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MANSOUL Houria

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Existence of Optimal Controls for Linear Forward-Backward Doubly Stochastic Differential Equation of Mean-Field

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Committee Members of the Examination:

LAKHDARI Imad Eddine	MCA,	University of Biskra	President
GHERBAL Boulakhras	Prof,	University of Biskra	Advisor
KORICHI Fatiha	MCB,	University of Biskra	Examiner

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Dedicace

All praise and thanks be to Allah, for His countless blessings and for guiding me through this journey.

To my father, may Allah have mercy on him,

I dedicate this humble work to you, praying that Allah grants you its reward and makes it a light in your grave.

To my dear mother,

Your patience, love, and endless support mean the world to me. Thank you for standing by my side and for every heartfelt prayer.

To my beloved siblings,

You are my strength, my encouragement, and my constant support. Thank you for always being there.

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Contents

Introduction

The mathematical theory of stochastic differential equations was developed in the 1940s through the groundbreaking work of Japanese mathematician Kiyosi Itô, who introduced the concept of stochastic integral and initiated the study of nonlinear stochastic differential equations (SDEs). The linear backward stochastic differential equations (LBSDEs in short) related to the stochastic version of Pontryagin's maximum principle, has been studied by Bismut [5]. After that, the non linear BSDEs have been introduced by Pardoux and Peng [10]. Forward-backward stochastic differential equations (FBSDEs in short) were first studied by Antonelli (see [2]), where the system of such equations is driven by Brownian motion on a small time interval. The proof there relies on the fixed point theorem. There are also many other methods to study forward-backward stochastic differential equations on an arbitrarily given time interval. For example, the four-step scheme approach of Ma et al. [9], in which the authors proved the result of existence and uniqueness of solutions for fully coupled FBSDEs on an arbitrarily given time interval, where the diffusion coefficients were assumed to be nondegenerate and deterministic. Their work is based on continuation method.

A new class of stochastic differential equations with terminal condition, called backward doubly stochastic differential equation (BDSDE) have been introduced by Pardoux and Peng in [11]. The authors show existence and uniqueness for this kind of stochastic differential equation.

This memoir focuses on the existence of optimal controls for systems governed by a linear forward-backward doubly stochastic differential equations of mean-field type (MF-FBDSDE). These systems exhibit a coupling between forward and backward components, along with dependencies on the expected values of the processes, which makes their analysis both theoretically rich and technically challenging. In particular, we prove existence of strong optimal control (that is adapted to the initial $\sigma-$algebra) for the following linear forward-backward doubly stochastic differential equations,

$$\begin{cases} dX_t = & \left(AX_t + \hat{A}\mathbb{E}\left[X_t\right] + bu_t\right)dt + \left(CX_t + \hat{C}\mathbb{E}\left[X_t\right] + \hat{b}u_t\right)dW_t, \\ dY_t = & -\left(DX_t + \hat{D}\mathbb{E}\left[X_t\right] + EY_t + \hat{E}\mathbb{E}\left[Y_t\right] + FZ_t + \hat{F}\mathbb{E}\left[Z_t\right] + Gu_t\right)dt \\ & -\left(HX_t + \hat{H}\mathbb{E}\left[X_t\right] + KY_t + \hat{K}\mathbb{E}\left[Y_t\right] + MZ_t + \hat{M}\mathbb{E}\left[Z_t\right] + \hat{G}u_t\right)\overleftarrow{dB}_t + Z_t dW_t, \\ & X_0 = x, Y_T = \xi, \end{cases}$$

The objective is to find a control u_t that minimizes the following cost functional:

$$\mathbb{J}\left(u_\cdot\right) := \mathbb{E}\left[\alpha\left(X_t, \mathbb{E}\left[X_t\right]\right) + \beta\left(Y_0, \mathbb{E}\left[Y_0\right]\right) + \int_0^T \ell\left(t, X_t, \mathbb{E}\left[X_t\right], Y_t, \mathbb{E}\left[Y_t\right], Z_t, \mathbb{E}\left[Z_t\right], u_t\right)dt\right].$$

With real coefficients and non linear functional cost. The control domain and the cost function were assumed convex. The proof strategy relies on weak convergence results of admissible controls and the application of Mazur's theorem to extract strong convergence, which allows us to prove the existence of an optimal control.

The first chapter presents the theoretical background of stochastic calculus, including filtration, adaptedness, Brownian motion, martingales, and stochastic integration, along with essential results used in building such systems.

The second chapter is devoted to the existence of an optimal controls for systems governed by a linear forward-backward doubly stochastic differential equations of mean-field type (MF-FBDSDE), under suitable assumptions. It uses analytical tools and convergence techniques to establish the main result.

Chapter 1

Stochastic Calculus

1.1 Stochastic Process

Stochastic processes describe dynamical systems whose time-evolution is of probabilistic nature. The precise definition is given below :

Definition 1.1.1 *(Stochastic process): Let T be an ordered set, $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space and $(E, \mathcal{G})$ a measurable space. A stochastic process is a collection of random variables $\mathbf{X} = \{X_t; t \in T\}$ where, for each fixed $t \in T$, X_t is a random variable from $(\Omega, \mathcal{F}, \mathbb{P})$ to $(E, \mathcal{G})$. Ω is known as the sample space, where E is the state space of the stochastic process X_t.*

The set T can be either discrete, for example the set of positive integers $\mathbb{Z}_+$, or continuous, $T = \mathbb{R}$. The state space E will usually be $\mathbb{R}^d$ equipped with the σ-algebra of Borel sets.

A stochastic process X may be viewed as a function of both $t \in T$ and $\omega \in \mathbb{R}$. We will sometimes write: $X(t), X(t, \omega)$ or $X_t(\omega)$ instead of X_t. For a fixed sample point $\omega \in \Omega$, the function

$$X_t(\omega) : T \to E,$$

is called a (realization, trajectory) of the process X.

Definition 1.1.2 *(Filtration): We will be interested in phenomena that depend on time. What is known at date t is gathered in a $\sigma-$algebra $\mathbf{F}_t$, which is the information at date t.*

A filtration is an inceasing family of sub-σ-algabras of $\mathcal{F}$, meaninig that $\mathcal{F}_t \subset \mathcal{F}_s$ for all $t \leq s$.

It is often required that negligible sets are contained in $\mathcal{F}_0$.

We speak of usual hypotheses if:

- The negligible sets are contained in $\mathcal{F}_0$.

- The filtration is right-continuous in the sense that $\mathcal{F}_t = \cap_{s \geq t} \mathcal{F}_s$.

A filtration $\mathbf{G}$ *is said to be larger than* $\mathbf{F}$ *if* $\mathcal{F}_t \subset \mathcal{G}_t, \forall t$.

Definition 1.1.3 *(Measurability): Let* $(\Omega, \mathcal{F})$ *and* (E, ε) *be two measurable spaces. A function* f *from* Ω *to* E *is said to be* $(\mathcal{F}, \varepsilon)$*-measurable if* $f^{-1}(A) \in \mathcal{F}$ *for all* $A \in \varepsilon$*, where*

$$f^{-1}(A) \overset{def}{=} \{\omega \in \Omega | f(\omega) \in A\}.$$

When there is no ambiguity about the sigma-algebras used, we simply say that f *is measurable.*

A function f *from* $\mathbb{R}$ *to* $\mathbb{R}$ *is* ***Borel measurable*** *if it is* $(\mathcal{B}_{\mathbb{R}}, \mathcal{B}_{\mathbb{R}})$*-measurable, meaning* $f^{-1}(A) \in \mathcal{B}_{\mathbb{R}}$ *for all* $A \in \mathcal{B}_{\mathbb{R}}$*. It suffices that this property be verified for intervals* A*. Continuous functions are Borel measurable.*

Definition 1.1.4 *Let* $(\Omega, \mathcal{F})$ *be a measurable space. A real random variable (r.v.)* X *is a measurable function from* $(\Omega, \mathcal{F})$ *to* $\mathbb{R}$ *(such that* $X^{-1}(A) \in \mathcal{F}, \forall A \in \mathcal{B}_{\mathbb{R}}$*).*

A constant is a random variable in the same way as an indicator function of a set in the σ*-algebra* $\mathcal{F}$.

Proposition 1.1.1 *If* X *is a real* $\mathcal{G}$*-measurable random variable and* f *is a Borel function, then* $f(X)$ *is* $\mathcal{G}$*-measurable.*

A $\mathcal{G}$*-measurable random variable is an increasing limit of random variables of the type*

$$\sum_{i=1}^{n} a_i \mathbb{I}_{A_i},$$

with $A_i \in \mathcal{G}$*, A Borel function is an increasing limit of functions of the type* $\sum_{i=1}^{n} a_i \mathbb{I}_{A_i}$*, where* A_i *is an interval.*

Definition 1.1.5 *(Adapted process): A sochastic processes* $X_t, t \in [0, T]$ *is a adapted to filtration* $\mathcal{F}_t$ *if* X_t *is measurable for any* $t \in [0, T]$.

Let $\{X_n\}_{n \in \mathbb{N}}$ *be a stochastic process on* $(\Omega, \mathcal{F}, \mathbb{P})$*. It is said that the process is adapted to the filtration* $\mathcal{F}_n$ *if* X_n *is measurable with respect to* $\mathcal{F}_n$ *for every* n*.*

A minimal choice of adapted filtration is the canonical (or natural) filtration:

$$\mathcal{F}_n = \sigma\left(X_0, X_1, ..., X_n\right).$$

In this case, $\mathcal{F}_n$ represents the information available at time n if the stochastic process is observed.

1.2 Conditional Expectation

Let X, Y be random variables (integrable) defined on $(\Omega, \mathcal{F}, P)$ and let $\mathcal{G}$ be a sub-sigma-algabra of $\mathcal{F}$.

Definition 1.2.1 *The conditional expectation $\mathbb{E}\left[X|\mathcal{G}\right]$ of X, given $\mathcal{G}$, is the unique random variable that satisfies:*

- It is $\mathcal{G}-$measurable.
- Such that

$$\int_A \mathbb{E}\left[X|\mathcal{G}\right] dP = \int_A X dP, \quad \forall A \in \mathcal{G}.$$

Propreties of Conditional Expectation

1 Linearity. Let a and b be two constants, we have

$$\mathbb{E}\left[aX + bY|\mathcal{G}\right] = a\mathbb{E}\left[X|\mathcal{G}\right] + b\mathbb{E}\left[Y|\mathcal{G}\right].$$

2 Monotonicity. Let X and Y be two random variables such that $X \leq Y$. Then:

$$\mathbb{E}\left[X|\mathcal{G}\right] \leq \mathbb{E}\left[Y|\mathcal{G}\right].$$

3 We have also

$$\mathbb{E}\left[\mathbb{E}\left(X|\mathcal{G}\right)\right] = \mathbb{E}\left[X\right].$$

4 If X is $\mathcal{G}-$measurable, then:

$$\mathbb{E}\left[X|\mathcal{G}\right] = X.$$

5 If Y is $\mathcal{G}-$measurable, then:

$$\mathbb{E}\left[XY|\mathcal{G}\right] = Y\mathbb{E}\left[X|\mathcal{G}\right].$$

6 If X is independent of $\mathcal{G}$, then:

$$\mathbb{E}\left[X|\mathcal{G}\right] = \mathbb{E}\left[X\right].$$

7 If $\mathcal{G}$ and $\mathcal{H}$ are two sigma-algabras such that $\mathcal{H} \subseteq \mathcal{G}$, then:

$$\mathbb{E}\left[X|\mathcal{H}\right] = \mathbb{E}\left[\mathbb{E}\left[(X|\mathcal{G})\right]\mathcal{H}\right].$$

1.3 Martingales

Definition 1.3.1 *(Martingales): A martingale is a model of a fair game. We will let $\{\mathcal{F}_n\}$ denote an increasing collection of information. By this we mean for each n, we have a collection of random variables $\mathcal{A}_n$ such that $\mathcal{A}_m \subset \mathcal{A}_n$ if $m < n$. The information that we have at time n is the value of all of the variables in $\mathcal{A}_n$. The assumption $\mathcal{A}_m \subset \mathcal{A}_n$ means that we do not lose information. A random variable X is $\mathcal{F}_n$-measurable if we can determine the value of X if we know the value of all the random variables in $\mathcal{A}_n$. The increasing sequence of information $\mathcal{F}_n$ is often called a filtration. We say that a sequence of random variables $M_0, M_1, M_2, ...$ with $\mathbb{E}\left[|M_i|\right] < \infty$, is a martingale with respect to $\{\mathcal{F}_n\}$ if each M_n is measurable with respect to $\mathcal{F}_n$, and for each $m < n$,*

$$\mathbb{E}\left[M_n|\mathcal{F}_m\right] = M_m.$$

Or equivalently,

$$\mathbb{E}\left[M_n - M_m|\mathcal{F}_m\right] = 0.$$

The condition $\mathbb{E}\left[|M_i|\right] < \infty$ is needed to guarantee that the conditional expectations are well defined. If $\mathcal{F}_n$ is the information in random variables $X_1, ..., X_n$, then we will also say that $M_0, M_1, ...$ is a martingale with respect to $X_0, X_1, ...,$. Sometimes we will just say $M_0, M_1, ...$ is a martingale without making reference to the filtration $\mathcal{F}_n$. In this case it will mean that the sequence M_n is a martingale with respect to itself (in which case the first condition is trivially true). In order to verify it suffices to prove that for all n,

$$\mathbb{E}\left[M_{n+1}|\mathcal{F}_m\right] = M_n.$$

Since if this holds,

$$\begin{aligned}\mathbb{E}\left[M_{n+2}|\mathcal{F}_m\right] &= \mathbb{E}\left[\mathbb{E}\left[M_{n+2}|\mathcal{F}_{n+1}\right]|\mathcal{F}_n\right] \\ &= \mathbb{E}\left[M_{n+1}|\mathcal{F}_m\right] = M_n,\end{aligned}$$

And so on.

