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Physics 3 Course

VIBRATIONS AND WAVES

Part One

Second year of the Science and Technology specialty

« LMD »

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PREFACE

This guide is specifically designed for second-year students in Science and Technology programs at Algerian universities and engineering schools. It forms part of the official curriculum for the "Mechanical Vibrations and Waves" course, taught during the second year (L2-S3) for students majoring in Science and Technology (ST) and Materials Science (SM).

The primary aim of this program is to introduce second-year students to key concepts in physics and mechanics related to vibration and mechanical wave phenomena, while clearly distinguishing between the two. The unit is divided into two main sections: the first section focuses on Vibrational Motion, and the second on Waves. This guide is crafted to effectively meet the recommendations outlined in the official curriculum.

In accordance with the official curriculum, Lagrange's equations and differential equations are emphasized for the study of mechanical vibrations, without relying on the laws of classical mechanics. These concepts are highlighted in alignment with the official curriculum for both Mechanical Engineering and Methods Engineering majors. For Electrical Engineering students, however, the focus is placed on the second section, which covers Mechanical Waves.

In the context of this scientific guide, the first part covers five fundamental topics related to mechanical vibrations, along with a summary of the necessary mechanical and mathematical concepts needed to support students throughout the course.

We have simplified Lagrange's equations as much as possible for a single-particle system with one degree of freedom. The classification of various types of constraints has been omitted, as the primary focus is on the concept of generalized forces and, in the case of dissipative systems, the introduction of the dissipation function. The generalization to systems with multiple degrees of freedom is presented without formal proof. The analysis of oscillations is intentionally limited to low-amplitude oscillations and is covered in the chapter titled "Free Oscillations of Systems with One Degree of Freedom." Only viscous friction, proportional to velocity, is considered in this guide.

The subsequent chapter, dedicated to forced oscillations in systems with one degree of freedom, explores the concepts of resonance and mechanical impedance. The study of oscillatory systems with multiple degrees of freedom is confined to systems with two degrees of freedom, approached through the superposition of specific sinusoidal solutions. This section facilitates the integration of resonance and impedance concepts and introduces the phenomenon of antiresonance.

The formal approach also recommends representing the propagation of mechanical waves using a linear string model, ensuring a smooth transition from a discrete medium to a continuous medium. This transition will be explored in detail in the second part of the guide.

Finally, we trust that this guide will provide valuable insights and be a useful resource for students and the broader academic community.

Dr : DJOUDI Tarek

Contents

Chapter I

Basic Concepts in Mechanics and Mathematics

The study of vibrational and wave mechanics, a core branch of physics, is intricately grounded in mathematical principles that offer a precise and systematic framework for understanding physical phenomena. Mastery of the fundamental concepts in both mechanics and mathematics is undeniably essential, as they constitute the foundation for more advanced topics and applications across various scientific and engineering disciplines.

This chapter is designed to introduce the fundamental concepts and the necessary mathematical and mechanical tools required for analyzing and solving problems related to vibrational and wave motions. It covers foundational principles such as Newton's laws of motion, as well as key concepts including force, work, energy, and momentum. Additionally, the course delves into essential mathematical techniques, such as vector analysis and differential equations. A comprehensive understanding of these core concepts facilitates a deeper insight into vibrational and wave systems, equipping students with the intellectual tools needed for more complex future studies.

Through the exploration of these fundamental principles, the course will serve as a bridge between theoretical knowledge and practical applications. This will prepare students for an in-depth examination of more advanced topics in vibrational and wave mechanics, thereby enhancing their problem-solving abilities

1. Newton's Laws

Newton's laws of motion form the foundation of classical mechanics. They describe the relationship between a body and the forces acting upon it, and the body's motion in response to those forces.

1.1. Principle of Inertia

This principle states that an object at rest will remain at rest, and an object in motion will continue to move with constant velocity, unless acted upon by an external force. For example, a book resting on a table will remain stationary unless a force is applied to move it.

$$\sum \overrightarrow{\boldsymbol{F}_{ext}} = \vec{\boldsymbol{0}} \qquad \text{(I.1)}$$

1.2. Principle of Action and Reaction

This law states that for every action, there is an equal and opposite reaction. An example of this is when you push a wall; the wall pushes back with the same force, which is why you feel resistance.

$$\overrightarrow{\boldsymbol{F}_{a/b}} = \overrightarrow{\boldsymbol{F}_{b/a}} \qquad \text{(I.2)}$$

1.3. Principle of Motion

This law asserts that the acceleration of an object is directly proportional to the net force acting on the object and inversely proportional to its mass. Mathematically, this is expressed as F = ma, where F is the force, m is the mass, and a is the acceleration. For example, pushing a sled will cause it to accelerate depending on how hard you push and the sled's mass.

$$\sum \overrightarrow{\boldsymbol{F}_{ext}} = \boldsymbol{m}\vec{\boldsymbol{a}} \qquad \text{(I.3)}$$

2. Definition of Kinetic Energy

Kinetic energy (Ec) is the energy possessed by an object due to its motion. The formula to calculate the kinetic energy of an object is:

$$\boldsymbol{E_c} = \frac{1}{2}\boldsymbol{mV^2} \qquad \text{(I.4)}$$

Where:
m: is the mass of the object;
V: is the velocity of the object.

For instance, a car moving at 60 km/h has kinetic energy proportional to its mass and speed. The faster the car moves, the higher its kinetic energy.

In the case of a rigid body, the total kinetic energy is the sum of its translational kinetic energy and rotational kinetic energy. The translational kinetic energy is associated with the motion of the center of mass, while the rotational kinetic energy is associated with the rotation of the body about its center of mass. Mathematically, the total kinetic energy (***Ec***) of a rigid body can be expressed as:

$$E_c = E_{CT} + E_{CR} \qquad (I.5)$$

Where:
m : is the mass of the rigid body,
Vm : is the velocity of the center of mass,
J : is the moment of inertia of the body about its axis of rotation,
ω : is the angular velocity.
The first term, $E_{CT} = \frac{1}{2} m V_m^2$, represents the **translational kinetic energy**.
The second term, $E_{CR} = \frac{1}{2} J \omega^2$, represents the **rotational kinetic energy**.

3. Definition of Potential Energy

Potential energy (E_P) is the energy stored in an object due to its position or configuration. The most common example is gravitational potential energy, which depends on an object's height above the ground:

$$E_P = mgh \qquad (I.6)$$

Where:
m: is the mass of the object;
g: is the acceleration due to gravity;
h: is the height above the reference point.

An example is a rock held at the top of a hill. When the rock is released, its potential energy is converted into kinetic energy as it falls.

4. Mechanical Energy

Mechanical energy is the sum of an object's kinetic and potential energies. It is conserved in the absence of non-conservative forces (like friction). The total mechanical energy of an object can be expressed as:

$$E_M = E_c + E_P \qquad (I.7)$$

For example, a roller coaster at the top of a hill has high potential energy, and as it descends, this energy is converted into kinetic energy.

5. Conservative System

A conservative system is one where mechanical energy (kinetic + potential) is conserved. No energy is lost to external forces such as friction or air resistance. For example, an ideal pendulum in a vacuum (without air resistance) would be a conservative system because its total mechanical energy remains constant throughout the oscillation.

$$\Delta \boldsymbol{E_M} = \mathbf{0} \qquad (I.8)$$

$$\boldsymbol{E_{M2}} - \boldsymbol{E_{M1}} = \mathbf{0} \qquad (I.9)$$

6. Equilibrium Condition of a Mechanical System

An equilibrium condition exists when the net force and net torque acting on a system are zero, resulting in no acceleration. This implies that the body has no kinetic energy and is in a state of rest, with no movement occurring in either translational or rotational forms. Mathematically, this is:

$$\sum \overrightarrow{\boldsymbol{F_{ext}}} = \vec{\mathbf{0}} \;\; \boldsymbol{and} \;\; \sum \overrightarrow{\boldsymbol{M_{/0}}} = \vec{\mathbf{0}} \qquad (I.10)$$

For example, when a book is resting on a flat surface, the gravitational force pulling it downward is balanced by the normal force from the surface pushing upward, resulting in no motion.

7. Definition of Moment

A moment (also called torque) is the rotational equivalent of force. It measures the tendency of a force to rotate an object around an axis or fixed point. It is calculated as:

$$\boldsymbol{M(F)_{/\Delta}} = \boldsymbol{F.r}.\sin\boldsymbol{\theta} \qquad (I.11)$$

Where:

$M(F)/_{\Delta}$: It is the moment or torque of a force relative to the axis of rotation;

r : is the distance from the axis of rotation;

F : is the force applied;

θ : is the angle between the force vector and the lever arm.

For example, when you use a wrench to loosen a bolt, the force you apply at the end of the wrench creates a torque that turns the bolt.

8. Moment of Inertia

The moment of inertia ($\boldsymbol{J}$) is a measure of an object's resistance to rotational motion about an axis. It depends on the mass distribution relative to the axis of rotation.

The formula for some known shapes is:

shapes	moment of inertia (J)
Material point is r away from the center of rotation.	mr^2
Ring with diameter R	mR^2
Cylinder with diameter R	$\frac{1}{2}mR^2$
Leg with length L	$\frac{1}{2}mL^2$

Where:
m : is the mass of the object;
r and R : is the distance from the axis of rotation;
L: length of Leg.

9. Huygens' Theorem

Huygens' Theorem states that the moment of inertia J_{Δ_2} of a body about any axis is related to the moment of inertia J_{Δ_1} bout the center of mass and the mass distribution relative to that axis. The relationship is given by:

$$\boldsymbol{J_{\Delta_2} = J_{\Delta_1} + Md^2} \qquad \text{(I.12)}$$

Where:
J_{Δ_2} : is the moment of inertia about any axis (not necessarily through the center of mass);
J_{Δ_1} : is the moment of inertia about the center of mass;
M : is the mass of the rigid body;
d : is the distance between the center of mass and the axis of rotation.

10. Definition of degree of freedom

The degree of freedom refers to the number of independent parameters or coordinates needed to uniquely define the position and configuration of a system. In mechanical systems, it corresponds to the number of independent ways a system can move or the number of independent variables required describing its motion.

For example:

A particle moving in a straight line has 1 degree of freedom, as only its position along the line is needed to describe its motion.

A particle moving in a plane has 2 degrees of freedom because its position requires two independent coordinates (e.g., x and y coordinates).

A rigid body moving in three-dimensional space has 6 degrees of freedom: 3 for translation (motion along the x, y, and z axes) and 3 for rotation (rotation about the x, y, and z axes).

In general, the degree of freedom of a system is determined by the number of independent motion components that the system can exhibit, considering any constraints it may have.

11. Springs in Series

When springs are connected in series, the total spring constant (k_{total}) is found by summing the reciprocals of the individual spring constants:

$$\frac{1}{k_{total}} = \frac{1}{k_1} + \frac{1}{k_2} + \cdots \qquad (I.13)$$

This arrangement causes the system to have a lower spring constant compared to the individual springs, resulting in greater elongation for a given force. For example, if two springs of different stiffness are connected in series, the combined system will stretch more than either of the springs alone.

12. Springs in Parallel

When springs are arranged in parallel, the total spring constant (k_{total}) of the system is the sum of the individual spring constants. This is in contrast to the series arrangement where the spring constants are combined differently.

$$k_{total} = k_1 + k_2 + \cdots + k_n \qquad (I.14)$$

Where:

k1,k2,…,kn are the spring constants of the individual springs in parallel.

13. Springs with Branches

Springs with branches involve more complex arrangements where one spring divides into two or more paths, each with its own spring constant. The total behavior of the system depends on how the branches are arranged, and the equivalent spring constant can be calculated using a combination of series and parallel spring formulas. For instance, a network of springs in parallel will have an equivalent spring constant that is the sum of the individual spring constants, while a combination of series and parallel can be used to find the effective spring constant for more complex setups.

