People's Democratic Republic of Algeria

Ministry of Higher Education and Scientific Research

MOHAMED KHIDER UNIVERSITY, BISKRA

FACULTY of EXACT SCIENCES and SIENCES of NATURE and LIFE

DEPARTMENT OF MATHEMATICS

Thesis Presented with a View to Obtaining the Diploma:

Master in Mathematics

Option: Analysis

Presented by:

GAMAR Abdellatif

Title:

Numerical Integration :Newton-cotes formula

Review Committee Members:

Pr.	Khelil Naceur	UMKB	President
Dr.	Ouaar Fatima	UMKB	Supervisor
Dr.	Radjeh Fouzia	UMKB	Examinator

June 2025

بسم الله الرحمن الرحيم

Dedicace

I dedicate this humble work to

my familly, my father who taught me that patience is the key for success

To my mother,

To all my brothers and sisters,

To all my colleagues,

To all the dearest people in my heart, to my professors.

And to everyone who knows me.

Acknowledgements

First of all I would like to thank....

First of all, I thank my God for helping me and giving me the will to reach this stage and complete this work.

May I thank all those who, in one way or another, directly or indirectly, have contributed to this.

My thanks in particular to:

my supervisor, Dr. Ouaar Fatima.

for their wise scientific advice and support during the work time.

I would like to thank the members of the Jury who did me the honor of participating in my thesis.

And I would like to thank all the professors in the Mathematics Department at Mohamed Kheider University.

A big thanks to all.

Contents

List of Figures

Introduction

The objective of this work is to compute the value of the integral:

$$I = \int_a^b f(x)\, dx.$$

We shall assume that the value of $f(x)$ can be easily computed for all values of x between a and b, but somtimes the finding an antiderivative of f is impossible or too complicated. for exemple when we want dinding an antiderivative of

$$\int_0^1 e^{x^2} dx \qquad \text{or} \qquad \int_{-1}^1 \sqrt{1+x^3} dx$$

The second scenario arises when the function is obtained from scientific observations or collected data. There may be no formula for this function. In both cases we need to use a numerical methods for calculate the values of the integrals.

This thesis is organized as follows:

Chapter 1: Has a theoritical aspects about the simple and double integrals like definitions, properties, methods of calculus...etc.

Chapter 2: Includes the most used numerical methods to calculate the simple integrals and double integrals like The Newton-cots formula, The Midpoint rule, Trapezoidal and Simpson's rules we discuss also the efficiency and the errors.

Chapter 3: Concentrates on Matlab excution of the above methods with a selected example by showing their numerical and graphical results and comparing the obtaining results to the exact results in order to compare between the performance of these methods in terms of solution quality, time of calculus and convergence.

Chapter 1

Theoretical generalities on Integration

1.1 Simple integral

Definite Integral:

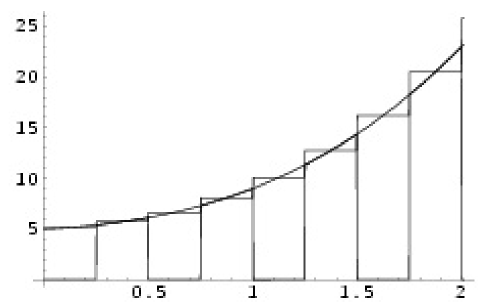

Figure 1.1: Reimann integral

let f is a continuous function defined for $a \leqslant x \leqslant b$. We divide the interval $[a, b]$ into n subintervals of equal width Δx. where $\Delta x = (b-a)/n$.We choose sample points $x_0^* = a, x_1^*, x_2^*, ..., x_n^* = b$ in these subintervals and asume that $x_0 = a, x_1, x_2, ..., x_n = b$ is the endpoints of these subintervals , where x_i^* be in the i^{th} subinterval $[x_{i-1}, x_i]$.

Then the definite integral of f from a to b is:

$$\int_a^b f(x)\,dx = \lim_{n\longrightarrow+\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x.$$

(we calle **Riemann sums** for the integral the formula $\sum_{i=1}^{n} f(x_i^*)\,\Delta x$, see figure (1.1)).

Graphical interpretation: $\int_a^b f(x)\,dx$ is the area inserted betwin the graph $y = f(x)$ and the $x - axis$ and the vertical lines $x = a, x = b$.

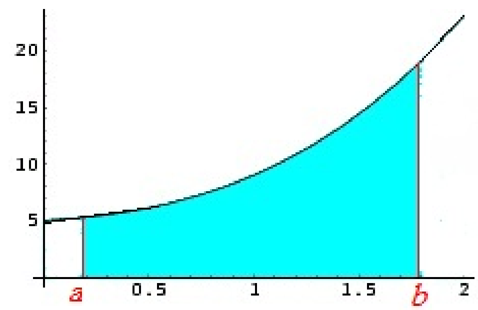

Figure 1.2: Graphical interpretation

Example 1.1.1 *We want calculate* $\int_0^1 e^x dx$.

$$\int_0^1 e^x dx = \lim_{n\longrightarrow+\infty} \sum_{i=1}^{n} e^{x_i^*}.\Delta x \quad with \quad \Delta x = \frac{1-0}{n} = \frac{1}{n} \quad and \quad x_i^* = \frac{i}{n}.$$

so

$$\int_0^1 e^x dx = \lim_{n\longrightarrow+\infty} \sum_{i=1}^{n} \frac{e^{\frac{i}{n}}}{n} = \lim_{n\longrightarrow+\infty} \frac{1}{n} \sum_{i=1}^{n} e^{\frac{i}{n}} = \lim_{n\longrightarrow+\infty} \frac{1}{n} \sum_{i=1}^{n} \left(e^{\frac{1}{n}}\right)^i = \lim_{n\longrightarrow+\infty} \frac{1}{n}.e^{\frac{1}{n}}.\frac{\left(e^{\frac{1}{n}}\right)^n - 1}{e^{\frac{1}{n}} - 1}$$
$$= \lim_{n\longrightarrow+\infty} \frac{1}{n}.e^{\frac{1}{n}}.\frac{e^{\frac{1}{n}\times n} - 1}{e^{\frac{1}{n}} - 1} = e - 1.$$

Theorem 1.1.1 (The Fundamental Theorem of Calculus) *Suppose f is continuous on $[a, b]$:*

1. *if* $g(x) = \int_a^x f(t)\,dt \quad a \leqslant x \leqslant b$. *Then* $g'(x) = f(x)$.
2. $\int_a^b f(x)\,dx = F(b) - F(a)$. *Where* F *is an antiderivative of* f , *that is* $F\prime = f$.

Example 1.1.2 *Let's take the previous example:*

$$\int_0^1 e^x dx = e^1 - e^0 = e - 1.$$

If we reverse a and b, then Δx changes from $(b-a)/n$ to $(a-b)/n$. By the definition of the definit integral as a limit of Riemann sums have a sense even if $a > b$. Therefore

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx.$$

If $a = b$, then $\Delta x = 0$ and so:

$$\int_a^b f(x)\,dx = 0$$

Properties 1.1.1 *Suppose that* f *and* g *are continuous functions then.*

1. $\int_a^b c dx = c(b-a)$ *where* c *is a costant.*
2. $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx.$
3. $\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$ *where* c *is any costant.*
4. $\int_a^b [f(x) - g(x)]\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx.$
5. $\int_a^c f(x)\,dx + \int_c^b f(x)\,dx = \int_a^b f(x)\,dx$, *where* $c \in [a, b]$.

1.2 The Substitution Rule

The Substitution Rule for Definite Integrals: This method, which is preferred, it consists transforming the limits of integral when the variable is changed.

