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Regularity of the Solution of Volterra Integral Equation.

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Notation

The following symbols are used frequently throughout this thesis:

- $u(t)$ Unknown solution function.
- $f(t)$ Given data function (right-hand side).
- $K(t,s)$ Kernel of the integral equation.
- $V_{\alpha,\beta}$ Weakly singular Volterra integral operator.
- $C^{\alpha}[0,T]$ Hölder space of order α.
- $\|u\|_{C^{\alpha}}$ Hölder norm.
- $\phi(t)$ Initial function (used in applications).
- T Final time or upper bound of integration.
- α Hölder continuity exponent $(0 < \alpha < 1)$.
- γ Singularity parameter in weakly singular kernels.

Introduction

Integral equations are considered one of the fundamental pillars of modern mathematical analysis, as they provide an effective tool for modeling many physical and engineering phenomena involving non-local effects or memory. These equations often arise when transforming differential problems into integral formulations in order to simplify the analysis or obtain more stable numerical solutions. In general, integral equations are written as a relation between an unknown function and an integral that involves this function. They are classified into various types depending on the kernel and the domain of integration (e.g., Fredholm or Volterra types).

The systematic emergence of integral equations dates back to the late 19^{th} century, where both the Swedish mathematician Erik Fredholm and the Italian mathematician Vito Volterra laid the theoretical foundations of this field. In 1896, Volterra introduced what are now known as Volterra equations of the first and second kinds, with the aim of studying time-dependent systems that rely on previous values of variables. Fredholm later developed the theory of integral equations with kernels defined over fixed domains, focusing on the existence and uniqueness of solutions in functional spaces. These contributions led to the formulation of functional analysis, which became the cornerstone of many applications in applied mathematics and theoretical physics.

During the 20^{th} century, the theory of integral equations experienced significant development, especially with the introduction of concepts from functional analysis such as Banach and Hilbert spaces. These advancements allowed for a deeper understanding of the mathematical structure of integral equations and the development of powerful tools to analyze them, including compact operators and the spectral theory of linear operators. As a result, integral equations found widespread application in fields like quantum mechanics, heat transfer, control theory, and biological systems.

With the evolution of research, interest extended beyond the existence and uniqueness of solutions to include what is known as the *regularity of solutions*, that is, the study of continuity, differentiability, and boundary behavior of the solutions. Regularity is a critical factor in understanding the nature of the modeled system and has a direct impact on the accuracy and efficiency of the numerical methods used for approximation.

The regularity of a solution is strongly influenced by the nature of the kernel:

- Smooth kernels typically lead to smooth solutions of the same or higher order.

- Weakly singular kernels (such as $(t-s)^{-\alpha}$ with $0 < \alpha < 1$) often result in solutions with limited regularity.
- Kernels with boundary singularities may degrade the regularity of the solution or lead to singular-like behavior at boundary points.

This variability in solution behavior necessitates the classification of kernels according to their properties (smooth, weakly singular, non-smooth, or with boundary singularities) and the study of solution regularity in each case separately. Such analysis is fundamental in evaluating the numerical accuracy of integral equation solvers, since many numerical methods rely on the existence of certain solution derivatives, such as spectral methods or quadrature-based techniques. Several academic references have addressed this topic in depth.

This thesis is organized as follows: In the first chapter we discus about the deferent type of integral equations and we present same operator notions. In the second one we present the proof of existence and uniqueness of the solution of Volterra integral equation, and same theorems about the regularity of the solution which depends on the regularity of the known parts of the equations. In the third chapter we present same example of Volterra integral equation of the second kind.

Chapter 1

Integral Equations

Integral equations are equations in which the unknown function appears under an integral sign. They are typically classified into two main categories: linear and nonlinear. In general, an integral equation has the form where the solution function is integrated over a given interval, and the resulting expression is set equal to a known function:

$$f(x) = \phi(x)u(x) + \int_{a(x)}^{b(x)} k(x,t)F(u(t))\,dt \tag{1.1}$$

where:

- f, F and K are known functions.
- F takes u as a variable.
- a and b are the bounds of the integral, which can either be constants or functions of x.
- K is a function of two variables, known as the **kernel** of the integral equation.

We say that an integral equation is **linear** if F is a linear function, and it is written in the form:

$$f(x) = \phi(x)u(x) + \lambda \int_{a(x)}^{b(x)} K(x,t)u(t)dt \tag{1.2}$$

The parameter λ is a constant.
On the other hand, and a nonlinear integral equation when F is nonlinear.
If $f = 0$, it is called a homogeneous equation, whereas if $f \neq 0$, it is called a non-homogeneous equation

1.1 Types of integral equations

Fredholm Equations

A Fredholm equation is an integral equation where the limits of integration are fixed.

$$\Phi(x)u(x) = f(x) + \lambda \int_a^b k(x,t)F(u(t))dt, \tag{1.3}$$

If the function $\Phi(x) = 0$, then the equation (1.3) becomes:

$$f(x) + \int_a^b k(x,t)F(u(t))dt = 0,$$

This simpler equation is called the linear Fredholm integral equation of the first kind. If the function $\Phi(x) \neq 0$, then equation (1.3) becomes simply:

$$u(x) = f(x) + \int_a^b k(x,t)F(u(t))dt,$$

this is called the linear Fredholm integral equation of the second kind.

Examples of Fredholm Integral Equations of the Second Kind

Consider the equation:

$$u(x) = 1 + \int_0^1 xt\, u(t)\, dt$$

$$u(x) = \sin(x) + \int_0^1 e^{xt}\, u(t)\, dt$$

Volterra Equations

A Volterra equation is an integral equation where the upper or lower limit of integration depends on the independent variable. It can be of the first or second kind. [7]

Volterra equation of the first kind:

$$f(x) = \int_a^x k(x,t)F(u(t))dt \tag{1.4}$$

Volterra equation of the second kind:

$$u(x) = f(x) + \lambda \int_a^x k(x,t)F(u(t))dt \tag{1.5}$$

It is observed that the limits of integration in the Fredholm integral equation are constant, whereas in the Volterra equation one limit is constant and the other is variable.

Remark 1.1.1 *Consider the equations of the second kind:*

$$u(x) = f(x) + \int_G k(x,t)F(u(t))\,dt \tag{1.6}$$

Let G be a closed bounded interval (i.e., $G = [a,b]$ in the case of Fredholm equations, and $G = [a,x]$ in the case of Volterra equations).
It is clarified here that Volterra equations are special cases of Fredholm equations if we assume that the kernel $k(x,t)$ equals zero when $t > x$. This means that the effects from values after x do not enter into the equation, and thus the integration is limited to the interval from a to x only.

Some examples of non linear integral equations:

$$u(x) = f(x) + \int_a^x k(x,t)[u(t)]^2 dt$$
$$u(x) = f(x) + \int_a^x k(x,t)[u(t)f(t)]dt$$

If f(x)=0 in both cases we say that the resulted equation is a homogeneous equation or it will be called a non-homogeneous equation.
Example:

$$u(x) = 1 + \int_0^x (x-t)[u(t)]^3 dt$$

Here, the unknown function $u(t)$ appears raised to the power of 3 (i.e., $u(t)^3$).
The equation represents a nonlinear case because the unknown function appears in a nonlinear form inside the integral.

Deviant Integral Equations

Deviant integral equations: We say that integral equations are deviant if one of the integration limits is infinity, or if the kernel k(x,t) tends to infinity at one or more points within the integration domain. Its general form is

$$u(x) = f(x) + \lambda \int_{-\infty}^{\infty} u(t)dt. \tag{1.7}$$

Example of deviant integral equations

$$u(t) - \int_0^{\infty} e^{st}u(s)ds = g(t),$$
$$u(t) - \int_{-\infty}^{\infty} e^{st}u(s)ds = g(t),$$
$$u(t) - \int_a^b \ln|s-t|u(s)ds = g(t).$$

Abel's Equation

Let the following integral equation be:

$$f(x) = \int_0^x \frac{u(t)}{(x-t)^\alpha} dt, \tag{1.8}$$

where $\alpha \in (0,1)$.

This equation is of Abel's type, and such equations can be solved using the Laplace transform.