Definition 1.3.2 ***Example 1.3.1*** *Let $X_1, X_2, ...$ be independent random variables each with mean μ. Let $S_0 = 0$ and for $n > 0$ let S_n be the partial sum of $X_0, X_1, ...$*

$$S_n = X_1 + ... + X_n.$$

Then $M_n = S_n - n\mu$ is a martingale with respect to $\mathcal{F}_n$, the information contained in $X_0, ..., X_n$. This can easily be checked,

$$\begin{aligned}\mathbb{E}\left[M_{n+1}|\mathcal{F}_n\right] = \mathbb{E}\left[S_{n+1} - (n+1)\mu|\mathcal{F}_n\right] &= \mathbb{E}\left[S_{n+1}|\mathcal{F}_n\right] - (n+1)\mu \\ = (S_n + \mu) - (n+1)\mu &= M_n.\end{aligned}$$

In particular, if $\mu = 0$, then S_n is a martingale with respect to $\mathcal{F}_n$.

Definition 1.3.3 *(Martingale, Submartingale and Supermartingale): Let $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}, \mathbb{P})$ be a filtered probability space. A martingale with respect to the filtration $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ is a stochastic process $\{X_n\}_{n\in\mathbb{N}}$ such that:*

1. $\mathbb{E}\left[|X_n|\right] < \infty$ *for all* $n \in \mathbb{N}$;
2. $\{X_n\}_{n\in\mathbb{N}}$ *is adapted to the filtration* $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$;
3. *And*

$$\mathbb{E}\left[X_{n+1}|\mathcal{F}_n\right] = X_n,$$

For all $n \in \mathbb{N}$.

If the last condition is replaced by $\mathbb{E}\left[X_{n+1}|\mathcal{F}_n\right] \leq X_n$, then $\{X_n\}_{n\in\mathbb{N}}$ is called a ***supermartingale****, and if it is replaced by $\mathbb{E}\left[X_{n+1}|\mathcal{F}_n\right] \geq X_n$, it is called a* ***submartingale****.*

It should be noted that, in terms of gambling, a ***supermartingale*** *is unfavorable to the player,* *whereas a **submartingale** is favorable to them (this terminology originates from the concept of a subharmonic function).*

1.4 Brownian Motion

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a probability space. To simplify things, we assume that our interval of study is bounded (we will often take $[0, T]$ with $T = 1$, but this does not change anything).

Brownian motion is a continuous process $(B_t)_{t\in[0,1]}$ whose increments are independent, stationary, and Gaussian. More precisely.

Definition 1.4.1 *(Brownian Motion): A standard Brownian montion (Wiener process) is a process $(B_t)_{t\in[0,1]}$ satisfying :*

1. $B_0 = 0 \ P - a.s.$
2. *B is B_t continuous, i.e. $t \to B_t(\omega)$ is continuous for $\mathbb{P}$-almost every ω.*
3. *B has independent increments: If $t > s$, then $B_t - B_s$ is independent of $\mathcal{F}_s^B = \sigma(B_u, u \leq s)$.*
4. *The increments of B are stationary and Gaussian: If $t \geq s$ then $B_t - B_s$ follows a normal distribution $\mathcal{N}(0, t-s)$.*

Theorem 1.4.1 *(Properties of Brownian motion): The following symmetries exist:*

1. ***Time-homogeneity**: For any $s > 0$, the process $\widetilde{B}_t = B_{t+s} - B_s$ is a Brownian motion independent of $\sigma(B_u, u \leq s)$.*
2. ***Reflection symmetry**: The process $\widetilde{B}_t = -B_t$ is a Brownian motion.*
3. ***Brownian scaling**: For every $c > 0$, the process $\widetilde{B}_t = cB_{1/c^2}$ is a Brownian motion.*
4. ***Time inversion**: The process $\widetilde{B}_0 = 0$, $\widetilde{B}_t = tB_{1/t}$, $t > 0$ is a Brownian motion.*

Proof. Properties (1),(2) and (3): in each case, $\widetilde{B}_t$ is a continuous centered Gaussian process with continuous paths, independent increments and variance t.

(4) $\widetilde{B}$ is a centered Gaussian process with covariance

$$
\begin{aligned}
cov\left[\widetilde{B}_s, \widetilde{B}_t\right] = \mathbb{E}\left[\widetilde{B}_s, \widetilde{B}_t\right] &= st.\mathbb{E}\left[B_{1/s}, B_{1/t}\right] \\
&= st.\inf\left(\frac{1}{s}, \frac{1}{t}\right) = \inf\left(s, t\right).
\end{aligned}
$$

Continuity of $\widetilde{B}_t$ is obvious for $t > 0$, We have to check continuity only for $t = 0$, but since

$$
\mathbb{E}\left[\widetilde{B}_s^2\right] = s \to 0,
$$

For $s \to 0$, we know that $\widetilde{B}_s \to 0$ almost everywhere. ■

1.5 Stochastic Integral

The Ito integral is defined in a way that is similar to the Riemann integral. The Ito integral is taken with respect to infinitesimal increments of a Brownian motion, dB_t, which are random variables, while the Riemann integral considers integration with respect to the predictable infinitesimal changes dt. It is worth noting that the Ito integral is a random variable, while the Riemann integral is just a real number. Despite this fact, there are several common properties and relations between these two types of integrals.

In this section we will briefly review the definition and some properties of the stochastic integral.

1.5.1 Construction of Itô's Integral

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$ be a fixed filtered probability space satisfying the usual condition.

Let us denote by $L^2_{\mathcal{F}}(0, T, \mathbb{R})$ the space of all stochastic processes $f(t, \omega)$, $0 \leq t \leq T$, $\omega \in \Omega$. satisfying the following conditions

1. $f(t, \omega)$ is adapted to the filtration $\{\mathcal{F}_t\}$,
2. $\int_0^T \mathbb{E}\left[f(t)\right]^2 dt < \infty$.

For any f and $g \in L^2_{\mathcal{F}}(0, T, \mathbb{R})$ we define aninner product $\langle f, g\rangle = \mathbb{E}\left[\int_0^T f(t)\, g(t)\, dt\right]$ and it's norm $\|f\| = \langle f, f\rangle^{\frac{1}{2}} = \sqrt{\mathbb{E}\left[\int_0^T f^2(t)\, dt\right]}$. It can be shown that $L^2_{\mathcal{F}}(0, T, \mathbb{R})$ is a Hilbert space with norm. Thus

We want to define the stochastic integral

$$\int_0^T f(t,\omega)\, dB_t,$$

For elements f of $L^2_{\mathcal{F}}(0,T,\mathbb{R})$. We start with a definition for a simple class of functions f

Definition of the Ito Integral for Step Functions (step1)

Divide the interval $[0,T]$ into n subintervals using the partition points

$$0 = t_0 < t_1 < ... < t_{n-1} < t_n = T,$$

Suppose f is a step stochastic process given by

$$f = \sum_{k=1}^{n} f_{k-1} 1_{[t_{k-1},t_k]},$$

Where f_{k-1} is $\mathcal{F}_{k-1}-$measurble and $\mathbb{E}[f_{k-1}]^2 < \infty$.we define the following linear operator

$$I(f) = \sum_{k=1}^{n} f_{k-1}\left(B(t_k) - B(t_{k-1})\right).$$

Lemma 1.5.1 *Let $I(f)$ be a linear random variable with mean $\mathbb{E}[I(f)] = 0$, and variance $\mathbb{E}\left[|I(f)|^2\right] = \int_0^T \mathbb{E}\left[|f|^2\right] dt$.*

Proof. For each $0 \leq k \leq n$, we have

$$\begin{aligned}
\mathbb{E}\left[f_{k-1}\left(B(t_k) - B(t_{k-1})\right)\right] &= \mathbb{E}\left[\mathbb{E}\left[f_{k-1}\left(B(t_k) - B(t_{k-1})\right)/\mathcal{F}_{k-1}\right]\right] \\
&= \mathbb{E}\left[f_{k-1}\mathbb{E}\left[\left(B(t_k) - B(t_{k-1})\right)/\mathcal{F}_{k-1}\right]\right] \\
&= \mathbb{E}\left[f_{k-1}\mathbb{E}\left[\left(B(t_k) - B(t_{k-1})\right)\right]\right] \\
&= 0.
\end{aligned}$$

Hence $\mathbb{E}[I(f)] = 0$. On the other hand, we have

$$\mathbb{E}\left[|I(f)|^2\right] = \sum_{k,l=1}^{n} f_{k-1}f_{l-1}\left(B(t_k) - B(t_{k-1})\right)\left(B(t_l) - B(t_{l-1})\right).$$

If $k \neq l$, where $k < l$

$$\begin{aligned}
&\mathbb{E}\left[f_{k-1}f_{l-1}\left(B\left(t_k\right)-B\left(t_{k-1}\right)\right)\left(B\left(t_l\right)-B\left(t_{l-1}\right)\right)\right]\\
&\qquad=\mathbb{E}\left[\mathbb{E}\left[f_{k-1}f_{l-1}\left(B\left(t_k\right)-B\left(t_{k-1}\right)\right)\left(B\left(t_l\right)-B\left(t_{l-1}\right)\right)/\mathcal{F}_{l-1}\right]\right]\\
&\qquad=\mathbb{E}\left[f_{k-1}f_{l-1}\left(B\left(t_k\right)-B\left(t_{k-1}\right)\right)\mathbb{E}\left[\left(B\left(t_l\right)-B\left(t_{l-1}\right)\right)/\mathcal{F}_{l-1}\right]\right]\\
&\qquad=\mathbb{E}\left[f_{k-1}f_{l-1}\left(B\left(t_k\right)-B\left(t_{k-1}\right)\right)\mathbb{E}\left[\left(B\left(t_l\right)-B\left(t_{l-1}\right)\right)\right]\right]\\
&\qquad=0.
\end{aligned}$$

On the other hand, for $k=l$ we have from the independence of $B\left(t_k\right)-B\left(t_{k-1}\right)$ of $\mathcal{F}_{k-1}$,

$$\begin{aligned}
\mathbb{E}\left[f_{k-1}^2\left(B\left(t_k\right)-B\left(t_{k-1}\right)\right)^2\right] &=\mathbb{E}\left[\mathbb{E}\left[f_{k-1}^2\left(B\left(t_k\right)-B\left(t_{k-1}\right)\right)^2/\mathcal{F}_{k-1}\right]\right]\\
&=\mathbb{E}\left[f_{k-1}^2\mathbb{E}\left[\left(B\left(t_k\right)-B\left(t_{k-1}\right)\right)^2/\mathcal{F}_{k-1}\right]\right]\\
&=\mathbb{E}\left[f_{k-1}^2\mathbb{E}\left[\left(B\left(t_k\right)-B\left(t_{k-1}\right)\right)^2\right]\right]\\
&=\mathbb{E}\left[f_{k-1}^2\left(t_k-t_{k-1}\right)\right]\\
&=\left(t_k-t_{k-1}\right)\mathbb{E}\left[f_{k-1}^2\right].
\end{aligned}$$

So, we get

$$\mathbb{E}\left[\left|I\left(f\right)\right|^2\right]=\sum_{k=1}^{n}\left(t_k-t_{k-1}\right)\mathbb{E}\left[f_{k-1}^2\right].$$

■

An approximation lemma (step2)

We need to prove an approximation lemma in this step in order to be able to define the stochastic integral $\int_0^T f\left(t\right)dB_t$ for general stochastic processes $L^2_{\mathcal{F}}\left(0,T,\mathbb{R}\right)$.