. If g' is continuous on $[a, b]$ and f is continuous on the range of $u = g(x)$, then:

$$\int_a^b f(g(x))\, g'(x)\, dx = \int_{g(a)}^{g(b)} f(u)\, du.$$

Example 1.2.1 *Evaluate* $\int_0^4 \sqrt{2x+1}dx$

Solution: We use the follwoing substitution $u = 2x+1$ and $dx = \frac{du}{2}$. Then the new limits of integral as follows , when $x = 0, u = 1$ and when $x = 4, u = 9$. Therefore:

$$\begin{aligned}\int_0^4 \sqrt{2x+1}dx &= \int_1^9 \frac{1}{2}\sqrt{u}du \\ &= \left[\frac{1}{2}.\frac{2}{3}u^{\frac{3}{2}}\right]_1^9 \\ &= \frac{1}{3}\left(9^{\frac{3}{2}} - 1^{\frac{3}{2}}\right) = \frac{26}{3}.\end{aligned}$$

1.3 The integration by part rule

We assume that f and g are differentiable functions, then:

$$\frac{d}{dx} f(x)\, g(x) = f'(x)\, g(x) + f(x)\, g'(x).$$

We can evaluate definite integrals by parts. Evaluating both sides between a and b, assuming $f\prime$ and $g\prime$ are continuous, and using the Fundamental Theorem, we obtain:

$$\int_a^b f(x)\, g'(x)\, dx = [f(x)\, g(x)]_a^b - \int_a^b f'(x)\, g(x)\, dx$$

Example 1.3.1 *We wante to calculate* $\int_0^1 \arctan(x)\, dx$

so we integrating by part

$$\begin{aligned}\int_0^1 \arctan(x)\,dx &= [x\arctan(x)]_0^1 - \int_0^1 \frac{x}{1+x^2}dx \\ &= 1.\arctan(1) - 0.\arctan(0) - \int_0^1 \frac{x}{1+x^2}dx \\ &= \frac{\pi}{4} - \int_0^1 \frac{x}{1+x^2}dx\end{aligned}$$

To evaluate this integral we use the substitution $t = 1 + x^2$. *Then* $dt = 2xdx$, *so* $xdx = dt/2$. *When* $a = 0, t = 1$; *when* $x = 1, t = 2$; *so*

$$\int_0^1 \frac{x}{1+x^2}dx = \frac{1}{2}\int_1^2 \frac{dt}{t} = \frac{1}{2}[\ln|t|]_1^2 = \frac{1}{2}(\ln(2) - \ln(1)) = \frac{1}{2}\ln(2)$$

Therefore

$$\int_0^1 \arctan(x)\,dx = \frac{\pi}{4} - \frac{\ln(2)}{2}$$

1.4 Double integrals

Considering f is a function of two variables defined on a closed rectangle:

$$R = [a,b] \times [c,d] = \left\{(x,y) \in \mathbb{R}^2, a \leqslant x \leqslant b, c \leqslant y \leqslant d\right\}.$$

We begining with divide the rectangle R into subrectangles. Then dividing the interval $[a,b]$ into m subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = (b-a)/m$ and dividing $[c,d]$ into n subintervals $\left[y_{j-1}, y_j\right]$ of equal width $\Delta y = (d-c)/n$. Let $\Delta A = \Delta x \Delta y$. Then

$$R_{ij} = [x_{i-1}, x_i] \times \left[y_{j-1}, y_j\right] = \left\{(x,y) \in \mathbb{R}^2, x_{i-1} \leqslant x \leqslant x_i, y_{j-1} \leqslant y \leqslant y_j\right\}.$$

we choose a sample point (x^*, y^*) in each R_{ij}.

Definition 1.4.1 *The double integral of over the rectangle R is:*

$$\int\int_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dxdy = \lim_{m,n\longrightarrow+\infty} \sum_{i=1}^{m}\sum_{j=1}^{n} f\left(x_{ij}^*, y_{ij}^*\right)\Delta x \Delta y.$$

*($\sum_{i=1}^{m}\sum_{j=1}^{n} f\left(x_{ij}^*, y_{ij}^*\right)\Delta x \Delta y$ is called a **double Riemann sum** and is used as an approximation to the value of the double integral).*

1. $\int\int_R [f(x,y)+g(x,y)]\,dA = \int\int_R f(x,y)\,dA + \int\int_R g(x,y)\,dA.$
2. $\int\int_R c.f(x,y)\,dA = c.\int\int_R c.f(x,y)\,dA$ where c is a constant.
3. If $(x,y) \geqslant g(x,y)$ for all (x,y) in $\mathbb{R}$, then $\int\int_R f(x,y)\,dA \geqslant \int\int_R g(x,y)\,dA.$

Theorem 1.4.1 (Fubini's Theorem) *If f is continuous on the rectangle $R = \{(x,y) \in \mathbb{R}, a \leqslant x \leqslant b, c \leqslant y \leqslant d\}$, then*

$$\int\int_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dxdy = \int_c^d \int_a^b f(x,y)\,dydx$$

More generally, this is true if we assume that f is bounded on $\mathbb{R}$, f is discontinuous only on a finite number of smooth curves, and the iterated integrals exist.

Example 1.4.1 *Evaluate the double integral $\int\int_R x - 3y^2 dA$, where $R = \{(x,y) \in \mathbb{R}^2, 0 \leqslant x \leqslant 2, 1 \leqslant$*

Solution 1.4.1 *Fubini's Theorem gives:*

$$\begin{aligned}\int\int_R x - 3y^2 dA &= \int_0^2 \int_1^2 x - 3y^2 dydx \\ &= \int_0^2 \left[xy - y^3\right]_{y=1}^{y=2} dx \\ &= \int_0^2 x - 7dx = \left[\frac{x^2}{2} - 7x\right]_0^2 = -12\end{aligned}$$

Again applying Fubini's Theorem, but this time integrating with resp first, we have:

$$\int\limits_R\int x-3y^2dA=\int_0^2\int_1^2 x-3y^2dydx$$
$$=\int_1^2\left[\frac{x^2}{2}-3xy^2\right]_{x=0}^{x=2}dx$$
$$=\int_0^2 2-6y^2dy=\left[2y-2y^2\right]_1^2=-12$$

<u>**Double Integrals over General Regions**</u>: Here we have two cases,the first case is when yje region D lies between the graphs of two continue functions of x, that is,

$$D=\left\{(x,y)\in\mathbb{R}^2, a\leqslant x\leqslant b, g_1(x)\leqslant y\leqslant g_2(x)\right\},$$

where g_1 and g_2 are continuous on $[a,b]$. then:

$$\int\int\limits_D f(x,y)\,dA=\int_a^b\int_{g_1(x)}^{g_2(x)} f(x,y)\,dydx.$$

The integral on the right side is an iterated integral that is similar to the ones we considered in the preceeding section, except that in the inner integral we regard x as being constant not only in $f(x,y)$ but also in the limits of integration, $g_1(x)$and $g_2(x)$.

For the second case, we consider plane regions as the following:

$$D=\left\{(x,y)\in\mathbb{R}^2, c\leqslant y\leqslant d, h_1(y)\leqslant x\leqslant h_2(y)\right\},$$

where h_1 and h_2 are tow continuous functions of y .

In a similar manner in the case one,using the same methods , we can show that:

$$\int\int\limits_D f(x,y)\,dA=\int_c^d\int_{h_1(y)}^{h_2(y)} f(x,y)\,dxdy.$$

Example 1.4.2 *Suppose that $D1$ is the region bounded by the parabolas $y = x^2$ and $y = 1 + x^2$.Evaluate the following integral $\int\int_{D1} (x + 2y)\, dA$.*

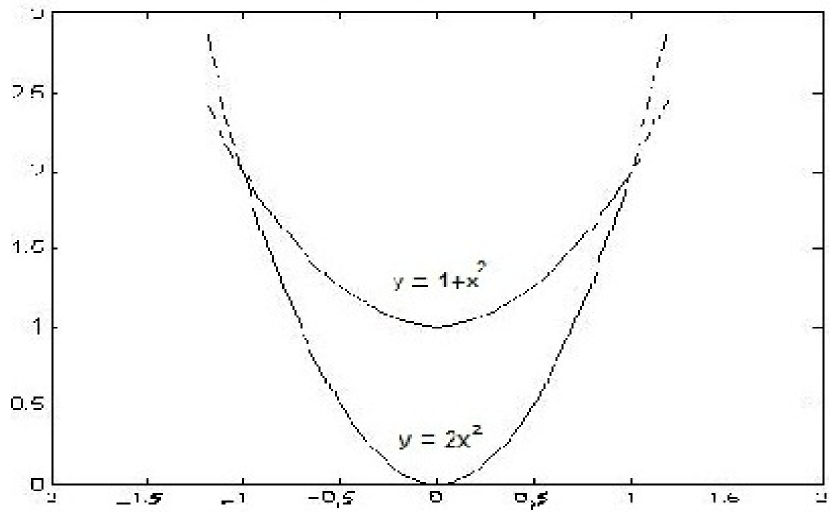

Figure 1.3: Area1

The parabolas intersect when $2x^2 = 1 + \pi 2$, that is, $x^2 = 1$, so $x = \pm 1$. We note that the region $D1$, sketched in figure (1.3) is a type (1) region but not a type(2) region and we can write:

$$D1 = \left\{(x, y) \in \mathbb{R}^2, -1 \leqslant x \leqslant 1, 2x^2 \leqslant y \leqslant 1 + x^2\right\}$$

Notice that the lower boundary of $D1$ is $y = 2x^2$and the upper boundary is $1 + x^2$ then

