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Dedicace

"And the conclusion of their call will be, Praise be to Allah, Lord of the Worlds".

Praise be to Allah at the beginning and at the end.

Whoever says 'I am up to it' will achieve it.

I would like to express my deepest gratitude and appreciation to my beloved family for their unconditional love, support, and encouragement. You have been the true foundation of my journey, and after Allah, I owe what I have achieved to you.

I dedicate this work to my dear father, **Atia**, and my precious mother, **Khadra**, in recognition of your limitless generosity, patience, and countless sacrifices. This work is a token of my love, gratitude, and appreciation for your constant support and encouragement.

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Notations

- ODEs — Ordinary differential equations.
- PDE — Partial differential equation.
- CP — Cauchy problem.
- IVP — Initial value problem.
- $\mathbb{R}$ — Set of real numbers.
- $\mathbb{N}$ — Set of natural numbers .
- C^k — Space of k times continuously differentiable functions.
- ∂y — Partial derivative with respect to y.
- y' — First order derivative of y.
- $\tilde{y}$ — Extended solution.
- T — Open set in $\mathbb{R}^m$.
- K — Compact subset of $\mathbb{R} \times \mathbb{R}^m$.
- $d(x, y)$ — Distance between x and y in a metric space.

Introduction

When studying natural phenomena from the motion of planets in the sky to the flow of water in rivers, and even the spread of diseases within societies we find that these phenomena are most accurately described by differential equations. Furthermore, differential equations play a pivotal role in modeling complex diseases, including cancer and infectious outbreaks.

The origins of differential equations trace back to the development of calculus in the 17th century, with Isaac Newton among the first scientists to employ them. These equations played a crucial role in the mathematical modeling of natural processes, relating variables (functions) to their rates of change. Consequently, differential equations fundamentally involve unknown functions subject to differentiation.

Differential equations can be classified into several types. Those that depend on a single variable are called ordinary differential equations (ODEs), they are used to understand the evolution of systems over time in applications such as chemical reactions and population dynamics. Typically, ODEs are posed as **initial value problems (Cauchy problems)**, where an initial condition is imposed to ensure the selection of a unique solution. This research addresses two fundamental questions related to the Cauchy problem:

- **Existence:** Does a solution to the problem exist?
- **Uniqueness:** If a solution exists, is it unique?

Existence and uniqueness are the mathematical cornerstones of any reliable model. Without them, predictions derived from a model are merely illusions without a sound basis. Thus, many mathematicians including Picard and Cauchy–Lipschitz have established precise conditions under which solutions to ODEs are guaranteed to exist and be unique. This work is structured into three main chapters:

- **Chapter 1**: Foundational concepts of ordinary differential equations, including key definitions, solution characteristics, and the formulation of the Cauchy problem.
- **Chapter 2**: Analysis of existence theorems, beginning with Peano's theorem, followed by existence and uniqueness criteria under Lipschitz conditions. This chapter also examines solution extensions to the Cauchy problem, supplemented by practical modeling application.
- **Chapter 3**: Investigation of solution uniqueness under different conditions, leading to the same conclusion as the Lipschitz condition.

Chapter 1

Generalities on Ordinary Differential Equations

In this chapter, we will discuss fundamental concepts, to prove the existence and uniqueness of solutions to ordinary differential equations. Understanding this chapter is essential for reading the subsequent ones and will help simplify this work.

1.1 Basic Definitions

Definition 1.1.1. *An ordinary differential equation (ODE) of order n is equation that relates an independent variable t, the unknown function $y(t)$ and its derivatives $y', y'', \ldots, y^{(n)}$, defined by:*

$$G\left(t, y, y', \ldots, y^{(n)}\right) = 0, \tag{1.1}$$

where $G : U \to \mathbb{R}^m$ is a function defined on an open set $U \subset \mathbb{R} \times (\mathbb{R}^m)^{n+1}$.

Remark 1.1.1. *:*

- *Term ordinary for the differential equation (1.1) means that the unknown function y depends on a single variable t.*
- *When the equation (1.1) involves multiple variables t_i, it is referred to as a partial differential equation (PDE).*
- *Equation (1.1) is* ***scalar*** *when $m = 1$. Otherwise, it is called* ***vectorial****.*

- *Order of a differential equation is the highest derivative that appears in the differential equation.*

Example 1.1.1. *:*

- $y'' + yy' = 0$ *is the ordinary differential equation of order 2, where* $G(t, y, y', y'') = y'' + yy'$.

- $y''' - ty'' + 2y' = 0$ *is the ordinary differential equation of order 3, where* $G(t, y, y', y'') = y''' - ty'' + 2y'$.

Definition 1.1.2. *A normal differential equation of order n can also be written in its solved form as:*

$$y^{(n)} = g(t, y, y', \dots, y^{(n-1)}), \tag{1.2}$$

where $g : U \to \mathbb{R}^m$ *is a function defined on an open set* $U \subseteq \mathbb{R} \times (\mathbb{R}^m)^n$.

Example 1.1.2. $y' = -\frac{1}{2} - \frac{1}{2}ty$ *is the solved form of* $2y' + ty + 1 = 0$.

Definition 1.1.3. *An autonomous differential equation of order n is any equation of the form:*

$$y^{(n)} = g(y, y', \dots, y^{(n-1)}), \tag{1.3}$$

where $g : V \to \mathbb{R}^m$ *is a function defined on an open set* $V \subseteq (\mathbb{R}^m)^n$.

Example 1.1.3. $y' = g(y) = \frac{1}{2}y + \frac{3}{2}$ *is an autonomous differential equation.*

Definition 1.1.4. *An ordinary differential equation (ODE) of order n is said to be linear if it has the form:*

$$a_n(t)y^{(n)} + a_{n-1}(t)y^{(n-1)} + \dots + a_0(t)y(t) = f(t), \tag{1.4}$$

where:

- *The functions* $t \mapsto a_i(t)$, *for* $0 \leq i \leq n$ *are called the coefficients of (1.4).*

- *The functions* $t \mapsto f(t)$ *is called the second member of (1.4).*

Remark 1.1.2. *:*

- *The function $f(t)$ and the coefficients $a_i(t)$, for $0 \leq i \leq n$ are continuous on the interval $I \subseteq \mathbb{R}$.*
- *If $f(t) = 0$, the equation (1.4) is called a homogeneous linear equation.*

Example 1.1.4. *:*

- *$y''' + 2y'' - 3y' = 0$ is a homogeneous linear equation of order 3.*
- *$y''' - ty'' + 2y' = 4$ is a nonhomogeneous linear equation of order 3.*
- *$y'' - y^3 = 0$ is a homogeneous nonlinear equation of order 2.*

Writing in Coordinates

If $m \geq 2$, let us write the functions with values in $\mathbb{R}^m$ in terms of their component functions, that is to say: $y = (y_1, \ldots, y_m)$, $g = (g_1, \ldots, g_m)$. The equation (1.2) appears as a system of m scalar differential equations of order n with m unknown functions $y_1, \ldots, y_m$, we could then write:

$$\begin{cases} y_1^{(n)} = g_1(t, y, y', \ldots, y^{(n-1)}), \\ y_2^{(n)} = g_2(t, y, y', \ldots, y^{(n-1)}), \\ \vdots \\ y_m^{(n)} = g_m(t, y, y', \ldots, y^{(n-1)}). \end{cases} \tag{1.5}$$

This system is called a system of ordinary differential equations of order n.

1.2 Reduction of an Ordinary Differential Equation to Order 1

Before starting the study of differential equation (1.2) for any order, we can observe that it is possible to transform it into a system of first order differential equations by making some appropriate changes. If we set $Y_0 = y,\ Y_1 = y',\ Y_2 = y'', \ldots, Y_{n-1} = y^{(n-1)}$, we find that:

$$\begin{cases} Y_0' = y' = Y_1, \\ Y_1' = y'' = Y_2, \\ Y_2' = y''' = Y_3, \\ \vdots \\ Y_{n-1}' = y^{(n)} = g(t, y, y', \ldots, y^{(n-1)}), \\ \qquad\qquad = g(t, Y_0, Y_1, \ldots, Y_{n-1}). \end{cases} \tag{1.6}$$

The system (1.6) can still be written $Y' = G(t, Y)$, where:

- $Y = (Y_0, Y_1, \ldots, Y_{n-1}) \in (\mathbb{R}^m)^n$.
- $G = (G_0, G_1, \ldots, G_{n-1}) : U \to (\mathbb{R}^m)^n$, such that:

$$G_0(t, Y) = Y_1,\ G_1(t, Y) = Y_2 \ldots, G_{n-1}(t, Y) = g(t, Y).$$

Thus, every differential system (1.2) of order n in $\mathbb{R}^m$ is equivalent to a differential system (1.6) of order 1 in $\mathbb{R}^m$.

Example 1.2.1. *Consider the second order system $y'' = g(t, y, y')$, where:*

$$y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, g = \begin{bmatrix} g_1 \\ g_2 \end{bmatrix} = \begin{bmatrix} -2y_1 + 3y_2 + \sin t \\ 4y_1 - y_2' + e^t \end{bmatrix}.$$

Thus, $y'' = g(t, y, y')$ is given by:

$$y'' = \begin{bmatrix} y_1'' \\ y_2'' \end{bmatrix} = \begin{bmatrix} -2y_1 + 3y_2 + \sin t \\ 4y_1 - y_2' + e^t \end{bmatrix}.$$

We set $Z_0 = y,\ Z_1 = y',\ Z_2 = y''$, where:

$$Z_0 = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, \quad Z_1 = \begin{bmatrix} z_3 \\ z_4 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}'.$$

$$\begin{cases} z_1 = y_1, \\ z_2 = y_2, \\ z_3 = y_1', \\ z_4 = y_2'. \end{cases} \Rightarrow \begin{cases} z_1' = y_1' = z_3, \\ z_2' = y_2' = z_4, \\ z_3' = y_1'', \\ z_4' = y_2''. \end{cases}$$

We obtain

$$\begin{cases} z_1' = z_3, \\ z_2' = z_4, \\ z_3' = -2z_1 + 3z_2 + \sin t, \\ z_4' = 4z_1 - z_4 + e^t. \end{cases}$$

The second order equation $y'' = g(t, y, y')$ is equivalent to the first order system $Z' = G(t, Z)$, such that:

$$Z = (Z_0, Z_1),\ G(t, Z) = (G_0(t, Z), G_1(t, Z)) = (Z_1, Z_2).$$

In the later parts, we will study the differential equation (1.2) in the case $n = 1$. Let $U = I \times \Omega$, where I is an open interval in $\mathbb{R}$, Ω is an open subset of $\mathbb{R}^m$ and $g : U \to \mathbb{R}^m$ is a continuous function. Consider the following differential equation:

$$y' = g(t, y),\ (t, y) \in U. \tag{1.7}$$

1.3 Characteristics of the Solutions

1.3.1 Local Solution

Definition 1.3.1.1. *The solution of (1.7) on an interval $I \subset \mathbb{R}$, is a differentiable function $y : I \to \mathbb{R}^m$, such that:*

- $\forall t \in I,\ (t, y(t)) \in U.$
- $\forall t \in I,\ y'(t) = g(t, y(t)).$

Definition 1.3.1.2. *We say that y is a local solution of (1.7) if there exists a non empty interval $J \subset I$, such that:*

- $\forall t \in J,\ y(t) \in \Omega.$
- *y is differentiable on J.*
- $\forall t \in J,\ y'(t) = g(t, y(t)).$

Definition 1.3.1.3. ***(Extension)*** *Let $y : I \to \mathbb{R}^m$ and $\tilde{y}$: $\tilde{I} \to \mathbb{R}^m$ be solutions of (1.7), we say that $\tilde{y}$ is an extension of y if:*

- $I \subset \tilde{I}.$
- $\forall t \in I,\ \tilde{y}(t) = y(t).$

Example 1.3.1.1. *Consider the differential equation on $I = \mathbb{R}^*$:*

$$y' = \frac{2}{t}y. \tag{1.8}$$

The function $y : J =]3, +\infty[\to \mathbb{R}$ defined by $y(t) = t^2$ is a local solution of (1.8), because:

- $J \subset I.$
- $\forall t \in J,\ y'(t) = \frac{2}{t}y(t).$

The solution $y_1 : J_1 =]2, +\infty[\to \mathbb{R}$ defined by $y(t) = t^2$ is an extension of y, because:

- $J \subset J_1.$
- $t \in J,\ y(t) = y_1(t).$

The solution $y_2 : J_2 =]1, +\infty[\to \mathbb{R}$ defined by $y(t) = 0$ is not an extension of y, because $J \subset J_2$, but $y(t) \neq y_2(t)$.