Lemma 1.5.2 *Suppose* $f\in L^2_{\mathcal{F}}\left(0,T,\mathbb{R}\right)$. *Then there exists a sequence* $\left\{f_n,\ n\geq 1\right\}$ *of step processes in* $L^2_{\mathcal{F}}\left(0,T,\mathbb{R}\right)$ *such that*

$$\lim_{n\to\infty}\int_0^T \mathbb{E}\left[\left|f\left(t\right)-f_n\left(t\right)\right|^2\right]dt=0. \tag{1.1}$$

Stochastic integral (step3)

Now we can use what we proved in Step 1and Step 2 to define the stochastic integral

$$\int_0^T f\left(t\right)dB_t,$$

For $f\in L^2_{\mathcal{F}}\left(0,T,\mathbb{R}\right)$. Apply first Lemma 1.5.2 to get a sequence $\left\{f_n,\ n\geq 1\right\}$ of adapted step stochastic processes such that 1.1 holds.

For each n, $I(f_n)$ is defined by (Step1). By Lemma 1.5.1 we have

$$\mathbb{E}\left[|f_n(t) - f_m(t)|^2\right] = \int_0^T \mathbb{E}\left[|f_n(t) - f_m(t)|^2\right] dt \to 0,\ as\ n, m \to 0.$$

It follows that $\{I(f_n)\}$ is a Cauchy sequence in $L^2(\Omega, \mathcal{F}, \mathbb{P})$. Thus $\{I(f_n)\}$ has a unique limit in L^2, denoted by $I(f)$ is called the Itô integral, so

$$I(f) = \int_0^T f(t)\, dB_t.$$

The integral is independent of the choice of the approximating sequence $\{f_n,\ n \geq 1\}$

1.5.2 Some Properties of Itô Integral

1. Linearity: let $\alpha, \beta \in \mathbb{R}$ and $f, g \in L^2_{\mathcal{F}}(0, T, \mathbb{R})$. Then $\alpha f + \beta g \in L^2_{\mathcal{F}}(0, T, \mathbb{R})$

$$\int_0^T (\alpha f(t) + \beta g(t))\, dB_t = \alpha \int_0^T f(t)\, dB_t + \beta \int_0^T g(t)\, dB_t.$$

2. Partition property

$$\int_0^T f(t)\, dB_t = \int_0^c f(t)\, dB_t + \int_c^T f(t)\, dB_t,\ \forall 0 < c < T.$$

3. Zero mean

$$\mathbb{E}\left[\int_0^T f(t)\, dB_t\right] = 0.$$

4. Isometry

$$\mathbb{E}\left[\left(\int_0^T f(t)\, dB_t\right)^2\right] = \int_0^T \mathbb{E}\left[f^2(t)\, dt\right].$$

5. Covariance

$$\mathbb{E}\left[\left(\int_0^T f(t)\, dB_t\right)\left(\int_0^T g(t)\, dB_t\right)\right] = \mathbb{E}\left[\int_0^T f(t)\, g(t)\, dt\right].$$

Theorem 1.5.1 ***(Martingale Property)*** *Suppose* $L^2_{\mathcal{F}}(0, T, \mathbb{R})$ *Then the stochastic process*

$$X_t = \int_0^t f(s)\, dB_s,\ 0 \leq t \leq T,$$

Is a martingale with respect to the filtration $\{\mathcal{F}_t,\ 0 \leq t \leq T\}$.

1.6 Itô Processes

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be the canonical probability space and let $\mathcal{F}_t$ be the σ-algebra generated by the Brownian motion $B_t(\omega)$.

Definition 1.6.1 *(Itô Processes): $X_t(\omega)$ is an Itô processes if there exist stochastic processes $f(t,\omega)$ and $\sigma(t,\omega)$ such that*

1. *$f(t,\omega)$ and $\sigma(t,\omega)$ are $\mathcal{F}_t$-measurable ,*
2. *$\int_0^t |f|\, ds < \infty$ and $\int_0^t |\sigma|^2 \, ds < \infty$ almost surely,*
3. *$X_0(\omega)$ is $\mathcal{F}_0$-measurable ,*
4. *With probability one the following holds*

$$X_t(\omega) = \int_0^t f_s(\omega)\, ds + \int_0^t \sigma_s(\omega)\, dB_s(\omega).$$

The processes $f(t,\omega)$ and $\sigma(t,\omega)$ are referred to as drift and diffusion coefficients of X_t. For brevity, one often writes as

$$dX_t(\omega) = f_t(\omega)\, dt + \sigma_t(\omega)\, dB_t(\omega).$$

But this is just notation for the integral equation above.

Integral with respect to an Itô process

Let X be an Itô process with decomposition

$$dX_t = b_t dt + \sigma_t dB_t.$$

The intgral is defined as :

$$\int_0^t \theta_s dX_s \overset{def}{=} \int_0^t \theta_s b_s ds + \int_0^t \theta_s \sigma_s dB_s.$$

Definition 1.6.2 *A (generalized) Itô process is an adapted and continuous process on $[0,T]$ of the form*

$$X_t = X_0 + \int_0^t \psi_s ds + \int_0^t \phi_s dB_s,$$

Where $\psi \in \mathcal{H}^2_{loc}(\Omega \times [0,T])$, $\phi \in \mathcal{H}^1_{loc}(\Omega \times [0,T])$ *and* X_0 *is* $\mathcal{F}_0^B$ *measurable. We often adopt the following differential notation*

$$dX_t = \psi_s ds + \phi_s dB_s.$$

Itô's formula

Itô's formula is a basic tool of stochastic calculus. We first demonstrate this formula and then present several applications.

If $x : \mathbb{R} \to \mathbb{R}$ is a C^1 function and if $F : \mathbb{R} \to \mathbb{R}$ is also C^1, then

$$F(x(t)) = F(x(0)) + \int_0^t F'(x(s)) x'(s) ds.$$

More generally, if

$$x = (x_1, ..., x_d) : \mathbb{R} \to \mathbb{R}^d,$$

Is C^1, then

$$F(x(t)) = F(x(0)) + \sum_{i=1}^{d} \int_0^t \frac{\partial F}{\partial x_i}(x(s)) x_i'(s) ds.$$

Theorem 1.6.1 ***(Itô's formula)***

1. ***(One-dimensional case).*** *Let* X *be a continuous semimartingale and let* $F : \mathbb{R} \to \mathbb{R}$ *be a function of class* C^2*. Almost surely, for all* $t \geq 0$,

$$F(X_t) = F(X_0) + \int_0^t F'(X_s) dX_s + \frac{1}{2}\int_0^t F''(X_s) d\langle X \rangle_s.$$

2. ***(Multidimensional case).*** *Let* $X = (X^1, ..., X^d)$ *be continuos semi martingales and let* $F : \mathbb{R}^d \to \mathbb{R}$ *be a function of class* C^2*. Almost surely, for all* $t \geq 0$,

$$F(X_t) = F(X_0) + \sum_{i=1}^{d} \int_0^t \frac{\partial F}{\partial x_i}(X_s)\, dX_s^i + \frac{1}{2} \sum_{i,j=1}^{d} \int_0^t \frac{\partial^2 F}{\partial x_i \partial x_j}(X_s)\, d\left\langle X^i, X^j \right\rangle_s.$$

Where $X_t = \left(X_t^1, ..., X_t^d\right), \forall t \geq 0.$

1.7 Control Classes

Let $(\Omega, \mathcal{F}, \mathcal{F}_{t\geq 0}, \mathbb{P})$ be a complete filtred probability space.

1. **Admissible Control:** An admissible control is $\mathcal{F}_t-$adapted process u_t with values in a borelian $A \subset \mathbb{R}^n$

$$\mathcal{U} := \{u_. : [0,T] \times \Omega \to A : u_t \text{ is } \mathcal{F}_t - \text{adapted}\}.$$

2. **Optimal Control:** The optimal control problem consists to minimize a cost functional $J(u)$ over the set of admissible control $\mathcal{U}$. We say that the control $u_.^*$ is an optimal control if

$$J(u_t^*) \leq J(u_t), \text{ for all } u_. \in \mathcal{U}.$$

3. **Near-Optimal Control:** Let $\varepsilon > 0$, a control u^ε is a near-optimal control (or $\varepsilon-$optimal) if for all control $u \in \mathcal{U}$ we have

$$J(u_t^\varepsilon) \leq J(u_t) + \varepsilon, \text{ for all } u_. \in \mathcal{U}.$$

4. **Feedback Control:** We say that u is a feedback control if u depends on the state variable X.

   If $\mathcal{F}_t^X$ the natural filtration generated by the process X, then u is a feedback control if u is $\mathcal{F}_t^X-$adapted.

5. **Relaxed Control:** The idea is then to compactify the space of controls $\mathcal{U}$ by extending the defnition of controls to include the space of probability measures on U. The set of relaxed controls $\mu_t(du)\, dt$, where μ_t is a probability measure, is the closure under weak* topology of the measures $\delta_{u_t}(du)\, dt$ corresponding to usual, or strict, controls.

1.8 Preliminary Theorems and Inequalities

1.8.1 Probability spaces

Let Ω be a nonempty set and $\mathcal{F}$ be a collection of subsets of Ω.

Definition 1.8.1 ***The sample space*** Ω *of an experiment is the set of all possible outcomes.*

Definition 1.8.2 *We say that $\mathcal{F}$ is $\sigma-$algebra or $\sigma-$field if*

1. $\Omega \in \mathcal{F}$,
2. $A \in \mathcal{F} \Longrightarrow A^c \in \mathcal{F}$,
3. $(A_n)_{n\in\mathbb{N}} \subset \mathcal{F} \Longrightarrow \bigcup_{n\in\mathbb{N}} A_n \in \mathcal{F}$,

If $\mathcal{F}$ and $\mathcal{G}$ are both $\sigma-$fields on Ω and $\mathcal{G} \subseteq \mathcal{F}$, then $\mathcal{G}$ is called a sub $\sigma-$ field of $\mathcal{F}$.

The pair $(\Omega, \mathcal{F})$ is called a measurable space.

Example 1.8.1 *The following are always $\sigma-$fields*

1. $\mathcal{F}_0 = \{\phi, \Omega\}$ *(trivial $\sigma-$field),*
2. $\mathcal{P}(\Omega) = \{$*all subsets of* $\Omega\}$ *(complete $\sigma-$field).*

Let $\{\mathcal{F}_n\}$ be a family of $\sigma-$fields on Ω. Define

$$\bigvee_n \mathcal{F}_n = \sigma\left(\bigcup_n \mathcal{F}_n\right),$$

$$\bigwedge_n \mathcal{F}_n = \sigma\left(\bigcap_n \mathcal{F}_n\right).$$

It is easy to show that $\bigvee_n \mathcal{F}_n$ and $\bigwedge_n \mathcal{F}_n$ are both $\sigma-$fields and that they are the smallest $\sigma-$field containing all $\mathcal{F}_n$ and the largest $\sigma-$field contained in all $\mathcal{F}_n$ respectively.

Definition 1.8.3 *A* ***probability measure*** $\mathbb{P}$ *on* $(\Omega, \mathcal{F})$ *is a function*

$$\mathbb{P} : \mathcal{F} \to [0;1],$$

With the properties

1. $\mathbb{P}(\phi) = 0$ *and* $\mathbb{P}(\Omega) = 1$,

2. Let the family $(A_n)_{n\in\mathbb{N}} \subset \mathcal{F}$ *is disjoint (* $A_i \cap A_j = \phi$ *if* $i \neq j$*), then*

$$\mathbb{P}\left(\bigcup_{n\in\mathbb{N}} A_n\right) = \sum_{n\geq 0} \mathbb{P}(A_n).$$

The triple $(\Omega, \mathcal{F}, \mathbb{P})$ is called a probability space. The subsets A of Ω which belong to $\mathcal{F}$ are called $\mathcal{F}-$measurable sets. In a probability context these sets are called events and we use the interpretation

$$\mathbb{P}(A) = \text{"the probability that the event } A \text{ occurs"}.$$

In particular, if $\mathbb{P}(A) = 1$ we say that A occurs with probability 1, or "almost surely (a.s.)".

