People's Democratic Republic of Algeria

Ministry of Higher Education and Scientific Research

MOHAMED KHIDER, UNIVERSITY OF BISKRA

Faculty of Exact Sciences

Mathematics Department

Thesis Submitted in Partial Execution of the

Requirements of the Degree of

Master in **Probability and Statistics**

By

Djihad LARDJANI

Title:

Maximum principle for partially observed optimal control of Mckean-Vlasov FBSDEs

Examination Committee Members:

Pr. Adel CHALA. *University of Biskra*,________________ **President**

Dr. Imad Eddine LAKHDARI. *University of Biskra*,________________ **Supervisor**

Dr. HAKIMA MILOUDI. *University of Biskra*,________________ **Examiner**

2025

Dedication

To my dear parents—

To my beloved mother, Soumia Lakhal, who has always been my unwavering support and companion through every stage of my life.

And to the soul of my father, Ahmed Lardjani, who dedicated his life to us and encouraged me to pursue my studies until his very last breath — may Allah have mercy on him and grant him eternal peace.

To those who stood by me after the loss of my father:

To my grandfather, Said Lakhal, and my grandmother, Saliha Bekkour — thank you for your prayers, love, and tenderness. May God protect you both and bless you with health and long life.

To my beautiful, loving family —

To my dear uncle, Lakhal Abdel Fattah — thank you for all your support.

To my beloved sister, Issaad Heba Allah Lardjani —

And to my fiancé, Nacer Sassi — thank you for your constant support, love, and encouragement.

To my dearest friends, with whom I shared the most beautiful academic memories: Sarah Gheraf, Anfal Lesteb, Chaima Esbaâ, and Ibtissam Touati — thank you for being part of this unforgettable journey.

Acknowledgements

At the beginning of this work, I would like to express my heartfelt thanks and deep gratitude to Allah Almighty for His countless blessings and abundant generosity. He granted me determination, patience, and good health throughout my years of study. All praise is due to Him, now and always.

I would also like to express my sincere thanks and appreciation to my supervisor, Dr. Imad Eddine Lakhdari, for his valuable guidance, wise advice, and continuous patience during the preparation of this work. His support played a fundamental role in the completion of this thesis.

It is a great honor for me to extend my sincere thanks and deep appreciation to Professor Adel Chala, for kindly accepting to chair the defense committee and evaluate this work. I am truly grateful for his trust and support.

I also extend my heartfelt thanks to Dr. Hakima Miloudi, for accepting to serve as a member of the defense committee. I highly appreciate her scientific insights and valuable comments, which will undoubtedly enrich this research

Contents

Symbols and acronyms

The different abbreviations and ratings used throughout this thesis are explained below:

$(\Omega, \mathcal{F}, \mathbb{F}, P)$	Probability space.
$\left(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{F}}, \widehat{P}\right)$	Copy of the probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$.
$\mathcal{F}_t$	Filtration.
$\mathcal{F}_t^W$	Filtration generated by W.
$\mathcal{F}_t^Y$	Filtration generated by Y.
SDE	Stochastic differential equation.
$BSDE$	Backward stochastic differential equation.
$\mathbb{R}$	Real numbers.
$\mathbb{N}$	Natural numbers.
ODE	Ordinary differential equation.
$\mathcal{U}$	The set of the admissible control variables.
H	The Hamiltonian function.
$\mathbb{L}^2\left(\mathcal{F}; \mathbb{R}^d\right)$	The Hilbert space.
$\mathcal{L}_{\mathcal{F}}^2\left(0, T, \mathbb{R}^n\right)$	The set of all $\mathbb{R}^n$-valued square-integrable $\mathcal{F}_t$-adapted processes.
$\mathcal{L}_{\mathcal{F}}^2(\Omega, \mathbb{R}^n)$	The set of all $\mathbb{R}^n$-valued square-integrable $\mathcal{F}_T$-measurable random variables.
$Q_2\left(\mathbb{R}^d\right)$	The space of all probability measures.

Introduction

Introduction

Stochastic optimal control is a field of mathematics that deals with the optimization of control policies for systems that are subject to random disturbances or noise. It extends the principles of optimal control theory to systems that evolve probabilistically over time, rather than deterministically. Stochastic optimal control has a wide range of applications, including: finance, mechanics, biology, electricity, chemistry, economics, etc. One of the well-known approaches to solving the optimal control problem is the stochastic maximum principle (SMP).

The stochastic maximum principle for McKean-Vlasov systems without partial observation has been explored by numerous researchers. For instance, Buckdahn et al. [3] developed the stochastic maximum principle for general mean-field systems using the method of second-order derivatives concerning probability measures. Additionally, Carmona and Delarue [4] introduced a new version of the stochastic maximum principle for nonlinear stochastic dynamical systems of the McKean–Vlasov type and provided sufficient conditions for the existence of an optimal control.

However, the aforementioned studies all assume that controllers have access to complete information, an assumption that is not always realistic. In practice, controllers often have access only to partial information. Consequently, it is logical to investigate these types of optimal control problems under conditions of partial observation. There is extensive literature on the subject of partially observed optimal control problems; see, for example, [1, 2, 5, 7, 6, 8, 9, 10].

The objective of this thesis is to study the necessary conditions (also referred to as the stochastic maximum principle) as well as sufficient conditions of the partially observed optimal control problem of forward-backward stochastic differential equations (FBSDEs for short) of McKean-Vlasov type. More precisely, the parameters of the system and the cost function are influenced by the current state of the solution process as well as of its probability measures under the assumption that the control domain is necessarily convex. This study is based on the work of Abba and Lakhdari [1].

We present our work as follows:

- In the first chapter, we introduce some helpful concepts from stochastic analysis and differential calculus on Wasserstein space.

- In the second chapter, we present the necessary and sufficient conditions of optimality for the partially observed optimal control problem of forward-backward stochastic differential equations of McKean-Vlasov type.

- The last chapter is an application of our theoretical study, which is the partially observed linear-quadratic control problem.

Chapter §.1 Stochastic analysis and differntial calculus on Wasserstein space

Chapter 1

Stochastic analysis and differntial calculus on Wasserstein space

In this chapter, we introduce some helpful concepts from stochastic analysis and differential calculus on Wasserstein space.

1.1 Stochastic processes

Let $(\Omega, \mathcal{F}, P)$ be a probability space and T be a nonempty index set. A stochastic process is a set of random variables $\{X(t) : t \in T\}$ from $(\Omega, \mathcal{F}, P)$ to $\mathbb{R}^n$. For any $w \in \Omega$ the map $t \to X(t, w)$ is called a sample path.

1.2 Natural fitration

Consider the stochastic process $X = (X_t, t \geq 0)$ on the probability space $(\Omega, \mathcal{F}, P)$. denoted by $\mathcal{F}_t^X$ for the natural filtration of X which is defined by $\mathcal{F}_t^X = \delta(X_s, 0 \leq s \leq t)$. Also, we called the filtaration generated by X.

1.3 Brownian motion

A stochastic process $(W(t), t \geq 0)$ is called a standard Brownian motion if:

• $P[W(0) = 0] = 1$.

• $t \to W(t, w)$ is continuous. $P - p.s.$

• $\forall s \leq t, W(t) - W(s)$is normally distributed; center with variation $(t - s)$ i.e $W(t) - W(s) \sim N(0, t - s)$.

• $\forall n, \forall 0 \leq t_0 \leq t_1 \leq ... \leq t_n$, the variables$\left(W_{t_n} - W_{t_{n-1}}, ..., W_{t_1} - W_{t_0}, W_{t_0}\right)$ are independents.

1.4 Martingale

Definition 1.4.1 ***(Martingale)***:*$\{X_t\}$ is a martingale with respect to fltration $\{\mathcal{F}_t\}$ if for all $t > s$ we have*

i) X_t is $\mathcal{F}_t-$measurable.

ii) $\mathbb{E}\left[|X_t|\right] < \infty$.

iii) $\mathbb{E}\left[X_t/\mathcal{F}_t\right] = X_t$

i) and ii) above, and

$$\mathbb{E}\left[X_t/\mathcal{F}_t\right] \leq X_t, \mathbb{E}\left[X_t/\mathcal{F}_t\right] \geq X_t \quad P-\text{a.s.}$$

Proposition 1.4.1 *Let X_t be a stochastic process such that for any stopping time T, X_t is integral and*

$$\mathbb{E}\left[X_0\right] = \mathbb{E}\left[X_T\right],$$

then X_t is a martingale.

Definition 1.4.2 ***(Local martingale):*** *An adapted process X_t is a local martingale if there exists a sequence of stopping times $\{T_n\}$ such that*

$$\lim_{n\to\infty} T_n(\omega) = \infty \quad P-a.s.$$

and the stopped process $X_{T_n \wedge t}$ is a martingale for all n.

1.5 Notations and spaces

Let T be a fixed strictly positive real number and $(\Omega, \mathcal{F}, \mathbb{F}, P)$ be a complete filtered probability space equipped with two independent standard one-dimensional Brownian motions W and Y. Also assume that $\mathbb{F} = \{\mathcal{F}_t\}_{t\geq 0}$ and $\mathcal{F}_t := \mathcal{F}_t^W \vee \mathcal{F}_t^Y \vee \mathcal{N}$, where $\mathcal{N}$ denotes the totality of P-null set and $\mathcal{F}_t^W$ and $\mathcal{F}_t^Y$ denotes the P-completed natural filtration generated by W and Y respectively. We denote by $\mathbb{R}^n$ the n-dimensional Euclidean space, and by $(\cdot,\cdot)$ (resp. $|\cdot|$) the inner product (resp. norm). The set of the admissible control variables is denoted by $\mathcal{U}$.

Throughout what follows, we will use the following notations.

- $\mathcal{L}^2_{\mathcal{F}}(0, T, \mathbb{R}^n)$ the set of all $\mathbb{R}^n$-valued square-integrable $\mathcal{F}_t$-adapted processes.
- $\mathcal{L}^2_{\mathcal{F}}(\Omega, \mathbb{R}^n)$ the set of all $\mathbb{R}^n$-valued square-integrable $\mathcal{F}_T$-measurable random variables.
- $\mathbb{L}^2\left(\mathcal{F}; \mathbb{R}^d\right)$ is the Hilbert space with inner product $(x,y)_2 = \mathbb{E}[x.y]\,, x, y \in \mathbb{L}^2\left(\mathcal{F}; \mathbb{R}^d\right)$ and the norm $\|x\|_2 = \sqrt{(x,x)_2}$.
- $Q_2\left(\mathbb{R}^d\right)$ the space of all probability measures μ on $\left(\mathbb{R}^d, \mathcal{B}\left(\mathbb{R}^d\right)\right)$ with finite second moment, i.e, $\int_{\mathbb{R}^d} |x|^2 \mu(dx) < \infty$, endowed with the following 2-Wasserstein metric: for $\mu, \nu \in Q_2\left(\mathbb{R}^d\right)$, $\mathbb{D}_2(\mu_1, \mu_2) = \inf \left\{ \begin{array}{l} \left[\int_{\mathbb{R}^d} |x-y|^2 \rho(dx, dy)\right]^{\frac{1}{2}} \\ : \rho \in Q_2\left(\mathbb{R}^{2d}\right), \rho\left(\cdot, \mathbb{R}^d\right) = \mu_1, \rho\left(\mathbb{R}^d, \cdot\right) = \mu_2 \end{array} \right\}$.
- $\left(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{\mathbb{F}}, \widehat{P}\right)$ is a copy of the probability space $(\Omega, \mathcal{F}, \mathbb{F}, P)$.

- $\left(\widehat{\vartheta},\widehat{\alpha}\right)$ is an independent copy of the random variable (ϑ,α) defined on $\left(\widehat{\Omega},\widehat{\mathcal{F}},\widehat{\mathbb{F}},\widehat{P}\right)$, such that
$$(\vartheta,\alpha)\in\mathbb{L}^2\left(\mathcal{F};\mathbb{R}^d\right)\times\mathbb{L}^2\left(\mathcal{F};\mathbb{R}^d\right).$$

- $\left(\Omega\times\widehat{\Omega},\mathcal{F}\otimes\widehat{\mathcal{F}},\mathbb{F}\otimes\widehat{\mathbb{F}},P\otimes\widehat{P}\right)$ is the product probability space, such that
$$\left(\widehat{\vartheta},\widehat{\alpha}\right)(w,\widehat{w})=\left(\vartheta\left(\widehat{w}\right),\xi\left(\widehat{w}\right)\right)\text{ for any }(w,\widehat{w})\in\Omega\times\widehat{\Omega}.$$

Let $\left(\widehat{u}_t,\widehat{x}_t,\widehat{y}_t,\widehat{z}_t,\widehat{\bar{z}}_t\right)$ be an independent copy of $(u_t,x_t,y_t,z_t,\bar{z}_t,r_t)$ so that $P_{x_t}{=}\widehat{P}_{\widehat{x}_t}$, $P_{y_t}{=}\widehat{P}_{\widehat{y}_t}$, $P_{z_t}{=}\widehat{P}_{\widehat{z}_t}$ and $P_{\bar{z}_t}{=}\widehat{P}_{\widehat{\bar{z}}_t}$. We denote by $\widehat{\mathbb{E}}\left[\cdot\right]$ the expectation under probability measure $\widehat{P}$ and $P_X=P{\circ}X^{-1}$ denotes the law of the random variable X.

1.6 Differentiability with respect to probability measures

In the following, we introduce the basic notations of mean-field theory (the differentiability with respect to probability measures). The principal idea is to identify a distribution $\mu\in Q_2\left(\mathbb{R}^d\right)$ with a random variables $\vartheta\in\mathbb{L}^2\left(\mathcal{F};\mathbb{R}^d\right)$ so that $\mu=P_\vartheta$. To be more precise, we assume that probability space $(\Omega,\mathcal{F},\mathbb{F},P)$ is rich enough in the sense that for every $\mu\in Q_2\left(\mathbb{R}^d\right)$, there is a random variable $\vartheta\in\mathbb{L}^2\left(\mathcal{F};\mathbb{R}^d\right)$ such that $\mu=P_\vartheta$. It is well-known that the probability space $\left([0,1],\mathcal{B}\left[0,1\right],dx\right)$, where dx is the Borel measure, has this property.

Next, for any function $f:Q_2\left(\mathbb{R}^d\right)\to\mathbb{R}$, we induce a function $\widetilde{f}:\mathbb{L}^2\left(\mathcal{F};\mathbb{R}^d\right)\to\mathbb{R}$ such that $\widetilde{f}\left(\vartheta\right):=f\left(P_\vartheta\right),\ \vartheta\in\mathbb{L}^2\left(\mathcal{F};\mathbb{R}^d\right)$. Clearly, the function $\widetilde{f}$ called the lift of f in the literature, depends only on the law of $\vartheta\in\mathbb{L}^2\left(\mathcal{F};\mathbb{R}^d\right)$ and is independent of the choice of the representative ϑ.

