addresses these jumps, making it easier to estimate values within systems that include such dis- continuities, and simplifies the process of making estimates within these systems. This is particularly important as the values we are dealing with become larger, where traditional methods might become more complex or less accurate. This broad applicability ensures that it can be used in various situations without significant limitations.

Résumé

Cette dissertation de master introduit une méthode simple pour gérer les sauts dans les processus stochastiques. Voici une explication des points clés : L'approche que nous présentons est simple et facile à appliquer, ce qui la rend accessible à ceux qui travaillent avec des processus stochastiques qui peuvent connaître des changements soudains ou des "sauts". Cette méthode traite spécifiquement ces sauts, ce qui facilite l'estimation des valeurs au sein des systèmes incluant de telles discontinuités, et simplifie le processus d'estimation dans ces systèmes. Cela est particulièrement important lorsque les valeurs que nous traitons deviennent plus grandes, car les méthodes traditionnelles peuvent devenir plus complexes ou moins précises. Cette large applicabilité assure qu'elle peut être utilisée dans diverses situations sans limitations significatives.

الملخص:

تقدم هذه الرسالة طريقة بسيطة للتعامل مع القفزات في العمليات العشوائية. فيما يلي شرح للنقاط الرئيسية:

النهج الذي نقدمه بسيط وسهل التطبيق، مما يجعله متاحاً لأولئك الذين يعملون مع العمليات العشوائية .يمكن أن تتعرض هذه العمليات لتغيرات مفاجئة أو "قفزات". تتناول هذه الطريقة بشكل خاص هذه القفزات، مما يسهل تقدير القيم داخل الأنظمة التي تتضمن مثل هذه الانقطاعات، كما تبسط عملية إجراء التقديرات داخل هذه الأنظمة. هذا الأمر مهم بشكل خاص عندما تصبح القيم التي نتعامل معها أكبر، حيث يمكن أن تصبح الطرق التقليدية أكثر تعقيدا أو أقل دقة. يضمن هذا النطاق الواسع للتطبيق إمكانية $. استخدام هذه الطريقة في مختلف الحالات دون قيود كبيرة.