Definition 1.3.1.4. *We define $y : I \to \mathbb{R}^m$ as a maximal solution of (1.7) if there is no solution $\tilde{y}$: $\tilde{I} \to \mathbb{R}^m$ verifying $I \subset \tilde{I}$ and $\tilde{y} \restriction_I = y$.*

Example 1.3.1.2. *The function $y : \mathbb{R} \to \mathbb{R}$ defined by $y(t) = e^t$ is a maximal solution of $y' = y$. Its domain of definition $\mathbb{R}$ constitutes the maximal interval of existence for this solution, as it cannot be extended.*

1.3.2 Global Solution

Definition 1.3.2.1. *A global solution of (1.7) is a solution that is defined on the entire interval I.*

Example 1.3.2.1. *The function $y : \mathbb{R} \to \mathbb{R}$ defined by $y(t) = e^t$ is also a global solution of $y' = y$.*

Remark 1.3.2.1. *Every global solution is maximal, but the converse is false.*

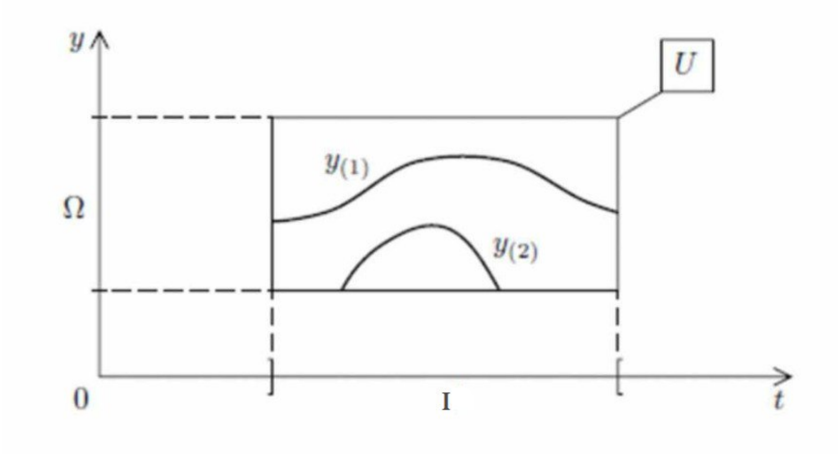

Figure 1.1: Global and Maximal Solution.

In the diagram above, for example $y_{(1)}$ is global, while $y_{(2)}$ is maximal but not global. Let us give an explicit example of this situation.

Example 1.3.2.2. *Consider on $\mathbb{R} \times \mathbb{R}$ the differential equation:*

$$y' = y^2. \tag{1.9}$$

If $y \neq 0$, we have:

$$\begin{aligned} y' = y^2 &\Rightarrow \frac{dy}{dt} = y^2, \\ &\Rightarrow \int \frac{dy}{y^2} = \int dt, \\ &\Rightarrow \int y^{-2} dy = \int dt, \\ &\Rightarrow -\frac{1}{y} = t + c, \\ &\Rightarrow y = \frac{1}{C - t}, \; C = -c \in \mathbb{R}. \end{aligned}$$

This formula defines two solutions, which are respectively defined on $]-\infty, C[$ *and* $]C, +\infty[$*, these solutions are maximal but not global. In this example,* $y(t) = 0$ *is the only global solution of (1.9).*

1.4 Regularity of the Solutions

The regularity of a solution depends on how smooth the function $g(t, y)$ is. In general, the more regular $g(t, y)$ is, the more regular solution will be. The next theorem explains this relationship clearly.

Theorem 1.4.1. *[4] If* $g : U \to \mathbb{R}^m$ *is of class* C^k*, then every solution of (1.7) is of class* C^{k+1}*.*

Proof. We prove the theorem by induction on k:

- In the case $k = 0$, g is continuous. By hypothesis, $y : I \to \mathbb{R}^m$ is differentiable, hence continuous. Consequently, $y'(t) = g(t, y(t))$ is continuous, so y is of class C^1.
- If the result is true at order $k - 1$, then y is of class C^k.
  Since g is of class C^k, it follows that $y' = g(t, y(t))$ is of class C^k as a composition of functions of class C^k. Therefore, y is of class C^{k+1}.

□

1.5 Cauchy Problem

Sometimes, the goal is not to find all the solutions of an ordinary differential equation but only those that satisfy certain constraints, known as Cauchy initial conditions.

Definition 1.5.1. *The Cauchy problem (CP) for an ordinary differential equation, also known as the initial value problem (IVP), consists of a differential equation (1.7) and an initial condition* $y(t_0) = y_0$*.*

$$\begin{cases} y' = g(t, y), \\ y(t_0) = y_0, \ (t_0, y_0) \in U. \end{cases} \tag{1.10}$$

Physical Interpretation: In many practical situations, the variable t represents time, and $y = (y_1, \dots, y_m)$ is a set of parameters describing the state of a given physical system. The equation (1.10) physically represents the law of evolution of the system as a function of time and the values of the parameters. Solving the Cauchy problem means predicting the system's evolution over time, knowing that at $t = t_0$, the system is described by the parameters $y_0 = (y_{0,1}, \dots, y_{0,m})$. The point (t_0, y_0) is called the initial data of the Cauchy problem.

Definition 1.5.2. ***(Classical Solution of (CP))*** *A function $y : I \to \mathbb{R}^m$ is said to be a solution of (1.10) if:*

- *y is of class C^1 on I.*
- *$\forall t \in I,\ y' = g(t, y(t))$.*
- *$y(t_0) = y_0$.*

The simple lemma below shows that the solution of (1.10) is equivalent to the solution of an integral equation.

lemma 1.5.1. *The function $y : I \to \mathbb{R}^m$ is solution of (1.10) if and only if:*

- *y is continuous and for all $t \in I,\ (t, y(t)) \in U$.*
- *$\forall t \in I,\ y(t) = y_0 + \int_{t_0}^{t} g(r, y(r))\, dr$.*

Proof. ($\Longrightarrow$) If y is a solution of (1.10), then it satisfies the following equations:

$$\begin{cases} y' = g(t, y), \ \forall (t, y) \in U, \\ y(t_0) = y_0. \end{cases}$$

By integrating from t_0 to t, we obtain:

$$\int_{t_0}^{t} y'(r)\, dr = \int_{t_0}^{t} g(r, y(r))\, dr, \quad \forall t \in I.$$

From the intial condition $y(t_0) = y_0$, we find:

$$y(t) = y_0 + \int_{t_0}^{t} g(r, y(r))\, dr. \tag{1.11}$$

($\Longleftarrow$) If y satisfies the integral equation (1.11), its differentiation leads to:

$$y' = g(t, y(t)),$$

and $y(t_0) = y_0$, this confirms that y is a solution of (1.10). □

1.6 Security Cylinder

Consider $\|\cdot\|$ as an arbitrary norm on $\mathbb{R}^m$, and we denote by $B(t,r)$ (respectively $\bar{B}(t,r)$) the open (respectively closed) ball centered at t with radius r in $\mathbb{R}^m$.
Since U is assumed to be open, there exists a cylinder $C_0 = [t_0 - a_0, t_0 + a_0] \times \bar{B}(y_0, r_0)$, of length $2a_0$ and radius r_0, such that $C_0 \subset U$. The set C_0 is closed and bounded in $\mathbb{R}^{m+1}$, hence compact. This implies that g is bounded on C_0, that is,

$$M = \sup_{(t,y)\in C_0} \|g(t,y)\| < +\infty.$$

Now, let us consider the cylinder $C = [t_0 - a, t_0 + a] \times \bar{B}(y_0, r_0) \subset C_0$ which has the same diameter as C_0 but length $a \leq a_0$.

Definition 1.6.1. *We say that C is a security cylinder for equation (1.7) if every solution $y : [t_0 - a, t_0 + a] \to \mathbb{R}^m$ of (1.10) remains contained in $\bar{B}(y_0, r_0)$.*

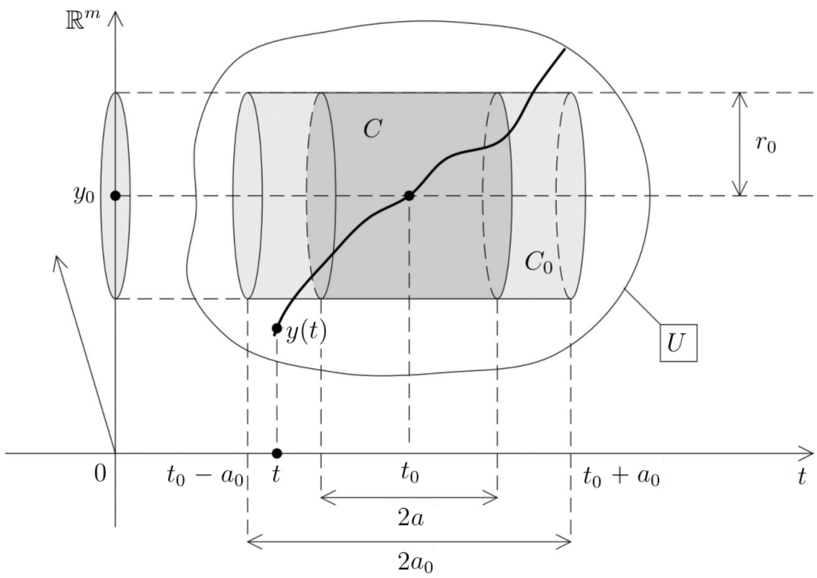

Figure 1.2: Security Cylinder.

Let us assume that the solution y escapes from C on the interval $[t_0, t_0 + a]$. Let τ be the first instant at which this occurs:

$$\tau = \inf\{t \in [t_0, t_0 + a]; \|y(t) - y_0\| > r_0\}.$$

By the definition of τ, we have $\|y(t) - y_0\| \leq r$ for all $t \in [t_0, \tau[$. Then, by the continuity of y, we obtain $\|y(\tau) - y_0\| = r_0$. Moreover, since $(t, y(t)) \in C \subset C_0$ for all $t \in [t_0, \tau]$, it follows that:

$$\|y'(t)\| = \|g(t, y(t))\| \leq M.$$

Thus, we have:

$$r_0 = \|y(\tau) - y_0\| = \|\int_{t_0}^{\tau} y'(u)\, du\| \leq M(\tau - t_0).$$

Therefore

$$\tau - t_0 \geq \frac{r_0}{M}.$$

Consequently, if $a \leq \dfrac{r_0}{M}$, no solution can escape from C over the interval $[t_0 - a, t_0 + a]$.