Definition 1.6.1 $\left(\textit{Differentiable function in } Q_2\left(\mathbb{R}^d\right)\right)$ *A function* $f:Q_2\left(\mathbb{R}^d\right)\to\mathbb{R}$ *is* *said to be differentiable at* $\mu_0 \in Q_2\left(\mathbb{R}^d\right)$ *if there exists* $\vartheta_0 \in \mathbb{L}^2\left(\mathcal{F};\mathbb{R}^d\right)$ *with* $\mu_0 = P_{\vartheta_0}$ *such that its lift* $\widetilde{f}$ *is Fréchet differentiable at* ϑ_0. *More precisely, there exists a continuous linear functional* $D\widetilde{f}\left(\vartheta_0\right) : \mathbb{L}^2\left(\mathcal{F};\mathbb{R}^d\right) \to \mathbb{R}$ *such that*

$$\widetilde{f}\left(\vartheta_0+\alpha\right)-\widetilde{f}\left(\vartheta_0\right)=\left\langle D\widetilde{f}\left(\vartheta_0\right),\alpha\right\rangle+O\left(\|\alpha\|_2\right)=D_\alpha f\left(\mu_0\right)+O\left(\|\alpha\|_2\right), \tag{1.1}$$

where $\langle\cdot,\cdot\rangle$ *is the dual product on* $\mathbb{L}^2\left(\mathcal{F};\mathbb{R}^d\right)$, *and we will refer to* $D_\alpha f\left(\mu_0\right)$ *as the Fréchet derivative of* f *at* μ_0 *in the direction* α. *In this case, we have*

$$D_\alpha f\left(\mu_0\right)=\left\langle D\widetilde{f}\left(\vartheta_0\right),\alpha\right\rangle=\frac{d}{dt}\widetilde{f}\left(\vartheta_0+t\alpha\right)\bigg|_{t=0},\ \textit{with } \mu_0=P_{\vartheta_0}.$$

Note that by Riesz's representation theorem, there is a unique random variable $\Lambda_0 \in \mathbb{L}^2\left(\mathcal{F};\mathbb{R}^d\right)$ such that $\left\langle D\widetilde{f}\left(\vartheta_0\right),\alpha\right\rangle = \left(\Lambda_0,\alpha\right)_2 = \mathbb{E}\left[\left(\Lambda_0,\alpha\right)_2\right]$, where $\alpha \in \mathbb{L}^2\left(\mathcal{F};\mathbb{R}^d\right)$. Then there exists a Boral function $h\left[\mu_0\right] : \mathbb{R}^d \to \mathbb{R}^d$, depending only on the law $\mu_0 = P_{\vartheta_0}$ but not on the particular choice of the representative ϑ_0 such that $\Lambda_0 = h\left[\mu_0\right]\left(\vartheta_0\right)$.
So, we can write equation (1.1) as

$$f\left(P_\vartheta\right)-f\left(P_{\vartheta_0}\right)=\left(h\left[\mu_0\right]\left(\vartheta_0\right),\vartheta-\vartheta_0\right)_2+O\left(\|\vartheta-\vartheta_0\|_2\right),\quad \forall\vartheta\in\mathbb{L}^2\left(\mathcal{F};\mathbb{R}^d\right).$$

We shall denote $\partial_\mu f\left(P_{\vartheta_0},x\right) = h\left[\mu_0\right](x)$, $x \in \mathbb{R}^d$. Moreover, we have the following identities:

$$\begin{aligned} D\widetilde{f}\left(\vartheta_0\right)&=\Lambda_0=h\left[\mu_0\right]\left(\vartheta_0\right)=\partial_\mu f\left(P_{\vartheta_0},\vartheta_0\right),\\ D_\alpha f\left(P_{\vartheta_0}\right)&=\left\langle\partial_\mu f\left(P_{\vartheta_0},\vartheta_0\right),\alpha\right\rangle,\end{aligned}$$

where $\alpha = \vartheta - \vartheta_0$, and for each $\mu \in Q_2\left(\mathbb{R}^d\right)$, $\partial_\mu f\left(P_\vartheta,\cdot\right) = h\left[P_\vartheta\right](\cdot)$ is only defined in a $P_\vartheta(dx) - a.e$ sense, where $\mu = P_\vartheta$.

Definition 1.6.2 *We say that the function* $f \in \mathbb{C}_b^{1,1}\left(Q_2\left(\mathbb{R}^d\right)\right)$ *if for all* $\vartheta \in \mathbb{L}^2\left(\mathcal{F};\mathbb{R}^d\right)$, *there exists a P_ϑ-modification of $\partial_\mu f(P_\vartheta,\cdot)$ such that $\partial_\mu f : Q_2(\mathbb{R}^d)\times\mathbb{R}^d \to \mathbb{R}^d$ is bounded and Lipchitz continuous. That is for some $C>0$, it holds that*

1. $|\partial_\mu f(\mu,x)| \leq C, \forall \mu \in Q_2(\mathbb{R}^d), \forall x \in \mathbb{R}^d$;
2. $|\partial_\mu f(\mu,x) - \partial_\mu f(\acute{\mu},\acute{x})| \leq C(\mathbb{D}_2(\mu,\acute{\mu}) + |x-\acute{x}|), \forall \mu,\acute{\mu} \in Q_2(\mathbb{R}^d), \forall x,\acute{x} \in \mathbb{R}^d$.

Remark 1.6.1 *If $f \in \mathbb{C}_b^{1,1}(Q_2(\mathbb{R}^d))$, the derivative $\partial_\mu f(P_\vartheta,\cdot), \vartheta \in \mathbb{L}^2(\mathcal{F};\mathbb{R}^d)$ indicated in the Definition 2.2 is unique.*

Definition 1.6.3 *Let U be a nonempty convex subset of $\mathbb{R}^k$. A control $v:\Omega\times[0,T]\to U$ is called admissible if it is $\mathcal{F}_t^Y$-adapted and satisfies $\sup_{0\leq t\leq T}\mathbb{E}|v_t|^2 < \infty$.*

Chapter §.2
Necessary and sufficient condition of optimality

Chapter 2

Necessary and sufficient conditions of optimality

In this chapter, we study the necessary and sufficient conditions of optimality for our system of McKean–Vlasov type, satisfied by a partially observed optimal control, assuming that the solution exists. The proof is based on convex perturbation and on some estimates of the state processes of the system and observed process.

We consider the following stochastic control system with general McKean–Vlasov FBSDEs

$$\begin{cases} dx_t^v = b\left(t, x_t^v, P_{x_t^v}, v_t\right) dt + g\left(t, x_t^v, P_{x_t^v}, v_t\right) dW_t + \sigma\left(t, x_t^v, P_{x_t^v}, v_t\right) d\widetilde{W}_t^v \\ -dy_t^v = f\left(t, x_t^v, P_{x_t^v}, y_t^v, P_{y_t^v}, z_t^v, P_{z_t^v}, \bar{z}_t^v, P_{\bar{z}_t^v}, v_t\right) dt - z_t^v dW_t - \bar{z}_t^v dY_t, \\ x_0^v = x_0, \quad y_T^v = \varphi\left(x_T^v, P_{x_T^v}\right), \end{cases} \tag{2.1}$$

where $P_{x_t}, P_{y_t}, P_{z_t}$ and $P_{\bar{z}_t}$ denotes the law of the random variable x, y, z and $\bar{z}$ respectively. The coefficients of the controlled system (2.1) are defined as follows

$$b : [0,T] \times \mathbb{R} \times Q_2(\mathbb{R}) \times U \to \mathbb{R}, \quad g, \sigma : [0,T] \times \mathbb{R} \times Q_2(\mathbb{R}) \times U \to \mathbb{R},$$

$$\varphi : \mathbb{R} \times Q_2(\mathbb{R}) \to \mathbb{R},$$

$$f : [0,T] \times \mathbb{R} \times Q_2(\mathbb{R}) \times \mathbb{R} \times Q_2(\mathbb{R}) \times \mathbb{R} \times Q_2(\mathbb{R}) \times \mathbb{R} \times Q_2(\mathbb{R}) \times U \to \mathbb{R}.$$

It is worth noting that the above forward-backward stochastic differential equation (2.1) of type McKean–Vlasov is very general, in that the dependence of the coefficients on the probability law of the solution $P_{x_t^v}, P_{y_t^v}, P_{z_t^v}$ and $P_{\bar{z}_t^v}$ could be genuinely nonlinear as an element of the space of probability measures.

We assume that the state processes $(x^v, y^v, z^v, \bar{z}^v)$ cannot be observed directly, but the controllers can observe a related noisy process Y, which is the solution of the following equation

$$\begin{cases} dY_t = \xi\left(t, x_t^v, P_{x_t^v}\right) dt + d\widetilde{W}_t^v, \\ Y_0 = 0, \end{cases} \tag{2.2}$$

where $\xi : [0, T] \times \mathbb{R} \times Q_2(\mathbb{R}) \to \mathbb{R}$ and $\widetilde{W}_t^v$ is stochastic processes depending on the control v.

Inserting (2.2) into (2.1), we have

$$\begin{cases} dx_t^v = \left[b\left(t, x_t^v, P_{x_t^v}, v_t\right) dt - \sigma\left(t, x_t^v, P_{x_t^v}, v_t\right) \xi\left(t, x_t^v, P_{x_t^v}\right)\right] dt \\ \qquad + g\left(t, x_t^v, P_{x_t^v}, v_t\right) dW_t + \sigma\left(t, x_t^v, P_{x_t^v}, v_t\right) dY_t, \\ -dy_t^v = f\left(t, x_t^v, P_{x_t^v}, y_t^v, P_{y_t^v}, z_t^v, P_{z_t^v}, \bar{z}_t^v, P_{\bar{z}_t^v}, v_t\right) dt - z_t^v dW_t - \bar{z}_t^v dY_t, \\ x_0^v = x_0, \quad y_T^v = \varphi\left(x_T^v, P_{x_T^v}\right). \end{cases} \tag{2.3}$$

Define $dP^v = Z_t^v dP$ with

$$Z_t^v = \exp\left\{\int_0^t \xi\left(s, x_s^v, P_{x_s^v}\right) dY_s - \frac{1}{2}\int_0^t \left|\xi\left(s, x_s^v, P_{x_s^v}\right)\right|^2 ds\right\},$$

where Z^v is the unique $\mathcal{F}_t^Y$-adapted solution of the SDE of McKean–Vlasov type

$$\begin{cases} dZ_t^v = Z_t^v \xi\left(t, x_t^v, P_{x_t^v}\right) dY_t, \\ Z_0^v = 1. \end{cases} \tag{2.4}$$

The associated cost functional is also of McKean–Vlasov type, defined as

$$J(v) = \mathbb{E}^v \left[\int_0^T l\left(t, x_t^v, P_{x_t^v}, y_t^v, P_{y_t^v}, z_t^v, P_{z_t^v}, \bar{z}_t^v, P_{\bar{z}_t^v}, r_t^v, P_{r_t^v}, v_t\right) dt \right] + \mathbb{E}^v \left[M\left(x_T^v, P_{x_T^v}\right) + h\left(y_0^v, P_{y_0^v}\right) \right], \tag{2.5}$$

where $\mathbb{E}^v$ denotes the expectation with respect to the probability space $(\Omega, \mathcal{F}, \mathbb{F}, P^v)$ and

$$M : \mathbb{R} \times Q_2(\mathbb{R}) \to \mathbb{R}, \quad h : \mathbb{R} \times Q_2(\mathbb{R}) \to \mathbb{R},$$
$$l : [0,T] \times \mathbb{R} \times Q_2(\mathbb{R}) \times \mathbb{R} \times Q_2(\mathbb{R}) \times \mathbb{R} \times Q_2(\mathbb{R}) \times \mathbb{R} \times Q_2(\mathbb{R}) \times \mathbb{R} \times Q_2(\mathbb{R}) \times U \to \mathbb{R}.$$

Our partially observed optimal control problem of general McKean–Vlasov FBSDE is to minimize the cost functional (2.5) over $v \in \mathcal{U}$ subject to (2.1) and (2.2), *i.e.*,

$$\min_{v \in \mathcal{U}} J(v).$$

If an admissible control u attains the minimum, we call u an optimal control and $(x, y, z, \bar{z})$ an optimal state, respectively. Obviously, cost functional (2.5) can be rewritten as

$$J(v) = \mathbb{E} \left[\int_0^T Z_t^v l\left(t, x_t^v, P_{x_t^v}, y_t^v, P_{y_t^v}, z_t^v, P_{z_t^v}, \bar{z}_t^v, P_{\bar{z}_t^v}, v_t\right) dt \right] + \mathbb{E} \left[Z_T^v M\left(x_T^v, P_{x_T^v}\right) + h\left(y_0^v, P_{y_0^v}\right) \right]. \tag{2.6}$$

Then the original problem (2.5) is equivalent to minimize (2.6) over $v \in \mathcal{U}$ subject to (2.1) and (2.4).

Let us impose some assumptions on the coefficients of the state and the performance cost functional.

2.1 Assumptions

Assumption (A1)

1. For any $t \in [0,T]$, the functions b, g and σ are continuously differentiable in (x,v) and they are bounded by $C(1+|x|+|v|)$. The function ξ is continuously differentiable in x.
2. The functions f and l are continuously differentiable in $(x,y,z,\bar{z},v)$, and they are bounded by $C(1+|x|+|y|+|z|+|\bar{z}|+|v|)$ and $C\left(1+|x|^2+|y|^2+|z^{|2}+|\bar{z}|^2+|v|^2\right)$ respectively. The derivatives of f and l with respect to $(x,y,z,\bar{z},v)$ are uniformly bounded.
3. The functions φ and M are continuously differentiable in x, and the function h is continuously differentiable in y. The derivatives M_x, h_y are bounded by $C(1+|x|)$ and $C(1+|y|)$ respectively.
4. The derivatives $b_x, b_v, g_x, g_v, \sigma_x, \sigma_v, \xi_x$ are continuous and uniformly bounded.

