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Hidden Attractors and Hidden Bifurcation in Continuous Chaotic Systems .

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DEDICACE

I praise "**Allah**" Almighty for His guidance and endless grace, by which He has granted me the success to complete this work. No words can fully express His right, for He is the Helper of the patient and the Light on the path, and by His grace alone are achievements realized.

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♡ I LOVE YOU ♡

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CONTENTS

LIST OF FIGURES

INTRODUCTION

In the study of dynamic systems, continuous chaotic systems represent a fascinating and intricate class of systems whose behavior can be both unpredictable and deterministic [1, 2]. A dynamic system is a mathematical model used to describe the evolution of a system over time. It is typically governed by differential equations that define how the system's state changes continuously with respect to time [3, 4].
These systems are used to model a wide variety of physical, biological, and engineered systems, ranging from fluid dynamics and population growth to electrical circuits and climate models [5, 6].
One of the most important concepts in the study of dynamical systems is stability [7]. Stability refers to the behavior of a system's solutions under small perturbations to the initial conditions. A system is considered stable if small changes in initial conditions lead to small changes in the system's future behavior. Conversely, instability occurs when even small perturbations can result in large, unpredictable changes in the system's state [8].
As dynamic systems evolve, they may undergo significant qualitative changes in behavior [9]. These changes are often captured in the concept of bifurcation, where a small change in a system's parameters can lead to a sudden and dramatic change in its long-term behavior. Bifurcations are a key phenomenon in understanding chaotic systems, as they mark the point at which a system transitions from periodic or regular behavior to chaotic dynamics [10].
At the heart of chaotic systems lies chaos, a form of deterministic unpredictability [11]. In chaotic systems, small differences in initial conditions can lead to vastly different outcomes, making long-term prediction practically impossible. This sensitive dependence on initial conditions, often referred to as the "butterfly effect," is one of the hallmarks of chaos [12]. Despite this unpredictability, chaotic systems are governed by deterministic laws, which means that their behavior is fully determined by their initial conditions and governing equations [13].
The concept of chaotic systems with stable equilibria has gained significant attention in recent research [14]. The term "chaotic system with hidden attractors" refers to systems that either have no equilibrium points or possess only a single stable equilibrium point [15, 16]. This novel class of attractors was first identified by Leonov and colleagues [17, 18, 19].
A key characteristic is that any unstable equilibrium point does not lie within its basin of attraction [20].To distinguish between different types of attractors, the conventional attractor is classified as "self-excited," while a hidden attractor forms in systems without equilibria [21]. The fundamental difference is that the attraction basin of a hidden attractor does not overlap with any small neighborhood surrounding any equilibrium point. In contrast, the attraction basin of a self-excited attractor will intersect with some unstable equilibrium points [22].
In 2016, the researchers enhanced the discrete parameter model by introducing hidden

bifurcations and generating multiple spiral patterns [23].Hidden bifurcation theory is established on the foundation of hidden attractor theory, which was developed by Leonov and colleagues [17]. To identify hidden bifurcations, the researchers maintained fixed system parameters while introducing a new control parameter ε in the nonlinear component, regulated by a homotopy parameter ε, while keeping the other parameters constant [25]. This parameter, ε, ranges from 0 to 1. When ε equals 0, the nonlinear part of the system is decoupled, resulting in a cycle-shaped attractor [26, 27]. Conversely, when ε equals 1, the system displays the attractor of the original system [28, 24].

This thesis explores three fundamental aspects of complex dynamics. The first chapter establishes a comprehensive foundation in continuous dynamic systems, covering essential mathematical frameworks, stability analysis, and bifurcation theory required to understand chaotic behavior in deterministic systems. The second chapter analyzes the novel concepts of hidden attractors and hidden bifurcations, distinguishing them from traditional self-excited attractors by examining how their basins of attraction relate to equilibrium points, with particular emphasis on Leonov's groundbreaking classification methodology and the homotopy-based approach for uncovering hidden structures in phase space. The final chapter applies these theoretical concepts to Chua's circuit system, demonstrating how hidden attractors and bifurcations manifest in this paradigmatic work.

CHAPTER 1

CONTINUOUS DYNAMIC SYSTEM

1.1 Introduction

Chaos theory is a branch of mathematics and physics that focuses on the study of dynamic systems characterized by extreme sensitivity to initial conditions. This fundamental property, commonly referred to as the "butterfly effect," causes small perturbations in starting conditions to lead to vastly different outcomes over time. Among the most widely studied in this domain are *continuous dynamic systems*, which are systems described by nonlinear ordinary differential equations and evolve over continuous time.

A continuous chaotic system typically exhibits irregular, non-repeating behavior while remaining deterministic in nature. The trajectories of such systems do not settle into fixed points or periodic orbits but instead approach complex structures known as strange attractors. These attractors possess a fractal geometry and exhibit sensitive dependence on initial conditions, which makes long-term prediction practically impossible despite the system being governed by deterministic rules.

One of the earliest and most influential examples of a continuous chaotic system is the Lorenz system [12], originally developed to model atmospheric convection. Since its discovery, many other continuous-time chaotic systems have been formulated Chua's circuit [17].

These systems have spurred extensive research due to their potential applications in areas such as secure communication, electrical engineering, neuroscience, weather forecasting, and financial systems.

The analysis of continuous chaotic systems involves various qualitative and quantitative tools. Phase space visualization, bifurcation diagrams, and Poincaré sections are commonly used to study the global behavior of the system. Additionally, indicators such as Lyapunov exponents help in distinguishing chaotic behavior from regular dynamics by quantifying the average rate of divergence of nearby trajectories.

In recent years, research has further expanded into complex behaviors such as hidden attractors, multistability, and coexisting attractors. These phenomena have significant implications for understanding the complete dynamic range of nonlinear systems and developing advanced control and synchronization techniques.

1.2 Important Definitions

Definition 1.2.1 (Dynamic Systems) *A dynamic system is one in which the state changes over time t. Evolution is regulated by a framework of principles that determine the system's state for discrete or continuous values of t.*

Dynamic system are classified into two categories:
- *Continuous dynamic systems.*
- *Discrete dynamic systems.*

Definition 1.2.2 (Continuous Dynamic Systems) *A continuous dynamic system can be represented by a differential equation of the form,*

$$\frac{dx}{dt} = f(x, \mu), x \in M \subset \mathbb{R}^m, \mu \in \mathbb{R}^r, m \text{ and } r \in \mathbb{N}. \tag{1.1}$$

Where f is continuous .

Example 1.2.1 *We consider a system of differential equations (Lotka-Volterra)*

$$\begin{cases} \dot{x} = \alpha x - \beta xy \\ \\ \dot{y} = \delta xy - \gamma y. \end{cases} \quad ,(x, y) \in \mathbb{R}^2,$$

Definition 1.2.3 (Discrete Dynamical Systems) *Discrete dynamical systems are known as regressive sequences,*

$$x_{k+1} = f(x_k, \mu), x_k \in M \subset \mathbb{R}^m, \mu \in \mathbb{R}^r \qquad k = 1, 2, ... \tag{1.2}$$

Where f is discrete .

Example 1.2.2 *We consider the discrete dynamical system (logistic map) :*

$$x_{n+1} = rx_n(1 - x_n), x_n \in [0, 1].$$

Definition 1.2.4 (Autonomous System) *An autonomous system is a system of differential equations in which the independent variable (typically time, t) does not explicitly appear in the equations. Mathematically, it is expressed as:*

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}), \tag{1.3}$$

where $\mathbf{x}$ is the state vector and $\mathbf{f}(\mathbf{x})$ is a function independent of t. Autonomous systems describe time-invariant dynamics, meaning their behavior is solely determined by their state and not by explicit time dependence.

Definition 1.2.5 (Non-Autonomous System) *A non-autonomous system is a system of differential equations where the independent variable explicitly appears in the governing equations. It has the general form:*

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, t), \tag{1.4}$$

Here, the evolution of $\mathbf{x}$ depends both on the current state and explicitly on time t. Such systems often model time-dependent external forces or perturbations.

Definition 1.2.6 (Dissipative System) *A dissipative system is a dynamical system in which the total energy decreases over time due to dissipation (e.g., friction, resistance, or damping). It is characterized by the existence of an attractor, towards which trajectories converge. The phase-space volume of such a system contracts over time. Mathematically, if V represents the phase-space volume, a dissipative system satisfies:*

$$\frac{dV}{dt} < 0, \tag{1.5}$$

for most cases. Dissipative systems are commonly encountered in thermodynamics, fluid dynamics, and chaos theory.

Definition 1.2.7 (Conservative System) *A conservative system is a system in which the total energy remains constant over time, meaning there is no energy loss due to dissipation. The system's Hamiltonian (total energy function) is preserved:*

$$\frac{dV}{dt} = 0. \tag{1.6}$$

Conservative systems maintain phase-space volume (Liouville's theorem) and typically describe systems with purely reversible dynamics, such as ideal mechanical systems without friction.

Definition 1.2.8 (Linear Dynamic System) *A linear dynamic system or linear differential system is an equation relating to a vector function x(t) , which can be written as*

$$\dot{x} = A(t)x(t). \tag{1.7}$$

Where more generally

$$\dot{x} = A(t)x(t) + g(t),$$

Where A is a square matrix and g a vector whose elements are functions of t. The linear word only concerns the dependence on x, the elements of $A(t)$ and $g(t)$ do not have to be linear in t.

Definition 1.2.9 (Nonlinear Dynamic System) *A nonlinear dynamical system can always be written by a differential equation from the following form :*

$$\dot{x} = f(t, x(t)) = \begin{pmatrix} f_1(x(t)) \\ f_2(x(t)) \\ . \\ . \\ . \\ f_n(x(t)) \end{pmatrix},$$

or $x(t) \in \mathbb{R}^n$, and f is a nonlinear function ,We note $f(t, x(t)) = F(x(t))$.

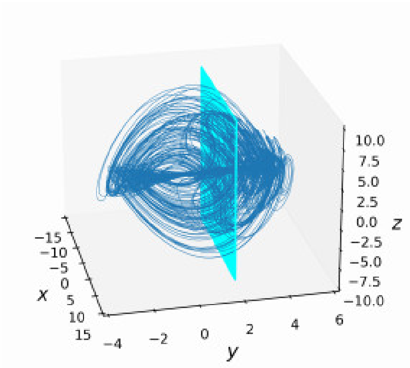
1.2.1 Phase Space

Definition 1.2.10 *A graph representing a solution of a system of differential equations is called its integral curve, and a projection of the integral curve onto the phase space along the* t *axis is referred to as a phase curve (trajectory, or orbit).*

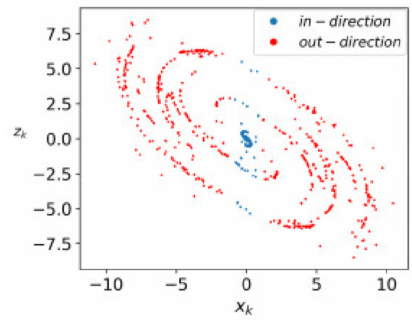
Definition 1.2.11 *A limit cycle is considered orbitally asymptotically stable (or simply stable) if, for any perturbation within its small neighborhood* U *, all trajectories that start near the cycle remain within this neighborhood and eventually tend toward the cycle over time* ($t \longrightarrow \infty$).