14. Second-Order Differential Equations

A second-order differential equation is an equation that involves the second derivative of an unknown function. These equations are widely used to describe various physical phenomena, such as motion, electrical circuits, and heat transfer.

A general second-order differential equation is given by:

$$\boldsymbol{f}(\boldsymbol{x}, \boldsymbol{y}, \frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}}, \frac{\partial^2 \boldsymbol{y}}{\partial \boldsymbol{x}^2})=\mathbf{0} \quad \text{(I.15)}$$

Where:
y: is the unknown function dependent on x
dy/dx :is the first derivative
d^2y/dx^2 : is the second derivative of y with respect to x.

The general solution to a second-order linear differential equation depends on the nature of the system and its parameters, such as damping, forcing, and initial conditions. The solutions can be classified based on the damping conditions or external forces applied to the system. Below are the primary categories of solutions for second-order differential equations .The solution process to a second-order differential equation typically follows these steps:

1.Write the equation in standard form: Begin by expressing the equation in the form:

$$\frac{\partial^2 \boldsymbol{y}}{\partial \boldsymbol{x}^2} + \boldsymbol{p}(\boldsymbol{x})\frac{\partial \boldsymbol{y}}{\partial \boldsymbol{x}} + \boldsymbol{q}(\boldsymbol{x})\boldsymbol{y} = \boldsymbol{g}(\boldsymbol{x}) \quad \text{(I.16)}$$

2.Identify the type of equation: Determine if the equation is homogeneous or non-homogeneous. In a homogeneous equation, g(x) = 0, while in a non-homogeneous equation, g(x) ≠ 0.
3.Solve the homogeneous equation: For homogeneous equations, solve the corresponding equation (g(x) = 0) by finding the characteristic equation. The solution will depend on the discriminant of the characteristic equation.
4.Consider damping conditions: For physical systems like oscillators, identify if the system is underdamped, critically damped, or overdamped. This will affect the form of the solution.
*Underdamped: The solution includes sinusoidal terms multiplied by an exponential decay factor.
*Critically Damped: The solution consists of a decaying exponential without oscillation.
*Overdamped: The solution consists of two decaying exponential terms.
5.Solve the non-homogeneous equation: If the equation is non-homogeneous, find a particular solution that satisfies the non-homogeneous term (g(x)). Methods like undetermined coefficients or variation of parameters can be used to find the particular solution.
Let the second-order differential equation for oscillatory motion be of the form:

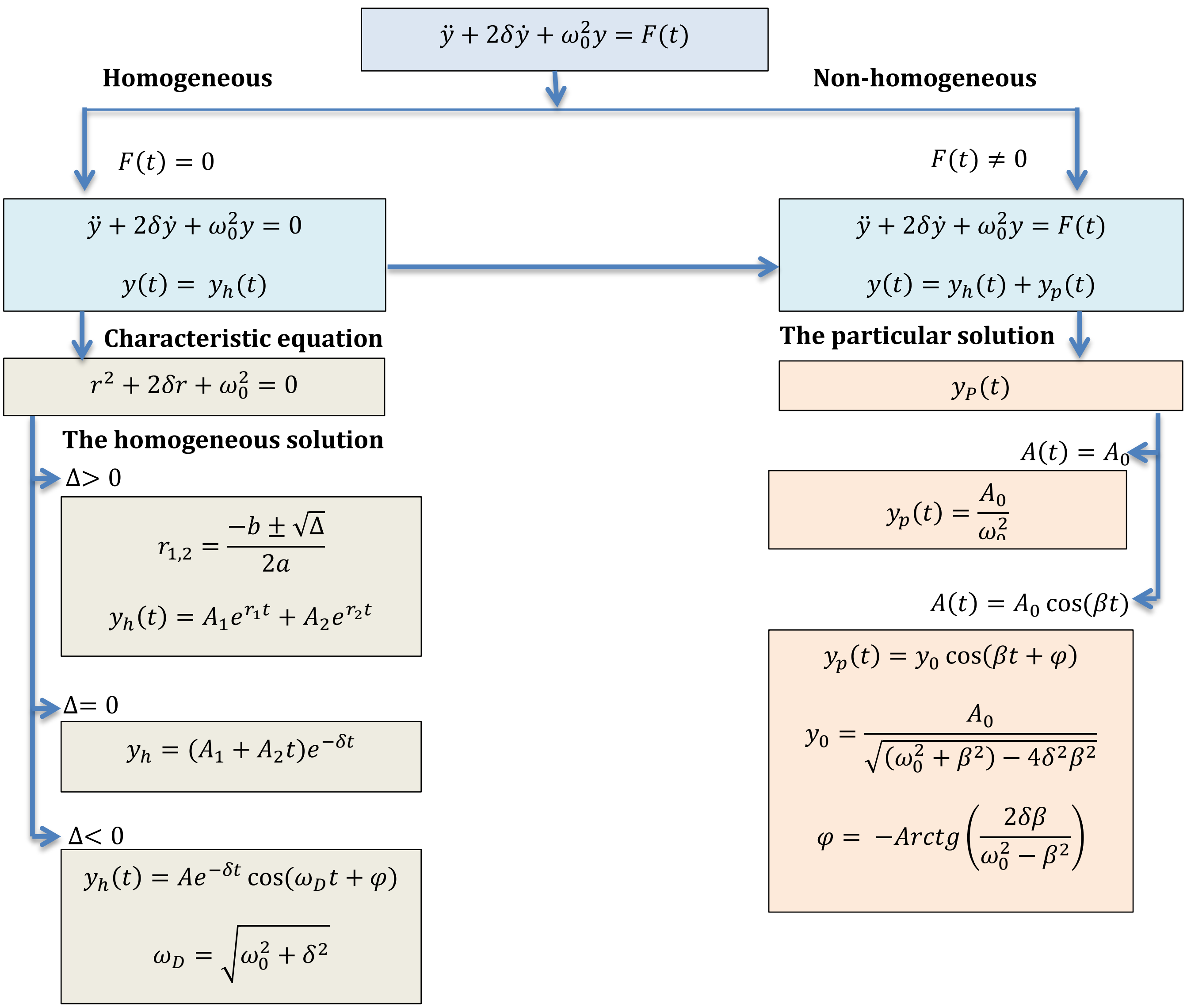
$$\ddot{y} + 2\delta\dot{y} + \omega_0^2 y = F(t) \tag{I.17}$$

Whereas:

x=t, $p(x) = 2\delta$, $q(x) = \omega^2$, $g(x) = F(t)$, $\frac{\partial^2 y}{\partial x^2} = \ddot{y}$ **and** $\frac{\partial y}{\partial x} = \dot{y}$.

The solutions to the equation are as follows:

Figure I.1. Flowchart of the solution of a second-order differential equation



Chapter II

Introduction to Lagrange's Equations

Lagrange's equations form a cornerstone of classical mechanics and provide a powerful framework for analyzing the motion of systems. Developed by Joseph Louis Lagrange in the 18th century, these equations are derived from the principle of least action, also known as Hamilton's principle, which states that the path taken by a system between two states is the one that minimizes the action.

At the core of Lagrange's approach is the concept of generalized coordinates, which allow the description of a system's configuration in a way that simplifies the equations of motion. This is particularly useful for systems with constraints, complex geometries, or non-Cartesian coordinate systems where Newtonian mechanics may be cumbersome or inadequate.

Lagrange's equations are particularly powerful for dealing with systems that involve oscillations, vibrations, and waves. In such systems, where forces are often derived from potential energy, Lagrange's formalism allows for an elegant derivation of the equations of motion without needing to explicitly account for forces at each point in space.

1.1 Lagrange's Equations for a Single Particle

1.1.1 Lagrange's Equations

A Lagrange equation in a dynamical system is a scalar function of dynamical variables that allows writing the differential equations of motion of a mechanical system without resorting to the laws of classical Newtonian mechanics. Lagrange's equations of motion provide a powerful framework for analyzing the dynamics of physical systems, particularly when dealing with complex constraints and generalized coordinates. They are derived from the principle of least action, which states that the actual path taken by a system is the one for which the action integral is minimized. The basic form of Lagrange's equation for a particle is given by:

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = F_{ext,q} \quad \text{(II.1)}$$

where $\boldsymbol{L}$ is the Lagrangian, defined as the difference between the kinetic energy E_c and the potential energy $\boldsymbol{E_p}$ of the system,

$$L = E_c - E_p. \quad \text{(II.2)}$$

The coordinates q represent the generalized coordinates, while $\dot{q}$ denotes their time derivatives, also referred to as the generalized velocities.

$F_{ext,q}$: Generalized external forces.

1.1.2 The Case of Conservative Systems

For conservative systems, the potential energy E_p depends only on the generalized coordinates and not on time or velocities. This implies that the total mechanical energy E_M is conserved over time.

$$E_M = E_c + E_P = Cst \quad \text{(II.3)}$$

In such systems, the Lagrangian does not explicitly depend on time, and the generalized forces are derived from the gradient of the potential energy.

$$F_{ext,q} = -\frac{\partial E_P}{\partial q} \quad \text{(II.4)}$$

The equation of motion then becomes:

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}}\right)-\frac{\partial L}{\partial q}=0 \qquad \text{(II.5)}$$

This simplified form is particularly useful for systems like a mass on a spring or a simple pendulum, where the forces involved are purely conservative (no friction or other non-conservative forces).

1.1.3 The Case of Velocity-Dependent Friction Forces

In many practical systems, the forces acting on a particle are not purely conservative. For instance, forces like friction, which are often velocity-dependent, can significantly affect the dynamics of the system. These forces can be modeled as functions of the generalized velocities;

Such as $\overrightarrow{F_f}=-\alpha\vec{V}$ (II.6)

Where: α is a damping coefficient

$\vec{V}$:The vector represents the linear velocity.

In this case, the Lagrangian must be modified to account for the work done by these non-conservative forces. The generalized force arising from the friction is included in the Lagrange equation as an additional term on the right-hand side:

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}}\right)-\frac{\partial L}{\partial q}=F_q \qquad \text{(II.7)}$$

$$F_q=-\beta\dot{q} \qquad \text{(II.8)}$$

F_q: The generalized force and β: constant positive.

For a velocity-dependent friction force, this leads to damping effects that gradually dissipate the system's energy.

The generalized viscosity friction force is related to the dissipation function according D to the following relationship:

$$F_q=-\frac{\partial D}{\partial \dot{q}} \qquad \text{(II.9)}$$

With $D=\frac{1}{2}\beta\dot{q}^2$ (II.10)

By substitution in the Lagrange equation, the relationship when there is viscous friction becomes:

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} + \frac{\partial D}{\partial \dot{q}} = 0 \qquad \text{(II.11)}$$

1.1.4 The Case of Time-Dependent External Forces

When external forces, such as electromagnetic forces, gravitational fields, or time-varying constraints, are introduced, they add a time-dependence to the system's dynamics. These external forces are represented in the generalized force term of the Lagrange equation, which may now depend explicitly on time:

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} + \frac{\partial D}{\partial \dot{q}} = F_{ext,q}(t) \qquad \text{(II.12)}$$

Here, $F_{ext,\,q}(t)$ represents any external force that varies with time, such as a magnetic field that changes with time or a driving force applied to the system.

1.2 Systems with Multiple Degrees of Freedom

In systems with multiple particles or components, each of which can move independently, we introduce generalized coordinates for each degree of freedom. For example, a double pendulum has two degrees of freedom, represented by two generalized coordinates, q_1 and q_2. In this case, the Lagrangian formulation is extended to accommodate the interactions between the various degrees of freedom.

