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Ghofrane KHARFALLAH

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Hopf Bifurcation in Delay Differential Equations.

Examination Committee Members:

Dr.	Assia SENOUCI	MKUB	President
Dr.	Baya LAADJAL	MKUB	Supervisor
Dr.	Ouarda TABRHA	UMKB	Examiner

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Dedication

Praise be to Allah for the blessings of writing and thinking,
and for faith and patience until the dream came true.

I dedicate this thesis to: the dearest and most cherished people in my life, my beloved parents, ***Rabeh*** *and* ***Fatima****. Your love, devotion, and sacrifices have been the foundation of my academic success.*
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Abstract

This thesis focuses on the study of Hopf bifurcation in delay differential equations (DDEs), which depend on both the current and past states of the system. The work aims to examine the existence and uniqueness of solutions, and to analyze linear and nonlinear stability using tools such as the characteristic equation and Lyapunov functions, in addition to adopting a geometric approach to the analysis. The study focuses on the conditions under which Hopf bifurcation occurs and shows how changes in parameters can lead to the emergence of limit cycles. The theoretical part is supported by numerical applications that illustrate the system's behavior before and after the bifurcation.
Keywords: Delay Differential Equations, History Function, Stability Analysis, Hopf Bifurcation.

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Notations

– ODEs	Ordinary differential equations.
– DDEs	Delay differential equations.
– $y(t)$	The state of the system at time t.
– $y(t-\tau)$	The state of the system at a previous time $t-\tau$.
– τ	The delay.
– $t-\tau$	The time at which the function was in the past.
– $\varphi(t)$	The history function.
– $D(\lambda)$	The transcendental characteristic equation.
– $R(\lambda)$	The real part of a complex number λ.
– V	Lyapunov function.
– y_t	A past state of the system.
– μ_c	Parameter at which a Hopf bifurcation occurs.
– RFDE	Retarded Functional Differential Equation.

Introduction

Differential equations are fundamental tools in the mathematical modeling of natural phenomena and dynamic systems in various fields such as physics, economics, biology, and engineering. While ordinary differential equations (ODEs) have achieved great success in this field, their reliance solely on the current state of the system often proves insufficient to accurately represent many real-world systems that are influenced by past states, such as epidemic models, neural networks, and control systems.

This deficiency has led to the emergence of delay differential equations, which depend on both the current and past states of a system. By incorporating delay, these equations offer a more realistic description. Delay differential equations (DDEs) have been widely used in modeling physical and biological phenomena that exhibit time delays in their dynamics. For instance, DDEs are commonly used to model the dynamics of populations with time delays in their reproduction, the spread of infectious diseases with incubation periods [9]. Delay differential equations were initially introduced in the 18th century by Laplace and Condorcet. However, the rapid development of the theory and applications of those equations did not come until after the Second World War, and continues today[6].

They have development over history, especially in the year 1908, during the international conference of mathematicians, picard emphasized the significance of accounting for past effects when constructing models of physical systems, and In 1931, Volterra wrote a fundamental book on the role of hereditary effects on models for the interaction of species. DDEs gained momentum post 1940, driven by engineering and control challenges. During the 1950's, there was considerable activity in the subject which led to important publications by Myshkis (1951), Krasovskii (1959), in the 1960's, Bellman and Cooke (1963), Halanay (1966)[9].

Among the critical dynamical phenomena observed in such systems is Hopf bifurcation, which marks a qualitative change in the system's behavior. At a Hopf bifurcation point, a slight variation in a parameter can cause the system to transition from a stable steady state to exhibiting periodic oscillations. This phenomenon is particularly important in the study of stability and the effects of time delays on system dynamics.

The motivation for studying Hopf bifurcation in DDEs stems from the desire to understand the emergence of oscillations in real systems with memory, such as glucose-insulin regulation and population dynamics, where time delays play a crucial role in shaping long-term system behavior and stability.

This thesis aims to explore the behavior of delay differential equations near Hopf bifurcation points by analyzing stability, deriving the conditions under which the bifurcation occurs, and providing mathematical interpretation through rigorous analytical tools. We place special emphasis on the role of delay as a control parameter that may destabilize the system and lead to the formation of limit cycles.

The work is structured into two main chapters. In the **first chapter**, we introduce the fundamental concepts of delay differential equations, including their definition and resolution methods such as the method of steps and Laplace transform with a focus on existence and uniqueness theorems. Additionaly we study the nature of solutions by analyzing stability in both linear and nonlinear cases using the characteristic equation and Lyapunov functions, respectively. Finally, we highlight the impact of delay on system stability.

In the **second chapter**, we will focus on Hopf bifurcation. We begin by presenting the general concept and types of bifurcation, then focus on the Hopf bifurcation theorem, and present bifurcation diagrams that show how the behaviour of the system changes as the coefficients change. The chapter concludes with numerical simulations of one- and two-dimensional systems, providing a deeper understanding of the dynamics of limit cycle emergence.

Chapter 1

Preliminaries and Basic Properties

In many scientific applications, such as biology, medicine, economics and chemistry, phenomena are often modeled using ordinary or partial differential equations, where it is assumed that the future state of a system depends only on its current state, without any influence from the past. However, this assumption is not always accurate, as many real-world processes are affected by previous states. For example, in epidemic models, an individual infected with a virus does not become infectious immediately but goes through an incubation period before showing symptoms or transmitting the disease. Therefore, it is essential to use more realistic models that account for the influence of past states on the system's evolution, leading to the development of the theorem of delay differential equations [14].
In this chapter, we provide a formal definition of delay differential equations and discuss their fundamental properties. We also explore solution methods, the existence and uniqueness of solutions, stability analysis.

1.1 General Results on Delay Differential Equations (DDEs)

1.1.1 Definition of DDEs

Definition 1.1.1 ***Delay differential equations(DDEs)*** *are equations in which the change of state depends on time* t*, the present state* $y(t)$*, and the past state* $y(t-\tau)$*, where the delay* τ *is constant [7]. They are also called difference-differential equations and are considered a type of functional differential equations[6]. They have the form :*

$$\dot{y}(t) = f(t, y(t), y(t-\tau)) \quad for \quad t \geq t_0, \quad y \in \mathbb{R}^n, \tag{1.1}$$

where $\tau > 0$ *is the delay term and* $f : \mathbb{R} \times \mathbb{R}^{2n} \to \mathbb{R}^n$ *is a given function[9].*

Remark 1.1.1 *:*

- *Equation (1.1) is a first-order delay differential equation.*

- *If $\tau = 0$, then the equation (1.1) becomes an ordinary differential equation (ODE).*
- *If equation (1.1) does not explicitly contain t, i.e*

$$\dot{y}(t) = f(y(t), y(t-\tau)) \quad for \quad t \geq t_0, \quad y \in \mathbb{R}^n,$$

we call it an autonomous constant delay DDE.

The use of delay differential equations (DDE) creates more realistic models that arise in many fields such as (biology, chemistry, physics, economics and neuroscience..). There are different types of DDEs where the delay takes various forms :

Definition 1.1.2 ***(Types of DDEs)*** *[9]*

1. *If τ is constant, the equations are called delay differential equations with constant delay, see equation (1.2).*
2. *If τ depends on time, $\tau = \tau(t)$, we are talking about DDEs with time-dependent delay, see equation (1.3).*
3. *If τ depends on $y(t)$, $\tau = \tau(t, y(t))$, we are talking about DDEs with state-dependent delay, see equation (1.4).*

   There are other types of DDEs (such DDEs with distributed delays etc, DDEs of neutral type, etc...).

Example 1.1.1 *:*

1. ***(Mackey-Glass equation)*** *A model of circulating white blood cell numbers*

$$\dot{y}(t) = -\gamma y(t) + \beta \frac{y(t-\tau)}{1 + y(t-\tau)^n}, \quad y \in \mathbb{R}. \tag{1.2}$$

2. ***(Pantograph equation)*** *Originates from modelling pantographs*

$$\dot{y}(t) = ay(t) + by(kt), \quad y \in \mathbb{R}^n, \tag{1.3}$$

   where a, b and k are parameters with $k \in]0, 1[$, and $kt = t - \tau(t) \Longrightarrow \tau(t) = (1-k)t$.

3. ***(Sawtooth equation)*** *A model problem introduced by Mallet-Paret and Nussbaum*

$$\varepsilon \dot{y}(t) = -\gamma y(t) - ky(t - a - cy(t)), \quad y \in \mathbb{R}^n, \tag{1.4}$$

   where $\varepsilon, a, c > 0$ and $\gamma + k > 0$. This model gets its name from stable period solutions seen in $\varepsilon \longrightarrow 0$ limit, and $t - a + cy(t) = t - \tau(t, y(t)) \Longrightarrow \tau(t, y(t)) = a + cy(t)$.

1.1.2 Existence and uniqueness of an initial value problem

In ordinary differential equations (ODEs), the initial value is used because it is necessary to determine a unique solution to the equation, thus transforming it into an initial value problem (or Cauchy problem). From this perspective, we conclude that it is also essential to provide the initial value in delay differential equations (DDEs) to determine a unique solution, thus transforming it into an initial value problem. What do we mean by the solution?

Definition 1.1.3 ***(Solution of DDE)*** *A solution $y(t)$ of system (1.1) is a continuous function that satisfies the delay differential equation (1.1).*

Definition 1.1.2.1 ***(History function)*** *The history function in the delayed differential equations (1.1) represents the values of the function at previous points of time (the past values of the solution)*

$$y(t) = \varphi(t), \quad t \in [t_0 - \tau, t_0],$$

which provides the information required to calculate the derivative at the present time.

The history function determines the initial conditions of the delayed differential equations. In this example, we will demonstrate why the history function was chosen as an initial condition for DDE.

Example :[9] We consider the following ordinary differential equation:

$$\frac{dy}{dt} = ry(t), \tag{1.5}$$

where $r > 0$ is the growth rate.

This model predicts that the population will either grow or decline exponentially. Since the past affects the present, we consider the following DDE:

$$\frac{dy}{dt} = ry(t - \tau), \tag{1.6}$$

1. The term $ry(t - \tau)$ represents the population growth rate at time t, which depends on the population size $y(t - \tau)$ at a previous time $t - \tau$.

2. Here τ is the delay and it accounts for the time it takes for changes in resource availability to affect population growth.

By integrating (1.6), we obtain the following integral equation:

$$y(t) = y(t_0) + \int_{t_0}^{t} ry(s - \tau)ds,$$

Change of variable : $s-\tau=s \Longrightarrow ds=ds, s\longrightarrow t \Longrightarrow s=t-\tau$ end $s\longrightarrow t_0 \Longrightarrow s=t_0-\tau$.

$$y(t)=y(t_0)+\int_{t_0-\tau}^{t-\tau} ry(s)ds. \tag{1.7}$$

From equation (1.7), we conclude that to compute $y(t)$ for $t\geq t_0$, knowing only the value of $y(t_0)$ is not sufficient, we also need to know the values of $y(t)$ for $t\in[t_0-\tau,t_0]$.
Therefore, the initial conditions used in DDEs are different from those used in ODEs. In DDEs, an initial function must be specified over a certain interval of length τ, specifically on $t\in[t_0-\tau,t_0]$, and then we attempt to find the solution of equation (1.1) for all $t\geq t_0$.

Definition 1.1.4 *(**The initial value problem**) The problem involves finding a solution $y(t)$ that satisfies:*

$$\dot{y}(t)=f(t,y(t),y(t-\tau)), \quad t\geq t_0, \quad y\in\mathbb{R}^n,$$

where $\tau>0$ is the fixed delay, with an initial condition given on an earlier time interval:

$$y(t)=\varphi(t), \qquad\qquad t\in[t_0-\tau,t_0].$$

Is called the initial value problem (IVP) for the differential delay equation.

$$(IPV)\quad \begin{cases} \dot{y}(t)= & f(t,y(t),y(t-\tau)), \quad t\geq t_0 \quad \tau>0, \quad y\in\mathbb{R}^n. \\ y(t)= & \varphi(t), \qquad\qquad t\in J_{=}[t_0-\tau,t_0]. \end{cases} \tag{1.8}$$

The solution of the delay differential equation (1.1) with the initial condition $\varphi(t)$ is denoted by $y(t,\varphi)$.

Theorem 1.1.2.1 *Let $\tau>0$ be a constant in $J=[t_0,t_0+a]$, where $t_0>0$ and $a>0$. Let's consider the initial value problem (1.8). Assume that $f(t,y,x)$ and $f'_y(t,y,x)$, $f'_x(t,y,x)$ are continuous on $\mathbb{R}\times\mathbb{R}^{2n}$ and φ is a given continuous function on $\mathbb{R}$. Then the initial value problem (1.8) has exactly one solution.*

1.1.3 Analytical resolution of DDEs

The methods for solving delay differential equations vary depending on the type of delay and the nature of the equation, whether it is linear or nonlinear. Finding the solution can be complex in some cases. Therefore, in this section, we will present two fundamental methods for solving certain delay differential equations, with illustrative examples for each.

1. Method of Steps : This method is primarily used to solve linear or simple delay differential equations with a constant delay. Its main idea is to convert delay differential equations into a series of ordinary differential equations over small time intervals, where the solution in each state interval depends on the solution in the previous intervals.

1. First step, we solve DDE in the interval $[t_0, t_0 + \tau]$ using the history function (initial condition).

2. Second step, we use the solution obtained from the first step as an initial condition to find the solution in interval $[t_0 + \tau, t_0 + 2\tau]$.

3. We continue repeating the same method to find the solution in interval $[t_0 + (k-1)\tau, t_0 + k\tau] \quad (k = 3, 4, \dots)$.

By following these steps, a unique solution to the initial value problem(1.8) can be determined.

2. Laplace Transform : The main idea of this method is to transform delay differential equations into algebraic equations using Laplace transforms. This method is specifically applicable to linear delay differential equations with constant delay.

1. First, we apply the Laplace transform to the delay differential equation (DDE).

2. Second, we solve the resulting algebraic equation using the properties of the Laplace transform.

3. Finally, we use the inverse Laplace transform to find the solution $y(t)$ in the time domain.

Properties 1.1.1 *The Laplace transform has the following properties:*

(a) $\mathcal{L}[y(t)](s) = Y(s)$.

(b) $\mathcal{L}[y(t-\tau)](s) = e^{-s\tau}\left[\int_{-\tau}^{0} e^{-st}\varphi(t)dt + Y(s)\right]$, *where* $\tau > 0$.

(c) $\mathcal{L}[\frac{dy}{dt}](s) = -y(0) + sY(s)$.

(d) $\forall \lambda, \quad \mathcal{L}[\lambda y(t)](s) = \lambda Y(s)$.

Example:

Consider a delayed differential equation given by

$$\begin{cases} \dfrac{dy}{dt} = & -2y(t) + 3y(t-1), \quad t > 0. \\ y(t) = & e^t, \qquad\qquad\qquad \forall t \in [-1, 0]. \end{cases} \tag{1.9}$$

We want to solve this example using the method of steps and the Laplace transform.