Example 1.4.3

$$\iint_D (x+2y)\,dA = \int_{-1}^{1}\int_{2x^2}^{1+x^2} x + 2ydydx$$
$$= \int_{-1}^{1} \left[xy + 2y^2\right]_{2x^2}^{1+x^2} dydx$$
$$= \int_{-1}^{1} \left[x\left(1+x^2\right) + \left(1+x^2\right)^2 - x\left(2x^2\right) - \left(2x^2\right)^2\right] dx$$
$$= \int_{-1}^{1} \left(-3x^4 - x^3 + 2x^2 + x + 1\right) dx$$
$$= \left[-3\frac{x^5}{5} - \frac{x^4}{4} + 2\frac{x^3}{3} + \frac{x^2}{2} + x\right]_{-1}^{1} = \frac{32}{15}$$

Example 1.4.4 *For* $D2 = \{(x,y) \in \mathbb{R}^2, 0 \leqslant y \leqslant 1, y \leqslant x \leqslant 5\}$ *Calculate the integral* $\iint_{D2} (x+1)\,dA$*. The region D2 sketched in figure (1.4).*

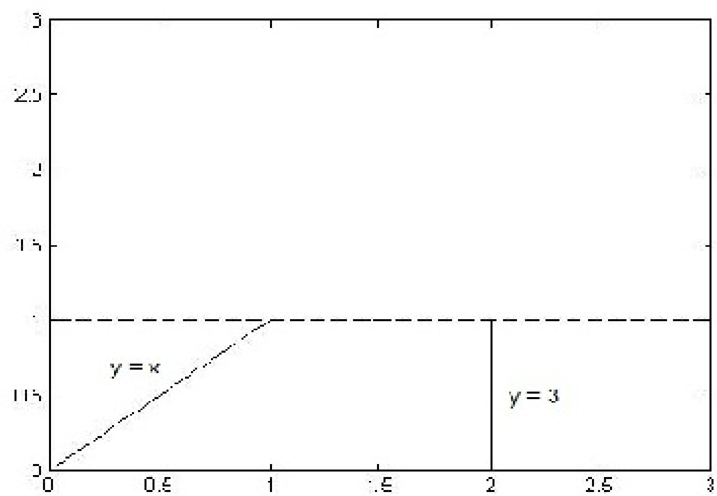

Figure 1.4: Area2

$$\begin{aligned}
\int\int_{D} (x+1)\,dA &= \int_0^1 \int_y^3 (x+1)\,dxdy \\
&= \int_0^1 \left[\frac{x^2}{2} + x\right]_y^3 dy \\
&= \int_0^1 \left[\frac{9}{2} + 3 - \frac{y^2}{2} - y\right] dy \\
&= \left[\frac{15}{2}y - \frac{y^3}{6} - \frac{y^2}{2}\right]_0^1 \\
&= \frac{15}{2} - \frac{1}{6} - \frac{1}{2} = \frac{41}{6}
\end{aligned}$$

Chapter 2

Review on some numerical methods to solve integrals

2.1 Methodes for simple integrals

2.1.1 The Newton-cotes rule

We want to evaluate the integral $\int_a^b f(x)\,dx$,we begin with the Lagrange interpolation,where $x_0, x_1, ..., x_n$ is nodes in $[a, b]$:

$$l_i(x) = \prod_{\substack{j=0 \\ j \neq i}}^{n} \frac{x - x_j}{x_i - x_j}, \qquad 0 \leqslant i \leqslant n.$$

We take up polynomial interpolation,these are the fondamental polynmials for interpolation. The polynomial of degree $\leqslant n$ that interpolate f at the nodes is:

$$p(x) = \sum_{i=0}^{n} f(x_i)\, l_i(x).$$

Then, integrate the both sides:

$$\int_a^b f(x)\,dx \;\approx \int_a^b p(x)\,dx = \sum_{i=0}^{n} f(x_i) \int_a^b l_i(x)\,.$$

Then we obtain a formula that can be used on any f, it been as follows:

$$\int_a^b f(x)\,dx \approx \sum_{i=0}^{n} A_i f(x_i)\,,$$

where

$$A_i = \int_a^b l_i(x)\,.$$

If the nodes are equally spaced, a formula of previousely forme is called a Newton-cotes formula. There are more than way for determine the error of Newton-cotes formula, we chose using the following theorem:

Theorem 2.1.1 (Error of Newton-cotes formula) *(a) For n even, assume $f(x)$ is $n+2$ times continuously differentiable on $[a,b]$. Then*

$$I(f) - I_n(f) = C_n h^{n+3} f^{(n+2)}(\eta) \quad \textit{some} \quad \eta \in [a,b]$$

with

$$C_n = \frac{1}{(n+2)!} \int_0^n \eta^2(\eta-1)...(\eta-n)d\eta$$

(b) For n odd, assume $f(x)$ is $n+1$ times continuously differentiable on $[a,b]$. Then

$$I(f) - I_n(f) = C_n h^{n+2} f^{(n+1)}(\eta) \quad \textit{some} \quad \eta \in [a,b]$$

with

$$C_n = \frac{1}{(n+1)!} \int_0^n \eta^2(\eta-1)...(\eta-n)d\eta$$

2.1.2 The Midpoint rule

There are additional Newton-Cotes formulas in which one or both of the endpoints of integration are deleted from the interpolation (and integration) node points. The best known of these is also the simplest, the midpoint rule. It is based on interpolation of the integrand $I(x)$ by the constant.

$$\int_a^b f(x)\,dx = (b-a)\,f\left(\frac{a+b}{2}\right) + \frac{(b-a)^3}{24} f''(\eta) \quad \text{some} \quad \eta \in [a,b].$$

Composite midpoint rule: For composite form of midpoint rule, we define:

$$x_j = a + \left(j - \frac{1}{2}\right)h, \; j = 1, 2, ..., n.$$

We obtain the midpoints of the intervals $[a + (j-1)\,h, a + jh]$. Therfore:

$$\int_a^b f(x)\,dx = I_n(f) + E_n(f)$$

$$I_n(f) = h\,[f_1 + f_2 + ... + f_n]$$

$$E_n(f) = \frac{h^2\,(b-a)''}{24}(\eta) \quad \text{some} \quad \eta \in [a,b].$$

2.1.3 Trapezoidal Rule

Starting with the simplest case results when choosing $n = 1$ and the nodes are $x_0 = 1, x_1 = b$. The fundamental polynomials for interpolation are

$$l_0(x) = \frac{b-x}{b-a} \qquad l_1(x) = \frac{x-a}{b-a}$$

Then:

$$A_0 = \int_{a0}^b l_0(x)\,dx = \frac{1}{2}(b-a) = \int_{a0}^b l_1(x)\,dx = A_1$$

And the corresponding quadrature formula is

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2}\left[f(a)+f(b)\right]$$

this is known as **The trapezidal rule**. It is exact for all $f \in R_1[x]$ (that is ,polynomials of degree at most 1). Moreovere it's error term can be detrmined by the previous theorem so:

$$I(f) - I_1(f) = C_1 h^3 f^{(2)}(\eta) \quad \text{some} \quad \eta \in [a,b]$$

$$E_1(f) = \tfrac{1}{2}h^3 f''(\eta)\int_0^1 \mu(\mu-1)\,d\mu \quad \text{some} \quad a \leqslant \eta \leqslant b.$$
$$\left[\tfrac{1}{2}f''(\eta)\right]\left[-\tfrac{1}{6}(b-a)^3\right] \quad \text{some} \quad \eta \in [a,b]\text{ .Whith } \ h = b-a.$$
$$E_1(f) = -\tfrac{(b-a)^3}{12}f''(\eta) \quad \text{some} \quad \eta \in [a,b].$$

Compostite trapezoidal rule: Now,we divide the interval $[a,b]$ into n subintervals like this:

$$a = x_0 < x_1 < \ldots < x_n = b$$

And appling the trapezoid rule to each subinterval. Here the nodes are not necessarly equally spaced. Then, we obtain the composite trapezoid rule:

$$\int_a^b f(x)\,dx = \sum_{i=1}^{n}\int_{x_{i-1}}^{x_i} f(x)\,dx \approx \frac{1}{2}\sum_{i=1}^{n}(x_i - x_{i-1})\left[f(x_{i-1}) + f(x_i)\right].$$

Remark 2.1.1 (A compostite rule) *It is one obtained by applying an integration formula for a single inteval to each subinterval of partitioned interval.The composite trapezoid rule also arises if the integrand f is replaced by an interpolating spline function of degree 1,that is broken line. With uniform spacing $h = (b-a)/n$ and*