Definition 1.8.4 *A family of events* $\{A_i;\ i \in I\}$ *is independent if*

$$\mathbb{P}(\bigcap_{i\in J} A_i) = \prod_{i\in J} \mathbb{P}(A_i),$$

For all finite subsets J *of* I*.*

Definition 1.8.5 *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space*

An event A *is said to be independent of a* $\sigma-$*field* $\mathcal{F}$ *if* A *is independent of any* $B \in \mathcal{F}$*.*

Two $\sigma-$algebras $\mathcal{F}_1$ and $\mathcal{F}_2$ are said to be independent if any event $A \in \mathcal{F}_1$ is independent of $\mathcal{F}_2$.

Definition 1.8.6 *An event* A *is a* $\mathbb{P}-$*null event if* $\mathbb{P}(A) = 0$.

A probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be complete if for any $\mathbb{P}-$null set $A \in \mathcal{F}$, one has $B \in \mathcal{F}$ whenever $A \subseteq B$ (thus, it is necessary that B is also a P-null set).

For any given probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we define

$$\mathcal{N} = \{B \subset \Omega /\ \exists A \in \mathcal{F},\ \mathbb{P}(A) = 0 \text{ and } A \subseteq B\}.$$

1.8.2 Types of convergence

The space $(\Omega, \mathcal{F}, \mathbb{P})$ is fixed. Random variables are defined on this space. We distinguish several types of convergence:

Almost sure convergence

A sequence of random variables X_n converges almost surely a.s. to X if for almost every ω,

$$X_n(\omega) \to X(\omega), \qquad as\ n \to \infty.$$

This is denoted as:

$$X_n \overset{a.s.}{\to} X.$$

This notion of convergence depends on the choice of the probability $\mathbb{P}$. If the probability Q is equivalent to $\mathbb{P}$ such that

$$X_n \overset{\mathbb{P}.a.s.}{\to} X,$$

then we have

$$X_n \overset{Q.a.s.}{\to} X.$$

Convergence in Probability

A sequence of random variable X_n converges in probability to X if:

$$\forall \epsilon > 0, \quad \mathbb{P}(|X_n - X| \geq \epsilon) \to 0, \ \text{ as } \ n \to \infty.$$

Remark 1.8.1 *We remark that*

Almost sure (a.s.) convergence implies convergence in probability.

Convergence in probability implies that there exists a subsequence which converges almost surely.

Convergence in Distribution

A sequence of random variable X_n converges in distribution to X if:

$$\mathbb{E}[\Phi(X_n)] \to \mathbb{E}[\Phi(X)], \ \text{ as } \ n \to \infty.$$

For every function Φ that is continuous and bounded.

The convergence in distribution is denoted by

$$X_n \overset{d}{\to} X.$$

The convergence in distribution is also defined by the simple convergence of the characteristic function, i.e. $\Psi_n(t) \to \Psi(t)$ for all t, where Ψ_n is the characteristic function of X_n and Ψ that of X.

Remark 1.8.2 *1. If X is a continuous random variable with distribution function F, and if X_n is a sequence of random variable with distribution function F_n such that $F_n(x)$ converges to $F(x)$ for all x, then X_n converges in distribution to X and conversely.*

2. Convergence in probability implies convergence in distribution.

1.8.3 Strong and Weak Convergence

Definition 1.8.7 *A sequence (x_n) of element in a pre-Hilbert space H **converges strongly** to x in H (denoted $x_n \to x$) if it converges in norm in H.*

Definition 1.8.8 *A sequence (x_n) of element in a pre-Hilbert space H **converges weakly** to x in H*

(denoted $x_n \xrightarrow{w} x$) if:

$$\lim_{n\to\infty} (x_n, y) = (x, y) \qquad \forall y \in H.$$

One may ask what is the relationship between these two types of convergence. We have the following result

Theorem 1.8.1 *Strong convergence implies weak convergence.*

1.8.4 Mazur's Theorem

If the sequence (x_n) converges weakly to x, then there exists a sequence of convex combination c_n such that:

$$c_n = \sum_{i\geq 0} \alpha_{in} x^i, \quad \text{where } \alpha_i \geq 0 \text{ and } \sum_{i\geq 0} \alpha_i = 1,$$

Which converges strongly to x :

$$\|c_n - x\| \to 0.$$

1.8.5 Fubini's theorem

Let $f : \mathbb{R}^{d_1} \times \mathbb{R}^{d_2} \to \mathbb{R}$ be a measurable function. If f is integrable over $\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}$, i.e.,

$$\int_{\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}} |f(x,y)|\, dxdy < \infty,$$

$$\int_{\mathbb{R}^{d_1} \times \mathbb{R}^{d_2}} f(x,y)\, dxdy = \int_{\mathbb{R}^{d_2}} \left(\int_{\mathbb{R}^{d_1}} f(x,y)\, dx \right) dy = \int_{\mathbb{R}^{d_1}} \left(\int_{\mathbb{R}^{d_2}} f(x,y)\, dy \right) dx$$

1.8.6 Burkholder-Davis-Gundy Inequality (BDG)

Proposition 1.8.1 *Let $p > 0$ be a real number. There exist two constants c_p and C_p such that, for every continuonus local martingale X, null at zero, we have*

$$c_p \mathbb{E}\left[\langle X, X \rangle_\infty^{\frac{p}{2}} \right] \leq \mathbb{E}\left[\sup_{t \geq 0} |X_t^p| \right] \leq C_p \mathbb{E}\left[\langle X, X \rangle_\infty^{\frac{p}{2}} \right].$$

Remark 1.8.3 *In particular, if $T > 0$,*

$$c_p \mathbb{E}\left[\langle X, X \rangle_T^{\frac{p}{2}} \right] \leq \mathbb{E}\left[\sup_{0 \leq t \leq T} |X_t^p| \right] \leq C_p \mathbb{E}\left[\langle X, X \rangle_T^{\frac{p}{2}} \right].$$

1.8.7 Gronwall's Inequality

Lemma 1.8.1 *If g is a continuous function such that,*

$$\forall t \geq 0 \qquad g(t) \leq a_1 + a_2 \int_0^t g(s)\, ds, \text{ with } a_2 \geq 0,$$

Then

$$g(t) \leq a_1 (1 - \exp(a_2 t)).$$

Chapter 2

Existence of Optimal Controls for Linear Forward-Backward Doubly SDE of Mean-Field

In this chapter establishes the existence of strong optimal solutions of a control problems in which the control systems are governed by linear forward backward doubly stochastic differential equations of mean-field type (MF-FBDSDEs), with real coefficients and non linear functional cost. Under the convexity of the control domain and the cost functions, the existence of strong optimal control, adapted to the initial $\sigma-$algebra is proved.

2.1 Formulation of The Problem and Assumptions

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space. Let $(W_t)_{t\in[0,T]}$ and $(B_t)_{t\in[0,T]}$ be two Brownian motions taking their values in $\mathbb{R}^d$ and $\mathbb{R}^l$ respectively, defined on this space.

Let $\mathcal{N}$ denote the class of $\mathbb{P}-$null sets of $\mathcal{F}$. For each $t \in [0, T]$, we define $\mathcal{F}_t \triangleq \mathcal{F}_t^W \vee \mathcal{F}_{t,T}^B$, where for any process $\{\delta_t\}$, we set $\mathcal{F}_{s,t}^\delta = \sigma(\delta_r - \delta_s; s \leq r \leq t) \vee \mathcal{N}$, $\mathcal{F}_t^\delta = \mathcal{F}_{0,t}^\delta$.

Note that the collection $\{\mathcal{F}_t, t \in [0, T]\}$ is neither increasing nor decreasing, then it does not constitute a classical filtration.

Given δ a square integrable and $\mathcal{F}_T-$measurable process, x a square integrable and $\mathcal{F}_0-$measurable process and for any admissible control u, we consider an optimal control problem driven by the following controlled linear FBDSDE of mean-field type:

$$
\begin{cases}
dX_t = & \left(AX_t + \hat{A}\mathbb{E}\left[X_t\right] + bu_t\right)dt + \left(CX_t + \hat{C}\mathbb{E}\left[X_t\right] + \hat{b}u_t\right)dW_t, \\
dY_t = & -\left(DX_t + \hat{D}\mathbb{E}\left[X_t\right] + EY_t + \hat{E}\mathbb{E}\left[Y_t\right] + FZ_t + \hat{F}\mathbb{E}\left[Z_t\right] + Gu_t\right)dt \\
& -\left(HX_t + \hat{H}\mathbb{E}\left[X_t\right] + KY_t + \hat{K}\mathbb{E}\left[Y_t\right] + MZ_t + \hat{M}\mathbb{E}\left[Z_t\right] + \hat{G}u_t\right)\overleftarrow{dB}_t + Z_t dW_t, \\
& X_0 = x, Y_T = \xi,
\end{cases}
\tag{2.1}
$$

With

$$
\begin{aligned}
b\left(t, X_t, \mathbb{E}\left[X_t\right], u_t\right) &= AX_t + \hat{A}\mathbb{E}\left[X_t\right] + bu_t, \\
\sigma\left(t, X_t, \mathbb{E}\left[X_t\right], u_t\right) &= CX_t + \hat{C}\mathbb{E}\left[X_t\right] + \hat{b}u_t, \\
f\left(t, X_t, \mathbb{E}\left[X_t\right], Y_t, \mathbb{E}\left[Y_t\right], Z_t, \mathbb{E}\left[Z_t\right], u_t\right) &= DX_t + \hat{D}\mathbb{E}\left[X_t\right] + EY_t + \hat{E}\mathbb{E}\left[Y_t\right] \\
&\quad + FZ_t + \hat{F}\mathbb{E}\left[Z_t\right] + Gu_t, \\
g\left(t, X_t, \mathbb{E}\left[X_t\right], Y_t, \mathbb{E}\left[Y_t\right], Z_t, \mathbb{E}\left[Z_t\right], u_t\right) &= HX_t + \hat{H}\mathbb{E}\left[X_t\right] + KY_t + \hat{K}\mathbb{E}\left[Y_t\right] \\
&\quad + MZ_t + \hat{M}\mathbb{E}\left[Z_t\right] + \hat{G}u_t, \\
h\left(X_T, \mathbb{E}\left[X_T\right]\right) &= \xi,
\end{aligned}
$$

And a cost functional:

$$
\mathbb{J}\left(u_{.}\right) := \mathbb{E}\left[\alpha\left(X_T, \mathbb{E}\left[X_T\right]\right) + \beta\left(Y_0, \mathbb{E}\left[Y_0\right]\right) + \int_0^T \ell\left(t, X_t, \mathbb{E}\left[X_t\right], Y_t, \mathbb{E}\left[Y_t\right], Z_t, \mathbb{E}\left[Z_t\right], u_t\right)dt\right], \tag{2.2}
$$

Where $A, \hat{A}, b, \hat{b}, C, \hat{C}, D, \hat{D}, E, \hat{E}, F, \hat{F}, G, \hat{G}, H, \hat{H}, K, \hat{K}, M$ and $\hat{M}$ are real-valued matrices of suitable sizes. The solution (X, Y, Z) takes values in $\mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m+d}$ and $u_{.}$ is the control variable values in subset U of $\mathbb{R}^k$. α, β, ℓ are a given functions define by

$$\ell : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times \mathbb{R}^{m\times d} \times U \to \mathbb{R},$$
$$\alpha : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n,$$
$$\beta : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^m.$$

Definition 2.1.1 *An admissible control $u_.$ is a square integrable, $\mathcal{F}_t-$measurable process with values in some subset $U \subseteq \mathbb{R}^k$. We denote by $\mathcal{U}_L$ the set of all admissible controls.*

Note that we have an additional constraint that a control must be square integrable just to ensure the existence of solutions of 2.1 under $u_.$. We say that an admissible control $(u^*_.) \in \mathcal{U}_L$ is an optimal control if

$$\mathbb{J}(u^*_.) = \inf_{v_. \in \mathcal{U}_L} \mathbb{J}(v_.). \tag{2.3}$$

The following notations are needed.