The equations of motion for a system with multiple degrees of freedom are obtained by writing down the Lagrangian for the entire system, which is the sum of the kinetic and potential energies of all components. The Lagrange equations for each generalized coordinate are then derived in a similar fashion as for a single particle, but now they involve the interactions between the coordinates:

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} + \frac{\partial D}{\partial \dot{q}_i} = F_{ext,q_i}(t) \qquad \text{(II.13)}$$

For each i = 1, 2, ..., n, where n is the total number of degrees of freedom in the system. In such cases, the interactions between the different components of the system are explicitly included in the Lagrangian, which may lead to coupled equations of motion.

2. An applied example

Determine the equation of motion of a simple pendulum, as shown in FigureII.1, consisting of a mass m and a massless string of length l, for small oscillations, using Lagrange's method.

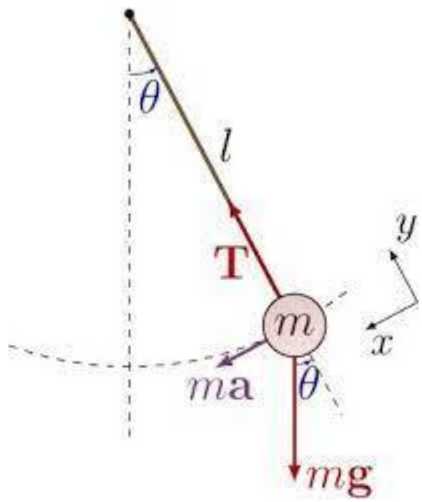


Figure II.1 *simple pendulum.*

The rotational motion, therefore, implies that the coordinates of the system are given by q=θ.

1. Kinetic energy and potential energy

$$E_c = \frac{1}{2}J\dot{\theta}^2 = \frac{1}{2}ml^2\dot{\theta}^2 \quad \text{(II.14)}$$

$$E_P = -mgh = -mgl\cos\theta \quad \text{(II.15)}$$

2. Lagrange equation

The Lagrange equation that characterizes the system is expressed as:

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} = 0 \quad \text{(II.16)}$$

$$L = E_c - E_p \quad \text{(II.17)}$$

$$L = \left(\frac{1}{2}ml^2\dot{\theta}^2\right) - (-mgl\cos\theta) \quad \text{(II.18)}$$

For weak oscillations ($\sin\theta \approx \theta$).

Applying Lagrange's equation:

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = ml^2\ddot{\theta} \tag{II.19}$$

$$\frac{\partial L}{\partial \theta} = mlg\sin\theta = mgl\theta \tag{II.20}$$

Substituting into the Lagrange equation (II.6):

$$ml^2\ddot{\theta} + mgl\sin\theta = 0 \tag{II.21}$$

The differential equation of motion is as follows:

$$\ddot{\theta} + \frac{g}{l}\theta = 0 \tag{II.22}$$

And the free pulse $\omega_0 = \sqrt{\frac{g}{l}}$ (II.23)

Chapter III

Free Oscillations of Single-Degree-of-Freedom Systems

The study of free oscillations in mechanical systems is essential for understanding the fundamental dynamics of oscillatory motion. A single-degree-of-freedom (SDOF) system is a simple yet powerful model for describing many mechanical systems, where motion can be characterized by a single coordinate or displacement. These systems are widely used in engineering to model various physical systems such as vibrating beams, mass-spring systems, and even complex mechanical structures.

In a free oscillation scenario, no external forces are applied to the system once it is displaced from its equilibrium position. Instead, the system's motion is governed by its inherent properties—such as mass, stiffness, and damping—and its natural tendency to oscillate. The study of these oscillations helps us understand the system's response to initial disturbances and provides insights into its behavior over time.

For an SDOF system, the equation of motion is typically a second-order differential equation that describes how the system's displacement evolves with time. The solution to this equation depends on the system's parameters, such as mass, spring constant, and damping coefficient, which influence the nature of the oscillations. Depending on the level of damping, the system may undergo oscillations that decay over time, or it may return to equilibrium without oscillating.

Understanding these free oscillations is crucial for the design and analysis of structures and mechanical systems, especially when considering factors such as resonance, stability, and energy dissipation. In this context, free oscillations provide the foundation for more complex dynamic analyses, including forced oscillations and the study of wave propagation.

This section introduces the key concepts and mathematical formulations necessary to analyze free oscillations in SDOF systems, laying the groundwork for a deeper understanding of vibration analysis in engineering and physics.

1 Undamped Oscillations

The undamped oscillations occur in systems where the mechanical energy is not dissipated by friction or other non-conservative forces. In such systems, the mechanical system is displaced from its equilibrium position and oscillates freely without any external excitation force. As a result, the total mechanical energy, this is the sum of kinetic and potential energies, remains constant over time, and the motion exhibits periodicity.

The fundamental equation of simple harmonic motion (SHM) is derived from either Newton's second law or Lagrange's equations, resulting in a differential equation that describes the motion of a free system with one degree of freedom in generalized coordinates:

$$\ddot{q} + \omega^2 q = 0 \tag{III.1}$$

The differential equation is obtained from the Lagrange equation for a conservative system with a first degree of freedom, considering the two conditions that characterize oscillatory motion (The equilibrium condition and the stability condition).

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} = 0 \tag{III.2}$$

Where:

$$L = E_c - E_p \tag{III.3}$$

The equilibrium condition

$$\left.\frac{\partial E_p}{\partial q}\right|_{q=0} = 0 \tag{III.4}$$

And the stability condition is given by

$$\left.\frac{\partial^2 E_p}{\partial q^2}\right|_{q=0} > 0 \tag{III.5}$$

This differential equation represents the dynamics of a free system without damping or external forces. Its solution describes simple harmonic motion, a sinusoidal equation that determines the position of the body at any instant, and takes the following form:

$$q(t) = A\cos(\omega_0 t + \varphi) \tag{III.6}$$

Where:

A is the amplitude of oscillation;

$\omega_0 = \sqrt{\frac{k}{m}}$ is the angular frequency;

φ is the phase constant.

The period of oscillation, T, is given by:

$$T = 2\pi/\omega_0 = 2\pi\sqrt{(m/k)} \quad \text{(III.7)}$$

This result shows that the period of oscillation depends only on the mass and the stiffness of the system, and it is independent of the amplitude. The motion is sinusoidal, with constant amplitude, and the system oscillates indefinitely without any loss of energy.

To determine the values of the angular frequency $\boldsymbol{A}$ and phase constant $\boldsymbol{\varphi}$, it is necessary to refer to the initial conditions of the motion described in the following equation:

$$\begin{cases} q(t=0) = q_0 \\ \dot{q}(t=0) = \dot{q_0} \end{cases} \quad \text{(III.8)}$$

Thus we get the following values:

$$A = \sqrt{\frac{q_0^2\omega_0^2 + \dot{q}_0^2}{\omega_0^2}} \quad \text{(III.9)}$$

And

$$\varphi = -Arctg\left(\frac{\dot{q_0}}{q_0\,\omega_0}\right) \quad \text{(III.10)}$$

The free oscillatory motion of a material point can be represented in Cartesian coordinates as follows:

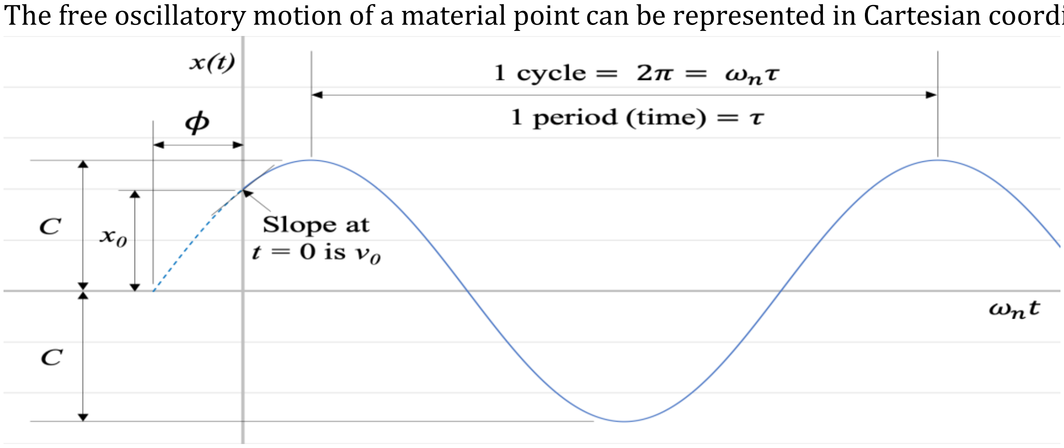


***Figure III.1**: Displacement response of the mass spring system (solution to the differential equation).*

τ *:* The period is defined as the time interval required for the oscillating mass to return to the same position in its motion, that is, the duration between two successive passages through an identical point, as illustrated in the figure III.1.

$$\tau = \frac{2\pi}{\omega_0} = \frac{1}{f} \qquad \text{(III.11)}$$

Where: f denotes the frequency of the oscillatory motion; Its unit is hertz **[Hz]** or **[S^{-1}].**

2 Free Oscillations of Damped Systems

In practical systems, mechanical systems are invariably subjected to external resistive forces, such as friction, which introduces damping effects. This damping leads to a gradual reduction in the amplitude of oscillations over time. Damped oscillations are characterized by the presence of these resistive forces, which are typically directly proportional to the velocity of the oscillating body, thereby dissipating energy from the system. In this section, we will specifically examine viscous friction, commonly referred to as damping.

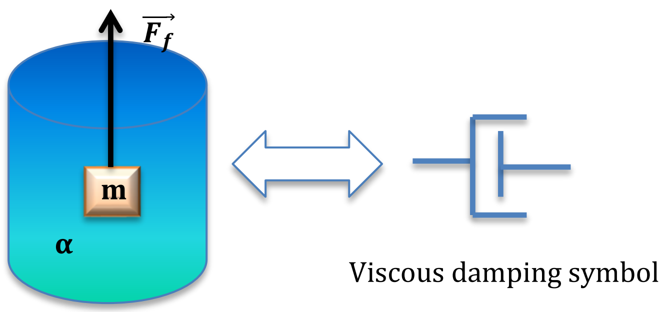


Figure III.2 *: Viscous friction.*

The Lagrange equation that governs the oscillations damped by the frictional force takes the following form:

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}}\right) - \frac{\partial L}{\partial q} + \frac{\partial D}{\partial \dot{q}} = 0 \qquad \text{(III.12)}$$

From the Lagrange equation for a free system undergoing damped vibrations, the second-order homogeneous differential equation is derived as follows:

$$\ddot{q} + 2\delta\dot{q} + \omega_0^2 q = 0 \qquad \text{(III.13)}$$

Where:
δ : is the damping coefficient [s^{-1}];
ω_0: is the free pulsation [rad/s].

The resulting second-order linear differential equation represents the dynamics of a damped harmonic oscillator. The solution to this equation varies based on the damping condition, which can be categorized as under-damped, critically damped, or over-damped.
$\delta < \omega_0$: Under-damped case;
$\delta = \omega_0$: Critically damped case;
$\delta > \omega_0$: Over-damped case;

2.1 Underdamped Case (Weak Damping)

When the damping is small ($c^2 < 4mk$), the system exhibits oscillations, but the amplitude decreases exponentially over time. The solution to the equation for underdamped motion is:

$$q(t) = Ae^{-\delta t}\cos(\omega_a t + \varphi) \qquad \text{(III.14)}$$

Where A and φ are integration constants determined from the initial conditions. ω_a is the pseudo-frequency defined by:

$$\omega_a = \sqrt{\omega_0^2 - \delta^2} \qquad \text{(III.15)}$$

The system oscillates with decreasing amplitude, and the time between peaks (the period) increases slightly due to the damping. The exponential factor $e^{-\delta t}$ represents the gradual decay of energy in the system. The frequency of oscillation ω_a is less than the natural frequency ω due to the damping.