$x_i = a + ih$, the compostite trapezoid rule takes the form:

$$\int_a^b f(x)\,dx \approx I_{1,n}(f) = \frac{h}{2}\left[f(a) + 2\sum_{i=1}^{n-1} f(a+ih) + f(b)\right],$$

and the erorr term is

$$E_{1,n}(f) = \sum_{i=1}^{n} -\frac{h^3}{12} f''(\eta_i) = -\frac{h^3 n}{12}\left[\frac{1}{n}\sum_{i=1}^{n} f''(\eta_i)\right],$$

for the term in braket

$$\min_{a\leqslant x\leqslant b} f''(\eta_i) \leqslant \frac{1}{n}\sum_{i=1}^{n} f''(\eta_i) \leqslant \max_{a\leqslant x\leqslant b} f''(\eta_i),$$

since $\overset{''}{\mathrm{f}}(x)$ is cntinuous for $a \leqslant x \leqslant b$, it must attain all values between its munimum and maximum at some point of $[a,b]$; thus $f"(\eta) = M$ for some $\eta \in [a,b]$. Thus we can write:

$$E_{1,n}(f) = -\frac{(b-a)h^2}{12} f"(\eta) \quad \text{some} \quad \eta \in [a,b].$$

2.1.4 The Simpson's rule

We use a quadratic interpolating polynomial $p_2(f)$ to approximate $f(x)$ on $[a,b]$. Let $c = \dfrac{a+b}{2}$, and define:

$$I_2(f) = \int_a^b \frac{(x-c)(x-b)}{(a-c)(a-b)} f(a) + \frac{(x-a)(x-b)}{(c-a)(c-b)} f(c) + \frac{(x-c)(x-c)}{(b-a)(b-c)} f(b)\,dx$$

carring out the integration, we obtain:

$$I_2(f) = \frac{h}{3}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] \quad h = \frac{b-a}{2}$$

This is called *Simson's rule.* An illustration is given in figure, with shaded region denoting the area under the graph of $y = p_2(x)$. We can calculat he Simpsn's error from the error theorem as:

$$\begin{aligned} I(f) - I_2(f) &= C_2 h^5 f^{(4)}(\eta) \quad \text{some} \quad \eta \in [a,b] \\ E_2(f) &= \frac{1}{24} h^5 f^{(4)}(\eta) \int_0^2 \mu^2(\mu-1)\ (\mu-2) d\mu \quad \text{some} \quad a \leqslant \eta \leqslant b. \\ &= \left[\frac{1}{24} h^5 f^{(4)}(\eta)\right]\left[-\frac{4}{15}\right] \quad \text{some} \quad \eta \in [a,b]. \\ &= -\frac{h^5}{90} f^{(4)}(\eta) \quad \eta \in [a,b]. \end{aligned}$$

Composite simpson's rule: Again we create a composite rule. For $n > 2$ and even, define $h = (b-a)/n$, $xj = a + jh$ for $j = 0, 1, ..., n$. Then

$$\int_a^b f(x)\,dx = \sum_{j=1}^{\frac{n}{2}} \int_{x_{2i-2}}^{x_{2i}} f(x)\,dx \approx \sum_{j=1}^{\frac{n}{2}} \left\{\frac{h}{3}\left[f_{2j-2} + 4f_{2j-1} + f_{2j}\right] - \frac{h^5}{90} f^{(4)}(\eta_j)\right\}$$

with $x_{2j-2} \leqslant \eta_j \leqslant x_{2j}$. Simplifying the first terms in the sum, we obtain the composite Simpson rule:

$$I_n(f) = \frac{h}{3}\left[f_0 + 4f_1 + 2f_2 + 4f_3 + 2f_4 + ... + 2f_{n-2} + 4f_{n-1} + f_n\right]$$

As before, we will simply call this Simpson's rule. It is probably the most well-used numerical integration rule. It is simple, easy to use, and reasonably accurate for a wide variety of integrals. For the error, as with the trapezoidal rule:

$$E_n(f) = I(f) - I_n(f) = -\frac{h^5(n/2)}{90}\frac{2}{n}\sum_{j=1}^{\frac{n}{2}} f^{(4)}(\eta_j)$$
$$E_n(f) = -\frac{h^4(b-a)}{180}f^{(4)}(\eta) \quad \text{some} \quad \eta \in [a,b]$$

2.2 Methods for double integrals

2.2.1 The Midpoint rule for double integrals

We can using the same methods for approximating single integrals (the Midpoint Rule, the Trapezoidal Rule, Simpson's Rule) to approximate the double integrals. In a similar manner we define the Midpoint Rule for double integrals. This means that we use a double Riemann sum to approximate the double integral, where the sample point (x^*, y^*) in R_{ij} is chosen to be the center $\left(\bar{x_i}, \bar{y_j}\right)$ of R_{ij}.

$$\int_a^b \int_c^d f(x,y)\,dxdy = \sum_{i=1}^{m}\sum_{j=1}^{n} f\left(x_{ij}^*, y_{ij}^*\right)\Delta x \Delta y$$

where $\bar{x_i}$ is the midpoint of $\left[x_{i-1}, x_i\right]$ and $\bar{y_j}$ is the midpoint of $\left[y_{j-1}, y_j\right]$.

2.2.2 Composite Simpson's Rule for double integrals

Consider the double integral

$$\int_R f(x,y)\,dxdy \quad \text{where} \quad R = \left\{(x,y) \in \mathbb{R}^2, a \leqslant x \leqslant b, c \leqslant y \leqslant d\right\}.$$

for some constants a, b, c, and d, is a rectangular region in the plane. To apply the Composite Simpson's rule, we divide the region R by partitioning both $[a,b]$ and $[c,d]$ into an even number of subintervals. To simplify the notation, we choose even integers n and m and partition $[a, b]$ and $[c, d]$ with the evenly spaced mesh points $x_0, x_1, ..., x_n$ and $y_0, y_1, ..., y_m$ respectively. These subdivisions determine step sizes $h = (b-a)/n$ and $k = (d-c)/m$. Writing the double integral as the iterated integral

$$\int_R f(x, y)\, dxdy = \int_a^b \left(\int_c^d f(x, y)\, dy \right) dx$$

we first use the Composite Simpson's rule to approximate:

$$\int_c^d f(x, y)\, dy.$$

treating x as a constant. Let $y_j = c + jk$, for each $j = 0, 1, ..., m$. Then

$$\int_c^d f(x, y)\, dy = \frac{k}{3} \left[f(x, y_0) + 2 \sum_{j=1}^{(m/2)-1} f(x, y_{2j}) + 4 \sum_{j=1}^{m/2} f(x, y_{2j-1}) + f(x, y_m) \right] - \frac{(d-c)\, k^4}{180} \frac{\partial^4 f(x, \mu)}{\partial y^4}.$$

for some μ, in (c, d).Thus:

$$\int_a^b \int_c^d f(x, y)\, dydx = \frac{k}{3} \left[\int_a^b f(x, y_0)\, dx + 2 \sum_{j=1}^{(m/2)-1} \int_a^b f(x, y_{2j})\, dx + 4 \sum_{j=1}^{m/2} \int_a^b f(x, y_{2j-1})\, dx + \int_a^b f(x, y_m)\, dx \right] - \frac{(d-c)\, k^4}{180} \int_a^b \frac{\partial^4 f(x, \mu)}{\partial y^4} dx.$$

The Composite Simpson's rule is now employed on the integrals in this equation. Let $x_i = a + ih$, for each $i = 0, 1, ..., n$. Then for each$j = 0, 1, ..., m$, we have:

$$\int_a^b f(x, y_j)\, dx = \frac{h}{3} \left[f(x_0, y_j) + 2 \sum_{i=1}^{(n/2)-1} f(x_{2i}, y_j) + 4 \sum_{i=1}^{n/2} f(x_{2i-1}, y_j) + f(x_n, y_j) \right] - \frac{(d-c)\, h^4}{180} \frac{\partial^4 f(\xi_j, y_j)}{\partial x^4}.$$

for some ξ_j in (a, b). The resulting approximation has the form:

$$\begin{aligned}
\int_a^b \int_c^d f(x,y)\,dydx \approx \tfrac{hk}{3} \Bigg\{ & \Bigg[f(x_0,y_0) + 2\sum_{j=1}^{(m/2)-1} f(x_{2i},y_0) + 4\sum_{j=1}^{m/2} f(x_{2i-1},y_0) + f(x_n,y_0) \Bigg] \\
& +2\Bigg[\sum_{j=1}^{(m/2)-1} f(x_0,y_j) + 2\sum_{j=1}^{(m/2)-1}\sum_{i=1}^{(n/2)-1} f(x_{2i},y_j) + 4\sum_{j=1}^{(m/2)-1}\sum_{i=1}^{n/2} f(x_{2i-1},y_j) \\
& \quad + \sum_{j=1}^{(m/2)-1} f(x_n,y_j) \Bigg] \\
& +4\Bigg[\sum_{j=1}^{m/2} f(x_0,y_{2j-1}) + 2\sum_{j=1}^{m/2}\sum_{i=1}^{(n/2)-1} f(x_{2i},y_{2j-1}) + 4\sum_{j=1}^{m/2}\sum_{i=1}^{n/2} f(x_{2i-1},y_{2j-1}) \\
& \quad + \sum_{j=1}^{m/2} f(x_n,y_{2j-1}) \Bigg] \\
& + \Bigg[f(x_0,y_m) + 2\sum_{j=1}^{(m/2)-1} f(x_{2i},y_m) + 4\sum_{j=1}^{m/2} f(x_{2-1},y_m) + f(x,y_m) \Bigg] \Bigg\}.
\end{aligned}$$