1. We apply the **steps method** to equation (1.9).
   For all $t \in [0, 1]$, we integrate (1.9) on the interval $[0, t]$ leading to:

$$y(t) = y(0) - 2\int_0^t y(s)ds + 3\int_0^t y(s-1)ds,$$

Change of variable : s=s-1,

$$
\begin{aligned}
y(t) =& y(0) - 2\int_0^t y(s)ds + 3\int_{-1}^{t-1} y(s)ds,\\
=& 1 - 2\int_0^t y(s)ds + 3\int_{-1}^{t-1} e^s ds,\\
=& 1 + 3e^{t-1} - 3e^{-1} - 2\int_0^t y(s)ds,\\
y(t) =& y_1(t).
\end{aligned}
$$

Where, $t \in [0,1]$:

$$y_1(t) = y(0) - 2\int_0^t y_1(s)ds + 3\int_{-1}^{t-1} y(s)ds.$$

We repeat the same step to find of $y(t)$ on the intervals $[1,2]$,

$$
\begin{aligned}
y(t) =& y(1) - 2\int_1^t y(s)ds + 3\int_1^t y(s-1)ds,\\
=& y(1) - 2\int_1^t y(s)ds + 3\int_0^{t-1} y(s)ds,\\
=& y_1(1) - 2\int_1^t y(s)ds + 3\int_0^{t-1} y_1(s)ds,\\
y(t) =& y_2(t).
\end{aligned}
$$

Where, $t \in [1,2]$:

$$y_2(t) = y_1(1) - 2\int_1^t y_2(s)ds + 3\int_0^{t-1} y_1(s)ds.$$

$y_k(t)$ can be calculated in the same way on the intervals $[(K-1), K]$, for all $k = 3, 4, \ldots$. Where,$t \in [(K-1), K]$:

$$y_K(t) = y_{K-1}(t) - 2\int_{K-1}^t y_K(s)ds + 3\int_{K-2}^{t-1} y_{K-1}(s)ds.$$

The solutions obtained using the method of steps above are plotted in Figure 1.1, using the MATLAB solver `dde23`.

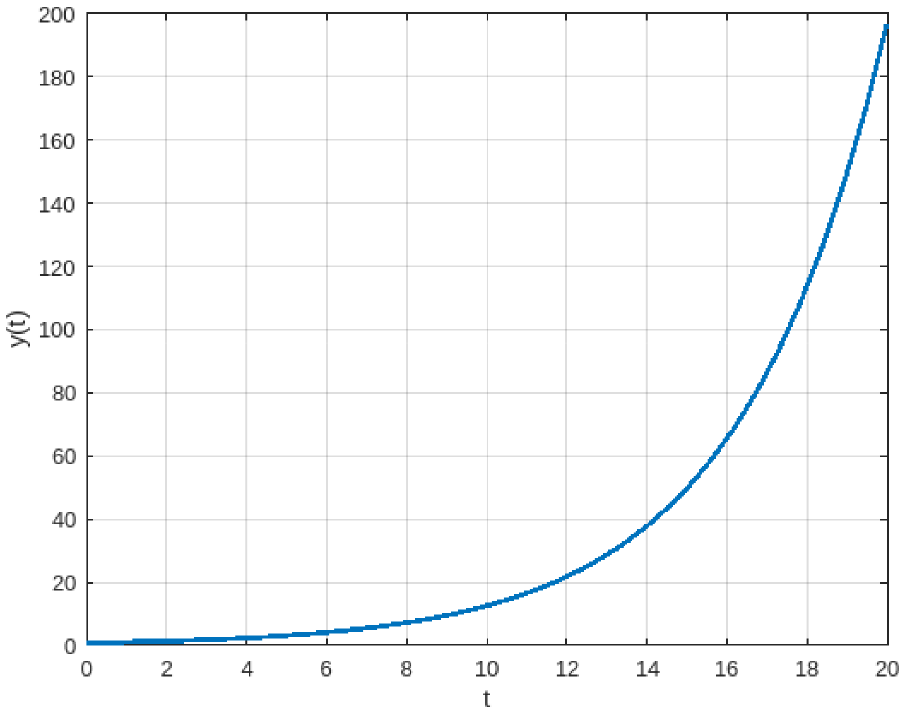


Figure 1.1: Exact solution of (1.9) using the method of steps.

2. We apply the **Laplace transform** method to equation (1.9).
   Applying the Laplace transform to both sides of equation (1.9), we obtain:

$$\mathcal{L}[\frac{dy}{dt}](s) = \mathcal{L}[-2y(t) + 3y(t-1)](s), \tag{1.10}$$

by using the linear properties of the Laplace transform.

$$\mathcal{L}[\frac{dy}{dt}](s) = -2\mathcal{L}[y(t)](s) + 3\mathcal{L}[y(t-1)](s). \tag{1.11}$$

After substituting (a), (b), and (c) into (1.11), we obtain.

$$\begin{aligned} -y(0) + sY(s) =& -2Y(s) + 3\mathcal{L}[y(t-1)](s), \\ =& -2Y(s) + \frac{3e^{-s}}{1-s} + \frac{3e^{-1}}{s-1} + 3e^{-s}Y(s), \\ sY(s) + 2Y(s) - 3e^{-s}Y(s) =& 1 + \frac{3(e^{-1} - e^{-s})}{s-1}, \\ Y(s) =& \frac{1 + \dfrac{3(e^{-1} - e^{-s})}{s-1}}{s + 2 - 3e^{-s}}. \end{aligned}$$

Where

$$\mathcal{L}[y(t-1)](s) = \frac{e^{-s}}{1-s} + \frac{e^{-1}}{s-1} + e^{-s}Y(s).$$

To find $y(t)$, we apply the inverse Laplace transform.

$$\mathcal{L}^{-1}[Y(s)](t) = \mathcal{L}^{-1}\left[\frac{1 + \dfrac{3(e^{-1}-e^{-s})}{s-1}}{s+2-3e^{-s}}\right](t),$$

$$y(t) = \mathcal{L}^{-1}\left[\frac{1 + \dfrac{3(e^{-1}-e^{-s})}{s-1}}{s+2-3e^{-s}}\right](t).$$

The inverse Laplace transform is difficult to find Because of the presence exponentials e^{-s}.

Comparison Between the Method Steps and the Laplace Transform Method :
1. Method of Steps : It's an easy-to-understand method that calculates the solution step by step over each time interval. We use past values to find new values, which is why it is very suitable for numerical and graphical applications. Its advantages include accurate results that can be easily represented graphically.
2. Laplace Transform : In the Laplace method, we transform the differential equation into a simpler algebraic form that is easier to manipulate mathematically, using the Laplace transform. This allows us to solve equations in a systematic and clear manner, especially in cases involving initial conditions. However, while this method is theoretically powerful, returning to the original solution in real time (i.e., finding the inverse transform) can be complex and difficult, especially if the resulting function contains unfamiliar exponential or fractional terms. Therefore, despite its great usefulness in analysis, the Laplace method is not always practical for numerical calculations or when representing solutions graphically.

1.2 Stability of Delay Differential Equations (DDEs)

In this section, we study the stability analysis of equilibrium points in autonomous delay differential equations, whether linear or nonlinear, with a particular focus on equations that contain a single delay, which is given by.

$$\begin{cases} \dot{y}(t) = f(y(t), y(t-\tau)) & for \quad t \geq 0, \\ y(t) = \varphi(t) & -\tau \leq t \leq 0, \end{cases} \tag{1.12}$$

where $\tau > 0$, $f \in C^1(E \times E, \mathbb{R}^n)$, $E \subseteq \mathbb{R}^n$ and $\varphi \in C([-\tau, 0], \mathbb{R}^n)$.
The solution of DDE (1.12) with initial data $\varphi(t)$ is denoted by $y(t, \varphi(t))$.

Definition 1.2.1 *Equilibrium points y^* are the values at which the system does not change over time. So, the points y^* are solutions of the equation*

$$f(y^*, y^*) = 0.$$

The equilibrium points are considered solutions to the system (1.12), and are called steady states.

Definition 1.2.2 ***(Periodic solution)*** *[14]*
A solution $y(t)$ of (1.12) is called a periodic solution if there exists a real number $T > 0$, such that

$$y(t + T) = y(t), \quad \forall t \geq 0.$$

Definition 1.2.3 ***(limit cycle)***
A limit cycle in a delay differential equation is a periodic solution that repeats over time and is isolated, meaning no other periodic solutions are close to it.

In dynamical systems theory, we know that the stability of the system's solutions is closely related to the stability of the equilibrium points.
After determining the equilibrium points, we study their stability using eigenvalues, which are the solutions of the characteristic equation. So, what is **the characteristic equation**?

1.2.1 Characteristic equation of DDE

In ordinary differential equations (ODEs), the characteristic equation is an algebraic equation in the form of a polynomial, which means that the number of its roots is finite and can be determined using the fundamental theorem of algebra.
On the other hand, in delay differential equations (DDEs), the characteristic equation is transcendental because it contains exponential functions ($e^{-\lambda\tau}$), making it non-polynomial. As a result, the fundamental theorem of algebra does not apply. For this reason, there is no general Theorem that determines the number of its roots, which may be infinite. This makes the study of characteristic roots more complex, and the equation is referred to as transcendental due to the presence of delay [11].
It is true that in ODEs, we need $n \geq 2$ to study complex stability, while in DDEs, stability can be complex even in the case of $n = 1$ due to the delay.

- **Characteristic equation in case of linear scalar DDEs** ($n = 1$) [12]
  A linear autonomous delay differential equation with a single delay is given as follows:

$$\dot{y}(t) = ay(t) + by(t - \tau), \quad x \in \mathbb{R}, \tau > 0, \tag{1.13}$$

where a and b are two real number.
Same logic as what we do with ODEs, we seek exponential solutions of the form:

$$y(t) = ve^{\lambda t}, \quad where \quad v \neq 0 \quad and \quad \lambda \in \mathbb{C}.$$

We plug it into (1.13) and get

$$\lambda e^{\lambda t} v = e^{\lambda t} va + e^{\lambda(t-\tau)} vb,$$

then we obtain

$$(\lambda - a - be^{-\lambda\tau})v = 0. \tag{1.14}$$

The equation (1.14) has non-zero solution if and only if

$$\lambda - a - be^{-\lambda\tau} = 0. \tag{1.15}$$

The equation (1.15) is called the transcendental characteristic equation of equation (1.13), and there are infinite number of solutions to this equation and the complex solution $\lambda = \alpha + i\beta$, lies on the curve

$$\beta = \pm\sqrt{b^2 e^{-2\alpha\tau} - (\alpha - a)^2}.$$

- **Characteristic equation in case of linear DDEs system** ($n \geq 2$) [14]
  A linear autonomous delay differential equation with a single delay is given as follows:

$$\dot{y}(t) = Ay(t) + By(t - \tau), \quad x \in \mathbb{R}^n, \tau > 0, \tag{1.16}$$

where A and B are two matrix.
To find the characteristic equation, we assume a solution of the form $y(t) = ce^{\lambda t}$, where $e^{\lambda t}$ represents the exponential function and $c \in \mathbb{C}^n$ is a constant vector ($c \neq 0$). When we apply this solution to the system (1.16), we obtain:

$$\lambda ce^{\lambda t} = Ace^{\lambda t} + Bce^{\lambda(t-\tau)},$$

By dividing by $e^{\lambda t}$, we obtain

$$(\lambda I - A - e^{-\lambda\tau}B)c = 0, \tag{1.17}$$

(1.17) has non-zero solution if and only if

$$D(\lambda) := det(\lambda I - A - e^{-\lambda\tau}B) = 0. \tag{1.18}$$

We call $D(\lambda)$ the transcendental characteristic equation of (1.16), and its roots are said to be characteristics or eigenvalues of (1.16).

1.2.2 Stability of the linear DDE

Consider a homogeneous linear autonomous system of delay differential equations with a single delay [14], which can be written as follows:

$$\dot{y}(t) = Ay(t) + By(t - \tau), \tag{1.19}$$

A is a constant coefficient matrix that determines the evolution of the system without delay.
B is a constant coefficient matrix representing the effect of delayed states.
To find the equilibrium points of the system (1.19), we set the derivative of the variable to zero and then look for the values that satisfy this condition.

$$\dot{y}(t) = 0 \Longrightarrow Ay(t) + By(t - \tau) = 0.$$

To find equilibrium points, we shall solve the equation :

$$Ay^* + By^* = 0.$$

We note that if the matrix $A + B$ is singular, then every $y^* \in \mathbb{R}^n$ is a solution of the previous equation, but if the matrix $A + B$ is non-singular, then the system (1.19) has a unique equilibrium point, which is $y^* = 0$. Therefore, all solutions of equation (1.19) are (stable) or (asymptotically stable) if and only if the equilibrium point $y^* = 0$ is (stable) or (asymptotically stable)[14].
In what follows, we present some theorems related to the stability of linear delay differential equations.

Theorem 1.2.1 *[14] Let $D(\lambda)$ be the characteristic equation (1.18), then the equilibrium $y^* = 0$ of (1.19) is **asymptotically stable** if all the roots of $D(\lambda)$ have negative real parts. In fact, there exist constants $\exists \sigma > 0, \exists K > 0$ such that*

$$\mid y(t, \varphi) \mid < Ke^{-\sigma t} \|\varphi\|, \quad t \geq 0, \varphi \in C,$$

where

$$Re(\lambda) < -\sigma,$$

and $y(t, \varphi)$ is the solution of (1.19) with initial condition $y_0 = \varphi$.
*On the other hand, if there exists a root with positive real part, it is **unstable**.*

Theorem 1.2.2 *[12]The following hold for the system (1.13).*

1. *If $a + b > 0$, then $y = 0$ is* ***unstable****.*

2. *If $a + b < 0$ and $b \geq a$, then $y = 0$ is* ***asymptotically stable****.*

3. *If $a + b < 0$ and $b < a$, then there $\exists \tau^* > 0$ such that $y = 0$ is* ***asymptotically stable*** *for $0 < \tau < \tau^*$ and* ***unstable*** *for $\tau > \tau^*$, where:*

$$\tau^* = \frac{1}{\sqrt{b^2 - a^2}} \cos^{-1}(\frac{-a}{b}) > 0.$$

1.2.3 Local stability of nonlinear system of delay differential equations

The local stability of nonlinear delay differential equation (DDE) systems is analyzed using various techniques, including:

- Linearization Method [15].
- Lyapunov Method [1].

1.2.3.1 Linearization of nonlinear DDE

Consider a nonlinear autonomous system of DDEs with a single delay (1.1). That system is equivalent to:

$$\begin{cases} \dot{y}_1(t) = f_1(y_1(t), \ldots, y_n(t), y_1(t-\tau), \ldots, y_n(t-\tau)), \\ \vdots \\ \dot{y}_n(t) = f_n(y_1(t), \ldots, y_n(t), y_1(t-\tau), \ldots, y_n(t-\tau)). \end{cases} \tag{1.20}$$

Where $\tau > 0$ and $f = (f_1, \ldots, f_n)$, $y = (y_1, \ldots, y_n)$, $f \in C^1(E \times E, \mathbb{R}^n)$, $E \subseteq \mathbb{R}^n$.
Let $y(t)$ be a solution of DDE (1.20) and $y^* = (y_1^*, \ldots, y_n^*)$ is an equilibrium (i.e.$f(y^*) = 0$).
We define :

$$x(t) = y(t) - y^*, \tag{1.21}$$

we obtain

$$\begin{aligned} \dot{x}(t) =& \dot{y}(t) - \frac{dy^*}{dt}, \\ =& \dot{y}(t) - 0, \\ =& f(y(t), y(t-\tau)), \end{aligned} \tag{1.22}$$

with (1.21) in (1.22) we find

$$\dot{x}(t) = f(y^* + x(t), y^* + x(t-\tau)). \tag{1.23}$$

To study the stability of y^*, we need to investigate the behavior of the solution of (1.20) around y^*, i.e. the behavior of solution of (1.23) near $x(t) = 0$. For this purpose we expand the right hand side using the first-order Taylor expansion

$$\dot{x}(t) = \frac{\partial f}{\partial y(t)}|_{y=y^*} x(t) + \frac{\partial f}{\partial y(t-\tau)}|_{y=y^*} x(t-\tau) + R(x(t), x(t-\tau)),$$

where R contains terms $x(t)$ and $x(t-\tau)$ of orber≥ 2, and $f(y^*, y^*) = 0$.
The linearization of the delay differential equations (DDE) at y^* is given as follows:

$$\dot{x}(t) = Ax(t) + Bx(t-\tau), \tag{1.24}$$

where A and B are $n \times n$ matrix given by :

$$A = \left(\frac{\partial f_i}{\partial y_j(t)} \right)_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}} \Bigg|_{(y^*, y^*)} = \left(\frac{\partial f}{\partial y(t)} \right)_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}} \Bigg|_{(y^*, y^*)},$$

and

$$B = \left(\frac{\partial f_i}{\partial y_j(t-\tau)} \right)_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}} \Bigg|_{(y^*, y^*)} = \left(\frac{\partial f}{\partial y(t-\tau)} \right)_{\substack{1 \leq i \leq n \\ 1 \leq j \leq n}} \Bigg|_{(y^*, y^*)}.$$

The following result show that there is an equivalence between linear and nonlinear systems, in case the matrix $A + B$ is non-singular.