Applying the Laplace transform to the equation

We apply the Laplace transform to both sides of equation (1.8):

$$\mathcal{L}[f(x)](s) = \mathcal{L}\left[\int_0^x \frac{u(t)}{(x-t)^\alpha} dt\right](s).$$

Using the convolution property of the Laplace transform, where the integral is the convolution of $u(x)$ and $k(x) = x^{-\alpha}$:

$$\mathcal{L}[f(x)](s) = \mathcal{L}[u(x)](s) \cdot \mathcal{L}[x^{-\alpha}](s).$$

We know that:

$$\mathcal{L}[x^{-\alpha}](s) = \frac{\Gamma(1-\alpha)}{s^{1-\alpha}}.$$

So we can solve for $\mathcal{L}[u(x)](s)$:

$$\mathcal{L}[u(x)](s) = \frac{\mathcal{L}[f(x)](s)}{\frac{\Gamma(1-\alpha)}{s^{1-\alpha}}} = \mathcal{L}[f(x)](s) \cdot \frac{s^{1-\alpha}}{\Gamma(1-\alpha)}.$$

Using the inverse Laplace transform

Applying the inverse Laplace transform:

$$u(x) = \mathcal{L}^{-1}\left[\mathcal{L}[f(x)](s) \cdot \frac{s^{1-\alpha}}{\Gamma(1-\alpha)}\right](x).$$

This can be rewritten as:

$$u(x) = \frac{1}{\Gamma(1-\alpha)} \mathcal{L}^{-1}\left[s \cdot (s^{-\alpha}\mathcal{L}[f(x)](s))\right](x).$$

Using the property $\mathcal{L}^{-1}[sF(s)](x) = \frac{d}{dx}\mathcal{L}^{-1}[F(s)](x)$ (assuming conditions for initial values are met) and the convolution theorem for the inner part:

$$\begin{aligned}
u(x) &= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \mathcal{L}^{-1} \left[s^{-\alpha} \mathcal{L}[f(x)](s) \right] (x) \\
&= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \left(\mathcal{L}^{-1}[s^{-\alpha}](x) * f(x) \right) \\
&= \frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \left(\frac{x^{\alpha-1}}{\Gamma(\alpha)} * f(x) \right) \\
&= \frac{1}{\Gamma(\alpha)\Gamma(1-\alpha)} \frac{d}{dx} \int_0^x (x-t)^{\alpha-1} f(t) dt.
\end{aligned}$$

Using the reflection formula $\Gamma(\alpha)\Gamma(1-\alpha) = \dfrac{\pi}{\sin(\alpha\pi)}$, the solution is:

$$u(x) = \frac{\sin(\alpha\pi)}{\pi} \frac{d}{dx} \int_0^x \frac{f(t)}{(x-t)^{1-\alpha}} dt.$$

This is the standard solution to Abel's integral equation.

Theorem: Leibniz Rule For Differentiation Under The Integral Sign

Let $f(x, t)$ be a function defined over a region where all necessary derivatives exist and are continuous differentiable. Suppose the limits of integration are functions of x, denoted as $a(x)$ and $b(x)$. Then, the derivative of the integral with respect to x is given by [8]

$$\frac{d}{dx} \int_{a(x)}^{b(x)} f(x, t)\, dt = \int_{a(x)}^{b(x)} \frac{\partial f(x, t)}{\partial x}\, dt + f(x, b(x)) \frac{db(x)}{dx} - f(x, a(x)) \frac{da(x)}{dx}.$$

This formula accounts for both the variation of the integrand and the movement of the integration limits. It is widely used in applied mathematics, physics, and engineering.

1.2 The Relationship Between Differential Equations And Integral Equations

The relationship between integral equations and differential equations is very strong, and some differential equations can be transformed into integral equations, and the inverse is true in some cases. In fact, these differential equations can be used, especially when analytical solutions to differential equations are complex or difficult to obtain.

Conversion of a First-Order Differential Equation to an Integral Equation

Let us consider the following first-order differential equation:

$$F(x) = a(x)y(x) + \frac{dy}{dx}, \tag{1.9}$$

with the initial condition:

$$y(0) = C_0, \quad \text{where } C_0 \in \mathbb{R} \text{ or } C_0 \in \mathbb{C}$$

Rearranging the equation, we get:

$$\frac{dy}{dx} = -a(x)y(x) + F(x). \tag{1.10}$$

We define:

$$\varphi(x) = -a(x)y(x) + F(x). \tag{1.11}$$

Now we integrate both sides from 0 to x:

$$y(x) = \int_0^x \varphi(t)\, dt + C_0. \tag{1.12}$$

Substituting back the expression of $\varphi(t)$, we obtain:

$$y(x) = \int_0^x [-a(t)y(t) + F(t)]\, dt + C_0. \tag{1.13}$$

Thus, the final integral equation is:

$$y(x) = C_0 + \int_0^x [-a(t)y(t) + F(t)]\, dt. \tag{1.14}$$

Once the differential equation is converted into an integral equation, we can use methods to solve integral equations, such as numerical or symbolic analysis techniques to solve the equation, and this may be easier in some cases.

1.3 Notions about operators

A bounded linear operator

Definition (linear operator) 2.1.1: Let X and Y be two vector spaces. A map $T : X \to Y$ is called a linear operator if: For all $x, y \in X$ and $\alpha \in \mathbb{R}$ or $\mathbb{C}$,

$$T(x + y) = T(x) + T(y), \quad \text{and} \quad T(\alpha x) = \alpha T(x).$$

Recall the two fundamental spaces associated with a linear operator:
The kernel of T is the subspace of X:

$$N(T) = \{x \in X : T(x) = 0\}.$$

The image of T is the subspace of Y:

$$R(T) = \{y \in Y : \exists x \in X, T(x) = y\}.$$

Definition(Bounded Operators) 2.1.2 Let X and Y be two normed spaces. A linear operator $T : X \to Y$ is continuous if there exists a constant $C > 0$ such that:

$$\|Tx\|_Y \leq C\|x\|_X$$

for all $x \in X$.

Using the linearity of the operator T, it is easy to see that T is bounded if and only if

$$\|T\| = \sup_{\|x\|_X \leq 1} \|Tx\|_Y < \infty.$$

The number $\|T\|$ is called the norm of T.

We denote by $L(X; Y)$ the space of bounded linear operators from X to Y.

A linear operator $T : X \to Y$ is said to be of finite rank if and only if its image $R(T)$ is a subspace of Y of finite dimension.

Definition 2.1.3: A linear operator $T : X \to Y$ from a normed space X to a normed space Y is said to be *compact* if it transforms every bounded subset of X into a relatively compact subset of Y.

Theorem 2.1.1: A linear operator $T : X \to Y$ is compact if and only if, for every bounded sequence (ϕ_n) in X, one can extract a convergent subsequence from $(T\phi_n)$ in Y, i.e., if every sequence in the set $\{T\phi : \phi \in X, \|\phi\| \leq 1\}$ contains a convergent subsequence.

Theorem 2.1.2 : Riesz Theory for Compact Linear Operators of the Second Kind

We consider in this context the study of linear equations of the form:

$$\varphi - T\varphi = f,$$

where $T : X \to X$ is a compact linear operator defined on a normed vector space X. We denote $L = I - T$, where I is the identity operator. This equation is sometimes referred to as a Riesz equation of the second kind and plays a central role in the analysis of integral equations with continuous or weakly singular kernels.

The Riesz theory in this context states the following:

- If the associated homogeneous equation:

  $$\varphi - T\varphi = 0$$

  admits only the trivial solution $\varphi = 0$, then the operator L is invertible, i.e., L^{-1} exists and is defined on the whole space X. Consequently, the inhomogeneous equation:

$$\varphi - T\varphi = f$$

admits a unique solution for every $f \in X$, and this solution is given explicitly by:

$$\varphi = L^{-1}f.$$

- On the other hand, if the homogeneous equation admits non-trivial solutions, then the inhomogeneous equation:

  $$\varphi - T\varphi = f$$

  is solvable if and only if a compatibility condition is satisfied, namely, that f is orthogonal to all solutions $\psi \in X^*$ of the adjoint equation:

  $$\psi - T^*\psi = 0,$$

  where T^* is the adjoint operator of T in the dual space X^*.

where T is a compact linear operator $T : X \to X$ in a normed space X. We denote $L = I - T$, where I represents the identity operator.

Theorem 2.1.3: Let $T : X \to X$ be a compact linear operator. Then $I - T$ is injective if and only if it is surjective. Moreover, if $I - T$ is injective (hence bijective), then the inverse operator $(I - T)^{-1} : X \to X$ is bounded.

Theorem 2.1.4: Every linear operator $T : X \to Y$ from a normed space X of finite dimension to a space Y is bounded.

Theorem :(Neumann Series)

Let E be a Banach space, and let $T \in L(E)$ with $\|T\| < 1$. Let Id_E be the identity operator on E. Then:

$$(\mathrm{Id}_E - T)^{-1} = \sum_{n=0}^{\infty} T^n \tag{1.15}$$

Furthermore,

$$\left\|(\mathrm{Id}_E - T)^{-1}\right\| \leq \frac{1}{1 - \|T\|} \tag{1.16}$$

Proof. See [6].

Chapter 2

Regularity of The Solution of Volterra Integral Equation

2.1 Existence And Uniqueness of The Solution of A Volterra Integral Equation of The Second Kind

In this chapter, we focus on studying the existence and uniqueness of the solution for the Volterra integral equations of the second kind. If the Volterra integral equations admit a solution, what are the methods used to find the explicit form of this solution? There are analytical methods such as the Adomian method, the series method, and others, and numerical methods such as the trapezoidal method and the Simpson method.