The figure represents the time-reduced decay curve of the system in Cartesian coordinates, illustrating the variation of the system's response over time

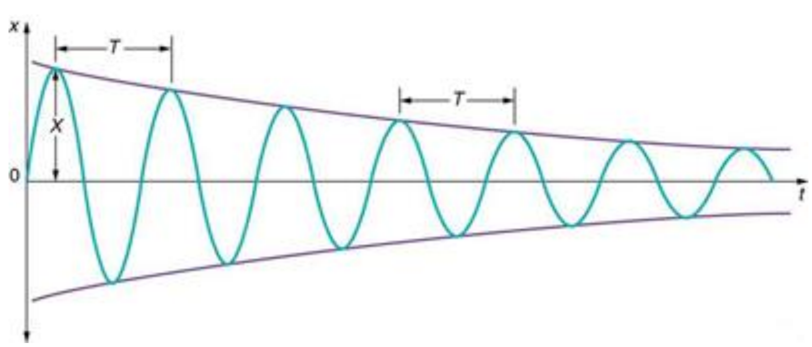


Figure III.3 *: The time-reduced decay curve of the system in cartesian coordinates.*

2.2 Critically Damped Case

At critical damping, the damping force is sufficiently large to prevent oscillations, causing the system to return to equilibrium in the shortest time possible. In this scenario, it is important to note that the motion is non-periodic. The general solution to the differential equation for a critically damped system is as follows:

$$q(t) = (A_1 + A_2 t)e^{-\delta t} \tag{III.16}$$

Where A_1 and A_2 are constants determined by the initial conditions.

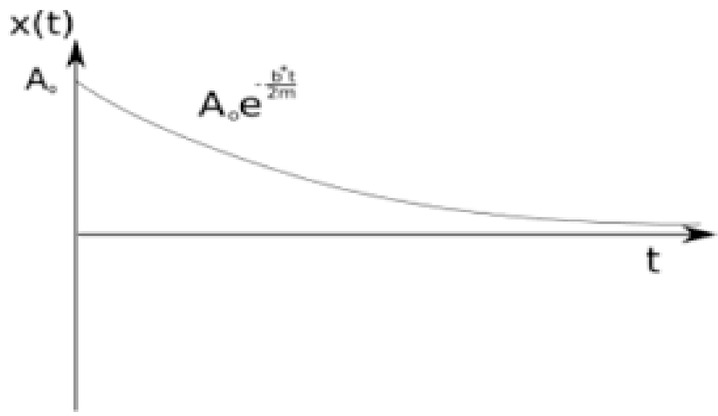


***Figure III.4** : Variation in capacitance over time for a critically damped system.*

The figure illustrates the variation in capacitance over time for a critically damped system, demonstrating how the system gradually returns to rest as time progresses

2.3 Overdamped Case (Strong Damping)

If the damping is large ($c^2 > 4mk$), the system does not oscillate, and the displacement gradually decays to zero without any oscillation. The solution to the equation for overdamped motion is:

$$q(t) = A_1 e^{r_1 t} + A_2 e^{r_2 t} \tag{III.17}$$

Where: r_1 and r_2 are the roots of the characteristic equation, both real and negative.

$$r_1 = -\delta - \sqrt{\delta^2 - \omega_0^2} \tag{III.18}$$

$$r_2 = -\delta + \sqrt{\delta^2 - \omega_0^2} \tag{III.19}$$

The system returns to equilibrium slowly without oscillating, and the rate of decay depends on the value of the damping coefficient.

3. An applied example

A rod OC with negligible mass is hinged without friction at point O and carries a mass m at its end C (Figure III.5). Two springs with stiffness constants k1 and k2 are attached to the rod at points A and B, respectively. In equilibrium, the rod is horizontal and is displaced by a small angle θ (assumed to be very small, corresponding to small oscillations).

1- Calculate the kinetic and potential energy of the system.
2- Deduce the Lagrangian.
3-Derive the equation of motion.
4- Find the solution θ(t) given that $\theta(0) = \frac{\pi}{\;}$ and $\dot{\theta(0)} = 0$.

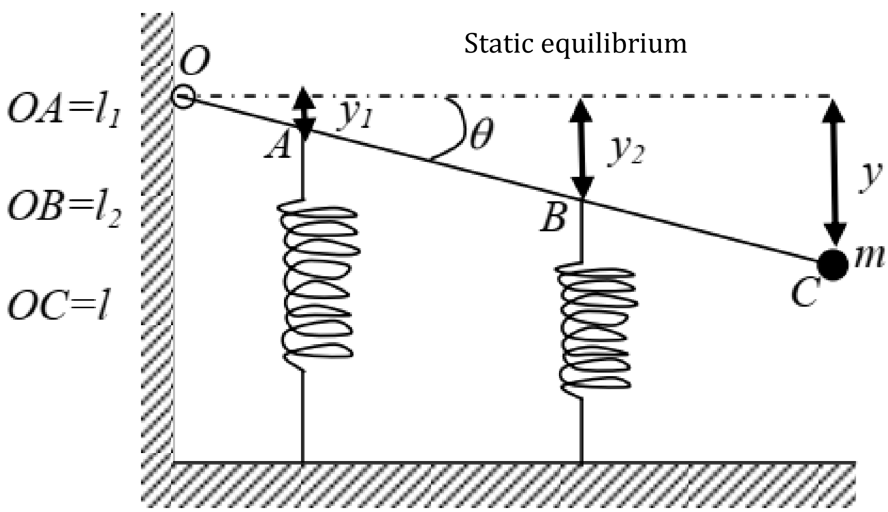


***Figure III.4** Pendulum-spring mechanical system.*

The rotational motion, therefore, implies that the coordinates of the system are given by q=θ.

Degree of freedom : D=N-L=4-(2+1)=1

1. Kinetic energy and potential energy

For weak oscillations ($\sin\theta \approx \theta$).

$$E_c = \frac{1}{2}m\dot{y}^2 = \frac{1}{2}m\left(l\dot{\theta}\right)^2 \tag{III.20}$$

$$E_P = E_{PR1} + E_{PR2} \tag{III.21}$$

$$E_P = \frac{1}{2}k_1y_1^2 + \frac{1}{2}k_2y_2^2 = \frac{1}{2}k_1(l_1\theta)^2 + \frac{1}{2}k_2(l_2\theta)^2 \tag{III.22}$$

2. Lagrange equation

The Lagrange equation that characterizes the system is expressed as:

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} = 0 \tag{III.23}$$

$$L = E_c - E_p \tag{III.24}$$

$$L = \frac{1}{2}m\left(l\dot{\theta}\right)^2 - \frac{1}{2}k_1(l_1\theta)^2 - \frac{1}{2}k_2(l_2\theta)^2 \tag{III.25}$$

Applying Lagrange's equation:

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = ml^2\ddot{\theta} \tag{III.26}$$

$$\frac{\partial L}{\partial\theta} = (k_1l_1^2 + k_2l_2^2)\theta \tag{III.27}$$

Substituting into the Lagrange equation (III.23):

$$\ddot{\theta} + \frac{(k_1l_1^2+k_2l_2^2)}{ml^2}\theta=0 \tag{III.28}$$

And the free pulse $\omega_0 = \sqrt{\frac{(k_1l_1^2+k_2l_2^2)}{ml^2}}$ (III.29)

$$\begin{cases} \theta(t) = A\cos(\omega_0 t + \varphi) \\ \dot{\theta}(t) = -A\omega_0\sin(\omega_0 t + \varphi) \end{cases}$$

$$\begin{cases} \theta(t=0) = A\cos(\varphi) = \frac{\pi}{15} \\ \dot{\theta}(t=0) = -A\omega_0\sin(\varphi) = 0 \end{cases} \Rightarrow \begin{cases} A = \frac{-\pi}{15} \\ \varphi = \pi \end{cases} \tag{III.30}$$

The equation of motion as a function of time is:

$$\theta(t) = -\frac{\pi}{15}\cos\left(\sqrt{\frac{(k_1 l_1^2 + k_2 l_2^2)}{ml^2}}\ \ t + \pi\right) \tag{III.31}$$

4 An applied example (Free damped system)

A mass mmm is welded to the end of a rod of length l and negligible mass (FigureIV.2). The other end of the rod is hinged at point O. The rod is connected at point A to a frame (B1) by a spring with stiffness k_1. At point B, the rod is connected to another frame (B2) by a spring with stiffness k_2. The mass m is connected to frame (B2) by a damper with a friction coefficient α. The distances are *OA=l/3* and *OB=2l/3*.

1- Find the differential equation of motion.
2- Determine the solution of the differential equation in the case of weak damping, the damping coefficient, the natural frequency, and the pseudo-frequency.

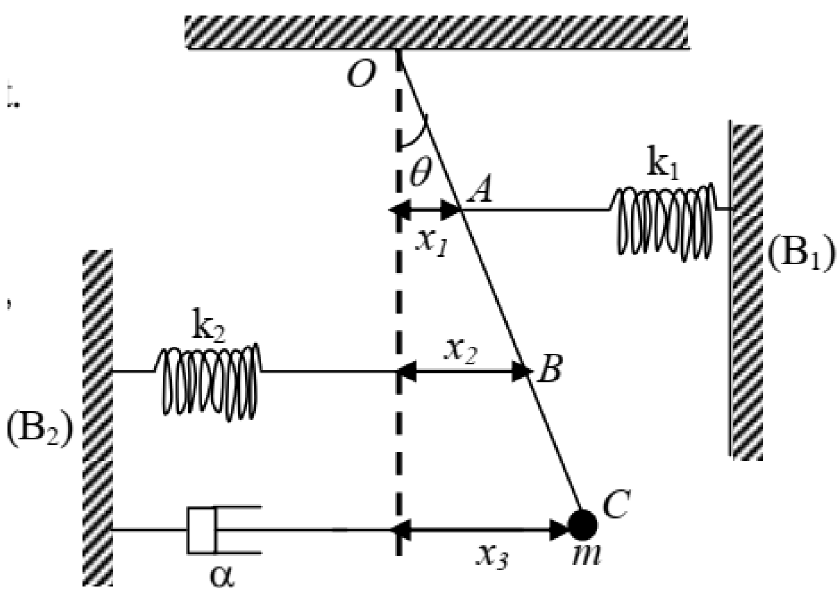


Figure IV.2 *Mechanical mass-shock absorber-spring system.*

The rotational motion, therefore, implies that the coordinates of the system are given by q=θ.