Theorem 1.2.3 *(Local stability of nonlinear autonomous DDEs) [14]: Let $D(\lambda)$ be the characteristic equation corresponding to (1.24), then y^* is locally asymptotically stable if every root of $D(\lambda)$ has negative real part. In fact, there exist $\varepsilon > 0, k > 0$ such that*

$$\| \varphi - y^* \| < \varepsilon \Longrightarrow \| y_t(\varphi) - y^* \| \leq k \| \varphi - y^* \| e^{-\sigma t}, \quad t \geq 0,$$

where

$$-\sigma := supRe(\lambda) < 0.$$

On the other hand, y^* is unstable if one of the roots of $D(\lambda)$ has positive real part.

Remark 1.2.1 *If the equilibrium point $x^* = 0$ of system (1.24) is locally asymptotically stable, then the equilibrium point y^* of system (1.20) is also locally asymptotically stable. Similarly, if 0 is unstable, then y^* is also unstable.*

Definition 1.2.4 *(**Classification of equilibrium points [15]**)*

1. *If all roots λ of (1.15) satisfy $Re(\lambda) < 0$ then equilibrium y^* is called a **sink**.*

2. *If all roots λ of (1.15) satisfy $Re(\lambda) > 0$ then equilibrium y^* is called a **source**.*

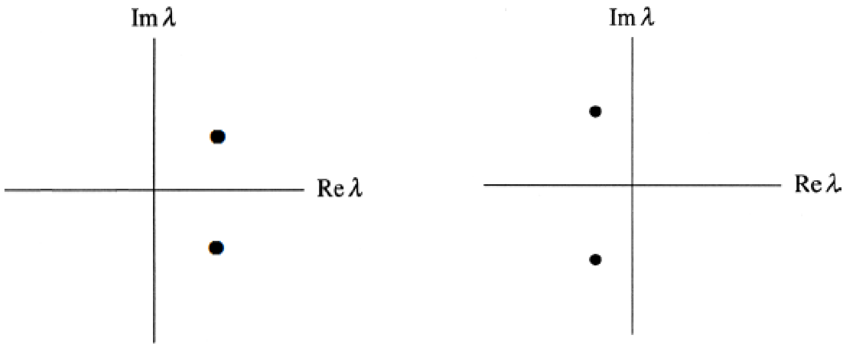


Figure 1.2: Eigenvalue locations for a stable equilibrium point.

3. *If there exists a root λ_1 with $Re(\lambda_1) < 0$ and a root λ_2 with $Re(\lambda_2) > 0$ and no any root with zero real part then y^* is called a* ***saddle****.*

Theorem 1.2.4 *:*

1. *All* ***sinks*** *are asymptotically stable equilibrium points.*

2. *All* ***sources*** *and* ***saddles*** *are* ***unstable*** *equilibrium points.*

1.2.3.2 Stability by Lyapunov function

This method is based on searching for a Lyapunov function that corresponds to one of the two theories. Razumikhin and Krasovskii modified and extended Lyapunov's Theorem to fit delay differential equations, where this Theorem is used to analyze the stability of the equilibrium point $y^* = 0$. However, these theories can be generalized to study the stability of any other equilibrium point y^* by changing variables so that the new equilibrium point is at zero. This is done by redefining $z(t) = y(t) - y^*$, which transforms the equation into a form where the equilibrium point is at z^*.

Theorem 1.2.5 (Razumikhin) *Suppose that $u, v, w : [0, \infty) \to [0, \infty)$ are continuous non-decreasing functions, and that $u(s), v(s)$ are positive for $s > 0$ with $u(0) = v(0) = 0$, where v is strictly increasing. If there is a continuous function $V : \mathbb{R}^n \to \mathbb{R}$ such that $u(|y|) \leq V(y) \leq v(|y|), \quad \forall y \in \mathbb{R}^n$, and $\dot{V}(\varphi(0)) \leq -w(|\varphi(0)|)$, if $V(\varphi(\theta)) \leq V(\varphi(0)), \quad \theta \in [-\tau, 0]$, then the equilibrium point 0 is* ***stable****.*
In addition, if there is a continuous nondecreasing function $p(s) > s$ for $s > 0$ such that

$$\dot{V}(\varphi(0)) \leq -w(|\varphi(0)|)$$

if

$$V(\varphi(\theta)) \leq p(V(\varphi(0))), \quad \theta \in [-\tau, 0]$$

, then 0 is ***asymptotically stable****.*

Theorem 1.2.6 ***(Krasovskii)***
Suppose that $v, u, w : [0, \infty) \longrightarrow [0, \infty)$ are continuous nondecreasing functions, $u(s), v(s)$ positive for $s > 0, u(0) = v(0) = 0$.

If there exists a continuous function $V : \mathbb{C} \to \mathbb{R}$ such that

$$u(|\varphi(0)|) \leq V(\varphi) \leq v(|\varphi|), \quad \forall \varphi \in \mathbb{C},$$

$$\dot{V}(\varphi) = \lim_{t \longrightarrow 0} \sup \frac{1}{t}[V(y_t(., \varphi)) - V(\varphi)] \leqslant -w(| \varphi(0) |)$$

then 0 is ***stable****. If, in addition, $w(s) > 0$ for $s > 0$, then 0 is* ***asymptotically stable****.*

1.2.4 A geometric approach for stability

A geometric approach to study the stability system (1.16)[15].
Supposes that the characteristic equation of the DDE system (1.18) at the equilibrium y^* is given by:

$$P(\lambda) + Q(\lambda)e^{-\lambda\tau} = 0, \tag{1.25}$$

where Q and P are polynomial in λ.
As previors, we know that the steady state y^* is asymptotically stable if all the roots λ of (1.25) satisfy $Re(\lambda) < 0$, and is unstable if there exists a root λ such that $Re(\lambda) > 0$. Therefore the change in stability can occur only if a root λ of equation (1.25) crosses the imaginary axis. i.e $\lambda = i\beta, \quad \beta > 0$, Substituting into equation (1.25), we get:

$$\frac{P(i\beta)}{Q(i\beta)} = -e^{-i\tau\beta}. \tag{1.26}$$

If the equilibrium point y* is asymptotically stable with $\tau = 0$ then the change of in stability can occur only if existe some positive β and $\tau > 0$ for which the equation (1.26) holds.
As $\beta\tau$ incresed from 0 to 2π the right-hand side of (1.26) traces 0 at a unit circle in the complex plane. On the other hand, the left-hand side of (1.26) also defines a curve called ratio curve. If there is a change in stability then the ratio curve must intersect the unit circle.
We find β such that:

$$\left| \frac{P(i\beta)}{Q(i\beta)} \right| = 1,$$

and the critical value τ_c is defined by:

$$\tau_c = \frac{-1}{i\beta} \log \left[\frac{-P(i\beta)}{Q(i\beta)} \right].$$

Then for $0 < \tau < \tau_c$, the equilibrium point is asymptotically stable.
To see how this approach applies, you can find it in resolution of applications in Chapter 2.

Chapter 2

Hopf Bifurcation Analysis

The concept of bifurcation was first introduced by the French mathematician Henri Poincaré in 1885. The Hopf bifurcation theorem holds a prominent position in mathematical research, as it serves as a powerful tool for studying the existence of periodic solutions of differential equations that depend on parameters, and which arise in various scientific applications such as physics, chemistry, biology, and other fields.

In this chapter, our study focuses on the phenomenon of bifurcation in delayed differential equations, with a particular focus on Hopf bifurcation. We discuss the proof of Hopf bifurcation, as well as its design, and conclude the chapter by examining thee dynamical systems, the first is one-dimensional system and the last are two-dimensional systems.

2.1 Bifurcation Analysis

Before introducing the concept of bifurcation, we will present a real-life phenomenon to help better understand this idea.

One real-life example that helps in understanding the phenomenon of bifurcation in delay differential equations (DDEs) is the temperature control system in a room using a thermostat. In this system, the device measures the current temperature and compares it with a desired reference temperature T_{set}, then activates heating or cooling depending on the difference between the two values. However, the process of measurement and response does not occur instantaneously; there is a time delay due to factors such as the slowness of sensors or a lag in the actual response of the device [3].

This system can be represented by the following equation:

$$\frac{dT}{dt} = -k(T(t-\tau) - T_{set}), \quad k, \tau > 0,$$

where:

- $T(t)$: Temperature at time t.

- k : Sensitivity coefficient (represents the strength of the system's response).
- τ : Time delay.
- T_{set} : Reference temperature.

For small values of the coefficient k and the delay τ, the system is stable, and the temperature gradually approaches the desired value T_{set}. However, when k or τ is increased, the system can reach a critical value at which the nature of the solutions changes, and periodic oscillations begin to appear instead of stability. This sudden change in the system's behavior with respect to the parameter is known as bifurcation, and specifically, it can be a Hopf bifurcation when periodic solutions emerge.

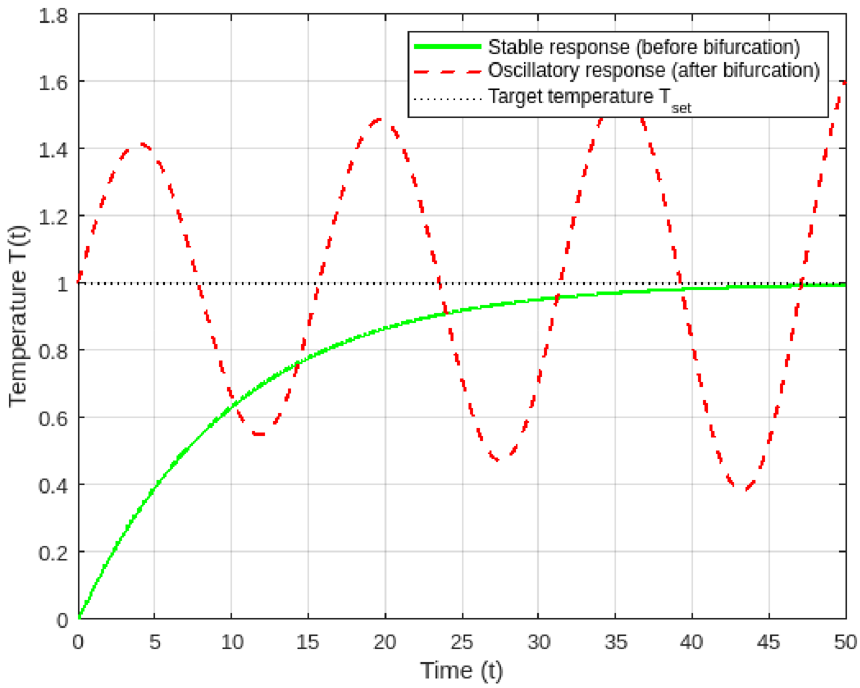


Figure 2.1: Change in System Behavior Before and After Bifurcation.

2.1.1 Definition of Bifurcation and Its Types

We consider the following nonlinear autonomous delay differential equation that depends on a real parameter μ :

$$\begin{cases} \dot{y}(t) = f(y(t), y(t-\tau), \mu) & for \quad t \geq 0, \\ y(t) = \varphi(t) & -\tau \leq t \leq 0, \end{cases} \tag{2.1}$$

where $\tau > 0$, $\mu \in \mathbb{R}$ is a real parameter, $f : E \times E \times \mathbb{R} \to \mathbb{R}^n$ is of class C^k with $k \geq 2$, $E \subset \mathbb{R}^n$, and $\varphi \in C([-\tau, 0], \mathbb{R}^n)$ is a given initial function.

A **bifurcation** occurs when a continuous change in the parameter μ leads to a qualitative change in the behavior of solutions to equation (2.1), typically at a critical value $\mu = \mu_c$ [16]. This change can manifest in two primary ways :

1. A change in the *number of equilibrium points* of the system.
2. A change in the *stability* of an equilibrium point, often due to eigenvalues of the characteristic equation crossing the imaginary axis.

There are different types of bifurcation that can occur in differential equations with delay:

- **Saddle-node bifurcation**: where two equilibrium points (one stable and one unstable) merge and annihilate each other.
- **Pitchfork bifurcation**: where symmetry in the system causes a stable equilibrium to become unstable and give rise to two new equilibria.
- **Hopf bifurcation**: where a pair of complex conjugate eigenvalues cross the imaginary axis, leading to the emergence of periodic (oscillatory) solutions.

The type of bifurcation and instability depends on how the parameter μ changes. In this work, we will focus particularly on the **Hopf bifurcation**.

Remark 2.1.1 *In ordinary differential equations (ODEs), both Saddle-node and Pitchfork bifurcations occur in one dimension, while Hopf bifurcation requires a two-dimensional system to occur, since the appearance of complex eigenvalues is not possible in one dimension. However, in delay differential equations (DDEs), the situation is different. Although the system may be of only one dimension, the characteristic equation is non-algebraic and includes exponential terms, which allows complex eigenvalues to appear even in one dimension. For this reason, all types of bifurcations, including Hopf bifurcation, can occur in one-dimensional delay differential equations.*

2.1.2 Hopf Bifurcation Theorem

In book [12], the Hopf bifurcation Theorem in retarded functional differential equations (RFDEs) is discussed. Meanwhile, in reference [8], the possibility of generalizing bifurcation Theorem in ordinary differential equations (ODEs) to include delay differential equations (DDEs) is explored by studying retarded functional differential equations (RFDEs). Through these two references, we can formulate the Hopf bifurcation Theorem for delay differential equations.

Theorem 2.1.1 *Consider the system (2.1) where we assume that $y^* = 0$ is an equilibrium for all value of μ, i.e, $f(0,0,\mu) = 0$. The linearization process leads to:*

$$\dot{y}(t) = A(\mu)y(t) + B(\mu)y(t-\tau), \tag{2.2}$$

where A and B matrix, $A(\mu) = \frac{\partial f}{\partial y(t)} |_{(y^*,y^*,\mu)}$ and $B(\mu) = \frac{\partial f}{\partial y(t-\tau)} |_{(y^*,y^*,\mu)}$.
Let $D(\lambda,\mu)$ be the characteristic function associated with equation (2.2), which has an infinite

number of roots λ*. Let* $\lambda_{1,2}(\mu) = \alpha(\mu) \pm i\beta(\mu)$ *be a pair of roots.*
We assume that the following two conditions are satisfied :

1. *There exists a simple pure imaginary root at the critical value* μ_c *such that :*

$$\lambda_{1,2}(\mu_c) = \pm i\beta_0, \beta_0 > 0.$$

   There are no other eigenvalues of the form $\pm ik\beta_0$ *for* $k = 2, 3, \ldots$*.*

2. *Transversality condition*

$$\left.\frac{dRe(\lambda(\mu))}{d\mu}\right|_{\mu=\mu_c} \neq 0.$$

Thus, the system (2.1) undergoes a Hopf bifurcation, where the stability of the equilibrium point changes, leading to the emergence of isolated periodic solutions (limit cycles). The stability of the bifurcating periodic orbit is referred to as a supercritical hopf bifurcation, and the instability of the bifurcating periodic orbit is referred to as a subcritical hopf bifurcation.