Definition 1.2.12 *The phase curve (trajectory) of the periodic solution of the system 1.2.2 is closed and is referred to as a cycle. Furthermore, any cycle (the closed phase curve of the system 1.2.2 defines a periodic solution with a certain period.*

Definition 1.2.13 (The Poincaré map) *The Poincaré map, or first return map, introduced by Henri Poincaré in* 1881*, is a fundamental tool for studying the stability and bifurcations of periodic orbits. The concept is simple: if* r *is a periodic orbit of the system 1.2.2 passing through the point ,*x_0 *and* $\sum$ *is a hyperplane perpendicular to* r *at* x_0*, then for any point* x *in* P *close to* x_0*, the solution of 1.2.2 starting from* x *at* $t = 0$*, denoted* $\phi(x)$*, will cross* P *again at a point* $P(x)$ *near* x_0 *see Fig 1.1.*



(a)



(b)

Figure 1.1: Poincaré map

Definition 1.2.14 *Let* x_0 *be a singular point (fixed point, equilibrium point, stationary point) of a differentiable vector field* $F(x)$ *so* $F(x_0) = 0$.

Definition 1.2.15 *A singular point of a vector field is a point in phase space where the vector of the field vanishes.*

Definition 1.2.16 *The periodic solution* $x(t)$ *of an autonomous system of differential equations 1.2.2 exists if there exists a constant* T*; such that* $x(t + T) = x(t)$ *for all* t*. The period of the solution* $x(t)$ *is named after the minimal such value* T *and the solution* $x(t)$ *is called* T*-periodic solution.*

Definition 1.2.17 *A stationary solution of an autonomous system of differential equations (a solution that is constant at a singular point) is called Lyapunov stable if all solutions with initial conditions from a sufficiently small neighborhood of the singular point are defined for all positive time and, as time progresses, converge uniformly to the stationary solution when the initial conditions approach the singular point.*

Definition 1.2.18 *The isolated closed trajectory is referred to as a limit cycle in an autonomous system of ordinary differential equations.*

1.2.2 Attractors of Dissipative Systems

Definition 1.2.19 *In relation to a flow φ^t set $B \subset M$ compact invariant if there is an attractive set in its neighbourhood U (the open set containing B) such, that $B \subset \omega(U)$ and for almost all*

$$x \in U, \varphi^t(x) \longrightarrow B, t \longrightarrow \infty,$$

(i.e.dist($\varphi^t(x), B) = inf_{y \in B} = \| \varphi^t(x) - y \| \longrightarrow 0$, when $t \longrightarrow \infty$). The greatest set, U, satisfying this definition, is called a attraction of field for B.

Definition 1.2.20 *The indecomposable attractive set is referred to as an attractor.*

Remark 1.2.1 *Not all attractive sets are attractors; only those that have the property of indecomposability, meaning they cannot be divided into two separate compact invariant subsets, are considered attractors.*

Different Types of Attractors

There are two types of attractors: regular attractors and strange attractors, or chaoticattractors.

1.Regular attractors:

Regular attractors characterize the evolution of non-chaotic systems and can be of three kinds.

•The fixed point:
A simple type of attraction in which the system moves towards a fixed point. This is the most usual case. It is important to remember that the real draw is mostly through the docks. There is always at least one "output path" for other types of fixed points. Each value in the Jacobian matrix with a positive real part is linked to a special vector that shows the direction the path goes away from the fixed point.
•The Periodic Limit Cycle
Due to perpetual oscillations, the phase path may close on itself, making time evolution cyclic. In a dissipative physical system, a strong equation component compensates for average losses from dissipation.
•The pseudo-periodic limit cycle(Invariant tori)
This represents a nearly specific instance of the preceding case. The system offers a minimum of two concurrent periods with an irrational ratio. The phase path does not self-close; instead, it surrounds a two-dimensional polysurface.
2. Strange attractors
A surface containing divergent paths is called an unstable surface, while a surface containing converging paths is called a stable surface. It is worth noting that this cannot be visualized in a phase space of only two dimensions. Strange attractors are properties of the evolution of chaotic systems: after a certain period of time, all points in phase space (belonging to the attraction basin of the attractor) give paths tending to form the strange attractor.

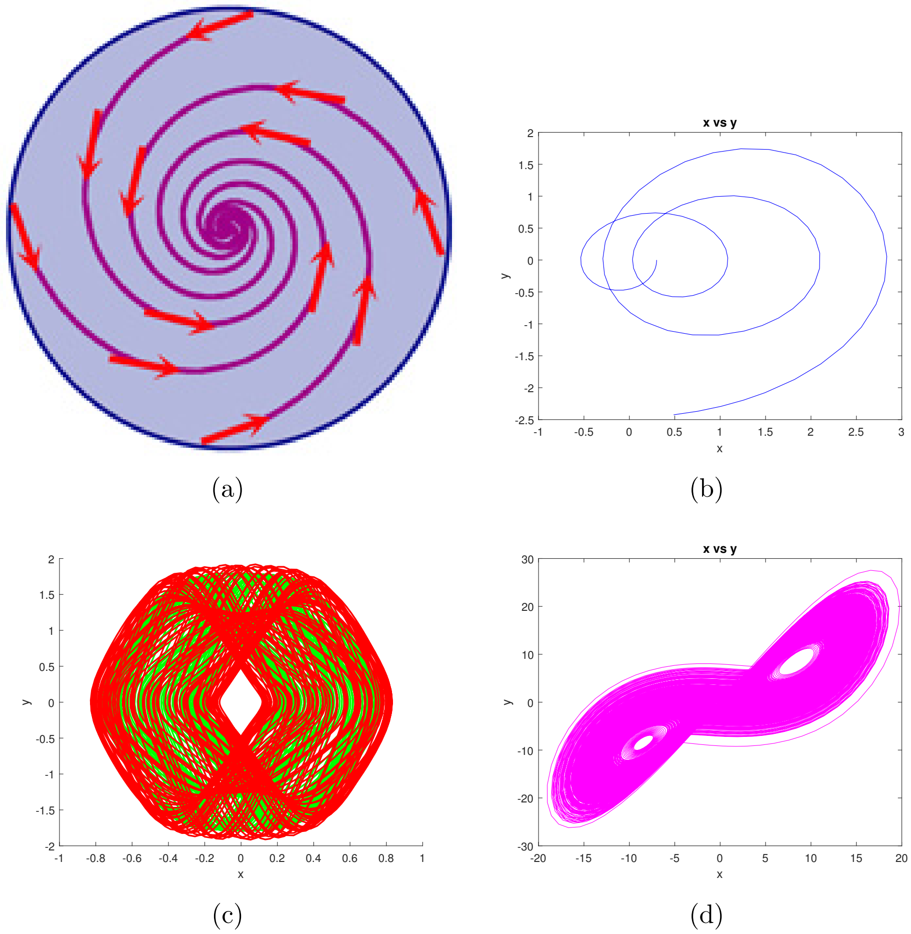


Figure 1.2: Different types of attractors (a) fixed point; (b) periodic limit cycle; (c) pseudo-periodic limit cycle(Invariant tori); (d) strange attractor (Lorenz attractor).

Qualitative Study of Dynamic Systems
Qualitative studies help us understand solution behaviour with out solving the differential equation. Especially useful for analysing solutions at equilibrium points. For a comprehensive understanding of a dynamic system, we want the transition function or vector field to cancel all transient occurrences, resulting in stable behaviour. In this scenario, the system will enter one of two states:
1)The case of equilibrium (fixed points, periodic points).
2) The case of chaotic.
Linear algebra is used to simplify dynamic system equations in this study. We need linearize most nonlinear dynamic systems connected to events in nature.

1.2.3 Linearization of Dynamic Systems

Consider the nonlinear dynamic system defined by:

$$\dot{X} = F(X),\ X = (x_1, x_2, \dots, x_n), \qquad F = (f_1, f_2, \dots, f_n), \tag{1.8}$$

were X_0 a fixed point (equilibrium) of this system.

Suppose a small upset $\varepsilon(t)$ is applied in the neighborhood of the fixed point. The function f can be developed in series of Taylor in the neighborhood of point X_0 as follows:

$$\varepsilon(t) + X_0 = F(\varepsilon(t) + X_0) \simeq F(X_0) + J_F(X)\varepsilon(t), \tag{1.9}$$

with $J_F(X_0)$ is the Jacobian matrix of the function F defined by

$$J_F(X_0) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}_{X=X_0} \tag{1.10}$$

As $F(X_0) = X_0$, then equation (1.9) becomes again:

$$\varepsilon(t) = J_F(X_0)\varepsilon(t). \tag{1.11}$$

The writing (1.11) means that the system (1.8) is linearized.

1.3 Concept of Stability

Stability in the sense of Lyapunov

Consider the following dynamic system:

$$\frac{dx}{dt} = f(x, t), \tag{1.12}$$

with f a nonlinear function.

Definition 1.3.1 *the equilibrium point x_0 of the system (1.12) is:*

1. ***Stable** if*

$$\forall \varepsilon > 0, \exists\ \delta > 0 :\ \|x(t_0) - x\| < \delta \Rightarrow \|x(t, x(t_0)) - x_0\| < \varepsilon, \forall\ t \geq t_0. \tag{1.13}$$

2. ***Asymptotically stable*** *if:*

$$\exists\ \delta > 0\ :\ \|x(t_0) - x\| < \delta \Rightarrow \lim_{t\to\infty} \|x(t, x(t_0)) - x_0\| = 0. \tag{1.14}$$

3. ***Exponentially stable*** *if:*

$$\forall\varepsilon > 0, \exists\ \delta > 0\ :\ \|x(t_0) - x\| < \delta \Rightarrow \|x(t, x(t_0)) - x_0\| < a\,\|x(t_0) - x\|\exp(-bt),\ \ \forall\ t > t_0. \tag{1.15}$$

4. ***Unstable*** *if*

$$\forall\varepsilon > 0, \exists\ \delta > 0\ :\ \|x(t_0) - x\| < \delta \Rightarrow \|x(t, x(t_0)) - x_0\| > \varepsilon, \forall\ t \geq t_0. \tag{1.16}$$

wich mean that equation (1.13) is not satisfied see Fig. 1.3.

