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Dedication

To myself...

To you, who endured long nights, silent struggles, and heavy responsibilities alone yet continued the journey with strength and hope.

You were your own support, your own comfort, and your own true hero. You are the one who truly deserves this dedication, for everything you've overcome, and because you never gave up on yourself.

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Notations and symbols

Symbol	Definition
$\mathbb{R}$	The set of real numbers
iid	Independent and identically distributed
$P(\cdot)$	Probability function
$F(\cdot)$	The cumulative distribution function
$F_n(\cdot)$	The empirical (or sample) distribution function
$f(\cdot)$	density function
$B_{\mathbb{R}}$	borel tribe
$\phi(\cdot)$	characteristic function
$Q(u)$	quantile function
λ_r	L-moment
$P_r(u)$	Legendre polynomial
τ_3	L-skewness
τ_4	L-kurtosis
PWM	probability weighted moment
l_r	estimator of L-moment
$\lambda_r^{(t_1,t_2)}$	TL-moment
t_3	Estimator of L-skewness
t_4	Estimator ofL-kurtosis
$\tau_3^{(t_1,t_2)}$	TL-skewness
t_4	Estimator of L-kurtosis
$\tau_4^{(t_1,t_2)}$	TL-kurtosis
$t_3^{(t_1,t_2)}$	Estimators of TL-skewness
$t_4^{(t_1,t_2)}$	Estimators of TL-kurtosis

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Introduction

L-moments are considered one of the modern statistical innovations that emerged in response to a pressing need in statistical science, namely the limitations of classical momentswhen dealing with certain types of probability distributions especially those that do not possess defined moments, or whose moments are highly sensitive to the influence of outliers. [2]. [3]. [6].[7]. [?]. [15]. The Cauchy distribution is the most prominent example of these challenges, as it lacks both the first moment (mean) and the second moment (variance), rendering classical statistical approaches inapplicable or unreliable when analyzing such distributions. Moreover, many real world datasets particularly in fields such as hydrology, finance, natural risk analysis, and economics often

contain outliers, asymmetry, or heavy tails, which further reduce the reliability of estimates derived from classical moments.

In response to this limitation, J. R. M. Hosking introduced the concept of L-moments in1990 as a robust and reliable alternative that allows for estimating different distribution parameters, and their characteristics such as location, scale, skewness, and kurtosis using linear combinations of order statistics instead of raw powers of random variables. This method gives L-moments greater resilience to noise and outliers, making them especially stable in small sample sizes. One of their most significant properties is that they are well-defined for all distributions with a finite mean, including those lacking second or third classical moments. They also offer a more intuitive and graphical interpretation compared to traditional moments.

As the application of L-moments expanded, a further refinement emerged in the form of Trimmed L-moments, which enhance the robustness of the measure by trimming a specific number of the smallest and largest values in the sample during estimation. This trimming process increases resistance to extreme outliers and enhances the accuracy and reliability of statistical analysis, particularly in fields characterized by strong deviations or non-standard be-

havior, such as income distribution, environmental catastrophes, and volatile time series data. Trimmed L-moments provide flexibility, allowing researchers to control the level of trimming based on the nature of the dataset, which ultimately improves both the precision and interpretability of results.[8].[?].[10].[13].[14]
Theoretically, L-moments belong to the broader class of robust estimators, and they represent an extension of the foundational concept of order statistics central to nonparametric statistics.

The practical applications of both L-moments and Trimmed L-moments have become widespread across multiple disciplines, including rainfall and flood modeling, insurance loss estimation, financial market analysis, and environmental data characterization. Notably, they have been integrated into advanced statistical software such as R and Python, with dedicated libraries for their computation, reflecting their growing adoption in both academic research and applied settings.

These methods enabled the analysis of more realistic and complex datasets with greater flexibility and precision, establishing a new standard for descriptive estimation beyond the constraints of classical moments, and solidifying their place as one of the most impactful contributions to contemporary statistics.
Our work was divided into three chapters:
-The first includes general information on probability and order statistics
-Second in this part we present the definition of L-moment, their respective properties, representation by covariance, linear function ...,their estimator, and especially the use of their estimator through the calculation of location and scale parameters, as well as L-skewness, L-kurtosis, and their comparison with classical parameters.The method based on the use of L-moment is also introduced in this chapter.
-final chapter we present the definition of TL-moment, their respective properties, their estimator and method. of TL-moment for estimate parameter, finaly application in R of Cauchy distribution followed Generalized Pareto Distribution.

Chapter 1

Random vector and Order Statistics

This chapter covers the basics of probability and random variables, followed by random vectors, covariance, and characteristic functions. It also introduces order statistics, their distributions, and moments.

References used: [2]. [3]. [6].[7]. [?]. [15].

1.1 Random variable and Random vector

1.1.1 Probability space

Definition 1.1.1.

Let a set Ω be nonempty, $\mathcal{F}$ a σ-field on Ω and a map $P : \mathcal{F} \to [0.1]$ called a probability measure on $(\Omega, \mathcal{F})$ if

$$\begin{cases} P(\emptyset) = 0, \quad P(\Omega) = 1 \\ A_i \in \mathcal{F}, \quad A_i \bigcap A_j = \emptyset, \quad i, j = 1, 2, ..., i \neq j \\ \Rightarrow P(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} P(A_i) \end{cases}$$

The triplet $(\Omega, \mathcal{F}, P)$ is called a probability space.

1.1.2 Random variable

Definition 1.1.2.

Let $(\Omega, \mathcal{F})$ and $(\mathbb{R}, \mathbb{B}_{\mathbb{R}})$ be two measurable spaces and $X : \Omega \to \mathbb{R}$ an $\mathcal{F}/\mathbb{B}_{\mathbb{R}}$-measurable map. We call X an $\mathcal{F}/\mathbb{B}(\mathbb{R})$-random variable, or simply a random variable if there would be no confusion.

For a random variable $X : \Omega \to R$, $X^{-1}(\mathbb{R})$ is a sub-σ-field of $\mathcal{F}$, which is called the σ-field generated by X, denoted by $\sigma(X)$. This is the smallest σ-field in Ω under which X is measurable. X is said to be independent of Y if $\sigma(X)$ and $\sigma(Y)$ are independent.

Now we look at some mesurability properties for random variables [3]

1.1.3 Probability distributions

Let X be a random variable, taking values that are real numbers. The relative frequency with which these values occur defines the frequency distribution or probability distribution of $X$and is specified by the cumulative distribution function

$$F(x) = P[X \leqslant x]. \tag{1.1}$$

Where $P[A]$ denotes the probability of the event$A$. $F(x)$ is an increasing function of x, and $0 \leqslant F(x) \leqslant 1$ for all x. We shall normally be concerned with continuous random variables, for which $P[X = t] = 0$ for all t, that is, no single value has nonzero probability. In this case, $F(.)$ is a continuous function and has an inverse function $\mathbb{F}^{-1}(.)$, the quantile function of X. Given any $p, 0 < p < 1, x(p)$ is the unique value that satisfies

$$F(x(p)) = p \tag{1.2}$$

If $F(x)$ is differentiate, its derivative

$$\frac{\partial F(x)}{\partial x} = f(x)$$

Is the probability density function of X.[?]

1.1.4 The moment of a random variable

Discrete random variable

Mathematical expectation

Definition 1.1.3.

Let X be a discrete random variable taking values in $\{x_1, x_2, \ldots, x_n\}$ and let P be the probability distribution of X. the expectation of X is define as :

$$E(X) = \sum_{i=1}^{n} x_i P(x_i).$$

Variance

Definition 1.1.4.

Let X be a discrete random variable taking values in $\{x_1, x_2, \ldots, x_n\}$ with P be the probability distribution of X.is defined as:

$$V(x) = E(X - E(X))^2$$

.

Properties 1.

$$V(X) = E(X^2) - (E(X))^2,$$

where

$$E(X^2) = \sum_{i=1}^{n} x_i^2 P(x_i).$$

Moment of order r

Let X be a discrete random variable with value $\{x_1, x_2, \ldots, x_n\}$ with P the probability distribution of X

Definition 1.1.5.

We call the moment of order r of X with respect to its mean the quantity:

$$M_r(X) = E(X^r) = \sum_{i=1}^{n} x_i^r P(x_i).$$

Remark 1.1.1.

$r = 1$ *, we have* $M_1(X) = E(X)$

$r = 2$ *, we have* $M_2(X) = E(X^2)$

Continuous variable

Mathematical expectation

Definition 1.1.6.

Let X be a continuous random variable with probability density function f,then the mathematical expectation is defined as:

$$E(X) = \int_{-\infty}^{+\infty} x f(x) dx$$

Variance

Definition 1.1.7.

Let X be a continuous random variable, the variance of X is given by :

$$V(X) = \int_{-\infty}^{+\infty} (x - E(X))^2 f(x)dx$$

Practical property

$$V(X) = E(X^2) - (E(X))^2$$

where

$$E(X^2) = \int_{-\infty}^{+\infty} x^2 f(x)dx$$

Standard deviation

The standard deviation of random variable X is defined as:

$$\sigma_X = \sqrt{V(X)}$$

Special case

A continuous random variable is said to be standardized if its expected value is zero, $E(X) = 0$, and its standard deviation is equal to 1: $\sigma_x = 1$

Moment of order r

Let X be a continuous random variable, The moment of order r of X with respect to the origin the quantity:

$$E(X^r) = \int_{-\infty}^{+\infty} x^r f(x)dx.$$

1.1.5 Random vector

Definition 1.1.8.

Let $X_1, X_2, \ldots, X_n$ be a random variables .The n-tuple of random variables $(X_1, X_2, \ldots, X_n)$is called a ***random vector***

Law of a random vector

Let $(\Omega^n, \mathcal{F}^{\otimes n}, P)$ be a probability space if X is a random vector taking values in $\mathbb{R}^n$, the law of X,denoted as P is defined by

$$\forall\ B \in \mathbb{B}_{\mathbb{R}},\ P_X(B) = P(X^{-1}(B)) = P(\omega \in \Omega : X(\omega) \in B). \tag{1.3}$$

Definition 1.1.9.

If $X = (X_1, X_2, \dots, X_n)$ *is a vector taking values in* $\mathbb{R}^n$*, the law of the random variable* X_i $(1 \leqslant i \leqslant n)$ *is called the* i^{th} *marginal law*

1. The law P of X is the joint law of the n-tuple $(X_1, X_2, \dots, X_n)$, which is a probability measure on $\mathbb{R}^n$

2. The law P of X_i is the i^{th} marginal probability law on $\mathbb{R}$

Definition 1.1.10.

The law of X *admit a density function* f *that is positive, integrate over* R^n*, and has an integral of 1 if and only if*

$$\int_{-\infty}^{+\infty} \cdots \int_{-\infty}^{+\infty} f(y_1 \dots y_n) dy_1 \dots dy_n = 1. \tag{1.4}$$

Propsition 1.1.1.