Remark 2.1.2 *The delay can serve as a bifurcation parameter.*

2.1.3 Explanation of Hopf Bifurcation Theorem in Delay Differential Equations

The goal of Hopf bifurcation Theorem is to determine the critical value of a parameter, denoted by μ_c, which is the point at which Hopf bifurcation occurs in the system. To reach this value, we analyze the roots of the system's characteristic equation, tracking how its sign changes as μ increases. The key point in this analysis is to identify the value at which purely imaginary roots of the form $\lambda_{1,2}(\mu) = \pm i\beta(\mu)$ appear.
In order to guarantee the existence of a Hopf bifurcation, two basic conditions must be met:

- **Condition 1 (Existence of Purely Imaginary Roots):**
  The system must have, at $\mu = \mu_c$, exactly one pair of purely imaginary roots:

$$\lambda_{1,2}(\mu_c) = \pm i\beta(\mu_c),$$

  and

$$\alpha(\mu_c) = 0,$$

  where $\alpha(\mu)$ is the real part of the root. Having exactly one such pair guarantees that the system will oscillate with one frequency only, indicating the formation of a single limit cycle. If there is more than one pair of purely imaginary roots, then the system can oscillate with multiple frequencies.

- **Condition 2 (Transversality Condition):**
  The purely imaginary root must cross the imaginary axis transversally as μ changes, so that the derivative of the real part with respect to μ at μ_c is not zero:

$$\left.\frac{dRe(\lambda(\mu))}{d\mu}\right|_{\mu=\mu_c} \neq 0.$$

  1. If the derivative is positive, the root moves from left to right (from stability to instability).
$$\left.\frac{dRe(\lambda(\mu))}{d\mu}\right|_{\mu=\mu_c} > 0.$$
  2. If it is negative, the root moves from right to left (from instability to stability).
$$\left.\frac{dRe(\lambda(\mu))}{d\mu}\right|_{\mu=\mu_c} < 0.$$

  This situation gives us information about the way the root changes position in the complex plane when μ varies. If the derivative is zero, there is no alteration of the behavior of the root and hence no bifurcation (Hopf)(see Figure 2.2).
  However, if the real part of the root becomes positive after the critical value, this indicates that the solutions begin to grow, signifying a loss of stability and the appearance of a stable limit cycle.
  Hence, if the previous two conditions are met, we can conclude that the system undergoes a Hopf bifurcation at $\mu = \mu_c$, giving rise to a limit cycle [14][5].

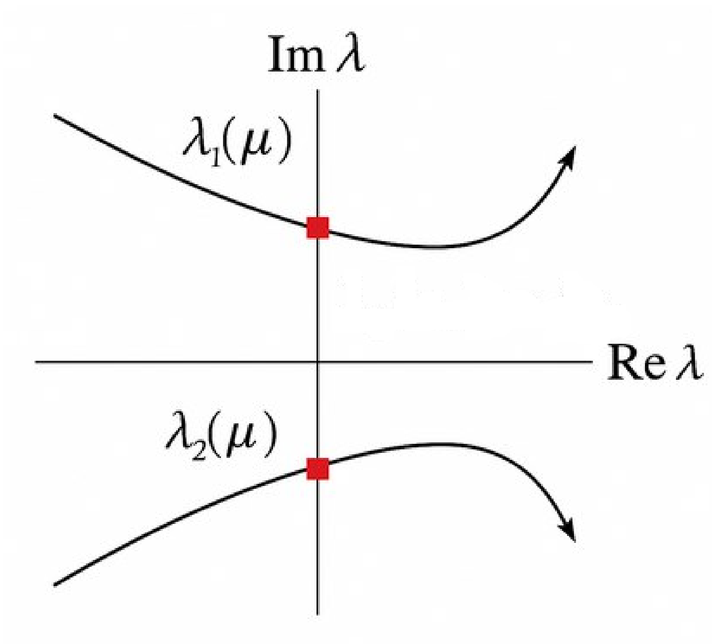


Figure 2.2: Diagram showing the transition of eigenvalues across the imaginary axis at the critical point μ_c in a Hopf bifurcation. (The two red points represent $\lambda_{1,2}(\mu_c) = \pm i\beta_0$).

2.1.4 Hopf Bifurcation Diagrams

Definition 2.1.1 *A bifurcation diagram for a delay differential system is a partitioning of the parameter space based on changes in the system's behavior (such as the number of equilibrium points, their stability, or the emergence of limit cycles) when parameters like delay or other parameters change, with the system's behavior in each region represented by an appropriate phase or graphical portrait.*

To gain a deeper understanding of this concept, we present in the following paragraph the three main stages that a dynamical system undergoes as the bifurcation parameter approaches its critical value.

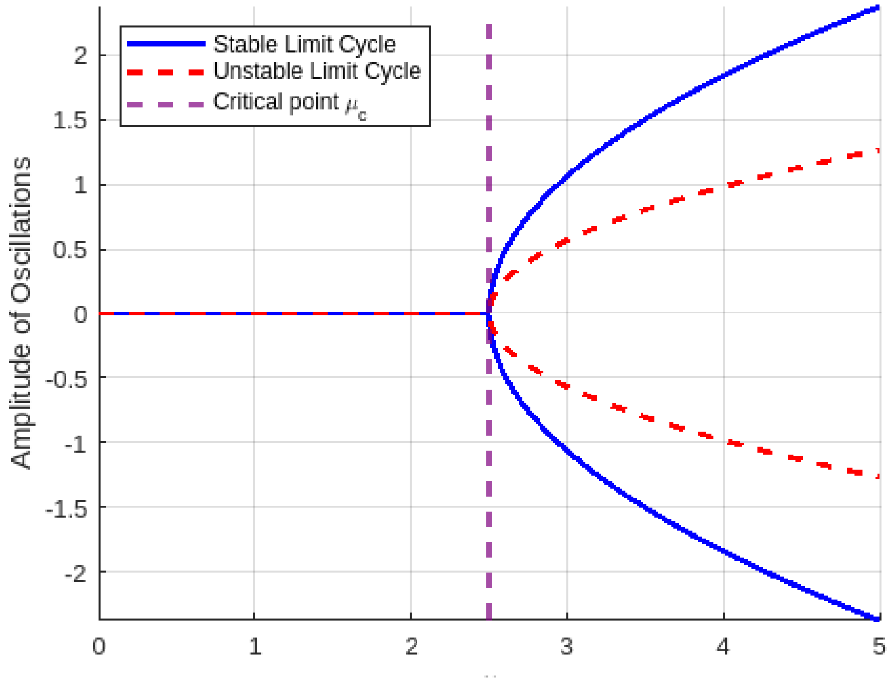


Figure 2.3: Hopf Bifurcation Diagram.

The accompanying illustration (Figure 2.3) illustrates the behavior of system (2.1) as the bifurcation parameter μ varies and shows the Emergence of stable and unstable limit cycles at the critical value μ_c. This behavior can be divided into three main stages :

1. Before the bifurcation :($\mu < \mu_c$)
   In this stage, the equilibrium point (often $y^* = 0$) is asymptotically stable, meaning that all solutions are attracted to it over time.
   This stability is due to the fact that all roots of the characteristic equation (resulting from the analysis of the corresponding linearized system) have negative real parts, i.e.:

$$Re(\lambda_i) < 0, \quad \forall i.$$

2. At the bifurcation :($\mu = \mu_c$)
   When the bifurcation parameter reaches the critical value μ_c, the system loses its stability, and a pair of purely imaginary roots appear in the characteristic equation:

$$\lambda_{1,2}(\mu_c) = \pm i\beta_0, \quad Re(\lambda(\mu_c)) = 0.$$

   This transition indicates that the roots have reached the imaginary axis a critical state that precedes the emergence of periodic oscillations. At this point, the equilibrium point loses its stability, leading to the appearance of small-amplitude periodic solutions the beginning of a limit cycle.

3. After the bifurcation :($\mu > \mu_c$)
   When μ exceeds the critical value, the system exhibits a periodic solution in the form of a limit cycle, which may be stable or unstable depending on the nature of the bifurcation:

   - If the equilibrium point becomes unstable, the resulting limit cycle is stable. This is in the **supercritical bifurcation** type.
   - If the equilibrium point before the bifurcation is unstable and after the bifurcation it becomes stable, the limit cycle is unstable and this type of bifurcation is called a **subcritical bifurcation**.

   This behavior reflects the transition of some roots to having positive real parts:

$$\exists\lambda \quad such\ that \quad Re(\lambda) > 0,$$

   which leads to the complete loss of stability of the equilibrium point and the emergence of oscillations.

Example: (Hutchinson's Delayed Logistic Model) [3] Consider the following scalar delay differential equation, which depends on a single parameter μ:

$$\frac{dy}{dt} = \mu y(t)(1 - y(t-1)), \tag{2.3}$$

This system is nonlinear and has two equilibrium points: $y_1^* = 0$ and $y_2^* = 1$. By linearizing the system around the nontrivial equilibrium $y_2^* = 1$, and using Theorem (1.2.2), we find that a change in stability indeed occurs at a critical value $\mu_c = \dfrac{\pi}{2} \approx 1,57$. When $\mu > \mu_c$, the system transitions from a steady state to a stable periodic solution. We can represent the maxima and minima of y as a function of μ, (see Figure 2.4). We observe that the amplitude of oscillations increases smoothly from zero. This is an example of a bifurcation diagram that illustrates a Hopf bifurcation at $\mu = \mu_c$.

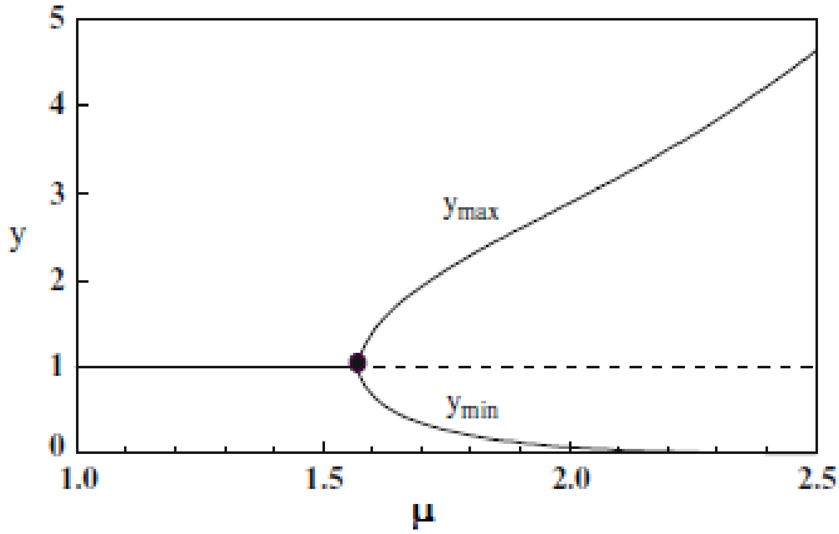


Figure 2.4: Bifurcation diagram of the stable solutions. A Hopf bifurcation to sustained oscillations appears at $\mu = \mu_c \approx 1,57$ (black dot).

2.2 Examples of Hopf Bifurcation Applications

In this section, we will study a various delayed differential systems with one and two-dimension.

2.2.1 Hopf bifurcation in Population Growth Model with Delay

We consider a nonlinear autonomous delay differential equation:

$$\frac{dN}{dt} = \mu N(t) \left(1 - \frac{N(t-\tau)}{r}\right), \tag{2.4}$$

where:
$N(t)$: Number of individuals (population) at the current time t.
μ: Growth Rate.
τ: delay.
r:Carrying Capacity.
This model, represented by equation (2.4), is illustrated in the Figure 2.5 where population growth depends on both the current population size (reproductive growth) and the availability of resources. The delayed logistic equation (2.4) represents this abstraction well [10].

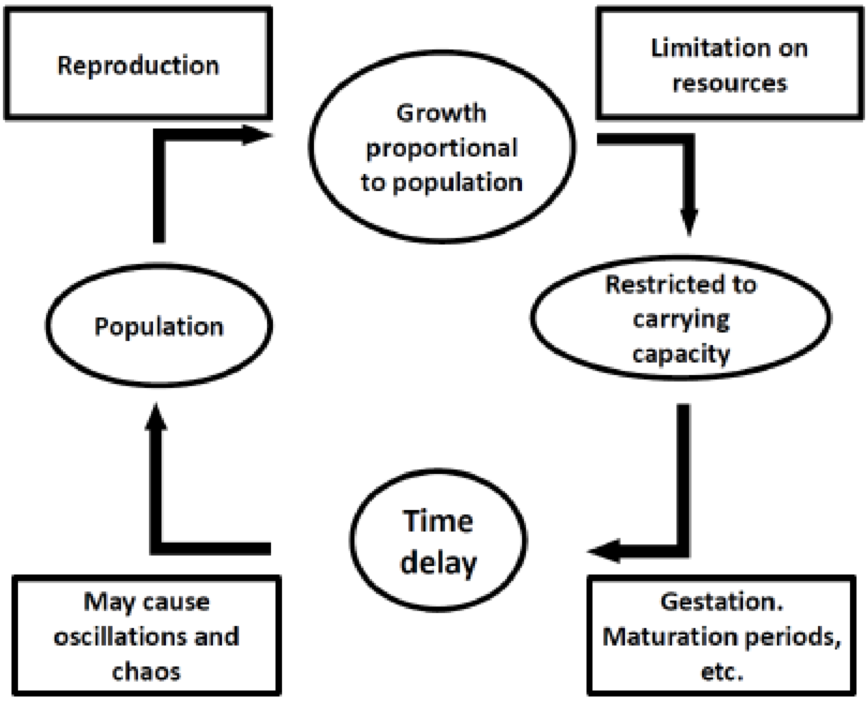


Figure 2.5: Schematic diagram of the model (2.4).

In this schematic, the study of population evolution over time is illustrated, where growth depends on several interrelated factors. Initially, reproduction leads to an increase in the population, and the growth rate is proportional to the current population size; the larger the number of individuals, the faster the reproduction. However, this growth cannot continue indefinitely due to the limitation of resources (such as food and shelter), which restricts the population size to what is known as the environmental carrying capacity (the maximum number the environment can support). Additionally, there is a time delay resulting from gestation and maturation periods, newly born individuals do not contribute to the growth rate until after a certain period has passed. This delay can lead to oscillations in the population size or even chaotic and irregular behavior.
To simplify fomulation, we change a state $N(t)$ by $y(t)$, then we get:

$$\frac{dy}{dt} = \mu y(t)\left(1 - \frac{y(t-\tau)}{r}\right). \tag{2.5}$$

1. To identify the equilibrium points:
   We have to solve the equation
   $$\mu y^*\left(1 - \frac{y^*}{r}\right) = 0,$$
   so,we obtain
   $$y^* = 0, \quad y^* = r,$$
   Therefore, the system (2.5) has two equilibrium points.