Degree of freedom : D=N-L=4-(2+1)=1

1. Kinetic energy and potential energy

For weak oscillations ($\sin\theta \approx \theta$).

$$E_c = \frac{1}{2}m\dot{x}_3^2 = \frac{1}{2}m\left(l\dot{\theta}\right)^2 \qquad \text{(III.32)}$$

$$E_P = E_{PR1} + E_{PR2} \qquad \text{(III.33)}$$

$$E_P = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2x_2^2 = \frac{1}{2}k_1\left(\frac{l}{3}\theta\right)^2 + \frac{1}{2}k_2\left(\frac{2l}{3}\theta\right)^2 \qquad \text{(III.34)}$$

$$D = \frac{1}{2}\alpha l^2\dot{\theta}^2 \qquad \text{(III.35)}$$

2. Lagrange equation

The Lagrange equation that characterizes the system is expressed as:

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} + \frac{\partial D}{\partial\dot{\theta}} = 0 \qquad \text{(III.36)}$$

$$L = E_c - E_p \qquad \text{(III.37)}$$

$$L = \frac{1}{2}m\left(l\dot{\theta}\right)^2 - \frac{1}{2}k_1(l_1\theta)^2 - \frac{1}{2}k_2(l_2\theta)^2 \qquad \text{(III.38)}$$

Applying Lagrange's equation:

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial\dot{\theta}}\right) = ml^2\ddot{\theta} \qquad \text{(III.39)}$$

$$\frac{\partial L}{\partial\theta} = -\left(\frac{k_1l^2}{9} + \frac{4k_2l^2}{9}\right)\theta \qquad \text{(III.40)}$$

$$\frac{\partial D}{\partial\dot{\theta}} = \alpha l^2\dot{\theta} \qquad \text{(III.41)}$$

Substituting into the Lagrange equation (IV.26):

$$\ddot{\theta} + \frac{\alpha}{m}\dot{\theta} + \left(\frac{k_1}{9m} + \frac{4k_2}{9m}\right)\theta = 0 \qquad \text{(III.42)}$$

$$\ddot{\theta} + 2\delta\dot{\theta} + \omega_0^2\theta = 0 \qquad \text{(III.43)}$$

For weak damping:

$$\theta(t) = Ae^{-\delta t}\cos(\omega_D t + \varphi) \qquad \text{(III.44)}$$

The natural frequency:

$$\omega_0 = \sqrt{\frac{k_1+4k_2}{9m}} \tag{III.45}$$

The damping coefficient:

$$\delta = \frac{\alpha}{2m} \tag{III.46}$$

The pseudo frequency:

$$\omega_D = \sqrt{\omega_0^2 - \delta^2} \tag{III.47}$$

$$\omega_D = \frac{1}{6m}\sqrt{4(k_1 + 4k_2) - 9\alpha^2} \tag{III.48}$$

Thus, the equation of motion with respect to time takes the form of:

$$\theta(t) = Ae^{-\frac{\alpha}{2m}t} \cos\left(\frac{1}{6m}\sqrt{4(k_1 + 4k_2) - 9\alpha^2}\, t + \varphi\right) \tag{III.49}$$

The amplitude and phase are calculated from the initial conditions of the motion.

5. Summary

In summary, undamped oscillations represent idealized systems that exhibit continuous periodic motion with constant amplitude. In contrast, damped oscillations are more realistic, where energy is dissipated over time, causing the amplitude to decrease. The nature of the damping (underdamped, critically damped, or overdamped) determines the system's behavior in response to its initial displacement. Damped oscillations are common in mechanical systems, electrical circuits, and even biological systems, where energy dissipation plays a significant role in the dynamics.

Chapter IV

Forced Ocillations of One-Degree-of-Freedom Systems

in the study of mechanical systems, the analysis of forced vibrations plays a crucial role in understanding how systems respond to external excitations. Unlike free vibrations, which occur without any external influence once a system is displaced from its equilibrium position, forced vibrations arise when an external periodic or time-varying force is applied to the system. These external forces can come from various sources, such as motors, engines, or environmental factors like wind or seismic activity. Understanding how systems react to such forces is essential for designing and analyzing structures, machinery, and other engineered systems.

In the context of first-order systems, the equations governing forced vibrations are typically simpler than those of more complex systems with higher degrees of freedom, yet they provide foundational insights that can be extended to more intricate problems. A first-order forced vibration system is typically modeled using a simple differential equation, where the system's displacement is influenced by both the internal restoring forces (such as spring stiffness) and the external forcing function.

Through the analysis of forced vibrations, we gain valuable insights into key phenomena such as resonance, where the frequency of the external force matches the natural frequency of the system, leading to potentially large oscillations that can result in failure if not properly managed. The study of resonance, damping, and the amplitude of vibrations under different external forcing conditions provides the foundation for understanding the dynamic behavior of engineering systems.

This chapter delves into the principles behind forced vibrations in first-order systems, focusing on how external forces influence the system's motion. By exploring key concepts like amplitude response, damping, and resonance, we aim to equip the reader with the tools needed to analyze and predict the behavior of simple systems under external excitation. While first-order systems may seem simple, they serve as the cornerstone for more complex analyses, providing essential knowledge applicable to a wide range of practical engineering problems.

In this chapter, we will discuss forced oscillations in a one-degree-of-freedom system, which is a fundamental topic in mechanical vibrations. We will cover the differential equations governing the system, explore a specific mass-spring-damper system, and analyze different types of excitations that can force oscillations. Finally, we will introduce the concept of mechanical impedance and its role in these systems.

1 Differential Equation

A one-degree-of-freedom (1-DOF) system is a mechanical system that can be modeled with a single coordinate, usually denoted as q(t), which represents the displacement of the system. The general form of the equation of motion for forced oscillations is derived from Newton's Second Law or Lagrange's Equation.

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} + \frac{\partial D}{\partial \dot{q}_i} = F_{ext,q_i}(t) \tag{IV.1}$$

For a system subject to external force ***F(t),*** the equation of motion is given by:

$$\ddot{q} + 2\delta\dot{q} + \omega_0^2 q = F_{ext,q_i}(t) \tag{IV.2}$$

Where:

δ : is the damping coefficient [s^{-1}];

ω_0: is the free pulsation [rad/s];

$F_{ext,q_i}(t)$: is the external force applied to the system.

This is a second-order linear differential equation that governs the forced oscillations of the system. Solving this equation provides the displacement q(t)=x(t) as a function of time in the case of retraction motion relative to the shaft x.

2 Mass-Spring-Damper System

One of the most common examples of a 1-DOF system is the mass-spring-damper system, which consists of:

A mass ***m*** attached to a spring with a stiffness constant ***k***,

A damper with damping coefficient $\boldsymbol{\beta}$ which resists motion.

The force applied to the system may be external and time-dependent, like a periodic force or a harmonic excitation. The system's natural behavior (unforced response) is determined by its natural frequency and damping ratio, but when subjected to external excitation, the response changes depending on the frequency and form of the excitation.

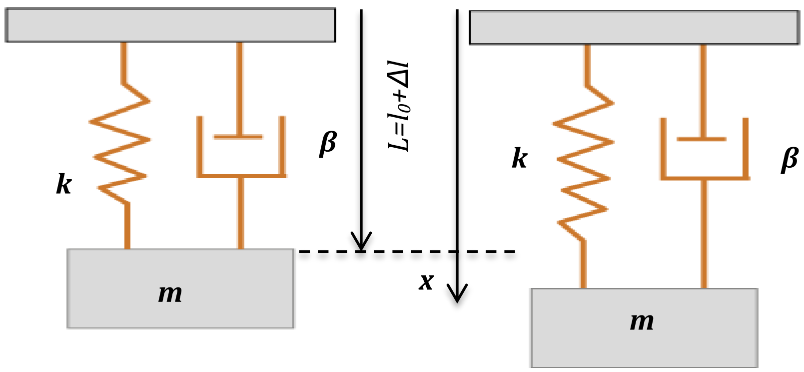


Figure IV.1 *Mass-Spring-Damper System.*

2.1 Kinetic Energy

$$E_c = \frac{1}{2} m\dot{x}^2 \quad \text{(IV.3)}$$

2.2 Potential Energy

$$E_P = E_{P_m} + E_{P_e} \quad \text{(IV.4)}$$

Where:

E_{P_m} : Gravitational potential energy,

E_{P_e} : Elastic potential energy.

$$E_P = -mgx + \frac{1}{2} k(x + \Delta l_0)^2 \quad \text{(IV.5)}$$

2.3 Lagrangian

$$L = E_c - E_p \quad \text{(IV.6)}$$

$$L = \frac{1}{2} m\dot{x}^2 - \left(-mgx + \frac{1}{2} k(x + \Delta l_0)^2\right) \quad \text{(IV.7)}$$

2.4 Dissipation function

$$D = \frac{1}{2} \beta\dot{x}^2 \quad \text{(IV.8)}$$

2.5 Equilibrium condition

$$\left.\frac{\partial E_P}{\partial x}\right|_{x=0} = 0 \quad \text{(IV.9)}$$

$$\left.\frac{\partial E_P}{\partial x}\right|_{x=0} = -mg + k(x + \Delta l_0) = 0 \rightarrow\ mg = k(x + \Delta l_0) \qquad \text{(IV.10)}$$

2.6 Lagrange's Equations

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} + \frac{\partial D}{\partial \dot{x}} = F_{ext,q}(t) \qquad \text{(IV.11)}$$

After substituting the values and differentiating equation (IV.11), we obtain the following differential equation:

$$\ddot{x} + \frac{\beta}{m}\dot{x} + \frac{k}{m}x = \frac{F_{ext,q}(t)}{m} \qquad \text{(IV.12)}$$

Where:

$\delta = \frac{\beta}{2m}$: is the damping coefficient [s^{-1}];

$\omega_0 = \sqrt{\frac{k}{m}}$: is the free pulsation [rad/s].

3 Solution of the Differential Equation

The differential equation governing the forced oscillation motion is a second-order equation with a non-homogeneous term. The solution to equation (IV.12) is expressed as the sum of the homogeneous and particular solutions, as given by:

$$x(t) = x_h(t) + x_P(t) \qquad \text{(IV.13)}$$

Where:

$x_h(t)$: is the homogeneous solution;

$x_P(t)$: is the particular solution.

The homogeneous solution $x_h(t)$, as indicated in the previous section, accepts three cases directly proportional to an exponential limit, which are the following:

$\delta > \omega_0$: This corresponds to the over-damped condition.

$$x_h(t) = (A_1 e^{\omega_0 t} + A_2 e^{-\omega_0 t})e^{-\delta t} \qquad \text{(IV.14)}$$

$\delta = \omega_0$: This represents the critically damped case.

$$x_h(t) = (A_1 + A_2 t)e^{-\delta t} \qquad \text{(IV.15)}$$

$\delta < \omega_0$: This illustrates the under-damped scenario.

$$x_h(t) = Ae^{-\delta t}\cos(\omega_a t + \varphi) \qquad \text{(IV.16)}$$

This solution describes a transient motion that progressively approaches equilibrium or diminishes to zero as time elapses. Ultimately, the particular solution $x_P(t)$ persists, representing the permanent or forced response of the system. The particular solution is a non-homogeneous complementary solution that mirrors the form of the non-homogeneous term on the right-hand side of the differential equation. It is important to note that the constants in this solution are independent of the initial conditions of the pulse.

$$x(t) = x_P(t) \tag{IV.17}$$

3.1 Harmonic Excitation

When the external force is sinusoidal, we say the system is subject to harmonic excitation. A typical form of harmonic excitation is:

$$F_{ext,q}(t) = F_0 \cos \omega_F t \tag{IV.18}$$

Where:

F_0 :is the amplitude of the force,

ω_F :is the frequency of excitation.

For this type of excitation, the solution to the equation of motion is found using the method of undetermined coefficients or Laplace transforms, yielding a solution of the form:

$$x_P(t) = A_0 \cos(\omega_F t + \varphi) \tag{IV.19}$$

Where:

A_0 :is the amplitude of the oscillation.

φ :is the phase angle.

The amplitude of the response depends on the resonance of the system, which occurs when the excitation frequency ω_F matches the system's natural frequency; $\omega_0 = \sqrt{\frac{k}{m}}$.

At resonance, the system experiences maximum displacement due to the accumulation of energy and damping is critical in controlling the response.

4 Mechanical Impedance

Mechanical impedance is an important concept when analyzing forced oscillations. It is defined as the ratio of the force applied to the system to the velocity of the system's motion:

$$Z_m(\omega) = \frac{F}{V} \tag{IV.20}$$

Where:

$Z_m(\omega)$:is the mechanical impedance at a frequency ω,

V: is the velocity of the mass.

$F(t) = F_0 e^{i\omega_F t}$ and $V = \dot{x}$.

Impedance gives insight into how the system resists motion at various frequencies. The mechanical impedance of a mass-spring-damper system can be expressed as:

$$Z(\omega) = m\omega^2 + \beta\omega + k \quad \text{(IV.21)}$$

5. An applied example

Consider the inverted pendulum in Figure V.6. (J = mL^2). At rest, the rod OC is vertical and the springs are undeformed.