If X_i *have a cumulative distribution function* F_{X_i} *where* $(1 \leqslant i \leqslant n)$ *given by:*

$$\begin{aligned} F_{X_i}(x) =& P(X_1 \in \mathbb{R}, \dots X_i \in B, \dots, X_n \in \mathbb{R}) \\ =& \int_{-\infty}^{+\infty} f_{X_1}(t)dt \int_{-\infty}^{+\infty} f_{X_2}(t)dt, \cdots \int_{-\infty}^{x} f_{X_i}(t)dt \cdots \int_{-\infty}^{+\infty} f_{X_n}(t)dt, \end{aligned}$$

when we differentiate the cumulative distribution function $F_{X_i}(x)$ *with respect to* x*, we obtain the probability density function given by:*

$$\frac{\partial F_{X_i}(x)}{\partial x} = f_{X_i(x)} = \int_{-\infty}^{+\infty} \dots \int_{-\infty}^{+\infty} f(x_1, x_2, \dots x_{i-1}, x, x_{i+1} \dots x_n) dx_1 \dots dx_{i-1} dx_{i+1} \dots dx_n.$$

1.1.6 Moment of random vector

Just like with random variables, we define the first and second-order moments for random vectors.

Mathematical expectation

Definition 1.1.11.

Let $X = (X_1, X_2, \dots, X_n)$ *be a random vector where each component is non-negative or zero -valued random variable.The expectation (or first-order mean) of* X *is defined as the vector in* R^d *given by:*

$$E(X) = (E(X_1), \dots, E(X_n)). \tag{1.5}$$

If X is integrate, then each component X_i is also integrate, ensuring that the definition is valid. the expectation $E(X_i)$ can be computed using the distribution of X or the marginal distribution of X_i:

$$E(X_i) = \int_R x dP_{X_i}(x)$$

Propsition 1.1.2.

If $(X_1, X_2, \ldots, X_n)$ are square integrate random variables,their variance follows the formula:

$$V(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} var(X_i) + 2 \sum_{1 \le i < j \le n} cov(X_i, X_j).$$

1.1.7 Covariance matrix

Definition 1.1.12.

The covariance matrix of the vector X is a square matrix $\mathbb{K}_{\mathbb{X}}$ of size n,where each element is defined as follow (provided it exists):

$$\forall i, j = 1, \ldots, n \quad K_{ij} = Cov(X_i, X_j)$$

. The covariance between two variables X_i and X_j is given by :

$$cov(X_i, X_j) = E[(X_i - E(X_i))(X_j - E(X_j)) = E(X_i X_j) - E(X_i)E(X_j).$$

The matrix K_X, is ***positive semi- definite****.*

Remark 1.1.2.

1. *For a pair of real random variables (X, Y) the covariance matrix is given by:*

$$K_{(X,Y)} = \begin{pmatrix} Var(X) & Cov(X, Y) \\ Cov(X, Y) & Var(Y) \end{pmatrix}$$

2. *The real number r defined as:*

$$r = \frac{Cov(X, Y)}{\sigma_X \sigma_Y},$$

where

$$\sigma_X = \sqrt{Var(X)}, \quad \sigma_Y = \sqrt{Var(Y)},$$

the value r referred to as the correlation coefficient between X and Y

3.For a pair of real random variables (X, Y) the correlation matrix is given by:

$$\rho = \begin{pmatrix} \dfrac{Cov(X,X)}{\sigma_X\sigma_X} & \dfrac{Cov(X,Y)}{\sigma_X\sigma_Y} \\ \\ \dfrac{Cov(X,Y)}{\sigma_X\sigma_Y} & \dfrac{Cov(Y,Y)}{\sigma_Y\sigma_Y} \end{pmatrix} = \begin{pmatrix} 1 & \rho_{XY} \\ \\ \rho_{XY} & 1 \end{pmatrix}$$

Properties 2.

1.symmetry: *$Cov(X,Y) = Cov(Y,X) = E(XY) - E(X)E(Y)$*

2.bi-linearity: *For all $a,b,c,d \in \mathbb{R}$*

$$\begin{aligned} Cov(aX_1 + bX_2, cY_1 + dY_2) =& aCov(X_1, cY_1 + dY_2) + bCov(X_2, cY_1 + dY_2) \\ =& acCov(X_1, Y_1) + adCov(X_1, Y_2) \\ +& bcCov(X_2, Y_1) + bdCov(X_2, Y_2) \end{aligned}$$

3.independence: *If the random vectors X and Y are independent, then their covariance zero:*

$$Cov(X,Y) = 0$$

4.variance of linear combination: *The variance of a linear combination of two random vectors is given by:*

$$Var(aX + bY) = a^2Var(X) + b^2Var(Y) + 2abCov(X,Y)$$

5.Cauchy-Schwartz inequality and correlation: *According to the Cauchy-Schwartz inequality ,we have:*

$$|Cov(X,Y)| \leqslant \sqrt{Var(X)}.\sqrt{Var(Y)} = \sigma_X\sigma_Y$$

then $-1 \leqslant r \leqslant 1$

To see why,consider any $\lambda \in \mathbb{R}$. since variance is always non-negative, we have:

$$Var(\lambda X + Y) \geq 0$$

$$P(\lambda) = \lambda^2 Var(X) + 2\lambda Cov(X,Y) + Var(Y) \geq 0$$

where $P(\lambda)$ is a polynomial of degree 2 with respect to, therefore

$$P(\lambda) \geq 0 \iff (Cov(X,Y))^2 - Var(X)Var(Y) \leqslant 0$$

$$(Cov(X,Y))^2 \leqslant Var(X)Var(Y) \iff -1 \leq \rho \geq 1$$

1.1.8 Independence and identically distributed of two variables

Independence:

We say that the random variables X and Y are independent if for all $i = 1, 2, \ldots, n$ and all $j = 1, 2 \ldots, m$:

$$P(X = x_i, Y = y_j) = P(X = x_i)P(Y = y_j).$$

The independent of X and Y means that for every $i = 1, 2, \ldots, n$ and every $j = 1, 2, \ldots, m$ the event $X = x_i$ and $Y = y_j$ are independent.In other words,the value taken by one of the random variables has no influence on the probabilities of the values that the other variable can take we can also express the independence of X and Y using the following equation:

$$P(X = x_i \mid Y = y_j) = P(X = x_i) \quad (i = 1, 2, \ldots, n \quad ; \quad j = 1, 2, \ldots, m).$$

Identically distributed:

Meaning that all the random variables in the sequence share the same probability distributed

1.1.9 Characteristic Function

Definition 1.1.13.
Let X be a random vector taking values in R^n, $n \geq 2$. The characteristic function of X, denoted by $\phi_X(.)$, is a function from $\mathbb{R}^n$ to $\mathbb{C}$ defined as [6] :

$$\phi_X(t) = E[\exp(i\langle t, X\rangle)],$$

where $\langle t, X\rangle$represents the inner product of t and X $\mathbb{R}^n$, given by:

$$\langle t, X\rangle = \sum_{i=1}^{n} t_i X_i.$$

Since the function $\exp(i\langle t, X\rangle)$has a modulus of 1, it is p-integrate. If X denotes the probability distribution of X, the characteristic function ϕ_X can also be expressed as:

$$\phi_X(t) = \int_R^n \exp(i\langle t, X\rangle) dP_X(x).$$

The characteristic function of X is the Fourier transform of the distribution of X it is a continuous function bounded by 1.

If $f(x)$ is a density function of X then:

$$\phi_X(t) = \int_{\mathbb{R}} \exp(i\langle t, X\rangle) f(x)\, dx$$

and the characteristic function of X the Fourier transform of the function f

Propsition 1.1.3.

Two random vectors X and Y have the same distribution if and only if :

$$\phi_X(.) = \phi_Y(.).$$

Properties 3.

Let $= (X_1, \ldots, X_n)^T$ be a random vector of dimension n the components of X are independent if and only if the characteristic function of X is the product of the characteristic functions of its components, i.e.:

$$t = (t_1, \ldots, t_n)^T \in \mathbb{R}^n, \quad \phi_X(t) = \prod_{k=1}^{n} \phi_{X_k}(t_k).$$

1.2 Order statistics

Definition 1.2.1.

Let $X_1, X_2, \ldots, X_n$ be a sequence of independent and identically distributed (iid) random variable over space (Ω,$\mathcal{F}$).The order statistics, denoted by

$$X_{1,n}, X_{2,n}, \ldots, X_{n,n},$$

are the random variables arranged in ascending as follows: [2]

$$X_{1,n} \leqslant X_{2,n} \leqslant \cdots \leqslant X_{n,n}$$

Definition 1.2.2.

- *The vector $X = (X_{1,n}, X_{2,n}, \ldots, X_{n,n})$ is called the **associated ordered sample** $(X_1, X_2, \ldots, X_n)$ where $X_{k,n}$ represent the K^{th} **order statistics***
- *The minimum and maximum of the iid.n-sample correspond more closely to the concept of extreme values.*

$$X_{1,n} = min(X_1, X_2, \ldots, X_n) = min_{1 \leqslant i \leqslant n} X_i$$

$$X_{n,n} = max(X_1, X_2, \dots, X_n) = max_{1 \leqslant i \leqslant n} X_i$$

- *The empirical(or sample) distribution function $F_n(x)$ is defined as :*

$$F_n(x) = \frac{1}{n} \sum_{i=1}^{n} \mathbf{1}_{(X \leqslant x_i)}$$

where this function represent the proportion of values in the sample that are less than or equal to x. then [2]

$$F_n(x) = \begin{cases} 0, & \text{if } x < X_{1,n} \\ \frac{i}{n}, & \text{if } X_{i,n} \leq x \leq X_{i+1,n} \text{ and } 1 \leq i \leq n-1 \\ 1, & \text{if } x > X_{n,n} \end{cases}$$

Remark 1.2.1.

- *If the law of variable X is absolutely continuous one can conclude that*

$$P(X_{1,n} \leqslant X_{2,n} \leqslant \cdots \leqslant X_{n,n}) = 1$$

- *the order statistics of minimum and maximum $X_{1,n}$ and $X_{n,n}$ are dependent on two different law.*
- *$X_{1,n}$ and $X_{n,n}$ play a major role in the study of extreme values.*

1.2.1 law of the i-th order statistics

Propsition 1.2.1.

The cumulative distribution function (CDF) of the order statistics $X_{k,n}$ is given for all $x \in \mathbb{R}$ as follow[2] :

$$F_{X_{k:n}}(x) = P(X_{k:n} \leq x) = \sum_{t=k}^{n} C_n^t [P(X \leq x)]^t [1 - P(X \leq x)]^{n-t}, \quad x \in R$$

Propsition 1.2.2.

In the case where the distribution function F is continuous and derivable almost everywhere from derivative f, order statistics have a law that admits a density relative to the Lebesgue measurement. The density of $X_{k,n}$ is given by [2]

$$f_{k,n}(x) = \frac{n!}{(k-1)!(n-k)!} f(x)[F(x)]^{k-1}[1 - F(x)]^{n-k}$$

As another direct consequence of this result, it is easily shown that if $U_1, U_2, \dots, U_n$ are random variables independent of uniform law then $U_{k,n}$ follows a Beta law of R and $n-k+1$ parameters. It is recalled that the density of a Beta law of parameter $a > 0$ and $b > 0$ is

$$f_{a,b}(x) = \frac{1}{B(a,b)} x^{a-1}(1-x)^{b-1}, \quad x \in (0,1).$$

In particular, for all $s \in \mathbb{N}$,

$$E(U_{k,s}) = \frac{1}{(k+s-1)!} \frac{(k+s-1)!}{(k-1)!(b-1)!}.$$

Then

$$f_{u_{k,n}}(x) = \frac{1}{B(k,n-k+1)} x^{k-1}(1-x)^{n-k}, \quad x \in (0,1).$$

Propsition 1.2.3.