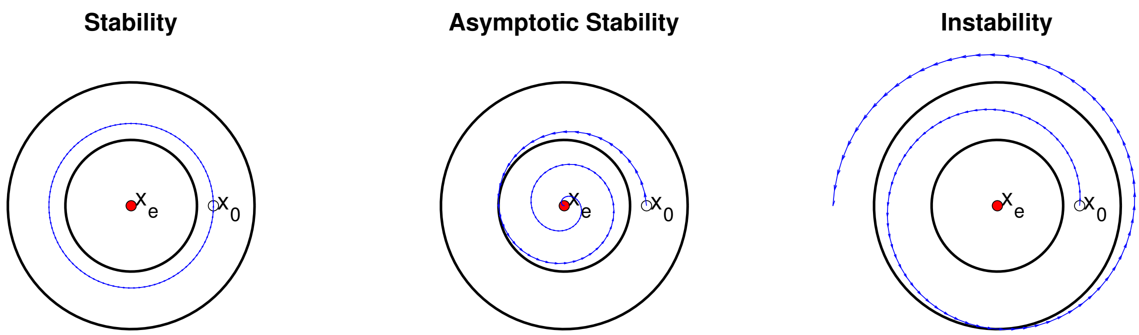


Figure 1.3: The various classifications of stability according to Lyapunov's theory.

Lyapunov's first method (indirect method)

Lyapunov's first method is based on examining the linearization around the equilibrium point x_0 of the system (1.12). More precisely, we examine the eigenvalues λ_i of the Jacobian matrix evaluated at the equilibrium point. According to this method, the properties of stability of x_0 are expressed as follows:

1- If all the eigenvalues of the Jacobian matrix have a strictly real part negative, x_0 is exponentially stable.

2- If the Jacobian matrix has at least one eigenvalue with a strictly positive real part, x_0 is unstable.

Remark 1.3.1 *We cannot determine equilibrium stability using this method if the matrix Jacobian has at least one zero eigenvalue and no eigenvalue with a positive real portion. The system's trajectories converge to a manifold equal to the number of zero eigenvalues of the Jacobian matrix. The stability of the equilibrium can be studied in this subspace using the second technique.*

Lyapunov's Second Technique (Direct Technique)

Lyapunov's first technique is straight forward to implement; yet, it permits only a limited analysis of equilibrium stability. Furthermore, she provides no indication of the dimensions of the basins of attraction. The second technique is more challenging to execute; yet, it is extensive and more comprehensive. The concept relies on the definition of a particular function, referred to as the Lyapunov function, denoted as $V(x)$, which diminishes along the system's paths with in the attraction basin. This theorem will encapsulate this technique.

Theorem 1.3.1 *The equilibrium point x_0 of the system (1.12) is stable if there exists a function $V(x) : D \to \mathbb{R}$ continuously differentiable having the following properties :*

1. *D is an open of $\mathbb{R}^n$ and $x_0 \in D$.*
2. *$V(x) > V(x_0)\ \forall\ x \neq x_0$ in D.*
3. *$\dot{V}(x) \leq 0\ \forall\ x \neq x_0$ in D.*

   There is no method to find a Lyapunov function. But in mechanics and for electrical systems one can often use the total energy as a Lyapunov function.

1.4 Bifurcation and Chaos

1.4.1 The Bifurcation Theory

This section addresses a differential system influenced by auxiliary parameters, focussing on the variations in the phase portrait as these parameters change. This issue is examined by catastrophe theory, especially in the context of dissipative systems reliant on a potential, where the positions of equilibria and their bifurcations are regarded as the key characteristics of the phase portrait.

To construct the phase portrait at bifurcation values, where qualitative changes occur, specific tools are required. This study focusses on local bifurcations associated with an equilibrium point in a continuous system. The bifurcation diagram will provide geometric assistance for both methods, highlighting the significance of utilising appropriate coordinates.

Definition 1.4.1

$$\frac{dx}{dt} = g(x, t, \eta), \tag{1.17}$$

Let x_0 be the solution to the problem of dimension n and control parameter η .
A bifurcation is a qualitative change in the solution of the system x_0 when we modify it, specifically including the disappearance or change of stability and the emergence of new solutions see Fig.1.4.

Definition 1.4.2 *The "common dimension" of misalignment is the smallest number of parameters required to achieve desingularity. In a system of differential equations when bifurcation takes place, the common dimension is the number of parameters. The system's bifurcation becomes more complex as the common dimension increases beyond one.*

Definition 1.4.3 *A bifurcation diagram represents a specific segment of the parameter space, illustrating all branch points within that segment.*

Definition 1.4.4 (Bifurcations in Codimension 1) *A codimension-1 bifurcation occurs when a qualitative change in the system's behavior happens by varying a single parameter. These bifurcations are the most common in practical applications and are typically classified into:*

- ***Saddle-node bifurcation**: Two fixed points (one stable, one unstable) collide and annihilate each other.*
- ***Transcritical bifurcation**: Two equilibrium points exchange stability.*
- ***Pitchfork bifurcation**: A symmetric bifurcation where one equilibrium splits into three (super critical) or three merge into one (subcritical).*
- ***Hopf bifurcation**: A stable fixed point becomes unstable, giving rise to a periodic orbit.*

Definition 1.4.5 (Bifurcation in Codimension 2) *A codimension-2 bifurcation occurs when a qualitative change requires the variation of two independent parameters. These bifurcations act as organizing centers for more complex dynamical behaviors and often give rise to bifurcation diagrams with multiple bifurcation. Examples include:*

- ***Bogdanov-Takens bifurcation**: A saddle-node bifurcation and a Hopf bifurcation co-occur at a critical point.*
- ***Cusp bifurcation**: A saddle-node bifurcation unfolds with an additional parameter.*
- ***Takens-Bogdanov bifurcation**: A system has a degenerate equilibrium where two eigenvalues are zero, leading to complex dynamic behaviors.*

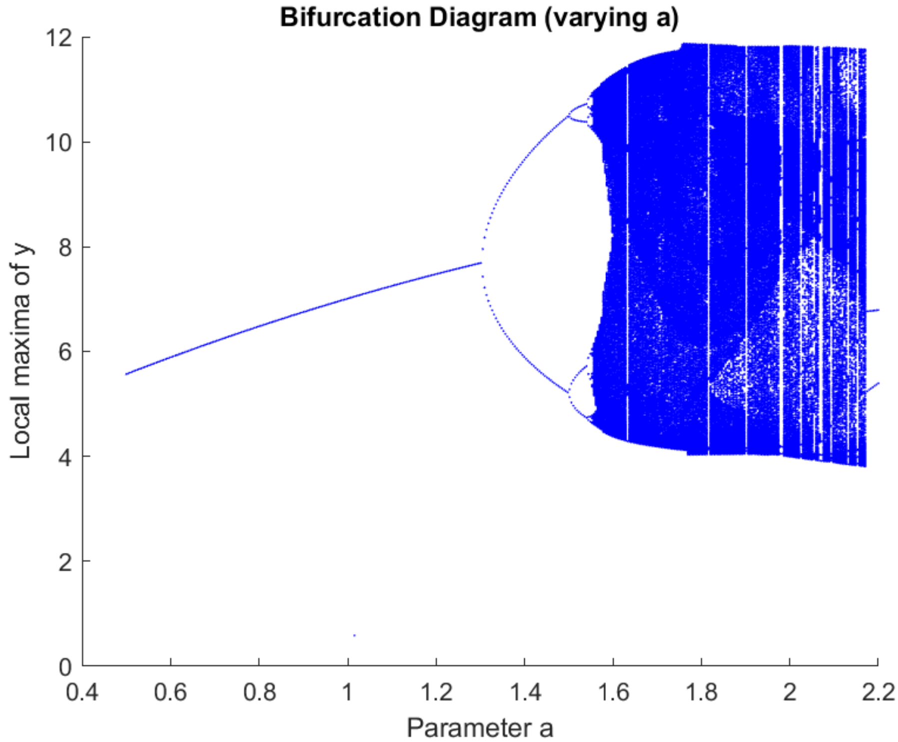


Figure 1.4: Bifurcation diagram

1.4.2 Chaos Theory

Nonlinear dynamic systems, including piecewise linear ones, can display intricate and ostensibly unpredictable behaviours that may initially seem random. Nonetheless, despite this seemingly arbitrary nature, these behaviours are wholly deterministic, indicating that they adhere to exact mathematical principles devoid of any external randomness or noise affecting their progression. The intrinsic unpredictability in these systems is termed chaos.

Chaos emerges when little variations in initial conditions result in significantly divergent outcomes over time, a phenomenon referred to as sensitive dependency on original conditions. This indicates that even minor fluctuations in the initial conditions of a system can lead to significantly divergent outcomes, rendering long-term predictions virtually unattainable despite the system's deterministic characteristics.

The mathematical discipline that systematically investigates and delineates chaotic phenomena is known as chaos theory. This discipline aims to comprehend and delineate the long-term progression of dynamical systems, especially those regulated by nonlinear equations. Chaos theory typically seeks to analyse qualitative aspects, such as stability, bifurcations, attractors, and the existence of fractal structures, rather than concentrating on identifying precise

solutions see Fig. 1.4.2. It has extensive applications in disciplines such as physics, engineering, meteorology, biology, and economics, where intricate systems often display chaotic behaviour .

Chaos Properties

1. Sensitivity to Initial Conditions

In a chaotic system, even a minute mistake in determining the beginning state x_0 in phase space would (nearly always) be rapidly magnified. From a mathematical point of view, we say that the function f demonstrates a sensitive dependency on initial conditions when the following conditions are met:

$$\exists \beta > 0, \forall (x,d) \in M, \forall \varepsilon > 0, \exists (y,q) \in M : \begin{cases} \| x - y \| < \varepsilon, \\ \|f^q(x) - f^d(y)\| > \beta. \end{cases} \tag{1.18}$$

2. The strange attractor
A strange attractor, or at least one variant, exists within every dissipative chaotic system by repeatedly stretching and folding a phase space cycle an infinite number of times, one would geometrically obtain this attractor. The attractor exists with in a bounded space, despite its "length" being infinite. As a result, the definition is as follows:

Definition 1.4.6 *A bounded subset H of phase space attracts a weird or chaotic transformation P if it has a neighbourhood G that contains every point of H and a ball with the following qualities:*
a) Attraction: G is a capture zone, thus any orbit by P with a beginning point in G is totally contained in it. Additionally, any orbit can be as near to H as desired.
b)- Its space is limited. Zero volume. The dimension is fractal.
c)- Each trajectory on the attractor is nearly certainly aperiodic, as it never passes again over the same location.
d) Two trajectories near together increase their distance exponentially (sensitivity to initial conditions).