1- Give the differential equation of motion of this system (in the case of small oscillations).

Given: m = 0.2 kg, k1 = 9 N/m, k2 = 5 N/m, β = 0.9 kg/s, L = 0.5 m, and f(t) = cos(2t).

2- Give the natural angular frequency, the damped angular frequency (pseudo-angular frequency), and the logarithmic decrement.

3- Give the solution for the steady state.

4- Give the solution for the transient state.

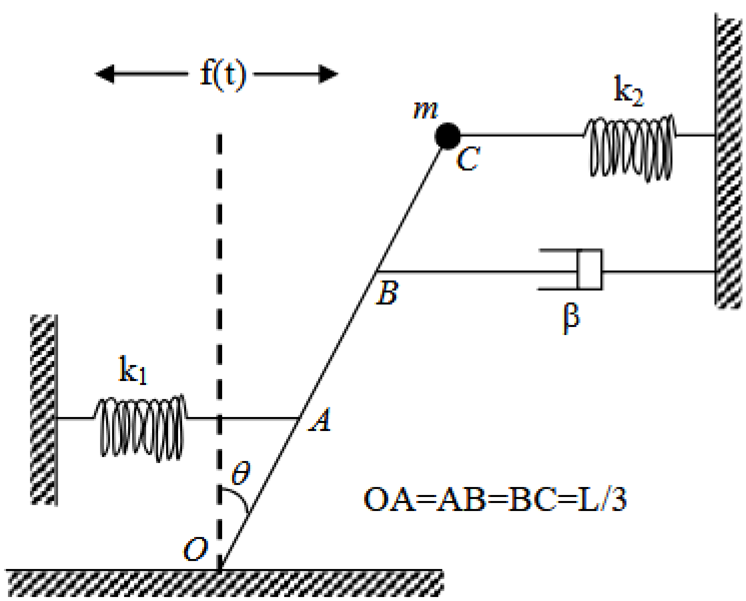


Figure V.6 *Inverted Pendulum system.*

The rotational motion, therefore, implies that the coordinates of the system are given by q=θ.

Degree of freedom : D=N-L=4-(2+1)=1

1. Kinetic energy and potential energy

For weak oscillations ($\sin\theta \approx \theta$).

$$E_c = \frac{1}{2}m\left(L\dot{\theta}\right)^2 \quad \text{(IV.22)}$$

$$E_P = E_{PR1} + E_{PR2} \quad \text{(IV.23)}$$

$$E_P = \frac{1}{2}k_1\left(\frac{L\theta}{3}\right)^2 + \frac{1}{2}k_2(L\theta)^2 + mgL\cos\theta \quad \text{(IV.24)}$$

$$D = \frac{1}{2}\beta\left(\frac{2L}{3}\right)^2\dot{\theta}^2 \quad \text{(IV.25)}$$

2. Lagrange equation

The Lagrange equation that characterizes the system is expressed as:

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial\dot{\theta}}\right) - \frac{\partial L}{\partial\theta} + \frac{\partial D}{\partial\dot{\theta}} = F(t) \quad \text{(IV.26)}$$

$$L = E_c - E_p \quad \text{(IV.27)}$$

$$L = \frac{1}{2}m\left(L\dot{\theta}\right)^2 - \left(\frac{1}{2}k_1\left(\frac{L\theta}{3}\right)^2 + \frac{1}{2}k_2(L\theta)^2 + mgL\cos\theta\right) \quad \text{(IV.28)}$$

After differentiation and replacing term by term in the Lagrange equation, we obtain:

$$mL^2\ddot{\theta} + \frac{4L^2}{9}\beta\dot{\theta} + \left(\frac{L^2k_1}{9} + L^2k_2 - mgL\right)\theta = F(t) \quad \text{(IV.29)}$$

$$\ddot{\theta} + \frac{4}{9m}\beta\dot{\theta} + \left(\frac{k_1}{9m} + \frac{k_2}{m} - \frac{g}{L}\right)\theta = \frac{1}{mL^2}\cos 2t \quad \text{(IV.30)}$$

And the free pulse:

$$\omega_0 = \sqrt{\frac{k_1+9k_2}{9m} - \frac{g}{L}} = 3.16\, rad/s \quad \text{(IV.31)}$$

The pseudo frequency:

$$\omega_D = \sqrt{\omega_0^2 - \delta^2} = 2.78\, rad/s \quad \text{(IV.32)}$$

Logarithmic decrement: $d = \delta T = 094$

Steady-state solution:

$$\theta_P(t) = \theta_0 \cos(2t + \varphi) \tag{IV.33}$$

$$\theta_0 = \frac{1/_{mL^2}}{\sqrt{\left(\omega_0^2 - \Omega^2\right)^2 - 4\delta^2\Omega^2}} \tag{IV.34}$$

$$\varphi = -artg\left(\frac{2\delta\Omega}{\omega_0^2 - \Omega^2}\right) \tag{IV.35}$$

Thus, the steady-state solution is expressed by the following equation:

$$\theta_P(t) = 0.18\cos(2t - 0.2\pi) \tag{IV.36}$$

Transient-state solution:

$$\theta(t) = \theta_h(t) + \theta_P(t) \tag{IV.37}$$

The differential equation without the forcing term is written as:

$$\ddot{\theta} + 1.5\dot{\theta} + 10\theta = 0 \tag{IV.38}$$

$$\Delta = -17.5 < 0$$

$$\theta_h(t) = Ae^{-\frac{3}{2}t}\cos(2.78t + \varphi) \tag{IV.39}$$

A and φ are constants of integration calculated from the initial conditions.

The impedance shows how the system's response changes with frequency. When the system is subjected to a force at resonance, the impedance is minimized, and the displacement is maximized. For non-resonant frequencies, the impedance is higher, and the system resists oscillations more.

In this chapter, we explored the dynamics of forced oscillations in a 1-DOF system, starting from the governing differential equation to specific solutions under harmonic and periodic excitation. We also introduced the concept of mechanical impedance and discussed its significance in understanding system behavior at different frequencies.

Chapter V

Free Oscillations of Two-Degree-of-Freedom Systems

Understanding free oscillations is a fundamental topic in vibration mechanics, providing fundamental insights into the behavior of dynamic vibrating systems. The two-degree-of-freedom system is a critical model in the study of mechanical vibrations, embodying the fundamental properties of oscillatory behavior in systems with more than one mode of motion. This chapter focuses on analyzing these systems, exploring their natural frequencies, modes, and corresponding dynamic responses.

The primary objective of this chapter is to provide students with the mathematical and physical tools necessary to analyze free oscillations in two-degree-of-freedom systems. By the end of the chapter, students will gain a comprehensive understanding of the fundamental principles governing the motion of two-degree-of-freedom systems, including how to calculate natural frequencies, mode shapes, and time responses. These skills are essential in engineering applications where understanding vibration behavior is critical, such as structural dynamics, machine design, and vibration control.

This course also aims to enhance students' analytical skills, enabling them to determine the natural frequencies and time responses of two-degree-of-freedom mechanical systems. These skills are essential for the design and analysis of mechanical systems subjected to non-forced vibrations.

1 Introduction

In this chapter, we will focus on the analysis of free oscillations in systems with two degrees of freedom (2-DOF). A two-degree-of-freedom system is a more complex mechanical system than the one-degree-of-freedom systems discussed earlier. Such systems are often encountered in engineering, where they may represent interconnected mechanical elements such as beams, frames, or multi-body systems.

In the case of second-order free oscillations, the system is considered the sum of two first-order systems, which are interconnected. Therefore, the general coordinates are q_1 and q_2.

The analysis of free oscillations in two-degree-of-freedom systems involves understanding the coupled motions of the system's components, each of which may exhibit its own natural frequency. The system's behavior can be described using the principle of superposition, and the natural frequencies and modes of vibration are important for determining the system's response.

2 Mechanical Joints

Mechanical joints are critical components in the design and analysis of mechanical systems. They facilitate the connection between various system components while regulating their relative motion. Despite the diversity of mechanical joints, we can categorize them into three basic types in this chapter, as follows:

2.1 Flexible Joints

These joints permit deformation while maintaining contact under the influence of applied forces. Elastic springs, for example, are commonly used to absorb shocks and vibrations in mechanical systems. These joints are important for accommodating slight movements and providing elasticity in response to dynamic loads.

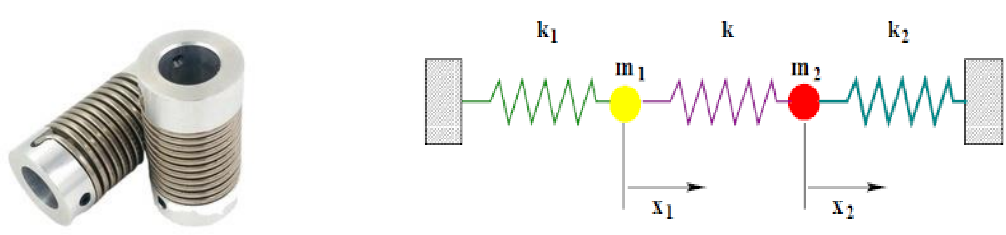


Figure V.1 *Elastic coupling.*

2.2 Inertial Joints

This category includes all hinged joints that allow rotational motion, often coupled with multiple displacements. These joints enable the rotation of components around a fixed point, and they are essential in systems where such motion is necessary, like in joints of mechanical arms or robotic systems.

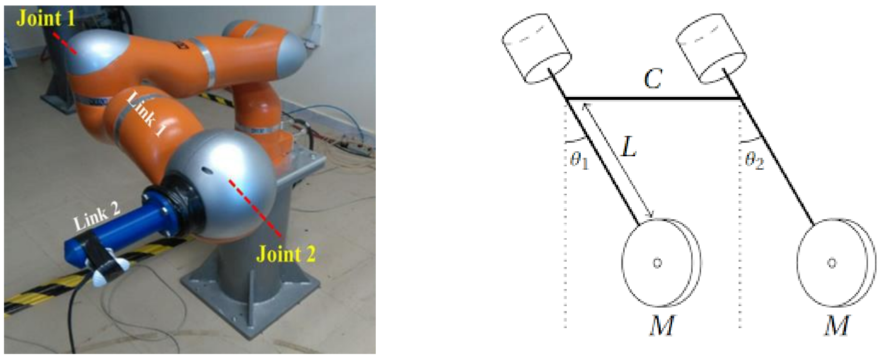


Figure V.2 *Inertial coupling.*

2.2 Viscous Joints

Viscous joints utilize the viscosity of a material to absorb deformations while ensuring a connection and movement between the bodies. Typically found in dampers, these joints are designed to dissipate energy through fluid resistance, making them crucial for controlling vibrations and stabilizing dynamic systems.

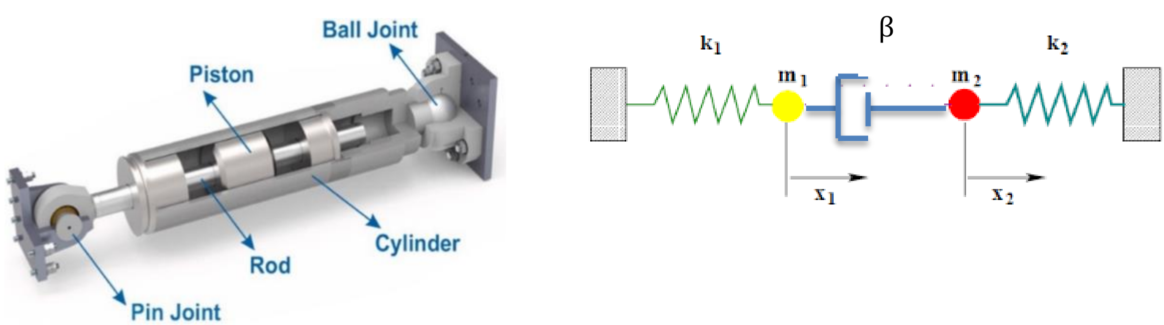


Figure V.3 *Viscous coupling.*

These mechanical joints play a vital role in the modeling and analysis of mechanical systems because they dictate the types of relative motions that can occur between the different components. Proper understanding and implementation of these joints are key in the design of efficient and reliable mechanical systems across various applications.