If the distribution function F is continuous and derivable almost everywhere from derivative f, the density of the vector $(X_1, X_2, \ldots, X_n)$ is

$$f_n(x_1, \ldots, x_n) = n! \prod_{i=1}^{n} f(x_i), \quad x_1 \leq x_2 \leq \cdots \leq x_n.$$

Propsition 1.2.4.

*It can therefore be concluded from the statistics of the minimum that the cumulative distribution and the probability density function are respectively[**?**]:*

$$F_{1,n}(x) = F_{X_{1,n}}(x) = 1 - [1 - F(x)]^n, \qquad f_{1,n}(x) = f_{X_{1,n}}(x) = n[1 - F(x)]^{n-1} f(x).$$

For the maximum statistics, we have

$$F_{n,n}(x) = F_{X_{n,n}}(x) = [F(x)]^n$$

$$f_{n,n}(x) = f_{X_{n,n}}(x) = n[F(x)]^{n-1} f(x).$$

Proof

Using the independent property of the random variables $X_1, X_2, \ldots, X_n$, we deduce that,

$$\begin{aligned} F_{1,n}(x) =& P\{X_{1,n} \leq x\} = 1 - P\{X_{1,n} > x\} = 1 - P\left\{\bigcap_{i=1}^{n} X_i > x\right\} \\ =& 1 - \prod_{i=1}^{n} P\{X_i > x\} = 1 - \prod_{i=1}^{n} [1 - P\{X_i \leq x\}] = 1 - [1 - F(x)]^n. \\ F_{n,n}(x) =& P\{X_{n,n} \leq x\} = \prod_{i=1}^{n} P\{X_i \leq x\} = [F(x)]^n. \end{aligned}$$

1.2.2 Moment of order statistics

Definition 1.2.3.

Let $X_{i,n}$ be the $i-th$ order statistic associated with a sample of size n with probability density *function* $f(x)$ *and cumulative distribution function* $F(x)$*,which is continuous with the quantile function [2]*

$$Q(u) = F_X^{-1}(u).$$

The $k-th$ *moment of the*$i-th$ *order statistic is defined by*

$$E(X_{i,n}^k) = u_{i,n}^k = \int_{-\infty}^{\infty} x^k f_{X_{i,n}}(x)\,dx$$

$$= \frac{n!}{(i-1)!(n-i)!}\int_{-\infty}^{\infty} x^k \{F(x)\}^{i-1}\{1-F(x)\}^{n-i} f(x)\,dx.$$

Alternatively,

$$E(X_{i,n}^k) = \frac{n!}{(i-1)!(n-i)!}\int_0^1 Q(u)^i u^{i-1}(1-u)^{n-i}\,du.$$

Chapter 2

L-moments

This chapter presents L-moments as robust alternatives to classical moments, highlighting their definitions, properties, estimation methods, and use in parameter estimation for various distributions Reference used [8].[10].[13].[14]

2.1 Definitions and basic properties

Definition 2.1.1.
Let X be a real valued random variable with cumulative distribution function $F(X)$ and quantile function $Q(u) = F^{-1}(u)$, and let $X_{1,n} \leqslant X_{2,n} \leqslant \cdots \leqslant X_{n,n}$ be the order statistics of a random sample of size n, then the L-moments as defined by Hosking[10], are written as follow:

$$\lambda_r = r^{-1} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} E(X_{r-k;r}), \quad r = 1, 2, \ldots \tag{2.1}$$

The letter L in "L-moments" signifies that is a linear function of the expected order statistics. Furthermore, as noted by Hosking 1990[10], the natural estimator of λ_r, derived from an observed data sample, is a linear combination of the ordered data values, making it an L-statistic By replacing the expression of the expectation of an order statistic in the formula of the quantities as given above, we have:

$$\lambda_r = \frac{1}{r} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} \frac{r!}{(r-k-1)!k!} \int_0^1 Q(u)^{r-k} u^{r-k-1} (1-u)^k du, \quad \text{for } r \geq 1, \text{ and } 0 \leq u \leq 1.$$

Thus, the first L-moments for a probability distribution are given by:

$$\begin{aligned}
\lambda_1 =& E(X) &= \int x \, dF, \\
\lambda_2 =& \frac{1}{2} E(X_{2,2} X_{1,2}) &= \int x \, (2F - 1) \, dF, \\
\lambda_3 =& \frac{1}{3} E(X_{3,3} - 2X_{2,3} + X_{1,3}) &= \int x \, (6F^2 - 6F + 1) \, dF, \\
\lambda_4 =& \frac{1}{4} E(X_{4,4} - 3X_{3,4} + 3X_{2,4} - X_{1,4}) &= \int x \, (20F^3 - 30F^2 + 12F - 1) \, dF.
\end{aligned}$$

2.1.1 Examples

Uniform distribution over (0,1)

$$\begin{cases} \lambda_1 = \frac{1}{2}, \\ \lambda_2 = \frac{1}{6}, \end{cases}$$

Normal distribution with mean 0 and variance 1

$$\begin{cases} \lambda_1 = 0, \\ \lambda_2 = \frac{1}{\sqrt{\pi}}, \end{cases}$$

Exponential distribution with mean 1

$$\begin{cases} \lambda_1 = 1, \\ \lambda_2 = \frac{1}{2}, \end{cases}$$

2.2 L-parameters

Proportions of L-moments represent the ratios between different L-moments. The parameters λ_1 and λ_2 are used to characterize the properties of distributions.[14],[?] These proportions are defined as follows:

$$\tau_r := \frac{\lambda_r}{\lambda_2}, \quad \text{for } r = 3, 4, \ldots$$

Such that:

The ratio between the first L-moment λ_1 and the second L-moment λ_2 represents the measure of L-variation, defined as:

$$\tau = L - CV = \frac{\lambda_1}{\lambda_2} \tag{2.2}$$

In particular, the ratio between the third L-moment and the second L-moment provides a measure of skewness, known as *L-skewness*, which is given by:

$$\tau_3 := \frac{\lambda_3}{\lambda_2}$$

Additionally, the ratio of the fourth L-moment to the second L-moment serves as the measure of kurtosis, referred to as "L-kurtosis." It is given by:

$$\tau_4 = \frac{\lambda_4}{\lambda_2}.$$

Properties 4.

Numerical values [?] are follows

- λ_1 *can take any value.*
- $\lambda_2 \geq 0$
- *For a distribution that takes only positive values,* $0 \leq \tau < 1$*. L-moment ratios satisfy* $|\tau_r| < 1$ *for all* $r \geq 3$*. Tighter bounds can be found for individual* τ_r *quantities. For example, bounds for* τ_4 *given* τ_3 *are*

$$\frac{1}{4}(5\tau_3^2 - 1) \leq \tau_4 < 1.$$

- *For a distribution that takes only positive values, bounds for* τ_3 *given* τ *are*

$$2\tau - 1 \leq \tau_3 < 1.$$

- ***Linear transformation.*** *Let* X *and* Y *be random variables with L-moments* λ_r *and* λ_r^**, respectively, and suppose that* $Y = aX + b$*. Then*

$$\begin{aligned} \lambda_1^* &= a\lambda_1 + b \\ \lambda_2^* &= |a|\lambda_2 \\ \tau_r^* &= (\text{sign}(a))\,\tau_r, \quad r \geq 3 \end{aligned}$$

2.3 L-moment using polynomials

The L-moment can also be written in terms of Legendre polynomials, as follows[10]:

$$\lambda_r = \int_0^1 Q(u) P^*_{r-1}(u) du, \quad 0 \leq u \leq 1, \quad r = 1, 2, \ldots$$

$$P^*_r(u) = \sum_{k=0}^{r} p^*_{r,k} u^k \tag{2.3}$$

$$p^*_{r,k} = (-1)^r \binom{r}{k} \binom{r+k}{k} \tag{2.4}$$

For a positive integer r, and for $r = 1, 2$, and 3, we have:

$$\begin{aligned}
P^*_0(u) &= 1 \\
P^*_1(u) &= (2u - 1) \\
P^*_2(u) &= (6u^2 - 6u + 1) \\
P^*_3(u) &= (20u^3 - 30u^2 + 12u - 1)
\end{aligned}$$

The first L-moments are given as follows:

$$\begin{aligned}
\lambda_1 &= \int_0^1 Q(u)\, du. \\
\lambda_2 &= \int_0^1 (2u - 1) Q(u)\, du. \\
\lambda_3 &= \int_0^1 (6u^2 - 6u + 1) Q(u)\, du. \\
\lambda_4 &= \int_0^1 (20u^3 - 30u^2 + 12u - 1) Q(u)\, du.
\end{aligned} \tag{2.5}$$

- $u = 1$, $P^*_r(1) = 1$.
- $r \neq s$, then $\int_0^1 P^*_r(u) P^*_s(u)\, du = 0$.

The polynomials form an orthonormal basis on $[0, 1]$.

2.3.1 Example (Weibull distribution)

The **Weibull distribution** with scale parameter λ and shape parameter k as the cumulative distribution function (CDF) and quantile function defined as follows:

$$\begin{cases} F_{\lambda,k}(x) = 1 - e^{-\left(\frac{x}{\lambda}\right)^k} \\ Q(u) = \lambda\left(-\ln(1-u)\right)^{\frac{1}{k}} \end{cases}$$

After calculations, we obtain:

$$\lambda_1 = \lambda\Gamma(1+1/k)$$

$$\lambda_2 = \lambda\frac{\Gamma(1+2/k)}{\Gamma(1+1/k)}$$

$$\lambda_3 = \frac{\Gamma(1+3/k)}{\Gamma(1+1/k)} - 3\lambda_2 + 2$$

$$\lambda_4 = \frac{\Gamma(1+4/k)}{\Gamma(1+1/k)} - 4\lambda_3 + 6\lambda_2 - 3$$

Thus, the L-moment ratios are also:

$$\begin{cases} \tau_3 = \dfrac{\lambda_3}{\lambda_2} \\ \tau_4 = \dfrac{\lambda_4}{\lambda_2} \end{cases}$$

2.4 L-moment using covariance

As indicated by **Hosking & Wallis (1997)**,[?] by using the orthogonality of the functions P_r^*, we easily obtain:

$$\lambda_r = \begin{cases} E(X), & r = 1 \\ \mathrm{Cov}(X, P_{r-1}^*(F(X))), & r \geq 2 \end{cases} \tag{2.6}$$

The first **L-moments** are given as follows:

$$\begin{aligned}
\lambda_1 =& E(X), \\
\lambda_2 =& 2\text{Cov}(X, F(X)) = \text{Cov}(X, 2F(X) - 1), \\
\lambda_3 =& -6\text{Cov}(X, F(X)(1 - F(X))) = \text{Cov}(X, 6F(X)^2 - 6F(X) + 1), \\
\lambda_4 =& \text{Cov}(X, 20F(X)^3 - 30F(X)^2 + 12F(X) - 1).
\end{aligned}$$