Solution of Volterra Integral Equation

Given the integral equation with a separable kernel:

$$\varphi(x) = f(x) + \lambda \int_0^x k(x,t)\varphi(t)\,dt$$

We seek a solution in the form of an infinite series expansion in terms of λ:

$$\varphi(x) = \varphi_0(x) + \lambda\varphi_1(x) + \lambda^2\varphi_2(x) + \lambda^3\varphi_3(x) + \dots$$

Substituting this into the equation:

$$\begin{aligned}\varphi_0(x) + \lambda\varphi_1(x) + \lambda^2\varphi_2(x) + \ldots &= f(x) + \lambda \int_0^x k(x,t)\left(\varphi_0(t) + \lambda\varphi_1(t) + \lambda^2\varphi_2(t) + \ldots\right) dt \\ &= f(x) + \lambda \int_0^x k(x,t)\varphi_0(t)\, dt \\ &+ \lambda^2 \int_0^x k(x,t)\varphi_1(t)\, dt \\ &+ \lambda^3 \int_0^x k(x,t)\varphi_2(t)\, dt + \ldots\end{aligned}$$

By substituting into the given equation and matching coefficients of λ, we get:

$$\varphi_0(x) = f(x)$$

$$\varphi_1(x) = \int_0^x k(x,t)\varphi_0(t)\, dt = \int_0^x k(x,t) f(t)\, dt \tag{2.1}$$

$$\varphi_2(x) = \int_0^x k(x,t)\varphi_1(t)\, dt = \int_0^x k(x,t)\left(\int_0^t k(t,s) f(s)\, ds\right) dt \tag{2.2}$$

$$\varphi_3(x) = \int_0^x k(x,t)\varphi_2(t)\, dt$$

The series converges uniformly.

To establish the uniform convergence of the series solution

$$\varphi(x) = \sum_{n=0}^{\infty} \lambda^n \varphi_n(x),$$

we utilize the Neumann series theorem and Weierstrass' M-test.

We consider the Volterra integral equation of the second kind:

$$\varphi(x) = f(x) + \lambda \int_0^x k(x,t)\, \varphi(t)\, dt. \tag{2.3}$$

This can be rewritten as:

$$(I - \lambda K)\varphi = f, \tag{2.4}$$

where K is the Volterra integral operator defined by:

$$(K\varphi)(x) = \int_0^x k(x,t)\, \varphi(t)\, dt. \tag{2.5}$$

The solution can be expressed using the Neumann series:

$$\varphi = (I - \lambda K)^{-1} f = \sum_{n=0}^{\infty} \lambda^n K^n f. \tag{2.6}$$

We define the terms φ_n as follows:

$$\begin{aligned}
\varphi_0 &= f, \\
\varphi_1 &= Kf, \\
\varphi_2 &= K(Kf) = K^2 f, \quad \text{and so on.}
\end{aligned}$$

Therefore, the solution can be written as:

$$\varphi(x) = \sum_{n=0}^{\infty} \lambda^n (K^n f)(x). \tag{2.7}$$

We now relate this to the iterated kernels. The n-th iteration of K applied to f can be written as:

$$(K^n f)(x) = \int_0^x k_n(x,t)\, f(t)\, dt, \quad \text{for } n \geq 1, \tag{2.8}$$

where the iterated kernel $k_n(x,t)$ is defined recursively by:

$$\begin{aligned}
k_1(x,t) &= k(x,t), \\
&\textit{and} \\
k_{n+1}(x,t) &= \int_t^x k(x,s)\, k_n(s,t)\, ds.
\end{aligned}$$

Thus, the series solution becomes:

$$\varphi(x) = f(x) + \sum_{n=1}^{\infty} \lambda^n \int_0^x k_n(x,t)\, f(t)\, dt. \tag{2.9}$$

We aim to prove the **uniform convergence** of this series on the interval $[0,T]$, assuming that f is continuous and the kernel $k(x,t)$ is continuous (and hence bounded) on the domain:

$$D = \{(x,t) \,|\, 0 \leq t \leq x \leq T\}.$$

Let $M = \sup_{(x,t)\in D} |k(x,t)|$ and $\|f\|_\infty = \sup_{t\in[0,T]} |f(t)|$. Assume M and $\|f\|_\infty$ are finite.

Using an inductive argument, we estimate $|k_n(x,t)|$:

Base case $n = 1$:

$$|k_1(x,t)| = |k(x,t)| \leq M = \frac{(x-t)^0}{0!} M^1. \tag{2.10}$$

Inductive step: Assume that

$$|k_n(s,t)| \leq M^n \frac{(s-t)^{n-1}}{(n-1)!}. \tag{2.11}$$

Then:

$$\begin{aligned}
|k_{n+1}(x,t)| &= \left| \int_t^x k(x,s)k_n(s,t)\,ds \right| \\
&\le \int_t^x |k(x,s)| \cdot |k_n(s,t)|\,ds \\
&\le \int_t^x M \cdot M^n \frac{(s-t)^{n-1}}{(n-1)!}\,ds \\
&\le \frac{M^{n+1}}{(n-1)!} \int_t^x (s-t)^{n-1}\,ds \\
&\le \frac{M^{n+1}}{(n-1)!} \left[\frac{(s-t)^n}{n} \right]_t^x \\
&\le \frac{M^{n+1}}{(n-1)!} \cdot \frac{(x-t)^n}{n} \\
&\le M^{n+1} \frac{(x-t)^n}{n!}.
\end{aligned}$$

So the bound holds:

$$|k_n(x,t)| \le M^n \frac{(x-t)^{n-1}}{(n-1)!} \quad \text{for } n \ge 1.$$

Now consider the terms in the series solution: $|\lambda^n (K^n f)(x)| = |\lambda^n \int_0^x k_n(x,t) f(t) dt|$

$$\begin{aligned}
|\lambda^n (K^n f)(x)| &\le |\lambda|^n \int_0^x |k_n(x,t)||f(t)|dt \\
&\le |\lambda|^n \|f\|_\infty \int_0^x |k_n(x,t)|dt \\
&\le |\lambda|^n \|f\|_\infty \int_0^x M^n \frac{(x-t)^{n-1}}{(n-1)!} dt \\
&\le |\lambda|^n \|f\|_\infty \frac{M^n}{(n-1)!} \int_0^x (x-t)^{n-1} dt
\end{aligned}$$

Let $u = x - t$, $du = -dt$. When $t = 0, u = x$. When $t = x, u = 0$.

$$\int_0^x (x-t)^{n-1} dt = \int_x^0 u^{n-1}(-du) = \int_0^x u^{n-1} du = \frac{x^n}{n}.$$

So,

$$|\lambda^n (K^n f)(x)| \le |\lambda|^n \|f\|_\infty \frac{M^n}{(n-1)!} \frac{x^n}{n} = \|f\|_\infty \frac{(|\lambda| M x)^n}{n!}.$$

Since $x \in [0, T]$, we have $x \le T$, so:

$$|\lambda^n (K^n f)(x)| \le \|f\|_\infty \frac{(|\lambda| M T)^n}{n!}.$$

We examine the series for $\varphi(x) = f(x) + \sum_{n=1}^{\infty} \lambda^n (K^n f)(x)$. We use the Weierstrass M-test for the series $\sum_{n=1}^{\infty} \lambda^n (K^n f)(x)$ on the interval $[0, T]$. Let

$$M_n = \|f\|_\infty \frac{(|\lambda| MT)^n}{n!}.$$

The series $\sum_{n=1}^{\infty} M_n$ is $\|f\|_\infty \sum_{n=1}^{\infty} \frac{(|\lambda| MT)^n}{n!}$. This is related to the Taylor series for the exponential function: $e^y = \sum_{n=0}^{\infty} \frac{y^n}{n!}$. The series $\sum_{n=1}^{\infty} M_n$ converges if $\|f\|_\infty$ and $|\lambda| MT$ are finite. The series converges to $\|f\|_\infty (e^{|\lambda| MT} - 1)$. Since the series $\sum M_n$ converges, by the Weierstrass M-test, the series

$$\sum_{n=1}^{\infty} \lambda^n (K^n f)(x)$$

converges uniformly on $[0, T]$.

Since the series converges uniformly, the sum $\varphi(x) = f(x) + \sum_{n=1}^{\infty} \lambda^n (K^n f)(x)$ is continuous on $[0, T]$ (as f is continuous and uniform limit of continuous functions is continuous). This establishes the existence of a unique continuous solution for any finite λ, M, T and continuous f. The condition $|\lambda| \|K\| < 1$ required for Neumann series in general Banach spaces is not needed here because the Volterra operator K is quasinilpotent on $C[0, T]$. Its spectral radius is 0.