2. Linearization around the equilibrium point system (2.5) and stability analysis:

$$\dot{x}(t) = Ax(t) + Bx(t-\tau), \tag{2.6}$$

we consider

$$f(y(t), y(t-\tau)) = \mu y(t)\left(1 - \frac{y(t-\tau)}{r}\right).$$

So,

$$\begin{aligned} A =& \frac{\partial f(y(t), y(t-\tau))}{\partial y(t)}\bigg|_{(y^*,y^*)} &= \mu\left(1 - \frac{y^*}{r}\right), \\ B =& \frac{\partial f(y(t), y(t-\tau))}{\partial y(t-\tau)}\bigg|_{(y^*,y^*)} &= \frac{-\mu y^*}{r}, \end{aligned} \tag{2.7}$$

At equilibrium points $y^* = 0$ by substituting (2.7) into (2.6), we get:

$$\dot{x}(t) = \mu x(t). \tag{2.8}$$

Stability study of a first-order ordinary differential equation by derivative (dim $= 1$).

- if $\mu < 0$, then $y^* = 0$ is locally asymptotically stable.
- if $\mu > 0$, then $y^* = 0$ is unstable.

Hopf bifurcation does not occur in ODE of dim$= 1$.
At equilibrium points $y^* = r$ by substituting (2.7) into (2.6), we get:

$$\dot{x}(t) = -\mu x(t-\tau). \tag{2.9}$$

The characteristic equation, also known in delay differential equations as a transcendental equation, is obtained by substituting the solution $x(t) = e^{\lambda t}$ into equation (2.9):

$$\lambda + \mu e^{-\lambda\tau} = 0. \tag{2.10}$$

When $\tau = 0$ equation (2.10) has only one root $\lambda = -\mu$ hence we discuss two cases:

- **Case $\mu < 0$:** In this case, $y^* = r$ is unstable $(Re(\lambda) > 0)$ for $\tau = 0$.
- **Case $\mu > 0$:** In this case, $y^* = r$ is locally asymptotically stable $(Re(\lambda) < 0)$ for $\tau = 0$.

As the value of τ increases, the stability of the equilibrium point $y^* = r$ may change, if the real part of one of the eigenvalues crosses the imaginary axis from left to right.

(a) So, we look for the critical value that makes the real part of the eigenvalue equal to zero, that is, we look for $\lambda = i\beta$, where β is a non-zero real number $(\beta > 0)$.

By substituting $\lambda = i\beta$ into equation (2.10), we obtain:

$$\begin{aligned} i\beta =& -\mu e^{-i\tau\beta}, \\ =& -\mu(\cos(\tau\beta) - i\sin(\tau\beta)), \\ =& -\cos(\tau\beta) + i\mu\sin(\tau\beta), \end{aligned}$$

By identifacotion we get:

$$\begin{cases} \mu\cos(\tau\beta) = 0, \\ \beta = \mu\sin(\tau\beta), \end{cases} \tag{2.11}$$

Therefore, we obtain:

$$\tau\beta = (2k+1)\frac{\pi}{2}, \quad k = 0, 1, 2, \dots. \tag{2.12}$$

By substituting (2.12) into (2.11):

$$\begin{aligned} \beta =& \mu\sin((2k+1)\frac{\pi}{2}), \quad k = 0, 1, 2, \dots, \\ =& \mu(-1)^k \end{aligned}$$

Therefore, we obtain:

$$\beta = \begin{cases} \mu, \quad k = 0, 2, \dots, \\ \\ -\mu, \quad k = 1, 3, \dots, \quad rejected, \end{cases} \implies \beta = \mu, \quad k = 0, 2, \dots. \tag{2.13}$$

By substituting (2.13) into equation (2.12), we obtain:

$$\tau\mu = (2k+1)\frac{\pi}{2}, \quad k = 0, 2, \dots. \tag{2.14}$$

All these values make the real part of the characteristic root equal to zero. Therefore, we choose only one value, since one of the conditions for the occurrence of a Hopf bifurcation is that the critical value corresponds to a single pair of purely imaginary roots. When $k = 0$, the critical point becomes:

$$\tau\mu = \frac{\pi}{2}.$$

If we consider μ as the bifurcation parameter, then the critical point is

$$\mu_c = \frac{\pi}{2\tau}. \tag{2.15}$$

If we consider τ as the bifurcation parameter, then the critical point is

$$\tau_c = \frac{\pi}{2\mu}. \tag{2.16}$$

Accordingly, we conclude that when the delay value lies within the interval $0 < \tau < \tau_c$, the equilibrium point $y^* = r$ is locally asymptotically stable. However, when the delay exceeds the critical value, i.e., $\tau > \tau_c$, the equilibrium point loses its stability. The value $\tau = \tau_c$ is considered a bifurcation point, at which a qualitative change in the system's behavior occurs.

(b) Transversality condition.

$$\left.\frac{dRe(\lambda(\mu))}{d\mu}\right|_{\mu=\mu_c} \neq 0.$$

Since $\lambda \in \mathbb{C}$:

$$\left.\frac{dRe(\lambda(\mu))}{d\mu}\right|_{\mu=\mu_c} = Re\left(\left.\frac{d\lambda}{d\mu}\right|_{\mu=\mu_c}\right).$$

When considering τ as the bifurcation parameter, equation (2.10) takes the following form:

$$\lambda(\tau) = -\mu e^{-\lambda(\tau)\tau}. \tag{2.17}$$

By differentiating equation (2.17) with respect to the delay, we obtain:

$$\begin{aligned}
\frac{d\lambda}{d\tau} =& \frac{d(-\mu e^{-\lambda(\tau)\tau})}{d\tau}, \\
=& -\mu\frac{d(-\lambda(\tau)\tau)}{d\tau}e^{-\lambda(\tau)\tau}, \\
=& \mu\left(\frac{d\lambda}{d\tau}\tau + \lambda(\tau)\right)e^{-\lambda(\tau)\tau},
\end{aligned}$$

so, we get:

$$\begin{aligned}
\frac{d\lambda}{d\tau}\left(1 - \tau\mu e^{-\lambda(\tau)\tau}\right) =& \mu e^{-\lambda(\tau)\tau}\lambda(\tau), \\
\frac{d\lambda}{d\tau} =& \frac{\mu\lambda(\tau)e^{-\lambda(\tau)\tau}}{1 - \mu\tau e^{-\lambda(\tau)\tau}}.
\end{aligned}$$

We know $\lambda(\tau_c) = i\beta(\tau_c)$, $\alpha(\tau_c) = 0$.

$$\left.\frac{d\lambda}{d\tau}\right|_{\tau=\tau_c} = \frac{i\mu\beta(\tau_c)(\cos(\beta(\tau_c)\tau_c) - i\sin(\beta(\tau_c)\tau_c))}{1 - \mu\tau_c(\cos(\beta(\tau_c)\tau_c) - i\sin(\beta(\tau_c)\tau_c))},$$

$$= \frac{i\beta(\tau_c)\mu\cos(\beta(\tau_c)\tau_c) + \mu\beta(\tau_c)\sin(\beta(\tau_c)\tau_c)}{1 - \mu\tau_c\cos(\beta(\tau_c)\tau_c) + i\mu\tau_c\sin(\beta(\tau_c)\tau_c)},$$

we set:

$$A = \mu\beta(\tau_c)\sin(\beta(\tau_c)\tau_c).$$

$$B = \beta(\tau_c)\mu\cos(\beta(\tau_c)\tau_c).$$

$$C = 1 - \mu\tau_c\cos(\beta(\tau_c)\tau_c).$$

$$D = \mu\tau_c\sin(\beta(\tau_c)\tau_c).$$

Therefore,

$$Re\left(\frac{A+iB}{C+iD}\right) = \frac{AC+BD}{C^2+D^2}.$$

Since,

$$\frac{A+iB}{C+iD} = \left.\frac{d\lambda}{d\tau}\right|_{\tau=\tau_c},$$

then,

$$Re\left(\frac{A+iB}{C+iD}\right) = Re\left(\left.\frac{d\lambda}{d\tau}\right|_{\tau=\tau_c}\right).$$

After some simplifications, we find that:

$$Re\left(\left.\frac{d\lambda}{d\tau}\right|_{\tau=\tau_c}\right) = \frac{\mu\beta(\tau_c)\sin(\beta(\tau_c)\tau_c)}{1+\mu^2\tau_c^2-2\mu\tau_c\cos(\beta(\tau_c)\tau_c)},$$

After simplification, we find:

$$Re\left(\left.\frac{d\lambda}{d\tau}\right|_{\tau=\tau_c}\right) = \frac{\mu^2}{1+\frac{\pi^2}{4}} > 0. \tag{2.18}$$

From equations (2.16) and (2.18), and according to Hopf bifurcation theorem (2.1.1), we conclude that the system (2.5) undergoes a Hopf bifurcation at $\tau_c = \frac{\pi}{2\mu}$, and a cycle limit emerges as a result of this bifurcation.

Since the crossing condition is positive, the bifurcation type is supercritical.

When considering μ as the bifurcation parameter, the equation (2.10) takes the following form:

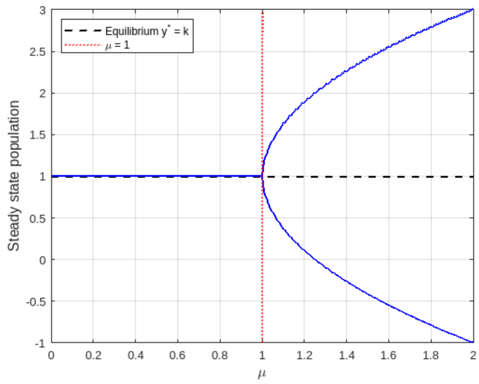
$$\lambda(\mu) = -\mu e^{-\lambda(\mu)\tau}.$$

We follow the same steps used in the simplification when τ was the bifurcation parameter, where we find that:

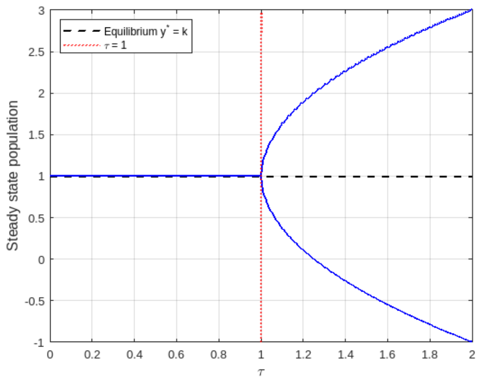
$$Re\left(\left.\frac{d\lambda}{d\mu}\right|_{\mu=\mu_c}\right) = \frac{\frac{\pi}{2}}{1+\frac{\pi^2}{4}}. \tag{2.19}$$

From equations (2.16) and (2.19) , and according to Hopf bifurcation theorem (2.1.1), we conclude that the system (2.5) undergoes a Hopf bifurcation at $\mu_c = \dfrac{\pi}{2\tau}$, and a limit cycle emerges as a result of this bifurcation.

3. Hopf Bifurcation Diagrams



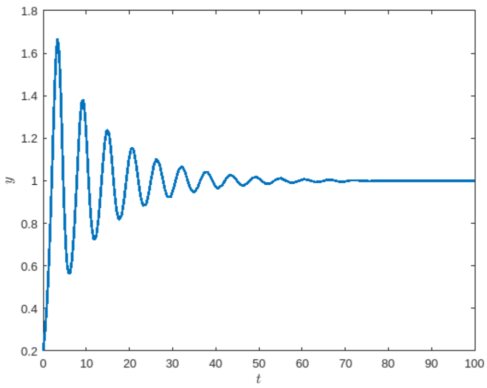
(a) $\mu_c = \dfrac{\pi}{2}$.



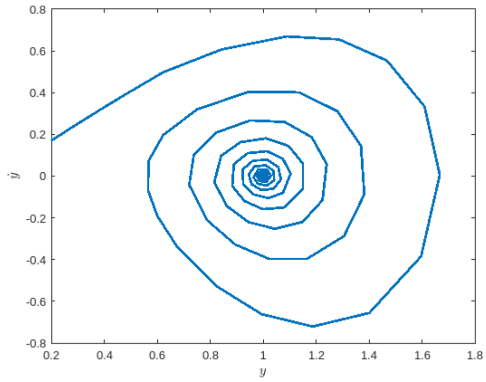
(b) $\tau_c = \dfrac{\pi}{2}$.

Figure 2.6: Bifurcation diagram of system (2.5) (a). wrt the growth rate μ and (b). wrt delay τ.

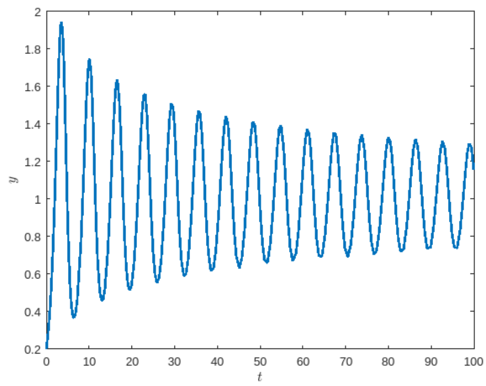
Illustrative plots (see Figure 2.7) showing the behavior of the equilibrium point and the emergence of the limit cycle at the critical value $\tau_c = \pi/2$, in the case where $\mu = 1$ and $r = 1$. The Figure 2.7 shows that (a), (c), and (e) represent the behavior of the solution, while (b), (d), and (f) show the phase planes.



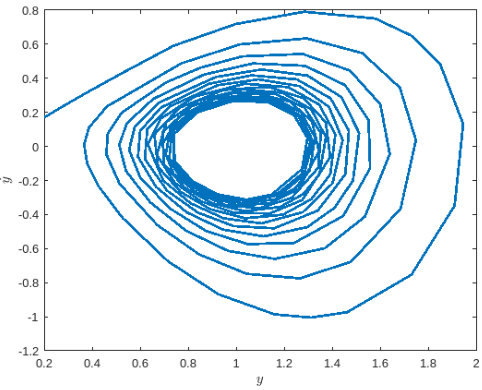
(a) $\tau_c - 0.2$



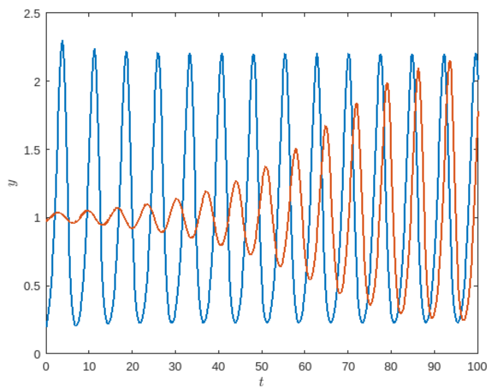
(b) $\tau_c - 0.2$



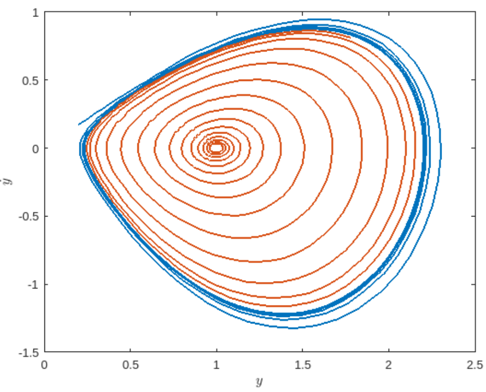
(c) τ_c



(d) τ_c



(e) $\tau_c + 0.2$



(f) $\tau_c + 0.2$

Figure 2.7: Phase portraits and time series plot of system (2.5) for various values of τ using history function $\varphi(t) = 0.2 + 0.05sin(t)$.

2.2.2 Hopf bifurcation in Simple Control Model

Consider the two dimensional system of delayed differential equations:

$$\begin{cases} \dot{y}_1(t) = -y_1^2(t) - y_2(t-\tau). \\ \\ \dot{y}_2(t) = y_1(t) - 3y_2(t) + 2y_1^2(t-\tau). \end{cases} \tag{2.20}$$

1. To determine the equilibrium points, we must solve the equation:

$$\begin{cases} -y_1^{*^2} - y_2^* = 0, \\ \\ y_1^* - 3y_2^* + 2y_1^{*^2} = 0, \end{cases}$$

so,we obtain:

$$\begin{cases} y_1^* = 0 \quad \text{ and } \quad y_2^* = 0. \\ \qquad\qquad or \\ y_1^* = \frac{-1}{5} \quad \text{ and } \quad y_2^* = \frac{-1}{25}. \end{cases}$$

Therefore the system (2.20) has two equilibrium points $E_1(0,0)$ and $E_2(\frac{-1}{5}, \frac{-1}{25})$.