3.One of the most important characteristics of chaotic motion in a system is the presence of broad spectra. The value of one of the variables in a dynamic system at regular intervals is frequently used to describe the temporal evolution of the system. This type of representation is known as the time series.

3.Lyapunov's Exponents
Various techniques exist to assess whether a nonlinear system demonstrates chaotic behaviour; however, these methods are frequently constrained in quantity and necessitate extended observation periods. We selected two widely utilised and complementary methods for analysing chaotic behaviour: the fractal dimension and Lyapunov exponents. The fractal dimension quantifies the geometric complexity of an attractor, whereas Lyapunov exponents assess sensitivity to initial conditions, which is a fundamental characteristic of chaotic systems.

Lyapunov exponents were introduced by Aleksandr Mikhailovich Lyapunov in his doctoral thesis, "The General Problem of the Stability of Motion", presented on "October 12, 1892", at the University of Moscow. This study presents a method for quantifying the divergence of proximate trajectories. If this divergence increases exponentially over time, the system exhibits a high sensitivity to initial conditions, which is characteristic of chaotic behaviour.

A positive Lyapunov exponent indicates chaotic behaviour, while negative or zero values suggest stability or periodic motion. The integration of Lyapunov exponent analysis and fractal dimension provides an effective means to characterise chaotic systems. These methods are essential in chaos theory and have extensive applications in physics, engineering, and various complex systems.

Paths to Chaos
A dynamic system generally possesses one or more "control" parameters that affect the attributes of the transition function. The value of the control parameter influences the initial conditions, resulting in trajectories that may represent distinct dynamic regimes.

The ongoing modification of the control parameters frequently leads to a progressive escalation in the complexity of the system's dynamic regime. Multiple scenarios illustrate the transition from a fixed point to chaotic behaviour. The transition from a fixed point to chaotic behaviour is characterised by discontinuous changes, commonly known as bifurcations, rather than a smooth progression. A bifurcation signifies a sudden transition from one dynamic regime to another, characterised by distinct qualitative differences. Three scenarios of transition to chaos can be identified:

1. Intermittency Towards Chaos

Periodic motion remains stable until interrupted by turbulence bursts. With an increase in the control parameter, the frequency of turbulence bursts escalates, ultimately resulting in the predominance of turbulence.

2.The Period-doubling

This phenomenon is defined by multiple bifurcations. As stress escalates, the period of a forced system doubles, subsequently quadruples, and then octuples, continuing in this manner. The period doublings occur with increasing proximity, and as the period approaches infinity, the system transitions into chaos.

3.Quasi-Periodicity

Occurs when a secondary system disrupts a previously periodic system. When the ratio of the periods of two systems is irrational, the system is classified as quasi-periodic.

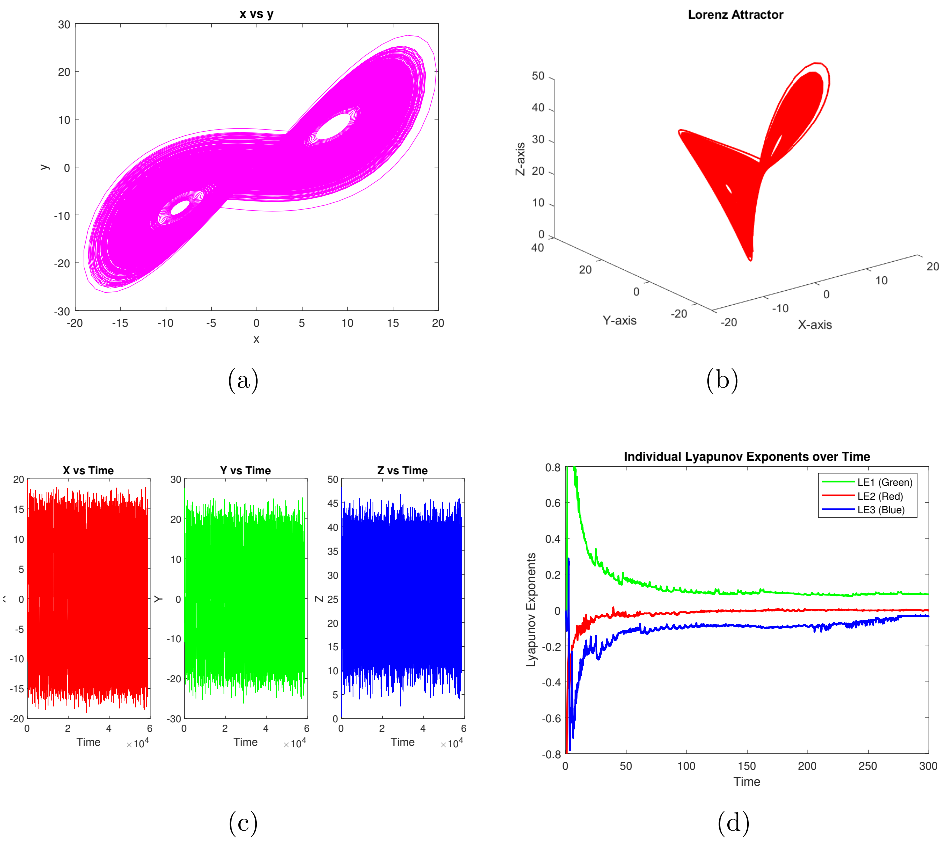


Figure 1.5: Chaotic behavior of the Lorenz system (a) phase diagram on $(x-y)$ plane; (b) phase diagram on $(x-y-z)$ plane; (c) the state varaible x, y, and z with time; (d) the Lyapunov exponents with the initial values $X(0) = (1, -1, 2)$.

CHAPTER 2

HIDDEN ATTRACTORS AND HIDDEN BIFURCATIONS

2.1 Introduction

Nonlinear dynamic systems research has recently uncovered crucial phenomena that challenge traditional analytical methods. While conventional self-excited attractors can be located through trajectories near unstable equilibria, hidden attractors whose basins do not intersect with neighborhoods of equilibrium points remain elusive to standard computational approaches. These hidden attractors represent significant regimes in system behavior that cannot be detected through equilibrium analysis alone. The identification of hidden chaotic attractors in Chua's circuit by Kuznetsov et al. (2010) and Leonov et al. (2011) invigorated research into hidden oscillations. Although numerous research have examined Chua's circuit over the years, they primarily focused on self-excited attractors [18, 19]. The necessity for efficient analytical and numerical techniques to investigate hidden attractors persists as a fundamental challenge. This survey seeks to underscore recent progress in these strategies, mirroring contemporary trends in both theoretical and computational approaches. Menacer et al. [23] altered the paradigm of discrete parameters by inserting hidden bifurcations, resulting in multiscrolls within a family of systems characterised by a continuous bifurcation parameter. Subsequently, all traditional theories of dynamical systems and their robust methodologies can be employed to investigate multiscrolls. This hidden bifurcation theory is founded on the hidden attractor theory proposed by Leonov et al.

2.2 Hidden attractors

The analysis and synthesis of oscillating systems, in which the problem of the existence of oscillations could be resolved with relative ease, received a lot of attention during the early stages of the foundation of the theory of nonlinear oscillations, which took place in the first half of the twentieth century. This was the time when the theory was being developed. This strategy was supported by the investigation of periodic oscillations in a variety of practical domains, including as biology, electronics, chemistry, and mechanics.

Definition 2.2.1 (self-excited) *An attractor is called a self-excited attractor if its basin of attraction intersects with any open neighborhood of an unstable fixed point see Fig. 2.1, [15].*

Example 2.2.1 *In* 1963, *The Lorenz system was the first well-known example of a visualization of chaotic attractor in a dynamical system corresponding to the excitation of chaotic attractor from unstable equilibria [12].*

Consider Lorenz system

$$\begin{cases} \dot{x} &= a(y-x), \\ \dot{y} &= x(c-z)-y, \\ \dot{z} &= xy-bz, \end{cases} \tag{2.1}$$

and make it's simulation with standard parameters. $a = 10,\ b = \frac{8}{3},\ c = 28.$

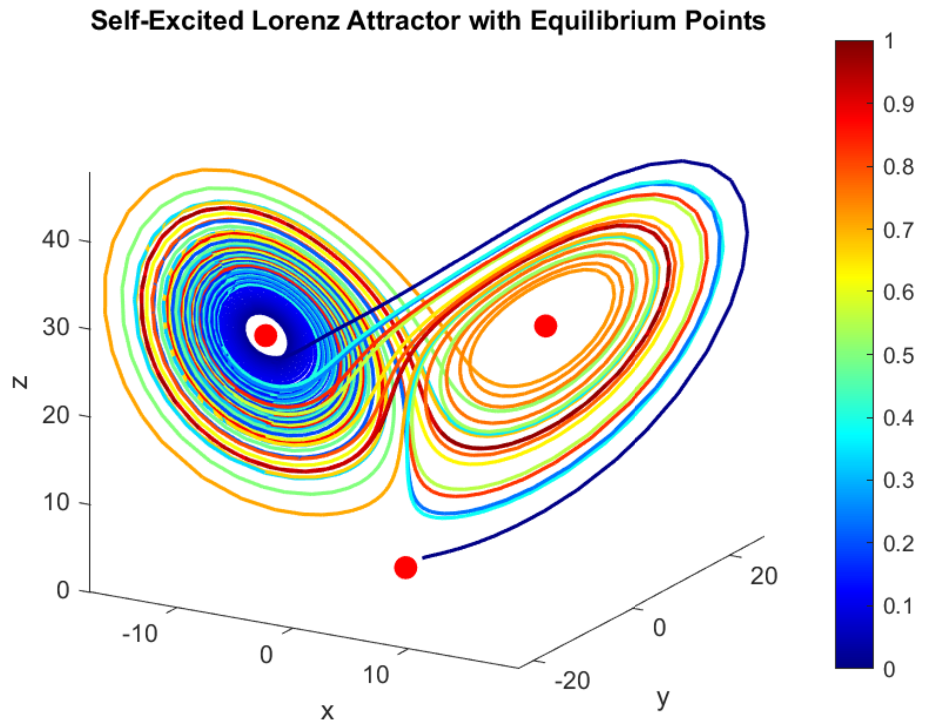


Figure 2.1: Self-excited Lorenz attractor with equilibrium points

Definition 2.2.2 (Hidden attractors) *If an attractor's basin of attraction does not cut off with small regions of equilibria, it is referred to as a hidden attractor see Fig2.3.*

Definition 2.2.3 *hidden attractors are attractors in systems with out equilibria or with only one stable equilibrium [15].*

Remark 2.2.1 *The hidden vs. self-excited classification of attractors was introduced inconnection with the discovery of the first hidden Chua attractor.*