$\mathcal{S}^2_{\mathcal{F}}(0,T;\mathbb{R}^m)$: The set of process $\pi_.$, $\mathcal{F}_t-$adapted with values in $\mathbb{R}^m$ such that

Definition 2.1.2

$$\mathbb{E}\left[\int_0^T |\pi_t|^2 \, dt\right] < \infty,$$

$\mathcal{M}^2_{\mathcal{F}}(0,T;\mathbb{R}^n)$: The set of process $\eta_.$, $\mathcal{F}_t-$adapted and $\mathbb{R}^n-$valued continuous processes such that

$$\mathbb{E}\left[\sup_{0\le t\le T} |\eta_t|^2\right] < \infty,$$

$$\mathcal{U}_L \triangleq \left\{ v_. \in \mathcal{S}^2_{\mathcal{F}}(0,T;\mathbb{R}^n) / v_t \in U,\ a.e.t \in [0,T], \mathbb{P}-a.s. \right\}.$$

We shall consider in this chapter, the following assumptions

(**H1**): The set $U \subseteq \mathbb{R}^k$ is convex and compact and the functions ℓ, α and β are continuous, bounded and convex,

(**H2**): $A, \hat{A}, b, \hat{b}, C, \hat{C}, D, \hat{D}, E, \hat{E}, F, \hat{F}, G, \hat{G}, H, \hat{H}, K$ and $\hat{K}$ are positive real numbers and, $M, \hat{M} \in \left]0, \frac{1}{2}\right[$.

Proposition 2.1.1 *Under assumptions (**H1**)–(**H2**) the system of linear FBDSDE of mean-field type 2.1, has a unique strong solution.*

Proof. The proof of this proposition is established in Zhu and Shi [34], by using a method of continuation, and the fact that our system 2.1 is a special case of the one given in [34]. ■

Remark 2.1.1 *A special case is that in which both α, β are ℓ are convex quadratic functions. The control problem (2.1, 2.2, 2.3) is then reduced to a stochastic linear quadratic optimal control problem.*

2.2 Existence of Optimal Control

In this section, we prove the existence of a strong strict optimal control which is adapted to the initial $\sigma-$algebra, under the convexity of the cost functions and the action space U.

Theorem 2.2.1 *Under either (**H1**)–(**H2**), if the control problem (2.1, 2.2, 2.3) is finite, then it admits an optimal solution.*

Proof. *Assume that (**H1**)–(**H2**) holds. Let $(u^n_\cdot)$ be a minimizing sequence, that is*

$$\lim_{n\to\infty} \mathbb{J}\left(u^n_\cdot\right) = \inf_{v_\cdot\in\mathcal{U}_L} \mathbb{J}\left(v_\cdot\right).$$

With associated trajectories (X^n_t, Y^n_t, Z^n_t) satisfies the linear FBDSDE of mean-field type 2.1 i.e.,

$$\begin{cases} dX^n_t = & \left(AX^n_t + \hat{A}\mathbb{E}\left[X^n_t\right] + bu^n_t\right)dt + \left(CX^n_t + \hat{C}\mathbb{E}\left[X^n_t\right] + \hat{b}u^n_t\right)dW_t, \\ dY^n_t = & -\left(DX^n_t + \hat{D}\mathbb{E}\left[X^n_t\right] + EY^n_t + \hat{E}\mathbb{E}\left[Y^n_t\right] + FZ^n_t + \hat{F}\mathbb{E}\left[Z^n_t\right] + Gu^n_t\right)dt \\ & -\left(HX^n_t + \hat{H}\mathbb{E}\left[X^n_t\right] + KY^n_t + \hat{K}\mathbb{E}\left[Y^n_t\right] + MZ^n_t + \hat{M}\mathbb{E}\left[Z^n_t\right] + \hat{G}u^n_t\right)\overleftarrow{dB}_t + Z^n_t dW_t, \\ & X^n_0 = x, Y^n_T = \xi, \end{cases}$$

From the fact that U is a compact set, the sequence $(u^n_\cdot)_{n\geq 0}$ is relatively compact.

Thus, there exists a subsequence (which is still labeled by $(u^n_\cdot)_{n\geq 0}$) such that

$$u^n_\cdot \to \bar{u}_\cdot, \text{ weakly in } \mathcal{S}^2\left([0,T];\mathbb{R}^k\right).$$

Applying Mazur's theorem, there is a sequence of convex combinations defined by

$$\widetilde{U}_{\cdot}^{n} = \sum_{\mathcal{J}\geq 0} \theta_{\mathcal{J}n} u_{\cdot}^{\mathcal{J}+n} \ (\textit{with } \theta_{\mathcal{J}n} \geq 0, \ \textit{and} \sum_{\mathcal{J}\geq 0} \theta_{\mathcal{J}n} = 1),$$

Such that

$$\widetilde{U}_{\cdot}^{n} \to \bar{u}_{\cdot} \textit{strongly in } \mathcal{S}^2\left([0,T];\mathbb{R}^k\right). \tag{2.4}$$

Since the set $U \subseteq \mathbb{R}^k$ is convex and compact, it follows that $\bar{u}_{\cdot} \in \mathcal{U}_L$.
Let $\left(\widetilde{X}_{\cdot}^{n}, \widetilde{Y}_{\cdot}^{n}, \widetilde{Z}_{\cdot}^{n}\right)$ and $\left(\bar{X}_{\cdot}, \bar{Y}_{\cdot}, \bar{Z}_{\cdot}\right)$ be the solutions of the linear MF-FBDSDE 2.1, associated with $\widetilde{U}_{\cdot}^{n}$ and $\bar{u}_{\cdot}$ respectively i.e.,

$$\begin{cases} d\widetilde{X}_t^n = & (A\widetilde{X}_t^n + \hat{A}\mathbb{E}\left[\widetilde{X}_t^n\right] + b\widetilde{U}_t^n)dt + (C\widetilde{X}_t^n + \hat{C}\mathbb{E}\left[\widetilde{X}_t^n\right] + \hat{b}\widetilde{U}_t^n)dW_t \\ d\widetilde{Y}_t^n = & -\left(D\widetilde{X}_t^n + \hat{D}\mathbb{E}\left[\widetilde{X}_t^n\right] + E\widetilde{Y}_t^n + \hat{E}\mathbb{E}\left[\widetilde{Y}_t^n\right] + F\widetilde{Z}_t^n + \hat{F}\mathbb{E}\left[\widetilde{Z}_t^n\right] + G\widetilde{U}_t^n\right)dt \\ & -\left(H\widetilde{X}_t^n + \hat{H}\mathbb{E}\left[C\widetilde{X}_t^n\right] + K\widetilde{Y}_t^n + \hat{K}\mathbb{E}\left[\widetilde{Y}_t^n\right] + M\widetilde{Z}_t^n + \hat{M}\mathbb{E}\left[\widetilde{Z}_t^n\right] + \hat{G}\widetilde{U}_t^n\right)\overleftarrow{dB}_t \\ & +\widetilde{Z}_t^n dW_t, \\ \widetilde{X}_0^n = x, \widetilde{Y}_T^n = \xi, \end{cases} \tag{2.5}$$

And

$$\begin{cases} d\bar{X}_t = & (A\bar{X}_t + \hat{A}\mathbb{E}\left[\bar{X}_t\right] + b\bar{u}_t)dt + (C\bar{X}_t + \hat{C}\mathbb{E}\left[\bar{X}_t\right] + \hat{b}\bar{u}_t)dW_t \\ d\bar{Y}_t = & -\left(D\bar{X}_t + \hat{D}\mathbb{E}\left[\bar{X}_t\right] + E\bar{Y}_t + \hat{E}\mathbb{E}\left[\bar{Y}_t\right] + F\bar{Z}_t + \hat{F}\mathbb{E}\left[\bar{Z}_t\right] + G\bar{u}_t\right)dt \\ & -\left(H\bar{X}_t + \hat{H}\mathbb{E}\left[\bar{X}_t\right] + K\bar{Y}_t + \hat{K}\mathbb{E}\left[\bar{Y}_t\right] + M\bar{Z}_t + \hat{M}\mathbb{E}\left[\bar{Z}_t\right] + \hat{G}\bar{u}_t\right)\overleftarrow{dB}_t \\ & +\bar{Y}_t dW_t, \\ \bar{X}_0 = x, \bar{Z}_T = \xi. \end{cases} \tag{2.6}$$

Then let us prove

$$\left(\widetilde{X}_t^n, \widetilde{Y}_t^n, \int_0^T \widetilde{Z}_s^n dW_s\right) \textit{ converges strongly to } \left(\bar{X}_t, \bar{Y}_t, \int_0^T \bar{Z}_s dW_s\right), \tag{2.7}$$

In $\mathcal{M}^2\left([0,T];\mathbb{R}^{n+m}\right) \times \mathcal{S}^2\left([0,T];\mathbb{R}^{m+d}\right)$.

Firstly, we have

$$\begin{aligned}\left|\widetilde{X}_t^n-\bar{X}_t\right|^2 \leq\ & 2\left|\int_0^t\left(A\left(\widetilde{X}_s^n-\bar{X}_s\right)+\hat{A}\left(\mathbb{E}\left[\widetilde{X}_s^n\right]-\mathbb{E}\left[\bar{X}_s\right]\right)+b\left(\widetilde{U}_s^n-\bar{u}_s\right)\right)ds\right|^2 \\ & +2\left|\int_0^t\left(C\left(\widetilde{X}_s^n-\bar{X}_s\right)+\hat{C}\left(\mathbb{E}\left[\widetilde{X}_s^n\right]-\mathbb{E}\left[\bar{X}_s\right]\right)+\hat{b}\left(\widetilde{U}_s^n-\bar{u}_s\right)\right)dW_s\right|^2,\end{aligned}$$

Which gives

$$\begin{aligned}&\left(\sup_{0\leq s\leq t}\left|\widetilde{X}_s^n-\bar{X}_s\right|^2\right) \\ &\leq\int_0^t\left(A^2\left(\sup_{0\leq r\leq s}\left|\widetilde{X}_r^n-\bar{X}_r\right|^2\right)+\hat{A}^2\left(\sup_{0\leq r\leq s}\left|\mathbb{E}\left[\widetilde{X}_r^n\right]-\mathbb{E}\left[\bar{X}_r\right]\right|^2\right)\right. \\ &\left.+b^2\left|\widetilde{U}_s^n-\bar{u}_s\right|^2\right)ds+\sup_{0\leq s\leq t}\left(\left|\int_0^t\left(C\left(\widetilde{X}_s^n-\bar{X}_s\right)+\hat{C}\left(\mathbb{E}\left[\widetilde{X}_s^n\right]-\mathbb{E}\left[\bar{X}_s\right]\right)\right.\right.\right. \\ &\left.\left.\left.+\hat{b}\left(\widetilde{U}_s^n-\bar{u}_s\right)\right)dW_s\right|\right)^2,\end{aligned}$$

Passing to the expectation and using fubini's theorem, we get

$$\begin{aligned}&\mathbb{E}\left[\sup_{0\leq s\leq T}\left|\widetilde{X}_s^n-\bar{X}_s\right|^2\right] \\ &\leq\int_0^t\left(A^2\mathbb{E}\left[\sup_{0\leq r\leq s}\left|\widetilde{X}_r^n-\bar{X}_r\right|^2\right]+\hat{A}^2\mathbb{E}\left[\left(\sup_{0\leq r\leq s}\left|\mathbb{E}\left[\widetilde{X}_r^n\right]-\mathbb{E}\left[\bar{X}_r\right]\right|\right)^2\right]\right. \\ &\left.+b^2\mathbb{E}\left[\left|\widetilde{U}_s^n-\bar{u}_s\right|^2\right]\right)ds+\mathbb{E}\left[\sup_{0\leq s\leq t}\left(\left|\int_0^t\left(C\left(\widetilde{X}_s^n-\bar{X}_s\right)+\hat{C}\left(\mathbb{E}\left[\widetilde{X}_s^n\right]-\mathbb{E}\left[\bar{X}_s\right]\right)\right.\right.\right.\right. \\ &\left.\left.\left.\left.+\hat{b}\left(\widetilde{U}_s^n-\bar{u}_s\right)\right)dW_s\right|\right)^2\right],\end{aligned}$$

Using the Burkholder-Davis-Gundy inequality to the martingale part, we get

$$\mathbb{E}\left[\sup_{0\leq s\leq T}\left|\widetilde{X}_s^n-\bar{X}_s\right|^2\right]\leq k\int_0^t\mathbb{E}\left[\sup_{0\leq r\leq s}\left|\widetilde{X}_r^n-\bar{X}_r\right|^2\right]ds+k'\mathbb{E}\left[\int_0^t\left|\widetilde{U}_s^n-\bar{u}_s\right|^2ds\right].$$