2.4.1 Example(Uniform distribution)

The cumulative distribution function and quantile function of a random variable X following the **Uniform** distribution on $(0, 1)$are given by:

$$F_X(x) = \begin{cases} 0, & x < 0 \\ x, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$$

$$Q(u) = u, \quad 0 < u < 1$$

Calculation of L-moments

The L-moments and the L-moment ratios are calculated as follows:

$$\begin{aligned}
\lambda_1 =& \mathbb{E}(X) = \frac{1}{2}. \\
\lambda_2 =& \text{Cov}(X, 2F(X) - 1) = \text{Cov}(X, 2X - 1) \\
=& \mathbb{E}(X(2X - 1)) - \mathbb{E}(X)\mathbb{E}(2X - 1).
\end{aligned}$$

For expectation, we get:

$$E(X) = \int_0^1 x\,dx = \frac{1}{2}.$$

$$E(X(2X - 1)) = \int_0^1 x(2x - 1)dx = \frac{1}{6}.$$

$$E(2X - 1) = 2E(X) - 1 = 0.$$

Thus,

$$\lambda_2 = \frac{1}{6} - \frac{1}{2} \times 0 = \frac{1}{6}.$$

Calculation of λ_3 and λ_4

$$\lambda_3 = E(X(6X^2 - 6X + 1)) - E(X)E(6X^2 - 6X + 1).$$

We compute:

$$E(X(6X^2 - 6X + 1)) = \int_0^1 x(6x^2 - 6x + 1)dx.$$

After integration:

$$E(X(6X^2 - 6X + 1)) = \frac{1}{12}.$$

$$E(6X^2 - 6X + 1) = 6E(X^2) - 6E(X) + 1.$$

With:

$$E(X^2) = \int_0^1 x^2 dx = \frac{1}{3}.$$

$$E(6X^2 - 6X + 1) = 6 \times \frac{1}{3} - 6 \times \frac{1}{2} + 1 = 0.$$

Thus,

$$\lambda_3 = \frac{1}{12} - \frac{1}{2} \times 0 = \frac{1}{12}.$$

For λ_4

$$\lambda_4 = E(X(20X^3 - 30X^2 + 12X - 1)) - \mathbb{E}(X)E(20X^3 - 30X^2 + 12X - 1).$$

We compute:

$$E(X(20X^3 - 30X^2 + 12X - 1)) = \int_0^1 x(20x^3 - 30x^2 + 12x - 1)dx.$$

After integration:

$$E(X(20X^3 - 30X^2 + 12X - 1)) = \frac{1}{20}.$$

$$E(20X^3 - 30X^2 + 12X - 1) = 20E(X^3) - 30E(X^2) + 12E(X) - 1.$$

With:

$$E(X^3) = \int_0^1 x^3 dx = \frac{1}{4}.$$

$$E(20X^3 - 30X^2 + 12X - 1) = 20 \times \frac{1}{4} - 30 \times \frac{1}{3} + 12 \times \frac{1}{2} - 1 = 0.$$

Thus,

$$\lambda_4 = \frac{1}{20} - \frac{1}{2} \times 0 = \frac{1}{20}.$$

L-moment Ratios

$$\tau_2 = \frac{\lambda_2}{\lambda_1} = \frac{1/6}{1/2} = \frac{1}{3}.$$

$$\tau_3 = \frac{\lambda_3}{\lambda_2} = \frac{1/12}{1/6} = \frac{1}{2}.$$

$$\tau_4 = \frac{\lambda_4}{\lambda_3} = \frac{1/20}{1/12} = \frac{3}{5}.$$

2.5 L-moment in terms of probability weighted moments

Probability Weighted Moments (PWM) serve as a generalized form of the conventional moments of a probability distribution.[8]

Definition 2.5.1.

Let Y a continuous random variable. The probability-weighted moments $M_{l,j,k}$ are defined as:

$$M_{l,j,k} = E[Y^l F^j (1-F)^k] = \int_0^1 Q(u)^l u^j (1-u)^k du, \tag{2.7}$$

where F and $Q(u)$ represent the cumulative distribution function and the quantile function of Y, with $Q(u) = F_Y^{-1}(u)$.
The parameters l, k, j are real numbers.

For specific cases of j and k we obtain the following expressions:

- When $j = k = 0$ and l positive integer:

$$M_{l,0,0} = E[Y^l] = \int_0^1 Q(u)^l du \tag{2.8}$$

which correspond to the classical moments of order l with respect to the origin.

- When $j = 0$ and l, k are positive integers, or when $k = 0$ and j, l are positive integers:

$$M_{l,0,k} = E[Y^l(1-F)^k] = \int_0^1 Q(u)^l(1-u)^k du \tag{2.9}$$

$$M_{l,j,0} = E[Y^l F^j] = \int_0^1 Q(u)^l u^j du \tag{2.10}$$

These two quantities are the most commonly used in practice for calculating L-moments. Moreover, just like the method of moments, the distribution parameters can be expressed in terms of weighted probability moments, which provides us with a new estimation method as an alternative to the method of moments. This method yields unbiased estimators and is more robust than the method of moments. The estimation procedure based on and was developed by Landweher et al. (1979) [8]

For $l = 1$, we have:

$$\lambda_r = \sum_{k=0}^{r}(-1)^k p^*_{r,k} a_k = \sum_{j=0}^{r} p_{r,j} B_j \tag{2.11}$$

where the coefficients $p^*_{r,k}$ correspond to those of the shifted Legendre polynomial.

Additionally, we define:

$$a_k = M_{1,0,k}, \quad B_j = M_{1,j,0}$$

Specifically, the first four L-moments are given by:

$$\begin{aligned} \lambda_1 &= a_0 = B_0 \\ \lambda_2 &= a_0 - 2a_1 = 2B_1 - B_0 \\ \lambda_3 &= a_0 - 6a_1 + 6a_2 = 6B_2 - 6B_1 - B_0 \\ \lambda_4 &= a_0 - 12a_1 + 30a_2 - 20a_3 = 20B_3 - 30B_2 + 12B_1 - B_0 \end{aligned} \tag{2.12}$$

2.5.1 Example(exponential distribution)

1/exponential distribution Let X a random variable following an exponential distribution with parameter $\lambda > 0$. Its cumulative distribution function (CDF) is:

$$F(x) = 1 - e^{-\lambda x}, \quad x \geq 0$$

The corresponding quantile function is given by:

$$Q(u) = -\frac{1}{\lambda}\ln(1-u)$$

Calculation of L-moments

We use the definition of L-moments:

$$\beta_j = \int_0^1 Q(u)u^j\,du$$

Replacing $Q(u)$

$$\beta_j = -\frac{1}{\lambda}\int_0^1 \ln(1-u)u^j\,du$$

Using the identity:

$$\int_0^1 \ln(1-u)u^j\,du = -\frac{1}{(j+1)^2}$$

We obtain:

$$\beta_j = \frac{1}{\lambda(j+1)^2}$$

First L-moments

In particular, the first fourth L-moments are:

$$\begin{aligned}\lambda_1 =& \frac{1}{\lambda}\\ \lambda_2 =& \frac{1}{2\lambda}\\ \lambda_3 =& \frac{1}{3^2\lambda} = \frac{1}{9\lambda}\\ \lambda_4 =& \frac{1}{4^2\lambda} = \frac{1}{16\lambda}\end{aligned}$$

L-moment Ratios

The L-skewness: $\tau_3 = \frac{\lambda_3}{\lambda_2} = \frac{1/9\lambda}{1/2\lambda} = \frac{2}{9}$

The L-kurtosis: $\tau_4 = \frac{\lambda_4}{\lambda_2} = \frac{1/16\lambda}{1/2\lambda} = \frac{2}{16} = \frac{1}{8}$

2.5.2 Examples(usuels distribution)

The table (2.1) represents the first two L-moments, the L-moment ratios, and the quantile function of some distributions.

Distribution	**Quantile**	λ_1	λ_2	τ_3	τ_4
Uni	$a+(b-a)u$	$\frac{1}{2}(a+b)$	$\frac{1}{6}(b-a)$	0	0
G Exp	$\xi - a\log(1-u)$	$\xi - a$	$\frac{1}{2}a$	$\frac{1}{3}$	$\frac{1}{6}$
Normal	$\mu+\sigma\sqrt{2}\,\mathrm{erf}^{-1}(2u-1)$	μ	$\sigma\frac{\pi}{\sqrt{2}}$	0.1226	0.1226
G Pareto	$\xi+\frac{a}{1-\lambda}(u^{-\lambda}-1)$	$\frac{\xi-a}{1-\lambda}$	$\frac{a}{(1-\lambda)(2-\lambda)}$	$\frac{(1-\lambda)(2-\lambda)}{(3-\lambda)(1-\lambda)}$	$\frac{(1-\lambda)(2-\lambda)}{(3-\lambda)(1-\lambda)}$

Table 2.1: L-moments of some distributions.

2.6 L-moment λ_k by Linear Functions

We have the following equation[13]:

$$\forall r \leq n: \quad E(X_{r,n}) := r\binom{n}{r}\int_0^1 F^{-1}(u)u^{r-1}(1-u)^{n-r}\,du \tag{2.13}$$

$$= n\binom{n-1}{r-1}\int_{-\infty}^{\infty} x[F(x)]^{r-1}[1-F(x)]^{n-r}\,dF(x) \tag{2.14}$$

By applying this equation along with the definition of β_j from David & Nagaraja (2003)[13], we derive: by using finite different

$$(1-F(x))^{n-r} = \sum_{j=0}^{n-r}\binom{n-r}{j}(-1)^j[F(x)]^j$$

By substituting in (2.14)

$$
\begin{aligned}
E(X_{r,n}) =& n\binom{n-1}{r-1}\int_{-\infty}^{\infty} x[F(x)]^{r-1}\sum_{j=0}^{n-r}\binom{n-r}{j}(-1)^j[F(x)]^j dF(x) \\
=& n\binom{n-1}{r-1}\sum_{j=0}^{n-r}\binom{n-r}{j}(-1)^j\int_{-\infty}^{\infty} x[F(x)]^{r+j-1}dF(x) \\
=& n\binom{n-1}{r-1}\sum_{j=0}^{n-r}\binom{n-r}{j}(-1)^{n-j-1}\beta_{n-j-1} \quad ,\forall r\leq n
\end{aligned}
$$

Thus, for $k = 1, 2, \ldots, n$, it follows that:

$$\beta_k = \sum_{j=1}^{n-1} n^{-1}\binom{n-1}{k}\sum_{j=1}^{r-1}\frac{j-1}{k}E(X_{j,n})$$

By substituting the above equation into the earlier formulation, we obtain:

$$
\begin{aligned}
\lambda_k =& \sum_{j=0}^{k-1} p_{k-1,j}\beta_j \qquad (2.15)\\
=& \sum_{j=0}^{k-1} p_{k-1,j} n^{-1}\left(\binom{n-1}{j}\right)^{-1}\sum_{j=1}^{n}\binom{r-1}{j}E(X_{r,n}) \\
=& n\sum_{r=1}^{n}\omega_{r,n}^{(k)}E(X_{r,n})
\end{aligned}
$$

where

$$\omega_{r,n}^{(k)} := \sum_{j=0}^{\min\{r-1,k-1\}}(-1)^j\binom{k-1}{j}\binom{k-1-j}{n-1}\binom{r-1}{j} \qquad (2.16)$$