Remark 1.6.1. *:*

- *For C to be a security cylinder, it is sufficient to take:*

$$a \leq \min\left(a_0, \frac{r_0}{M}\right).$$

- *If $C \subset C_0$ is a security cylinder, then every solution of the Cauchy problem (CP) $y : [t_0 - a, t_0 + a] \to \mathbb{R}^m$ satisfies $\|y'(t)\| \leq M$, so y is Lipschitz continuous with constant M.*

Chapter 2

Fundamental Theorems

In this chapter, we study the local existence of solutions to problem (1.10) using Peano's theorem, as well as the possibility of extending these solutions. Additionally, we examine the existence and uniqueness theorem for the system described by (1.10) within the domain $U = I \times \Omega$, where I is an open interval in $\mathbb{R}$ and Ω is an open subset of $\mathbb{R}^m$. We also consider the security cylinder $C = [t_0 - a, t_0 + a] \times \bar{B}(y_0, r_0)$, where:

$$M = \sup_{(t,y)\in C} \|g(t,y)\| \quad \text{and} \quad a \leq \min\left(a_0, \frac{r_0}{M}\right).$$

2.1 Peano's Theorem: Existence of a Local Solution

Theorem 2.1.1. *[4] Suppose that g is continuous in U. For every point $(t_0, y_0) \in U$, the CP (1.10) admits a solution in the cylinder $C = [t_0 - a, t_0 + a] \times \bar{B}(y_0, r_0)$.*

The proof of Peano's theorem is based on the following steps:

1. We begin by subdividing the interval $[t_0, t_0 + a]$ with step h: $t_0 < t_1 < \cdots < t_N = t_0 + a$. An approximate solution to problem (1.10) is constructed using explicit Euler method. Specifically, consider the piecewise affine function $y : [t_0, t_0 + a] \to \mathbb{R}^m$ defined by:

$$\begin{cases} y(t_0) = y_0, \\ y(t_{i+1}) = y(t_i) + (t_{i+1} - t_i) g(t_i, y(t_i)), \ \forall i \in \{0, \ldots, N-1\}. \end{cases}$$

By induction for i, prove that the approximated values $\forall t \in [t_0, t_i], \ y(t) \in \bar{B}(y_0, r_0)$:

- For $i = 0,\ y_0 \in \bar{B}(y_0, r_0)$.
- Assume that $\forall t \in [t_0, t_i],\ y(t) \in \bar{B}(y_0, r_0)$.
- We prove it $\forall t \in [t_i, t_i + 1]$:

$$\begin{aligned}
\|y(t) - y_0\| &= \|y(t) - y_i + y_i - y_0\|, \\
&\leq \|y(t) - y_i\| + \|y_i - y_0\|, \\
&\leq (t - t_i)\|g(t_i, y(t_i))\| + (t_i - t_0)\|g(t_i, y(t_i))\|, \\
&\leq M(t - t_0), \\
&\leq Ma \leq r_0.
\end{aligned}$$

It follows that $\forall t \in [t_0, t_i + 1],\ y(t) \in \bar{B}(y_0, r_0)$.

2. We now aim to estimate the error $\|y'(t) - g(t, y(t))\|$ for $t \in [t_0, t_0 + a]$. Let us introduce the **modulus of uniform continuity** of the function g, defined for all $u > 0$ as follows:

$$\omega(u) = \sup\left\{\|g(u_1, v_1) - g(u_2, v_2)\|\ ;\ |u_1 - u_2| + \|v_1 - v_2\| \leq u\right\}. \tag{2.1}$$

Since g is uniformly continuous on the compact set C, we have:

$$\lim_{u \to 0} \omega(u) = 0.$$

On this subinterval, the function y satisfies: $y'(t) = g(t_i, y(t_i))$. Therefore:

$$\|y(t) - y(t_i)\| = \|\int_t^{t_i} y'(r)\, dr\| = \|\int_t^{t_i} g(t_i, y(t_i))\| \leq Mh.$$

Using the definition of $\omega(u)$, we find:

$$\|y'(t) - g(t, y(t))\| = \|g(t_i, y(t_i)) - g(t, y(t))\| \leq \omega(Mh + h).$$

Proof. For all $p \geq 1$, let $h_p = \dfrac{a}{p}$, and consider an approximate solution y_p constructed as before, using a subdivision of $[t_0, t_0 + a]$ and $[t_0 - a, t_0]$ with step size less than or equal to h_p. Since each y_p is Lipschitz, the family $(y_p)_{p \geq 1}$ is equicontinuous. Moreover, for all $p \geq 1$ and all $t \in [t_0 - a, t_0 + a]$, we have $y_p(t) \in \bar{B}(y_0, r_0)$. The hypotheses of the Arzelà–Ascoli Theorem 3.4.2 are thus satisfied: we can extract subsequence $(y_{\varphi(p)})_{p\geq 1}$ which converges uniformly to a function y on $[t_0 - a, t_0 + a]$, satisfying $y(t_0) = y_0$. To complete the proof it suffices to show that y is a solution of (1.10). From the previous analysis, for all $p \geq 1$ and for all $t \in [t_0 - a, t_0 + a]$, we have:

$$\|y'_{\varphi(p)}(t) - g(t, y(t))\| \leq \omega(Mh_p + h_p).$$

It follows that:

$$\begin{aligned}
\left\| y_{\varphi(p)}(t) - y_0 - \int_{t_0}^{t} g(s, y_{\varphi(p)}(s))\, ds \right\| &= \left\| \int_{t_0}^{t} y'_{\varphi(p)}(s)\, ds - \int_{t_0}^{t} g(s, y_{\varphi(p)}(s))\, ds \right\|, \\
&\leq \int_{t_0}^{t} \|y'_{\varphi(p)}(s) - g(s, y_{\varphi(p)}(s)\|\, ds, \\
&\leq |t - t_0|\omega(Mh_p + h_p), \\
&\leq a \cdot \omega(Mh_p + h_p).
\end{aligned}$$

By taking the limit as $p \to \infty$, we have $h_p \to 0$, since $\omega(Mh_p + h_p) \to 0$, the right-hand side of the inequality tends to zero. Moreover, $y_{\varphi(p)} \to y$ uniformly, then $g(s, y_{\varphi(p)}(s)) \to g(s, y(s))$ and conclude that:

$$y(t) = y_0 + \int_{t_0}^{t} g(s, y(s))\, ds.$$

Thus, y is a solution of (1.10). □

Example 2.1.1. *Consider the differential problem:*

$$\begin{cases} y' = t^2 + e^{-y^2}, \\ y(0) = 0, \end{cases} \tag{2.2}$$

where $g(t, y) = t^2 + e^{-y^2}$ is a continuous in $\mathbb{R}^2$. For every point $(t_0, y_0) \in \mathbb{R}^2$, the CP (2.2) admits a local solution in $R = [t_0 - a, t_0 + a] \times [y_0 - r_0, y_0 + r_0]$.

For example, we take $a_0 = \frac{1}{2}$ and $r_0 = 1$, then $R_0 = \left[-\frac{1}{2}, \frac{1}{2}\right] \times [-1, 1]$, we get

- $M = \sup\limits_{(t,y)\in R} \left(t^2 + e^{-y^2}\right) = \frac{1}{4} + e^0 = \frac{1}{4} + 1 = \frac{5}{4}$.

- $a \leq \min\left\{a_0, \frac{r_0}{M}\right\} = \min\left\{\frac{1}{2}, \frac{4}{5}\right\} = \frac{1}{2}$.

Then, the CP (2.2) admits a local solution defined on the interval $R = \left[-\frac{1}{2}, \frac{1}{2}\right] \times [-1, 1]$.

2.2 Extending the Solution Domain

Since there is a local solution, can you extend it to a larger domain.

Theorem 2.2.1. *[5] Every solution of (1.10) can be extended to a maximal solution.*

Proof. Let $y : J \to \mathbb{R}^m$ be a solution of (1.10) on an interval $J \subset I$. Define the set of extensions of y as:

$$E := \{x : J_x \to \mathbb{R}^m,\ J \subset J_x \subset I,\ x \text{ solves (1.10)},\ y(t) = x(t)\ \forall t \in J\}.$$

Note that $y \in E$, and so $E \neq \varnothing$ satisfies the first condition of Zorn's Lemma 3.4.1. We define a partial ordering $\preceq$ on E by:

$$w \preceq x \iff J_w \subset J_x \text{ and } w(t) = x(t), \quad \forall t \in J_w.$$

To prove the theorem, it suffices to show that E has a maximal element (that is, an element $z \in E$ such that, if $x \in E$ and $z \preceq x$, then $x = z$). This we do by Zorn's Lemma 3.4.1. To this end, let T be any totally ordered subset of E, $J_z := \bigcup_{x \in T} J_x$ and define the function $z : J_z \to \mathbb{R}^m$ by the property:

$$z|_{J_x} = x, \quad \forall x \in T.$$

1. Prove that J_z is a domain:

Let $a, b \in J_z$ be arbitrary with $a \leq b$. Since $J_z = \bigcup_{x \in T} J_x$, there exist $x_a, x_b \in T$ such that $a \in J_{x_a}, b \in J_{x_b}$. In the other hand, T is totally ordered, we have two possible cases:

- If $x_a \preceq x_b$, then $J_{x_a} \subset J_{x_b}$, so $a \in J_{x_b}$ and $b \in J_{x_b}$. Therefore, $[a, b] \subset J_{x_b} \subset J_z$.
- If $x_b \preceq x_a$, then $J_{x_b} \subset J_{x_a}$, so $a \in J_{x_a}$ and $\in J_{x_a}$. Therefore, $[a, b] \subset J_{x_a} \subset J_z$.

Thus, in both cases, we obtain $[a, b] \subset J_z$, so J_z is an interval.

2. Prove that z is well defined:

Let $t \in J_z$, $t \in J_x$ for $x \in T$ and $\bar{x} \in T$ such that $t \in J_{\bar{x}}$. This leads to two different cases:

- $x \preceq \bar{x} \iff J_x \subset J_{\bar{x}}$ and $x(t) = \bar{x}(t), \quad \forall t \in J_x \subset J_z$.
- $\bar{x} \preceq x \iff J_{\bar{x}} \subset J_x$ and $x(t) = \bar{x}(t), \quad \forall t \in J_{\bar{x}} \subset J_z$.