Example 2.2.2 *Consider the behavior of classical Chua circuit [Chua,* 1992*].*
In the dimension less coordinates a dynamic model of this circuit is as follows [21]

$$\begin{cases} \dot{x} &= a(y-x)-af(x), \\ \dot{y} &= x-y+z, \\ \dot{z} &= -(by+cy). \end{cases} \tag{2.2}$$

Here the function

$$f(x) = \alpha_1 x + \frac{1}{2}(\alpha_0 - \alpha_1)(|x+1| - |x-1|). \tag{2.3}$$

For simulation of this system, we use the following parameters, $a = 15.6$, $b = 28$, $c = 0.016$, $\alpha_0 = -\frac{8}{7}$, $\alpha_1 = \frac{5}{7}$.*(see Fig. 2.2)*

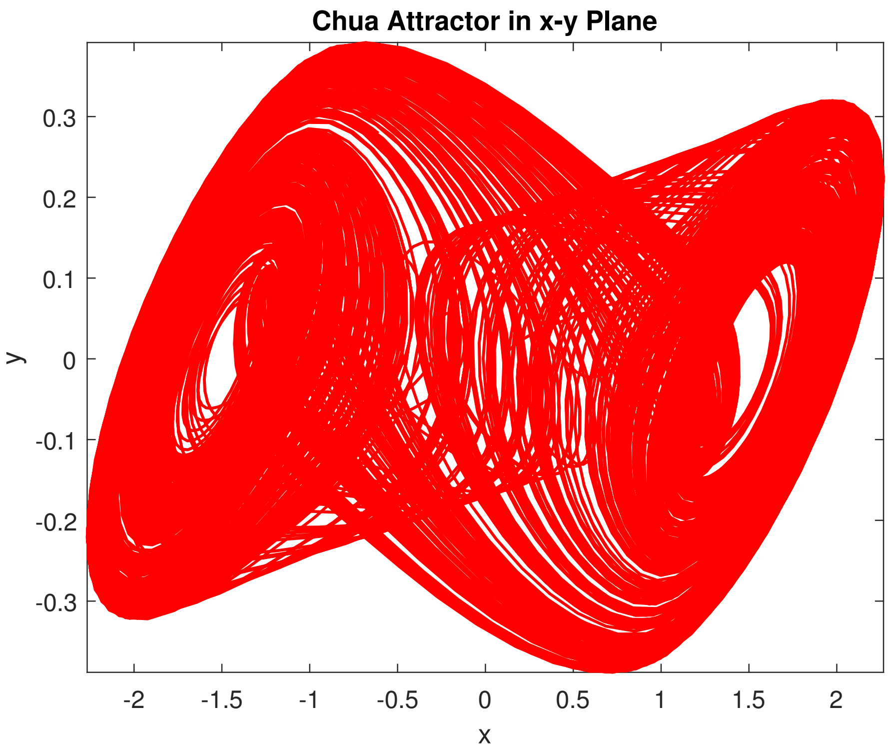


Figure 2.2: Chua attractor in $x - y$ plane

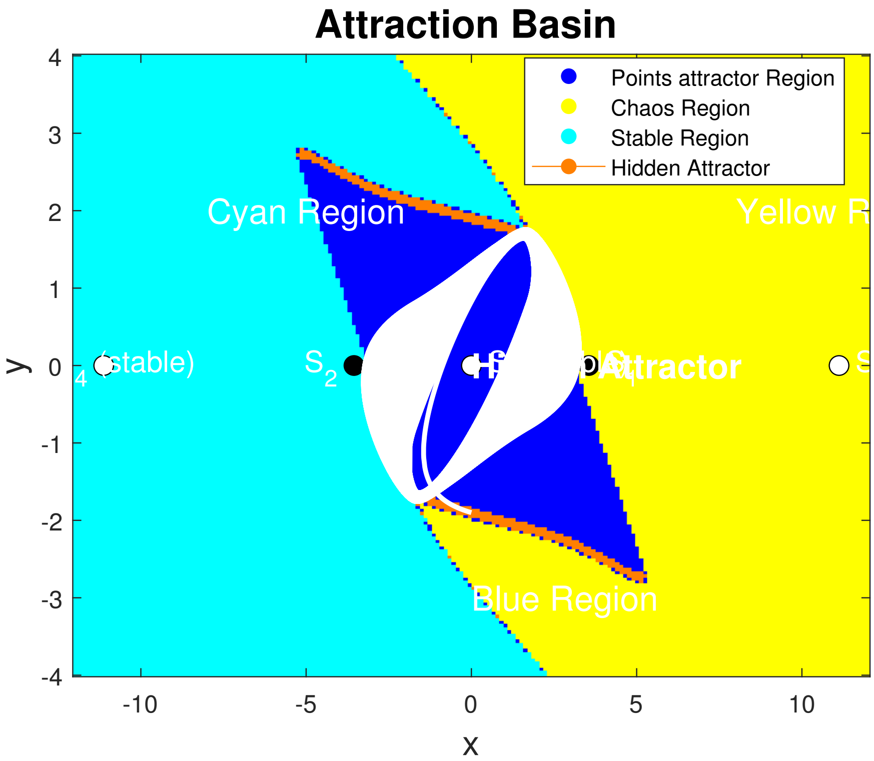


Figure 2.3: Attractions Basin in $x - y$ plane

2.2.1 Analytical-Numerical Method for Hidden Attractor Localization

New concepts like self-excited and veiled attractors have been introduced[11, 18], and [20]. Self-excited attractors have basins of attraction near equilibrium points, while hidden attractors do not. In systems without equilibria, with one stable equilibrium, or with infinite stable equilibria, hidden attractors are attractors. Hidden attractors are hard to find because their basins of attraction do not connect with local equilibria. The term "hidden" comes from computational complexity. Quantifying their existence was developed by Leonov et al. [17], [19], and [20]. This method is for Chua attractors.

$$\frac{dX}{dt} = PX + g\Psi(r^T X), \qquad X \in \mathbb{R}^3. \tag{2.4}$$

Let P be a constant $(n \times n)$-matrix, and let g and r be constant n-dimensional vectors. The operation T denotes transposition, while $\Psi(\sigma)$ represents a continuous piecewise-differentiable scalar function, satisfying the condition $\Psi(0) = 0$. Define the coefficient k^* of harmonic linearization such that the matrix

$$P0 = P + k^* g r^T. \tag{2.5}$$

The system possesses a pair of purely imaginary eigenvalues, $\pm i\omega_0$, where $\omega_0 > 0$, while the remaining eigenvalues exhibit negative real parts. It is assumed that such k^* exists. Reformulate system (2.4) as

$$\frac{dX}{dt} = P_0 X + g\varphi(r^T X). \tag{2.6}$$

We define $\varphi(\sigma) = \Psi(\sigma) - k^*\sigma$ and introduce a finite sequence of functions $\varphi^0(\sigma), \varphi^1(\sigma), \cdots, \varphi^m(\sigma)$ such that the graphs of neighboring functions $\varphi^j(\sigma)$ and $\varphi^{j+1}(\sigma)$, for $j = 0, \cdots, m-1$, exhibit slight differences. The function $\varphi^0(\sigma)$ is small, and we have $\varphi^m(\sigma) = \varphi(\sigma)$.

By employing a limited function, we can implement the method of harmonic linearization, also known as the describing function method, for the system.

$$\frac{dX}{dt} = P_0 X + \varphi^0(r^T X), \tag{2.7}$$

and identify a stable nontrivial periodic solution $X^0(t)$.

To localize the attractor of the original system (2.6), we will numerically track the transformation of this periodic solution. All points of this stable periodic solution reside with in the domain of attraction of the stable periodic solution $X^1(t)$ of the system.

$$\frac{dX}{dt} = P_0 X + \varphi^j(r^T X). \tag{2.8}$$

When $j = 1$, or moving from (2.7) to system (2.8) with $j = 1$, the instability bifurcation that disrupts the periodic solution can be observed. In the initial scenario, one can determine $X^1(t)$ numerically by using any point from the stable periodic solution $X^0(t)$ as the initial condition for system (2.8) with $j = 1$. Beginning with this initial condition, the trajectory attains the periodic solution $X^1(t)$ following a transient phase. Subsequent to the computation of $X^1(t)$, one can derive a periodic trajectory $X^2(t)$ for system (2.8) with $j = 2$, initiated from any point on the stable periodic solution $X^1(t)$. This process can be continued to obtain a periodic solution for system (2.6), contingent upon the existence of such a solution.

Remark 2.2.2 *In certain instances, obtaining such a solution may prove impossible due to the observation of an instability bifurcation at a specific step, which undermines the periodic solution.*

To identify the initial condition $X^0(0)$ of the periodic solution, system(2.7) can be modified through a linear nonsingular transformation. F $(X = FY)$ to the form :

$$\begin{cases} \dot{y_1} = -\omega_0 y_2 + b_1 \varphi^0(y_1 + u_3^t Y_3), \\ \dot{y_2} = \omega_0 y_1 + b_2 \varphi^0(y_1 + u_3^t Y_3), \\ \dot{Y_3} = A_3 Y_3 + B_3 \varphi^0(y_1 + u_3^t Y_3). \end{cases} \tag{2.9}$$

In this context, y_1 and y_2 represent scalar values, while Y_3 denotes a $(n-2)$-dimensional vector, B_3 and u_3 $(n-2)-$dimensional vector, where b_1 and b_2 represent real numbers; A_3 is a $(n-2)\times(n-2)$ matrix, characterized by the property that all of its eigenvalues possess negative real parts. It can be assumed, without loss of generality, that for the matrix A_3, there exists a positive number d_2 such that $d_2 > 0$.

$$Y_3^t (A_3 + A_3^t) Y_3 \leq -2d_2 \left|Y_3\right|^2, \ \forall Y_3 \in \mathbb{R}^{n-2}. \tag{2.10}$$

In the scalar instance, let us introduce the characterizing function Φ of a real variable m:

$$\Phi\left(m\right) = \int_0^{2\pi/\omega_0} \varphi(\cos(\omega_0 t)m)\cos(\omega_0 t)dt. \tag{2.11}$$

Theorem 2.2.1 *[11]If a positive m_0 such that*

$$\Phi(m_0) = 0, \ b_1 \frac{d\Phi(m)}{dm} \mid_{m=m_0} < 0, \tag{2.12}$$

then for the initial condition of the periodic solution $X^0(0) = F(y_1(0),\, y_2(0),\, Y_3(0))^T$ at the first step of algorithm we have

$$y_1(0) = m_0 + O(\varepsilon), \ y_2(0) = 0, \ Y_3(0) = O_{n-2}(\varepsilon), \tag{2.13}$$

were $O_{n-2}(\varepsilon)$ is an $(n-2)-$dimensional vector such that all its components are $O(\varepsilon)$.