In particular, the following cases hold:

$$\omega_{r,n}^{(1)} = 1$$

$$\omega_{r,n}^{(2)} = \frac{(2r-n-1)}{(n-1)}, \quad r = 1, \ldots, n$$

Special Cases

$$\omega_{r,n}^{(k)} = (-1)^{k-1}\omega_{n-r+1,n}^{(k)}$$

Through the previous equations, the sequences $\{\lambda_k\}$ and $\{\beta_j\}$ are equivalent and can also be expressed in terms of $\{E(X_{k,k})\}$. From an earlier result, we have:

$$\beta_j = E(XF'(X)) = (j+1)^{-1}E(X_{j+1,j+1})$$

Thus, we can express λ_k in terms of expected extreme values as follows:

$$\lambda_k = \sum_{j=0}^{k-1} p_{k-1,j}(j+1)^{-1}E(X_{j+1,j+1})$$

2.7 Estimation of L-moment

L-moments are used to describe probability distributions, but when dealing with limited data, they must be estimated from a given sample. This estimation is based on a sample of size n, arranged in ascending order. Let: $X_{1,n} \leqslant X_{2,n} \leqslant \cdots \leqslant X_{n,n}$ represent the ordered sample. It is preferable to start with an estimator of the probability-weighted moment β_r. An unbiased estimator of β_r is given as follows[?]:

$$b_r = n^{-1}\binom{n-1}{r}^{-1}\sum_{j=r+1}^{n}\binom{j-1}{r}x_{j,n} \tag{2.17}$$

This may alternatively be written as

$$\begin{aligned} b_0 &= n^{-1}\sum_{j=1}^{n} x_{j,n} \\ b_1 &= n^{-1}\sum_{j=2}^{n}\frac{(j-1)}{(n-1)}x_{j,n} \\ b_2 &= n^{-1}\sum_{j=3}^{n}\frac{(j-1)(j-2)}{(n-1)(n-2)}x_{j,n} \end{aligned}$$

and in general

$$b_r = n^{-1}\sum_{j=r+1}^{n}\frac{(j-1)(j-2)\dots(j-r)}{(n-1)(n-2)\dots(n-r)}x_{j,n}$$

$$\ell_1 = b_0$$
$$\ell_2 = 2b_1 - b_0$$
$$\ell_3 = 6b_2 - 6b_1 + b_0$$
$$\ell_4 = 20b_3 - 30b_2 + 12b_1 - b_0$$

and in general

$$\ell_{r+1} = \sum_{k=0}^{r} p^*_{r,k} b_k, \quad r = 0, 1, \ldots, n-1 \tag{2.18}$$

from eqs (2.17)and (2.18) l_r is a linear combination of the ordered sample values $x_1, x_2, \quad x_n$

$$\ell_r = n^{-1} \sum_{j=1}^{n} w^{(r)}_{j:n} x_{j:n} \quad r = 0, 1, \ldots, n-1$$

Definition 2.7.1. *the sample L-moment ratios are defined by*

$$t_r = \frac{\ell_r}{\ell_2}$$

and the sample L-CV by

$$t = \frac{\ell_2}{\ell_1}.$$

They are natural estimators of τ_r and τ, respectively.

Remark 2.7.1.

- *The estimators t_r and t are not biased, but their biases are very small in moderate or large samples*
- *Bias for smaller samples can be evaluated by simulation. It is generally the case that the bias of the sample L-CV, t, is negligible in samples of size 20 or more*

2.8 Estimation of parameters by L-moment

The role of the L-moment method is similar to that of the calsical method of moments, which is used to estimate the parameters of a distribution. Thus, estimation using the L-moment method is based on the same idea, meaning that the theoretical L-moments are assumed to be equal to the empirical L-moments Let X be a random variable of size n with as the distribution function containing unknown parameters. These unknown parameters are estimated by solving the system of equations derived from the first theoretical$k$ L-moments, which are assumed to be equal to the first empirical$k$ L-moments, i.e., λ_r for $.r = 1, \quad , k$

It is more robust compared to other methods and provides more reliable results, especially in cases of small samples.

2.8.1 Example(Weibull distribution)

In this example, we apply the L-moment method to the Weibull distribution, following , We derive the L-moments using the Probability-Weighted Moments (PWM) approach and then estimate the parameters using the L-moment method.

Definition of L-moments for the Weibull Distribution

The moments of order j are defined as follows:

$$\beta_j = \int_0^1 Q(u)u^j \, du$$

where $Q(u)$ the inverse cumulative distribution function (CDF) of the Weibull distribution:

$$Q(u) = \lambda(-\ln u)^{1/k}$$

Substituting this into the equation:

$$\beta_j = \int_0^1 \left(\lambda(-\ln u)^{1/k}\right) u^j \, du$$

Using the variable substitution $v = -\ln(u)$ and $w = (j+1)v$ we obtain:

$$\beta_j = \frac{\lambda}{j+1} \int_0^\infty v^{1/k} e^{-v} dv$$

Using the standard integral result:

$$\beta_j = \frac{\lambda}{j+1} \Gamma(1 + 1/k)$$

where $\Gamma(x)$is the Gamma function.

Computation of the L-moments

From the above derivation, the L-moments are obtained as:

- **First L-moment** λ_1

$$\lambda_1 = \frac{\lambda}{2}\Gamma(1 + 1/k)$$

- **Second L-moment** λ_2:

$$\lambda_2 = \frac{\lambda}{6}\Gamma(1 + 1/k)$$

- **Third L-moment** λ_3:

$$\lambda_3 = \frac{\lambda}{12}\Gamma(1 + 1/k)$$

- **Fourth L-moment** λ_1:

$$\lambda_4 = \frac{\lambda}{20}\Gamma(1 + 1/k)$$

2.8.2 L-moment Ratios

The L-moment ratios are derived as follows:

- **L-CV (Coefficient of Variation)**:

$$\tau = \frac{\lambda_2}{\lambda_1} = \frac{1}{3}$$

- **L-skewness**:

$$\tau_3 = \frac{\lambda_3}{\lambda_2} = \frac{1}{2}$$

- **L-kurtosis**:

$$\tau_4 = \frac{\lambda_4}{\lambda_2} = \frac{5}{12}$$

Numerical Example for $k = 2$

and let X random variable composed of the following sample

$X(i) = [0.8, 1, 1.2, 1.5, 1.8, 2.1, 2.2, 2.6, 2.9, 3]$

$,n = 10$

For numerical values:

$$\lambda_1 = 1.91, \quad \lambda_2 = 0.57, \quad \lambda_3 = -0.036, \quad \lambda_4 = 0.035$$

Estimation using the L-moment Method

- using the adjusted value for λ

  To estimate the parameters λ and k using the L-moment method, we solve the following system:

$$\begin{cases} \lambda_1 = \lambda \dfrac{\Gamma(1 + 1/k)}{2} \\ \lambda_2 = \lambda \dfrac{\Gamma(1 + 2/k)}{\Gamma(1 + 1/k)} \end{cases}$$

  Rearranging:

$$\lambda = \frac{2\lambda_1}{\Gamma(1 + 1/k)}$$

$$\lambda = \frac{2\lambda_1}{\Gamma(1 + 1/2)} = \frac{2\lambda_1}{\Gamma(1.5)} = \frac{4\lambda_1}{\sqrt{\pi}} = 4.31$$

- using estimation of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$

$$b_0 = \frac{1}{n} \sum_{j=1}^{n} x_{j,n} = 1.91$$

$$b_1 = \frac{1}{n} \sum_{j=2}^{n} \frac{(j-1)}{(n-1)} x_{j,n} = 1.1$$

$$b_2 = \frac{1}{n} \sum_{j=3}^{n} \frac{(j-1)(j-2)}{(n-1)(n-2)} x_{j,n} = 0.88$$

$$\begin{aligned}
\ell_1 =& b_0 = 1.91 \\
\ell_2 =& 2b_1 - b_0 = -2.28 \\
\ell_3 =& 6b_2 - 6b_1 + b_0 = -0.61 \\
\ell_4 =& 20b_2 - 30b_1 + 12b_0 - b_0 = 1.21
\end{aligned}$$

Then

$$\hat{\lambda} = \frac{2l_1}{\Gamma(1 + 1/k)} = 4.31$$

We notice that when using the exact values for λ_1 or the approximate values, we get the same result

Chapter 3

Trimmed L-moment

This chapter explores Trimmed L-moments, an extension of L-moments designed to reduce the influence of outliers. It covers their definitions, properties, estimation methods, and applications to distributions like Cauchy and Generalized Pareto using R Reference used [5].[1].[10].[9]. [4]

3.1 Definitions and basic properties

Definition 3.1.1.

Let $X_1, X_2, \ldots, X_n$ be a random sample drawn from a continuous distribution with quantile function $Q(u)$, and cumulative distribution function $F(x)$. Let the corresponding order statistics be denoted by $X_{1:n}, \ldots, X_{n:n}$.

defined the trimmed L-moments (TL-moments) in terms of expected values as follows[5]:

$$\lambda_r^{(t_1,t_2)} = \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k \binom{r-1}{k} E\left(X_{r+t_1-k:r+t_1+t_2}\right), \quad r = 1, 2, \ldots; \quad t_1, t_2 = 0, 1, \ldots$$

As observed, TL-moments involve two additional parameters, t_1 and t_2, which represent the amount of trimming and need to be specified.

first four of TL-moments $(1, 0)$ can be written as [1]:

$$\begin{aligned}
\lambda_1^{(1,0)} &= E(Y_{2:2}) \\
\lambda_2^{(1,0)} &= \frac{1}{2}E(Y_{3:3} - Y_{2:3}) \\
\lambda_3^{(1,0)} &= \frac{1}{3}E(Y_{4:4} - 2Y_{3:4} + Y_{2:4}) \\
\lambda_4^{(1,0)} &= \frac{1}{4}E(Y_{5:5} - 3Y_{4:5} + 3Y_{3:5} - Y_{2:5})
\end{aligned}$$

In particular, $\lambda_1^{(1,0)}$ provides a measure of the location of the distribution, $\lambda_2^{(1,0)}$ is the measure of the scale, $\lambda_3^{(1,0)}$ is a measure of the skewness and $\lambda_4^{(1,0)}$ is a measure of kurtosis respectively.