Therefore, $\forall t \in J_z$, we may associate a unique solution $z(t)$ of $\mathbb{R}^m$ defined by $z(t) = x(t)$, where x is any element of T such that $t \in J_x$. The function $z : J_z \to \mathbb{R}^m$ is well defined and has the property:

$$z|_{J_x} = x \quad \forall x \in T.$$

3. Prove that z is an upper bound for T:

We have $J_x \subset J_z$ and $z(t) = x(t)$ for $t \in J_x$. Thus, according to the definition f the relation $\preceq$, it follows that z is an upper bound for T. By Zorn's Lemma 3.4.1, it follows that E has a maximal element. □

Theorem 2.2.2. *For every point $(t_0, y_0) \in U$, there exists a maximal solution of (1.10). Moreover, the interval I of any maximal solution is open (but in general, there is no uniqueness of these maximal solutions).*

Proof. Based on what we previously established, there exists a maximal solution y of (1.10). We want to prove that y is defined on an open interval $[b, c[\subset \mathbb{R} \to \mathbb{R}^m$. Suppose that y is defined at the point c, then according to Peano's existence theorem, we can find another solution y_1 of (1.10), but with the condition (c, y_c), defined on an interval of the form $[c - \varepsilon, c + \varepsilon]$. Now, we define a new function:

$$\tilde{y}(t) = \begin{cases} y(t), & \text{if } t \in [b, c[, \\ y_1(t), & \text{if } t \in [c, c + \varepsilon]. \end{cases}$$

$\tilde{y}$ is a strict extension of y, which contradicts the maximality of y. □

Example 2.2.1. *The fonctions $y_1(t) = 0$ and $y_2(t) = t^2$ are two solutions of the following differential problem defined on $\mathbb{R}^+ \times \mathbb{R}$:*

$$\begin{cases} y' = 2\sqrt{|y|}, \\ y(0) = 0. \end{cases} \tag{2.3}$$

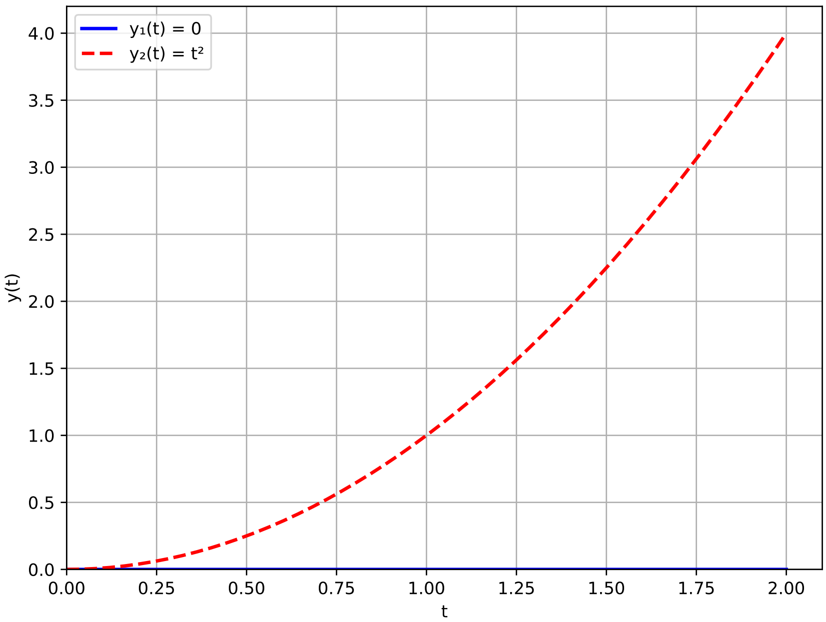

Figure 2.1: Solutions of CP (2.3).

2.3 Maximality Criterion

Theorem 2.3.1. *[4] Let $g : U \to \mathbb{R}^m$ be a continuous function and $y : I = [t_0, c[$ be a solution of (1.10). y can be extended beyond c if and only if there exists a compact set $K \subset I \times \mathbb{R}^m$ such that the curve $t \mapsto (t, y(t))$, remains contained in K. In other words, y is not extendable beyond time c if and only if $(t, y(t))$ escapes from every compact set K as $t \mapsto c^-$.*

Proof. If we extend the solution y to the interval $[t_0, c]$, then the image of this interval, being compact and mapped by a continuous function, will be a compact set. Conversely, suppose there exists a compact set $K \subset U$ such that $(t, y(t)) \in K$ for all $t \in [t_0, c[$ and since the function g is continuous, then:

$$M = \sup_{(t,y)\in K} \|g(t, y)\| < +\infty.$$

It follows that the solution $y(t)$ is Lipschitz continuous on the interval $[t_0, c[$, implying its uniform continuity on the same interval. Consequently, $\lim_{t\to c^-} y(t) = \ell$ exists. We can extend y continuously to c by setting $y(c) = \ell$, and we have $(c, y(c)) \in K \subset U$. The relation $y'(t) = g(t, y)$ shows that $y \in C^1([t_0, c])$. By virtue of the local existence theorem, there exists a local solution z to the CP (1.10) with initial condition $z(c) = \ell$, defined on

$[c-\varepsilon, c+\varepsilon]$.

$$\tilde{y}(t) = \begin{cases} y(t), & \text{if } t \in [t_0, c[, \\ z(t), & \text{if } t \in [c, c+\varepsilon], \end{cases}$$

where $\tilde{y}$ is an extension of y. □

Remark 2.3.1. *A solution $y :]b, c[\to \mathbb{R}^m$ of (1.10) is maximal if and only if $t \mapsto (t, y(t))$ escapes from every compact set K of U as $t \to b^+$ or $t \to c^-$. This also means that $(t, y(t))$ either approaches the boundary of U or tends to infinity.*

Example 2.3.1. *Consider the following differential problems:*

1. $y' = y^2, \ y(0) = 1$.

2. $y' = \frac{1}{2y}, \ y(0) = 1$.

Equations	**Domain**	**Solution**	**Behavior**	**Maximality check**	**Reason**
1	$U = \mathbb{R}^2$	$y(t) = \dfrac{1}{1-t}$ on $]-\infty, 1[$	As $t \to 1^-$, $y(t) \to +\infty$	Maximal	Escapes to infinity
2	$U = \mathbb{R} \times]0, \infty[$	$y(t) = \sqrt{t+1}$ on $]-1, \infty[$	As $t \to -1^+$, $y(t) \to 0$	Maximal	Approaches boundary

Table 2.1: Examples of Maximal Solutions.

As we have seen in the previous Example 2.2.1, the uniqueness property is not satisfied, which calls for addressing the theories that guarantee the existence of a unique solution of (1.10). Therefore, we will present below the sufficient condition to ensure uniqueness, among which is represented by the Cauchy–Lipschitz.

2.4 Cauchy–Lipschitz Theorem: Existence and Uniqueness

2.4.1 Local Existence and Uniqueness

Definition 2.4.1.1. *We say that g is locally Lipschitz continuous with respect to its second variable y, uniformly in t, if for every point $(t_0, y_0) \in U$, there exist a constant $k > 0$ and* *a cylinder* $C_0 = [t_0 - a_0, t_0 + a_0] \times \bar{B}(y_0, r_0)$, *such that* g *be* k*–Lipschitz continuous in* y *on* C_0*:*

$$\forall (t, y_1), (t, y_2) \in C_0, \; \|g(t, y_1) - g(t, y_2)\| \leq k\|y_1 - y_2\|.$$

Example 2.4.1.1. *The function* $g(t, y) = y^2$ *is locally Lipschitz continuous with respect to its second variable* y*, uniformly in* t *on* $\mathbb{R}^2$*. Indeed, let* $(t_0, y_0) \in \mathbb{R}^2$*. For any* $t \in [t_0 - a_0, t_0 + a_0]$ *and* $y_1, y_2 \in \bar{B}(y_0, r_0)$*, we have:*

$$\begin{aligned} \|g(t, y_1) - g(t, y_2)\| = \|y_1^2 - y_2^2\| &= \|(y_1 + y_2)(y_1 - y_2)\|, \\ &\leq (\|y_1\| + \|y_2\|) \cdot \|y_1 - y_2\|. \end{aligned}$$

But

$$\|y_1\| \leq \|y_1 - y_0 + y_0\| \leq \|y_1 - y_0\| + \|y_0\| \leq r_0 + \|y_0\|.$$

And

$$\|y_2\| \leq \|y_2 - y_0 + y_0\| \leq \|y_2 - y_0\| + \|y_0\| \leq r_0 + \|y_0\|.$$

Then

$$\|g(t, y_1) - g(t, y_2)\| \leq 2(r_0 + \|y_0\|) \cdot \|y_1 - y_2\|.$$

Thus, g *is* k*–Lipschitz with* $k = 2(r_0 + \|y_0\|)$*.*

Remark 2.4.1.1. *If* g *is a continuous function of class* C^1 *(meaning that all partial derivatives of* g *are continuous) in* U*, then this function is locally Lipschitz continuous. To illustrate this, we use the mean value theorem on each component* g_i *of* g*:*

$$g_i(t, y_1) - g_i(t, y_2) = \sum_j \frac{\partial g_i}{\partial y_j}(t, \xi)(y_{1,j} - y_{2,j}),$$

with $\xi \in]y_1, y_2[$*. Taking the absolute value:*

$$|g_i(t, y_1) - g_i(t, y_2)| \leq \sum_j \left| \frac{\partial g_i}{\partial y_j}(t, \xi) \right| \cdot |y_{1,j} - y_{2,j}|.$$

Let

$$A = \max_{i,j} \sup_{(t,y) \in C_0} \left| \frac{\partial g_i}{\partial y_j}(t, y) \right|.$$

We find

$$\begin{aligned}|g_i(t,y_1)-g_i(t,y_2)| &\le A\cdot\sum_j |y_{1,j}-y_{2,j}|,\\ &\le mA\cdot\max_j |y_{1,j}-y_{2,j}|.\end{aligned}$$

Taking the maximum over i, leads to:

$$\|g(t,y_1)-g(t,y_2)\| \le mA\cdot\|y_1-y_2\|.$$

Thus, g is locally Lipschitz in y.

Theorem 2.4.1.1. *Let g be a function that is locally Lipschitz continuous with respect to y. Then, for every security cylinder $C=[t_0-a,t_0+a]\times\bar{B}(y_0,r_0)$, the (1.10) admits a unique solution y.*

Proof. Let $E=C([t_0-a,t_0+a],\bar{B}(y_0,r_0))$ be the set of continuous functions from $[t_0-a,t_0+a]$ into $\bar{B}(y_0,r_0)$, $\forall y\in\bar{B}(y_0,r_0)$, we associate the function $\phi(y)$ defined by:

$$\Phi(y)(t)=y_0+\int_{t_0}^{t} g(u,y(u))\,du. \tag{2.4}$$

The proof of the Lipschitz theorem consists of two steps:

1. Prove that Φ is a mapping from E into E: If $t\in[t_0-a,t_0+a]$, then

$$\begin{aligned}\|\Phi(y)(t)-y_0\| = \left\|y_0+\int_{t_0}^{t} g(s,y(s))\,ds-y_0\right\| &= \left\|\int_{t_0}^{t} g(s,y(s))\,ds\right\|,\\ &\le M|t-t_0|\le r_0.\end{aligned}$$

   Which implies $\Phi(y)(t)\in\bar{B}(y_0,r_0)$ for all $t\in[t_0-a,t_0+a]$, and hence $\Phi(y)\in E$.