Applying Gronwall's lemma, we get

$$\mathbb{E}\left[\sup_{0\leq s\leq T}\left|\widetilde{X}_s^n-\bar{X}_s\right|^2\right]\leq k'\mathbb{E}\left[\int_0^t\left|\widetilde{U}_s^n-\bar{u}_s\right|^2ds\right]+k\int_0^t\mathbb{E}\left[\sup_{0\leq r\leq s}\left|\widetilde{X}_r^n-\bar{X}_r\right|^2\right]ds,$$
$$\leq k'\mathbb{E}\left[\int_0^t\left|\widetilde{U}_s^n-\bar{u}_s\right|^2ds\right]\exp\left(\int_0^t kds\right),$$
$$\leq k'e^{kt}\mathbb{E}\left[\int_0^t\left|\widetilde{U}_s^n-\bar{u}_s\right|^2ds\right].$$

With

$$g(s)=\mathbb{E}\left[\sup_{0\leq s\leq T}\left|\widetilde{X}_s^n-\bar{X}_s\right|^2\right], a_1=k'\mathbb{E}\left[\int_0^t\left|\widetilde{U}_s^n-\bar{u}_s\right|^2ds\right], a_2=k.$$

And the fast that $\left(\widetilde{U}_\cdot^n\right)$ *converges strongly to* $\bar{u}_\cdot$ *in* $\mathcal{S}^2\left([0,T];\mathbb{R}^k\right)$ *(from 2.4), we get*

$$\lim_{n\to\infty}\mathbb{E}\left[\sup_{0\leq s\leq T}\left|\widetilde{X}_s^n-\bar{X}_s\right|^2\right]=0. \tag{2.8}$$

Secondly, we have

$$\begin{aligned}
&d\left|\widetilde{Y}_t^n-\bar{Y}_t\right|\\
&=-\left[D\left(\widetilde{X}_t^n-\bar{X}_t\right)+\hat{D}\left(\mathbb{E}\left[\widetilde{X}_t^n\right]-\mathbb{E}\left[\bar{X}_t\right]\right)+E\left(\widetilde{Y}_t^n-\bar{Y}_t\right)+\hat{E}\left(\mathbb{E}\left[\widetilde{Y}_t^n\right]-\mathbb{E}\left[\bar{Y}_t\right]\right)\right.\\
&\left.+F\left(\widetilde{Z}_t^n-\bar{Z}_t\right)+\hat{F}\left(\mathbb{E}\left[\widetilde{Z}_t^n\right]-\mathbb{E}\left[\bar{Z}_t\right]\right)+G\left(\widetilde{U}_t^n-\bar{u}_t\right)\right]dt+\left(\widetilde{Z}_t^n-\bar{Z}_t\right)dW_t\\
&-\left[H\left(\widetilde{X}_t^n-\bar{X}_t\right)+\hat{H}\left(\mathbb{E}\left[\widetilde{X}_t^n\right]-\mathbb{E}\left[\bar{X}_t\right]\right)+K\left(\widetilde{Y}_t^n-\bar{Y}_t\right)+\hat{K}\left(\mathbb{E}\left[\widetilde{Y}_t^n\right]-\mathbb{E}\left[\bar{Y}_t\right]\right)\right.\\
&\left.+M\left(\widetilde{Z}_t^n-\bar{Z}_t\right)+\hat{M}\left(\mathbb{E}\left[\widetilde{Z}_t^n\right]-\mathbb{E}\left[\bar{Z}_t\right]\right)+\hat{G}\left(\widetilde{U}_t^n-\bar{u}_t\right)\overleftarrow{dB}_t\right],
\end{aligned}$$

Applying Itô's formula (integration by part) to $\left|\widetilde{Y}_t^n-\bar{Y}_t\right|^2$, *we get*

$$
\begin{aligned}
& d\left|\widetilde{Y}_t^n-\bar{Y}_t\right|^2 \\
& =2\left|\widetilde{Y}_t^n-\bar{Y}_t\right| d\left|\widetilde{Y}_t^n-\bar{Y}_t\right|+d\left\langle\widetilde{Y}^n-\bar{Y}, \widetilde{Y}^n-\bar{Y}\right\rangle_t \\
& =2\left|\widetilde{Y}_t^n-\bar{Y}_t\right|\left[-\left[D\left(\widetilde{X}_t^n-\bar{X}_t\right)+\hat{D}\left(\mathbb{E}\left[\widetilde{X}_t^n\right]-\mathbb{E}\left[\bar{X}_t\right]\right)\right.\right. \\
& +E\left(\widetilde{Y}_t^n-\bar{Y}_t\right)+\hat{E}\left(\mathbb{E}\left[\widetilde{Y}_t^n\right]-\mathbb{E}\left[\bar{Y}_t\right]\right)+F\left(\widetilde{Z}_t^n-\bar{Z}_t\right) \\
& \left.+\hat{F}\left(\mathbb{E}\left[\widetilde{Z}_t^n\right]-\mathbb{E}\left[\bar{Z}_t\right]\right)+G\left(\widetilde{U}_t^n-\bar{u}_t\right)\right] dt+\left(\widetilde{Z}_t^n-\bar{Z}_t\right) dW_t \\
& +\left[H\left(\widetilde{X}_t^n-\bar{X}_t\right)+\hat{H}\left(\mathbb{E}\left[\widetilde{X}_t^n\right]-\mathbb{E}\left[\bar{X}_t\right]\right)+K\left(\widetilde{Y}_t^n-\bar{Y}_t\right)\right. \\
& +\hat{K}\left(\mathbb{E}\left[\widetilde{Y}_t^n\right]-\mathbb{E}\left[\bar{Y}_t\right]\right)+M\left(\widetilde{Z}_t^n-\bar{Z}_t\right)+\hat{M}\left(\mathbb{E}\left[\widetilde{Z}_t^n\right]-\mathbb{E}\left[\bar{Z}_t\right]\right) \\
& \left.\left.+\hat{G}\left(\widetilde{U}_t^n-\bar{u}_t\right)\right] \overleftarrow{dB}_t\right]+\left\|\widetilde{Z}_t^n-\bar{Z}_t\right\|^2 dt-\left|H\left(\widetilde{X}_t^n-\bar{X}_t\right)\right. \\
& +\hat{H}\left(\mathbb{E}\left[\widetilde{X}_t^n\right]-\mathbb{E}\left[\bar{X}_t\right]\right)+K\left(\widetilde{Y}_t^n-\bar{Y}_t\right)+\hat{K}\left(\mathbb{E}\left[\widetilde{Y}_t^n\right]-\mathbb{E}\left[\bar{Y}_t\right]\right) \\
& \left.+M\left(\widetilde{Z}_t^n-\bar{Z}_t\right)+\hat{M}\left(\mathbb{E}\left[\widetilde{Z}_t^n\right]-\mathbb{E}\left[\bar{Z}_t\right]\right)+\hat{G}\left(\widetilde{U}_t^n-\bar{u}_t\right)\right|^2 dt,
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\int_t^T d\left|\widetilde{Y}_s^n-\bar{Y}_s\right|^2 ds\\
&=-2\int_t^T\left\langle \widetilde{Y}_s^n-\bar{Y}_s, D\left(\widetilde{X}_s^n-\bar{X}_s\right)+\hat{D}\left(\mathbb{E}\left[\widetilde{X}_s^n\right]-\mathbb{E}\left[\bar{X}_s\right]\right)\right.\\
&+E\left(\widetilde{Y}_s^n-\bar{Y}_s\right)+\hat{E}\left(\mathbb{E}\left[\widetilde{Y}_s^n\right]-\mathbb{E}\left[\bar{Y}_s\right]\right)+F\left(\widetilde{Z}_s^n-\bar{Z}_s\right)\\
&\left.+\hat{F}\left(\mathbb{E}\left[\widetilde{Z}_s^n\right]-\mathbb{E}\left[\bar{Z}_s\right]\right)+G\left(\widetilde{U}_s^n-\bar{u}_s\right)\right\rangle ds\\
&+2\int_t^T\left\langle \widetilde{Y}_s^n-\bar{Y}_s,\widetilde{Z}_s^n-\bar{Z}_s\right\rangle dW_s-2\int_t^T\left\langle \widetilde{Y}_s^n-\bar{Y}_s, H\left(\widetilde{X}_s^n-\bar{X}_s\right)\right.\\
&+\hat{H}\left(\mathbb{E}\left[\widetilde{X}_s^n\right]-\mathbb{E}\left[\bar{X}_s\right]\right)+K\left(\widetilde{Y}_s^n-\bar{Y}_s\right)+\hat{K}\left(\mathbb{E}\left[\widetilde{Y}_s^n\right]-\mathbb{E}\left[\bar{Y}_s\right]\right)\\
&\left.+M\left(\widetilde{Z}_s^n-\bar{Z}_s\right)+\hat{M}\left(\mathbb{E}\left[\widetilde{Z}_s^n\right]-\mathbb{E}\left[\bar{Z}_s\right]\right)+\hat{G}\left(\widetilde{U}_s^n-\bar{u}_s\right)\right\rangle \overleftarrow{dB}_s\\
&+\int_t^T\left\|\widetilde{Z}_s^n-\bar{Z}_s\right\|^2 ds-\int_t^T\left|H\left(\widetilde{X}_s^n-\bar{X}_s\right)+\hat{H}\left(\mathbb{E}\left[\widetilde{X}_s^n\right]-\mathbb{E}\left[\bar{X}_s\right]\right)\right.\\
&+K\left(\widetilde{Y}_s^n-\bar{Y}_s\right)+\hat{K}\left(\mathbb{E}\left[\widetilde{Y}_s^n\right]-\mathbb{E}\left[\bar{Y}_s\right]\right)+M\left(\widetilde{Z}_s^n-\bar{Z}_s\right)\\
&\left.+\hat{M}\left(\mathbb{E}\left[\widetilde{Z}_s^n\right]-\mathbb{E}\left[\bar{Z}_s\right]\right)+\hat{G}\left(\widetilde{U}_s^n-\bar{u}_s\right)\right|^2 ds,
\end{aligned}
$$

Which implies that

$$
\begin{aligned}
&\left|\widetilde{Y}_t^n-\bar{Y}_t\right|^2+\int_t^T\left\|\widetilde{Z}_s^n-\bar{Z}_s\right\|^2 ds\\
&=2\int_t^T\left\langle \widetilde{Y}_s^n-\bar{Y}_s, D\left(\widetilde{X}_s^n-\bar{X}_s\right)+\hat{D}\left(\mathbb{E}\left[\widetilde{X}_s^n\right]-\mathbb{E}\left[\bar{X}_s\right]\right)\right.\\
&+E\left(\widetilde{Y}_s^n-\bar{Y}_s\right)+\hat{E}\left(\mathbb{E}\left[\widetilde{Y}_s^n\right]-\mathbb{E}\left[\bar{Y}_s\right]\right)+F\left(\widetilde{Z}_s^n-\bar{Z}_s\right)\\
&\left.+\hat{F}\left(\mathbb{E}\left[\widetilde{Z}_s^n\right]-\mathbb{E}\left[\bar{Z}_s\right]\right)+G\left(\widetilde{U}_s^n-\bar{u}_s\right)\right\rangle ds\\
&-2\int_t^T\left\langle \widetilde{Y}_s^n-\bar{Y}_s,\widetilde{Z}_s^n-\bar{Z}_s\right\rangle dW_s+2\int_t^T\left\langle \widetilde{Y}_s^n-\bar{Y}_s, H\left(\widetilde{X}_s^n-\bar{X}_s\right)\right.\\
&+\hat{H}\left(\mathbb{E}\left[\widetilde{X}_s^n\right]-\mathbb{E}\left[\bar{X}_s\right]\right)+K\left(\widetilde{Y}_s^n-\bar{Y}_s\right)+\hat{K}\left(\mathbb{E}\left[\widetilde{Y}_s^n\right]-\mathbb{E}\left[\bar{Y}_s\right]\right)\\
&\left.+M\left(\widetilde{Z}_s^n-\bar{Z}_s\right)+\hat{M}\left(\mathbb{E}\left[\widetilde{Z}_s^n\right]-\mathbb{E}\left[\bar{Z}_s\right]\right)+\hat{G}\left(\widetilde{U}_s^n-\bar{u}_s\right)\right\rangle \overleftarrow{dB}_s\\
&+\int_t^T\left|H\left(\widetilde{X}_s^n-\bar{X}_s\right)+\hat{H}\left(\mathbb{E}\left[\widetilde{X}_s^n\right]-\mathbb{E}\left[\bar{X}_s\right]\right)\right.\\
&+K\left(\widetilde{Y}_s^n-\bar{Y}_s\right)+\hat{K}\left(\mathbb{E}\left[\widetilde{Y}_s^n\right]-\mathbb{E}\left[\bar{Y}_s\right]\right)+M\left(\widetilde{Z}_s^n-\bar{Z}_s\right)\\
&\left.+\hat{M}\left(\mathbb{E}\left[\widetilde{Z}_s^n\right]-\mathbb{E}\left[\bar{Z}_s\right]\right)+\hat{G}\left(\widetilde{U}_s^n-\bar{u}_s\right)\right|^2 ds.
\end{aligned}
$$