The error term E is given by:

$$\begin{aligned}
E = \frac{-k(b-a)h^2}{540} & \Bigg[\frac{\partial^4 f(\xi_0,y_0)}{\partial x^4} + 2\sum_{j=1}^{(m/2)-1} \frac{\partial^4 f(\xi_{2j},y_{2j})}{\partial x^4} + 4\sum_{j=1}^{m/2} \frac{\partial^4 f(\xi_{2j-1},y_{2j-1})}{\partial x^4} dx \\
& + \frac{\partial^4 f(\xi_m,y_m)}{\partial x^4} \Bigg] - \frac{(d-c)k^4}{180} \int_a^b \frac{\partial^4 f(x,\mu)}{\partial y^4} dx.
\end{aligned}$$

If $\partial^4 f/\partial x^4$ is continuous, the Intermediate Value Theorem can be repeatedly applied to show that the evaluation of the partial derivatives with respect to x can be replaced by a common value and that:

$$E = \frac{-k(b-a)h^2}{540} \left[3m \frac{\partial^4 f(\bar{\eta},\bar{\mu})}{\partial x^4} \right] - \frac{(d-c)k^4}{180} \int_a^b \frac{\partial^4 f(x,\mu)}{\partial y^4} dx.$$

for some $(\bar{\eta}, \bar{\mu})$ in R.]f d4f/dy4 is also continuous, the Weighted Mean Value Theorem for Integrals implies that:

$$\int_a^b \frac{\partial^4 f(x,\mu)}{\partial y^4} dx = (b-a) \frac{\partial^4 f(\hat{\eta},\hat{\mu})}{\partial y^4}.$$

for some $(\hat{\eta},\hat{\mu})$ in R. Since $m = (d-c)/k$, the error term has the form:

$$E = \frac{-k(b-a)h^2}{540}\left[3m\frac{\partial^4 f(\bar{\eta},\bar{\mu})}{\partial x^4}\right] - \frac{(d-c)}{180}k^4(b-a)\frac{\partial^4 f(\hat{\eta},\hat{\mu})}{\partial y^4}.$$

or

$$E = -\frac{(d-c)(b-a)}{180}\left[h^4\frac{\partial^4 f(\bar{\eta},\bar{\mu})}{\partial x^4} - k^4\frac{\partial^4 f(\hat{\eta},\hat{\mu})}{\partial y^4}\right].$$

for some $(\bar{\eta},\bar{\mu})$ and $(\hat{\eta},\hat{\mu})$ in R.

The use of approximation methods for double integrals is not limited to integrals with rectangular regions of integration. The techniques previously discussed can be modified to approximate double integrals of the form

$$\int_a^b \int_{c(x)}^{d(x)} f(x,y)\, dydx.$$

or

$$\int_c^d \int_{a(y)}^{b(y)} f(x,y)\, dydx.$$

In fact, integrals on regions not of this type can also be approximated by performing appropriate partitions of the region. To describe the technique involved with approximating an integral in the form:

$$\int_a^b \int_{c(x)}^{d(x)} f(x,y)\, dydx.$$

we will use the basic Simpson's rule to integrate with respect to both variables. The step size for the variable x is $h = (b-a)/2$, but the step size for y varies with x and is written:

$$k(x) = \frac{d(x) - c(x)}{2}.$$

Consequently,

$$\begin{aligned}
\int_a^b \int_{c(x)}^{d(x)} f(x,y)\, dydx &\approx \int_a^b \tfrac{k(x)}{3} \left[f(x, c(x)) + 4f(x, c(x) + k(x)) + f(x, d(x)) \right] dx \\
&\approx \tfrac{h}{3} \Big\{ \tfrac{k(a)}{3} \left[f(a, c(a)) + 4f(a, c(a) + k(a)) + f(a, d(a)) \right] \\
&+ \tfrac{4k(a+h)}{3} \left[f(a+h, c(a+h)) + 4f(a+h, c(a+h) + k(a+h)) \right. \\
&\left. + f(a+h, d(a+h)) \ \right] \\
&+ \tfrac{k(b)}{3} \left[f(b, c(b)) + 4f(b, c(b) + k(b)) + f(b, d(b)) \right].
\end{aligned}$$

Chapter 3

Application by Matlab software

3.1 Simple integral

In this section we try to apply the previous methods to the following example:

$$I(f) = \int_0^{\pi} \sin(x)\, dx$$

3.1.1 The Composite Midpoint rule with matlab

In the command window, we find the follwoing result: The definite integral of $\sin(x)$ from 0 to π has an exact value equal 2. Using the composite midpoint rule, MATLAB computes an approximate value of 2.0333. The second result is an aproximate value of the area under the curve.

The midpoint rule estimates area using rectangles whose height is determined at the midpoint of each subinterval and whose width is the length of subinterval. Fewer intervals typically result in a greater error, while more intervals reduce it. In this case, the error is about 0.0333, wich almost 1.67% of the true value. This is a reasonably small error for such a simple approximation method.

The midpoint rule is generally more accurate than the left or right Riemann sum. It strikes a good balance between simplicity and efficiency.

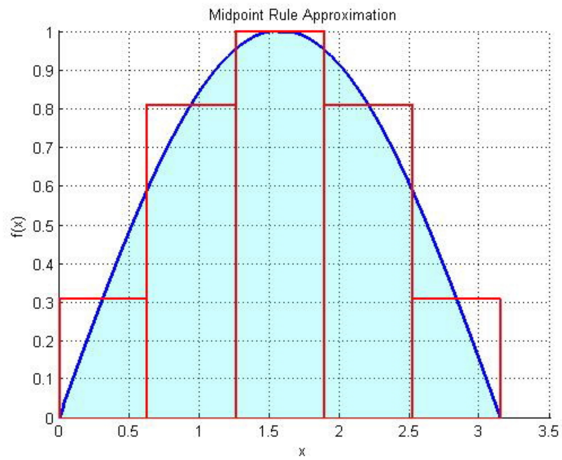

Figure 3.1: Midpoint rule approximation.

Graphically, the midpoint rectangles cover the area under the curve quite closely see fig (3.1). However, near the peaks of the sine curve, some over-coverage occurs. This visual gap helps explain the slightly higher or somtimes is lower result. Increasing the number of subintervals would bring the estimate closer to 2. The midpoint rule is a good choice for fast, approximate solutions where high precision isn't critical.

3.1.2 The Composite Trapezoidal rule with matlab

Trapezoidal Rule Approximation: 1.9338, while the exact integral value is 2. Using the trapezoidal rule, the integral of $\sin(x)$ from 0 to π is approximated as 1.9338. So the trapezoidal method underestimates the true area. This is expected, since $\sin(x)$ is a concave function on the interval $[0, \pi]$. In such cases, the trapezoids formed by straight lines between function values lie below the actual curve.

The gap between the curved area and the linear approximation accounts for the negative error. The relative error here is small but noticeable about 0.0662. This underestimation can be reduced by dividing the interval into more, smaller trapezoids.

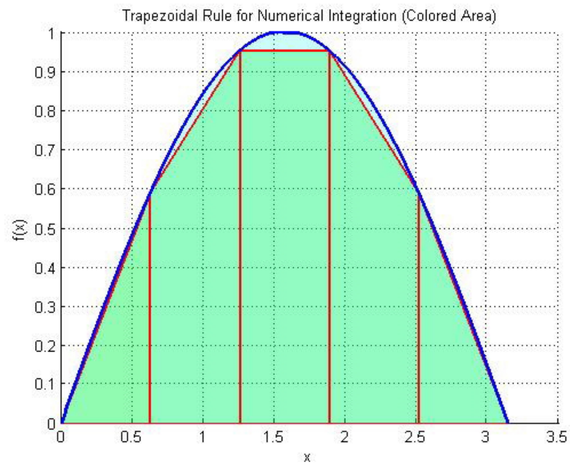

Figure 3.2: Trapezoidal rule approximation.