Assumption (A2)

1. The functions $b, g, \sigma, f, l, \xi, M, h, \varphi \in \mathbb{C}_b^{1,1}\left(Q_2\left(\mathbb{R}\right)\right)$.
2. The derivatives $\partial_\mu^{P_x} b, \partial_\mu^{P_x} g, \partial_\mu^{P_x} \sigma, \partial_\mu^{P_x} \xi, \left(\partial_\mu^{P_x}, \partial_\mu^{P_y}, \partial_\mu^{P_z}, \partial_\mu^{P_{\bar{z}}}\right)(f,l)$ are bounded and Lipchitz continuous, such that, for some $C>0$, it holds that

(i) For $\rho = b, g, \sigma, \xi$, and $\forall \mu, \mu' \in Q_2\left(\mathbb{R}\right), \forall x, x' \in \mathbb{R}$,

$$\left|\partial_\mu^{P_x} \rho\left(t,x,\mu\right)\right| \leq C,$$
$$\left|\partial_\mu^{P_x} \rho\left(t,x,\mu\right) - \partial_\mu^{P_x} \rho\left(t,x',\mu'\right)\right| \leq C\left(\mathbb{D}_2\left(\mu,\mu'\right) + |x-x'|\right),$$

(ii) For $\rho = M, \varphi$, and $\forall \mu, \mu' \in Q_2\left(\mathbb{R}\right), \forall x, x' \in \mathbb{R}$,

$$\left|\partial_\mu^{P_x} \rho\left(x,\mu\right)\right| \leq C,$$
$$\left|\partial_\mu^{P_x} \rho\left(x,\mu\right) - \partial_\mu^{P_x} \rho\left(x',\mu'\right)\right| \leq C\left(\mathbb{D}_2\left(\mu,\mu'\right) + |x-x'|\right);$$

(iii) For $\rho = f, l$, and $\forall \mu_1, \mu_1', \mu_2, \mu_2', \mu_3, \mu_3', \mu_4, \mu_4' \in Q_2\left(\mathbb{R}\right)$ and $\forall x, x', y, y', z, z', \bar{z}, \bar{z}' \in \mathbb{R}$,

$$
\begin{aligned}
&\left|\left(\partial_\mu^{P_x},\partial_\mu^{P_y},\partial_\mu^{P_z},\partial_\mu^{P_{\bar z}}\right)\rho\left(t,x,\mu_1,y,\mu_2,z,\mu_3,\bar z,\mu_4\right)\right|\leq C,\\
&\left|\left(\partial_\mu^{P_x},\partial_\mu^{P_y},\partial_\mu^{P_z},\partial_\mu^{P_{\bar z}}\right)\rho\left(t,x,\mu_1,y,\mu_2,z,\mu_3,\bar z,\mu_4\right)\right.\\
&\left.-\left(\partial_\mu^{P_x},\partial_\mu^{P_y},\partial_\mu^{P_z},\partial_\mu^{P_{\bar z}}\right)\rho\left(t,x',\mu_1',y',\mu_2',z',\mu_3',\bar z',\mu_4'{}'\right)\right|\\
&\leq C\left(|x-x'|+|y-y'|+|z-z'|+|\bar z-\bar z'|+\mathbb{D}_2\left(\mu_1,\mu_1'\right)\right.\\
&\left.+\mathbb{D}_2\left(\mu_2,\mu_2'\right)+\mathbb{D}_2\left(\mu_3,\mu_3'\right)+\mathbb{D}_2\left(\mu_4,\mu_4'\right)\right).
\end{aligned}
$$

Clearly, under assumptions (**A1**) and (**A2**), for each $v\in\mathcal{U}$, there is a unique solution $(x,y,z,\bar z)\in\mathcal{L}^2_{\mathcal{F}}(0,T,\mathbb{R})\times\mathcal{L}^2_{\mathcal{F}}(0,T,\mathbb{R})\times\mathcal{L}^2_{\mathcal{F}}(0,T,\mathbb{R})\times\mathcal{L}^2_{\mathcal{F}}(0,T,\mathbb{R})$ which solves

$$
\begin{cases}
x_t^v=x_0+\displaystyle\int_0^t\left[b\left(s,x_s^v,P_{x_s^v},v_s\right)-\sigma\left(s,x_s^v,P_{x_s^v},v_s\right)\xi\left(s,x_s^v,P_{x_s^v}\right)\right]ds\\
\qquad\qquad+\displaystyle\int_0^t g\left(s,x_s^v,P_{x_s^v},v_s\right)dW_s+\int_0^t\sigma\left(s,x_s^v,P_{x_s^v},v_s\right)dY_s,\\
y_t^v=y_T^v-\displaystyle\int_t^T f\left(s,x_s^v,P_{x_s^v},y_s^v,P_{y_s^v},z_s^v,P_{z_s^v},r_s^v,P_{r_s^v},v_s\right)dt\\
\qquad\qquad+\displaystyle\int_t^T z_s^v dW_s+\int_t^T\bar z_s^v dY_s,
\end{cases}
$$

To simplify our notations, we denote for ξ,c and $\psi=b,g,\sigma$

$$
\begin{aligned}
\xi(t)&=\xi\left(t,x_t,P_{x_t}\right), &\psi(t)&=\psi\left(t,x_t,P_{x_t},u_t\right),\\
\xi_x(t)&=\xi_x\left(t,x_t,P_{x_t}\right), &\psi_\rho(t)&=\psi_\rho\left(t,x_t,P_{x_t},u_t\right),
\end{aligned}
$$

and the derivative processes

$$
\begin{aligned}
\partial_\mu^{P_x}\xi\left(t\right)&=\partial_\mu^{P_x}\xi\left(t,\widehat{x}_t,P_{x_t};x_t\right), &\partial_\mu^{P_x}\psi\left(t\right)&=\partial_\mu^{P_x}\psi\left(t,\widehat{x}_t,P_{x_t},\widehat{u}_t;x_t\right),\\
\partial_\mu^{P_x}\xi\left(t,\widehat{x}_t\right)&=\partial_\mu^{P_x}\xi\left(t,x_t,P_{x_t};\widehat{x}_t\right), &\partial_\mu^{P_x}\psi\left(t,\widehat{x}_t\right)&=\partial_\mu^{P_x}\psi\left(t,x_t,P_{x_t},u_t;\widehat{x}_t\right),
\end{aligned}
$$

Similarly, we denote for $\Psi=f,l$ and $\rho=x,y,z,\bar z,v$

$$
\begin{aligned}
\Psi(t)&=\Psi\left(t,x_t,P_{x_t},y_t,P_{y_t},z_t,P_{z_t},\bar z_t,P_{\bar z_t},u_t\right),\\
\Psi_\rho(t)&=\Psi_\rho\left(t,x_t,P_{x_t},y_t,P_{y_t},z_t,P_{z_t},\bar z_t,P_{\bar z_t},u_t\right).
\end{aligned}
$$

Finally, we denote for $\zeta = x, y, z, \bar{z}$

$$\partial_\mu^{P_\zeta} \Psi(t) = \partial_\mu^{P_\zeta} \Psi\left(t, \widehat{x}_t, P_{x_t}, \widehat{y}_t, P_{y_t}, \widehat{z}_t, P_{z_t}, \widehat{\bar{z}}_t, P_{\widehat{\bar{z}}_t}, \widehat{u}_t; \zeta\right),$$
$$\partial_\mu^{P_\zeta} \Psi\left(t, \widehat{\zeta}_t\right) = \partial_\mu^{P_\zeta} \Psi\left(t, x_t, P_{x_t}, y_t, P_{y_t}, z_t, P_{z_t}, \bar{z}_t, P_{\bar{z}_t}, u_t; \widehat{\zeta}_t\right).$$

Now, we introduce the following variational equations which is a linear FBSDEs

$$\begin{cases} dx_t^1 = [(b_x(t) - \sigma_x(t)\xi(t) - \sigma(t)\xi_x(t))\, x_t^1 + [b_v(t) - \sigma_v(t)\xi(t)]\, v_t \\ \qquad + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} b(t, \widehat{x}_t)\, \widehat{x}_t^1\right] - \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} \sigma(t, \widehat{x}_t)\, \widehat{x}_t^1\right]\xi(t) - \sigma(t)\, \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} \xi(t, \widehat{x}_t)\, \widehat{x}_t^1\right]\Big]\, dt \\ \qquad + \left[g_x(t)\, x_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} g(t, \widehat{x}_t)\, \widehat{x}_t^1\right] + g_v(t)\, v_t\right] dW_t \\ \qquad + \left[\sigma_x(t)\, x_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} \sigma(t, \widehat{x}_t)\, \widehat{x}_t^1\right] + \sigma_v(t)\, v_t\right] dY_t \\ -dy_t^1 = \left[f_x(t)\, x_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} f(t, \widehat{x}_t)\, \widehat{x}_t^1\right] + f_y(t)\, y_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_y} f(t, \widehat{y}_t)\, \widehat{y}_t^1\right]\right. \\ \qquad + f_z(t)\, z_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_z} f(t, \widehat{z}_t)\, \widehat{z}_t^1\right] + f_{\bar{z}}(t)\, \bar{z}_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_{\bar{z}}} f\left(t, \widehat{\bar{z}}_t\right) \widehat{\bar{z}}_t^1\right] \\ \qquad + f_v(t)\, v_t]\, dt - z_t^1 dW_t - \bar{z}_t^1 dY_t, \\ x_0^1 = 0, \quad y_T^1 = \varphi_x(x_T, P_{x_T})\, x_T^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} \varphi(x_T, P_{x_T}, \widehat{x}_t)\, \widehat{x}_T^1\right], \end{cases} \tag{2.7}$$

and a linear SDE

$$\begin{cases} dZ_t^1 = \left[Z_t^1 \xi(t) + Z_t \xi_x(t)\, x_t^1 + Z_t \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} \xi(t, \widehat{x}_t)\, \widehat{x}_t^1\right]\right] dY_t, \\ Z_0^1 = 0. \end{cases} \tag{2.8}$$

Set $\vartheta = Z^{-1} Z^1$, using Itô's formula, we have

$$\begin{cases} d\vartheta_t = \left[\xi_x(t)\, x_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} \xi(t, \widehat{x}_t)\, \widehat{x}_t^1\right]\right] d\widetilde{W}_t, \\ \vartheta_0 = 0. \end{cases} \tag{2.9}$$

Next, we introduce the following adjoint equations of McKean–Vlasov type

$$\begin{cases} -dp_t = \Big[b_x(t)p_t + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} b\left(t\right)\widehat{p}_t\right] - \sigma\left(t\right)\xi_x\left(t\right)p_t - \sigma\left(t\right)\widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\xi\left(t\right)\widehat{p}_t\right] \\ \qquad -\sigma_x\left(t\right)\xi\left(t\right)p_t - \xi\left(t\right)\widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\sigma\left(t\right)\widehat{p}_t\right] + g_x(t)k_t + \widehat{\mathbb{E}}\Big[\partial_\mu^{P_x} g\left(t\right)\widehat{k}_t\Big] \\ \qquad +\sigma_x(t)\bar{k}_t + \widehat{\mathbb{E}}\Big[\partial_\mu^{P_x}\sigma\left(t\right)\widehat{\bar{k}}_t\Big] + \xi_x\left(t\right)Q_t + \widehat{\mathbb{E}}\Big[\partial_\mu^{P_x}\xi\left(t\right)\widehat{Q}_t\Big] - f_x(t)q_t \\ \qquad - \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} f\left(t\right)\widehat{q}_t\right] + l_x(t) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} l\left(t\right)\right]\Big] dt - k_t dW_t - \bar{k}_t d\widetilde{W}_t, \\ dq_t = \Big[f_y(t)q_t + \widehat{\mathbb{E}}\Big[\partial_\mu^{P_y} f\left(t\right)\widehat{q}_t\Big] - l_y\left(t\right) - \widehat{\mathbb{E}}\Big[\partial_\mu^{P_y} l\left(t\right)\Big]\Big] dt \\ \qquad + \Big[f_z\left(t\right)q_t + \widehat{\mathbb{E}}\left[\partial_\mu^{P_z} f\left(t\right)\widehat{q}_t\right] - l_z\left(t\right) - \widehat{\mathbb{E}}\left[\partial_\mu^{P_z} l\left(t\right)\right]\Big] dW_t \\ \qquad + \Big[f_{\bar{z}}\left(t\right)q_t + \widehat{\mathbb{E}}\left[\partial_\mu^{P_{\bar{z}}} f\left(t\right)\widehat{q}_t\right] - \xi\left(t\right)q_t - l_{\bar{z}}\left(t\right) - \widehat{\mathbb{E}}\left[\partial_\mu^{P_{\bar{z}}} l\left(t\right)\right]\Big] d\widetilde{W}_t, \\ p_T = M_x\left(x_T, P_{x_T}\right) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} M\left(\widehat{x}_T, P_{x_T}, x_T\right)\right] \\ \qquad -\varphi_x\left(x_T, P_{x_T}\right)q_t - \widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\varphi\left(\widehat{x}_T, P_{x_T}, x_T\right)\widehat{q}_t\right], \\ q_0 = -h_y(y_0, P_{y_0}) - \widehat{\mathbb{E}}\Big[\partial_\mu^{P_y} h(\widehat{y}_0, P_{y_0}, y_0)\Big]. \end{cases} \tag{2.10}$$

It is clear that, under assumptions (**A1**) and (**A2**), there exists a unique $\left(p, k, \bar{k}, q\right) \in \mathcal{L}_{\mathcal{F}}^2\left(0, T, \mathbb{R}\right) \times \mathcal{L}_{\mathcal{F}}^2\left(0, T, \mathbb{R}\right) \times \mathcal{L}_{\mathcal{F}}^2\left(0, T, \mathbb{R}\right) \times \mathcal{L}_{\mathcal{F}}^2\left(0, T, \mathbb{R}\right)$ satisfying the FBSDE (2.10) of McKean–Vlasov type.