By taking expectation, we get

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\widetilde{Y}_t^n-\bar{Y}_t\right|^2\right]+\mathbb{E}\left[\int_0^T\left\|\widetilde{Z}_s^n-\bar{Z}_s\right\|^2ds\right]$$
$$\leq 2\mathbb{E}\left[\int_0^T\left\langle\widetilde{Y}_t^n-\bar{Y}_t,D\left(\widetilde{X}_s^n-\bar{X}_s\right)+\hat{D}\left(\mathbb{E}\left[\widetilde{X}_s^n\right]-\mathbb{E}\left[\bar{X}_s\right]\right)\right.\right.$$
$$+E\left(\widetilde{Y}_s^n-\bar{Y}_s\right)+\hat{E}\left(\mathbb{E}\left[\widetilde{Y}_s^n\right]-\mathbb{E}\left[\bar{Y}_s\right]\right)+F\left(\widetilde{Z}_s^n-\bar{Z}_s\right)$$
$$\left.\left.+\hat{F}\left(\mathbb{E}\left[\widetilde{Z}_s^n\right]-\mathbb{E}\left[\bar{Z}_s\right]\right)+G\left(\widetilde{U}_s^n-\bar{u}_s\right)\right\rangle ds\right]$$
$$+\mathbb{E}\left[\int_0^T\left|H\left(\widetilde{X}_s^n-\bar{X}_s\right)+\hat{H}\left(\mathbb{E}\left[\widetilde{X}_s^n\right]-\mathbb{E}\left[\bar{X}_s\right]\right)\right.\right|$$
$$+K\left(\widetilde{Y}_s^n-\bar{Y}_s\right)+\hat{K}\left(\mathbb{E}\left[\widetilde{Y}_s^n\right]-\mathbb{E}\left[\bar{Y}_s\right]\right)+M\left(\widetilde{Z}_s^n-\bar{Z}_s\right)$$
$$\left.+\hat{M}\left(\mathbb{E}\left[\widetilde{Z}_s^n\right]-\mathbb{E}\left[\bar{Z}_s\right]\right)+\hat{G}\left(\widetilde{U}_s^n-\bar{u}_s\right)\right|^2ds.$$

*According to the assumption **(H2)** and by using the Young's formula, we obtain*

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\widetilde{Y}_t^n-\bar{Y}_t\right|^2\right]+\mathbb{E}\left[\int_0^T\left\|\widetilde{Z}_s^n-\bar{Z}_s\right\|^2ds\right]\leq\frac{1}{\rho_1}\mathbb{E}\left[\int_0^T\left|\widetilde{Y}_s^n-\bar{Y}_s\right|^2ds\right]$$
$$+7\rho_1\lambda^2\mathbb{E}\left[\int_0^T\left(\left|\widetilde{X}_s^n-\bar{X}_s\right|^2+\left|\mathbb{E}\left[\widetilde{X}_s^n\right]-\mathbb{E}\left[\bar{X}_s\right]\right|^2\right.\right.$$
$$+\left|\widetilde{Y}_s^n-\bar{Y}_s\right|^2+\left|\mathbb{E}\left[\widetilde{Y}_s^n\right]-\mathbb{E}\left[\bar{Y}_s\right]\right|^2+\left\|\widetilde{Z}_s^n-\bar{Z}_s\right\|^2$$
$$\left.\left.+\left\|\mathbb{E}\left[\widetilde{Z}_s^n\right]-\mathbb{E}\left[\bar{Z}_s\right]\right\|^2+\left|\widetilde{U}_s^n-\bar{u}_s\right|^2\right)ds\right]$$
$$+5\lambda^2\mathbb{E}\left[\int_0^T\left(\left|\widetilde{X}_s^n-\bar{X}_s\right|^2+\left|\mathbb{E}\left[\widetilde{X}_s^n\right]-\mathbb{E}\left[\bar{X}_s\right]\right|^2\right.\right.$$
$$\left.\left.+\left|\widetilde{Y}_s^n-\bar{Y}_s\right|^2+\left|\mathbb{E}\left[\widetilde{Y}_s^n\right]-\mathbb{E}\left[\bar{Y}_s\right]\right|^2+\left|\widetilde{U}_s^n-\bar{u}_s\right|^2\right)ds\right]$$
$$+2\gamma^2\mathbb{E}\left[\int_0^T\left(\left\|\widetilde{Z}_s^n-\bar{Z}_s\right\|^2+\left\|\mathbb{E}\left[\widetilde{Z}_s^n\right]-\mathbb{E}\left[\bar{Z}_s\right]\right\|^2\right)ds\right]$$
$$+2\lambda\gamma\mathbb{E}\left[\int_0^T\left\langle\left(\widetilde{X}_s^n-\bar{X}_s\right)+\left(\mathbb{E}\left[\widetilde{X}_s^n\right]-\mathbb{E}\left[\bar{X}_s\right]\right)+\left(\widetilde{Y}_s^n-\bar{Y}_s\right)\right.\right.$$
$$\left.\left.+\left(\mathbb{E}\left[\widetilde{Y}_s^n\right]-\mathbb{E}\left[\bar{Y}_s\right]\right)+\left(\widetilde{U}_s^n-\bar{u}_s\right),\left(\widetilde{Z}_s^n-\bar{Z}_s\right)+\left(\mathbb{E}\left[\widetilde{Z}_s^n\right]-\mathbb{E}\left[\bar{Z}_s\right]\right)\right\rangle ds\right].$$

Where

$$\lambda = \max\left(A, \hat{A}, b, \hat{b}, C, \hat{C}, D, \hat{D}, E, \hat{E}, F, \hat{F}, G, \hat{G}, H, \hat{H}, K, \hat{K}\right),$$
$$\gamma = \max\left(M, \hat{M}\right).$$

Hence

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le t\le T}\left|\tilde{Y}_t^n - \bar{Y}_t\right|^2\right] + \mathbb{E}\left[\int_0^T \left\|\tilde{Z}_s^n - \bar{Z}_s\right\|^2 ds\right] \le \\
&\frac{1}{\rho_1}\mathbb{E}\left[\int_0^T \left|\tilde{Y}_s^n - \bar{Y}_s\right|^2 ds\right] + 14\rho_1\lambda^2\mathbb{E}\left[\int_0^T \left(\left|\tilde{X}_s^n - \bar{X}_s\right|^2 + \left|\tilde{Y}_s^n - \bar{Y}_s\right|^2\right.\right. \\
&\left.\left. + \left\|\tilde{Z}_s^n - \bar{Z}_s\right\|^2 + \frac{1}{2}\left|\tilde{U}_s^n - \bar{u}_s\right|^2\right) ds\right] + 10\lambda^2\mathbb{E}\left[\int_0^T \left(\left|\tilde{X}_s^n - \bar{X}_s\right|^2\right.\right. \\
&\left.\left. + \left|\tilde{Y}_s^n - \bar{Y}_s\right|^2 + \frac{1}{2}\left|\tilde{U}_s^n - \bar{u}_s\right|^2\right) ds\right] + 4\gamma^2\mathbb{E}\left[\int_0^T \left\|\tilde{Z}_s^n - \bar{Z}_s\right\|^2 ds\right] \\
&+ \frac{5\lambda\gamma}{\rho_2}\mathbb{E}\left[\int_0^T \left(\left|\tilde{X}_s^n - \bar{X}_s\right|^2 + \left|\mathbb{E}\left[\tilde{X}_s^n\right] - \mathbb{E}\left[\bar{X}_s\right]\right|^2 + \left|\tilde{Y}_s^n - \bar{Y}_s\right|^2\right.\right. \\
&\left.\left. + \left|\mathbb{E}\left[\tilde{Y}_s^n\right] - \mathbb{E}\left[\bar{Y}_s\right]\right|^2 + \left|\tilde{U}_s^n - \bar{u}_s\right|^2\right) ds\right] \\
&+ 2\rho_2\lambda\gamma\mathbb{E}\left[\int_0^T \left(\left\|\tilde{Z}_s^n - \bar{Z}_s\right\|^2 + \left\|\mathbb{E}\left[\tilde{Z}_s^n\right] - \mathbb{E}\left[\bar{Z}_s\right]\right\|^2\right) ds\right],
\end{aligned}$$

And therefore

$$\begin{aligned}
&\mathbb{E}\left[\sup_{0\le t\le T}\left|\tilde{Y}_t^n - \bar{Y}_t\right|^2\right] + \mathbb{E}\left[\int_0^T \left\|\tilde{Z}_s^n - \bar{Z}_s\right\|^2 ds\right] \le \\
&\quad\left(\frac{1}{\rho_1} + 14\rho_1\lambda^2 + 10\lambda^2 + \frac{10\lambda\gamma}{\rho_2}\right)\mathbb{E}\left[\int_0^T \left|\tilde{Y}_t^n - \bar{Y}_t\right|^2 ds\right] \\
&\quad+ \left(14\rho_1\lambda^2 + 4\gamma^2 + 4\rho_2\lambda\gamma\right)\mathbb{E}\left[\int_0^T \left\|\tilde{Z}_s^n - \bar{Z}_s\right\|^2 ds\right] \\
&\quad+ \left(14\rho_1\lambda^2 + 10\lambda^2 + \frac{10\lambda\gamma}{\rho_2}\right)\mathbb{E}\left[\int_0^T \left|\tilde{X}_t^n - \bar{X}_t\right|^2 ds\right] \\
&\quad+ \left(7\rho_1\lambda^2 + 5\lambda^2 + \frac{5\lambda\gamma}{\rho_2}\right)\mathbb{E}\left[\int_0^T \left|\tilde{U}_s^n - \bar{u}_s\right|^2 ds\right],
\end{aligned}$$

Choosing

$$\rho_1 = \frac{1-4\gamma^2}{28\lambda^2} > 0 \ \textit{and} \ \rho_2 = \frac{1-4\gamma^2}{12\lambda\gamma} > 0 \ \textit{because} \ 0 < \gamma < \frac{1}{2},$$

The previous inequality becomes

$$\begin{aligned} &\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\widetilde{Y}_t^n - \bar{Y}_t\right|^2\right] + \mu_1\mathbb{E}\left[\int_0^T \left\|\widetilde{Z}_s^n - \bar{Z}_s\right\|^2 ds\right] \\ &\leq \mu_2\mathbb{E}\left[\int_0^T \left|\widetilde{Y}_t^n - \bar{Y}_t\right|^2 ds\right] + \mu_3\mathbb{E}\left[\int_0^T \left|\widetilde{X}_s^n - \bar{X}_s\right|^2 ds\right] \\ &+ \mu_4\mathbb{E}\left[\int_0^T \left|\widetilde{U}_s^n - \bar{u}_s\right|^2 ds\right], \end{aligned} \tag{2.9}$$

Where

$$\begin{aligned} \mu_1 &= \frac{1-4\gamma^2}{6} > 0, \\ \mu_2 &= \frac{28\lambda^2}{1-4\gamma^2} + \frac{1-4\gamma^2}{2} + 10\lambda^2 + \frac{120\left(\lambda\gamma\right)^2}{1-4\gamma^2} > 0, \\ \mu_3 &= \frac{1-4\gamma^2}{2} + 10\lambda^2 + \frac{120\left(\lambda\gamma\right)^2}{1-4\gamma^2} > 0, \\ \mu_4 &= \frac{1-4\gamma^2}{4} + 5\lambda^2 + \frac{60\left(\lambda\gamma\right)^2}{1-4\gamma^2} > 0. \end{aligned}$$

We derive two inequalities from 2.9

$$\begin{aligned} \mathbb{E}\left[\sup_{0\leq t\leq T}\left|\widetilde{Y}_t^n - \bar{Y}_t\right|^2\right] &\leq \mu_2\mathbb{E}\left[\int_0^T \left|\widetilde{Y}_t^n - \bar{Y}_t\right|^2 ds\right] \\ &+ \mu_3\mathbb{E}\left[\int_0^T \left|\widetilde{X}_s^n - \bar{X}_s\right|^2 ds\right] + \mu_4\mathbb{E}\left[\int_0^T \left|\widetilde{U}_s^n - \bar{u}_s\right|^2 ds\right], \end{aligned} \tag{2.10}$$

And

$$\mu_1 \mathbb{E}\left[\int_0^T \left\|\widetilde{Z}_s^n - \bar{Z}_s\right\|^2 ds\right] \leq \mu_2 \mathbb{E}\left[\int_0^T \left|\widetilde{Y}_t^n - \bar{Y}_t\right|^2 ds\right] + \mu_3 \mathbb{E}\left[\int_0^T \left|\widetilde{X}_s^n - \bar{X}_s\right|^2 ds\right] + \mu_4 \mathbb{E}\left[\int_0^T \left|\widetilde{U}_s^n - \bar{u}_s\right|^2 ds\right], \tag{2.11}$$

Applying Gronwall's lemma to 2.10 and passing to the limit as $n \to \infty$, and using the convergence 2.4 and 2.8, we obtain

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\left|\widetilde{Y}_t^n - \bar{Y}_t\right|^2\right] = 0. \tag{2.12}$$

Then, one can shows directly from 2.4,2.8 and 2.12 that

$$\mathbb{E}\left[\int_0^T \left\|\widetilde{Z}_s^n - \bar{Z}_s\right\|^2 ds\right] \to 0, \text{ as } n \to \infty,$$

Which gives the result by applying the isometry of Itô.