The solution can be expressed using the iterated kernels $k_n(x, t)$. Define $k_1(x, t) = k(x, t)$ and

$$k_{n+1}(x, t) = \int_t^x k(x, z) k_n(z, t)\, dz \quad \text{for } n \geq 1.$$

Then $\varphi_n(x) = \int_0^x k_n(x, t) f(t) dt$ for $n \geq 1$, and $\varphi_0(x) = f(x)$. The solution is:

$$\varphi(x) = \sum_{n=0}^{\infty} \lambda^n \varphi_n(x) = f(x) + \sum_{n=1}^{\infty} \lambda^n \int_0^x k_n(x, t) f(t)\, dt$$

This can be written using the resolvent kernel $R(x, t; \lambda) = \sum_{n=1}^{\infty} \lambda^{n-1} k_n(x, t)$:

$$\varphi(x) = f(x) + \lambda \int_0^x \left(\sum_{n=1}^{\infty} \lambda^{n-1} k_n(x, t) \right) f(t)\, dt$$

$$\varphi(x) = f(x) + \lambda \int_0^x R(x, t; \lambda) f(t)\, dt$$

The series used in the text $\sum_{i=1}^{\infty} \lambda^i k_i(x,t)$ seems to be $\lambda R(x,t;\lambda)$.

$$\varphi(x) = f(x) + \int_0^x \left(\sum_{i=1}^{\infty} \lambda^i k_i(x,t) \right) f(t)\, dt$$

2.2 Volterra Integral Equations with Smooth Kernels

Let the Volterra integral equation of the second kind be of the form:

$$u(t) = f(t) + \int_0^t K(t,s)u(s)\, ds, \quad t \in I = [0,T], \tag{2.12}$$

The solution can be expressed using the resolvent kernel $R(t,s)$ of the given kernel $K(t,s)$ (assuming $\lambda = 1$ here for simplicity, or absorbing λ into K), as follows:

$$u(t) = f(t) + \int_0^t R(t,s)f(s)\, ds, \quad t \in I. \tag{2.13}$$

Here, the resolvent kernel $R(t,s)$ is defined by the Neumann series (with $\lambda = 1$):

$$R(t,s) = \sum_{n=1}^{\infty} K_n(t,s), \tag{2.14}$$

where the iterated kernels are given by the recurrence relation:

$$K_{n+1}(t,s) = \int_s^t K(t,v)K_n(v,s)\, dv, \quad (t,s) \in D = \{(t,s)|0 \le s \le t \le T\}, \quad (n \ge 1), \tag{2.15}$$

with the initial condition $K_1(t,s) = K(t,s)$.

If the kernel $K(t,s)$ is smooth, meaning that $K \in C^m(D)$ for some $m \ge 1$, then the corresponding iterated kernels inherit this regularity. The uniform convergence of the Neumann series ensures that the resolvent kernel $R(t,s)$ maintains the same smoothness, i.e., $R \in C^m(D)$.

Theorem 2.2.1

Assume that $K \in C^m(D)$ for some $m \ge 1$. Then its resolvent kernel R has the same degree of regularity, namely $R \in C^m(D)$. Hence, for any $f \in C^m(I)$, the solution of the Volterra integral equation (2.12) belongs to $C^m(I)$.

Proof[5]

Proof (Regularity of the Solution to the Volterra Equation with Smooth Kernel):

Assume that the kernel $K(t,s) \in C^m(D)$, where $D = \{(t,s) \in [0,T]^2 \mid 0 \le s \le t\}$, i.e., K is a smooth function of class C^m on the triangular domain D.

We define the iterated kernels recursively as:

$$K_1(t,s) = K(t,s), \quad K_{n+1}(t,s) = \int_s^t K(t,v)K_n(v,s)\,dv$$

These kernels are constructed iteratively using the original kernel K. By the principle of mathematical induction and the smoothness of K, we can prove that each $K_n(t,s) \in C^m(D)$, since the integral of two functions in C^m yields a function of the same smoothness class.

The resolvent kernel is defined by the series:

$$R(t,s) = \sum_{n=1}^{\infty} K_n(t,s)$$

This series converges uniformly on D, and its derivatives also converge uniformly. Hence, the resulting function satisfies $R(t,s) \in C^m(D)$.

Now let $f \in C^m([0,T])$, and consider the solution given by:

$$u(t) = f(t) + \int_0^t R(t,s)f(s)\,ds$$

Since both f and R are of class C^m, the integral $\int_0^t R(t,s)f(s)\,ds$ also belongs to $C^m([0,T])$. Therefore, we conclude that:

$$u(t) \in C^m([0,T])$$

This proves that the solution u inherits the same regularity as the given function f, provided the kernel K is sufficiently smooth.

Example: Computing the Resolvent Kernel for a Smooth Kernel

We consider the second-kind Volterra integral equation:

$$u(t) = f(t) + \int_0^t K(t,s)u(s)\,ds, \quad t \in [0,T]. \tag{2.16}$$

We choose a simple smooth kernel:

$$K(t,s) = e^{t-s}. \tag{2.17}$$

Here, we assume $\lambda = 1$.

The solution can be written using the resolvent kernel $R(t,s)$:

$$R(t,s) = \sum_{n=1}^{\infty} K_n(t,s), \tag{2.18}$$

where the iterated kernels are defined recursively as:

$$K_{n+1}(t,s) = \int_s^t K(t,v)K_n(v,s)dv. \tag{2.19}$$

with the initial condition:

$$K_1(t,s) = K(t,s) = e^{t-s}. \tag{2.20}$$

Let's compute $K_2(t,s)$:

$$K_2(t,s) = \int_s^t K(t,v)K_1(v,s)dv = \int_s^t e^{t-v}e^{v-s}dv. \tag{2.21}$$

Simplifying the integral:

$$K_2(t,s) = \int_s^t e^{t-s}dv = e^{t-s}\int_s^t 1dv = e^{t-s}(t-s). \tag{2.22}$$

Let's compute $K_3(t,s)$:

$$K_3(t,s) = \int_s^t K(t,v)K_2(v,s)dv = \int_s^t e^{t-v}[e^{v-s}(v-s)]dv. \tag{2.23}$$

$$K_3(t,s) = e^{t-s}\int_s^t (v-s)dv = e^{t-s}\left[\frac{(v-s)^2}{2}\right]_s^t = e^{t-s}\frac{(t-s)^2}{2}. \tag{2.24}$$

Using mathematical induction, we can show the general form is:

$$K_n(t,s) = e^{t-s}\frac{(t-s)^{n-1}}{(n-1)!}. \tag{2.25}$$

Now, we sum the series:

$$R(t,s) = \sum_{n=1}^{\infty} K_n(t,s) = \sum_{n=1}^{\infty} e^{t-s}\frac{(t-s)^{n-1}}{(n-1)!}. \tag{2.26}$$

Factoring out e^{t-s}:

$$R(t,s) = e^{t-s}\sum_{n=1}^{\infty}\frac{(t-s)^{n-1}}{(n-1)!}. \tag{2.27}$$

Let $k = n-1$. As n goes from 1 to ∞, k goes from 0 to ∞.

$$\sum_{n=1}^{\infty}\frac{(t-s)^{n-1}}{(n-1)!} = \sum_{k=0}^{\infty}\frac{(t-s)^k}{k!} = e^{t-s}. \tag{2.28}$$

Therefore:

$$R(t,s) = e^{t-s}e^{t-s} = e^{2(t-s)}. \tag{2.29}$$

Thus, the resolvent kernel for this example is:

$$R(t,s) = e^{2(t-s)}. \tag{2.30}$$

Since $e^{2(t-s)}$ is a smooth function of class $C^\infty(D)$, the solution $u(t) = f(t) + \int_0^t e^{2(t-s)}f(s)ds$ belongs to $C^\infty(I)$ as long as $f(t)$ belongs to $C^\infty(I)$.

Hölder Space Definition

The Hölder space $C^\alpha(\Omega)$ consists of continuous functions that satisfy the Hölder condition of order α on an open set $\Omega \subseteq \mathbb{R}^n$. A function $f : \Omega \to \mathbb{R}$ belongs to $C^\alpha(\Omega)$ if there exists a constant $C > 0$ such that:

$$|f(x) - f(y)| \leq C|x - y|^\alpha, \quad \forall x, y \in \Omega.$$

where α is the Hölder exponent, taking values in $(0, 1]$. When $\alpha = 1$, the space $C^1(\Omega)$ consists of functions whose first derivatives are continuous (if bounded domain, implies Lipschitz continuity)

. **Theorem 2.2.2 :** Assume that $f \in C^m(I)$ and $K \in C^m(D)$ with $K(t,t) = 0$ on I. Then:

For $\alpha \in (0, 1)$ and any $m \geq 1$, the unique (continuous) solution of the weakly singular Volterra integral equation

$$u(t) = f(t) + \int_0^t (t - s)^{\alpha-1} K(t, s)u(s)ds, \quad t \in I,$$

lies in the Hölder space $C^\alpha(I)$ but not in $C^1(I)$ (unless $f(0) = 0$ and $K(t,t)$ vanishes sufficiently fast). More precisely, its regularity is given by

$$u \in C^m((0, T]) \cap C(I), \quad \text{with} \quad u'(t) \sim C_\alpha t^{\alpha-1} \quad \text{as} \quad t \to 0^+,$$

where C_α is a nonzero constant (typically proportional to $f(0)\Gamma(\alpha)$).