Remark 2.1.1 *Note that the mean-field nature of FBSDE* (2.10) *comes from the terms involving Fréchet derivatives* $\partial_\mu^{P_x} b\left(t\right), \partial_\mu^{P_x} g\left(t\right), \partial_\mu^{P_x}\sigma\left(t\right), \partial_\mu^{P_x}\xi\left(t\right)$ *and* $\left(\partial_\mu^{P_x}, \partial_\mu^{P_y}, \partial_\mu^{P_z}, \partial_\mu^{P_{\bar{z}}}\right)(f, l)$, *which will reduce to a standard BSDE if the coefficients do not explicitly depend on law of the solution.*

Now, we introduce the following BSDE involved in the stochastic maximum principle

$$\begin{cases} -dP_t = l\left(t, x_t, P_{x_t}, y_t, P_{y_t}, z_t, P_{z_t}, \bar{z}_t, P_{\bar{z}_t}, u_t\right)dt \\ \qquad -\bar{Q}_t dW_t - Q_t d\widetilde{W}_t, \\ P_T = M(x_T, P_{x_T}). \end{cases} \tag{2.11}$$

Under assumptions (**A1**) and (**A2**), it is easy to prove that BSDE (2.11) admits a unique strong solution, given by

$$P_t = M(x_T, P_{x_T}) - \int_t^T l\left(s, x_s, P_{x_s}, y_s, P_{y_s}, z_s, P_{z_s}, \bar{z}_s, P_{\bar{z}_s}, v_s\right) ds$$
$$+ \int_t^T \bar{Q}_s dW_s + \int_t^T Q_s d\widetilde{W}_s.$$

Let us now, define the Hamiltonian H associated with the McKean–Vlasov stochastic control problem (2.1)-(2.6) by

$$\begin{aligned} & H(t, x, P_x, y, P_y, z, P_z, \bar{z}, P_{\bar{z}}, v, p, q, k, \bar{k}, Q) \\ & = p\left(b\left(t, x, P_x, v\right) - \sigma\left(t, x, P_x, v\right) \xi\left(t, x, P_x\right)\right) \\ & \quad -qf\left(t, x, P_x, y, P_y, z, P_z, \bar{z}, P_{\bar{z}}, v\right) + Q\xi\left(t, x, P_x\right) \\ & \quad +kg\left(t, x, P_x, v\right) + \bar{k}\sigma\left(t, x, P_x, v\right) \\ & \quad +l\left(t, x, P_x, y, P_y, z, P_z, \bar{z}, P_{\bar{z}}, v\right). \end{aligned} \tag{2.12}$$

The main result of this paper is stated in the following section.

2.2 Necessary conditions of optimality and some estimates

Suppose that u is an optimal control with the optimal trajectory $(x, y, z, \bar{z})$ of FBSDE (2.1). For any $0 \leq \theta \leq 1$ and $v + u \in \mathcal{U}$, we define a perturbed control $u_t^\theta = u_t + \theta v_t$. Our first result below, is related to the estimate of tajectory $(x, y, z, \bar{z})$ and the observation Z_t.

Lemma 2.2.1 *Let assumptions (**A1**) and (**A2**) hold. Then, we have*

$$\lim_{\theta\to 0}\mathbb{E}\left[\sup_{0\le t\le T}\left|x_t^\theta - x_t\right|^2\right] = 0, \tag{2.13}$$

$$\lim_{\theta\to 0}\mathbb{E}\left[\sup_{0\le t\le T}\left|y_t^\theta - y_t\right|^2 + \int_0^T \left(\left|z_t^\theta - z_t\right|^2 + \left|\bar{z}_t^\theta - \bar{z}_t\right|^2\right) ds\right] = 0, \tag{2.14}$$

$$\lim_{\theta\to 0}\mathbb{E}\left[\sup_{0\le t\le T}\left|Z_t^\theta - Z_t\right|^2\right] = 0. \tag{2.15}$$

Proof. We first prove (2.13). From standard estimates and by using the Burkholder-Davis-Gundy (BDG) inequality, we get

$$\begin{aligned}\mathbb{E}\left[\sup_{0\le s\le t}\left|x_s^\theta - x_s\right|^2\right] \le{}& \mathbb{E}\int_0^t \left|b^\theta(s) - b(s)\right|^2 ds + \mathbb{E}\int_0^t \left|\sigma^\theta(s)\xi^\theta(s) - \sigma(s)\xi(s)\right|^2 ds \\ &+ \mathbb{E}\int_0^t \left|g^\theta(s) - g(s)\right|^2 ds + \mathbb{E}\int_0^t \left|\sigma^\theta(s) - \sigma(s)\right|^2 ds,\end{aligned}$$

where

$$\psi\left(s, x_s^\theta, P_{x_s^\theta}, u_s^\theta\right) = \psi^\theta(s), \text{ for } \psi = b, g, \sigma.$$

Then,

$$\begin{aligned}\mathbb{E}\left[\sup_{0\le s\le t}\left|x_s^\theta - x_s\right|^2\right] \le{}& \mathbb{E}\int_0^t \left|b^\theta(s) - b(s)\right|^2 ds + \mathbb{E}\int_0^t \left|\sigma^\theta(s)\left(\xi^\theta(s) - \xi(s)\right)\right|^2 ds \\ &+ \mathbb{E}\int_0^t \left|\xi(s)\left(\sigma^\theta(s) - \sigma(s)\right)\right|^2 ds + \mathbb{E}\int_0^t \left|g^\theta(s) - g(s)\right|^2 ds \\ &+ \mathbb{E}\int_0^t \left|\sigma^\theta(s) - \sigma(s)\right|^2 ds.\end{aligned}$$

From assumptions (**A1**) and (**A2**), we have

$$\begin{aligned}\mathbb{E}\left[\sup_{0\le t\le T}\left|x_t^\theta - x_t\right|^2\right] \le{}& C_T\mathbb{E}\int_0^t \left[\left|x_s^\theta - x_s\right|^2 + \left|\mathbb{D}_2\left(P_{x_s^\theta}, P_{x_s}\right)\right|^2\right] ds \\ &+ C_T\theta^2\mathbb{E}\int_0^t \left|v_s\right|^2 ds.\end{aligned} \tag{2.16}$$

Recall that for the 2-Wasserstein metric $\mathbb{D}_2(\cdot,\cdot)$, we obtain

$$\begin{aligned}\mathbb{D}_2\left(P_{x_s^\theta}, P_{x_s}\right) = \inf\Big\{ \left[\mathbb{E}\left|\widetilde{x}_s^\theta - \widetilde{x}_s\right|^2\right]^{\frac{1}{2}}, \text{ for all } \widetilde{x}^\varepsilon, \widetilde{x} \in \mathbb{L}^2(\mathcal{F};\mathbb{R}), \\ \text{with } P_{x_s^\theta} = P_{\widetilde{x}_s^\theta} \text{ and } P_{x_s} = P_{\widetilde{x}_s}\Big\}, \\ \leq \left[\mathbb{E}\left|x_s^\theta - x_s\right|^2\right]^{\frac{1}{2}}. \end{aligned} \tag{2.17}$$

From (2.16), (2.17), and Definition 1.6.1, we get

$$\mathbb{E}\left[\sup_{0\leq t\leq T}\left|x_t^\theta - x_t\right|^2\right] \leq C_T \mathbb{E}\int_0^t \sup_{r\in[0,s]}\left|x_r^\theta - x_r\right|^2 ds + M_T^2\theta^2.$$

Then, from Gronwall's Lemma, the result follows immediately by letting ε go to zero.

Next, we prove (2.14). By applying Itô's formula to $\left|y_t^\theta - y_t\right|^2$ and taking expectation, we get

$$\begin{aligned}&\mathbb{E}\left|y_t^\theta - y_t\right|^2 + \mathbb{E}\int_t^T \left|z_s^\theta - z_s\right|^2 ds + \mathbb{E}\int_t^T \left|\bar{z}_s^\theta - \bar{z}_s\right|^2 ds \\ &= \mathbb{E}\left|\varphi\left(x_T^\theta, P_{x_T^\theta}\right) - \varphi\left(x_T, P_{x_T}\right)\right|^2 + 2\mathbb{E}\int_t^T \left(y_s^\theta - y_s\right)\left[f^\theta(s) - f(s)\right] ds,\end{aligned}$$

where

$$f\left(s, x_s^\theta, P_{x_s^\theta}, y_s^\theta, P_{y_s^\theta}, z_s^\theta, P_{z_s^\theta}, \bar{z}_s^\theta, P_{\bar{z}_s^\theta}, u_s^\theta\right) = f^\theta(s).$$

For each $\varepsilon > 0$, and from Young's inequality, we have

$$\begin{aligned}&\mathbb{E}\left|y_t^\theta - y_t\right|^2 + \mathbb{E}\int_t^T \left|z_s^\theta - z_s\right|^2 ds + \mathbb{E}\int_t^T \left|\bar{z}_s^\theta - \bar{z}_s\right|^2 ds \\ &\leq \mathbb{E}\left|\varphi\left(x_T^\theta, P_{x_T^\theta}\right) - \varphi\left(x_T, P_{x_T}\right)\right|^2 + \frac{1}{\varepsilon}\mathbb{E}\int_t^T \left|y_s^\theta - y_s\right|^2 ds + \varepsilon\mathbb{E}\int_t^T \left|f^\theta(s) - f(s)\right|^2 ds.\end{aligned}$$

By applying the Lipschitz conditions on the coefficients φ, f with respect to x, y, z, μ and

v, we obtain

$$\begin{aligned}
&\mathbb{E}\left|y_t^\theta - y_t\right|^2 + \mathbb{E}\int_t^T \left|z_s^\theta - z_s\right|^2 ds + \mathbb{E}\int_t^T \left|\bar{z}_s^\theta - \bar{z}_s\right|^2 ds \\
&\leq \frac{1}{\varepsilon}\mathbb{E}\int_t^T \left|y_s^\theta - y_s\right|^2 ds + C\varepsilon\mathbb{E}\int_t^T \left[\left|y_s^\theta - y_s\right|^2 + \left|\mathbb{D}_2\left(P_{y_s^\theta}, P_{y_s}\right)\right|^2\right] ds \\
&+ C\varepsilon\mathbb{E}\int_t^T \left[\left|z_s^\theta - z_s\right|^2 + \left|\mathbb{D}_2\left(P_{z_s^\theta}, P_{z_s}\right)\right|^2\right] ds \\
&+ C\varepsilon\mathbb{E}\int_t^T \left[\left|\bar{z}_s^\theta - \bar{z}_s\right|^2 + \left|\mathbb{D}_2\left(P_{\bar{z}_s^\theta}, P_{\bar{z}_s}\right)\right|^2\right] ds + \alpha_t^\theta. \qquad (2.18)
\end{aligned}$$

Here α_t^θ is given by

$$\alpha_t^\theta = \mathbb{E}\left|\varphi\left(x_T^\theta, P_{x_T^\theta}\right) - \varphi\left(x_T, P_{x_T}\right)\right|^2 + C\varepsilon\mathbb{E}\left[\int_t^T \left|x_s^\theta - x_s\right|^2 + \left|\mathbb{D}_2\left(P_{x_s^\theta}, P_{x_s}\right)\right|^2\right] ds + C\varepsilon\theta^2.$$

Recall that for the 2-Wasserstein metric $\mathbb{D}_2(\cdot,\cdot)$, and by invoking (2.13) and sending θ to 0, we get $\lim_{\theta\to 0}\alpha_t^\theta = 0$. Now, we take $\varepsilon = \frac{1}{2C}$ and replacing in (2.18), we obtain

$$\begin{aligned}
&\mathbb{E}\left|y_t^\theta - y_t\right|^2 + \frac{1}{2}\mathbb{E}\int_t^T \left|z_s^\theta - z_s\right|^2 ds + \frac{1}{2}\mathbb{E}\int_t^T \left|\bar{z}_s^\theta - \bar{z}_s\right|^2 ds \\
&\leq 2C\mathbb{E}\int_t^T \left|y_s^\theta - y_s\right|^2 ds + \frac{1}{2}\mathbb{E}\int_t^T \left|y_s^\theta - y_s\right|^2 ds + \alpha_t^\theta.
\end{aligned}$$

Finally, applying Gronwall's lemma and letting θ goes to 0, we obtain the estimate (2.14).

Now, we proceed to estimate (2.15). Applying Itô's formula to $\left|Z_t^\theta - Z_t\right|^2$ and taking expectation, we get

$$\mathbb{E}\left|Z_t^\theta - Z_t\right|^2 \leq C\int_0^t \left|Z_s^\theta - Z_s\right|^2 ds + C\beta_t^\theta, \qquad (2.19)$$

where β_t^θ is given by

$$\beta_t^\theta = \mathbb{E}^u\int_0^t \left|\xi\left(s, x_s^\theta, P_{x_s^\theta}\right) - \xi\left(s, x_s, P_{x_s}\right)\right|^2 ds.$$

Also, from assumptions (**A1**) and (**A2**), we have $\lim\limits_{\theta\to 0}\beta_t^\theta = 0$.

The proof of (2.15) follows directly by using Gronwall's lemma and sending θ to 0. $\square$

Lemma 2.2.2 *Under the assumptions (**A1**) and (**A2**), the following estimations holds*

$$\lim_{\theta\to 0}\mathbb{E}\left[\sup_{0\le t\le T}\left|\widetilde{x}_t^\theta\right|^2\right] = 0, \tag{2.20}$$

$$\lim_{\theta\to 0}\mathbb{E}\left[\sup_{0\le t\le T}\left|\widetilde{y}_t^\theta\right|^2 + \int_0^T\left(\left|\widetilde{z}_t^\theta\right|^2 + \left|\widetilde{\bar{z}}_t^\theta\right|^2\right)dt\right] = 0, \tag{2.21}$$

$$\mathbb{E}\int_0^T\left|\widetilde{Z}_t^\theta\right|^2 dt = 0. \tag{2.22}$$

Proof. We start by proving the first limit. For notational ease, we introduce the following notations.