TL-moments in terms of quantile function

$$\lambda_r^{(t_1,t_2)} = r! \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} \frac{(r+t_1+t_2)!}{(r+t_1-k-1)!(t_2+k)!} \int_0^1 Q(u) u^{r+t_1-k-1} (1-u)^{t_2+k}\, du \quad (3.1)$$

As shown by [5] TL-moments is defined for heavy tailed distributions and eliminate the influence of the most extreme observations by giving them zero weights. For example, when $t_1 = t_2 = 1$, $E(X_{2:3})$ is the median of sample of size 3, which give zero weight for first and third value. Since $E(X_{2:3})$ exists for Cauchy distribution, the TL-moments is defined for this distribution. While $E(X_{1:1})$ does not exist for Cauchy distribution, therefore the L-moments can not be defined for this distribution.
Then the first TL-moments for $t_1 = 0$ and $t_2 = 1$ are given by [1]:

$$\begin{aligned}
\lambda_1^{(0,1)} &= 2 \int_0^1 Q(u)(1-u)\, du, \\
\lambda_2^{(0,1)} &= \left(\frac{3}{2}\right) \int_0^1 Q(u)(4u - 3u^2 - 1)\, du, \\
\lambda_3^{(0,1)} &= \left(\frac{4}{3}\right) \int_0^1 Q(u)(-10u^3 + 18u^2 - 9u + 1)\, du, \\
\lambda_4^{(0,1)} &= \left(\frac{15}{2}\right) \int_0^1 Q(u) u (1-u)(-35u^4 + 80u^3 - 60u^2 + 16u - 1)\, du.
\end{aligned}$$

Remark 3.1.1.

- *L-moments are a special case of TL-moments when $t_1 = t_2 = 0$, and can be derived accordingly.*

Optimal choices for trimming

$$\lambda_r = r^{-1} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} E(X_{r-k:r})$$

- *In the symmetric case, that is to say $t_1 = t_2 = t$, the $\lambda_r^{(t)}$ are given by:*

$$\lambda_r^{(t)} = \frac{1}{r} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} E\left(Y_{r+t-k:r+2t}\right)$$

In terms of the quantile function:

$$\lambda_r^{(t)} = \frac{1}{r}\sum_{k=0}^{r-1}(-1)^k \frac{(r-1)!(r+2t)!}{k!(r-k-1)!(r+t-k-1)!(t+k)!}\int_0^1 Q(u)u^{r+t-k-1}(1-u)^{t+k}\,du \tag{3.2}$$

Then, for $r = 1,2,3,4, the \lambda_r^{(1)}$ *are given by* :

$$\begin{aligned}
\lambda_1^{(1)} &= E(Y_{2:3}) \\
\lambda_2^{(1)} &= \frac{1}{2}E(Y_{3:4} - Y_{2:4}) \\
\lambda_3^{(1)} &= \frac{1}{3}E(Y_{4:5} - 2Y_{3:5} + Y_{2:5}) \\
\lambda_4^{(1)} &= \frac{1}{4}E(Y_{5:6} - 3Y_{4:6} + 3Y_{3:6} - Y_{2:6})
\end{aligned}$$

Using (3.1)*, we have:*

$$\begin{aligned}
\lambda_1^{(1)} =& 6\int_0^1 Q(u)u(1-u)\,du \\
\lambda_2^{(1)} =& 6\int_0^1 Q(u)u(1-u)(2u-1)\,du \\
\lambda_3^{(1)} =& \frac{3}{5}\int_0^1 Q(u)u(1-u)(20u^2-30u+12)\,du \\
\lambda_4^{(1)} =& \frac{1}{15}\int_0^1 Q(u)u(1-u)(-210u^3+480u^2-378u+108)\,du
\end{aligned}$$

Definition 3.1.2.

From [11] the TL-moments can be written in terms of Jacobi polynomial as

$$\lambda_{r+1}^{(t_1,t_2)} = \frac{r!\,\Gamma(r+t_1+t_2+2)}{(r+1)\Gamma(r+t_1+1)\Gamma(r+t_2+1)}\int_0^1 Q(u)u^{t_1}(1-u)^{t_2}p_r^{(t_1,t_2)}(u)\,du$$

where

$$p_r^{(t_1,t_2)}(u) = \sum_{k=0}^{r}(-1)^k\binom{r+t_1}{r-k}\binom{r+t_2}{k}u^k(1-u)^{r-k}$$

For $r = 1,2,3$ *and* $t_1 = 0, t_2 = 1$*, we obtain:*

$$P_0^{*(0,1)}(u) = 1;$$
$$P_1^{*(0,1)}(u) = 2 - 3u;$$
$$P_2^{*(0,1)}(u) = 10u^2 - 12u + 3;$$
$$P_3^{*(0,1)}(u) = -35u^3 + 60u^2 - 30u + 4.$$

are the shifted Jacobi polynomials.

Also, we could re-write TL-moments as

$$\lambda_{r+1}^{(t_1,t_2)} = \frac{r!\,\Gamma(r+t_1+t_2+2)}{(r+1)\Gamma(r+t_1+1)\Gamma(r+t_2+1)} \int_0^1 Q(u) T_r^{(t_1,t_2)}(u)\, du$$

Where

$$T_r^{(t_1,t_2)}(F) = \sum_{k=0}^{r} (-1)^{r-k} \binom{r+t_1}{r-k} \binom{r+t_2}{k} F^{k+t_1} (1-F)^{r-k+t_2}$$

are the shifted Jacobi system of orthogonal polynomials. Since the weight function $T_r^{(t_1,t_2)}(F)$ is orthogonal for different values of r, t_1 and t_2, the $\lambda_{r+1}^{(t_1,t_2)}$ captures different types of information about the underlying distribution of X.

3.2 TL-parameter

TL-Skewness and TL-Kurtosis

TL-Skewness and TL-Kurtosis are defined as [1]:

$$\tau_3^{(t_1,t_2)} = \frac{\lambda_3^{(t_1,t_2)}}{\lambda_2^{(t_1,t_2)}}$$

$$\tau_4^{(t_1,t_2)} = \frac{\lambda_4^{(t_1,t_2)}}{\lambda_2^{(t_1,t_2)}}$$

They play the same role as L-Skewness and L-Kurtosis

3.3 proprieties

- **Existence**

  Elamir and Seheult (2003)[5] demonstrated that a distribution can be characterized by its trimmed L-moments even when some of the conventional L-moments do not exist. However, they did not specify the exact conditions under which this applies. The following theorem provides sufficient conditions for the existence of trimmed L-moments:

Theoreme 3.3.1.
Let X be a real-valued random variable. If $\mathbb{E}[(\max(-X,0))^{1/(s+1)}]$ and $\mathbb{E}[(\max(X,0))^{1/(t+1)}]$ exist, then all the trimmed L-moments $\lambda_r^{(s,t)}$, for $r = 1, 2, \ldots,$ of X also exist.

For example, the trimmed L-moments $\lambda_r^{(1,1)}$ exist if $\mathbb{E}[|X|^{1/2}] < \infty$, which allows for analyzing distributions with heavy tails.

- **Uniqueness**

Hosking (1990) [10]showed that only one distribution can correspond to a given set of L-moments. The same applies to trimmed L-moments.

Theoreme 3.3.2.
A distribution for which the trimmed L-moments $\lambda_r^{(s,t)}$, $r = 1, 2, \ldots,$ exist is uniquely determined by these moments.

- **Relation to Orthogonal Polynomials**

The L-moments of a random variable with quantile function $Q(u)$ are given by[10]:

$$\lambda_{r+1} = \int_0^1 P_r^*(u)Q(u)\,du,$$

where $P_r^*(u)$ is the shifted Legendre polynomial (Hosking, 1990) [10].

For trimmed L-moments, analogous polynomials are used, which are orthogonal over $[0, 1]$ with weight $u^s(1-u)^t$. These are the shifted Jacobi polynomials, defined by:

$$P_r^{*(s,t)}(u) = \sum_{j=0}^{r} (-1)^{r-j} \binom{r+t}{j} \binom{r+s}{r-j} u^j (1-u)^{r-j}.$$

Note that:

$$P_r^{*(s,t)}(u) = P_r^{(s,t)}(2u-1),$$

where $P_r^{(s,t)}(x)$ is the unshifted Jacobi polynomial.

Hence, the trimmed L-moments are:

$$\lambda_{r+1}^{(s,t)} = \frac{r!(r+s+t+1)!}{(r+1)(r+s)!(r+t)!} \int_0^1 u^s(1-u)^t P_r^{*(s,t)}(u)Q(u)\,du.$$

Using integration by parts (details in Section 7), we obtain:

$$\lambda_{r+1}^{(s,t)} = \frac{(r-k)!(r+s+t+1)!}{(r+1)(r+s)!(r+t)!} \int_0^1 u^{s+k}(1-u)^{t+k} P_{r-k}^{*(s+k,t+k)}(u) Q^{(k)}(u)\, du.$$

This holds provided the derivatives of the quantile function exist. For regular L-moments, we have:

$$\lambda_{r+1} = \frac{(r-k)!}{r!} \int_0^1 u^k (1-u)^k P_{r-k}^{*(k,k)}(u) Q^{(k)}(u)\, du,$$

and when $k = r$, this simplifies to:

$$\lambda_{r+1} = \frac{1}{r!} \int_0^1 u^r (1-u)^r Q^{(r)}(u)\, du.$$

L-moments may also be interpreted as coefficients in the expansion of the quantile function using a series of shifted Legendre polynomials. Trimmed L-moments use shifted Jacobi polynomials analogously.

Theoreme 3.3.3.

Let X be a continuous real-valued random variable such that $\mathbb{E}[(\max(-X,0))^{2/(s+1)}]$ and $\mathbb{E}[(\max(X,0))^{2/(t+1)}]$ are finite. Suppose X has quantile function $Q(u)$ and trimmed L-moments $\lambda_r^{(s,t)}$, for $r = 1, 2, \ldots$. Then:

$$Q(u) = \sum_{r=0}^{\infty} \frac{r(2r+s+t+1)}{r+s+t} \lambda_{r+1}^{(s,t)} P_r^{(s,t)}(u)$$

This series converges in the weighted mean square sense with weight function $u^s(1-u)^t$.

- **Recurrence Relations Between Trimmed L-Moments**

Shifted Jacobi polynomials satisfy recurrence relations that can be used to derive relationships between trimmed L-moments with different trimming parameters[10]:

$$(2r+s+t-1)\lambda_r^{(s,t)} = (r+s+t)\lambda_r^{(s-1,t)} - \frac{1}{r}(r+1)(r+s)(r+t)\lambda_{r+1}^{(s-1,t)} \quad (3.3)$$

$$(2r+s+t-1)\lambda_r^{(s,t)} = (r+s+t)\lambda_r^{(s,t-1)} + \frac{1}{r}(r+1)(r+s)(r+t)\lambda_{r+1}^{(s,t-1)} \quad (3.4)$$

For small values of s and t, we get:

$$\lambda_r^{(0,1)} = \frac{r+1}{2r}(\lambda_r - \lambda_{r+1})$$
$$\lambda_r^{(0,2)} = \frac{(r+1)(r+2)}{2r(2r+1)}\lambda_r + \frac{r+2}{2r}\lambda_{r+1} + \frac{r+2}{2r(2r+1)}\lambda_{r+2}$$
$$\lambda_{1,1} = \frac{(r+1)(r+2)}{2r(2r+1)}(\lambda_r - \lambda_{r+2})$$

These relations are valid when all involved trimmed L-moments exist. Although trimmed L-moments like $\lambda_r^{(1,1)}$ are mainly useful when λ_r does not exist, the above can be used to refine L-moment-based shape measures. For instance, the quantity:

$$\frac{\tau_3 - \tau_4}{\tau_1 - \tau_3}$$

may serve as a scale-invariant measure of skewness, particularly when τ_3 is near 1.