Despite the error, the trapezoidal rule gives a reasonable first approximation. It is a commonly used method due to its simple logic and implementation. Unlike midpoint or Simpson's rule, it doesn't require evaluating the function at interior midpoints. Instead, it uses only the function values at the interval endpoints and subinterval points. This makes it computationally efficient, especially when data is discrete or limited. However, it is more sensitive to the curvature of the function than midpoint rule.

Graphically, the trapezoids lie beneath the curve of $\sin(x)$, especially near the peak. This results in missing area between the curve and the tops of the trapezoids. The straight edges connect the endpoints of each subinterval, forming slanted tops. Since the sine curve is arched, the trapezoids cannot fully capture its curvature. This visual gap clearly explains the underestimation in the numerical result.

Overall, the plot highlights how the trapezoidal rule simplifies curves into piecewise linear segments, see fig (3.2).

3.1.3 The Composite Simpson's rule with matlab

Approximate integral value using Simpson's rule: 2.0009. While the exact integral value is 2. The application of Simpson's Rule in this case produced an impressively accurate approximation of the definite integral. The computed value, 2.0009, is extremely close to the exact value, 2., differing by only 0.0009. This minimal error reflects the strength of Simpson's Rule. Since the method relies on fitting parabolas through intervals of the function, it tends to perform particularly well when the integrand has continuous derivatives and low curvature.

In this situation, the function's behavior seems to align well with these conditions, allowing the parabolic segments to closely follow the true shape of the curve. The visual representation likely confirms this, showing how the approximated areas align nearly perfectly with the actual region under the curve. Such a small discrepancy is often negligible in practical scenarios, including physics, engineering, and computational modeling.

This result also suggests that the number of subintervals chosen was sufficient to achieve high precision without excessive computational effort. Compared to methods like the trapezoidal or midpoint rule, Simpson's Rule generally offers superior accuracy with fewer subintervals, as it captures the function's curvature more effectively. The near-exact match between approximation and truth here is a testament to the method's reliability.

The use of Simpson's Rule in this case was both appropriate and successful. The extremely low error highlights the effectiveness of the technique and shows that, when its assumptions are met, it can serve as a powerful tool for numerical integration. The result serves as a strong example of how numerical methods can closely approximate exact mathematical values with efficiency and precision.

Graphicaly, the Simpson's rule plot shows parabolic curves closely matching the

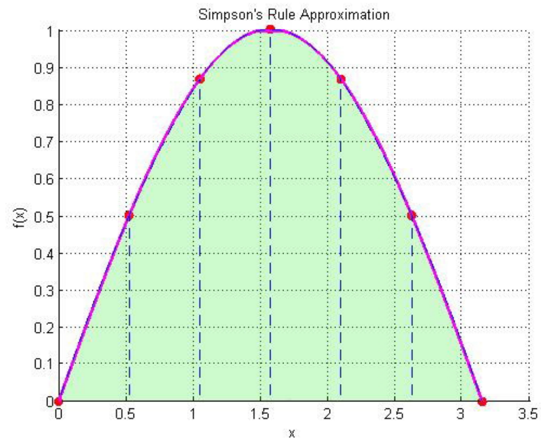

Figure 3.3: Simpson's rule approximation.

sine wave. Unlike trapezoids or rectangles, these parabolic segments hug the curve tightly. There is minimal visible gap between the approximation and the actual function. This explains the extremely small error of only 0.0009 in the result. Visually, the smooth arcs make Simpson's rule appear more accurate and refined. The plot effectively demonstrates why Simpson's rule outperforms simpler methods, see fig (3.3).

3.1.4 copmarison betwen methods

This is a comparative commentary between the Midpoint Rule, Trapezoidal Rule, and Simpson's Rule for approximating the integral of $\sin(x)$ over $[0, \pi]$. We applied three numerical integration methods to approximate:

$$I(f) = \int_0^{\pi} \sin(x)\, dx.$$

The exact value of the integral is 2 . The Midpoint Rule gave an approximation of 2.0333, slightly overestimating the true value. This occurs because the rectangles evaluated at midpoints overshoot the curve on concave intervals. Visually, the midpoint rectangles extend above the sine curve's arch. The Trapezoidal Rule produced an approximation of 1.9338, underestimating the actual area. Since the sine curve is concave on $[0, \pi]$ the trapezoids lie beneath the curve. The linear segments fail to match the curvature, creating visible gaps and a larger error. Simpson's Rule, on the other hand, gave a result of 2.0009, which is extremely close to the exact value. It uses parabolic arcs to approximate the function, capturing its curvature much more accurately.

Graphically, Simpson's method closely follows the sine curve with minimal deviation. In terms of accuracy, Simpson's rule is clearly superior. The midpoint rule performed better than the trapezoidal rule for this function but still had noticeable error. All three methods benefit from increasing the number of subintervals. However, Simpson's rule generally achieves higher precision with fewer intervals.

In conclusion, for smooth and well-behaved functions like $\sin(x)$ Simpson's rule is the most reliable. Midpoint and trapezoidal rules are useful for quick estimates but are more sensitive to function shape. The visual comparisons reinforce these observations, making the differences easy to interpret.

3.2 Double integrale

3.2.1 The midpoint rule for double integrale with matlab

In this section we try to apply the previous methods to the following example:

$$I(f) = \int_0^{\pi} \int_0^{\frac{\pi}{2}} \sin(x)\cos(y)\, dydx$$

Approximate value of the double integral is 2.028819 and the exact double integral value is 2.

The double integral $I(f) = \int_0^{\pi} \int_0^{\frac{\pi}{2}} \sin(x)\cos(y)\, dydx$ was evaluated both analytically and numerically using the midpoint rule in MATLAB. The exact value of the integral is 2 , which was obtained by first integrating $\cos(y)$ from 0 to $\frac{\pi}{2}$ and then integrating $\sin(x)$ from 0 to π . This confirms the exact area under the surface defined by the function over the given region is 2. In contrast, the approximate value obtained using the midpoint rule was 2.028819. This method estimates the integral by evaluating the function at the midpoint of each subrectangle in the integration domain. The result slightly overestimates the exact value, which is a typical behavior when using the midpoint rule for functions with curvature, such as sine and cosine. Despite the minor discrepancy, the approximation is quite close, demonstrating the method's effectiveness .

The error could be further reduced by increasing the number of subintervals used in both the x and y directions. This exercise illustrates how numerical integration techniques, like the midpoint rule, offer practical tools for evaluating definite integrals when analytical solutions are complecat or unavailable. MATLAB provides a straightforward platform for implementing such numerical methods. Overall, the comparison highlights both the accuracy and limitations of the midpoint rule. It also emphasizes the importance of validating numerical results against known analytical values when possible. The smooth nature of the sine and cosine functions contributed to the small error observed. Using a finer grid would enhance accuracy but at the cost of more computations. In real applications, such trade-offs must be balanced based on precision requirements. The example effectively demonstrates MATLAB's utility in approximating double integrals and reinforces fundamental concepts in numerical analysis.

The graph of the function $\sin(x)\cos(y)$ over the region $[0,\ \pi]\times\left[0,\frac{\pi}{2}\right]$ shows a smooth, wave-like surface. It rises and falls gently due to the sin variation in x and *cosine* variation in y. The surface reaches its maximum where both sine and cosine are near their peaks. The midpoint rule samples the height at the center of each subrectangle,

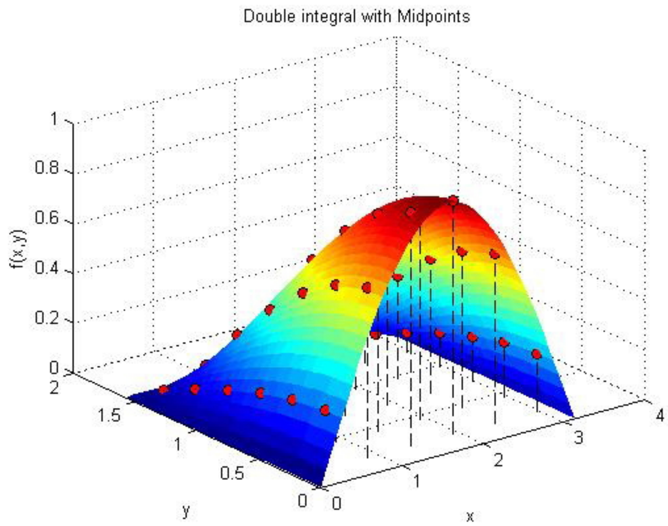

Figure 3.4: Midpoint rule approximation for double integral.

slightly overestimating areas under curved regions. This results in a small numerical error, as seen in the approximation. Overall, the graph visually confirms the smooth behavior that makes this integral well-suited for midpoint approximation, see fig (3.4).