2. Linearization at the equilibrium point E_1:

$$\dot{X}(t) = AX(t) + BX(t-\tau). \tag{2.21}$$

The Jacobian matrices are given by:

$$A = \begin{pmatrix} -2y_1(t) & 0 \\ 1 & -3 \end{pmatrix} \Big|_{E_1} = \begin{pmatrix} 0 & 0 \\ 1 & -3 \end{pmatrix}$$

And

$$B = \begin{pmatrix} 0 & -1 \\ 4y_1(t-\tau) & 0 \end{pmatrix} \Big|_{E_1} = \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix}$$

so,

$$\begin{cases} \dot{x}_1(t) = -x_2(t-\tau). \\ \dot{x}_2(t) = x_1(t) - 3x_2(t). \end{cases}$$

3. Stability analysis using the characteristic function.
We assume solutions of the form : $X(t) = Ve^{-\lambda t}$, $\quad V > 0$, by substituting the solution into equation (2.21), we obtain:

$$(A + Be^{-\tau\lambda} - \lambda I)V = 0, \quad V \neq 0 \Longrightarrow det(A + Be^{-\tau\lambda} - \lambda I) = 0.$$

$$\begin{vmatrix} -\lambda & -e^{-\tau\lambda} \\ 1 & -3-\lambda \end{vmatrix} = -\lambda(-3-\lambda) + e^{-\tau\lambda} = 0.$$

After simplification, we find:

$$\lambda^2 + 3\lambda + e^{-\tau\lambda} = 0. \tag{2.22}$$

   (a) When $\tau = 0$, equation (2.22) becomes as follows:

$$\lambda^2 + 3\lambda + 1 = 0.$$

$$\begin{cases} \lambda_1 = \dfrac{-3+\sqrt{5}}{2}. \\ \lambda_1 = \dfrac{-3-\sqrt{5}}{2}. \end{cases}$$

where $\lambda_1 = \bar{\lambda_2}$, $Re(\lambda_1) = Re(\lambda_2) < 0$.
Since the real part is less than 0, the equilibrium point E_1 locally asymptotically stable.

(b) When $\tau > 0$, to determine whether the system will remain stable as the delay increases, the imaginary values should be examined, as they indicate a change in stability.By substituting $\lambda = i\beta, \quad \beta > 0$ into equation (2.22), we obtain :

$$(i\beta)^2 + 3i\beta + e^{-i\tau\beta} = 0,$$

$$(i\beta)^2 + 3i\beta = -e^{-i\tau\beta}, \tag{2.23}$$

the right side of equation (2.23) is unit circle and the left side is a ratio carve. The ratio curve intersects the unit circle if:

$$\mid (i\beta)^2 + 3i\beta \mid = 1 \Longrightarrow \beta^4 + 9\beta^2 = 1,$$

$$\beta^4 + 9\beta^2 - 1 = 0. \tag{2.24}$$

Substituting $z = \beta^2$ into (2.24), we find:

$$\begin{cases} z_1 = \dfrac{-9+\sqrt{85}}{2} = 0.109772228. \\ \\ z_2 = \dfrac{-9-\sqrt{85}}{2} = -0.390227771. \end{cases}$$

And from it z_2 Rejected because it is impossible $\beta^2 \neq -0.390227771$, this means that $z_1 = \beta^2 = 0.109772228 \Longrightarrow \beta = 0.331318922$.
To find τ, we substitute $\beta = 0.331318922$ into (2.23) and since $e^{-i\tau\beta} = \cos(\tau\beta) - i\sin(\tau\beta)$, we get:

$$-0.109772228 + i0.993956766 = -\cos(0.331318922\tau) + i\sin(0.331318922\tau),$$

so,

$$\begin{cases} \cos(0.331318922\tau) = 0.109772228. \\ \\ \sin(0.331318922\tau) = 0.993956766 \approx 1. \end{cases} \tag{2.25}$$

Then,

$$\begin{aligned} 0.331318922\tau =& \frac{\pi}{2} + n\pi \\ \Rightarrow \tau =& \frac{1}{0.331318922}(\frac{\pi}{2} + n\pi), \quad n \in \mathbb{N}. \end{aligned}$$

At $n = 0$ the critical value of the delay:

$$\tau_c = \frac{\pi}{0.662637844}. \tag{2.26}$$

In particular, when τ_c equation (2.22) has a pair of purely imaginary roots $\pm i\beta$, which are simple and all other roots have negative real parts.
Therefore, when $0 < \tau < \tau_c$, all roots of (2.22) have strictly negative real parts. Denote $\lambda(\tau) = \alpha(\tau) + i\beta(\tau)$ the root of equation (2.22) satisfying,

$$\alpha(\tau_c) = 0, \quad n \in \mathbb{N}.$$

To find out if the eigenvalue $\lambda(\tau)$ crosses the imaginary axis, we calculate the

$$\left.\frac{dRe(\lambda(\tau))}{d\tau}\right|_{\tau=\tau_c} \neq 0.$$

From equation (2.30):

$$\lambda^2(\tau) + 3\lambda(\tau) + e^{-\tau\lambda(\tau)} = 0.$$

To calculate the rate of change of the eigenvalue λ with respect to the time delay τ, we implicitly differentiate the equation with respect to τ.

$$2\lambda(\tau)\frac{d\lambda(\tau)}{d\tau} + 3\frac{d\lambda(\tau)}{d\tau} + \frac{d(-\tau\lambda(\tau))}{d\tau}e^{-\tau\lambda(\tau)} = 0,$$

so,

$$\frac{d\lambda(\tau)}{d\tau} = \frac{\lambda(\tau)e^{-\tau\lambda(\tau)}}{2\lambda(\tau) + 3 - \tau e^{-\tau\lambda(\tau)}}.$$

We know $\lambda = i\beta$, we obtain:

$$\left.\frac{d\lambda(\tau)}{d\tau}\right|_{\tau=\tau_c} = \frac{i\beta e^{-i\beta\tau_c}}{2i\beta + 3 - \tau_c e^{-i\tau_c\beta}}.$$

To find the real part of this expression, we need to simplify it. We start by writing $e^{-i\beta\tau_c} = \cos(\beta\tau_c) - i\sin(\beta\tau_c)$:

$$\left.\frac{d\lambda(\tau)}{d\tau}\right|_{\tau=\tau_c} = \frac{i\beta e^{-i\beta\tau_c}}{2i\beta + 3 - \tau_c(\cos(\beta\tau_c) - i\sin(\beta\tau_c))}.$$

Starting from the previous equations (2.25), we can greatly simplify the expression.

After lengthy calculations, we obtain the following result.

$$Re\left(\frac{d\lambda}{d\tau}\bigg|_{\tau=\tau_c}\right) = 0.028883397 > 0. \qquad (2.27)$$

This implies that the branchs of eigenvalues $\lambda(\tau)$ crosse, at $\tau = \tau_c$, the imaginary axis and that the crossings are from left to right.
From equations (2.26) and (2.27), and according to Hopf bifurcation theorem (2.1.1), we conclude that the system (2.20) undergoes a Hopf bifurcation at τ_c.
Since the crossing condition is positive, the bifurcation type is supercritical.

Accordingly, we conclude that when the delay value lies within the interval $0 < \tau < \tau_c$, the equilibrium point E_1 is locally asymptotically stable. However, when the delay exceeds the critical value, i.e., $\tau > \tau_c$, the equilibrium point loses its stability. The value $\tau = \tau_c$ is considered a bifurcation point, at which a qualitative change in the system's behavior occurs.

4. Hopf Bifurcation Diagrams

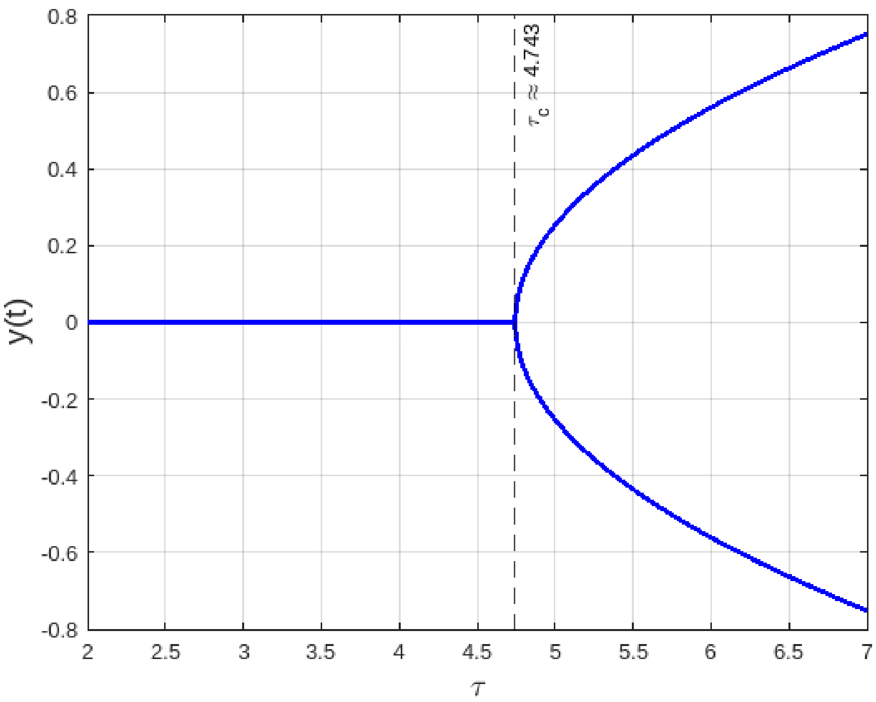
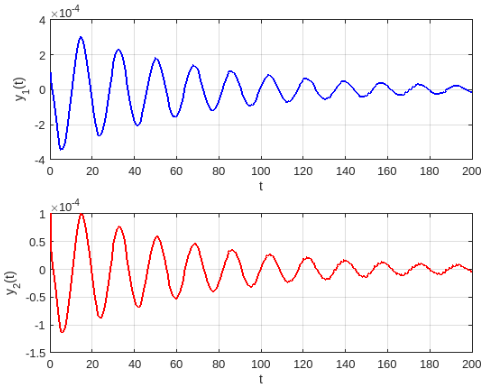
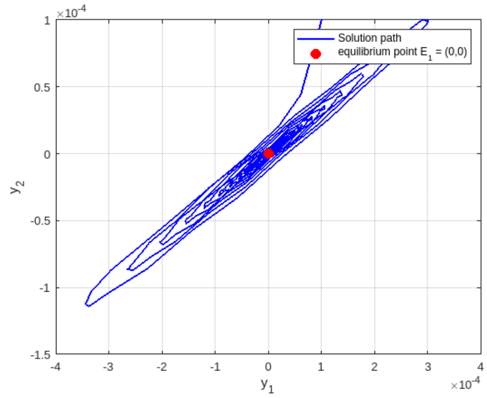


Figure 2.8: Bifurcation diagram of the system (2.20) wrt τ.

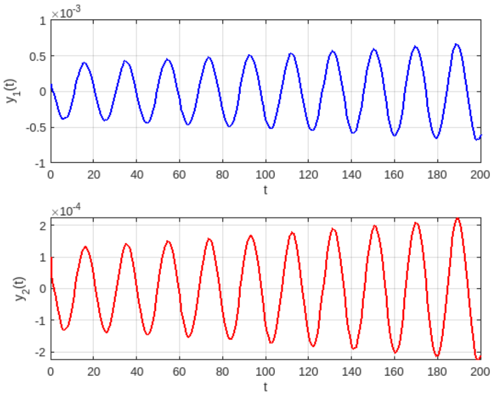
Be evidenced from the graph (see Figure 2.8), the equilibrium value is constant with respect to before the critical value $\tau \approx 4.743$. The amplitude increases sharply with changes in τ.



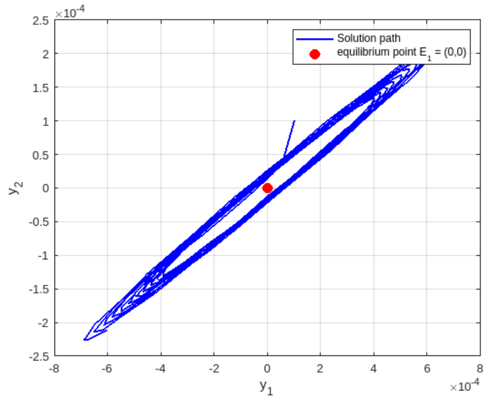
(a) $\tau_c - 0.7$



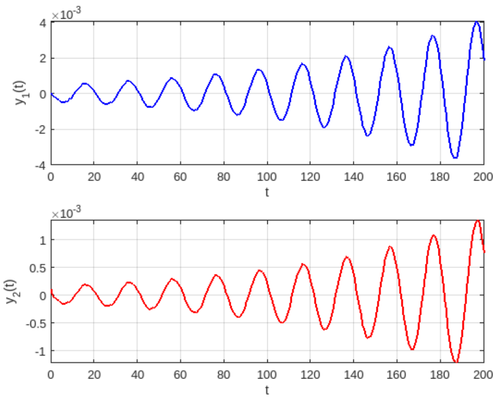
(b) $\tau_c - 0.7$



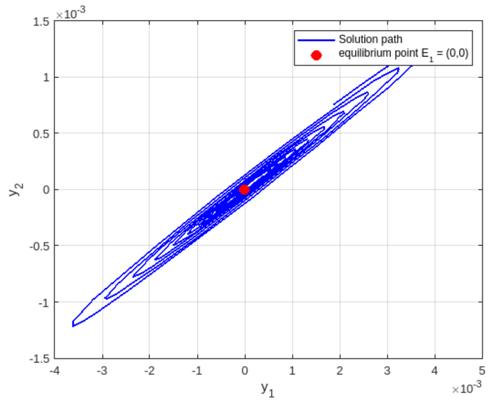
(c) τ_c



(d) τ_c



(e) $\tau_c + 0.7$



(f) $\tau_c + 0.7$

Figure 2.9: The phases of a Hopf bifurcation (2.20) using the history function $(y_1(t), y_2(t)) = (0.0001; 0.0001)$.

The (see Figure 2.9) shows that (a), (c), and (e) represent the behavior of the solution, while (b), (d), and (f) show the phase planes.

2.2.3 Hopf bifurcation in Glycaemic Regulation Model

We consider the glucose-insulin regulation system within the framework of two-dimensional delayed differential equations as follows:

$$\begin{cases} \dfrac{dG}{dt} = 3 - G(t) - 2I(t-\tau)G(t), \\ \\ \dfrac{dI}{dt} = dG(t-\tau) - I(t), \end{cases}$$

where:
$G(t)$:Glucose concentration in the Blood at time t ,$G(t) > 0$.
$I(t)$:Insulin concentration in the blood at time t ,$I(t) > 0$.
d :Positive coefficients.
τ :Delay.
Change of parameters where $y_1 = G$ and $y_2 = I$.

$$\begin{cases} \dfrac{dy_1}{dt} = 3 - y_1(t) - 2y_2(t-\tau)y_1(t), \\ \\ \dfrac{dy_2}{dt} = dy_1(t-\tau) - y_2(t), \end{cases} \tag{2.28}$$

1. To determine the equilibrium points, we must solve the equation:

$$\begin{cases} 3 - y_1^* - 2y_2^* y_1^* = 0, \\ \\ dy_1^* - y_2^* = 0, \end{cases}$$

so,we obtain:

$$\begin{cases} y_1^* = -\dfrac{(1+\sqrt{1+24d})}{4d} \quad and \quad y_2^* = -\dfrac{(1+\sqrt{1+24d})}{4}, \\ \\ y_1^* = -\dfrac{(1-\sqrt{1+24d})}{4d} \quad and \quad y_2^* = -\dfrac{(1-\sqrt{1+24d})}{4}, \end{cases}$$

Therefore, the system (2.28) has two equilibrium points $E_1(-\dfrac{(1+\sqrt{1+24d})}{4d}, -\dfrac{(1+\sqrt{1+24d})}{4})$ end $E_2(-\dfrac{(1-\sqrt{1+24d})}{4d}, -\dfrac{(1-\sqrt{1+24d})}{4})$.