- Bounds on Trimmed L-Moment Ratios

  While traditional L-moment ratios τ_r are bounded in absolute value by 1, the trimmed L-moment ratios $\tau_r^{(s,t)}$ are not. However, we can show the following bound:

$$|\tau_r^{(s,t)}| \leq \frac{2(m+1)!(r+s+t)!}{(m+r-1)!(2+s+t)!}, \quad \text{where } m = \min(s,t)$$

  Unless $s = t = 0$, this bound is greater than 1 for all $r > 2$ and increases with r. For instance, in the generalized Pareto distribution with quantile function

$$Q(u) = \alpha(1 - \frac{1-u^k}{k})$$

  , the skewness measure $\tau_3^{(1,1)}$ depends on the shape parameter k and exceeds 1 when $k < -\frac{35}{19} \approx -1.8421$.

3.4 Estimation of trimmed L-moments

TL-moment estimators are considered as linear combinations of order statistics

$$X_{1;n} \leq \cdots \leq X_{n;n}$$

Associated with a sample $X_1, \ldots, X_n$ of size n. These estimators are based on those introduced by Downton (1966)[4]. Thus, the estimator of TL-moments $l_r^{(t_1,t_2)}$ are defined by:

$$l_r^{(t_1,t_2)} = \frac{1}{r\binom{r+t_1+t_2}{t_1}} \sum_{i=t_1+1}^{n-t_2} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} \binom{r+t_1-k-1}{i-1} \binom{n-i}{t_2+k} X_{i:n} \tag{3.5}$$

It is easy to prove this expression, and first we have:

$$l_r^{(t_1,t_2)} = \frac{1}{r} \sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} \widehat{E}(Y_{r+t_1-k:r+t_1+t_2}) \tag{3.6}$$

where $\widehat{E}(X_{r+t_1-k:r+t_1+t_2})$ is the unbiased estimator of $E(X_{r+t_1-k:r+t_1+t_2})$, defined as:

$$\widehat{E}(Y_{r+t_1-k:r+t_1+t_2}) = \frac{1}{\binom{r+t_1+t_2}{t_1}} \sum_{i=1}^{n} \binom{r+t_1-k-1}{i-1} \binom{n-i}{t_2+k} X_{i:n} \tag{3.7}$$

Replacing equation (3.6) into (3.7) , we obtain the expression of $l_r^{(t_1,t_2)}$ given in (3.5).

In particular, for $t_1 = 0$, $t_2 = 1$, the first TL-moment estimators are:

$$l_1^{(0,1)} = \sum_{i=1}^{n-1} \binom{i-1}{0} \binom{n-i}{1} X_{i:n}$$
$$l_2^{(0,1)} = \frac{1}{2} \sum_{i=1}^{n-1} \left(\binom{i-1}{1} \binom{n-i}{1} - \binom{i-1}{0} \binom{n-i}{2} \right) X_{i:n}$$
$$l_3^{(0,1)} = \frac{1}{3} \sum_{i=1}^{n-1} \left(\binom{i-1}{2} \binom{n-i}{1} - 2\binom{i-1}{1} \binom{n-i}{2} + \binom{i-1}{0} \binom{n-i}{3} \right) X_{i:n}$$
$$l_4^{(0,1)} = \frac{1}{4} \sum_{i=1}^{n-1} \left(\binom{i-1}{3} \binom{n-i}{1} - 3\binom{i-1}{2} \binom{n-i}{2} + 3\binom{i-1}{1} \binom{n-i}{3} - \binom{i-1}{0} \binom{n-i}{4} \right) X_{i:n}$$

In the particular case where $t_1 = t_2 = t$, the estimators $l_r^{(t)}$ are written as:

$$l_r^{(t)} = \frac{1}{r} \sum_{i=t+1}^{n-t} \left[\sum_{k=0}^{r-1} (-1)^k \binom{r-1}{k} \binom{r+t-k-1}{i-1} \binom{n-i}{t+k} \right] X_{i:n} \tag{3.8}$$

And for $t = 1$, the first TL-moment estimators are:

$$l_1^{(1)} = \sum_{i=2}^{n-1} \left(\frac{\binom{i-1}{1}\binom{n-i}{1}}{\binom{n}{3}} \right) X_{i:n}$$

$$l_2^{(1)} = \frac{1}{2} \sum_{i=2}^{n-1} \left(\frac{\binom{i-1}{2}\binom{n-i}{1} - \binom{i-1}{1}\binom{n-i}{2}}{\binom{n}{4}} \right) X_{i:n}$$

$$l_3^{(1)} = \frac{1}{3} \sum_{i=2}^{n-1} \left(\frac{\binom{i-1}{3}\binom{n-i}{1} - 2\binom{i-1}{2}\binom{n-i}{2} + \binom{i-1}{1}\binom{n-i}{3}}{\binom{n}{5}} \right) X_{i:n}$$

$$l_4^{(1)} = \frac{1}{4} \sum_{i=2}^{n-1} \left(\frac{\binom{i-1}{4}\binom{n-i}{1} - 3\binom{i-1}{3}\binom{n-i}{2} + 3\binom{i-1}{2}\binom{n-i}{3} - \binom{i-1}{1}\binom{n-i}{4}}{\binom{n}{6}} \right) X_{i:n}$$

Finally, the estimators of TL-skewness and TL-kurtosis are:

$$t_3^{(t_1,t_2)} = \frac{l_3^{(t_1,t_2)}}{l_2^{(t_1,t_2)}}, \quad t_4^{(t_1,t_2)} = \frac{l_4^{(t_1,t_2)}}{l_2^{(t_1,t_2)}}$$

3.5 application

3.5.1 Exemple(cauchy)

since the cauchy distribution does not have moment, therefore, to approximate its parameter, we use method of TL-moment or L-moment

Distribution function of cauchy

A random variable X follows a Cauchy distribution, depending on two parameters ξ and $\alpha > 0$, is defined as follows:

Its cumulative distribution function is given by:

$$F_X(x) = \frac{1}{\pi} \arctan\left(\frac{x-\xi}{\alpha}\right) + 0.5$$

Its probability density function is:

$$f_X(x) = \frac{1}{\pi\alpha\left(1 + \left(\frac{x-\xi}{\alpha}\right)^2\right)}$$

The quantile function $Q(u) = F_X^{-1}(u) = x$ is defined by:

$$Q(u) = \xi + \alpha \tan\left(\pi(u - 0.5)\right)$$

And we have that the L-moments of the Cauchy distribution do not exist ($\lambda_1 = \lambda_2 = +\infty$).[9]

TL-moments and TL-moment estimators for the Cauchy distribution

We will apply the quantile function of the Cauchy distribution to find the first four TL-moments, they are given as follows:

$$\begin{aligned}
\lambda_1^{(1)} =& 6\int_0^1 Q(u)u(1-u)\,du = \xi \\
\lambda_2^{(1)} =& 6\int_0^1 Q(u)u(1-u)(2u-1)\,du = 0.698\alpha \\
\lambda_3^{(1)} =& \frac{20}{3}\int_0^1 Q(u)u(1-u)(5u^2-5u+1)\,du = 0 \\
\lambda_4^{(1)} =& \frac{15}{2}\int_0^1 Q(u)u(1-u)(14u^3-21u^2+9u-1)\,du = 0.239414\alpha
\end{aligned}$$

From these, the TL-moment ratios are deduced:

- **TL-skewness** is given by:

$$\tau_3^{(1)} = \frac{\lambda_3^{(1)}}{\lambda_2^{(1)}} = \frac{0}{0.698\alpha} = 0$$

- **TL-kurtosis** is given by:

$$\tau_4^{(1)} = \frac{\lambda_4^{(1)}}{\lambda_2^{(1)}} = \frac{0.239414\alpha}{0.698\alpha} = 0.343$$

It follows that:

$l_1^{(1)}$ is the estimator of $\lambda_1^{(1)}$, and $l_2^{(1)}$ is the estimator of $\lambda_2^{(1)}$, thus, the estimators of ξ and α are:

$$\begin{cases} l_1^{(1)} = \xi \\ l_2^{(1)} = 0.698\alpha \end{cases} \Leftrightarrow \begin{cases} \xi = l_1^{(1)} \\ \alpha = \dfrac{1}{0.698} l_2^{(1)} \end{cases}$$

application in R

```
# Parameters
n <- 20
epsilon <- 0

set.seed(123)

# Generate sample directly from the Cauchy distribution
x_sample <- rcauchy(n, location = epsilon, scale = alpha)

# Compute L1 to L4 using the sample directly
L_vals <- numeric(4)

for (r in 1:4) {
        sum_val <- 0
        for (k in 0:(r - 1)) {
                coeff <- (-1)^k * choose(r - 1, k) * choose(r + 1, r - k
                # Use empirical uniform values based on sorted ranks
                u <- (rank(x_sample) - 0.5) / n
                integrand_vals <- x_sample * u^(r - k) * (1 - u)^(1 + k)
                integral_approx <- mean(integrand_vals)
                sum_val <- sum_val + coeff * integral_approx
        }
        L_vals[r] <- sum_val / r
}

names(L_vals) <- paste0("L_", 1:4, "^(1)")
print("L1 to L4:")
print(L_vals)

# Estimate epsilon and alpha
epsilon_hat <- L_vals[1]
alpha_hat <- L_vals[2] / 0.698

cat("\nepsilon_hat =", epsilon_hat, "\n")
```

```
cat("alpha_hat =", alpha_hat, "\n")
```

The following table presents the TL-moments, their parameters (ξ, α), and their estimators based on TL-moments for the Cauchy distribution such that: $\xi = 0$ and $\alpha = 0.2$. We know

Sample sizeParameters	**The TL-moments**	**The estimators of the TL-moments**	**The estimators of the parameters**
$n = 20$	$\lambda_1^{(1)} = 0$ $\lambda_2^{(1)} = 0$ $\tau_3^{(1)} = 0$ $\tau_4^{(1)} = 0.1047882$	$l_1^{(1)} = -0.003534376$ $l_2^{(1)} = 0.125344552$ $t_3^{(1)} = 0.8204096657$ $t_4^{(1)} = 0.154907323$	$\hat{\xi} = -0.003534376$ $\hat{\lambda} = 0.1795337$
$n = 100$	$\lambda_1^{(1)} = 0$ $\lambda_2^{(1)} = 0$ $\tau_3^{(1)} = 0$ $\tau_4^{(1)} = 0.1047882$	$l_1^{(1)} = 0.04457335$ $l_2^{(1)} = 0.13227506$ $t_3^{(1)} = 0.8391543349$ $t_4^{(1)} = 0.6015680507$	$\hat{\xi} = 0.057335$ $\hat{\lambda} = 0.1895058$

Table 3.1: Estimation of parameters based on TL-moments

that the Bias is defined by:

$$\text{bias}(\hat{\gamma}) = \mathbb{E}(\hat{\gamma}) - \gamma.$$

And for the MSE:

$$\text{MSE} = \mathbb{E}(\hat{\gamma} - \gamma)^2.$$

Thus, the Bias and the Mean Squared Error for $\hat{\xi}$ and $\hat{\alpha}$ are:

Estimators	MSE	Bias
$\hat{\xi}$	0.0032873022	0.057335
$\hat{\alpha}$	0.00001101282	−0.0104942

We observe that the mean squared error and the bias are close to zero.