For $t\in[0,T]$, $\theta>0$, we set

$$\begin{aligned}
&\widetilde{x}_t^\theta = \theta^{-1}\left(x_t^\theta - x_t\right) - x_t^1, \quad \widetilde{y}_t^\theta = \theta^{-1}\left(y_t^\theta - y_t\right) - y_t^1,\\
&\widetilde{z}_t^\theta = \theta^{-1}\left(z_t^\theta - z_t\right) - z_t^1, \quad \widetilde{\bar{z}}_t^\theta = \theta^{-1}\left(\bar{z}_t^\theta - \bar{z}_t\right) - \bar{z}_t^1,\\
&\widetilde{Z}_t^\theta = \theta^{-1}\left(Z_t^\theta - Z_t\right) - Z_t^1.
\end{aligned}$$

We denote by

$$\begin{aligned}
&\widetilde{x}_t^{\lambda,\theta} = x_t + \lambda\theta\left(\widetilde{x}_t^\theta + x_t^1\right), \quad \widetilde{z}_t^{\lambda,\theta} = z_t + \lambda\theta\left(\widetilde{z}_t^\theta + z_t^1\right),\\
&\widetilde{y}_t^{\lambda,\theta} = y_t + \lambda\theta\left(\widetilde{y}_t^\theta + y_t^1\right), \quad \widetilde{\bar{z}}_t^{\lambda,\theta} = \bar{z}_t + \lambda\theta\left(\widetilde{\bar{z}}_t^\theta + \bar{z}_t^1\right),\\
&\gamma_t^{\lambda,\theta} = \left(\tilde{x}_t^{\lambda,\theta}, P_{\tilde{x}_t^{\lambda,\theta}}, u_t^\theta\right).
\end{aligned}$$

First, we have

$$\begin{cases}
d\tilde{x}_t^\theta = \left(\left[b_t^x - \sigma_t^x\xi_t - \sigma_t\xi_t^x\right]\widetilde{x}_t^\theta + \left[b_t^{\mu,x} - \sigma_t\xi_t^{\mu,x} - \xi_t\sigma_t^{\mu,x}\right] + \alpha_1^\theta\right)dt + \left(g_t^x\widetilde{x}_t^\theta dt + g_t^{\mu,x} + \alpha_2^\theta\right)dW_t\\
\qquad + \left(\sigma_t^x\widetilde{x}_t^\theta dt + \sigma_t^{\mu,x} + \alpha_3^\theta\right)dY_t,\\
\tilde{x}_0^\theta = 0,
\end{cases} \tag{2.23}$$

where

$$
\begin{aligned}
b_t^x &= \int_0^1 b_x\left(t,\gamma_t^{\lambda,\theta}\right)d\lambda, \quad b_t^{\mu,x} = \int_0^1 \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} b\left(t,\gamma_t^{\lambda,\theta},\widehat{\widetilde{x}_t^{\lambda,\theta}}\right)\widehat{\widetilde{x}}_t^\theta\right]d\lambda,\\
\sigma_t^x &= \int_0^1 \sigma_x\left(t,\gamma_t^{\lambda,\theta}\right)d\lambda, \quad \sigma_t^{\mu,x} = \int_0^1 \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} \sigma\left(t,\gamma_t^{\lambda,\theta},\widehat{\widetilde{x}_t^{\lambda,\theta}}\right)\widehat{\widetilde{x}}_t^\theta\right]d\lambda,\\
\xi_t^x &= \int_0^1 \xi_x\left(t,\gamma_t^{\lambda,\theta}\right)d\lambda, \quad \xi_t^{\mu,x} = \int_0^1 \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} \xi\left(t,\gamma_t^{\lambda,\theta},\widehat{\widetilde{x}_t^{\lambda,\theta}}\right)\widehat{\widetilde{x}}_t^\theta\right]d\lambda,\\
g_t^x &= \int_0^1 g_x\left(t,\gamma_t^{\lambda,\theta}\right)d\lambda, \quad g_t^{\mu,x} = \int_0^1 \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} g\left(t,\gamma_t^{\lambda,\theta},\widehat{\widetilde{x}_t^{\lambda,\theta}}\right)\widehat{\widetilde{x}}_t^\theta\right]d\lambda,
\end{aligned}
$$

and

$$
\begin{aligned}
\alpha_1^\theta &= \int_0^1 \left[b_x\left(t,\gamma_t^{\lambda,\theta}\right) - b_x(t)\right]d\lambda x_t^1\\
&\quad - \xi_t \int_0^1 \left[\sigma_x\left(t,\gamma_t^{\lambda,\theta}\right) - \sigma_x(t)\right]d\lambda x_t^1 - \sigma_t \int_0^1 \left[\xi_x\left(t,\gamma_t^{\lambda,\theta}\right) - \xi_x(t)\right]d\lambda x_t^1\\
&\quad + \int_0^1 \left[b_v\left(t,\gamma_t^{\lambda,\theta}\right) - b_v(t)\right]d\lambda v_t - \xi_t \int_0^1 \left[\sigma_v\left(t,\gamma_t^{\lambda,\theta}\right) - \sigma_v(t)\right]d\lambda v_t\\
&\quad + \int_0^1 \widehat{\mathbb{E}}\left[\left(\partial_\mu^{P_x} b\left(t,\gamma_t^{\lambda,\theta},\widehat{\widetilde{x}_t^{\lambda,\theta}}\right) - \partial_\mu^{P_x} b\left(t,\widehat{\widetilde{x}}_t\right)\right)\widehat{x}_t^1\right]d\lambda\\
&\quad - \xi_t \int_0^1 \widehat{\mathbb{E}}\left[\left(\partial_\mu^{P_x} \sigma\left(t,\gamma_t^{\lambda,\theta},\widehat{\widetilde{x}_t^{\lambda,\theta}}\right) - \partial_\mu^{P_x} \sigma\left(t,\widehat{\widetilde{x}}_t\right)\right)\widehat{x}_t^1\right]d\lambda\\
&\quad - \sigma_t \int_0^1 \widehat{\mathbb{E}}\left[\left(\partial_\mu^{P_x} \xi\left(t,\gamma_t^{\lambda,\theta},\widehat{\widetilde{x}_t^{\lambda,\theta}}\right) - \partial_\mu^{P_x} \xi\left(t,\widehat{\widetilde{x}}_t\right)\right)\widehat{x}_t^1\right]d\lambda,
\end{aligned}
$$

$$
\begin{aligned}
\alpha_2^\theta &= \int_0^1 \left[g_x\left(t,\gamma_t^{\lambda,\theta}\right) - g_x(t)\right]d\lambda x_t^1 + \int_0^1 \left[g_v\left(t,\gamma_t^{\lambda,\theta}\right) - g_v(t)\right]d\lambda v_t\\
&\quad + \int_0^1 \widehat{\mathbb{E}}\left[\left(\partial_\mu^{P_x} g\left(t,\gamma_t^{\lambda,\theta},\widehat{\widetilde{x}_t^{\lambda,\theta}}\right) - \partial_\mu^{P_x} g\left(t,\widehat{\widetilde{x}}_t\right)\right)\widehat{x}_t^1\right]d\lambda,\\
\alpha_3^\theta &= \int_0^1 \left[\sigma_x\left(t,\gamma_t^{\lambda,\theta}\right) - \sigma_x(t)\right]d\lambda x_t^1 + \int_0^1 \left[\sigma_v\left(t,\gamma_t^{\lambda,\theta}\right) - \sigma_v(t)\right]d\lambda v_t\\
&\quad + \int_0^1 \widehat{\mathbb{E}}\left[\left(\partial_\mu^{P_x} \sigma\left(t,\gamma_t^{\lambda,\theta},\widehat{\widetilde{x}_t^{\lambda,\theta}}\right) - \partial_\mu^{P_x} \sigma\left(t,\widehat{\widetilde{x}}_t\right)\right)\widehat{x}_t^1\right]d\lambda.
\end{aligned}
$$

Noting that under assumptions (**A1**) and (**A2**), we get

$$
\lim_{\theta\to 0} \mathbb{E}\left[\left|\alpha_1^\theta\right|^2 + \left|\alpha_2^\theta\right|^2 + \left|\alpha_3^\theta\right|^2\right] = 0.
$$

Applying Itô's formula to $\left|\widetilde{x}_t^\theta\right|^2$, we have

$$
\begin{aligned}
\mathbb{E}\left|\widetilde{x}_t^\theta\right|^2 &= 2\mathbb{E}\int_0^T \widetilde{x}_t^\theta\left(\left[b_t^x - \sigma_t^x\xi_t - \sigma_t\xi_t^x\right]\widetilde{x}_t^\theta + \left[b_t^{\mu,x} - \sigma_t\xi_t^{\mu,x} - \xi_t\sigma_t^{\mu,x}\right] + \alpha_1^\theta\right)dt \\
&\quad + \mathbb{E}\int_0^T \left|g_t^x\widetilde{x}_t^\theta + g_t^{\mu,x} + \alpha_2^\theta\right|^2 dt + \mathbb{E}\int_0^T \left|\sigma_t^x\widetilde{x}_t^\theta + \sigma_t^{\mu,x} + \alpha_3^\theta\right|^2 dt \\
&\leq C\mathbb{E}\int_0^T \left|\widetilde{x}_t^\theta\right|^2 dt + \int_0^T \mathbb{E}\left[\left|\alpha_1^\theta\right|^2 + \left|\alpha_2^\theta\right|^2 + \left|\alpha_3^\theta\right|^2\right]dt.
\end{aligned}
$$

Finally, estimate (2.20) now follows easily from the Gronwall inequality.

Let $\left(\widetilde{y}_t^\theta, \widetilde{z}_t^\theta, \widetilde{\bar{z}}_t^\theta\right)$ be the solution of the following BSDE

$$
\begin{cases}
d\widetilde{y}_t^\theta = \left[f_t^x\widetilde{x}_t^\theta + f_t^{\mu,x} + f_t^y\widetilde{y}_t^\theta + f_t^{\mu,y} + f_t^z\widetilde{z}_t^\theta + f_t^{\mu,z} + f_t^{\bar{z}}\widetilde{\bar{z}}_t^\theta + f_t^{\mu,\bar{z}} + \Upsilon_t^\theta\right]dt \\
\qquad\qquad + \widetilde{z}_t^\theta dW_t + \widetilde{\bar{z}}_t^\theta dY_t, \\
\widetilde{y}_T^\theta = \theta^{-1}\left[\varphi\left(x_T^\theta, P_{x_T^\theta}\right) - \varphi\left(x_T, P_{x_T}\right)\right] - \varphi_x\left(x_T, P_{x_T}\right)x_T^1 - \widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\varphi\left(x_T, P_{x_T}, \widehat{x}_T\right)\widehat{x}_T^1\right],
\end{cases}
$$

where $\tilde{x}_t^\theta$ satisfies SDE (2.23), and

$$
\begin{aligned}
f_t^\rho &= -\int_0^1 f_\rho\left(t, \chi_t^{\lambda,\theta}\right)d\lambda, \text{ for } \rho = x, y, z, \bar{z}, \\
\chi_t^{\lambda,\theta} &= \left(\widetilde{x}_t^{\lambda,\theta}, P_{\widetilde{x}_t^{\lambda,\theta}}, \widetilde{y}_t^{\lambda,\theta}, P_{\widetilde{y}_t^{\lambda,\theta}}, \widetilde{z}_t^{\lambda,\theta}, P_{\widetilde{z}_t^{\lambda,\theta}}, \widetilde{\bar{z}}_t^{\lambda,\theta}, P_{\widetilde{\bar{z}}_t^{\lambda,\theta}}, u_t^{\lambda,\theta}\right), \\
f_t^{\mu,\rho} &= -\int_0^1 \widehat{\mathbb{E}}\left[\partial_\mu^{P_\rho} f\left(t, \chi_t^{\lambda,\theta}, \widehat{\widetilde{\rho}_t^{\lambda,\theta}}\right)\widehat{\widetilde{\rho}}_t^\theta\right]d\lambda, \text{ for } \rho = x, y, z, \bar{z},
\end{aligned}
$$

and Υ_t^θ is given by

$$\begin{aligned}\Upsilon_t^\theta &= \int_0^1 \left[f_x\left(t,\chi_t^{\lambda,\theta}\right) - f_x(t)\right] d\lambda x_t^1 + \int_0^1 \widehat{\mathbb{E}}\left[\left(\partial_\mu^{P_x} f\left(t,\chi_t^{\lambda,\theta},\widehat{\widetilde{x}_t^{\lambda,\theta}}\right) - \partial_\mu^{P_x} f\left(t,\chi_t,\widehat{\widetilde{x}}_t\right)\right)\widehat{x}_t^1\right] d\lambda \\
&+ \int_0^1 \left[f_y\left(t,\chi_t^{\lambda,\theta}\right) - f_y(t)\right] d\lambda y_t^1 + \int_0^1 \widehat{\mathbb{E}}\left[\left(\partial_\mu^{P_y} f\left(t,\chi_t^{\lambda,\theta},\widehat{\widetilde{y}_t^{\lambda,\theta}}\right) - \partial_\mu^{P_y} f\left(t,\chi_t,\widehat{\widetilde{y}}_t\right)\right)\widehat{y}_t^1\right] d\lambda \\
&+ \int_0^1 \left[f_z\left(t,\chi_t^{\lambda,\theta}\right) - f_z(t)\right] d\lambda z_t^1 + \int_0^1 \widehat{\mathbb{E}}\left[\left(\partial_\mu^{P_z} f\left(t,\chi_t^{\lambda,\theta},\widehat{\widetilde{z}_t^{\lambda,\theta}}\right) - \partial_\mu^{P_z} f\left(t,\chi_t,\widehat{\widetilde{z}}_t\right)\right)\widehat{z}_t^1\right] d\lambda \\
&+ \int_0^1 \left[f_{\bar{z}}\left(t,\chi_t^{\lambda,\theta}\right) - f_{\bar{z}}(t)\right] d\lambda \bar{z}_t^1 + \int_0^1 \widehat{\mathbb{E}}\left[\left(\partial_\mu^{P_{\bar{z}}} f\left(t,\chi_t^{\lambda,\theta},\widehat{\widetilde{\bar{z}}_t^{\lambda,\theta}}\right) - \partial_\mu^{P_x} f\left(t,\chi_t,\widehat{\widetilde{\bar{z}}}_t\right)\right)\widehat{\bar{z}}_t^1\right] d\lambda \\
&+ \int_0^1 \left[f_v\left(t,\chi_t^{\lambda,\theta}\right) - f_v(t)\right] d\lambda v_t.\end{aligned}$$

Due the fact that $f_t^x, f_t^{\mu,x}, f_t^y, f_t^{\mu,y}, f_t^z, f_t^{\mu,z}, f_t^{\bar{z}}$ and $f_t^{\mu,\bar{z}}$ are continuous, we have

$$\lim_{\theta\to 0}\mathbb{E}\left|\Upsilon_t^\theta\right|^2 = 0. \tag{2.24}$$

Appying Itô's formula to $\left|\widetilde{y}_t^\theta\right|^2$, we have

$$\begin{aligned}&\mathbb{E}\left|\widetilde{y}_t^\theta\right|^2 + \mathbb{E}\int_t^T \left|\widetilde{z}_s^\theta\right|^2 ds + \mathbb{E}\int_t^T \left|\widetilde{\bar{z}}_s^\theta\right|^2 ds \\
&= \mathbb{E}\left|\widetilde{y}_T^\theta\right|^2 + 2\mathbb{E}\int_t^T \widetilde{y}_s^\theta\left(f_s^x\widetilde{x}_s^\theta + f_s^{\mu,x} + f_s^y\widetilde{y}_s^\theta + f_s^{\mu,y} + f_s^z\widetilde{z}_s^\theta + f_s^{\mu,z} + f_s^{\bar{z}}\widetilde{\bar{z}}_s^\theta + f_s^{\mu,\bar{z}} + \Upsilon_s^\theta\right) ds.\end{aligned}$$