2. Prove that y is a fixed point of Φ: Let's first show that Φ^p is a contraction i.e.,

$$\exists k\in]0,1[,\ \forall y,z\in E: \|\Phi^p(y)-\Phi^p(z)\|\le k\|y-z\|.$$

For $p = 1$:

$$\begin{aligned}
\|\Phi(y)(t) - \Phi(z)(t)\| &= \left\| y_0 + \int_{t_0}^{t} g(s, y(s))\, ds - y_0 - \int_{t_0}^{t} g(s, z(s))\, ds \right\|, \\
&= \left\| \int_{t_0}^{t} (g(s, y(s))\, ds - g(s, z(s)))\, ds \right\|, \\
&\leq \int_{t_0}^{t} \|g(s, y(s)) - g(s, z(s))\|\, ds, \\
&\leq \int_{t_0}^{t} k\|y(s) - z(s)\|\, ds, \\
&\leq k|t - t_0|\|y - z\|_\infty, \\
&\leq ka\|y - z\|_\infty.
\end{aligned}$$

For $p = 2$, we have:

$$\begin{aligned}
\|\Phi^2(y)(t) - \Phi^2(z)(t)\| &= \|\Phi(\Phi(y))(t) - \Phi(\Phi(z))(t)\|, \\
&= \left\| \int_{t_0}^{t} (g(s, \Phi(y)(s)) - g(s, \Phi(z)(s)))\, ds \right\|, \\
&\leq \int_{t_0}^{t} \|\Phi(y)(s) - \Phi(z)(s)\|\, ds, \\
&\leq \int_{t_0}^{t} k|t - t_0|\|y - z\|_\infty\, ds, \\
&\leq k^2|t - t_0|\|y - z\|_\infty \int_{t_0}^{t} ds, \\
&\leq \frac{k^2}{2}|t - t_0|^2\|y - z\|_\infty, \\
&\leq \frac{(ka)^2}{2}\|y - z\|_\infty.
\end{aligned}$$

By recurrence on p, we verify that:

$$\|\Phi^p(y) - \Phi^p(z)\|_\infty \leq \frac{(ka)^p}{p!}\|y - z\|_\infty.$$

There exists $p \in \mathbb{N}^*$ such that $\frac{(ka)^p}{p!} < 1$. It follows that Φ^p is contractive. According to the Picard fixed point Theorem 3.4.3, there exists a unique $y \in E$ such that $\Phi^p(y) = y$, it follows that:

$$\Phi(\Phi^p(y)) = \Phi^p(\Phi(y)) = \Phi(y).$$

The uniqueness of the fixed point implies that $\Phi(y)(t) = y(t) = y_0 + \int_{t_0}^{t} g(u, y(u))\, du$. Thus y is unique solution.

□

2.4.2 Existence of the Global Solution

Definition 2.4.2.1. *We say that a function g in $U = I \times \mathbb{R}^m$ is globally Lipschit with respect to its second variable y, if there exists a continuous function $k : \mathbb{R} \to \mathbb{R}_+$ such that:*

$$\|g(t, y_1) - g(t, y_2)\| \leq k(t)\|y_1 - y_2\|, \quad \forall (t, y_1), (t, y_2) \in U.$$

Theorem 2.4.2.1. *Let g is continuos and globally Lipschit in U, then every maximal solution of (1.10) is global.*

Proof. Since $U = I \times \mathbb{R}^m$, we can choose a security cylinder of radius $r_0 = +\infty$. Therefore, the mapping Φ defined in (2.4) operates on the complete space $E = C\left([t_0 - a,\, t_0 + a'],\, \mathbb{R}^m\right)$. Let $k = \max_{t \in [t_0 - a,\, t_0 + a']} k(t)$. By assumption, the function g is k–Lipschitz continuous in y over the domain $[t_0 - a,\, t_0 + a'] \times \mathbb{R}^m$.
According to the reasoning in (2.4.1), the mapping Φ^p is a Lipschitz function on E with Lipschitz constant:

$$\frac{1}{p!}\, k^p \left(\max(a, a')\right)^p,$$

thus becomes contractive for sufficiently large p. This implies that the unique solution to the CP (1.10) is defined over the entire interval $[t_0 - a,\, t_0 + a'] \subset I$. □

Corollary 1. *If the function g is continuous and globally Lipschitz with respect to y, then the CP (1.10) admits a unique global solution.*

Example 2.4.2.1. *Consider the differential problem:*

$$\begin{cases} y' = b(t)\, y, \\ y(0) = y_0, \end{cases}$$

where $g(t,y) = b(t)\,y$ is continuous on $\mathbb{R}^2$, we have:

$$\begin{aligned}\|g(t,y_1) - g(t,y_2)\| &= \|b(t)(y_1 - y_2)\|,\\ &= |b(t)|\|y_1 - y_2\|,\\ &\leq k(t)\|y_1 - y_2\|,\end{aligned}$$

such that $k(t) = |b(t)| + 1$ is continuous function on $\mathbb{R}$. Thus, g is globally Lipschitz function and the CP (2.4.2.1) admits a unique global solution.

Ordinary differential equations are fundamental tools for modeling dynamic processes in various scientific fields. A classic example is the modeling of infectious disease spread the SIR model. This model illustrates how the concepts of existence and uniqueness discussed in this chapter are crucial for ensuring that the model provides reliable and predictable results.

2.5 Practical Application (The SIR Model)

The SIR model divides a population into three compartments:

- $S(t)$: Number of individuals susceptible to the disease at time t.
- $I(t)$: Number of individuals infected with the disease and capable of transmitting it at time t.
- $R(t)$: Number of individuals who have recovered from the disease (or died) and are immune (or removed from the population) at time t.

The dynamics of these compartments are described by the following system of autonomous ordinary differential equations:

$$\begin{cases}\dfrac{dS}{dt} = -\dfrac{\beta}{N}SI,\\ \dfrac{dI}{dt} = \dfrac{\beta}{N}SI - \gamma I,\\ .\dfrac{dR}{dt} = \gamma I.\end{cases}$$

Where:

- N: Total population size ($N = S(t) + I(t) + R(t)$).
- $\beta > 0$: Average number of contacts per person per time, multiplied by the probability of disease transmission in a contact between a susceptible and an infectious subject (effective contact rate).
- $\gamma > 0$: Rate at which infected individuals recover or are removed ($1/\gamma$) is the average infectious period).

To study a specific scenario, we need initial conditions:

$$S(0) = S_0, \quad I(0) = I_0, \quad R(0) = R_0,$$

with $S_0 \geq 0$, $I_0 \geq 0$, $R_0 \geq 0$, and $S_0 + I_0 + R_0 = N$. Typically, $S_0 > 0$ and $I_0 > 0$ for an epidemic to start.

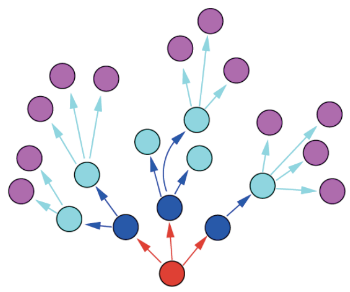

Figure 2.2: Exponential Growth of Infected Individuals at the Beginning of an Epidemic.

2.5.1 Applying Existence and Uniqueness Theorems

Let our state vector be $y(t) = (S(t), I(t), R(t))^T$. The system can be written as $y' = g(y)$, where:

$$g(y) = g(S, I, R) = \begin{pmatrix} -\dfrac{\beta}{N} SI \\ \dfrac{\beta}{N} SI - \gamma I \\ \gamma I \end{pmatrix}.$$

The system montions the domain $D = [0, N] \times [0, N] \times [0, N]$. This is closed and bounded subset of $\mathbb{R}^3$.

Existence (Peano's Theorem)

The function $g(S, I, R)$ is composed of polynomial terms in S and I. Polynomials are continuous functions everywhere therefore, $g(y)$ is continuous on $\mathbb{R}^3$, and certainly continuous on any compact subset of our domain D. According to the Peano's existence theorem, the system admits a solution $y(t) = (S(t), I(t), R(t))$ defined on some intervel $[-a, a]$ (around t_0). typically, $t \geq 0$, so the solution exists on $[0, a]$ for some $a \geq 0$. This means the model's equations properly describe fundamental epidemic behavior.

Uniqueness (Cauchy–Lipschitz Theorem)

To guarantee that the model gives a single, predictable outcome, we need uniqueness, which requires more than just continuity. We check if g is locally Lipschitz with respect to y. We can examine the partial derivatives of the components of g:

$$J_g(S, I, R) = \frac{\partial g}{\partial y} = \begin{pmatrix} -\frac{\beta I}{N} & -\frac{\beta S}{N} & 0 \\ \frac{\beta I}{N} & \frac{\beta S}{N} - \gamma & 0 \\ 0 & \gamma & 0 \end{pmatrix}.$$

All entries in the Jacobian matrix J_g are continuous functions of S and I on any compact set within D, the function $g(y)$ is locally Lipschitz continuous with respect to y on D (in fact, it is globally Lipschitz on the compact set D itself).

According to the Cauchy–Lipschitz theorem, g is continuous and locally Lipschitz, for any $y_0 \in D$, there exists a *unique* local solution $y(t)$ to the CP on some interval $[0, a]$. The uniqueness of the solution means there's only one possible epidemic behavior trajectory for given condition.

Global Existence and Positivity

Furthermore, one can show that solutions starting with non negative S_0, I_0, R_0 remain non negative for all $t > 0$. Since $S'(t) + I'(t) + R'(t) = 0$, the total population $S(t) + I(t) + R(t) = N$ remains constant. This means the solution $y(t)$ always stays within the compact set D. Therefore, the SIR model guarantees a unique, non negative, global solution for all $t \geq 0$.

This existence and uniqueness guarantee is essential. It means that for a given set of parameters (β, γ) and initial conditions (S_0, I_0, R_0), the SIR model predicts a single, well defined trajectory for the epidemic's progression over time.

2.5.2 Visualizing SIR Model Dynamics

The following figures illustrate the typical behaviors predicted by the SIR model using Python. These behaviors will be computed using more accurate and reliable numerical methods, such as the Runge-kutta method (3.4). The existence and uniqueness theorems provide the theoretical foundation that justifies the convergence of these numerical approximations to the true and unique solution as the step size decreases.

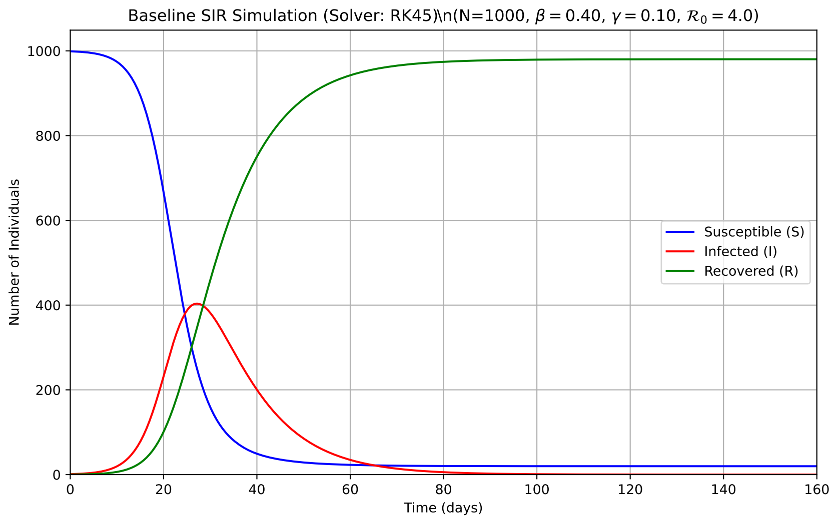

Figure 2.3: Number of Susceptible (S), Infected (I), and Recovered (R) Individuals Over Time Under Baseline Parameters Values.