2.3 Definition of Weakly Singular Kernels

Weakly singular kernels are those that exhibit a mild singularity at $t = s$, making them integrable while directly affecting the smoothness of solutions. These kernels are typically written in the form:

$$K_\alpha(t, s) = (t - s)^{\alpha-1} K(t, s), \quad 0 < \alpha < 1$$

where $(t - s)^{\alpha-1}$ represents the singular component, while $K(t, s)$ is a smooth function over the studied domain $D = \{(t, s)|0 \leq s \leq t \leq T\}$.

This type of kernel appears in various physical and engineering applications, such as models of materials with "long memory," where the present state depends on past values weighted by a decaying kernel.

Effect of Weakly Singular Kernels on Solutions

Continuity of Solutions

From the previous discussion, we have seen that smooth kernels typically ensure that the solution $u(t)$ has the same smoothness as the input function $f(t)$. However, when dealing with weakly singular kernels of the form $(t - s)^{\alpha-1} K(t, s)$, the solution $u(t)$ is generally less regular than $f(t)$, especially near $t = 0$. Specifically, even if $f \in C^\infty(I)$ and $K \in C^\infty(D)$, the solution $u(t)$ may not be differentiable at $t = 0$. It typically belongs to $C(I)$ but its derivative might behave like $t^{\alpha-1}$ near $t = 0$, which blows up as $t \to 0^+$ since $\alpha - 1 < 0$.

Behavior of the Solution as $t \to 0^+$

One of the key aspects of weakly singular kernels is their impact on solution regularity near the starting point $t = 0$. The solution may fail to be differentiable at $t = 0$ even if the initial function $f(t)$ is smooth. This behavior arises because the integral term $\int_0^t (t-s)^{\alpha-1} K(t,s) u(s) ds$ contributes a term behaving like t^α to $u(t)$, whose derivative behaves like $t^{\alpha-1}$.

Example:

Consider the integral equation:

$$u(t) = 1 + \int_0^t \frac{u(s)}{\sqrt{t-s}}\, ds$$

Here, the kernel is of the form:

$$K(t,s) = \frac{1}{\sqrt{t-s}} = (t-s)^{-1/2}$$

This corresponds to $\alpha = \frac{1}{2}$, so $\alpha - 1 = -\frac{1}{2} < 0$.

This integral leads to a solution of the form:

$$u(t) \sim t^{1/2}$$

and thus:

$$u'(t) \sim t^{-1/2}$$

This means that the derivative blows up as $t \to 0$, clearly illustrating the effect of weakly singular kernels.

2.4 Definition of Bounded but Non-Smooth Kernels

In Volterra integral equations, a kernel is called bounded but non-smooth if it is bounded over its domain but lacks sufficient smoothness (e.g., not continuously differentiable). This means it may have discontinuities or sharp variations in its values or derivatives. A general form involving such kernels might be combined with weak singularities, as studied in [2]:

$$u(t) = f(t) + \int_0^t (t-s)^{\alpha-1} K(t,s) u(s)\, ds, \quad t \in I. \tag{2.31}$$

where:

$K(t,s)$ is a bounded function, but not necessarily in $C^m(D)$ for $m \geq 1$. The factor $(t-s)^{\alpha-1}$ introduces a weak singularity. The regularity of the solution $u(t)$ will depend on the regularity of both $f(t)$ and $K(t,s)$, often inheriting the lowest regularity present.

Theorem 2.4.1 : Assume that $f \in C^m(I)$ and $K \in C^m(D)$, with $K(t,t) \neq 0$ for $t \in I$. Consider the equation:

$$u(t) = f(t) + \int_0^t (t-s)^{\nu-1} K(t,s) u(s)\, ds, \quad t \in I. \tag{2.32}$$

Then, for any integer $m \geq 1$ and integer $\nu \geq 1$: If $f \in C^{m+\nu-1}(I)$ and $K \in C^{m+\nu-1}(D)$, then the solution u of (2.32) lies in the space $C^{\nu-1,\alpha}(I)$ for some $\alpha \in (0,1]$ (related to Holder continuity of $(\nu-1)$-th derivative), and If f, K are sufficiently smooth (e.g., C^ν), then $u \in C^\nu(I)$ if $K(t,t) \neq 0$. The ν-th derivative might have a specific behavior near $t = 0$ depending on $f^{(\nu)}(0)$ and $K(0,0)$.

Theorem 2.4.2 (as originally stated): Assume that $f \in C^m(I)$ and $K \in C^m(D)$, with $K(t,t) \neq 0$ for $t \in I$. Then, for any $m \geq 1$, the solution of (2.31) with $\nu \geq 1$ lies in the space $C^{\nu-1,1-\alpha}(I)$, and

$$u^{(\nu)}(t) \sim C_{\nu,\alpha} t^{\alpha-1} \quad \text{as} \quad t \to 0^+,$$

where $C_{\nu,\alpha}$ denotes some non-zero constant. (This statement likely applies to a more specific form than (2.31) as written above, possibly involving fractional integrals or derivatives).

2.5 Kernels with Boundary Singularities

A representative example of a second-kind Volterra integral equation whose kernel contains both diagonal ($t = s$) and boundary ($s = 0$) singularities is

$$u(t) = f(t) + (V_{\alpha,\beta} u)(t), \quad t \in I, \tag{2.33}$$

where the operator $V_{\alpha,\beta}$ is defined as

$$(V_{\alpha,\beta} u)(t) := \int_0^t (t-s)^{\alpha-1} s^{-\beta} K(t,s) u(s) ds, \quad (0 < \alpha < 1, 0 \leq \beta < \alpha). \tag{2.34}$$

To ensure the integral converges and study regularity, we typically assume $K \in C(D)$ and often that K is smoother. We might assume $K(t,t) \neq 0$ (non-vanishing on diagonal) and $K(t,0) \neq 0$ (non-vanishing at boundary singularity source).

Theorem 2.5.1 : Assume that the kernel in (2.34) satisfies $K \in C(D)$, $K(t,t) \neq 0$, $K(t,0) \neq 0$ for $t \in I$, and $0 < \alpha < 1$, $0 \leq \beta < \alpha$. Then the VIE (2.33) possesses a unique solution u in $C(I)$ whenever $f \in C(I)$. Moreover, if $f \in C^m(I)$ and $K \in C^m(D)$, the solution u lies in $C^{m,\alpha-\beta}(I)$ (this is a more standard type of result, indicating Holder continuity), and there exists a non-zero constant $C_{\alpha,\beta}$ so that

$$u'(t) \sim C_{\alpha,\beta} t^{\alpha-\beta-1} \quad \text{as} \quad t \to 0^+.$$

(Note: The exact Holder space and derivative behavior can be quite complex and depend significantly on the interplay between α, β, and the behavior of K near $s = 0$ and $s = t$. The original statements $C^{\alpha+\beta}$ and $t^{\alpha+\beta-1}$ should be verified with the source.)

Consequence of the Theorem (Based on Original Statement)

According to the theorem (as originally stated), if the function f is continuous on the interval I, i.e., $f \in C(I)$, then the solution $u(t)$ is unique and belongs to the Hölder space $C^{\alpha+\beta}(I)$, which means that the solution has a certain degree of smoothness but is not necessarily fully differentiable, especially if $\alpha + \beta \leq 1$.

Regularity of the Solution (Based on Original Statement)

If f is more regular, i.e., $f \in C^m(I)$, and if the kernel K is also more regular, i.e., $K \in C^m(D)$, then the solution u has higher regularity and belongs to the space $C^{m,\alpha+\beta}(I)$. In other words, the solution becomes smoother; however, it is still not necessarily differentiable $m + 1$ times at $t = 0$ if $\alpha + \beta$ is not an integer.

Behavior of the Solution Near $t = 0$ (Based on Original Statement)

The theorem also states the existence of a non-zero constant $C_{\alpha,\beta}$ such that

$$u'(t) \sim C_{\alpha,\beta} t^{\alpha+\beta-1} \quad \text{as} \quad t \to 0^+.$$

This provides insight into how the solution behaves as it approaches zero. Since the exponent $\alpha + \beta - 1$ may be negative (if $\alpha + \beta < 1$), this implies that the first derivative of the solution may blow up at $t = 0$, indicating that the solution does not belong to $C^1(I)$.