By Young's inequality, for each $\varepsilon > 0$, we get

$$\begin{aligned}&\mathbb{E}\left|\widetilde{y}_t^\theta\right|^2 + \mathbb{E}\int_t^T \left|\widetilde{z}_s^\theta\right|^2 ds + \mathbb{E}\int_t^T \left|\widetilde{\bar{z}}_s^\theta\right|^2 ds \\
&\leq \mathbb{E}\left|\widetilde{y}_T^\theta\right|^2 + \frac{1}{\varepsilon}\mathbb{E}\int_t^T \left|\widetilde{y}_s^\theta\right|^2 ds \\
&+ \varepsilon\mathbb{E}\int_t^T \left|\left(f_s^x\widetilde{x}_s^\theta + f_s^{\mu,x} + f_s^y\widetilde{y}_s^\theta + f_s^{\mu,y} + f_s^z\widetilde{z}_s^\theta + f_s^{\mu,z} + f_s^{\bar{z}}\widetilde{\bar{z}}_s^\theta + f_s^{\mu,\bar{z}} + \Upsilon_s^\theta\right)\right|^2 ds\end{aligned}$$

$$
\begin{aligned}
&\leq \mathbb{E}\left|\widetilde{y}_T^\theta\right|^2 + \frac{1}{\varepsilon}\mathbb{E}\int_t^T \left|\widetilde{y}_s^\theta\right|^2 ds + C_\varepsilon \mathbb{E}\int_t^T \left|f_s^x \widetilde{x}_s^\theta\right|^2 ds + C_\varepsilon \mathbb{E}\int_t^T \left|f_s^{\mu,x}\right|^2 ds \\
&+ C_\varepsilon \mathbb{E}\int_t^T \left|f_s^y \widetilde{y}_s^\theta\right|^2 ds + C_\varepsilon \mathbb{E}\int_t^T \left|f_s^{\mu,y}\right|^2 ds + C_\varepsilon \mathbb{E}\int_t^T \left|f_s^z \widetilde{z}_s^\theta\right|^2 ds + C_\varepsilon \mathbb{E}\int_t^T \left|f_s^{\mu,z}\right|^2 ds \\
&+ C_\varepsilon \mathbb{E}\int_t^T \left|f_s^{\bar{z}} \widetilde{\bar{z}}_s^\theta\right|^2 ds + C_\varepsilon \mathbb{E}\int_t^T \left|f_s^{\mu,\bar{z}}\right|^2 ds.
\end{aligned}
$$

By the boundedness of $f_t^x, f_t^{\mu,x}, f_t^y, f_t^{\mu,y}, f_t^z, f_t^{\mu,z}, f_t^{\bar{z}}$ and $f_t^{\mu,\bar{z}}$, we obtain

$$
\begin{aligned}
&\mathbb{E}\left|\widetilde{y}_t^\theta\right|^2 + \mathbb{E}\int_t^T \left|\widetilde{z}_s^\theta\right|^2 ds + \mathbb{E}\int_t^T \left|\widetilde{\bar{z}}_s^\theta\right|^2 ds \\
&\leq \left(\frac{1}{\varepsilon} + C_\varepsilon\right)\mathbb{E}\int_t^T \left|\widetilde{y}_s^\theta\right|^2 ds + C_\varepsilon \mathbb{E}\int_t^T \left|\widetilde{z}_s^\theta\right|^2 ds + C_\varepsilon \mathbb{E}\int_t^T \left|\widetilde{\bar{z}}_s^\theta\right|^2 ds \\
&+ \mathbb{E}\left|\widetilde{y}_T^\theta\right|^2 + C_\varepsilon \mathbb{E}\int_t^T \left|f_s^x \widetilde{x}_s^\theta\right|^2 ds + C_\varepsilon \mathbb{E}\int_t^T \left|\Upsilon_s^\theta\right|^2 ds.
\end{aligned}
$$

Hence, in view of (2.20), (2.24), the fact that $f_t^x, f_t^{\mu,x}$ are continuous and bounded, by Gronwall's inequality, we obtain (2.21).

Now, we proceed to prove (2.22). It is plain to check that $\widetilde{Z}_t^\theta$ satisfies the following equality

$$
d\widetilde{Z}_t^\theta = \left[\widetilde{Z}_t^\theta \xi\left(t, x_t^\theta, P_{x_t^\theta}\right) + \bar{\Upsilon}_t^\theta\right] dY_t + Z_t\left[\xi_t^x \widetilde{x}_t^\theta + \xi_t^{\mu,x}\right] dY_t,
$$

where

$$
\begin{aligned}
\xi_t^x &= \int_0^1 \xi_x\left(t, \widetilde{x}_t^{\lambda,\theta}, P_{\widetilde{x}_t^{\lambda,\theta}}\right) d\lambda, \\
\xi_t^{\mu,x} &= \int_0^1 \widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\xi\left(t, \widetilde{x}_t^{\lambda,\theta}, P_{\widetilde{x}_t^{\lambda,\theta}}, \widehat{\widetilde{x}_t^{\lambda,\theta}}\right)\widehat{\widetilde{x}}_t^\theta\right] d\lambda,
\end{aligned}
$$

and $\bar{\Upsilon}_t^\theta$ is given by

$$\begin{aligned}
\bar{\Upsilon}_t^\theta &= Z_t \int_0^1 \left[\xi_x \left(t, \widetilde{x}_t^{\lambda,\theta}, P_{\widetilde{x}_t^{\lambda,\theta}}\right) - \xi_x\left(t\right)\right] d\lambda x_t^1 \\
&+ Z_t \int_0^1 \widehat{\mathbb{E}}\left[\left(\partial_\mu^{P_x}\xi\left(t, \widetilde{x}_t^{\lambda,\theta}, P_{\widetilde{x}_t^{\lambda,\theta}}, \widehat{\widetilde{x}_t^{\lambda,\theta}}\right) - \partial_\mu^{P_x}\xi\left(t, \widetilde{x}_t, P_{\widetilde{x}_t}, \widehat{\widetilde{x}}_t\right)\right)\widehat{x}_t^1\right] d\lambda \\
&+ Z_t^1 \left[\xi\left(t, x_t^\theta, P_{x_t^\theta}\right) - \xi\left(t\right)\right].
\end{aligned}$$

Taking into account the fact that ξ_t^x and $\xi_t^{\mu,x}$ are continuous, we deduce

$$\lim_{\theta\to 0}\mathbb{E}\left|\bar{\Upsilon}_t^\theta\right|^2 = 0. \tag{2.25}$$

Then, applying Itô's formula to $\left|\widetilde{Z}_t^\theta\right|^2$ and taking expectation, we have

$$\mathbb{E}\left|\widetilde{Z}_t^\theta\right|^2 \leq C\mathbb{E}\int_0^T \left|\widetilde{Z}_t^\theta\right|^2 dt + C\mathbb{E}\int_0^T \left|\widetilde{x}_t^\theta\right|^2 dt + C\mathbb{E}\int_0^T \left|\xi_t^{\mu,x}\right|^2 dt + C\mathbb{E}\int_0^T \left|\bar{\Upsilon}_t^\theta\right|^2 dt.$$

Finally, by Gronwall's inequality, estimates (2.20) and recall to the Wasserstein metric, the above convergence result (2.22) holds.

□

Since u is an optimal control, then, we have the following lemma.

Lemma 2.2.3 *Let assumptions (**A1**) and (**A2**) hold. Then, we have the following variational inequality*

$$\begin{aligned}
0 &\leq \mathbb{E}\left[Z_T M_x\left(x_T, P_{x_T}\right)x_T^1 + Z_T\widehat{\mathbb{E}}\left[\partial_\mu^{P_x} M\left(x_T, P_{x_T}, \widehat{x}_T\right)\widehat{x}_T^1\right]\right] \\
&+ \mathbb{E}\left[Z_T^1 M\left(x_T, P_{x_T}\right) + h_y\left(y_0, P_{y_0}\right)y_0^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_y} h(y_0, P_{y_0}, \widehat{y}_0)\widehat{y}_0^1\right]\right] \\
&+ \mathbb{E}\int_0^T \left[Z_t^1 l(t) + Z_t\left(l_x(t)x_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} l\left(t, \widehat{x}_t\right)\widehat{x}_t^1\right]\right) + Z_t\left(l_y(t)y_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_y} l\left(t, \widehat{y}_t\right)\widehat{y}_t^1\right]\right)\right. \\
&+ \left. Z_t\left(l_z(t)z_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_z} l\left(t, \widehat{z}_t\right)\widehat{z}_t^1\right]\right) + Z_t\left(l_{\bar{z}}(t)\bar{z}_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_{\bar{z}}} l\left(t, \widehat{\bar{z}}_t\right)\widehat{\bar{z}}_t^1\right]\right) + Z_t l_v(t)v_t\right] dt.
\end{aligned} \tag{2.26}$$

Proof. Using Lemmas 3.2 and Taylor expansion, we have

$$
\begin{aligned}
0 &\leq \frac{1}{\theta}\left[J\left(u_t^\theta\right)-J\left(u_t\right)\right] \\
&=\frac{1}{\theta}\mathbb{E}\left[Z_T^\theta M\left(x_T^\theta, P_{x_T^\theta}\right)-Z_T M\left(x_T, P_{x_T}\right)\right] \\
&+\frac{1}{\theta}\mathbb{E}\left[h\left(y_0^\theta\right)-h\left(y_0\right)\right] \\
&+\frac{1}{\theta}\mathbb{E}\int_0^T\left[Z_t^\theta l^\theta(t)-Z_t l(t)\right]dt \\
&=I_1+I_2+I_3,
\end{aligned}
$$

where $l^\theta(t)=l\left(t, x_t^\theta, P_{x_t^\theta}, y_t^\theta, P_{y_t^\theta}, z_t^\theta, P_{z_t^\theta}, \bar{z}_t^\theta, P_{\bar{z}_t^\theta}, u_t^\theta\right)$.

Then, from the results of (2.20), (2.21) and (2.22), we derive

$$
\begin{aligned}
I_1 &= \frac{1}{\theta}\mathbb{E}\left[Z_T^\theta M\left(x_T^\theta, P_{x_T^\theta}\right)-Z_T M\left(x_T, P_{x_T}\right)\right] \\
&=\frac{1}{\theta}\mathbb{E}\left[\left(Z_T^\theta-Z_T\right)M\left(x_T^\theta, P_{x_T^\theta}\right)\right] \\
&+\frac{1}{\theta}\mathbb{E}\left[Z_T\int_0^1 M_x\left(x_T+\lambda\left(x_T^\theta-x_T\right), P_{x_T+\lambda\left(\widehat{x}_T^\theta-\widehat{x}_T\right)}\right)\left(x_T^\theta-x_T\right)d\lambda\right] \\
&+\frac{1}{\theta}\mathbb{E}\left[Z_T\int_0^1 \widehat{\mathbb{E}}\left[\partial_\mu^{P_x}M\left(x_T+\lambda\left(\widehat{x}_T^\theta-\widehat{x}_T\right), P_{x_T+\lambda\left(\widehat{x}_T^\theta-\widehat{x}_T\right)}, \widehat{x}_T\right)\left(\widehat{x}_T^\theta-\widehat{x}_T\right)\right]d\lambda\right] \\
&\longrightarrow \mathbb{E}^u\left[\vartheta_T M\left(x_T, P_{x_T}\right)\right]+\mathbb{E}^u\left[\left(M_x\left(x_T, P_{x_T}\right)\right)x_T^1+\widehat{\mathbb{E}}\left[\partial_\mu^{P_x}M\left(x_T, P_{x_T}, \widehat{x}_T\right)\widehat{x}_T^1\right]\right].
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
I_2 &= \frac{1}{\theta}\mathbb{E}\left[h\left(y_0^\theta, P_{y_0^\theta}\right)-h\left(y_0, P_{y_0}\right)\right] \\
&=\frac{1}{\theta}\mathbb{E}\left[\int_0^1 h_y\left(y_0+\lambda\left(y_0^\theta-y_0\right), P_{y_0+\lambda\left(\widehat{y}_0^\theta-\widehat{y}_0\right)}\right)\left(y_0^\theta-y_0\right)d\lambda\right] \\
&+\frac{1}{\theta}\mathbb{E}\left[\int_0^1 \widehat{\mathbb{E}}\left[\partial_\mu^{P_y}h\left(y_0+\lambda\left(\widehat{y}_0^\theta-\widehat{y}_0\right), P_{y_0+\lambda\left(\widehat{y}_0^\theta-\widehat{y}_0\right)}, \widehat{y}_0\right)\left(\widehat{y}_0^\theta-\widehat{y}_0\right)\right]d\lambda\right] \\
&\longrightarrow \mathbb{E}^u\left[\left(h_y\left(y_0, P_{y_0}\right)\right)y_0^1+\widehat{\mathbb{E}}\left[\partial_\mu^{P_y}h(y_0, P_{y_0}, \widehat{y}_0)\widehat{y}_0^1\right]\right],
\end{aligned}
$$

and

$$I_3 = \frac{1}{\theta}\mathbb{E}\left[\int_0^T \left(Z_t^\theta l^\theta (t) - Z_t l(t)\right) dt\right]$$
$$\longrightarrow \mathbb{E}^u \int_0^T \left[\vartheta_t l(t) + l_x(t)x_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} l\left(t, \widehat{x}_t\right)\widehat{x}_t^1\right] + l_y(t)y_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_y} l\left(t, \widehat{y}_t\right)\widehat{y}_t^1\right]\right.$$
$$\left. + l_z(t)z_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_z} l\left(t, \widehat{z}_t\right)\widehat{z}_t^1\right] + l_{\bar{z}}(t)\bar{z}_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_{\bar{z}}} l\left(t, \widehat{\bar{z}}_t\right)\widehat{\bar{z}}_t^1\right] + l_v(t)v_t\right] dt.$$

Then, the variational inequality (2.26) can be rewritten as

$$\begin{aligned} 0 \leq\ & \mathbb{E}^u\left[M_x\left(x_T, P_{x_T}\right)x^1(T) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} M\left(x_T, P_{x_T}, \widehat{x}_T\right)\widehat{x}_T^1\right]\right] \\ & + \mathbb{E}^u\left[\vartheta_T M\left(x_T, P_{x_T}\right) + h_y\left(y_0, P_{y_0}\right)y^1(0) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_y} h(y_0, P_{y_0}, \widehat{y}_0)\widehat{y}_0^1\right]\right] \\ & + \mathbb{E}^u\int_0^T \left[\vartheta_t l(t) + l_x(t)x_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} l\left(t, \widehat{x}_t\right)\widehat{x}_t^1\right] + l_y(t)y_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_y} l\left(t, \widehat{y}_t\right)\widehat{y}_t^1\right]\right. \\ & \left. + l_z(t)z_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_z} l\left(t, \widehat{z}_t\right)\widehat{z}_t^1\right] + l_{\bar{z}}(t)\bar{z}_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_{\bar{z}}} l\left(t, \widehat{\bar{z}}_t\right)\widehat{\bar{z}}_t^1\right] + l_v(t)v_t\right] dt. \end{aligned} \tag{2.27}$$

□

Theorem 2.2.1 *(Partial necessary conditions of optimality) Let assumptions (**A1**) and (**A2**) hold. Let $(x, y, z, \bar{z}, u)$ be an optimal solution of our partially observed optimal control problem. Then, there are $(p, q, k, \bar{k})$ and $(P, \bar{Q}, Q)$ of $\mathbb{F}-$adapted processes that satisfy* (2.10), (2.11) *respectively, and that for all $v \in \mathcal{U}$, we have*

$$\mathbb{E}^u\left[H_v(t)(v_t - u_t)/\mathcal{F}_t^Y\right] \geq 0, a.e, a.s, \tag{2.28}$$

where the Hamiltonian function

$$H(t) = H\left(t, x_t, P_{x_t}, y_t, P_{y_t}, z_t, P_{z_t}, \bar{z}_t, P_{\bar{z}_t}, u_t, p_t, q_t, k_t, \bar{k}_t, Q_t\right),$$

is defined by (2.12).