For the stability of $X^0(t)$ (where stability is defined such that for all solutions with initial data sufficiently close to $X^0(0)$, the modulus of their difference with $X^0(t)$ remains uniformly bounded for all $t > 0$) It is necessary to ensure that the following condition holds:

$$b_1 \frac{d\Phi(m)}{dm} |_{m=m_0} < 0. \tag{2.14}$$

To find k^* and ω_0, one utilizes the transfer function $W(\lambda)$ of system (2.4):

$$W(s) = r^T (P - sI)^{-1} g, \tag{2.15}$$

where s represents a complex variable. The value of ω_0 is ascertained from the equation $\operatorname{Im} W(i\omega_0) = 0$. The values of and k^* are computed using the formula $k^* = \operatorname{Re} W(i\omega_0)^{-1}$.

2.2.2 Hidden Bifurcation

The number of scrolls or spirals produced in the phase space of all known multiscroll chaotic attractors is determined by a fixed integer value. This value generally relies on one or more discrete parameters that are explicitly integrated into the system's framework.
The parameters regulate the system's behavior in a piecewise manner, resulting in a stepwise increase or modification in the number of scrolls. Despite extensive research on multiscroll generation mechanisms, a comprehensive bifurcation analysis has not been performed in the traditional sense, especially concerning the variation in the number or topology of scrolls as a function of a continuous parameter.

This method was significantly altered by Menacer et al. [23], who proposed a novel approach that moves away from dependence on discrete parameters. The authors proposed the concept of hidden bifurcations within a family of systems, as described by equation (2.6), where the number of scrolls can vary continuously as a function of a bifurcation parameter. This new perspective is grounded in the concept of hidden attractors, a notion initially proposed by Leonov et al. [18, 19].

Hidden attractor theory examines attractors whose basins of attraction do not overlap with the neighborhoods of any equilibrium point, necessitating unconventional approaches for their detection and analysis. This concept is extended to encompass hidden bifurcations that do not correspond to observable changes in equilibria. Such hidden bifurcations are regulated by a homotopy parameter ε, while the system parameters remain constant.
This additional parameter, absent from the initial problem, is well-suited to elucidate the structure of the multispiral chaotic attractor.

CHAPTER 3

APPLICATION

3.1 Introduction

The investigation of nonlinear dynamic systems has been significantly enhanced by the analysis of chaotic phenomena, with Chua's circuit serving as a quintessential example of engineered chaos. The characterization of classical attractors and bifurcations in these systems is well-established; however, the emergence of hidden attractor basins that do not intersect with equilibrium neighborhoods, along with their corresponding hidden bifurcations, poses significant theoretical and practical challenges.

This chapter examines the complex interactions between hidden attractors and hidden bifurcations in Chua's system, utilizing methodologies from nonlinear dynamics, bifurcation theory, and computational simulations. The analytical-numerical method serves as an effective means for identifying hidden attractors. This approach combines mathematical analysis to clarify the system's structure with numerical simulations that track trajectories from carefully chosen initial conditions [16]. This method aims to identify trajectories that converge on non-obvious attractors and to define the parameter ranges in which these complex behaviors occur. This study applies the analytical-numerical method to continuous chaotic systems [19]. It explores the selection of appropriate initial conditions, identifies regions in phase space that may harbor hidden attractors and bifurcations, and demonstrates the method's importance in advancing our understanding of nonlinear dynamics [26]. This method is essential for applications necessitating the control and prediction of complex behaviors.

3.2 Hidden Attractor for Chua's System

In 2010, researchers made a notable finding by identifying a concealed chaotic attractor in Chua's circuit [Kuznetsov et al., 2011]. [17]-[11]. This groundbreaking discovery signified a significant progression in comprehending three-dimensional dynamic systems. The approach for localizing this hidden chaotic attractor in Chua's system will be illustrated later.

The researchers utilized the aforementioned method to effectively reveal this concealed attractor. To forward this inquiry, we developed a revised iteration of Chua's system:

$$\begin{cases} \dot{x} &= a_1(y-x) - a_1 f(x), \\ \dot{y} &= x - y + z, \\ \dot{z} &= -b_1 y. \end{cases} \tag{3.1}$$

Here the function

$$f(x) = \alpha_1 x + \frac{1}{2}(\alpha_0 - \alpha_1)(|x+1| - |x-1|). \tag{3.2}$$

This method can be employed to identify hidden attractors in Chua's system, which exhibits multiple spiral attractors produced by the the function (3.2). Examine the provided Chua's system is represented by equations (3.1) and (3.2). To achieve this, we reformulate system (3.1)–(3.2) into the structure of (3.3).

$$\frac{dX}{dt} = PX + g\Psi(r^T X), X \in \mathbb{R}^3. \tag{3.3}$$

Here,

$$P = \begin{pmatrix} -a_1(\alpha_1+1) & a_1 & 0 \\ 1 & -1 & 1 \\ 0 & -b_1 & 0 \end{pmatrix}, g = \begin{pmatrix} -a_1 \\ 0 \\ 0 \end{pmatrix}, r = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

and $\Psi(\sigma) = \varphi(\sigma)$.

Introduce the coefficent k^* and small parameter ε, and represent system as (3.3)

$$\frac{dX}{dt} = P_0 X + g\varepsilon\varphi(r^T X), \tag{3.4}$$

Where

$$P_0 = P + k^* g r^T = \begin{pmatrix} -a_1(\alpha_1+1+k^*) & a_1 & 0 \\ 1 & -1 & 1 \\ 0 & -b_1 & 0 \end{pmatrix}, \lambda_{1,2}^{P_0} = \pm i w_0 , \lambda_3^{P_0} = -d_1,$$

By nonsingular linear transformation $X = FY$ system (3.3) is compressed into the form

$$\frac{dy}{dt} = Qy + b\varepsilon\varphi(u^T Y), \tag{3.5}$$

Where

$$Q = \begin{pmatrix} 0 & -w_0 & 0 \\ w_0 & 0 & 0 \\ 0 & 0 & -d_1 \end{pmatrix}, B = \begin{pmatrix} b_1' \\ b_2' \\ 1 \end{pmatrix}, Y = \begin{pmatrix} y_1 \\ y_2 \\ Y_3 \end{pmatrix} \quad and \quad u = \begin{pmatrix} 1 \\ 0 \\ -h \end{pmatrix}$$

.

The transfer function $W_Q(s)$ of system (3.4) can be represented as

$$W_Q(s) = \frac{-b_1' s + b_2' w_0}{s^2 + w_0^2} + \frac{h}{s + d_1}.$$

Further, using the equality of transfer functions of systems (3.4) and (3.5), we obtain

$$W_Q(s) = W_{P_0} \longrightarrow \frac{-b_1' s + b_2' w_0}{s^2 + w_0^2} + \frac{h}{s + d_1} = \frac{-a_1}{p^3 + p^2(a_1(\alpha_1 + 1) + 1) + p(a_1(\alpha_1 + 1)) + b_1}.$$

This implies the relations indicated below:

$$\begin{aligned}
k^* &= \frac{a - w_0^2 - b}{a}, \\
d_1 &= a + w_0^2 - b + 1, \\
h &= \frac{a(b - d + d^2)}{w_0^2 + d^2}, \\
b_1' &= \frac{a(b - w_0^2 - d)}{w_0^2 + d^2}, \\
b_2' &= \frac{a(1 - d)(w_0^2 + bd)}{w_0(w_0^2 + d^2)}.
\end{aligned}$$

Since system (3.4) can be reduced to the form (3.5) by the nonsingular linear transformation $X = FY$, for the matrix F the following relations

$$Q = F^{-1} P F, B = F^{-1} g, C^T = r^T F, \tag{3.6}$$

are true. The entries of this matrix are obtained by solving these matrix equations:

$$F = \begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{pmatrix}$$

,

here

$F_{11} = 1$, $F_{12} = 0$, $F_{13} = -h$,

$F_{21} = \alpha_1 + 1 + k^*$, $F_{22} = \dfrac{-w_0^2}{a_1}$, $F_{23} = -\dfrac{h(a_1(\alpha_1 + 1 + k^*) - d_1)}{a_1 w_0}$,

$$F_{31} = \frac{a_1(\alpha_1 + k^*) - w_0^2}{a_1} \;,\; F_{32} = \frac{a_1 b_1(\alpha_1 + k^*) + a_1 b_1 - w_0^2}{a_1 w_0} \;,$$
$$F_{33} = h\frac{a_1(\alpha_1 + k^*)(d_1 - 1) + d(1 + a_1 - d_1)}{a_1}$$

We determine initial data for the first step of a multistage localization procedure for small enough ε ,as

$$X(0) = FY(0) = F\begin{pmatrix} m_0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} m_0 F_{11} \\ m_0 F_{21} \\ m_0 F_{31} \end{pmatrix}. \tag{3.7}$$

The starting condition for the system (3.1-3.2) is provided by this.

$$X^0(0) = (x^0(0) = m, y^0(0) = m(1 + \alpha_1 + k^*), z^0(0) = m\frac{a_1(1 + \alpha_1) - w_0^2}{a_1}). \tag{3.8}$$

Consider system (3.4) with the parameters

$$a_1 = 8.4562, b_1 = 12.0732, \alpha_0 = -0.1768, \alpha_1 = -1.1468. \tag{3.9}$$

There are three equilibria in the system for the parameter values under consideration: a locally stable zero equilibrium and two saddle equilibria. Let's now employ the hidden attractor localization process described above to Chua's system (3.3) with parameters(3.2). Calculate a beginning frequency and a harmonic linearization coefficient for this.

$$w_0 = 3.2483, k^* = 0.82006. \tag{3.10}$$

Then, we compute solutions of system (3.4) with the nonlinearity $\varepsilon(\phi(x) - k^* x)$ sequentially increasing ε from the value $\varepsilon_1 = 0.1$ to $\varepsilon_{10} = 1$ with step 0.1. By (3.2) and (3.8), the initial data can be obtained

$$x(0) = 0.3, y(0) = 0.1, z(0) = -0.1,$$

For the preliminary stage of a multi-step procedure. For $\varepsilon = 0.1$, the computation converges towards the initial oscillation $X^1(t)$ after undergoing a transitory process. Furthermore, the hidden set is computed for the original Chua's system (3.3) through numerical methods and the sequential transformation $X^j(t)$ with an increasing parameter ε_j. Figs.(3.2, 3.2) and Table. 3.2 presents this collection.