Example: Effect of Boundary Singularities

Consider the following Volterra integral equation of the second kind:

$$u(t) = 1 + \int_0^t (t - s)^{\alpha-1} s^{-\beta} u(s) ds, \tag{2.35}$$

where we choose $\alpha = 0.6$ and $\beta = 0.4$. These values satisfy the conditions of Theorem 2.5.1, namely $0 < \alpha < 1$ (i.e., $0 < 0.6 < 1$) and $0 \leq \beta < \alpha$ (i.e., $0 \leq 0.4 < 0.6$).

Here, the forcing term is $f(t) = 1$, which is continuous (in fact, $f \in C^\infty(I)$). The integral term corresponds to $(V_{\alpha,\beta}u)(t)$ with the smooth part of the kernel $K(t, s) = 1$. Since $f(t) = 1$ and $K(t, s) = 1$ are smooth functions (e.g., $m = 0$ for continuity, or $m \geq 1$ for higher smoothness), and $K(t, t) = 1 \neq 0$, $K(t, 0) = 1 \neq 0$, all conditions of Theorem 2.5.1 are met.

According to Theorem 2.5.1, the solution $u(t)$ exists uniquely in $C([0, T])$. The behavior of its derivative near $t = 0$ is given by $u'(t) \sim C_{\alpha,\beta} t^{\alpha-\beta-1}$.

In this example, with $\alpha = 0.6$ and $\beta = 0.4$: The exponent for the derivative's behavior is:

$$\alpha - \beta - 1 = 0.6 - 0.4 - 1 = 0.2 - 1 = -0.8.$$

So, the derivative is expected to behave like:

$$u'(t) \sim C_{\alpha,\beta} t^{-0.8} \quad \text{as } t \to 0^+.$$

This indicates that the derivative $u'(t)$ blows up as $t \to 0^+$, and thus the solution $u(t)$ is not in $C^1([0,T])$ at $t = 0$, even though it is continuous on $[0,T]$.

2.6 Integral Equations with Weakly Singular Kernels of the Form $(t^2 - s^2)^{-1/2}$

This study examines second-kind Volterra integral equations with weakly singular kernels, focusing on the equation:

$$u(t) = f(t) + \int_0^t (t^2 - s^2)^{-1/2} K(t,s) u(s) ds, \quad t \in I. \tag{2.36}$$

The effect of choosing the kernel $K(t,s)$ on the properties of the associated integral operator is analyzed. If the kernel has the form $K(t,s) = \kappa s$, where $\kappa \neq 0$, then $K(0,0) = 0$. In this case, the corresponding integral operator is compact on $C(I)$ with a spectrum consisting only of zero. This ensures the existence of a unique continuous solution $u \in C(I)$ for every $f \in C(I)$.

However, when the kernel is constant, i.e., $K(t,s) = \kappa \neq 0$, then $K(0,0) \neq 0$. The operator becomes non-compact on $C(I)$ and has an uncountable spectrum ($[-\lambda_0, \lambda_0]$ for some λ_0), potentially leading to a loss of uniqueness if 1 is in the spectrum. Uniqueness depends on whether 1 is an eigenvalue of the operator.

It is shown that if the equation possesses a unique solution, it inherits the regularity of f and K. A special case where $K(t,s) = \lambda s$ is also examined using the Picard iteration method (successive approximations), leading to the determination of the resolvent kernel $R(t,s;\lambda)$ in terms of the Mittag-Leffler function.

Theorem 2.6.1 : Assume that $f \in C^m(I)$ and $K \in C^m(D)$ for some $m \geq 0$. Consider the VIE (2.36). Then:

(a) If $K(0,0) = 0$, the integral equation (2.36) has a unique continuous solution $u \in C(I)$. Furthermore, $u \in C^m(I)$.

(b) If $K(0,0) \neq 0$, the continuous solution is unique if and only if 1 is not an eigenvalue of the Volterra integral operator associated with equation (2.36):

$$(Vu)(t) := \int_0^t (t^2 - s^2)^{-1/2} K(t,s) u(s) ds. \tag{2.37}$$

If equation (2.36) has a unique solution, then it belongs to $C^m(I)$.

Proof using Successive Approximation (Case K(t,s) = λ s)

Consider the Volterra integral equation:

$$u(t) = f(t) + \lambda \int_0^t (t^2 - s^2)^{-1/2} s u(s) ds, \quad t \in I. \tag{2.38}$$

Here $K(t,s) = \lambda s$, so $K(0,0) = 0$. Case (a) of the theorem applies, guaranteeing a unique solution. We can construct it using successive approximations.

We define the sequence of approximations $u_n(t)$ as follows:

- **Initial step:** Start with the initial approximation:

$$u_0(t) = f(t). \tag{2.39}$$

- **Iteration step:** Compute $u_{n+1}(t)$ using:

$$u_{n+1}(t) = f(t) + \lambda \int_0^t (t^2 - s^2)^{-1/2} s u_n(s) ds. \tag{2.40}$$

Step 1: Proving the sequence $\{u_n(t)\}$ converges

Let V be the integral operator: $(Vu)(t) = \lambda \int_0^t (t^2 - s^2)^{-1/2} s u(s) ds$. Then $u_1 = f + Vu_0$, $u_2 = f + Vu_1 = f + V(f + Vu_0) = f + Vf + V^2 f$, and generally $u_n = \sum_{k=0}^{n} V^k f$. The solution is $u = \sum_{k=0}^{\infty} V^k f$. We need to show this series converges. Since $K(0,0) = 0$, the operator V is compact on $C(I)$. We can show convergence using bounds on iterated kernels or operator norms in weighted spaces. For Volterra operators on $C[0,T]$, the spectral radius is typically 0, ensuring convergence of the Neumann series $\sum V^k f$ for all λ.

Convergence can be shown by bounding the operator norm. Let $M = \sup |K(t,s)| = \sup |\lambda s| = |\lambda| T$. Let L be the bound for the singular part $\int_0^t (t^2 - s^2)^{-1/2} ds = \int_0^1 (1 - z^2)^{-1/2} t dz = t \arcsin(1) = \frac{\pi}{2} t$. The norm $\|V^n\|$ can be shown to decay faster than any geometric progression, ensuring convergence. Specifically, $\|(V^k f)(t)\| \le C^k \frac{t^{k\epsilon}}{\Gamma(1 + k\epsilon)} \|f\|$ for some $C, \epsilon > 0$, leading to uniform convergence on $[0, T]$.

Step 2: Proving the regularity of the solution

If $f \in C^m(I)$, we need to show $u \in C^m(I)$. Since $u = f + Vu$, if $u \in C^k(I)$, then Vu involves integrating $su(s)$ against a kernel. If $K \in C^m$, the regularity of Vu is related to the regularity of u. For $K(t,s) = \lambda s$, which is C^∞, and the kernel $(t^2 - s^2)^{-1/2}$, the operator maps $C^k(I)$ to $C^k(I)$. $u_0 = f \in C^m(I)$. $u_1 = f + Vu_0$. Since $u_0 \in C^m$, Vu_0 involves $\int_0^t (t^2 - s^2)^{-1/2} s f(s) ds$. Differentiating this requires care due to the singularity. It can be shown that $Vu_0 \in C^m(I)$. By induction, if $u_n \in C^m(I)$, then $u_{n+1} = f + Vu_n \in C^m(I)$. Since the convergence $u_n \to u$ is uniform, and if the derivatives also converge uniformly, the limit u will be in $C^m(I)$. This holds for this type of equation.

Conclusion

Thus, we have established the existence and uniqueness of $u(t)$ in $C(I)$ and shown that if $f \in C^m(I)$ (and $K \in C^m(D)$, which holds for $K = \lambda s$), then the solution $u(t)$ belongs to $C^m(I)$, completing the proof outline for case (a) applied to $K(t,s) = \lambda s$.
Note: In a metric space (X, d), we say that the space is **complete** if every Cauchy sequence in X converges to a limit that is also in X.

1. **Metric Properties:** A function $d : X \times X \to [0, \infty)$ is a metric if for all $x, y, z \in X$:

   - **Non-negativity & Identity of Indiscernibles:** $d(x,y) \geq 0$, and $d(x,y) = 0 \iff x = y$.
   - **Symmetry:** $d(x,y) = d(y,x)$.
   - **Triangle Inequality:** $d(x,z) \leq d(x,y) + d(y,z)$.

2. **Completeness:** The metric space (X, d) is said to be **complete** if every Cauchy sequence converges in X.

   - A sequence $\{x_n\}$ in X is Cauchy if for every $\epsilon > 0$, there exists an integer N such that $d(x_n, x_m) < \epsilon$ for all $n, m > N$.
   - Completeness means that for any such Cauchy sequence $\{x_n\}$, there exists a limit $x \in X$ such that $\lim_{n\to\infty} d(x_n, x) = 0$.

   The space $C(I)$ with the sup-norm $\|u\|_\infty = \sup_{t \in I} |u(t)|$ is a complete metric space (it's a Banach space).