3.6 application

3.6.1 Exemple(generalized pareto)

The distribution function

We say that X follows a generalized Pareto distribution with parameters $\sigma > 0$ and $\delta \in \mathbb{R}$, if its cumulative distribution function is given by:

$$F_X(x) = \begin{cases} 1 - \left(1 - \dfrac{\delta}{\sigma}x\right)^{1/\delta} & \text{if } \delta \neq 0 \\ 1 - \exp\left(-\dfrac{x}{\sigma}\right) & \text{otherwise} \end{cases}$$

And the probability density function is given by:

$$f_X(x) = \frac{d}{dx}F(x) = \begin{cases} \frac{1}{\sigma}\left(1 - \frac{\delta}{\sigma}x\right)^{\frac{1}{\delta}-1} & \text{if } \delta \neq 0 \\ \frac{1}{\sigma}\exp\left(-\frac{x}{\sigma}\right) & \text{otherwise} \end{cases}$$

It is defined by:

$$\begin{cases} 0 \leq x & \text{if } \delta > 0 \\ 0 \leq x \leq -\frac{\sigma}{\delta} & \text{otherwise} \end{cases}$$

The Pareto distribution is defined by two parameters δ and σ (where δ is called the extreme value index and σ is the scale parameter). The quantile function $Q(u) = F_X^{-1}(u) = x$ is defined as:

$$Q(u) = \frac{\sigma}{\delta}(1 - (1-u)^{\delta}), \quad \text{if } \delta \neq 0$$

L-moments and Estimators for the Generalized Pareto Distribution

We apply the quantile function of the generalized Pareto distribution to derive the L-moments, the first four L-moments are[12]:

$$\lambda_1 = \int_0^1 Q(u)\,du = \frac{\sigma}{\delta}\int_0^1 \left(1 - (1-u)^{\delta}\right)\,du = \frac{\sigma(\delta+2)}{\delta(\delta+1)}$$
$$\lambda_2 = \int_0^1 (2u-1)\cdot\frac{\sigma}{\delta}(1-(1-u)^{\delta})\,du = \frac{\sigma}{(\delta+1)(\delta+2)}$$
$$\lambda_3 = \int_0^1 (6u^2-6u+1)\cdot\frac{\sigma}{\delta}(1-(1-u)^{\delta})\,du = \frac{\sigma(\delta-1)}{(\delta+1)(\delta+2)(\delta+3)}$$
$$\lambda_4 = \int_0^1 (20u^3-30u^2+12u-1)\cdot\frac{\sigma}{\delta}(1-(1-u)^{\delta})\,du = \frac{\sigma(\delta-1)(\delta-2)}{(\delta+1)(\delta+2)(\delta+3)(\delta+4)}$$

- **TL-skewness** is given by:

$$\tau_3 = \frac{\lambda_3}{\lambda_2} = \frac{\sigma(\delta-1)/(\delta+1)(\delta+2)(\delta+3)}{\sigma/(\delta+1)(\delta+2)} = \frac{\delta-1}{\delta+3}$$

- **TL-kurtosis** is given by:

$$\tau_4 = \frac{\lambda_4}{\lambda_2} = \frac{\sigma(\delta-1)(\delta-2)/(\delta+1)(\delta+2)(\delta+3)(\delta+4)}{\sigma/(\delta+1)(\delta+2)} = \frac{(\delta-1)(\delta-2)}{(\delta+3)(\delta+4)}$$

We denote l_1 as the estimator of λ_1, and t_3 as the estimator of τ_3, from which we derive the estimators of the other L-moment ratios. then deduce estimators of δ and σ :

$$\begin{cases} l_1 = \dfrac{\hat{\sigma}(\hat{\delta}+2)}{\hat{\delta}(\hat{\delta}+1)} \\ t_3 = \dfrac{\hat{\delta}-1}{\hat{\delta}+3} \end{cases} \iff \begin{cases} \hat{\delta} = \dfrac{3t_3+1}{1-t_3} \\ \hat{\sigma} = \dfrac{\hat{\delta}(\hat{\delta}+1)}{\hat{\delta}+2} \cdot l_1 \end{cases}$$

Then :

$$\hat{\delta} = \frac{3t_3+1}{1-t_3}, \quad \hat{\sigma} = \frac{\hat{\delta}(\hat{\delta}+1)}{\hat{\delta}+2} \cdot l_1$$

TL-moments and TL-moment estimators for the Generalized Pareto Distribution

We apply the quantile function of the Generalized Pareto Distribution to compute the first four TL-moments, they are given as follows:

$$\lambda_1^{(1)} = 6\int_0^1 Q(u)u(1-u)\,du = \frac{\delta(\delta+5)}{(\delta+2)(\delta+3)} \cdot 6\sigma$$
$$\lambda_2^{(1)} = 6\int_0^1 Q(u)u(1-u)(2u-1)\,du = \frac{20\sigma(1-\delta)}{(\delta+2)(\delta+3)(\delta+4)}$$
$$\lambda_3^{(1)} = \frac{20}{3}\int_0^1 Q(u)u(1-u)(5u^2-5u+1)\,du = \frac{15\sigma(\delta-1)(\delta-2)}{2(\delta+2)(\delta+3)(\delta+4)(\delta+5)}$$
$$\lambda_4^{(1)} = \frac{15}{2}\int_0^1 Q(u)u(1-u)(14u^3-21u^2+9u-1)\,du = \frac{15\sigma(\delta-1)(\delta-2)}{2(\delta+2)(\delta+3)(\delta+4)(\delta+5)(\delta+6)}$$

From these, we deduce the TL-moment ratios :

- **TL-kurtosis** is given by:

$$\tau_3^{(1)} = \frac{\lambda_3^{(1)}}{\lambda_2^{(1)}} = \frac{10(1-\delta)}{9(\delta+5)}$$

- **TL-kurtosis** is given by:

$$\tau_4^{(1)} = \frac{\lambda_4^{(1)}}{\lambda_2^{(1)}} = \frac{5(\delta-1)(\delta-2)}{4(\delta+5)(\delta+6)}$$

Given that $l_1^{(1)}$ is the estimator of $\lambda_1^{(1)}$ and $t_3^{(1)}$ is the estimator of $\tau_3^{(1)}$, we deduce the estimators of δ and σ as follows:

$$\begin{cases} l_1^{(1)} = \dfrac{\delta(\delta+5)}{(\delta+2)(\delta+3)} \cdot \sigma \\ t_3^{(1)} = \dfrac{10(1-\delta)}{9(\delta+5)} \end{cases} \iff \begin{cases} \hat{\delta} = \dfrac{10-45t_3^{(1)}}{10-9t_3^{(1)}} \\ \hat{\sigma} = \dfrac{(\hat{\delta}+2)(\hat{\delta}+3)}{\hat{\delta}+5} \cdot l_1^{(1)} \end{cases}$$

The following table presents the parameters (σ, δ), and their estimators based on TL-moments for the Cauchy distribution such that:$\sigma = 0.2, \delta = 1$

	n = 50	n = 100	n = 200	n = 500
$\hat{\sigma}_{TL}$	0.184359 (0.00024)	0.10255 (0.00001)	0.21019 (0.00010)	0.20475 (0.00002)
σ_L	0.15781 (0.00178)	0.13991 (0.003185)	0.25245 (0.00275)	0.36793 (0.02819)
$\hat{\delta}_{TL}$	0.645 (0.4233)	0.51186 (0.0082)	0.942701 (0.0032)	1.03928 (0.00154)
δ_L	0.90736 (0.0858)	0.64939 (0.2292)	0.62770 (0.13853)	0.52219 (0.2283)

Table 3.2: Corrected parameter estimates using TL-moment and L-moment methods.

We note that the squared error obtained by the TL-moments method is better than that obtained by the L-moments method

Conclusion

This study has highlighted the important role of L-moments and Trimmed L-moments as advanced statistical tools that serve as robust alternatives to classical moments, especially in contexts where data are skewed, contain outliers, or deviate from normality. Unlike conventional moments, L-moments are based on linear combinations of order statistics,Trimmed L-moments build upon this robustness by intentionally excluding a certain proportion of the most extreme values.

The estimation using the method of L-moments and trimmed L-moments is presented in the second and third chapters; their robustness and consistency are particularly more important than the classical method of moments. Finally, a simulation application on the estimation of the parameters of different distributions is based on the estimators of L-parameters such as L-skewness and L-kurtosis...

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ملخص

في هذا البحث الذي يتناول طرق تقدير اللحظات الخطية واللحظات الخطية المشذبة، قمنا بتذكير القارئ بالتعاريف والخصائص المتعلقة بالمتجهات العشوائية والإحصاءات الترتيبية الضرورية لفهم محتوى الدراسة. بعد ذلك، تم عرض طريقة تقدير اللحظات الخطية من خلال تقديم تعريفها، خصائصها، ومميزاتها الإحصائية. كما تناولنا اللحظات الخطية المشذبة، مبرزين مدى قوتها وثباتها، والتي تتفوق على الطريقة التقليدية للحظات بفضل قدرتها على التخلص من القيم المتطرفة والشاذة. وفي الختام، أُجري تطبيق عملي يعتمد على المحاكاة لتقدير معلمات توزيعات مختلفة باستخدام مقدّرات تعتمد على معلمات اللحظات الخطية مثل معامل الالتواء الخطي، ومعامل التفرطح الخطي...

Abstract

In this work, which focuses on estimation methods of L-moments and trimmed L-moments, we recalled the definitions and characteristics related to random vectors and order statistics that are necessary for understanding the content. Then, we presented the method for estimating L-moments by outlining its definition, properties, and statistical advantages. We also addressed the trimmed L-moments by highlighting their robustness and consistency, which surpass those of the classical method of moments thanks to the elimination of extreme and outlier values. Finally, an application based on simulation was carried out to estimate the parameters of different distributions using estimators of L-parameters such as L-skewness and L-kurtosis... In this work, which focuses on estimation methods of L-moments and trimmed L-moments, we recalled the definitions and characteristics related to random vectors and order statistics that are necessary for understanding the content. Then, we presented the method for estimating L-moments by outlining its definition, properties, and statistical advantages. We also addressed the trimmed L-moments by highlighting their robustness and consistency, which surpass those of the classical method of moments thanks to the elimination of extreme and outlier values. Finally, an application based on simulation was carried out to estimate the parameters of different distributions using estimators of L-parameters such as L-skewness and L-kurtosis...

Résumé

Dans ce travail portant sur les méthodes d'estimation des L-moments et des Trimmed L-moments, nous avons rappelé les définitions et les caractéristiques relatives aux vecteurs aléatoires et aux statistiques d'ordre nécessaires à la compréhension du contenu. Ensuite, nous avons présenté la méthode d'estimation des L-moments en exposant sa définition, ses propriétés et ses avantages statistiques. Nous avons également abordé les Trimmed L-moments en mettant en évidence leur robustesse et leur cohérence, qui surpassent celles de la méthode classique des moments grâce à l'élimination des valeurs extrêmes aberrantes. Enfin, une application basée sur la simulation a été réalisée pour estimer les paramètres de différentes distributions en utilisant des estimateurs des L-paramètres tels que le L-skewness, le L-kurtosis...