ε	0.1	0.5	0.95	1
Localization hidden	cycle	periodic	one attractor	original attractor

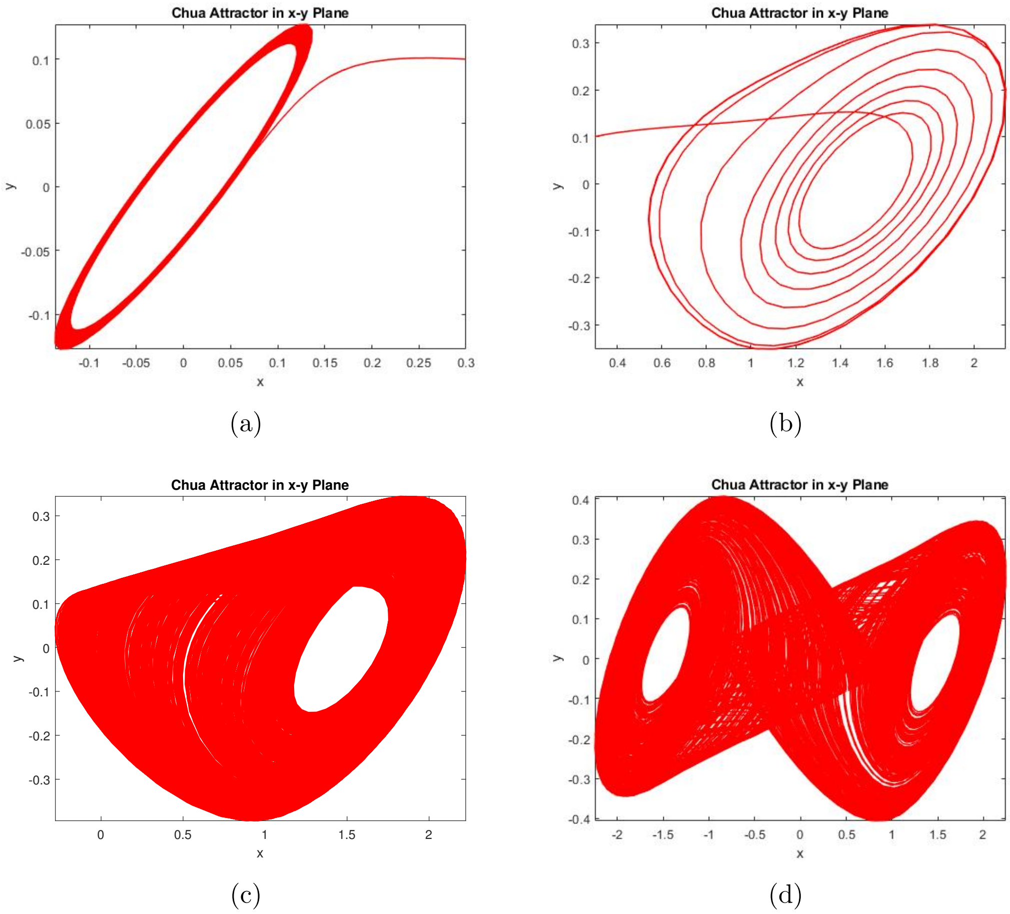


Figure 3.1: Localization hidden attractors in Chua's system where (a) $\varepsilon = 0.1$; (b) $\varepsilon = 0.5$; (c) $\varepsilon = 0.95$; (d) $\varepsilon = 1$.

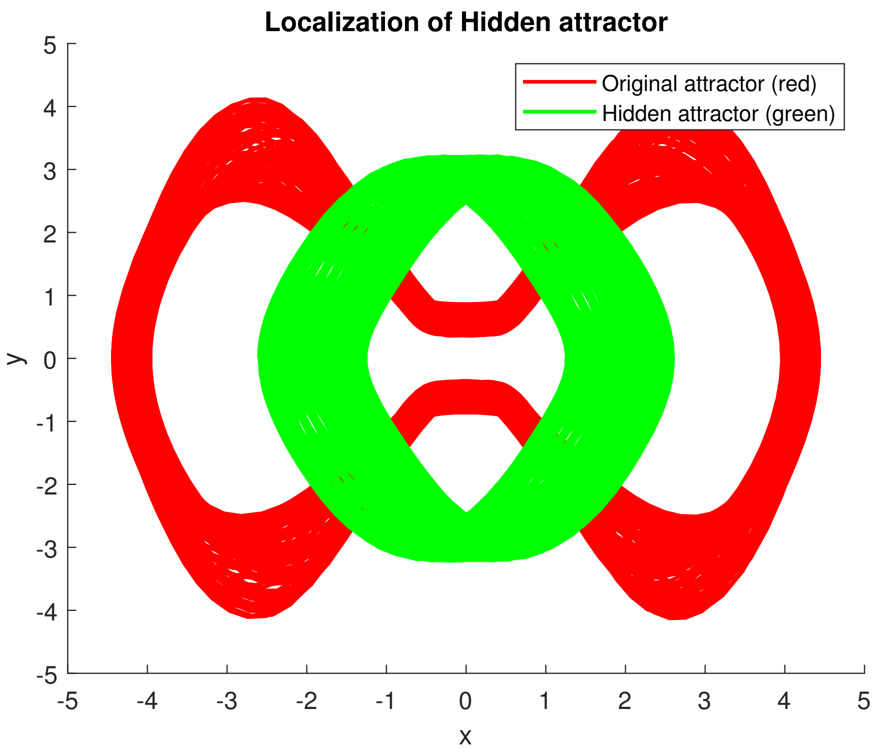


Figure 3.2: localization of hidden attractor in Chua's system

3.3 Hidden Bifurcation in Chua's System

The integer characteristic of the parameter c in the modified Chua circuit with a sine function (3.11)-(3.12) restricts its continuous variation, thereby precluding the observation of bifurcation of attractors from n to $n+2$ spirals as a function of parameter c. Additionally, non-integer real values for c are not permissible. Menacer et al [23] introduced a novel method for identifying hidden bifurcations by employing the central concepts of Leonov and Kuznetsov related to hidden attractors, specifically through homotopy and numerical continuation, there by addressing this challenge. A supplementary bifurcation parameter ε is introduced while keeping c constant [24].
A three-dimensional piecewise-linear nonlinear system of differential equations describes the behaviour of Chua's circuits.

$$\begin{cases} \dot{x} = \alpha(y - f(x)), \\ \dot{y} = x - y + z, \\ \dot{z} = -\beta y. \end{cases} \tag{3.11}$$

where

$$f(x) = \begin{cases} \frac{b_1\pi}{2a_1}(x - 2a_1c_1), & \text{if } x \geqslant 2a_1c_1, \\ -b_1 \sin\left(\frac{\pi x}{2a_1} + d_1\right), & \text{if } -2a_1c_1 < x < 2a_1c_1, \\ \frac{b_1\pi}{2a_1}(x + 2a_1c_1), & \text{if } x \leq -2a_1c_1 \end{cases} \tag{3.12}$$

In this context, α, β, a_1, b_1, and d_1 are parameters that are elements of $\mathbb{R}$ and will be defined subsequently for various applications, whereas c_1 is designated as an integer.
An n-spirals attractor is generated when

$$n = c_1 + 1, \tag{3.13}$$

where c and $n \in \mathbb{N}$, and

$$d_1 = \begin{cases} \pi, & \text{if } n \text{ is odd}, \\ 0, & \text{if } n \text{ is even}. \end{cases} \tag{3.14}$$

This method is applicable for detecting hidden bifurcations in Chua's system, which demonstrates multiple spiral attractors generated through sine function dynamics. Analysis should be conducted on the Chua's system as defined by equations (3.11) and (3.12).
To proceed with this investigation, we transform the system represented in equations (3.11)–(3.12) into the system outlined by equation (3.3).

$$\frac{dX}{dt} = PX + g\Psi(r^T X), X \in \mathbb{R}^3, \tag{3.15}$$

where

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\alpha & -\beta & 0 \end{pmatrix}, g = \begin{pmatrix} 0 \\ 0 \\ r_1 \end{pmatrix}, r = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},$$

and $\psi(\sigma) = \varphi(\sigma)$.

Introducing the coefficient k^* and a small parameter ε, system (3.15) can be transformed into the form of system (3.4) as follows:

$$\frac{dX}{dt} = P_0 X + g\varepsilon\varphi(r^T X), \tag{3.16}$$

where

$$P_0 = P + k^* g r^T = \begin{pmatrix} -\alpha k^* & \alpha & 0 \\ 1 & -1 & 1 \\ 0 & -\beta & 0 \end{pmatrix}$$

$$\varphi(\sigma) = \Psi(\sigma) - k^* r = -\alpha f(\sigma) - k^* r$$

.

The transfer function $W_{P_0}(s)$ of the system (3.16) isgiven by

$$W_{P_0}(s) = r^t (P - sI)^{-1} = \alpha \frac{s^2 + s + \beta}{s^3 + s^2 + (\beta - \alpha)s}.$$

For the parameter value $\alpha = 11$, $\beta = 15$, using the formulas $ImW(iw_0) = 0$ and $k^* = -(ReW(iw_0))^{-1}$, we calculated $w_0 = 2.1018$ and $k^* = 0.03796$. Via the nonsingular linear transformation $X = FY$ the system (3.16) is reduced to the form

$$\frac{dy}{dt} = QX + B\varepsilon\varphi(C^T Y), \tag{3.17}$$

Where

$$Q = F^{-1} P F, B = F^{-1} g, C^T = r^T S. \tag{3.18}$$

This implies

$$Q = \begin{pmatrix} 0 & -w_0 & 0 \\ w_0 & 0 & 0 \\ 0 & 0 & -d \end{pmatrix}, B = \begin{pmatrix} b_1' \\ b_2' \\ 1 \end{pmatrix}, Y = \begin{pmatrix} y_1 \\ y_2 \\ Y_3 \end{pmatrix}, C = \begin{pmatrix} 1 \\ 0 \\ -h \end{pmatrix} \; and \;\; F = \begin{pmatrix} F_{11} & F_{12} & F_{13} \\ F_{21} & F_{22} & F_{23} \\ F_{31} & F_{32} & F_{33} \end{pmatrix}$$

The transfer function $W_Q(s)$ of system (3.17) can be represented as

$$W_Q(s) = \frac{-b_1' s + b_2' w_0}{s^2 + w_0^2} + \frac{h}{s + d_1}.$$

Further, using the equality of transfer functions of systems (3.16) and (3.17), we obtain

$$\begin{aligned}
k^* &= \frac{\alpha - w_0^2 - \beta}{\alpha}, \\
d_1 &= \alpha + w_0^2 - \beta + 1, \\
h &= \frac{\alpha(\beta - d_1 + d_1^2)}{w_0^2 + d_1^2}, \\
b_1' &= \frac{\alpha(\beta - w_0^2 - d_1)}{w_0^2 + d_1^2}, \\
b_2' &= \frac{\alpha(1 - d)(w_0^2 + \beta d_1)}{w_0(w_0^2 + d^2)}.
\end{aligned}$$

When the parameters of system (3.15) are fixed at , $\alpha = 11$, $\beta = 15$, $a_1 = 2$, $b_1 = 0.2$, we obtain

$$k^* = 0.03796, d_1 = 1.4176, h = 26.686, b_1' = 15.686, b_2' = 3.1003$$

,

$$F = \begin{pmatrix} 1 & 0 & -26.686 \\ 0.03795 & -0.19107 & -2.4264 \\ -1.3636 & -0.27084 & -25.674 \end{pmatrix}$$

.