Remark : The regularity of solutions of the related VIEs

$$u(t) = f(t) + \int_0^t p(t,s) K(t,s) u(s) ds, \quad t \in I, \tag{2.41}$$

and

$$u(t) = f(t) + \int_0^t q(t,s) K(t,s) u(s) ds, \quad t \in I, \tag{2.42}$$

where

$$p(t,s) := \pi^{-1/2}(\log(t/s))^{-1/2} s t^{\mu} s^{-1} = \pi^{-1/2}(\log(t/s))^{-1/2} t^{\mu}, \tag{2.43}$$

and

$$q(t,s) := s t^{\mu} s^{-1} = t^{\mu}, \tag{2.44}$$

with $\mu > 0$, was analysed in Han (1994). As shown in Diogo, McKee & Tang (1991), the above VIEs are closely related: (2.42) can be obtained from (2.41) by a simple change of variables. A typical result states that the VIE (2.41) with $p(t,s)$ as in (2.43) and $\mu > 1$, and with $f \in C^m(I)$, $K \in C^m(D)$ $(m \geq 1)$ has a unique solution $u \in C^m(I)$. In other words, for such a weakly singular VIE, the solution inherits the regularity of the data on the closed interval I.

Chapter 3

Applications of Volterra Integral Equations

3.1 Application Example: The Renewal Equation as a Linear Volterra Integral Equation

The **Renewal equation**, which is a classical model in population dynamics and reliability theory, is given by:

$$u(t) = f(t) + \int_0^t k(t-s)u(s)\,ds = f(t) + \int_0^t k(s)u(t-s)\,ds, \quad t \geq 0, \tag{3.1}$$

where:

- $u(t)$ denotes the exit rate (e.g., death rate or service completion rate) at time t,
- $f(t)$ is the exit rate from the initial population,
- $k(s)$ is the probability density function of the lifetime s, i.e., the probability that an individual exits the system at age s , so that $k(s) \geq 0$ and $\int_0^\infty k(s)\,ds = 1$, as k represents a probability distribution. .
  This equation serves as a fundamental case upon which more complex models, such as nonlinear or delayed models, are built. Regularity analysis in this case is clear and tractable, offering a solid benchmark for exploring how changes in the kernel $k(t)$ affect the solution in more general cases. A dedicated section may be included to investigate different kernel types(smooth, weakly singular, or irregular)and their effect on the solution using this model.

3.1.1 Equation Properties

This equation is characterized by the following properties:

- It is **linear**, since $u(t)$ appears only linearly.
- It is a **Volterra integral equation of the second kind**, as $u(t)$ appears both inside and outside the integral.

- It has a **convolution kernel**, since $k(t-s)$ depends only on the difference $(t-s)$.

3.1.2 Regularity Analysis

To analyze the regularity of the solution:

- If $k \in C^\infty([0,T])$ and $f \in C^\infty([0,T])$, then the solution $u(t) \in C^\infty([0,T])$.
- If k is continuous or piecewise smooth, then the regularity of u depends on that of k and f via successive approximations.
- A Neumann series (successive approximations) can be used to construct the solution:

$$u(t) = f(t) + \int_0^t k(t-s)f(s)\,ds + \int_0^t k(t-s)\left(\int_0^s k(s-\tau)f(\tau)\,d\tau\right) ds + \cdots \tag{3.2}$$

This representation facilitates numerical and theoretical analysis of the solution.

In this figure:

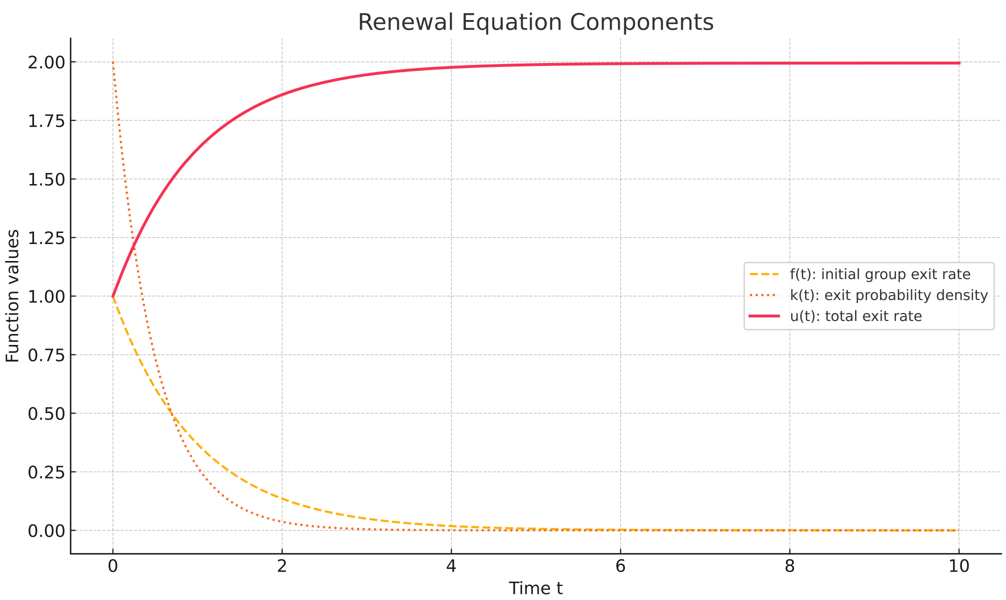

Figure 3.1: Graphical representation of the renewal equation and its components.

- The dashed curve represents $f(t) = e^{-t}$, which is the exit rate of individuals who were originally present in the system at time $t = 0$.
- The dotted curve corresponds to $k(t) = 2e^{-2t}$, a probability density function representing the likelihood that an individual leaves the system exactly t units of time after entering.
- The solid curve shows $u(t)$, the total exit rate at time t. This function combines the immediate exits from the initial population $(f(t))$ and the delayed exits from individuals

who entered the system at earlier times and are now leaving according to the distribution k.

Example: Linear Volterra Equation in Heat Transfer Modeling

Equation:

$$u(t) = f(t) + \int_0^t K(t-s)\, u(s)\, ds, \quad t \in [0, T]$$

This equation represents a simplified mathematical model for heat transfer in a rod, where the temperature at a given moment depends on both an external heat source and the cumulative effect of previous temperatures.

- $u(t)$: the temperature at a specific point in the rod at time t.
- $f(t)$: the external heat input (directly applied heat at time t).
- $K(t-s)$: the memory kernel, which describes the material's response to heat over the interval from time s to t, such as heat retention or diffusion rate.
- The term $\int_0^t K(t-s)\, u(s)\, ds$: represents the cumulative effect of past temperature states weighted by the material response.

Specific Kernel Choice (optional extension): If we choose the kernel $K(t-s) = e^{-(t-s)}$, the equation becomes:

$$u(t) = f(t) + \int_0^t e^{-(t-s)}\, u(s)\, ds$$

This form is consistent with the theory discussed in the chapter on smooth kernels, and its solution can be analyzed using the resolvent kernel method.

We consider the linear Volterra equation of the second kind, commonly known as the *renewal equation*:

$$u(t) = f(t) + \int_0^t k(t-s)\, u(s)\, ds$$

with the given functions:

$$f(t) = e^{-t}, \quad k(t) = 2e^{-2t}$$

Our goal is to determine the solution $u(t)$ using the **Laplace Transform method**.

Taking the Laplace transform of both sides of the equation:

$$\mathcal{L}[u](p) = \mathcal{L}[f](p) + \mathcal{L}[k * u](p)$$

Using the convolution property:

$$\mathcal{L}[k * u](p) = \mathcal{L}[k](p) \cdot \mathcal{L}[u](p)$$

So the equation becomes:

$$\mathcal{L}[u](p) = \mathcal{L}[f](p) + \mathcal{L}[k](p) \cdot \mathcal{L}[u](p)$$

Solving for $\mathcal{L}[u](p)$:

$$\mathcal{L}[u](p)(1 - \mathcal{L}[k](p)) = \mathcal{L}[f](p) \quad \Rightarrow \quad \mathcal{L}[u](p) = \frac{\mathcal{L}[f](p)}{1 - \mathcal{L}[k](p)}$$

$$\mathcal{L}[f](p) = \mathcal{L}[e^{-t}] = \frac{1}{p+1} \quad , \quad \mathcal{L}[k](p) = \mathcal{L}[2e^{-2t}] = \frac{2}{p+2}$$

Substituting into the formula:

$$\mathcal{L}[u](p) = \frac{1}{p+1} \cdot \frac{1}{1 - \frac{2}{p+2}} = \frac{1}{p+1} \cdot \frac{p+2}{p} = \frac{p+2}{p(p+1)}$$

We decompose:

$$\frac{p+2}{p(p+1)} = \frac{A}{p} + \frac{B}{p+1}$$

Solving for A and B gives:

$$p + 2 = A(p+1) + Bp$$

Substitute specific values:

$$p = 0 \Rightarrow A = 2 \quad , \quad p = -1 \Rightarrow B = -1$$

Hence:

$$\mathcal{L}[u](p) = \frac{2}{p} - \frac{1}{p+1}$$

Taking the inverse Laplace transform yields:

$$u(t) = 2 - e^{-t}$$

Final Result

$$\boxed{u(t) = 2 - e^{-t}}$$

This result represents the exact solution to the renewal equation, and may be compared with approximations obtained via the Neumann series.