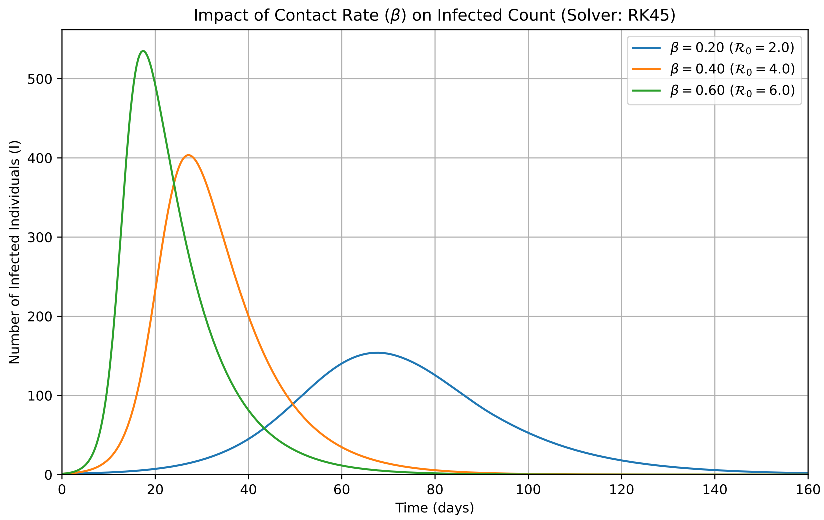

Figure 2.4: Comparison of the Infected Curve (I) for Different Values of β While Keeping γ and the Initial Conditions Constant.

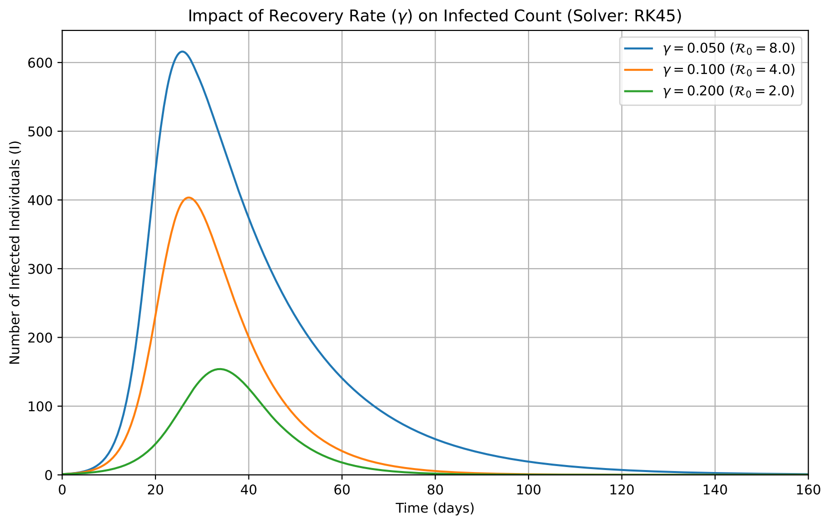

Figure 2.5: Comparison of the Infected curve $I(t)$ for Different Values of γ, While Keeping β.

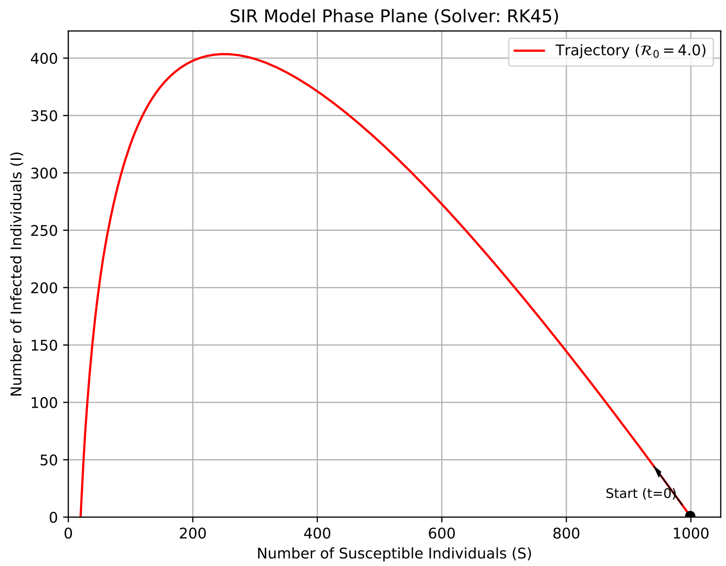

Figure 2.6: Trajectory of the Epidemic in Terms of Susceptible vs Infected Individuals.

A key concept derived from the SIR model is the basic reproduction number, $\mathcal{R}_0 = \frac{\beta}{\gamma}$ (at the start of the epidemic, when $S \approx S_0$). If $\mathcal{R}_0 > 1$, the number of infected individuals initially increases, leading to an epidemic. If $\mathcal{R}_0 < 1$, the infection dies out.

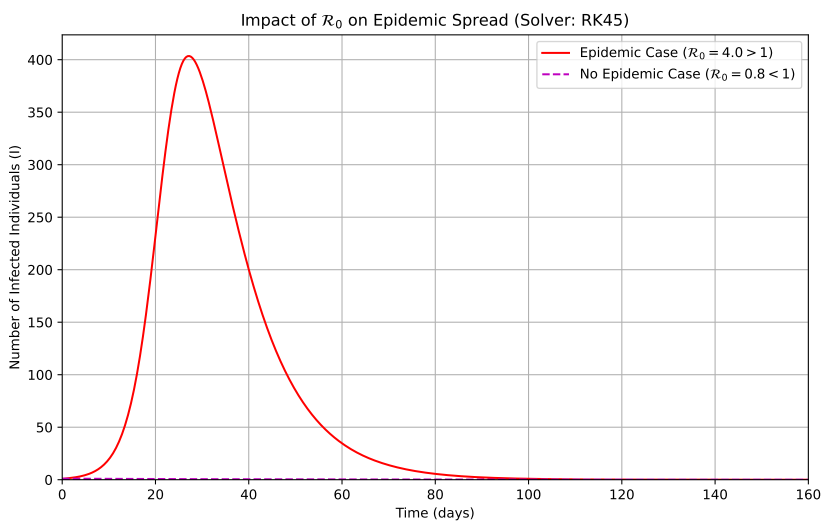

Figure 2.7: Epidemic Threshold Behavior $\mathcal{R}_0$ for the Infected $I(t)$.

In summary, the SIR model is a prime example where the theoretical guarantees of existence and uniqueness of solutions for ODEs, as provided by Peano's and Cauchy–Lipschitz theorems, are essential for the model's validity and utility in understanding and predicting real world phenomena like epidemic spread.

Chapter 3

Theories of Uniqueness

In the second chapter, we discussed the existence criterion, and since it is sufficient, we continued working based on the assumption that solutions always exist. This assumption leads us to study the uniqueness problem. We previously introduced the Cauchy Lipschitz theorem, which ensures uniqueness under certain assumptions, but it is not the only theorem. In this chapter, we will explore these theories related to uniqueness: if a solution exists, it is unique. Let us return to the notations used at the beginning of the Cauchy problem (1.10). We use the following notations:

$$C_+ = [t_0, t_0 + a] \times \bar{B}(y_0, r_0),\ C_- = [t_0 - a, t_0] \times \bar{B}(y_0, r_0),\ C = C_+ \cup C_-.$$

And since $y \in \mathbb{R}^m$, the norm is defined as follows $\|y\| = \sum_{i=1}^{m} |y_i|$.

3.1 Peano's Uniqueness Theorem

Theorem 3.1.1. *[1] Let $g(t, y)$ be continuous in C_+ and for all $(t, y), (t, \bar{y}) \in C_+$, it satisfies:*

$$(g(t, y) - g(t, \bar{y})) \cdot (y - \bar{y}) \leq 0. \tag{3.1}$$

Then, the CP(1.10) admits a unique solution in $[t_0, t_0 + a]$.

Proof. Suppose that $y, \bar{y}$ are solutions of (1.10), with $t \in [t_0, t_0 + a]$, then for the function

$v(t) = \|y(t) - \bar{y}(t)\|^2$, we find that:

$$\begin{aligned}
v'(t) &= \left(\|y(t) - \bar{y}(t)\|^2\right)', \\
&= \left(\sum_{i=1}^{m} (y_i(t) - \bar{y}_i(t))^2\right)', \\
&= 2\sum_{i=1}^{m} (y_i(t) - \bar{y}_i(t)) \cdot (y_i'(t) - \bar{y}_i'(t)), \\
&= 2\sum_{i=1}^{m} (y_i(t) - \bar{y}_i(t)) \cdot (g_i(t, y(t)) - g_i(t, \bar{y}(t))), \\
&= 2\left(g(t, y(t)) - g(t, \bar{y}(t))\right) \cdot (y(t) - \bar{y}(t)), \\
&\leq 0.
\end{aligned}$$

We have $v(t) \geq 0$, $v'(t) \leq 0$ and $v(t_0) = 0$, then:

$$v(t) = 0 \Rightarrow y(t) = \bar{y}(t).$$

Using the norm's definition, we find that the equation of (1.10) admits a unique solution according to Peano's uniqueness theorem. □

Remark 3.1.1. *:*

- *If the function g(t,y) is continuous in C_- and for all $(t, y), (t, \bar{y}) \in C_-$ it satisfies:*

$$(g(t, y) - g(t, \bar{y})) \cdot (y - \bar{y}) \geq 0. \tag{3.2}$$

Then, the CP (1.10) exists a unique solution in $[t_0 - a, t_0]$.

- *The function g(t,y) is continuous in C which satisfies (3.1) and (3.2) in C_+ and C_-, then the CP (1.10) admits a unique solution in $[t_0 - a, t_0 + a]$.*

Example 3.1.1. *Consider the differential problem:*

$$\begin{cases} y' = -y^3, \\ y(0) = y_0. \end{cases}$$

Here, the function $g(t, y) = -y^3$ satisfies:

$$(g(t, y) - g(t, \bar{y})) \cdot (y - \bar{y}) = (-y^3 + \bar{y}^3) \cdot (y - \bar{y}).$$

We know that

$$y^3 - \bar{y}^3 = (y - \bar{y}) \cdot (y^2 + y\bar{y} + \bar{y}^2),$$

we obtain

$$(g(t, y) - g(t, \bar{y})) \cdot (y - \bar{y}) = -(y - \bar{y})^2 \cdot (y^2 + y\bar{y} + \bar{y}^2) \leq 0.$$

Since this condition is satisfied, the uniqueness of the solution follows from Peano's uniqueness theorem.

3.2 Gard's Uniqueness Theorem

Theorem 3.2.1. *[1] Let $g(t, y)$ be continuous in C_+ and $\phi(t)$ be a continuous function defined in $[t_0, t_0 + a)$ and differentiable in $(t_0, t_0 + a)$, such that $\phi(t) > 0$ for $t > t_0$ and $\phi(t_0) = 0$. In addition to that:*
1) $\forall (t, y), (t, \bar{y}) \in C_+$:

$$(g(t, y) - g(t, \bar{y})) \cdot (y - \bar{y}) \leq \frac{\phi'(t)}{\phi(t)} \|y - \bar{y}\|^2.$$

2) $\exists L \in \mathbb{R}$, for $i \in \{1, \dots, m\}$, such that:

$$g_i(t, y) = L\phi'(t) + o(\phi'(t)) \quad \text{as } t \to t_0^+,\ y \to y_0.$$

Then, the CP (1.10) admits a unique solution in $[t_0, t_0 + a)$.

Proof. Suppose that $y(t)$ and $\bar{y}(t)$ be two solutions of (1.10), where:

$$v(t) = \begin{cases} \frac{1}{2} \left[\frac{\|y(t) - \bar{y}(t)\|}{\phi(t)} \right]^2, & t \in (t_0, t_0 + a), \\ 0, & t = t_0. \end{cases}$$

First, we study the continuity of $v(t)$ at t_0 using L'Hôpital's Rule and condition (2).