3 Two-Degree-of-Freedom Systems

A two-degree-of-freedom system is composed of two interconnected subsystems, such as two masses, springs, and potentially dampers, arranged in a way that enables two independent modes of motion for each mass. These systems can display intricate dynamics due to the interactions between their components. Generally, the system's motion can be characterized by two generalized coordinates, $q_1(t)$ and $q_2(t)$ which represent the displacements of the two masses. In the case of translational motion, these can be expressed as $x_1(t)$ and $x_2(t)$.

The governing equations of motion for a two-degree-of-freedom free system are derived from Newton's second law or Lagrange's equations. For a system with two masses m_1 and m_2, two springs with stiffness constants $k_1$and k_2, and a coupling spring with stiffness k (Figure V.4), The Lagrange differential equations of motion for a system of second-order coupled are given by the following form:

$$\begin{cases} \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x_1}}\right) - \frac{\partial L}{\partial x_1} = 0 \\ \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x_2}}\right) - \frac{\partial L}{\partial x_2} = 0 \end{cases} \quad \text{(V.1)}$$

Where: $\boldsymbol{L}$ is the Lagrangian, defined as the difference between the kinetic energy E_c and the potential energy $\boldsymbol{E_p}$ of the system,

$$L = E_c - E_p. \quad \text{(V.2)}$$

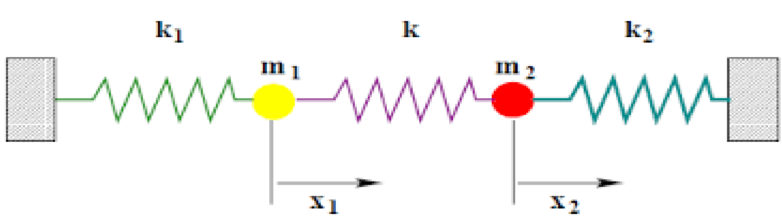


Figure V.4 *Two-degree-of-freedom system (coupling spring).*

The differential equations of free for the two-degree-of-freedom free system will be modified to account for the damping forces. The equations will take the following form:

$$\begin{cases} \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x_1}}\right) - \frac{\partial L}{\partial x_1} + \frac{\partial D}{\partial \dot{x}_1} = 0 \\ \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x_2}}\right) - \frac{\partial L}{\partial x_2} + \frac{\partial D}{\partial \dot{x}_2} = 0 \end{cases} \quad \text{(V.3)}$$

The differential equation of motion for a two-degree-of-freedom system is generally of the form:

$$\begin{cases} m_1\ddot{x}_1 + \beta(\dot{x}_1 - \dot{x}_2) + k_1x_1 + k(x_1 - x_2) = 0 \\ m_2\ddot{x}_2 + \beta(\dot{x}_2 - \dot{x}_1) + k_2x_2 + k(x_2 - x_1) = 0 \end{cases} \quad \text{(V.4)}$$

These equations (V.4) represent the interaction between the two masses and their respective forces. The solutions to these equations describe the free oscillations of the system, and the analysis focuses on determining the system's natural frequencies and mode shapes.

In the case of unamortized systems ($\beta = 0$)the system undergoes (free vibration) where the energies are conserved, and the system oscillates with its natural frequencies. The natural frequencies and mode shapes are determined by solving the characteristic equation obtained from the system's equations of motion.

We consider the free oscillations of the two-degree-of-freedom system in Figure V.5.

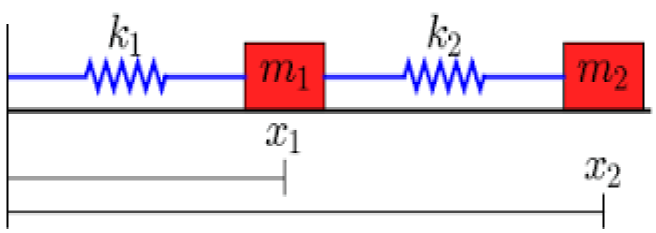


***Figure V.5**. Two-degree-of-freedom spring-mass system.*

1) Calculate the kinetic and potential energies of the system;

2) For k1 = k2 = k and m1 = m = m2/2, and using Lagrange's formula, establish the differential equations of motion. Deduce the system's natural pulsations.

1. The kinetic and potential energies of the system

$$E_c = E_{cm1} + E_{cm2} \quad \text{(V.5)}$$

$$E_c = \frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2 \quad \text{(V.6)}$$

$$E_P = E_{PR1} + E_{PR2} \quad \text{(V.7)}$$

$$E_P = \frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2(x_1 - x_2)^2 \quad \text{(V.8)}$$

2. Lagrangian

$$L = E_c - E_P \tag{V.9}$$

$$L = \left(\frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2\right) - \left(\frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2(x_1 - x_2)^2\right) \tag{V.10}$$

3. The differential equations

$$\begin{cases} \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x_1}}\right) - \frac{\partial L}{\partial x_1} = 0 \\ \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x_2}}\right) - \frac{\partial L}{\partial x_2} = 0 \end{cases} \tag{V.11}$$

By derivation, we find:

$$\begin{cases} m_1\ddot{x_1} + k_1x_1 + k_2(x_1 - x_2) = 0 \\ m_2\ddot{x_2} - k_2(x_1 - x_2) = 0 \end{cases} \tag{V.12}$$

After compensation we find:

$$\begin{cases} m\ddot{x_1} + 2kx_1 - kx_2 = 0 \\ 2m\ddot{x_2} - k(x_1 - x_2) = 0 \end{cases} \tag{V.13}$$

Suppose the solution is of the following form:

$$\begin{cases} x_1(t) = A_1\cos(\omega t + \varphi) \\ x_2(t) = A_2\cos(\omega t + \varphi) \end{cases} \tag{V.14}$$

After derivation and substitution in equation (V.13), we find:

$$\begin{cases} (-m\omega^2 + 2k)A_1 - kA_2 = 0 \\ (-2m\omega^2 + k)A_2 - kA_1 = 0 \end{cases} \tag{V.15}$$

For the equation to admit solutions, the determinant of the coefficient matrix must be zero:

$$det\begin{vmatrix} -m\omega^2 + 2k & -k \\ -k & -2m\omega^2 + k \end{vmatrix} = 0 \tag{V.16}$$

$$2m^2\omega^4 - 3mk\omega^2 + k^2 = 0$$

$$\Delta = m^2k^2 > 0 \tag{V.17}$$

The lowest frequency term corresponding to the angular frequency ω1 is called the fundamental. The other term, with angular frequency ω2, is called the harmonic. The two natural frequencies are:

$$\omega_1 = \sqrt{\frac{k}{m}} \quad \text{and} \quad \omega_2 = \sqrt{\frac{k}{2m}} \tag{V.18}$$

And the solutions are written as:

$$\begin{cases} x_1(t) = A_{11}\cos(\omega_1 t + \varphi_1) + A_{12}\cos(\omega_2 t + \varphi_2) \\ x_2(t) = A_{21}\cos(\omega_1 t + \varphi_1) + A_{22}\cos(\omega_2 t + \varphi_2) \end{cases} \tag{V.19}$$

A_{11}, A_{12}, A_{21}, A_{22}, φ_1 and φ_2 are constants of integration determined from the initial conditions.

The system oscillates in the first (fundamental) mode; the solutions are written as:

$$\begin{cases} x_1(t) = A_{11}\cos(\omega_1 t + \varphi_1) \\ x_2(t) = A_{21}\cos(\omega_1 t + \varphi_1) \end{cases} \tag{V.20}$$

The solutions of the oscillating system in the second mode (harmonic) are given by:

$$\begin{cases} x_1(t) = A_{12}\cos(\omega_2 t + \varphi_2) \\ x_2(t) = A_{22}\cos(\omega_2 t + \varphi_2) \end{cases} \tag{V.21}$$

A key concept in this chapter is the (coupling of modes). In a two-degree-of-freedom system, the motion of one mass influences the motion of the other mass, and the system may exhibit (normal modes). These are the inherent vibration patterns of the system, where each mass moves in a characteristic pattern without affecting the other mass. The system's total response can be represented as a combination of these normal modes.

Chapter VI

Forced Oscillations of Two-Degree-of-Freedom Systems

The study of forced oscillations in two-degree-of-freedom (2-DOF) systems is essential in the field of mechanical and structural engineering, as well as in many other disciplines where dynamic systems are present. Forced oscillations describe the behavior of a system subjected to external periodic forces, leading to oscillations that may differ significantly from those in free oscillations. Understanding the response of 2-DOF systems to such forces is crucial for designing stable, resilient structures and machinery, especially in scenarios where multiple interacting components are involved.

This chapter provides a comprehensive analysis of the forced oscillations in two-degree-of-freedom systems. It begins with the theoretical foundations, including the formulation of the equations of motion for such systems under external excitation. Subsequently, it explores various methods to solve these equations, emphasizing the significance of damping, resonance, and the role of the system's natural frequencies in shaping the response. Practical applications are also examined, highlighting how these systems are encountered in engineering systems such as vehicle suspension, multi-structure frameworks, and coupled mechanical systems. By delving into these aspects, this chapter aims to equip the reader with a deep understanding of forced oscillatory phenomena and their implications for system stability and performance.

1 Lagrange's Equations

In this chapter, we will derive the equations of motion for forced oscillations in a two-degree-of-freedom system using Lagrange's equations. Lagrange's formulation offers a systematic approach to deriving the equations of motion for complex systems with multiple degrees of freedom, particularly when external forces, periodic forces, or generalized coordinates are involved. By applying Lagrange's equations, we can account for the interactions between the system's components and the effects of external excitations, enabling a comprehensive understanding of the system's dynamic behavior under forced conditions.

Suppose a system consists of two masses, m_1 and m_2, connected by a spring of constant stiffness K, where each mass is subjected to an external force $F_1(t)$and $F_2(t)$,respectively. Each mass is also connected at its other end to an additional spring of constant stiffness $k_1$and k_2, respectively, and a damper with a damping coefficient β. The generalized coordinates, representing the displacements of the two masses along the ox-axis, are denoted by the symbols $x_1(t)$ and $x_2(t)$. The Lagrange differential equation for the system coordinates can be expressed as:

$$\begin{cases} \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x_1}}\right) - \frac{\partial L}{\partial x_1} + \frac{\partial D}{\partial \dot{x}_1} = F_1(t) \\ \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x_2}}\right) - \frac{\partial L}{\partial x_2} + \frac{\partial D}{\partial \dot{x}_2} = F_2(t) \end{cases} \quad \text{(VI.1)}$$

For a system with two masses m_1 and m_2,, two springs with stiffness constants k_1 and k_2, and a damping coefficient β, we can describe the generalized coordinates as $x_1(t)$ and $x_2(t)$, representing the displacements of the masses. Lagrange's equation for a generalized coordinate q is given by:

$$L = E_c - E_p \quad \text{(VI.2)}$$

Where:

L:is the Lagrangian, which is the difference between the kinetic energy (E_c) and potential energy (E_p),

E_c: is the total kinetic energy of the system,

E_p :is the total potential energy of the system.

For the two-degree-of-freedom system, the equations of motion are obtained by applying the principle of least action and solving the resulting Lagrangian equations. These equations describe the dynamics of the system and form the foundation for analyzing forced oscillations.