2. Linearization at the equilibrium point E_2:

$$\dot{X}(t) = AX(t) + BX(t-\tau). \tag{2.29}$$

The Jacobian matrices are given by:

$$A = \begin{pmatrix} -1-2y_2^* & 0 \\ 0 & -1 \end{pmatrix}$$

And

$$B = \begin{pmatrix} 0 & -2y_1^* \\ d & 0 \end{pmatrix}$$

so,

$$\begin{cases} \dot{x}_1(t) = (-1-2y_2^*)x_1(t) - 2y_1^* x_2(t-\tau). \\ \dot{x}_2(t) = -x_2(t) + dx_1(t-\tau). \end{cases}$$

3. Stability analysis using the characteristic function. We assume solutions of the form: $X(t) = Ve^{-\lambda t}, \quad V > 0$, by substituting the solution into equation (2.29), we obtain:

$$(A + Be^{-\tau\lambda} - \lambda I)V = 0, \quad V \neq 0 \Longrightarrow det(A + Be^{-\tau\lambda} - \lambda I) = 0.$$

$$\begin{vmatrix} -1-2y_2^*-\lambda & -2y_1^* e^{-\tau\lambda} \\ de^{-\tau\lambda} & -1-\lambda \end{vmatrix} = (1+2y_2^*+\lambda)(1+\lambda) + 2dy_1^* e^{-2\tau\lambda} = 0.$$

After simplification, we find:

$$\lambda^2 + (2+2y_2^*)\lambda + 1 + 2y_2^* + 2dy_1^* e^{-2\tau\lambda} = 0. \tag{2.30}$$

(a) When $\tau = 0$, equation (2.30) becomes as follows:

$$\lambda^2 + (2+2y_2^*)\lambda + 1 + 2y_2^* + 2dy_1^* = 0. \tag{2.31}$$

$$\Delta = i^2(4(\sqrt{1-24d} - (2+y_2^*2)^2).$$

$$\begin{cases} \lambda_1 = \dfrac{-(2+2y_2^*) + i\sqrt{4\sqrt{1-24d} - (2+2y_2^*)^2}}{2}, \\ \lambda_1 = \dfrac{-(2+2y_2^*) - i\sqrt{4\sqrt{1-24d} - (2+2y_2^*)^2}}{2}, \end{cases}$$

where $\lambda_1 = \bar{\lambda_2}$, $Re(\lambda_1) = Re(\lambda_2) = -(1+y_2^*) < 0, y_2^* < 0$.
Since the real part is less than 0, the equilibrium point E_2 locally asymptotically stable.

(b) When $\tau > 0$, to determine whether the system will remain stable as the delay increases, the imaginary values should be examined, as they indicate a change in stability.By substituting $\lambda = i\beta, \quad \beta > 0$ into equation (2.30), we obtain:

$$-\beta^2 + (2 + 2y_2^*)i\beta + 1 + 2y_*^2 + 2dy_1^* e^{-i2\tau\beta} = 0,$$

We know that $e^{-i2\tau\beta} = \cos(2\tau\beta) - i\sin(2\tau\beta)$.

$$-\beta^2 + (2 + 2y_2^*)i\beta + 1 + 2y_*^2 + 2dy_1^*(\cos(2\tau\beta) - i\sin(2\tau\beta)) = 0.$$

Simplification,

$$\begin{cases} -\beta^2 + 1 + 2y_2^* + 2y_1^* d\cos(2\beta\tau) = 0, \\ (2 + 2y_2^*)\beta - 2y_1^* d\sin(2\beta\tau) = 0, \end{cases}$$

so,

$$\begin{cases} \cos(2\beta\tau) = -\dfrac{(-\beta^2 + 1 + 2y_2^*)}{2y_1^* d}. \\ \\ \sin(2\beta\tau) = \dfrac{(2 + 2y_2^*)\beta}{2y_1^* d}. \end{cases} \tag{2.32}$$

To find β.

We know that $\sin^2(x) + \cos^2(x) = 1$. By squaring both sides and adding them, we obtain :

$$\left(\frac{-\beta^2 + 1 + 2y_2^*}{2dy_1^*}\right)^2 + \left(\frac{(2 + 2y_2^*)\beta}{2dy_1^*}\right)^2 = 1,$$

by multiplying both sides by $(2y_1^* d)^2$, we obtain.

$$(-\beta^2 + 1 + 2y_2^*)^2 + ((2 + 2y_2^*)\beta)^2 = (2dy_1^*)^2.$$

After simplifying, we find :

$$\beta^4 + ((y_2^*)^2 + 4y_2^* + 2)\beta^2 + (1 + 2y_2^*)^2 - 4d^2 y_1^* = 0.$$

To reduce the degree, we perform a change of variable by setting $z = \beta^2$,

$$z^2 + ((y_2^*)^2 + 4y_2^* + 2)z + (1 + 2y_2^*)^2 - 4d^2 y_1^* = 0.$$

Solution :

$$z = \frac{-((y_2^*)^2 + 4y_2^* + 2) \pm \sqrt{\Delta}}{2},$$

where $\Delta = (y_2^*)^4 + 8(y_2^*)^3 + (y_2^*)^2 + 16d^2(y_1^*)^2$, $y_2^* > 0$.

We need the positive square root of z because we set $z = \beta^2$, and β must be a positive real number. Where $\beta = \sqrt{z}$.

If τ is the bifurcation coefficient: Finding the value of the delay τ based on equation

(2.32).

$$2\beta\tau = \arcsin\left(\frac{(2+y_2^*)\beta}{2dy_1^*}\right) + 2\pi n, \quad n \in \mathbb{N}.$$
$$\tau = \frac{1}{2\beta}\left(\arcsin\left(\frac{(2+y_2^*)\beta}{2dy_1^*}\right) + 2\pi n\right),$$

At $n = 0$ the critical value of the delay:

$$\tau_c = \frac{1}{2\beta}\arcsin\left(\frac{(2+y_2^*)\beta}{2dy_1^*}\right). \tag{2.33}$$

In particular, when τ_c equation (2.30) has a pair of purely imaginary roots $\pm i\beta$, which are simple and all other roots have negative real parts.
Therefore, when $0 < \tau < \tau_c$, all roots of (2.30) have strictly negative real parts. Denote $\lambda(\tau) = \alpha(\tau) + i\beta(\tau)$ the root of equation (2.30) satisfying,

$$\alpha(\tau_c) = 0, \quad \beta(\tau_c) = \beta_c.$$

To find out if the eigenvalue $\lambda(\tau)$ crosses the imaginary axis, we calculate the

$$\left.\frac{dRe(\lambda(\tau))}{d\tau}\right|_{\tau=\tau_c} \neq 0.$$

From equation (2.30):

$$(\lambda(\tau))^2 + (2+2y_2^*)\lambda(\tau) + 1 + 2y_2^* + d2y_1^* e^{-2\tau\lambda(\tau)} = 0.$$

To calculate the rate of change of the eigenvalue λ with respect to the time delay τ, we implicitly differentiate the equation with respect to τ.

$$2\frac{d\lambda(\tau)}{d\tau}\lambda(\tau) + (2+2y_2^*)\frac{d\lambda(\tau)}{d\tau} + d2y_1^*\frac{d(-2\tau\lambda(\tau))}{d\tau}e^{-2\tau\lambda(\tau)} = 0,$$

so,

$$\frac{d\lambda}{d\tau} = \frac{d4y_1^*\lambda(\tau)e^{-2\tau\lambda(\tau)}}{2\lambda(\tau) + (2+2y_2^*) - d4y_1^*\tau e^{-2\tau\lambda(\tau)}}. \tag{2.34}$$

By substituting $\lambda = i\beta$, we obtain:

$$\frac{d\lambda}{d\tau} = \frac{d4y_1^* i\beta e - i2\tau\beta}{i2\beta + (2+2y_2^*) - d4y_1^*\tau e^{-i2\tau\beta}}.$$

To find the real part of this expression, we need to simplify it. We start by writing $e^{-2i\beta\tau} = \cos(2\beta\tau) - i\sin(2\beta\tau)$:

$$\frac{d\lambda}{d\tau} = \frac{d4y_1^* i\beta e - i2\tau\beta}{i2\beta + (2+2y_2^*) - d4y_1^*\tau(\cos(2\beta\tau) - i\sin(2\beta\tau))}. \tag{2.35}$$

Starting from the previous equations (2.32), we can greatly simplify the expression. After lengthy calculations, we obtain the following result.

$$Re\left(\left(\frac{d\lambda}{d\tau}\right)\bigg|_{\tau=\tau_c}\right) = \frac{2\beta^2(2+2y_2^*)^2 - 4\beta^2(-\beta^2+1+2y_2^*)}{((2+2y_2^*)+2\tau(-\beta^2+1+2y_1^*))^2+(2\beta+2\tau_c(2+2y_2^*)\beta)^2}.$$

we have:

$$Re\left(\frac{d\lambda}{d\tau}\bigg|_{\tau=\tau_c}\right) \neq 0. \tag{2.36}$$

The transversality condition is satisfied because this derivative is nonzero.
This implies that the branchs of eigenvalues $\lambda(\tau)$ crosse, at $\tau = \tau_c$, the imaginary axis.
From equations (2.33) and (2.36), and according to Hopf bifurcation theorem (2.1.1), we conclude that the system (2.5) undergoes a Hopf bifurcation at τ_c.

4. Hopf Bifurcation Diagrams

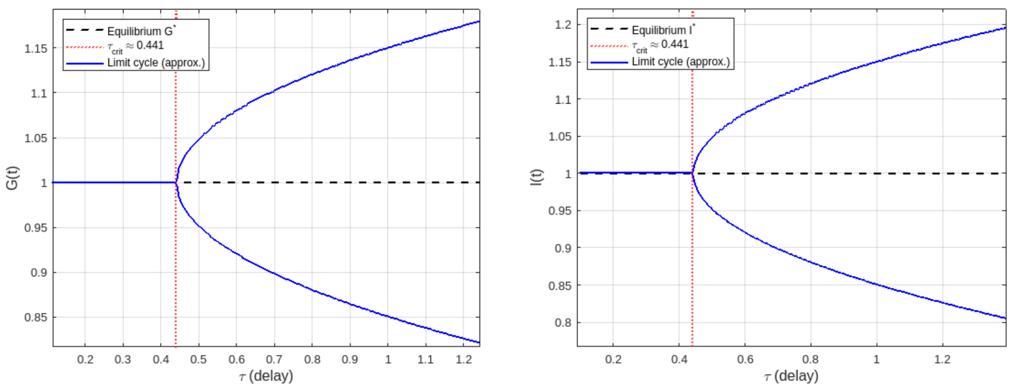


Figure 2.10: Bifurcation diagram of the system (2.20) wrt τ for $d = 1$.

For the numerical simulations, we take $d = 1$: As can be evidenced from the graph (see Figure 2.10), the equilibrium value is constant with respect to τ before the critical value $\tau_c \approx 0.441$. The amplitude increases sharply with changes in τ.
At $d = 1$,

$$\mathrm{Re}\left(\frac{d\lambda}{d\tau}\bigg|_{\tau=\tau_c}\right) \approx 1.022, \quad and \quad \beta_c \approx 1.292.$$

The sign of the computed derivative is positive, which means that when τ increases and exceeds τ_c, the real part becomes positive, therefore, the eigenvalues move from the stable region (negative real part) to the unstable region (positive real part).

Conclusion:

(a) The transversality condition is satisfied because the derivative is non-zero.

(b) The system undergoes a Hopf bifurcation at $\tau = \tau_c \approx 0.441$.

(c) The positive sign indicates that as τ increases, the real parts of the eigenvalues cross the imaginary axis from left to right, suggesting a supercritical bifurcation (a stable limit cycle appears).

In summary:

(a) At $\tau = \tau_c$, a limit cycle is born.

(b) For $\tau > \tau_c$, this cycle is stable, and the system oscillates around it.

(c) The equilibrium point becomes unstable.

Conclusion

Hopf bifurcation in delay differential equations was studied by analyzing the effect of the bifurcation parameter whether it is the time delay or another parameter on the stability of solutions and the emergence of periodic behavior. This work addressed both theoretical and applied aspects related to a specific type of delay differential equations with a single constant delay.

In **the first chapter**, we presented the fundamental properties of these equations, including solution methods and the existence and uniqueness theorem that ensures the existence of solutions. We also examined the stability of both linear and nonlinear systems based on the study of the characteristic (transcendental) equation. Since there is no general theorem that determines the number of its roots, which may be infinite, we adopted a geometric approach as a tool to analyze stability and assess the effect of the time delay.

In **the second chapter**, we presented the main theorem for the occurrence of the Hopf bifurcation in delay equations and introduced the Hopf bifurcation diagram that illustrates how the system behavior changes when crossing the critical values. Our study concluded with numerical simulations of three models: one-dimensional and two-dimensional systems.

It is important to note that using the Method of Steps in analytical solutions requires significant effort due to algebraic complexity. Alternatively, these equations can be solved numerically using the dde23 function in MATLAB, which has proven effective in tracking the system's behavior after the bifurcation point.

The results obtained confirm that time delay can lead to fundamental changes in the stability and dynamic behavior of systems, highlighting the importance of these equations in the mathematical modeling of biological and engineering phenomena. As future research directions, we suggest studying Hopf bifurcation using more advanced numerical tools such as DDE-Biftool, which allows accurate tracking of bifurcations in complex systems. Moreover, the research can be extended to study other types of delays (such as time-varying or distributed delays), or to apply bifurcation analysis in real biological models, thereby strengthening the link between mathematical analysis and real-world applications.

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Appendix A: Mathematical Tools

Laplace Transform

Definition 2.2.1 *Let $f : I \longrightarrow \mathbb{C}$, where $[0, +\infty) \subset I \subset \mathbb{R}$, admit a Laplace transform. For $s \in \mathbb{C}$, the function :*

$$F(s) := \int_0^{+\infty} f(t)e^{-st}dt,$$

is called the Laplace transform of f, and is denoted by $\mathcal{L}[f(t)](s)$.

Proposition 2.2.1 *:*

1. *Linear of the Laplace Transform*

$$\mathcal{L}[(\alpha f + \beta g)(t)](s) = \alpha\mathcal{L}[f(t)](s) + \beta\mathcal{L}[g(t)](s), \quad \alpha, \beta \in \mathbb{C}.$$

2. *Derivative*

$$\mathcal{L}[\frac{dy}{dt}](s) = -y(0) + sY(s).$$

3. *Convolution Theorem*

$$\mathcal{L}[(f * g)(t)](s) = F(s).G(s).$$

We need this definition and properties in the first chapter on solving differential equations with delay.