We have, for $1 \leq i \leq m$:

$$\lim_{t \to t_0^+} \frac{y_i(t) - \bar{y}_i(t)}{\phi(t)} = \lim_{t \to t_0^+} \frac{(y_i(t) - \bar{y}_i(t))'}{(\phi(t))'} = \lim_{t \to t_0^+} \frac{g_i(t, y(t)) - g_i(t, \bar{y}(t))}{\phi'(t)},$$
$$= \lim_{t \to t_0^+} \frac{g_i(t, y(t)) - L\phi'(t)}{\phi'(t)} - \frac{g_i(t, \bar{y}(t)) - L\phi'(t)}{\phi'(t)} = 0.$$

Then, for $v(t)$ is continuous for $t \geq t_0$, we will differentiate $v(t)$:

$$v'(t) = \frac{1}{2}\left(\frac{\|y(t) - \bar{y}(t)\|^2}{\phi(t)^2}\right)' = \frac{\left(\|y(t) - \bar{y}(t)\|^2\right)' \phi(t)^2 - (\phi(t)^2)'\|y(t) - \bar{y}(t)\|^2}{2\phi(t)^4},$$
$$= \frac{(y(t) - \bar{y}(t)) \cdot (g(t, y(t)) - g(t, \bar{y}(t)))\, \phi(t)^2 - \phi'(t)\phi(t)\|y(t) - \bar{y}(t)\|^2}{\phi(t)^4}.$$

By exploiting the condition (1), we further obtain:

$$v'(t) = \frac{1}{\phi(t)^2}\left[(g(t, y(t)) - g(t, \bar{y}(t))) \cdot (y(t) - \bar{y}(t)) - \frac{\phi'(t)}{\phi(t)}\|y(t) - \bar{y}(t)\|^2\right] \leq 0.$$

From the results and given data, we find that $v(t) = 0$, for all $t \in [t_0, t_0 + a]$. Using the same arguments in the proof of (3.1), we conclude that y is the unique solution. □

Example 3.2.1. *Consider the differential problem:*

$$\begin{cases} y' = \dfrac{y}{t - t_0 + 1}, \ t > t_0, \\ y(t_0) = 0. \end{cases}$$

Here, the function $g(t, y)$ is given by:

$$g(t, y) = \frac{y}{t - t_0 + 1}.$$

We find

$$(g(t, y) - g(t, \bar{y})) \cdot (y - \bar{y}) = \frac{(y - \bar{y})^2}{t - t_0 + 1}.$$

Thus

$$(g(t, y) - g(t, \bar{y})) \cdot (y - \bar{y}) \leq \frac{(y - \bar{y})^2}{t - t_0}.$$

We choose the function $\phi(t)$ in this inequality that satisfy the conditions of the Gard's *uniqueness theorem as follows:*

$$\phi(t) = t - t_0 \text{ and } \frac{\phi'(t)}{\phi(t)} = \frac{1}{t - t_0}.$$

Therefore

$$g(t, y) - g(t, \bar{y}) \leq \frac{\phi'(t)}{\phi(t)}(y - \bar{y})^2.$$

The first condition holds. On the other hand:

$$g(t, y) = L\phi'(t) + o(\phi'(t)) \quad \text{as } t \to t_0^+,\ y \to y_0.$$

Choosing $L = \dfrac{y_0}{t - t_0 + 1}$ *gives:*

$$g(t, y) = L + o(1) = L\phi'(t) + o(\phi'(t)).$$

Thus, the condition (2) holds. Since both conditions of the Gard's uniqueness theorem are satisfied, the solution is unique on the given interval.

3.3 Bownds Diaz's Uniqueness Theorem

Theorem 3.3.1. *[1] Let* g *be continuous in* C_+ *and for all* $(t, y), (t, \bar{y}) \in C_+$ *it satisfies:*

$$(g(t, y) - g(t, \bar{y})) \cdot (\phi(t, y) - \phi(t, \bar{y})) \leq 0.$$

For $1 \leq i \leq m$*:*

$$\phi_i(t, y) = \frac{\partial g_i(t, y)}{\partial t} + \sum_{j=1}^{m} \frac{\partial g_i(t, y)}{\partial y_j} g_j(t, y).$$

Thus

$$\sum_{i=1}^{m} (g_i(t, y) - g_i(t, \bar{y})) \cdot \left[\frac{\partial g_i(t, y)}{\partial t} + \sum_{j=1}^{m} \frac{\partial g_i(t, y)}{\partial y_j} g_j(t, y) - \frac{\partial g_i(t, \bar{y})}{\partial t} - \sum_{j=1}^{m} \frac{\partial g_i(t, \bar{y})}{\partial y_j} g_j(t, \bar{y}) \right] \leq 0.$$

Then, the CP (1.10) admits a unique solution in $[t_0, t_0 + a]$.

Proof. For $t_0 \leq t \leq t_0 + a$, we define the function $v(t)$ as:

$$v(t) = \frac{1}{2} \| g(t, y(t)) - g(t, \bar{y}(t)) \|^2.$$

We find

$$v'(t) = (g(t, y(t)) - g(t, \bar{y}(t))) \cdot \frac{d}{dt}(g(t, y(t)) - g(t, \bar{y}(t))).$$

Using the chain's rule, thus:

$$\begin{aligned}
\frac{d}{dt} g(t, y(t)) &= \frac{\partial g(t, y)}{\partial t}\frac{dt}{dt} + \frac{\partial g(t, y)}{\partial y_1}\frac{dy_1}{dt} + \cdots + \frac{\partial g(t, y)}{\partial y_m}\frac{dy_m}{dt}, \\
&= \frac{\partial g(t, y)}{\partial t} + \sum_{j=1}^{m} \frac{\partial g(t, y)}{\partial y_j}\frac{dy_j}{dt}, \\
&= \frac{\partial g(t, y)}{\partial t} + \sum_{j=1}^{m} \frac{\partial g(t, y)}{\partial y_j} g_j(t, y).
\end{aligned}$$

Then

$$\begin{aligned}
v'(t) &= (g(t, y) - g(t, \bar{y})) \cdot \left[\frac{\partial g(t, y)}{\partial t} + \sum_{j=1}^{m} \frac{\partial g(t, y)}{\partial y_j} g_j(t, y) + \frac{\partial g(t, \bar{y})}{\partial t} + \sum_{j=1}^{m} \frac{\partial g(t, \bar{y})}{\partial \bar{y}_j} g_j(t, \bar{y}) \right], \\
&= (g(t, y) - g(t, \bar{y})) \cdot (\phi(t, y) - \phi(t, \bar{y})) \leq 0.
\end{aligned}$$

As in the previous $v(t) = 0$, so:

$$\begin{aligned}
\frac{1}{2}\|g(t, y(t)) - g(t, \bar{y}(t))\|^2 = 0 &\Rightarrow g(t, y(t)) = g(t, \bar{y}(t)), \\
&\Rightarrow y'(t) = \bar{y}(t), \\
&\Rightarrow \int_{t_0}^{t} y'(x)\, dx = \int_{t_0}^{t} \bar{y}(x)\, dx, \\
&\Rightarrow y(t) - y(t_0) = \bar{y}(t) - \bar{y}(t_0).
\end{aligned}$$

Since $y, \bar{y}$ are solutions of (1.10) it means that $y(t_0) = \bar{y}(t_0)$ so admits a unique solution y. □

Example 3.3.1. *Consider the differential problem:*

$$\begin{cases} y' = -y + \sin(t), \\ y(0) = 1. \end{cases}$$

By definition:

$$\phi(t, y) = \frac{\partial g}{\partial t} + \frac{\partial g}{\partial y} g(t, y).$$

Thus

$$\phi(t, y) - \phi(t, \bar{y}) = (\cos(t) + y - \sin(t)) - (\cos(t) + \bar{y} - \sin(t)) = y - \bar{y}.$$

And

$$g(t, y) - g(t, \bar{y}) = (-y + \sin(t)) - (-\bar{y} + \sin(t)) = \bar{y} - y.$$

We find

$$(g(t, y) - g(t, \bar{y})) \cdot (\phi(t, y) - \phi(t, \bar{y})) = -(\bar{y} - y)^2 \leq 0.$$

The condition is satisfied, and thus the solution is unique.

3.4 Giuliano's Uniqueness Theorem

In some cases, the function g may not be fully continuous, but it can still be partially continuous. Therefore, before understanding the Giuliano's uniqueness theorem, we must first define some conditions and properties to prove it.

Carathéodory Conditions

In what follows, we present the existence results in the sense of Carathéodory.

Definition 3.4.0.1. *Let g verify Carathéodory conditions if:*

- *$g(t, y)$ is measurable in t for each fixed $y = (y_1, y_2, \dots, y_m)$.*
- *$g(t, y)$ is continuous in y for each fixed t.*
- *There exists an integrable function $M_i(t)$ in $\bar{I}$ such that:*

$$|g_i(t, y)| \leq M_i(t), \quad 1 \leq i \leq m, \quad \forall (t, y) \in U.$$

Theorem 3.4.1. *[1] Let $g(t, y)$ be a function that satisfies the Carathéodory conditions in I_0: $t_0 < t < t_0 + a, \|y\| < \infty$. For all $(t, y), (t, \bar{y}) \in I_0$ and the Giuliano's inequality:*

$$(g(t, y) - g(t, \bar{y})) \cdot (y - \bar{y}) \leq h(t) f(\|y - \bar{y}\|^2), \tag{3.3}$$

where $h(t) > 0$ is a Lebesgue integrable function on $[\alpha, \beta] \subset (t_0, t_0 + a)$ and f satisfies the following conditions

- *$f(z)$ be a continuous and non decreasing function in the interval $[0, \infty)$.*
- *$f(0) = 0, \;\; f(z) > 0$ for $z > 0$.*
- $\lim_{\varepsilon \to 0^+} \int_{\varepsilon} \frac{dt}{f(t)} = \infty.$

Then, the CP (1.10) admits a unique solution in $[t_0, t_0 + a]$.