Proof. Applying Itô's formula to $p_t x_t^1$ and $q_t y_t^1$ such that,

$$\begin{aligned} q_0 &= -h_y(y_0, P_{y_0}) - \widehat{\mathbb{E}}\left[\partial_\mu^{P_y} h(\widehat{y}_0, P_{y_0}, y_0)\right], \\ p_T &= M_x\left(x_T, P_{x_T}\right) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} M\left(\widehat{x}_T, P_{x_T}, x_T\right)\right] \\ &\quad -\varphi_x\left(x_T, P_{x_T}\right) q_T - \widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\varphi\left(\widehat{x}_T, P_{x_T}, x_T\right)\widehat{q}_T\right], \end{aligned}$$

and using Fubini's theorem, we get

$$\begin{aligned} \mathbb{E}^u\left[p_T x_T^1\right] &= \mathbb{E}^u \int_0^T \left[p_t\left(b_v(t) - \sigma_v(t)\xi(t)\right)v_t + \bar{k}_t \sigma_v(t) v_t + k_t g_v(t) v_t\right] dt \\ &+ \mathbb{E}^u \int_0^T x_t^1 \left[f_x(t) q_t + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} f(t)\widehat{q}_t\right] - l_x(t) - \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} l(t)\right]\right] dt \\ &- \mathbb{E}^u \int_0^T x_t^1 \left[\xi_x(t) Q_t + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\xi(t)\widehat{Q}_t\right]\right] dt, \end{aligned} \tag{2.29}$$

and

$$\begin{aligned} &\mathbb{E}^u\left[q_T y_T^1\right] + \mathbb{E}^u\left[h_y(y_0, P_{y_0}) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_y} h(\widehat{y}_0, P_{y_0}, y_0)\right]\right] \\ &= -\mathbb{E}^u \int_0^T q_t\left[f_v(t) v_t + f_x(t) x_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} f(t, \widehat{x}_t)\widehat{x}_t^1\right]\right] dt \\ &- \mathbb{E}^u \int_0^T y_t^1\left[l_y(t) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_y} l(t)\right]\right] dt - \mathbb{E}^u \int_0^T z_t^1\left[l_z(t) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_z} l(t)\right]\right] dt \\ &- \mathbb{E}^u \int_0^T \bar{z}_t^1\left[l_{\bar{z}}(t) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_{\bar{z}}} l(t)\right]\right] dt. \end{aligned} \tag{2.30}$$

Now, applying Itô's formula to $\vartheta_t P_t$ and using also Fubini's theorem, we have

$$\begin{aligned} \mathbb{E}^u\left[\vartheta_T M(x_T)\right] &= -\mathbb{E}^u \int_0^T \vartheta_t l(t)\, dt \\ &\quad + \mathbb{E}^u \int_0^T Q_t\left[\xi_x(t) x_t^1 + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\xi(t, \widehat{x}_t)\widehat{x}_t^1\right]\right] dt. \end{aligned} \tag{2.31}$$

From Eqs. (2.29), (2.30), and (2.31), we obtain

$$
\begin{aligned}
&\mathbb{E}^u\left[M_x\left(x_T, P_{x_T}\right)+\widehat{\mathbb{E}}\left[\partial_\mu^{P_x} M\left(x_T, P_{x_T}\right)\right]\right] \\
&+\mathbb{E}^u\left[h_y(y_0, P_{y_0})+\widehat{\mathbb{E}}\left[\partial_\mu^{P_y} h(\widehat{y}_0, P_{y_0}, y_0)\right]+\vartheta_T M(x_T)\right] \\
&=\mathbb{E}^u \int_0^T\left[p_t\left[b_v\left(t\right)-\sigma_v \xi(t)\right] v_t+\bar{k}_t \sigma_v\left(t\right) v_t+k_t g_v\left(t\right) v_t+\int_{\Theta} n_t\left(e\right) c_v\left(t, e\right) v_t-q_t f_v\left(t\right) v_t\right] dt \\
&-\mathbb{E}^u \int_0^T \vartheta_t l\left(t\right) dt-\mathbb{E}^u \int_0^T x_t^1\left[l_x(t)+\widehat{\mathbb{E}}\left[\partial_\mu^{P_x} l\left(t\right)\right]\right] dt \\
&-\mathbb{E}^u \int_0^T y_t^1\left[l_y\left(t\right)+\widehat{\mathbb{E}}\left[\partial_\mu^{P_y} l\left(t\right)\right]\right] dt-\mathbb{E}^u \int_0^T z_t^1\left[l_z\left(t\right)+\widehat{\mathbb{E}}\left[\partial_\mu^{P_z} l\left(t\right)\right]\right] dt \\
&-\mathbb{E}^u \int_0^T \bar{z}_t^1\left[l_{\bar{z}}\left(t\right)+\widehat{\mathbb{E}}\left[\partial_\mu^{P_{\bar{z}}} l\left(t\right)\right]\right] dt,
\end{aligned} \tag{2.32}
$$

thus

$$
\begin{aligned}
&\mathbb{E}^u\left[M_x\left(x_T, P_{x_T}\right)+\widehat{\mathbb{E}}\left[\partial_\mu^{P_x} M\left(x_T, P_{x_T}\right)\right]\right] \\
&+\mathbb{E}^u\left[h_y(y_0, P_{y_0})+\widehat{\mathbb{E}}\left[\partial_\mu^{P_y} h(\widehat{y}_0, P_{y_0}, y_0)\right]+\vartheta_T M(x_T)\right] \\
&=\mathbb{E}^u \int_0^T H_v\left(t\right) v_t-\mathbb{E}^u \int_0^T l_v(t) v_t dt-\mathbb{E}^u \int_0^T \vartheta_t l\left(t\right) dt-\mathbb{E}^u \int_0^T x_t^1\left[l_x(t)+\widehat{\mathbb{E}}\left[\partial_\mu^{P_x} l\left(t\right)\right]\right] dt \\
&-\mathbb{E}^u \int_0^T y_t^1\left[l_y\left(t\right)+\widehat{\mathbb{E}}\left[\partial_\mu^{P_y} l\left(t\right)\right]\right] dt-\mathbb{E}^u \int_0^T z_t^1\left[l_z\left(t\right)+\widehat{\mathbb{E}}\left[\partial_\mu^{P_z} l\left(t\right)\right]\right] dt \\
&-\mathbb{E}^u \int_0^T \bar{z}_t^1\left[l_{\bar{z}}\left(t\right)+\widehat{\mathbb{E}}\left[\partial_\mu^{P_{\bar{z}}} l\left(t\right)\right]\right] dt.
\end{aligned}
$$

This together with the variational inequality (2.27) imply (2.28), the proof is then completed. □

2.3 Sufficient conditions of optimality

In what follows, we will study that, under some additional convexity conditions, the above necessary condition of partially observed optimal control in Theorem 3.4 is also sufficient.

A function $\phi : \mathbb{R} \times Q_2(\mathbb{R}) \rightarrow \mathbb{R}$ is convex if, for every $(x^u, P_x^u), (x^v, P_x^v) \in \mathbb{R} \times Q_2(\mathbb{R})$,

$$\phi(x^v, P_x^v) - \phi(x^u, P_x^u) \geq \phi_x(x^u, P_x^u)(x^v - x^u) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\phi(x^u, P_x^u)(x^v - x^u)\right].$$

For this, we need an additional assumption condition (**A3**) as follows:

Assumption (**A3**)

1. The functions M, h are convex in (x, P_x) and (y, P_y) respectively.
2. $H\left(t, \cdot, \cdot, \cdot, \cdot, \cdot, \cdot, \cdot, \cdot, \cdot, p_t^u, q_t^u, k_t^u, \bar{k}_t^u, Q_t^u\right)$ is convex in $(x^u, P_x^u, y^u, P_{y^u}, z^u, P_{z^u}, \bar{z}^u, P_{\bar{z}^u}, u)$ for *a.e.* $t \in [0, T]$, $P - a.s.$

$$\begin{aligned}
H^v(t) - H^u(t) \geq\ & H_x^u(t)(x^v - x^u) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} H^u(t)(\widehat{x}^v - \widehat{x}^u)\right] \\
& + H_y^u(t)(y^v - y^u) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_y} H^u(t)(\widehat{y}^v - \widehat{y}^u)\right] \\
& + H_z^u(t)(z^v - z^u) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_z} H^u(t)(\widehat{z}^v - \widehat{z}^u)\right] \\
& + H_{\bar{z}}^u(t)(\bar{z}^v - \bar{z}^u) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_{\bar{z}}} H^u(t)\left(\widehat{\bar{z}}^v - \widehat{\bar{z}}^u\right)\right],
\end{aligned}$$

where

$$\begin{aligned}
H^v(t) &= H\left(t, x^v, P_x^v, y^v, P_{y^v}, z^v, P_{z^v}, \bar{z}^v, P_{\bar{z}^v}, v, p^u, q^u, k^u, \bar{k}^u, Q^u\right), \\
H^u(t) &= H\left(t, x^u, P_x^u, y^u, P_{y^u}, z^u, P_{z^u}, \bar{z}^u, P_{\bar{z}^u}, u, p^u, q^u, k^u, \bar{k}^u, Q^u\right).
\end{aligned}$$

Now, we can prove the sufficient conditions of optimality for our control problem of McKean–Vlasov FBSDEs with jumps, which is the third main result of this paper.

Theorem 2.3.1 *(Partial sufficient conditions of optimality) Suppose (**A1**),(**A2**) and (**A3**) hold. Let Z^v be $\mathcal{F}_t^Y$-adapted, $u \in \mathcal{U}$ be an admissible control, and $(x, y, z, \bar{z})$ be the corresponding trajectories. Let $\left(p, k, \bar{k}, q\right)$ and $\left(P, Q, \bar{Q}\right)$ satisfy* (2.10) *and* (2.11), *respectively.*

Moreover, the Hamiltonian H is convex in $(x, P_x, y, P_y, , z, P_z, \bar{z}, P_{\bar{z}}, v)$, and

$$\mathbb{E}^u \left[H_v(t)(v_t - u_t) / \mathcal{F}_t^Y \right] \geq 0, a.e, a.s, .$$

Then u is a partial observed optimal control for the problem $(2.1) - (2.6)$ subject to (2.4).

Proof. For any $v \in \mathcal{U}$, we have

$$\begin{aligned} J(v) - J(u) &= \mathbb{E}\left[Z_T^v M\left(x_T^v, P_{x_T^v}\right) - Z_T^u M\left(x_T^u, P_{x_T^u}\right)\right] \\ &+ \mathbb{E}\left[h\left(y_0^v, P_{y_0^v}\right) - h\left(y_0^u, P_{y_0^u}\right)\right] \\ &+ \mathbb{E}\int_0^T \left(Z_t^v l^v(t) - Z_t^u l^u(t)\right) dt, \end{aligned}$$

where

$$\begin{aligned} l^v(t) &= l\left(t, x_t^v, P_{x_t^v}, y_t^v, P_{y_t^v}, z_t^v, P_{z_t^v}, \bar{z}^v, P_{\bar{z}^v}, v_t\right), \\ l^u(t) &= l\left(t, x_t^u, P_{x_t^u}, y_t^u, P_{y_t^u}, z_t^u, P_{z_t^u}, \bar{z}^u, P_{\bar{z}^u}, u_t\right). \end{aligned}$$

By the convexity property of M and h, we get

$$\begin{aligned} \mathbb{E}\left[Z_T^v M\left(x_T^v, P_{x_T^v}\right) - Z_T^u M\left(x_T^u, P_{x_T^u}\right)\right] &\geq \mathbb{E}[(Z_T^v - Z_T^u) M\left(x_T^u, P_{x_T^u}\right)] \\ &+ \mathbb{E}^u[M_x\left(x_T^u, P_{x_T^u}\right)(x_T^v - x_T^u)] \\ &+ \mathbb{E}^u[\widehat{\mathbb{E}}\left[\partial_\mu^{P_x} M\left(x_T^u, P_{x_T^u}\right)\right](x_T^v - x_T^u)]. \end{aligned} \tag{2.33}$$

Similarly,

$$\begin{aligned} \mathbb{E}\left[h\left(y_0^v, P_{y_0^v}\right) - h\left(y_0^u, P_{y_0^u}\right)\right] &\geq \mathbb{E}\left[h_y\left(y_0^u, P_{y_0^u}\right)(y_0^v - y_0^u))\right] \\ &+ \mathbb{E}\left[\widehat{\mathbb{E}}\left[\partial_\mu^{P_y} h\left(y_0^u, P_{y_0^u}\right)\right](y_0^v - y_0^u)\right], \end{aligned} \tag{2.34}$$

and

$$\mathbb{E}\int_0^T \left(Z_t^v l^v(t) - Z_t^u l^u(t)\right) dt = \mathbb{E}\int_0^T Z_t^v \left(l^v(t) - l^u(t)\right) dt + \mathbb{E}\int_0^T \left(Z_t^v - Z_t^u\right) l^u(t)\, dt. \tag{2.35}$$