3.2.2 Simpson's rule for double integral with matlab

The approximate value of the double integral is: 2.000916. The double integral $I(f) = \int_0^{\pi} \int_0^{\frac{\pi}{2}} \sin(x)\cos(y)\,dydx$ was approximated using Simpson's rule in MATLAB. The result obtained was 2.000916, which is extremely close to the exact value of 2.000000. Simpson's rule is a higher-order numerical integration technique that fits parabolas through sets of data points, resulting in significantly better accuracy than methods like the midpoint or trapezoidal rules. In this case, the surface formed by $\sin(x)\cos(y)$ is smooth and wave-like, making it ideal for Simpson's rule.

The small error of less than 0.001 demonstrates the method's high precision. This level of accuracy is usually sufficient for engineering, physics, and applied mathematics problems. The graph of the function confirms the smoothness and symmetry in both variables, supporting the effectiveness of Simpson's method. MATLAB allows for efficient implementation of Simpson's rule using matrix operations and grid evaluations. As expected, the accuracy improves with increased subdivisions, particularly when they are even, which Simpson's rule requires. Compared to the midpoint rule's result of 2.028819, Simpson's result is far superior in accuracy.

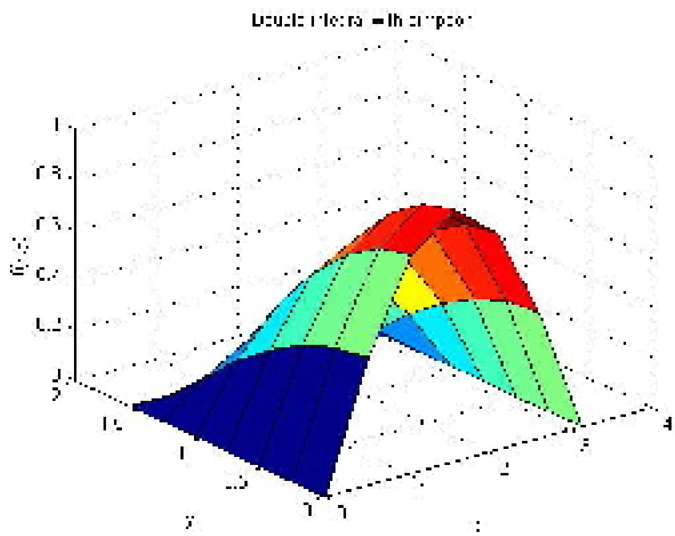

Figure 3.5: Simpson's rule approximation for doble integral.

This highlights the advantage of using higher-order interpolation in numerical integration. However, Simpson's rule can be slightly more complex to implement due to its requirement for an even number of intervals. In practice, this is a small price to pay for the accuracy gained. The method is especially useful when the exact solution is difficult or impossible to compute analytically. Overall, Simpson's rule provides a powerful and reliable tool for double integrals over rectangular domains. The MATLAB result illustrates this beautifully, showing nearly perfect agreement with the analytical solution. Such precision emphasizes the importance of choosing the right numerical method for a given problem.

The graph of the function $\sin(x)\cos(y)$ over the region $[0,\ \pi] \times \left[0, \frac{\pi}{2}\right]$ displays a smooth, undulating surface that gently rises and falls. This smoothness makes it ideal for Simpson's rule, which uses parabolic approximations to estimate area more accurately. The surface reaches its maximum where both sine and cosine values are high, creating a gentle peak. In the graph, the grid used for Simpson's rule aligns well with the curvature of the surface. This alignment explains the extremely small error in the result. The visual representation confirms that the method closely follows the shape of the function, leading to a highly accurate approximation, see fig (3.5).

Conclusion

Numerical integration is a powerful tool for approximating definite integrals when analytical solutions are difficult or impossible to obtain. It is widely used in engineering, physics, and computational sciences. Techniques such as the Trapezoidal Rule, Simpson's Rule, and more advanced adaptive methods offer varying trade-offs between accuracy and efficiency. While simple methods are easy to implement, they may lack precision for complex functions.

Adaptive and higher-order methods provide greater accuracy but at higher computational costs. The choice of method depends on the function behavior and required precision. Numerical integration is especially valuable in real-world applications involving experimental data or irregular functions. With advancements in computing, it is now feasible to integrate highly complex functions numerically. However, awareness of potential errors and convergence behavior remains critical.

Overall, numerical integration bridges the gap between theory and practice in modern applied mathematics.

Bibliography

[1] David Kincaid, Ward Cheney, 1991, numerical analysis.

[2] James Stewart, 1999, Calculus, forth edition.

[3] Kendall E. Atkinson, 1988, An introduction to numerical analysis, second edition.

[4] Richard L. Burden, J. Douglas Faires, 2001, Numerical analysis, seventh edition.

Annexe A: Abreviations and Notations

The various abbreviations and notations used throughout this thesis are explained below:

Symbol	Abreviation
$f(x), g(x)$	functions
$l_i(x)$	polynom of lagrange bases.
$E_n(f), E_{1,n}(f)$	the error
$p(x)$	the polynom of degree less than or equal n to interpolat f.
$f(x,y)$	unction of tow veriables.
$I(f)$	exact integral of f.
$I_n(f)$	aproch integral of f with n subdevisions.

Annexe B: What is Matlab?

MATLAB, short for MATrix LABoratory, is a high-level programming language and interactive environment developed by MathWorks, primarily designed for numerical computing and data analysis. Its core strength lies in matrix manipulation and linear algebra operations, making it especially powerful for engineers, scientists, and economists. MATLAB (see fig (**??**)) provides an intuitive syntax that resembles mathematical notation, simplifying tasks like algorithm development, simulation, and prototyping. It includes extensive built-in functions for applications such as signal processing, image analysis, machine learning, and control systems. Visualization is a key feature, with support for 2D and 3D plotting.

MATLAB integrates seamlessly with Simulink for model-based design and simulation, and it supports parallel computing, GPU acceleration, and code integration with languages like C, C++, Python, and Java. The environment includes a full-

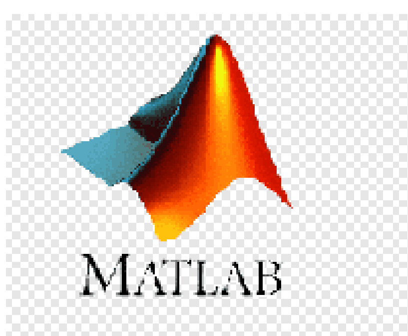

Figure 3.6: matlab icon

featured IDE with a script editor, debugger, and command window, as well as support for Live Scripts that combine code, output, and formatted text in a single document. Though it requires a paid license, MATLAB is widely used in both academia and industry for its robust capabilities and rich ecosystem of toolboxes tailored to specific domains.

Annexe C: Code Matlab used in chapter 3

The Composite Midpoint rule with matlab

```
% Define the function to integrate
f = @(x) sin(x); % Example function (f(x) = sin(x))
% Define the interval [a, b]
a = 0; b = pi;
% Number of subintervals (rectangles)
n = 5;
% Step size
h = (b - a) / n;
% Midpoints of the subintervals
x_mid = a + h/2 : h : b - h/2;
% Midpoint Rule Approximation
I_midpoint = h * sum(f(x_mid));
% Display the result of the integration
fprintf('Midpoint Rule Approximation: %.4f\n', I_midpoint);
% Plot the function and the rectangles
x = linspace(a, b, 100); % Fine x values for plotting the function
y = f(x); % Function values at x
```

```
figure;hold on;
% Plot the function
plot(x, y, 'b-', 'LineWidth', 2); % Plot the function in blue
% Color the area under the curve
fill([x, \ddag iplr(x)], [zeros(size(x)), \ddag iplr(y)], 'c',
'FaceAlpha', 0.2, 'EdgeColor', 'none');
% Plot the empty rectangles (with edges only)
for i = 1:n
% Coordinates of each rectangle
rect_x = [x_mid(i)-h/2, x_mid(i)+h/2, x_mid(i)+h/2, x_mid(i)-h/2];
rect_y = [0, 0, f(x_mid(i)), f(x_mid(i))];
% Plot the rectangle with no fill (empty rectangle)
plot(rect_x([1, 2]), rect_y([1, 2]), 'r-', 'LineWidth', 2); % Bottom edge
plot(rect_x([2, 3]), rect_y([2, 3]), 'r-', 'LineWidth', 2); % Right edge
plot(rect_x([3, 4]), rect_y([3, 4]), 'r-', 'LineWidth', 2); % Top edge
plot(rect_x([4, 1]), rect_y([4, 1]), 'r-', 'LineWidth', 2); % Left edge
end
% Add labels and title
xlabel('x'); ylabel('f(x)');
title('Midpoint Rule Approximation ');
grid on;
% Display the exact area under the curve for comparison
I_exact = integral(f, a, b);
fprintf('Exact Area under the Curve: %.4f\n', I_exact);
```