Finally, let us prove that $\bar{u}_.$ is an optimal control.

From the minimizing sequence, let $(X_.^n, Y_.^n, Z_.^n, u_.^n)$ be such that

$$\begin{aligned}\lim_{n\to\infty} \mathbb{J}(u_.^n) &= \lim_{n\to\infty} \mathbb{E}\left[\alpha\left(X_T^n, \mathbb{E}[X_T^n]\right) + \beta\left(Y_0^n, \mathbb{E}[Y_0^n]\right)\right. \\ &\quad \left.+ \int_0^T \ell\left(t, X_t^n, \mathbb{E}[X_t^n], Y_t^n, \mathbb{E}[Y_t^n], Z_t^n, \mathbb{E}[Z_t^n], u_t^n\right) dt\right] \\ &= \inf_{v_.\in\mathcal{U}_L} \mathbb{J}(v_.).\end{aligned}$$

Using the continuity of function α, β and ℓ, we get

$$\begin{aligned}\mathbb{J}\left(\bar{u}_.\right) &= \mathbb{E}\left[\alpha\left(\bar{X}_T, \mathbb{E}\left[\bar{X}_T\right]\right) + \beta\left(\bar{Y}_0, \mathbb{E}\left[\bar{Y}_0\right]\right)\right. \\ &\quad \left.+ \int_0^T \ell\left(t, \bar{X}_t, \mathbb{E}\left[\bar{X}_t\right], \bar{Y}_t, \mathbb{E}\left[\bar{Y}_t\right], \bar{Z}_t, \mathbb{E}\left[\bar{Z}_t\right], \bar{u}_t\right) dt\right] \\ &= \lim_{n\to\infty} \mathbb{E}\left[\alpha\left(\widetilde{X}_T^n, \mathbb{E}\left[\widetilde{X}_T^n\right]\right) + \beta\left(\widetilde{Y}_0^n, \mathbb{E}\left[\widetilde{Y}_0^n\right]\right)\right. \\ &\quad \left.+ \int_0^T \ell\left(t, \widetilde{X}_t^n, \mathbb{E}\left[\widetilde{X}_t^n\right], \widetilde{Y}_t^n, \mathbb{E}\left[\widetilde{Y}_t^n\right], \widetilde{Y}_t^n, \mathbb{E}\left[\widetilde{Z}_t^n\right], \widetilde{U}_t^n\right) dt\right].\end{aligned}$$

Thus

$$\begin{aligned}
&\mathbb{J}\left(\bar{u}_{\cdot}\right) \\
&= \lim_{n\to\infty} \mathbb{E}\left[\alpha\left(\sum_{\mathcal{J}\geq\prime}\theta_{\mathcal{J}n}X_T^{\mathcal{J}+n}, \mathbb{E}\left[\sum_{\mathcal{J}\geq\prime}\theta_{\mathcal{J}n}X_T^{\mathcal{J}+n}\right]\right) + \beta\left(\sum_{\mathcal{J}\geq\prime}\theta_{\mathcal{J}n}Y_0^{\mathcal{J}+n}, \mathbb{E}\left[\sum_{\mathcal{J}\geq\prime}\theta_{\mathcal{J}n}Y_0^{\mathcal{J}+n}\right]\right)\right. \\
&+ \int_0^T \ell\left(t, \sum_{\mathcal{J}\geq\prime}\theta_{\mathcal{J}n}X_t^{\mathcal{J}+n}, \mathbb{E}\left[\sum_{\mathcal{J}\geq\prime}\theta_{\mathcal{J}n}X_t^{\mathcal{J}+n}\right], \sum_{\mathcal{J}\geq\prime}\theta_{\mathcal{J}n}Y_t^{\mathcal{J}+n}, \mathbb{E}\left[\sum_{\mathcal{J}\geq\prime}\theta_{\mathcal{J}n}Y_t^{\mathcal{J}+n}\right],\right. \\
&\left.\left.\sum_{\mathcal{J}\geq\prime}\theta_{\mathcal{J}n}Z_t^{\mathcal{J}+n}, \mathbb{E}\left[\sum_{\mathcal{J}\geq\prime}\theta_{\mathcal{J}n}Z_t^{\mathcal{J}+n}\right], \sum_{\mathcal{J}\geq\prime}\theta_{\mathcal{J}n}u_t^{\mathcal{J}+n}\right) dt\right],
\end{aligned}$$

By the convexity of α, β *and* ℓ*, it follows that*

$$\begin{aligned}
\mathbb{J}\left(\bar{u}_{\cdot}\right) &\leq \lim_{n\to\infty} \sum_{\mathcal{J}\geq\prime}\theta_{\mathcal{J}n}\mathbb{E}\left[\alpha\left(X_T^{\mathcal{J}+n}, \mathbb{E}\left[X_T^{\mathcal{J}+n}\right]\right) + \beta\left(Y_0^{\mathcal{J}+n}, \mathbb{E}\left[Y_0^{\mathcal{J}+n}\right]\right)\right. \\
&\left.+ \int_0^T \ell\left(t, X_t^{\mathcal{J}+n}, \mathbb{E}\left[X_t^{\mathcal{J}+n}\right], Y_t^{\mathcal{J}+n}, \mathbb{E}\left[Y_t^{\mathcal{J}+n}\right], Z_t^{\mathcal{J}+n}, \mathbb{E}\left[Z_t^{\mathcal{J}+n}\right], u_t^{\mathcal{J}+n}\right) dt\right] \\
&= \lim_{n\to\infty} \sum_{\mathcal{J}\geq\prime}\theta_{\mathcal{J}n}\mathbb{J}\left(u_{\cdot}^{\mathcal{J}+n}\right), \\
&\leq \lim_{n\to\infty} \sum_{\mathcal{J}\geq\prime}\theta_{\mathcal{J}n} \operatorname*{Max}_{1\leq\mathcal{J}\leq i_n} \mathbb{J}\left(u_{\cdot}^{\mathcal{J}+n}\right) \\
&\leq \lim_{n\to\infty} \operatorname*{Max}_{1\leq\mathcal{J}\leq i_n} \mathbb{J}\left(u_{\cdot}^{\mathcal{J}+n}\right) \sum_{\mathcal{J}\geq\prime}\theta_{\mathcal{J}n} = \lim_{n\to\infty} \operatorname*{Max}_{1\leq\mathcal{J}\leq i_n} \mathbb{J}\left(u_{\cdot}^{\mathcal{J}+n}\right) = \inf_{v_{\cdot}\in\mathcal{U}_L} \mathbb{J}\left(v_{\cdot}\right).
\end{aligned}$$

■

Conclusion

In this memoir, we have proven the existence of an optimal control for linear forward-backward doubly stochastic differential equations of mean-field type, where the cost function is given in a general form.

The proof method is based on the fact that the set of admissible controls is convex and compact, and that the cost function is convex, along with the use of Mazur's theorem.

It would be interesting to explore how this existence result can be extended to more complex cases of mean-field forward-backward doubly stochastic differential equations (for example of McKean-Vlasov type).

Bibliography

[1] A. Agazzi, & J. C. Mattingly, *Introduction to Stochastic Calculus* (Math 545). Duke University.

[2] F. Antonelli. (1993). *Backward-forward stochastic differential equations*, Ann. Appl. Probab, 777–793.

[3] N. Berglund. (2009). *Martingales et calcul stochastique.* DEA, Université d'Orléans. [HAL Id: cel-00443085]. https:/cel.hal.science/cel-00443085v1.

[4] N. Berrouis, & B. Gherbal, *Stochastic optimal control for systems of a linear forward backward doubly SDEs of mean-field type.* Laboratory of Applied Mathematics, University of Mohamed Khider, Biskra, Algeria.

[5] J.M. Bismut. (1973). *Conjugate convex functions in optimal stochastic control*, J. Math. Anal. Appl. 44, 384–404.

[6] C. Chorro. (2006,Septembre). *Cours de calcul stochastique Mster M2 IRFA.*

[7] M. Jeanblanc. (2006,Septembre). *Cours de Calcul stochastique Master 2IF EVRY.*

[8] Korichi, F. (2024). *Contributions to the stochastic optimal control of Mckean-Vlasov stochastic differential systems via the derivatives with respect to measures with some applications.* [Doctoral thesis, University Mohamed Khider of Biskra]. Ministry of Higher Education and Scientific Research, People's Democratic Republic of Algeria.

[9] J.Ma, J. Yong. (1999). *Forward backward stochastic differential equations and their applications, in: Lecture notes in Mathematics*, vol. 1702, Springer, Berlin, Heidelberg.

[10] E. Pardoux and S. Peng. (1990). *Adapted solution of a backward stochastic differential equation, Systems and Control Letters.* no.1, 55–61 .

[11] E. Pardoux and S, Peng. (1994). *Backward doubly stochastic differential equations and system of quasilinear SPDEs.* Probab. Theory Related Fields 98, No.2, pp 209-227.

[12] Stein, E. M., & Shakarchi, R. *Real analysis: Measure theory, integration, & Hilbert spaces.* Princeton University Press.

[13] Yong, J., & Zhou, x. Y. *Stochastic Controls Hamiltonian Systems and HJB Equations. Springer.*

[14] Q. Zhu and Y. Shi. (2012). *Forward-backward doubly stochastic differential equations and related stochastic partial differential equations*, Sci. China Math., vol. 55, no. 12 , 2517–2534.

Annexe : Abreviations and Notations

The following is an explanation of the various abbreviations and notations which are in use throughout this report:

$(\Omega, \mathcal{F}, \mathbb{P})$	:	Probability space.
$(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_n, \mathbb{P})$	:	Filtered probability space.
$(\Omega, \mathcal{F})$	:	Measurable space.
$\mathcal{B}_{\mathbb{R}}$	:	Borel $\sigma-$algebra on $\mathbb{R}$.
B_t	:	Brownian motion.
$\mathbb{P}-p.s.$	:	Almost surely.
$\langle X \rangle_s$	:	Stochastic bracket.
$\mathbb{J}(u_.)$	:	Cost functional.
$u_.^n$	:	Minimizing sequence.
$\bar{u}_.$	:	Optimal control.
$FBDSED-MF$	:	Forward-Backwar Doubly Stochastic Differential Equations of Mean-Field.
$i.e.,$	:	That is.
U	:	Compact set.

ملخص

تهدف هذه المذكرة إلى دراسة وجود تحكمات مثلى لمعادلات تفاضلية تصادفية مزدوجة تقدمية-تراجعية من نوع متوسط الحقل، يعتمد البرهان على تقنيات التقارب القوي المطبقة على المعادلات ال تفاضلية تصادفية مزدوجة تقدميةتراجعية من نوع متوسط الحقل وكذلك على مبرهنة مازور.

Résumé

Ce mémoire étudie l'existence des contrôles optimaux pour des équations différentie l les stochastiques doublement progressive-rétrograde linéaire de type champ moyen. La démonstratio n repose sur des techniques de convergence forte appliquées aux FBDSDEs linéaires associées de type champ moyen, ainsi que sur le théorème de Mazur.

Abstract

This study investigates the existence of optimal controls for a linear forward-backward doubly stochastic differential equation of mean-field type. The proof is based on strong convergence techniques for the associated linear FBDSDEs of mean-field type and Mazur's theorem.