From (2.33), (2.34) and (2.35), we can write

$$\begin{aligned} J(v) - J(u) \geq\ & \mathbb{E}^u[M_x\left(x_T^u, P_{x_T^u}\right)(x_T^v - x_T^u)] + \mathbb{E}^u[\widehat{\mathbb{E}}\left[\partial_\mu^{P_x} M\left(x_T^u, P_{x_T^u}, \widehat{x}_T^u\right)\right](x_T^v - x_T^u)] \\ & + \mathbb{E}\left[h_y\left(y_0^u, P_{y_0^u}\right)(y_0^v - y_0^u))\right] + \mathbb{E}\left[\widehat{\mathbb{E}}\left[\partial_\mu^{P_y} h\left(y_0^u, P_{y_0^u}, \widehat{y}_0^u\right)\right](y_0^v - y_0^u))\right] \\ & + \mathbb{E}\int_0^T Z_t^v \left(l^v(t) - l^u(t)\right) dt + \mathbb{E}\int_0^T \left(Z_t^v - Z_t^u\right) l^u(t)\, dt \\ & + \mathbb{E}\left[(Z_T^v - Z_T^u)\left(\int_0^T l^u(t)\, dt + M\left(x_T^u, P_{x_T^u}\right)\right)\right]. \end{aligned}$$

Noting that

$$\begin{aligned} q_0 &= -h_y(y_0, P_{y_0}) - \widehat{\mathbb{E}}\left[\partial_\mu^{P_y} h(\widehat{y}_0, P_{y_0}, y_0)\right], \\ p_T &= M_x\left(x_T, P_{x_T}\right) + \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} M\left(\widehat{x}_T, P_{x_T}, x_T\right)\right] \\ &\quad -\varphi_x\left(x_T, P_{x_T}\right) q_T - \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} \varphi\left(\widehat{x}_T, P_{x_T}, x_T\right)\widehat{q}_T\right], \end{aligned}$$

we have

$$\begin{aligned} J(v) - J(u) \geq\ & \mathbb{E}^u[p_T^u\left(x_T^v - x_T^u\right)] + \mathbb{E}^u\left[\varphi_x\left(x_T, P_{x_T}\right) q_T\left(x_T^v - x_T^u\right)\right] \\ & + \mathbb{E}^u\widehat{\mathbb{E}}\left[\partial_\mu^{P_x}\varphi\left(\widehat{x}_T, P_{x_T}, x_T\right)\widehat{q}_T\left(x_T^v - x_T^u\right)\right] - \mathbb{E}\left[q_0^u\left(y_0^v - y_0^u\right)\right] \\ & + \mathbb{E}^u\int_0^T \left(l^v(t) - l^u(t)\right) dt + \mathbb{E}\left[(Z_T^v - Z_T^u)\left(\int_0^T l^u(t)\, dt + M\left(x_T^u, P_{x_T^u}\right)\right)\right]. \end{aligned}$$

Then, we can write

$$\begin{aligned} J(v) - J(u) \geq\ & \mathbb{E}^u[p_T^u\left(x_T^v - x_T^u\right)] + \mathbb{E}^u[q_T^u\left(y_T^v - y_T^u\right)] \\ & - \mathbb{E}\left[q_0^u\left(y_0^v - y_0^u\right)\right] + \mathbb{E}^u\int_0^T \left[l^v(t) - l^u(t)\right] dt \\ & + \mathbb{E}\left[(Z_T^v - Z_T^u)\left(\int_0^T l^u(t)\, dt + M\left(x_T^u, P_{x_T^u}\right)\right)\right]. \end{aligned}$$

Now, appliyng Ito's formula respectively to $p_t^u\left(x_t^v - x_t^u\right), q_t^u\left(y_t^v - y_t^u\right)$ and $P_t^u\left(Z_t^v - Z_t^u\right)$, and by taking expectations, we get

$$
\begin{aligned}
J(v) - J(u) \geq \mathbb{E}^u \int_0^T & \left(H^v\left(t\right) - H^u\left(t\right)\right) dt \\
& - \mathbb{E}^u \int_0^T H_x^u\left(t\right)\left(x_t^v - x_t^u\right) dt - \mathbb{E}^u \int_0^T \widehat{\mathbb{E}}\left[\partial_\mu^{P_x} H^u\left(t\right)\right]\left(x_t^v - x_t^u\right) dt \\
& - \mathbb{E}^u \int_0^T H_y^u\left(t\right)\left(y_t^v - y_t^u\right) dt - \mathbb{E}^u \int_0^T \widehat{\mathbb{E}}\left[\partial_\mu^{P_y} H^u\left(t\right)\right]\left(y_t^v - y_t^u\right) dt \\
& - \mathbb{E}^u \int_0^T H_z^u\left(t\right)\left(z_t^v - z_t^u\right) dt - \mathbb{E}^u \int_0^T \widehat{\mathbb{E}}\left[\partial_\mu^{P_z} H^u\left(t\right)\right]\left(z_t^v - z_t^u\right) dt \\
& - \mathbb{E}^u \int_0^T H_{\bar{z}}^u\left(t\right)\left(\bar{z}_t^v - \bar{z}_t^u\right) dt - \mathbb{E}^u \int_0^T \widehat{\mathbb{E}}\left[\partial_\mu^{P_{\bar{z}}} H^u\left(t\right)\right]\left(\bar{z}_t^v - \bar{z}_t^u\right) dt \\
& - \mathbb{E}^u \int_0^T H_v^u\left(t\right)\left(v_t - u_t\right) dt
\end{aligned}
$$

By the convexity of the functional H in $(x, P_x, y, P_y, z, P_z, \bar{z}, P_{\bar{z}}, v)$, we have

$$
\begin{aligned}
J(v) - J(u) &\geq \mathbb{E}^u \int_0^T H_v\left(t\right)\left(v_t - u_t\right) dt \\
&= \mathbb{E} \int_0^T Z_t^u \mathbb{E}\left[H_v\left(t\right)\left(v_t - u_t\right) / \mathcal{F}_t^Y\right] dt.
\end{aligned}
$$

Since $Z_t^u \geq 0$, and using condition (2.28), we have

$$
J(v) - J(u) \geq 0,
$$

i.e., u is a partial observed optimal control. $\square$

Chapter §.3
Partially observed Linear-Quadratic control problem

Chapter 3

Partially observed Linear–Quadratic control problem

In this chapter, we examine a linear-quadratic control problem under partial observation. By leveraging the findings from Chapter 2, we derive an explicit formula for the optimal control.

Examine a one-dimensional linear quadratic control problem under partial observation: Minimize the expected quadratic cost function

$$\begin{aligned} J\left(v\left(\cdot\right)\right) = \mathbb{E}^{u} \int_{0}^{T} \left[L_t^1 x_t^2 + L_t^2 \left(\mathbb{E}\left[x_t\right]\right)^2 + L_t^3 y_t^2 + L_t^4 \left(\mathbb{E}\left[y_t\right]\right)^2 + L_t^5 v_t^2\right] dt \\ + \mathbb{E}^{u}\left[M_1 x_T^2 + M_2 \left(\mathbb{E}\left[x_T\right]\right)^2 + h_t y_0^2\right], \end{aligned} \tag{3.1}$$

subject to

$$\begin{cases} dY_t = \gamma_t dt + d\widetilde{W}_t \\ Y_0 = 0, \end{cases} \tag{3.2}$$

and the state

$$
\begin{cases}
dx_t = \left(A_t^1 x_t + A_t^2 \mathbb{E}\left[x_t\right] + A_t^3 v_t - B_t^2 \gamma_t\right) dt + B_t^1 dW_t + B_t^2 dY_t, \\
-dy_t = \left(D_t^1 x_t + D_t^2 \mathbb{E}\left[x_t\right] + D_t^3 y_t + D_t^4 \mathbb{E}\left[y_t\right] + D_t^5 z_t + D_t^6 \mathbb{E}\left[z_t\right] + D_t^7 \overline{z}_t + D_t^8 \mathbb{E}\left[\overline{z}_t\right] \right. \\
\qquad \left. + D_t^9 r_t + D_t^{10} \mathbb{E}\left[r_t\right] + D_t^{11} v_t\right) dt - z_t dW_t - \overline{z} dY_t, \\
x\left(0\right) = x_0,\ y_T = \phi_1 x_T + \phi_2 \mathbb{E}\left[x_T\right],
\end{cases}
\tag{3.3}
$$

where

$$
\begin{aligned}
&A_t^1 x_t + A_t^2 \mathbb{E}\left[x_t\right] + A_t^3 v_t = b\left(t, x_t^v, P_{x_t^v}, v_t\right), \\
&B_t^1 = g\left(t, x_t^v, P_{x_t^v}, v_t\right), \\
&B_t^2 = \sigma\left(t, x_t^v, P_{x_t^v}, v_t\right), \\
&\gamma_t = \xi\left(t, x_t^v, P_{x_t^v}\right),
\end{aligned}
$$

and

$$
\begin{aligned}
f\left(t, x_t^v, P_{x_t^v}, y_t^v, P_{y_t^v}, z_t^v, P_{z_t^v}, \overline{z}_t^v, P_{\overline{z}_t^v}, v_t\right) &= D_t^1 x_t + D_t^2 \mathbb{E}\left[x_t\right] + D_t^3 y_t + D_t^4 \mathbb{E}\left[y_t\right] \\
&\quad + D_t^5 z_t + D_t^6 \mathbb{E}\left[z_t\right] + D_t^7 \overline{z}_t + D_t^8 \mathbb{E}\left[\overline{z}_t\right] + D_t^9 v_t.
\end{aligned}
$$

Here, all the coefficients $A^1(\cdot), A^2(\cdot), A^3(\cdot), B^1(\cdot), B^2(\cdot), \gamma(\cdot), D^i(\cdot)$ are bounded and deterministic functions for $i = 1, \cdots, 9$, $L^j(\cdot)$ is positive function and bounded for $j = 1, 2, 3, 4, 5$ and $M_1(\cdot), M_2(\cdot), h(\cdot)$ are positive constants. Then for any $v \in \mathcal{U}$, Eqs. (3.3) and (3.2) have unique solutions, respectively. Now, we introduce

$$
Z_t = \exp\left\{ \int_0^t \gamma_s dY_s - \frac{1}{2} \int_0^t \left|\gamma_s\right|^2 ds \right\},
$$

which is the unique $\mathcal{F}_t^Y$-adapted solution of the SDE:

$$
\begin{cases}
dZ_t = Z_t \gamma_t dY_t, \\
Z_0 = 1,
\end{cases}
$$

and we define the probability measure P^v by $dP^v = Z_t^v dP$.

In this setting, the Hamiltonian function is defined as

$$\begin{aligned}
&H(t,x,y,z,\bar{z},r,v,p,q,k,\bar{k},n,Q) \\
&= p\left(A_t^1 x_t + A_t^2 \mathbb{E}\left[x_t\right] + A_t^3 v_t - B_t^2 \gamma_t\right) - q\left(D_t^1 x_t + D_t^2 \mathbb{E}\left[x_t\right] + D_t^3 y_t + D_t^4 \mathbb{E}\left[y_t\right] + D_t^5 z_t\right. \\
&+ \left. D_t^6 \mathbb{E}\left[z_t\right] + D_t^7 \overline{z}_t + D_t^8 \mathbb{E}\left[\overline{z}_t\right] + D_t^9 v_t\right) + kB_t^1 + \bar{k} B_t^2 \\
&\quad + Q\gamma_t + L_t^1 x_t^2 + L_t^2 \left(\mathbb{E}\left[x_t\right]\right)^2 + L_t^3 y_t^2 + L_t^4 \left(\mathbb{E}\left[y_t\right]\right)^2 + L_t^5 v_t^2.
\end{aligned} \tag{3.4}$$

Further due to Eqs. (2.10) and (2.11), the corresponding adjoint equations will be given by

$$\begin{cases}
-dP_t = \left(L_t^1 x_t^2 + L_t^2 \left(\mathbb{E}\left[x_t\right]\right)^2 + L_t^3 y_t^2 + L_t^4 \left(\mathbb{E}\left[y_t\right]\right)^2 + L_t^5 v_t^2\right) dt \\
\qquad\quad -\bar{Q}_t dW_t - Q_t d\widetilde{W}_t, \\
P_T = M(x_T, P_{x_T}),
\end{cases} \tag{3.5}$$

and

$$\begin{cases}
-dp_t = \left[A_t^1 p_t + A_t^2 \mathbb{E}\left[p_t\right] - D_t^1 q_t - D_t^2 \mathbb{E}\left[q_t\right] + 2L_t^1 x_t + 2L_t^2 \mathbb{E}\left[x_t\right]\right] dt \\
\qquad\quad -k_t dW_t - \bar{k}_t d\widetilde{W}_t, \\
dq_t = \left(D_t^3 q_t + D_t^4 \mathbb{E}\left[q_t\right] - 2L_t^3 y_t - 2L_t^4 \mathbb{E}\left[y_t\right]\right) dt + \left(D_t^5 q_t + D_t^6 \mathbb{E}\left[q_t\right]\right) dW_t \\
\qquad\quad + \left[D_t^7 q_t + D_t^8 \mathbb{E}\left[q_t\right]\right] d\widetilde{W}_t \\
p_T = 2M_1 x_T + 2M_2 \mathbb{E}\left[x_T\right] \\
\qquad\quad -\phi_1 x_T - \phi_2 \mathbb{E}\left[x_T\right], \\
q_0 = -2h_t y_0.
\end{cases} \tag{3.6}$$

According to Theorem 2.2.1, the necessary condition for optimality (2.28) will be

$$\mathbb{E}^u\left[p_t A_t^3 - q_t D_t^9 + 2L_t^5 u_t / \mathcal{F}_t^Y\right] = 0. \ \ a.s.a.e.$$

If $u(\cdot)$ is partial observed optimal control, then

$$u_t = -\frac{1}{2L_t^5}\left(A_t^3\mathbb{E}^u\left[p_t/\mathcal{F}_t^Y\right] - D_t^{11}\mathbb{E}^u\left[q_t/\mathcal{F}_t^Y\right]\right). \tag{3.7}$$

Finally, for the sufficient conditions, let $u \in \mathcal{U}$ be a candidate to be optimal control. We suppose that $\left(\widetilde{x}, \widetilde{y}, \widetilde{z}, \widetilde{\bar{z}}\right)$ is the solution to the FBSDE (3.3) corresponding to u, and $\left(P, \bar{Q}, Q\right), (p, q, k, \bar{k})$ are the solution corresponding to Eqs. (3.5) and (3.6) respectively. It's easy to verify that the functional H is convex in $(x, y, z, \bar{z})$. So, if u satisfies (3.7) and the condition (2.28). Then by applying Theorem 2.2.2, we can check that u is an optimal control of our partially observed linear-quadratic control problem of McKean–Vlasov type.

Conclusion

In this thesis, we have studied the optimal control problem of McKean–Vlasov Forward Backward Stochastic Differential Equations (FBSDEs) under partial observation. By using the derivatives with respect to the probability law and integrating Girsanov's theorem with the classical convex variation technique, we derived both necessary and sufficient conditions for optimality. In order to illustrate these conclusions, we used a partially observed linear-quadratic control problem to apply the theoretical results.

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