2 Mass-Spring-Damper System

A two-degree-of-freedom system can be represented by a mass-spring-damper configuration, where two masses are connected by springs and dampers. This system is often used to model mechanical systems that exhibit vibrations under external forces. The masses m_1 and m_2 are connected by two springs, one between the first mass and a fixed point, and the second between the two masses, along with dampers that resist the motion.

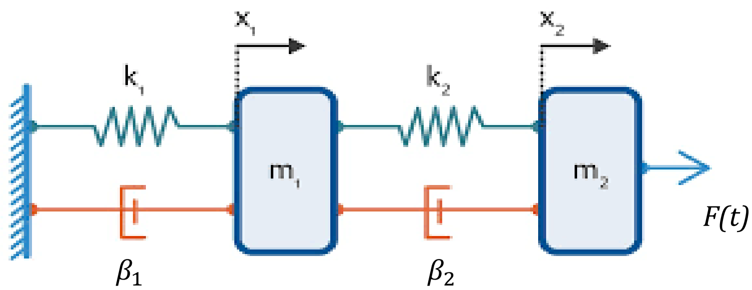


Figure VI.1 *Mass-Spring-Damper System*

Assumptions and system configuration:
Mass 1 (m_1): It is attached to a wall and connected to a spring and a damper. Its position is $x_1(t)$.
Mass 2 (m_2): It is free, connected to mass 1 by a spring and a damper. Its position is $x_2(t)$and it experiences an external force $F(t)$.

2.1 kinetic energy

The total kinetic energy of the system is the sum of the kinetic energies of the two masses

$$E_c = E_{cm1} + E_{cm2} \quad \text{(VI.3)}$$

$$E_c = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2 \quad \text{(VI.4)}$$

2.1 Potential energy

The potential energy arises from the restoring forces of the springs and the frictional forces exerted by the shock absorber

$$E_P = E_{PR_1} + E_{PR_2} \quad \text{(VI.5)}$$

$$E_P = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 (x_1 - x_2)^2 \quad \text{(VI.6)}$$

$$D = \frac{1}{2} \beta_1 \dot{x}_1^2 + \frac{1}{2} \beta_2 (\dot{x}_1 - \dot{x}_2)^2 \quad \text{(VI.7)}$$

2.1 Lagrangian

The Lagrangian is given by the difference between the kinetic energy and the potential energy.

$$L = E_c - E_p \tag{VI.8}$$

$$L = \left(\frac{1}{2}m_1\dot{x}_1^2 + \frac{1}{2}m_2\dot{x}_2^2\right) - \left(\frac{1}{2}k_1x_1^2 + \frac{1}{2}k_2(x_1 - x_2)^2\right) \tag{VI.9}$$

2.4. Équation differential de Lagrange

The Lagrange equation for each mass is obtained by applying the following formulas :

$$\begin{cases} \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}_1}\right) - \frac{\partial L}{\partial x_1} + \frac{\partial D}{\partial \dot{x}_1} = 0 \\ \frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}_2}\right) - \frac{\partial L}{\partial x_2} + \frac{\partial D}{\partial \dot{x}_2} = F(t) \end{cases} \tag{VI.10}$$

2.5 Les équations différentielles du système

The two coupled second-order differential equations describing the system are therefore:

$$\begin{cases} m_1\ddot{x}_1 + \beta_1\dot{x}_1 + k_2(x_1 + x_2) + \beta_2(\dot{x}_1 - \dot{x}_2) + k_1x_1 = 0 \\ m_2\ddot{x}_2 + \beta_2(\dot{x}_2 - \dot{x}_1) + k_2(x_2 - x_1) = F(t) \end{cases} \tag{VI.11}$$

These equations describe the interaction between the two masses and how the system responds to external forces. By solving these equations, we can analyze the free and forced oscillations of the system and determine its response to different types of external excitations. These equations are coupled and can be solved numerically or analytically (if possible) to obtain the displacements $x_1(t)$. and $x_2(t)$.as functions of time and the external force $F(t)$.

The solution is the sum of the homogeneous solution (transient regime) and the particular solution (steady state). However, the homogeneous solution diminishes over time due to the damping effect. Consequently, the general solution ultimately simplifies to the particular solution, which represents the motion's response to the applied force.

2.6 Case of the system is subjected to a sinusoidal force

The force applied to the system is of the form:

$$F(t) = F_0 \cos \omega_F t = F_0 e^{i\omega_F t} \tag{VI.12}$$

Hence, the particular solution takes the following form:

$$\begin{cases} x_1 = \bar{X}_1 \cos(\omega_F t + \varphi_1) \\ x_2 = \bar{X}_2 \cos(\omega_F t + \varphi_2) \end{cases} \tag{VI.13}$$

The complex solutions $x_1(t)$ and $x_2(t)$ are written as:

$$\begin{cases} x_1 = X_1 e^{i\omega_F t} \\ x_2 = X_2 e^{i\omega_F t} \end{cases} \tag{VI.14}$$

Where:

$$X_1 = \bar{X}_1 e^{i\varphi_1}$$
$$X_2 = \bar{X}_2 e^{i\varphi_2}$$

After substitution into the differential equation (VI.11) of motion, we find:

$$\begin{cases} (k_1 + k_2 - m_1\omega_F^2)X_1 + k_2X_2 = 0 \\ (k_2 - m_2\omega_F^2)X_2 + k_2X_1 = F_0 \end{cases} \tag{VI.15}$$

Damping is considered negligible in this case, or weak relative to the vibrations, and therefore has no significant effect.$\beta = 0$.

To solve this equation (VI.15), it is necessary for the determinant to be non-zero.

Where the determinant: $\Delta = \begin{vmatrix} k_1 + k_2 - m_1\omega_F^2 & k_2 \\ k_2 - m_2\omega_F^2 & k_2 \end{vmatrix} \neq 0$ (VI.16)

With:

$$X_1 = \frac{\begin{vmatrix} 0 & k_2 \\ F_0 & k_2 \end{vmatrix}}{\Delta} = \frac{F_0 k_2}{\Delta} \quad \text{and} \quad X_2 = \frac{\begin{vmatrix} k_1+k_2-m_1\omega_F^2 & 0 \\ k_2-m_2\omega_F^2 & F_0 \end{vmatrix}}{\Delta} = \frac{F_0(k_1+k_2-m_1\omega_F^2)}{\Delta} \tag{VI.17}$$

After obtaining the pulsation proper's w_1 and w_2 from Equation (VI.17), both the amplitude x_1, x_2 and phase φ_1, φ_2 can be subsequently determined.

5.3 Impedance

Impedance is a measure of how much a system resists the motion caused by an applied force. In the case of forced oscillations, impedance is a useful concept to describe the system's response to different frequencies of external excitation.

The mechanical impedance Z(ω) for a two-degree-of-freedom system is defined as the ratio of the applied force to the velocity of the system at a given frequency ω:

$$Z(\omega) = \frac{F(\omega)}{V(\omega)}$$

Where F(ω) is the applied force and V(ω) is the velocity of the system. The impedance is frequency-dependent and shows how the system's response changes with the excitation frequency. Impedance analysis helps to identify resonant frequencies where the system exhibits maximum displacement.

5.4 Applications

The concepts of forced oscillations and impedance in two-degree-of-freedom systems are applied in various engineering fields. Some common applications include:

- Vibration analysis of structures and mechanical systems,

- Suspension systems in vehicles,

- Seismic analysis of buildings and bridges,

- Modeling of multi-body systems in robotics.

In each of these applications, understanding the forced oscillations of the system allows engineers to design systems that can either avoid resonance or take advantage of it, depending on the application. For example, in vehicle suspension systems, the goal is often to minimize the amplitude of oscillations at certain frequencies, while in seismic analysis, engineers might want to analyze how a structure responds to earthquake-induced vibrations.

5.5 Generalization to N-Degree-of-Freedom Systems

The concepts introduced for two-degree-of-freedom systems can be extended to systems with more degrees of freedom (N-DOF). In an N-degree-of-freedom system, the system consists of multiple masses, springs, and dampers, and the motion is described by multiple generalized coordinates.

The equations of motion for an N-DOF system are derived from the Lagrangian formulation or Newton's second law, and they result in a system of coupled second-order differential equations. These equations describe the interaction between all components of the system and how the system responds to external forces. The response of the system can be analyzed by determining the system's normal modes, natural frequencies, and impedance.

Lagrange's equations give the equations of motion

$$\frac{\partial}{\partial t}\left(\frac{\partial L}{\partial \dot{x}_i}\right) - \frac{\partial L}{\partial x_i} = 0 \rightarrow \sum_{j=1}^{n} m_{i,j}\ddot{x}_j + \sum_{j=1}^{n} k_{i,j}x_j = 0 \qquad \text{(VI.18)}$$

Or:

$$\begin{cases} m_{11}\ddot{x}_1 + m_{12}\ddot{x}_2 + \cdots + m_{1n}\ddot{x}_n + k_{11}x_1 + k_{12}x_2 + \cdots + k_{1n}x_n = 0 \\ \downarrow \\ m_{n1}\ddot{x}_1 + m_{n2}\ddot{x}_2 + \cdots + m_{nn}\ddot{x}_n + k_{n1}x_1 + k_{n2}x_2 + \cdots + k_{nn}x_n = 0 \end{cases} \qquad \text{(VI.19)}$$

It is a system of n linear differential equations. It has been found that, in the case of systems with two degrees of freedom, sinusoidal solutions exist:

$$\begin{cases} x_1(t) = x_0 \cos \omega_1 t \quad and \quad x_2(t) = x_0 \cos \omega_1 t \\ x_1(t) = x_0 \cos \omega_2 t \quad and \quad x_2 = -x_0 \cos \omega_2 t \\ x_1(t) = \frac{x_0}{2} \cos \omega_1 t + \frac{x_0}{2} \cos \omega_2 t \quad and \quad x_2(t) = \frac{x_0}{2} \cos \omega_1 t - \frac{x_0}{2} \cos \omega_2 t \end{cases} \qquad \text{(VI.20)}$$

We can therefore seek, in the general case, solutions of the form:

$$x_j(t) = A_j \cos(\omega_j t + \varphi_j) \qquad \text{(VI.21)}$$

The system of differential equations (VI.19) is then transformed into a system of n linear and homogeneous algebraic equations:

$$\sum_{j=1}^{n} (k_{i,j} - \omega_j^2 m_{i,j}) A_j = 0 \qquad \text{(VI.22)}$$

For there to be non-zero solutions the determinant of the coefficients must be zero:

$$det\left|k_{i,j} - \omega_j^2 m_{i,j}\right| = 0 \qquad \text{(VI.23)}$$

We find an equation of degree n in ω_j^2 whose roots $\omega_1^2, \omega_2^2, \ldots \omega_\alpha^2 \ldots . \omega_n^2$, are the positive proper pulsations of the mechanical system.

Withω_α, we obtain the solution:

$$x_n(t) = A_n^\alpha \cos(\omega_\alpha t + \varphi_\alpha) \qquad \text{(VI.24)}$$

We can express the amplitudes A_j^αas a function of A_1^α, for example, and write:

$$x_j(t) = R_j^\alpha A_1^\alpha \cos(\omega_\alpha t - \varphi_\alpha) \qquad \text{(VI.25)}$$

The general solution is obtained by superimposing the different oscillation modes:

$$x_j(t) = \sum_{\alpha=1}^{n} R_j^{\alpha} A_1^{\alpha} \cos(\omega_\alpha t - \varphi_\alpha) \quad \text{(VI.26)}$$

The constants A_1^{α} and φ_α will be determined from the initial conditions.

In practice, N-DOF systems are common in complex engineering applications, such as in large structures, aerospace systems, and multi-body dynamics in robotics. The analysis of N-DOF systems often requires advanced mathematical techniques, such as matrix methods and numerical simulations.

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