3.2 Migration and Mortality Model: Volterra Equation with a Non-smooth Kernel

We consider a population model where the size varies due to migration and mortality. It is described by the following Volterra integral equation:

$$u(t) = f(t) + \int_0^t k(t,s)u(s)\,ds, \quad t \in [0,T], \tag{3.3}$$

where:

- $u(t)$ represents the number of individuals in the population at time t,
- $f(t)$ is the number of initial individuals who have survived up to time t,
- $k(t,s)$ is a kernel function representing the demographic changes (due to migration or mortality) occurring between times s and t.

The significance of this model lies in its presentation of a linear case with a non-smooth kernel, enriching the theoretical study on how the kernel's regularity affects the solution. It illustrates the contrast between smooth and weakly singular kernels in both analytical and numerical analysis. This model provides a realistic and mathematically tractable example that can be extended to more complex nonlinear models in later sections.
A typical example of the kernel is:

$$k(t,s) = \frac{\alpha}{t-s}, \quad \text{with } \alpha > 0,$$

which is a **weakly singular kernel** due to the singularity as $s \to t$.

3.2.1 Equation Properties

- The equation is **linear**, as $u(t)$ appears linearly inside and outside the integral.
- It is of the **second kind**.
- The kernel $k(t,s)$ is not smooth at $s = t$, and may exhibit a weak singularity.

3.2.2 Regularity Analysis of the Solution

- If $f \in C^r([0,T])$, the regularity of $u(t)$ depends on the smoothness of the kernel:
  - If $k(t,s)$ is smooth in the triangle $0 \le s \le t \le T$, then $u(t) \in C^r([0,T])$.
  - If $k(t,s)$ has a weak singularity (e.g., $k(t,s) \sim \frac{1}{t-s}$), then $u(t)$ may lose regularity near $t = 0$.
- Existence and uniqueness of the solution are guaranteed in function spaces such as $L^2([0,T])$ or $C([0,T])$, even in the presence of weak singularities in the kernel.

Example

Consider the Volterra integral equation:

$$u(t) = f(t) + \int_0^t \frac{1}{t-s} u(s)\, ds,$$

where the forcing function is given by $f(t) = 1$. The kernel

$$k(t, s) = \frac{1}{t-s}$$

has a weak singularity at $s = t$, while $f(t)$ is constant and continuous. This equation can be interpreted as describing the number of individuals at time t, where the value evolves due to the initial constant population $f(t)$ and a cumulative demographic effect represented by the integral term.

Example

Find an approximate solution to the Volterra integral equation:

$$u(t) = 1 + \int_0^t \frac{1}{t-s} u(s)\, ds$$

using the iterative approximation method (Neumann series). Compute $u_0(t)$ and $u_1(t)$.

Solution

We begin with the initial approximation:

$$u_0(t) = f(t) = 1.$$

Next, we compute $u_1(t)$ by substituting $u_0(t)$ into the integral:

$$\begin{aligned} u_1(t) &= 1 + \int_0^t \frac{1}{t-s} u_0(s) ds \\ &= 1 + \int_0^t \frac{1}{t-s} \cdot 1 ds \\ &= 1 + \int_0^t \frac{1}{t-s} ds. \end{aligned}$$

We perform a change of variable for the integral $I = \int_0^t \frac{1}{t-s} ds$: Let $v = t - s$, then $dv = -ds$. When $s = 0$, $v = t$. When $s = t$, $v = 0$. Rewriting the integral:

$$I = \int_{s=0}^{s=t} \frac{1}{t-s} ds = \int_{v=t}^{v=0} \frac{1}{v}(-dv) = \int_0^t \frac{1}{v} dv.$$

Evaluating this definite integral:

$$\int_0^t \frac{1}{v} dv = [\ln |v|]_0^t = \ln |t| - \lim_{v \to 0^+} \ln |v|.$$

Since $\lim_{v \to 0^+} \ln |v| = -\infty$, the integral I diverges to $+\infty$ for any $t > 0$. The integral $\int_0^t \frac{1}{t-s} ds$ does not evaluate to $\ln(t)$; instead, it is a divergent integral.

Therefore: The calculation for $u_1(t)$ involves a divergent integral.

Result

$u_0(t) = 1$. The first-order term $u_1(t)$ cannot be represented as $1 + \ln(t)$ because the integral $\int_0^t \frac{1}{t-s} ds$ diverges. This means that the first-order approximation, as calculated by this direct iterative step, is not a finite function. The nature of the singularity in the kernel $1/(t-s)$ prevents a straightforward evaluation of $u_1(t)$ using this method.

Aspect	**Renewal Model**	**Migration and Mortality Model**
Volterra Equation	$u(t) = f(t) + \int_0^t k(t-s)u(s)\, ds$	$u(t) = f(t) + \int_0^t k(t,s)u(s)\, ds$
Kernel Type	Smooth kernel depending only on $t-s$	Non-smooth (weakly singular) kernel depending on t and s
Example of Kernel	$k(t) = 2e^{-2t}$	$k(t,s) = \frac{\alpha}{t-s}$, $\alpha > 0$
Regularity	The solution $u(t)$ is smooth if f and k are smooth	The solution may lose regularity near $t = 0$
Solution Methods	Analytical methods such as Neumann series and Laplace transform	Requires advanced analytical or numerical techniques
Application Meaning	Models the time until exit or death after entering a system	Models population change over time due to cumulative migration and mortality

Table 3.1: Comparison between the Renewal Model and the Migration-Mortality Model in terms of their Volterra integral equation structure, kernel properties, and analytical implications.

Conclusion

In this thesis, we explored the field of integral equations, focusing specifically on linear Volterra equations of the second kind due to their significant role in the mathematical modeling of dynamic systems. We began by analyzing the ideal case where the kernel is smooth, a setting that typically allows for regular solution behavior and facilitates theoretical analysis.

We then examined more challenging cases where the behavior of the solution becomes more sensitive to the nature of the kernel, such as weakly singular kernels and kernels with boundary singularities. We highlighted how these features influence the regularity of the solution and discussed theoretical results that help to better understand these scenarios.

Our work extended beyond theoretical aspects, as we also linked these concepts to practical applications, most notably the renewal equation, which appears in several areas including statistics and biology. In some examples, we made use of the Laplace transform as a tool to simplify the solving process and analyze the structure of the equation.

The thesis concludes with a collection of theoretical exercises aimed at reinforcing the reader's understanding and encouraging deeper engagement with the material. We hope this work contributes to clarifying fundamental aspects of this important topic and serves as a helpful introduction for anyone interested in further studying this rich and complex field.

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الملخص

تعد المعادلات التكاملية أداة مهمة في نمذجة العديد من الظواهر الفيزيائية والهندسية. في هذه المذكرة، تناولنا المفاهيم الأساسية مع التركيز على معادلات فولتيرا الخطية من النوع الثاني، ودرسنا انتظامية الحلول تبعًا لطبيعة النواة، من النوى الملساء إلى النوى ذات التفرد. كما عرضنا تطبيقات عملية مثل معادلة التجديد، واستخدمنا تحويل لابلاس لتبسيط بعض التحاليل، واختتمنا المذكرة بتمارين تعزز الفهم النظري والتطبيقي.
الكلمات المفتاحية: النواة, معادلات فولتيرا, المعادلات التكاملية.

Abstract

Integral equations are an important tool for modeling many physical and engineering phenomena. In this thesis, we addressed the fundamental concepts with a focus on linear Volterra equations of the second kind, and we studied the regularity of solutions depending on the nature of the kernel, from smooth kernels to singular ones. We also presented practical applications such as the renewal equation, used the Laplace transform to simplify some analyses, and concluded the thesis with exercises that reinforce both theoretical and practical understanding.
Keywords: Kernel, Volterra equations, Integral equations.

Résumé

Les équations intégrales sont un outil important pour modéliser de nombreux phénomènes physiques et techniques. Dans ce mémoire, nous avons abordé les concepts fondamentaux en mettant l'accent sur les équations de Volterra linéaires du second type, et nous avons étudié la régularité des solutions en fonction de la nature du noyau, des noyaux réguliers aux noyaux singuliers. Nous avons également présenté des applications pratiques telles que l'équation de renouvellement, utilisé la transformée de Laplace pour simplifier certaines analyses, et conclu le mémoire par des exercices renforçant la compréhension théorique et pratique.
Mots clés: Noyau, Volterra linéaires, Equations intégrales.