Code Matlab of Trapezoidal Rule

```
% Define the function to be integrated
f = @(x) sin(x); % Example function f(x) = sin(x)
```

```
% Define the integration limits
a = 0; b = pi; % Lower limit an Upper limit
% Number of trapezoids
n = 5;
% Create a vector of x values for plotting
x = linspace(a, b, 100);
y = f(x);
% Trapezoidal Rule calculation
x_trap = linspace(a, b, n+1);   % n+1 points for n trapezoids
y_trap = f(x_trap);
trap_area = 0;  % Initialize area
% Loop to calculate the area using trapezoidal rule
for i = 1:n
    trap_area = trap_area + (y_trap(i) + y_trap(i+1)) *
(x_trap(i+1) - x_trap(i)) / 2;
end
% Display the result
disp(['Trapezoidal Rule Approximation: ', num2str(trap_area)]);
exact_integral = integral(f, a, b);
disp('Exact integral value:');
disp(exact_integral);
% Plot the function and the filled trapezoidal areas
figure;    hold on;
% Plot the function f(x)
plot(x, y, 'b-', 'LineWidth', 2);
% Plot the filled trapezoids (colored area under the curve)
for i = 1:n
```

```
    % Define the x and y coordinates of the trapezoid
    x_vals = [x_trap(i), x_trap(i), x_trap(i+1), x_trap(i+1)];
    y_vals = [0, y_trap(i), y_trap(i+1), 0];

    % Fill the trapezoid area with a transparent color
    fill(x_vals, y_vals, 'c', 'FaceAlpha', 0.3, 'EdgeColor', 'r',
'LineWidth', 1.5);  % Cyan with transparency
end
% Set labels and title
xlabel('x'); ylabel('f(x)');
title('Trapezoidal Rule for Numerical Integration (Colored Area)');
grid on;     hold off;
```

Code Matlab of Simpson's Rule

```
% Define the function to integrate
f = @(x) sin(x);
% Set integration limits
a = 0; b = 2*pi;
% Number of subintervals (even number)
n = 6;
% Step size
h = (b - a) / n; x = a:h:b; y = f(x);
% Simpson's Rule Calculation
integral_approx = (h/3)*(y(1)+y(end)+4*sum(y(2:2:end-1))+2*sum(y(3:2:end-2)));
% Display results
disp('Approximate integral value using Simpson"s rule:');
disp(integral_approx);
exact_integral = integral(f, a, b);
```

```
disp('Exact integral value:');
disp(exact_integral);
% Plot
x_fine = linspace(a, b, 1000);
y_fine = f(x_\ldots ne);
\ldots gure;
hold on;
plot(x_fine, y_fine, 'm', 'LineWidth',2);
scatter(x, y, 'ro', 'MarkerFaceColor', 'r');
title('Simpson"s Rule Approximation');
xlabel('x'); ylabel('f(x)');
fill([x_fine, \ddag iplr(x_fine)], [y_fine, zeros(1,
length(y_fine))], 'g', 'FaceAlpha', 0.2);
% Plot parabolic arcs for each pair of subintervals
for i = 1:2:n-1
% Three points for Simpson's rule segment
x0 = x(i); x1 = x(i+1); x2 = x(i+2);
y0 = f(x0); y1 = f(x1); y2 = f(x2);
% Generate fine x values between x0 and x2
x_parabola = linspace(x0, x2, 100);
% Lagrange interpolation
L0 = ((x_parabola - x1) .* (x_parabola - x2)) / ((x0 - x1) * (x0 - x2));
L1 = ((x_parabola - x0) .* (x_parabola - x2)) / ((x1 - x0) * (x1 - x2));
L2 = ((x_parabola - x0) .* (x_parabola - x1)) / ((x2 - x0) * (x2 - x1));
y_parabola = y0 * L0 + y1 * L1 + y2 * L2;
plot(x_parabola, y_parabola, 'b--', 'LineWidth', 0.5);
end
```

```
% Draw vertical lines from x-axis to points
for i = 1:length(x)
plot([x(i), x(i)], [0, y(i)], 'b--');
end
title('Simpson"s Rule Approximation');
xlabel('x'); ylabel('f(x)');
grid on;
hold on;
```

Code Matlab of Midpoint Rule for Double Integral

```
 % Midpoint Rule for Double Integral with Visualization
clc; clear; close all;
% Define the function to integrate
f = @(x, y) sin(x) .* cos(y); % Example function
% Define integration limits
a = 0; b = pi; % x-limits
c = 0; d = pi/2; % y-limits
% Number of subintervals
nx = 6; % Number of intervals in x
ny = 6; % Number of intervals in y
% Step sizes
hx = (b - a) / nx;
hy = (d - c) / ny;
% Initialize sum
I = 0;
% Prepare grid for plotting
[X, Y] = meshgrid(linspace(a, b, 50), linspace(c, d, 50));
Z = f(X, Y);
```

```
% Plot the surface
figure;
surf(X, Y, Z, 'EdgeColor', 'none');
hold on;
title('Double integral with Midpoints');
xlabel('x'); ylabel('y'); zlabel('f(x,y)');
% Loop through subrectangles
for i = 1:nx
 for j = 1:ny
 % Midpoint coordinates
 xi = a + (i - 0.5) * hx;
 yj = c + (j - 0.5) * hy;
 % Evaluate function at midpoint
 fxy = f(xi, yj);

 % Add contribution to total integral
 I = I + fxy * hx * hy;

 % Plot midpoint
 plot3(xi, yj, fxy, 'ko', 'MarkerFaceColor', 'r');
 plot3([xi xi], [yj yj], [0 fxy], 'k--', 'LineWidth',0.1);
 end
end
% Display result
fprintf('Approximate value of the double integral: %.6f\n', I);
I_exact = integral2(f, a, b, c, d);
fprintf('exact double integral value: %.6f\n', I_exact);
```

<u>Code Matlab of Simpsn's Rule for Double Integral</u>

```
clc;
clear;
% Define the function to integrate
f = @(x,y) sin(x) .* cos(y); % Example function
% Integration limits
a = 0; b = pi; % x-limits
c = 0; d = pi/2; % y-limits
% Number of subintervals (must be even)
n = 6; % x-direction
m = 6; % y-direction
% Step sizes
hx = (b - a) / n;
hy = (d - c) / m;
% Generate grid
x = linspace(a, b, n+1);
y = linspace(c, d, m+1);
[X, Y] = meshgrid(x, y);
% Evaluate the function over the grid
Z = f(X, Y);
% Simpson's weights
Wx = ones(1, n+1);
Wx(2:2:end-1) = 4;
Wx(3:2:end-2) = 2;
Wy = ones(1, m+1);
Wy(2:2:end-1) = 4;
Wy(3:2:end-2) = 2;
```

```
% Compute Simpson's double integral
I = 0;
for i = 1:m+1
 for j = 1:n+1
 I = I + Wx(j)*Wy(i)*Z(i,j);
 end
end
I = I * (hx * hy) / 9; % 9 comes from 3*3 (Simpson's 1/3 rule in 2D)
% Display result
fprintf('The approximate value of the double integral is: %.6f\n', I);
% Plotting the surface and the region
figure;
surf(X, Y, Z);
xlabel('x'); ylabel('y'); zlabel('f(x,y)');
title('Double integral with simpson');
grid on;
```

Abstract

In this work, we will study the numerical integration,we starting with some theoretical generalities on integration ,the definite integral ,theorems and properties, after that we pass to review some numerical methods on integration like Newton-cots formula for simple and double integral ,and we finsh with some applications by Matlab software.

Key words : Numerical integration, The Newton-cots rule, The Midpoint rule, Trapezoidal Rule, The Simpsons rule.

Résumé

Dans ce travail, nous étudierons l'intégration numérique, en commençant par quelques généralités théoriques sur l'intégration, l'intégrale définie, des théorèmes et des propriétés, après cela nous passerons en revue certaines méthodes numériques sur l'intégration comme la formule de Newton-cots pour l'intégrale simple et double, et nous terminerons par quelques applications par le logiciel Matlab.

Mots clés : Intégration numérique, la méthode de Newton-cots, la méthode de Midpoint, la méthode de Trapezoidal Rule, la méthode de Simpsons.

الملخص

في هذا العمل سوف ندرس التكامل العددي، نبدأ ببعض النظريات العامة حول التكامل ، نظريات وخصائص، بعد ذلك ننتقل إلى مراجعة بعض الطرق العددية في التكامل مثل صيغة نوتن-كوتس للتكامل البسيط والمزدوج، وننتهي ببعض التطبيقات باستخدام برنامج ماتلاب