Implicit Function Theorem

let $g : \Omega \subseteq \mathbb{R}^2 \longrightarrow \mathbb{R}$ be a function such that:

- $(x_0, y_0) \in \Omega$ and $g(x_0, y_0) = 0$.

If the following conditions hold:

1. $g(x_0, y_0) = 0$.

2. $\dfrac{\partial g}{\partial y}(x_0, y_0) \neq 0$.

3. $g \in C^1$.

Then, there exists an interval I_{x_0} around x_0 and an interval J_{y_0} around y_0 such that we can define a unique function $\psi : I_{x_0} \longrightarrow J_{y_0}$ satisfying:

- $\psi(x_0) = y_0$.
- for all $x \in I_{x_0}$, $g(x, \psi(x)) = 0$.
- The derivative of $\psi(x)$ is given by:

$$\psi'(x) = -\frac{g_x(x, \psi(x))}{g_y(x, psi(x))},$$

  where:

  - $g_x = \dfrac{\partial g}{\partial x}$.
  - $g_y = \dfrac{\partial g}{\partial y}$.

Theorem helps us study the Hopf bifurcation when applied to the characteristic equation $D(\lambda, \mu) = 0$, where the following conditions must be satisfied:

1. $D(\lambda, \mu) \in C^1$

2. There exists a specific root λ_0 at μ_c

$$D(\lambda_0, \mu_c) = 0, \frac{\partial D}{\partial \lambda}(\lambda_0, \mu_c) \neq 0.$$

If these conditions are satisfied, then Theorem guarantees the existence of $\lambda(\mu)$ such that:

$$D(\lambda(\mu), \mu) = 0, \lambda(\mu_c) = \lambda_0.$$

Thus, by applying Theorem, we ensure the existence of the function $\lambda(\mu)$, which determines how the eigenvalues change with μ, allowing us to understand when and how the Hopf bifurcation occurs.

Appendix B: Programming Codes in MATLAB

Figure 1.1

```
% Definition of an equation with a delay
dde = @(t, y, Z) -2*y + 3*Z;
% Delay definition
lags = 1;
%Definition of an Initial (Historical) Function
history = @(t) exp(t);
% Solve the equation using dde23
sol = dde23(dde, lags, history, [0 20]);
% Draw the solution
figure;
plot(sol.x, sol.y, 'LineWidth', 2);
xlabel('t');
ylabel('y(t)');
grid on;
```

Figure 2.1

```
% time
t = linspace(0, 50, 1000);
% Stable response before bifurcation (green curve)
T_stable = 1 - exp(-0.1 * t);
% Oscillatory response after bifurcation (red dashed curve)
T_oscillatory = 1 + 0.4 * sin(0.4 * t) .* exp(0.01 * t);
```

```
%Critical value (black dotted line)
T_set = ones(size(t));
% Plotting the curves
figure;
plot(t, T_stable, 'g', 'LineWidth', 1.8); hold on;
plot(t, T_oscillatory, 'r--', 'LineWidth', 1.5);
plot(t, T_set, 'k:', 'LineWidth', 1.2);
%Labels and features of the figure
xlabel('Time (t)');
ylabel('Temperature T(t)');
legend('Stable response (before bifurcation)',
'Oscillatory response (after bifurcation)',
'Target temperature T_{set}');
grid on;
```

Figure 2.3

```
% Hopf Bifurcation Diagram
% Define parameter
mu= linspace(0, 5, 400);
mu_c = 2.5;  % Critical point
% Define amplitudes
amp_stable = zeros(size(mu));
amp_unstable = zeros(size(mu));
% Only defined for mu > mu_c
amp_stable(mu > mu_c) = 1.5 * sqrt(mu(mu > mu_c) - mu_c);
amp_unstable(mu > mu_c) = 0.8 * sqrt(mu(mu > mu_c) - mu_c);
% Plotting
figure;
hold on;
% Plot upper and lower stable limit cycles (blue solid)
h1 = plot(mu, amp_stable, 'b', 'LineWidth', 2);
h2 = plot(mu, -amp_stable, 'b', 'LineWidth', 2);
% Plot upper and lower unstable limit cycles (red dashed)
h3 = plot(mu, amp_unstable, 'r--', 'LineWidth', 2);
h4 = plot(mu, -amp_unstable, 'r--', 'LineWidth', 2);
% Critical delay line (purple dashed)
h5 = xline(mu_c, '--', 'Color', [0.5 0 0.5], 'LineWidth', 2);
```

```
% Labels and title
xlabel('\mu');
ylabel('Amplitude of Oscillations');
title('Hopf Bifurcation Diagram with \mu');
% Proper legend (no repetition)
legend([h1, h3, h5], ...
{'Stable Limit Cycle', 'Unstable Limit Cycle', 'Critical point \
    mu_c'}, ...
'Location', 'northwest');
grid on;
axis tight;
```

Figure 2.6

```
%(a)
% Hopf bifurcation diagram for dy/dt = mu*y(t)*(1 - y(t-1))
clear; clc;
mu_critical=pi/2;
mu_values = linspace(1, 3, 100);% mu values around the critical
    value
equilibrium = ones(size(mu_values));
delay = 1;
ymax = zeros(size(mu_values));
ymin = zeros(size(mu_values));
for i = 1:length(mu_values)
mu = mu_values(i);
% The function
dde = @(t, y, Z) mu * y * (1 - Z);
% History function
history = @(t) 1 + 0.1*sin(2*pi*t);
% Solve the equation using dde23
sol = dde23(dde, delay, history, [0, 200]);
% Evaluating the solution in the last period
t_sample = linspace(180, 200, 1000);
y_sample = deval(sol, t_sample);
%We take the maximum values
ymax(i) = max(y_sample);
ymin(i) = min(y_sample);
```

```
end
% plot
figure;
hold on;
plot(mu_values, equilibrium, 'k--', 'LineWidth', 1.5);
xline(mu_critical, 'r:', 'LineWidth', 1.5);
plot(mu_values, ymax, 'b', 'LineWidth', 1.5);
plot(mu_values, ymin, 'b', 'LineWidth', 1.5);
xlabel('\mu');
ylabel('Steady state population', 'FontSize', 12);
legend('Equilibrium y^* = 1', '\mu = \pi/2', ...
'Location', 'northwest');
grid on;
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%(b)
% Hopf Bifurcation Diagram vs. tau (delay)
clear; clc;
mu = 1;
tau_values = linspace(0.1, 3, 100);
equilibrium = ones(size(tau_values));% Delay values
tau_critical = pi/2;
ymax = zeros(size(tau_values));
ymin = zeros(size(tau_values));
for i = 1:length(tau_values)
tau = tau_values(i);
% Definition of a function
dde = @(t, y, Z) mu * y * (1 - Z);
%History function
history = @(t) 1 + 0.1*sin(2*pi*t);
try
% Solve the equation
sol = dde23(dde, tau, history, [0, 200]);
% Extracting the solution in the last period
t_sample = linspace(180, 200, 1000);
y_sample = deval(sol, t_sample);
% Maximum and minimum values
ymax(i) = max(y_sample);
ymin(i) = min(y_sample);
end
```

```
end
figure;
hold on;
plot(tau_values, equilibrium, 'k--', 'LineWidth', 1.5);
xline(tau_critical, 'r--', 'LineWidth', 1.5);
plot(tau_values, ymax, 'b', 'LineWidth', 1.5);
plot(tau_values, ymin, 'b', 'LineWidth', 1.5);
xlabel('\tau');
ylabel('Steady state population', 'FontSize', 12);
legend('Equilibrium y^* = 1', '\tau= \pi/2', ...
'Location', 'northwest');
grid on;
```

Figure 2.7

```
% Right-hand side of the DDE
sys_rhs = @(y,Z,k) y * (1 - Z/k);
% Constants
k = 1;
history =@(t) 0.2 + 0.05 * sin(t); % Initial condition
% Just before the Hopf bifurcation (tau < pi/2)
tau1 =  pi/2 - 0.2;
sol1 = dde23(@(t,y,Z)sys_rhs(y,Z,k), tau1, history, [0 100]);
% time series plot
close all
plot(sol1.x,sol1.y, 'LineWidth', 2);
xlabel('$t$','Interpreter', 'latex', 'LineWidth', 2);
ylabel('$y$','Interpreter', 'latex', 'LineWidth', 2);
% plot in phase-plane
figure
plot(sol1.y,sol1.yp, 'LineWidth', 2);
xlabel('$y$','Interpreter', 'latex', 'LineWidth', 2);
ylabel('$\dot{y}$', 'Interpreter', 'latex', 'LineWidth', 2);
% at critical value
tau1 =  pi/2;
sol1 = dde23(@(t,y,Z)sys_rhs(y,Z,k), tau1, history, [0 100]);
% time series plot
plot(sol1.x,sol1.y, 'LineWidth', 2);
```

```
xlabel('$t$','Interpreter', 'latex', 'LineWidth', 2);
ylabel('$y$','Interpreter', 'latex', 'LineWidth', 2);
% plot in phase-plane
figure
plot(sol1.y,sol1.yp, 'LineWidth', 2);
xlabel('$y$','Interpreter', 'latex', 'LineWidth', 2);
ylabel('$\dot{y}$', 'Interpreter', 'latex', 'LineWidth', 2);
% just after the hopf bifurcation
tau1 =  (pi/2)+0.2;
options = ddeset('MaxStep',0.1);
sol1 = dde23(@(t,y,Z)sys_rhs(y,Z,k), tau1, history, [0 100],
    options)
sol2 = dde23(@(t,y,Z)sys_rhs(y,Z,k), tau1, 0.9722229,[0 100],
    options)
% time series plot
close all;
figure
plot(sol1.x,sol1.y, 'LineWidth', 2);
hold
plot(sol2.x,sol2.y, 'LineWidth', 2);
xlabel('$t$','Interpreter', 'latex', 'LineWidth', 2);
ylabel('$y$','Interpreter', 'latex', 'LineWidth', 2);
% plot in phase-plane
figure
plot(sol1.y,sol1.yp, 'LineWidth', 2);
hold
plot(sol2.y,sol2.yp, 'LineWidth', 2);
xlabel('$y$','Interpreter', 'latex', 'LineWidth', 2);
ylabel('$\dot{y}$','Interpreter', 'latex', 'LineWidth', 2);
```

Figure 2.9

```
function solve_dde_system()
%delay
% Just before the Hopf bifurcation
tau = 4;
% at critical value(tau=4.743)
% just after the hopf bifurcation (tau=4.8)
```

```
lags = tau;

history = [0.0001; 0.0001];
tspan = [0, 200];
%dde23
sol = dde23(@ddefun, lags, history, tspan);

figure;
subplot(2,1,1);
plot(sol.x, sol.y(1,:), 'b-', 'LineWidth', 1.5);
xlabel('t');
ylabel('y_1(t)');
grid on;

subplot(2,1,2);
plot(sol.x, sol.y(2,:), 'r-', 'LineWidth', 1.5);
xlabel('t');
ylabel('y_2(t)');
grid on;

figure;
plot(sol.y(1,:),sol.y(2,:), 'b-', 'LineWidth', 1.5);
xlabel('y_1');
ylabel('y_2');
grid on;
hold on;
%equlibrum point E1 = (0,0)
plot(0, 0, 'ro', 'MarkerSize', 8, 'MarkerFaceColor', 'r');
legend('Solution path','equilibrium point E_1 = (0,0)');
hold off;
end

function dydt = ddefun(t, y, Z)

y1 = y(1);
y2 = y(2);

y2_lag = Z(2,1);        % y2(t - tau)
y1_lag = Z(1,1);        % y1(t - tau)
```

```
dy1dt = -y1^2 - y2_lag;
dy2dt = y1 - 3*y2 + 2*y1_lag^2;

dydt = [dy1dt; dy2dt];
end
```

Figure 2.10

```
% Bifurcation Diagram for: tau is bifurcation parameter, d = 1
% System:
% dG/dt = 3 - G - 2*I(t - tau)*G
% dI/dt = d*G(t - tau) - I
% Fix d = 1, vary tau
% Parameters
d = 1;
tau = linspace(0.1, 1.5, 300);  % Range of delay values
equilibrium = zeros(size(tau));
amplitude = zeros(size(tau));
% Compute G* = (-1 + sqrt(1 + 24d)) / (4d)
sqrt_term = sqrt(1 + 24 * d);
G_star = (-1 + sqrt_term) / (4 * d);
I_star = d * G_star;
% Store equilibrium
equilibrium(:) = G_star;
% Estimated critical tau (from analysis or simulation)
tau_crit = 0.441;
% Amplitude appears only after tau_crit
amp_mask = tau > tau_crit;
% Example growth
amplitude(amp_mask) = 0.2 * sqrt(tau(amp_mask) - tau_crit);
% Plotting
figure;
hold on;
% Equilibrium
plot(tau, equilibrium, 'k--', 'LineWidth', 1.5);
%Bifurcation line
xline(tau_crit, 'r:', 'LineWidth', 1.5);
```

```
% Upper bound
plot(tau, equilibrium + amplitude, 'b', 'LineWidth', 1.5);
% Lower bound
plot(tau, equilibrium - amplitude, 'b', 'LineWidth', 1.5);
xlabel('\tau (delay)', 'FontSize', 12);
ylabel('G(t)', 'FontSize', 12);
legend('Equilibrium G^*', '\tau_{crit} \approx 0.441', ...
'Limit cycle (approx.)', 'Location', 'northwest');
title('Bifurcation Diagram with respect to \tau (d = 1)');
grid on;
box on;
```

الملخص

تتناول هذه المذكرة دراسة تشعب هويف في المعادلات التفاضلية ذات التأخير، وهي معادلات تعتمد على الحالة الحالية والماضية للنظام. يهدف العمل إلى دراسة وجود الحلول ووحدانيتها، وتحليل الاستقرار الخطي وغير الخطي باستخدام أدوات مثل المعادلة المميزة ودوال ليابونوف، إلى جانب اعتماد نهج هندسي في التحليل. تركز الدراسة على شروط حدوث تشعب هويف، وتُوضح كيف يؤدي تغير المعاملات إلى ظهور دورات حدية. وقد تم دعم الجانب النظري بتطبيقات عددية توضح سلوك النظام قبل وبعد التفرع.
الكلمات المفتاحية: المعادلات التفاضلية ذات التأخير, الدالة التاريخية, تحليل الاستقرار, تشعب هويف.

Abstract

This thesis focuses on the study of Hopf bifurcation in delay differential equations (DDEs), which depend on both the current and past states of the system. The work aims to examine the existence and uniqueness of solutions, and to analyze linear and nonlinear stability using tools such as the characteristic equation and Lyapunov functions, in addition to adopting a geometric approach to the analysis. The study focuses on the conditions under which Hopf bifurcation occurs, and shows how changes in parameters can lead to the emergence of limit cycles. The theoretical part is supported by numerical applications illustrating the system's behavior before and after the bifurcation.
Keywords: Delay Differential Equations, History Function, Stability Analysis, Hopf Bifurcation.

Résumé

Ce mémoire porte sur l'étude de la bifurcation de Hopf dans les équations différentielles à retard (DDEs), qui dépendent à la fois de l'état actuel et de l'état passé du système. Le travail vise à étudier l'existence et l'unicité des solutions, ainsi que l'analyse de la stabilité linéaire et non linéaire en utilisant des outils tels que l'équation caractéristique et les fonctions de Lyapunov, tout en adoptant une approche géométrique d'analyse. L'étude se concentre sur les conditions d'apparition de la bifurcation de Hopf, et montre comment la variation des paramètres peut conduire à l'apparition de cycles limites. La partie théorique est appuyée par des applications numériques illustrant le comportement du système avant et après la bifurcation..
Mots clés: Équations différentielles à retard, Fonction historique, Analyse de stabilité, Bifurcation de Hopf.