According to Theorem 2.2.1, for sufficiently small ε, we proceed to compute the initial condition for the first step of the multistage localisation procedure.

$$X(0) = FY(0) = F \begin{pmatrix} m_0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} m_0 F_{11} \\ m_0 F_{21} \\ m_0 F_{31} \end{pmatrix}. \tag{3.19}$$

3.3.1 Numerical Results

Consider the system (3.11) and (3.12) with parameter values

$$\alpha = 11, \beta = 15, a_1 = 2, b_1 = 0.2.$$

Now we apply the localization procedure described above to Chua's system (3.15) with multiple spiral attractors. For this purpose, we compute the following starting frequency w_0 and a coefficient of harmonic linearization k^*:

$$w_0 = 2.1018$$

and

$$k^* = 0.031084.$$

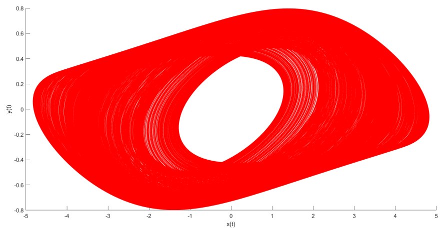
We proceed to calculate the solutions of system (3.16) with the nonlinearity $\varepsilon\varphi(x) = \varepsilon(\Psi(x) - k^*x)$. This involves incrementally increasing ε from 0.1 to 1 in steps of 0.1, and subsequently decreasing it to 0.001 for the range between $\varepsilon = 0.8$ and $\varepsilon = 1$. By utilizing (3.3), we derive the initial conditions for each integer value of c, as presented in Tables. 3.1 and 3.2, which serve as the first step in our multistage procedure for constructing the solutions. If the stable periodic solution $X^0(t)$ (associated with a very small ε) near the harmonic one is identified, all points of the stable periodic solution $X^0(t)$ reside within the domain of attraction of the stable periodic solution $X^1(t)$ of the system. The solution $X^1(t)$ can be determined numerically by analyzing one trajectory of system (3.15) with $\varepsilon = 0.1$ from the initial point $X^0(0)$. After a transient process, the computational procedure converges to the initial oscillation $X^1(t)$
We proceed by incrementing the parameter j and applying the numerical procedure, which yields the sequential transformation $X^j(t)$ for the original Chua system (3.11) see Fig. 3.3.1.

ε	$X^j(0)$	x_0	y_0	z_0
0.1	$U^1(0) = U^0(t_{max})$	3.6802	0.2698	-4.8435
0.2	$U^2(0) = U^1(t_{max})$	0.3087	-0.6999	-1.4931
0.3	$U^3(0) = U^2(t_{max})$	-2.1385	-0.6769	2.1216
0.4	$U^4(0) = U^3(t_{max})$	-2.6060	0.4095	4.2249
0.5	$U^5(0) = U^4(t_{max})$	1.1538	0.7447	-0.5236
0.6	$U^6(0) = U^5(t_{max})$	-3.5796	0.2991	4.9740
0.7	$U^7(0) = U^6(t_{max})$	2.0396	0.4259	-2.7065
0.8	$U^8(0) = U^7(t_{max})$	0.9087	0.6024	-0.6149
0.86	$U^9(0) = U^8(t_{max})$	-4.7786	0.1893	5.5193
0.9785	$U^9(0) = U^9(t_{max})$	-4.3940	0.1998	5.3250
0.993	$U^{10}(0) = U^{11}(t_{max})$	14.1756	0.1189	-12.4364
1	$U^{11}(0) = U^{14}(t_{max})$	17.5635	-0.4620	-19.4035

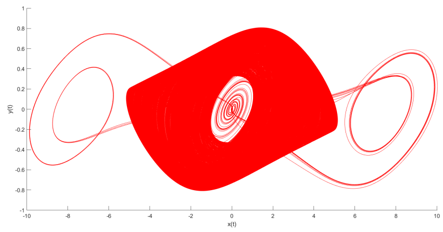
Table 3.1: Initial condition according to the values of ε.

ε	0.8	0.86	0.9785	0.989	0.993	1
Number of spirals	1 spiral	3 spirals	5 spirals	7 spirals	7 spirals	7 spirals

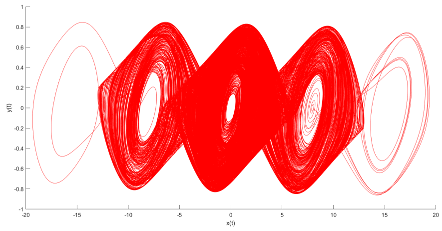
Table 3.2: Values of parameter ε at the bifurcation points for $c = 6$ (7 spirals).



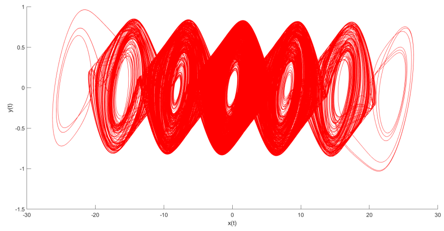
(a)



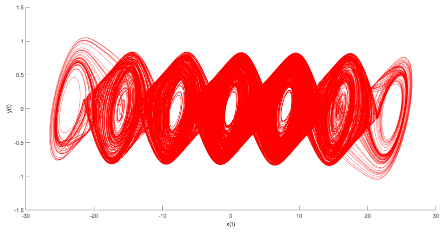
(b)



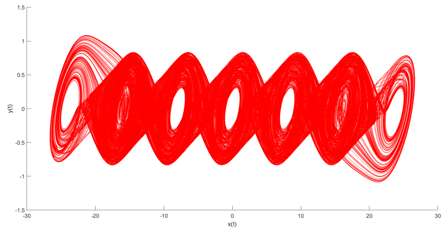
(c)



(d)



(e)



(f)

Figure 3.3: The increasing number of spirals of system (3.15) according to increasing ε values for 7 scroll ($c_1 = 6$). (a) 1 spiral for $\varepsilon = 0.8$; (b) 3 spirals for $\varepsilon = 0.86$; (c) 5 spirals for $\varepsilon = 0.9785$; (d) 7 spirals for $\varepsilon = 0.989$; (e) 7 spirals for $\varepsilon = 0.9994$; (f) 7 spirals for $\varepsilon = 1$.

CONCLUSION

This thesis has investigated hidden attractors and hidden bifurcations in continuous chaotic systems through theoretical analysis and numerical simulations. Our research establishes that these phenomena fundamentally differ from classical chaotic dynamics as they lack connections to equilibria that can be readily identified through standard linearization techniques. We have demonstrated that hidden attractors can exist in systems with no equilibria or with exclusively unstable equilibrium points, while hidden bifurcations represent qualitative transformations in system behavior that often remain undetected by conventional bifurcation analysis methods. The significance of this investigation lies in highlighting the necessity for developing advanced mathematical frameworks and computational algorithms specifically designed to reveal these hidden dynamical structures. Enhanced understanding of hidden attractors and bifurcations opens new pathways for chaos control strategies and performance optimization across diverse applications, including secure communication systems, nonlinear control engineering, and complex electrical networks.

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ملخص

تقدم هذه الأطروحة دراسة شاملة للأنظمة الديناميكية المستمرة، مع التركيز على الجواذب المخفية والتشعبات المخفية، منظمة في ثلاثة فصول مترابطة. يقدم الفصل الأول المفاهيم والخصائص الأساسية للأنظمة الديناميكية، ثم يصنف نقاط التوازن ويفحص استقرارها. يتم تناول مفهوم الدورات المحدودة كجانب أساسي من السلوك الديناميكي. يستكشف الفصل الثاني مفهوم الجواذب المخفية، بدءًا من الجواذب ذاتية الإثارة المرتبطة بنقاط التوازن غير المستقرة. ثم يقدم الجواذب المخفية التي لا ترتبط بأي توازن. يحدد الفصل التشعبات المخفية ويقدم طريقة فعالة للكشف عنها، موضحة بحالة عملية. ينفذ الفصل الأخير المنهجية المقترحة شاةشا نظاعلى م تشوا، من خلال إجراء تحليل رقمي وتشغيل محاكاة فيلتأكيد النتائج. يختتم الفصل بعرض وتحليل النتائج الرقمية المكتسبة.

الكلمات المفتاحية :

الديناميكا غير الخطية، الفوضى، الجواذب المخفية، التشعب المخفي، الجاذب الغريب.

Résumé

Cette thèse propose un examen complet des systèmes dynamiques continus, en se concentrant sur les attracteurs cachés et les bifurcations cachées, organisé en trois chapitres interconnectés. Le chapitre initial présente les concepts et caractéristiques essentiels des systèmes dynamiques, catégorisant ensuite les points d'équilibre et examinant leur stabilité. La notion de cycles limites est abordée comme un aspect fondamental du comportement dynamique. Le deuxième chapitre explore le concept d'attracteurs cachés, en commençant par les attracteurs auto-excités liés aux points d'équilibre instables. Il présente ensuite les attracteurs dissimulés qui n'ont aucune association avec l'équilibre. Le chapitre délimite les bifurcations cachées et introduit une méthode efficace pour leur détection, illustrée par un cas pratique. Le chapitre final met en œuvre la méthodologie proposée sur le système de Chua, effectuant une analyse numérique et exécutant des simulations dans MATLAB pour corroborer les résultats. Le chapitre se termine par la présentation et l'analyse des résultats numériques acquis.

Mot clés : Dynamique non linéaire, chaos, attracteurs cachés, bifurcation cachée, attracteur étrange.

Abstract

This thesis provides a comprehensive examination of continuous dynamical systems, focusing on hidden attractors and hidden bifurcations, organized into three interconnected chapters. The initial chapter presents the essential concepts and characteristics of dynamical systems, subsequently categorizing equilibrium points and examining their stability. The notion of limit cycles is addressed as a fundamental aspect of dynamic behavior. The second chapter explores the concept of hidden attractors, beginning with self-excited attractors linked to unstable equilibrium points. It subsequently presents concealed attractors that lack any association with equilibrium. The chapter delineates hidden bifurcations and introduces a proficient method for their detection, exemplified by a practical case. The final chapter implements the proposed method on the Chua system, doing a numerical analysis and running simulations in MATLAB to corroborate the results. The chapter closes with the presentation and analysis of the acquired numerical results.

Key-words : Nonlinear dynamics, chaos, hidden attractors, hidden bifurcation, strange attractor.