Proof. Suppose that $y, \bar{y}$ are two solution of (1.10) in $[t_0, t_0 + a]$. Consider the following function:

$$\phi(t) = \|y(t) - \bar{y}(t)\|^2 .$$

By using the inequality (3.3), we find:

$$\begin{aligned} \phi'(t) = (g(t, y) - g(t, \bar{y})) \cdot (y - \bar{y}) &\leq 2h(t)f(\|y - \bar{y}\|^2), \\ &\leq 2h(t)f(\phi(t)). \end{aligned}$$

Since $\phi(t) > 0 \Rightarrow f(\phi(t)) > 0$, implies:

$$\frac{\phi'(t)}{f(\phi(t))} \leq 2h(t).$$

$\forall \bar{t} \in [t_0, t_0 + \varepsilon]$, we integrate both sides:

$$\int_{\bar{t}}^{t_0+\varepsilon} \frac{\phi'(x)}{f(\phi(x))}\, dx \leq 2 \int_{\bar{t}}^{t_0+\varepsilon} h(x)\, dx.$$

We set $z = \phi(x), dz = \phi'(x)dx$, then:

$$\int_{\phi(\bar{t})}^{\phi(t_0+\varepsilon)} \frac{1}{f(z)}\, dz \leq 2 \int_{\bar{t}}^{t_0+\varepsilon} h(x)\, dx.$$

The above inequality, as $\bar{t} \to t_0$, the right-hand side remains bounded, whereas the left-hand side, according to the conditions on f, becomes unbounded. This leads to a contradiction. Therefore, we must have $\phi(t) = 0$ or $\phi(t) < 0$. From the definition of the norm, we conclude that $\phi(t) = 0$, which implies that $y(t) = \bar{y}(t)$. □

Example 3.4.1. *We consider the differential problem:*

$$\begin{cases} y' = g(t, y), \\ y(0) = 1, \end{cases}$$

where

$$g(t, y) = \begin{cases} t\sin(y), & t \geq 0, \\ \cos(y), & t < 0. \end{cases}$$

1. We check the measurability of** $g(t, y)$ **in** t **for a fixed** y**:

- *For* $t \geq 0$*,* $g(t, y) = t\sin(y)$*. Since* $\sin(y)$ *is a constant for a fixed* y*, and* t *is a measurable function, their product remains measurable.*
- *For* $t < 0$*,* $g(t, y) = \cos(y)$*, which is a constant function in* t *and therefore measurable.*

Thus, $g(t, y)$ *is measurable in* t *for each fixed* y*.*

2. We verify that** $g(t, y)$ **is continuous in** y **for each fixed** t**:

- *For* $t \geq 0$*,* $g(t, y) = t\sin(y)$*. Since* $\sin(y)$ *is continuous and* t *is constant,* $g(t, y)$ *is continuous in* y*.*
- *For* $t < 0$*,* $g(t, y) = \cos(y)$*, which is a well-known continuous function.*

Since the continuity in y *is only required for each fixed* t*, the function satisfies this condition.*
Thus, $g(t, y)$ *is continuous in* y *for each fixed* t*.*

3. We now show that there exists a function** $M(t)$ **such that** $\|g(t, y)\| \leq M(t)$ **for all** t**:

- *For* $t \geq 0$*, we have:*

$$\|g(t, y)\| = \|t\sin(y)\| \leq |t|, \quad \textit{since } \|\sin(y)\| \leq 1.$$

- *For $t < 0$, we have:*

$$\|g(t,y)\| = \|\cos(y)\| \leq 1.$$

We choose $M(t) = \max(|t|, 1)$. Then, for all $t \in \mathbb{R}$, we have $\|g(t,y)\| \leq M(t)$, where the function $M(t)$ is Lebesgue integrable on any finite interval. Since $g(t,y)$ satisfies Carathéodory's conditions, we now verify the Giuliano's inequality:

$$g(t,y) - g(t,\bar{y}) = \begin{cases} t(\sin(y) - \sin(\bar{y})), & t \geq 0, \\ \cos(y) - \cos(\bar{y}), & t < 0. \end{cases}$$

Using trigonometric identities:

$$\sin(y) - \sin(\bar{y}) = 2\cos\left(\frac{y+\bar{y}}{2}\right)\sin\left(\frac{y-\bar{y}}{2}\right),$$

$$\cos(y) - \cos(\bar{y}) = -2\sin\left(\frac{y+\bar{y}}{2}\right)\sin\left(\frac{y-\bar{y}}{2}\right).$$

Thus, we get:

$$(g(t,y) - g(t,\bar{y})) \cdot (y - \bar{y}) = \begin{cases} 2t\cos\left(\frac{y+\bar{y}}{2}\right)\sin\left(\frac{y-\bar{y}}{2}\right)(y-\bar{y}), & t \geq 0, \\ -2\sin\left(\frac{y+\bar{y}}{2}\right)\sin\left(\frac{y-\bar{y}}{2}\right)(y-\bar{y}), & t < 0. \end{cases}$$

We know that $\sin(a)a \leq a^2$, we find:

$$(g(t,y) - g(t,\bar{y})) \cdot (y - \bar{y}) \leq \begin{cases} 4t\cos\left(\frac{y+\bar{y}}{2}\right)\left(\frac{(y-\bar{y})^2}{4}\right), & t \geq 0, \\ -4\sin\left(\frac{y+\bar{y}}{2}\right)\left(\frac{(y-\bar{y})^2}{4}\right), & t < 0. \end{cases}$$

Hence, we obtain:

$$(g(t,y) - g(t,\bar{y})) \cdot (y - \bar{y}) \leq \begin{cases} |t|(y-\bar{y})^2, & t \geq 0, \\ (y-\bar{y})^2, & t < 0. \end{cases}$$

To satisfy Giuliano's inequality, we choose:

$$h(t) = |t|, \quad f(z) = z.$$

Where $h(t) > 0$ *is a Lebesgue integrable function on* $[\alpha, \beta]$ *and* $f(z)$ *satisfies the conditions of Giuliano's theorem. Therefore, the solution is unique. This example shows that it still applies even when* $g(t, y)$ *is not fully continuous.*

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Conclusion

In this work, the study initially focused on two fundamental problems in the theory of ODEs:

1. The existence of solutions under specified initial conditions.

2. The uniqueness of solutions, ensuring that no other solutions satisfy the same conditions.

To address these problems, we examined key theoretical results, including Peano's existence theorem, which guarantees that a continuous right-hand side in the ODE ensures the existence of at least one solution. However, continuity alone does not suffice to ensure uniqueness. Consequently, we expanded our study to the uniqueness problem and concluded that this property holds when certain conditions are met, such as the Cauchy–Lipschitz condition, Giuliano's condition, among others. In conclusion, this study resolves the central questions posed at the outset, which concern the criteria for the existence and uniqueness of solutions.

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Annex

Zorn's Lemma

Definition 3.4.1. *(Partial Order)*

A relation $\preceq$ on a set S is called a partial order if it satisfies the following properties for all elements $t, y, z \in S$:

*1. **Reflexivity:** For all $t \in S$, we have $t \preceq t$.*

*2. **Antisymmetry:** If $t \preceq y$ and $y \preceq t$, then $t = y$.*

*3. **Transitivity:** If $t \preceq y$ and $y \preceq z$, then $t \preceq z$.*

If $(S, \preceq)$ is a partially ordered set, there may exist elements $t, y \in C$ such that neither $t \preceq y$ nor $y \preceq t$, in which case they are said to be incomparable.

Remark 3.4.1. *If any two elements in S are always comparable (i.e., for all $t, y \in S$, either $t \preceq y$ or $y \preceq t$), then the partial order becomes a total order (or linear order).*

lemma 3.4.1. *(Zorn's) Let S be a non empty partially ordered set. If every totally ordered subset $T \subset S$ has an upper bound in S, then S contains at least one maximal element.*

Ascoli–Arzelà

Theorem 3.4.2. *Assume that (E, d) and (F, d') are compact metric spaces. Let $\varphi_n : E \to F$ be a sequence of functions. If the sequence $(\varphi_n)_{n\in\mathbb{N}}$ satisfies:*

- ***Boundedness:***

$$\exists M > 0,\ \forall x \in E,\ \forall n \in \mathbb{N},\ then\ \|\varphi_n\| \leq M.$$

- ***Equicontinuity:***

$$\forall \varepsilon > 0, \exists \delta > 0,\ \forall x, y \in E,\ \ d(x,y) \leq \delta \Rightarrow \forall n \in \mathbb{N},\ d(\varphi_n(x), \varphi_n(y)) < \varepsilon.$$

Then, there exists a subsequence $(\varphi_{n_k})_{k\in\mathbb{N}}$ that converges uniformly, and the limit function is continuous.

Picard Fixed Point

Theorem 3.4.3. *Let (E,d) be a complete metric space, and let $T : E \to E$ be a contractive mapping, i.e.,*

$$\exists\, k \in]0,1[,\ d(Tx,Ty) \leq k\, d(x,y), \quad \forall x, y \in E.$$

Then, there exists a unique $x \in E$ such that:

$$T(x) = x.$$

Numerical Methods for Solving Differential Equations

Consider the differential problem:

$$\begin{cases} y' = g(t,y), \\ y(t_0) = y_0, \end{cases} \tag{3.4}$$

where $g : U = I \times \Omega \to \mathbb{R}^m$, with I an open interval in $\mathbb{R}$, and Ω an open subset of $\mathbb{R}^m$. In many cases, it is impossible to find an explicit solution to a differential equation. Therefore, numerical methods are employed to approximate solutions. One of the simplest and most fundamental numerical methods:

Fourth Order Runge-Kutta Method (RK4)

Methods are a family of explicit multi stage schemes for approximating the solution of the CP (3.4). The classical fourth order Runge-kutta method (RK4) computes the next value y_{i+1} from the current value y_i using a step size $h = t_{i+1} - t_i$. The update involves calculating:

$$\begin{aligned}
k_1 &= hg(t_i, y_i), \\
k_2 &= hg\left(t_i + \frac{h}{2}, y_i + \frac{k_1}{2}\right), \\
k_3 &= hg\left(t_i + \frac{h}{2}, y_i + \frac{k_2}{2}\right), \\
k_4 &= hg(t_i + h, y_i + k_3).
\end{aligned}$$

Then

$$y_{i+1} = y_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4).$$

This process is repeated for subsequent steps.

Here are brief definitions of Python with the libraries relevant to the provided scientific computing code:

Python: A high level, interpreted, general purpose programming language known for its clear syntax and readability. It provides the core structure for executing the scientific simulation code presented with the help of libraries such as NumPy, SciPy for the heavy computation and Matplotlib for ploting.

الملخص

الهدف الرئيسي من هذه المذكرة هو دراسة الشروط التي تضمن وجود ووحدانية الحلول في المعادلات التفاضلية العادية، وذلك من خلال طرح السؤالين: هل يوجد حل للمعادلة؟ وإذا وجد، هل يكون هذا الحل وحيداً؟ للإجابة عن هذين السؤالين، تم عرض وإثبات المبرهنات الأساسية، مثل مبرهنة بيانو ومبرهنة كوشي-ليبسيتز للوجود والوحدانية، كما دُعّمت هذه النتائج بتطبيق على نموذج انتشار الأوبئة. إضافة إلى ذلك، تم تقديم نظريات أخرى حول الوحدانية، كما قدمت في بداية مفاهيم رياضية أساسية تتعلق بالمعادلات التفاضلية، وأنواع الحلول.
الكلمات المفتاحية: المعادلات التفاضلية العادية، الوجود، الوحدانية، مشكلة كوشي.

Abstract

The main objective of this thesis is to study the conditions that guarantee the existence and uniqueness of solutions to ordinary differential equations, by addressing the following two questions: Does a solution to the equation exist? And if so, is this solution unique? To answer these questions, the fundamental theorems, such as Peano's existence theorem and the Cauchy–Lipschitz theorem on existence and uniqueness, were presented and proved. These results were supported by an application to an epidemic spread model. In addition, other theories related to uniqueness were introduced. At the beginning, basic concepts and definitions related to differential equations and types of possible solutions were provided.
Key words: Ordinary Differential Equations, Existence, Uniqueness, Cauchy Problem.

Résumé

L'objectif principal de ce mémoire est d'étudier les conditions qui garantissent l'existence et l'unicité des solutions des équations différentielles ordinaires, à travers les deux questions suivantes : une solution existe-t-elle pour l'équation ? Et si elle existe, est-elle unique ? Pour répondre à ces deux questions, les théorèmes fondamentaux ont été présentés et démontrés, tels que le théorème de Peano et le théorème de Cauchy–Lipschitz sur l'existence et l'unicité. Ces résultats ont été appuyés par une application à un modèle de propagation d'épidémie. En outre, d'autres théorèmes relatifs à l'unicité ont été présentés. Des notions et définitions de base concernant les équations différentielles et les types de solutions ont également été introduites au début.
Mots clés: Équations différentielles ordinaires, Existence, Unicité, Problème de Cauchy.