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Dedication

In the name of Allah, the Most Gracious, the Most Merciful.

All praise is due to Allah, who granted me strength, patience, and the ability to complete this journey.

To the one who walked a path that was never easy, yet kept going anyway.

To the one who cried in secret, smiled in public, and hid her pain behind a hopeful heart.

To the one who chose patience when despair was closer, and chose to keep moving when giving up seemed easier.
I dedicate this work to you, in recognition of your struggle, in honor of your strength, and in pride of every step you took toward the light.
You were strong, and still are,
To myself.

To my beloved parents,
Your prayers, sacrifices, and unwavering support have been my greatest blessings.
May Allah preserve you and reward you with Jannah.

To my beloved siblings,
You were my comfort and motivation in moments of hardship.
Thank you for being part of my journey.

To those who were the light in the darkness of the days, and the support in times of fatigue,
I dedicate this work to you.

May Allah bless you all and grant you happiness in both worlds.

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Introduction

Partial differential equations (PDEs) occupy a central place in applied mathematics, representing the mathematical framework within which a wide range of natural phenomena and physical and engineering systems can be described. Thanks to its ability to relate spatial and temporal variations of phenomena, PDEs have become indispensable tool in the study of fields as diverse as mechanics of continuous media, fluid dynamics, heat diffusion, and reaction and diffusion models. A large part of its importance lies in the fact that it enables researchers to understand the complex behaviour of real systems and predict their evolution over time.

Among the special types of partial differential equations, damped equations are of particular interest, as they play a crucial role in modelling the effects of the gradual loss of energy within a system. The introduction of damping terms into the equations enables a more accurate analysis of non-ideal dynamic situations, where phenomena such as vibration reduction, gradual stabilisation, and extinction with time emerge. The study of damped equations is not only limited to describing the behaviour of physical and mechanical systems, but extends to issues of stability, existence, and extremes of solutions, making it a topic rich in both theoretical results and practical applications.

The Euler-Bernoulli beam equation is an important classical model that describes the curvature of elastic beams under different loads, and was developed in the 18th century by Leonhard Euler and Daniel Bernoulli. As engineering applications and industrial techniques evolved, this simple model was no longer sufficient to represent the behaviour of complex materials and modern systems, and had to be developed to include more realistic effects such as memory, viscosity and automated control.

One of the recent extensions of the model is the inclusion of a memory term that expresses the response of a material depending on its deformation history, as in viscoelastic materials such as polymers, which are characterised not only by their immediate response to forces, but also by their past history. This effect is represented in the mathematical model by a time integral term that expresses this temporal dependence.

Furthermore, output-dependent boundary feedback control is used as an effective mechanism to stabilise the girder, where forces or moments are applied at the edges based on local mea-

surements only without the need to monitor all internal points. This advanced theoretical framework is of great importance in many applications such as aerospace engineering, flexible robot arms, smart structures, and precision devices, as it provides high accuracy in control and vibration prevention.
The main contribution of this work is to investigate the well-posedness and exponential stability of the Euler–Bernoulli beam equation with a memory term and boundary output feedback control. The memory term reflects the dependence of the material on its past deformations (for more details, see [5] [7]). Our work is based on [17].
In this thesis, we consider the following problem:

$$\omega_{tt}(x,t) + \omega_{xxxx}(x,t) - \int_0^t \kappa(t-\tau)\omega_{xxxx}(x,\tau)\,d\tau + g(\omega_t(x,t)) = 0, \quad x \in [0,L],\ t > 0, \tag{1}$$

where κ represents the kernel of the memory term, $g : \mathbb{R} \to \mathbb{R}$ is a given function, $v : \mathbb{R}_+ \to \mathbb{R}$ the boundary control force applied at the and of the beam and $\omega_{out}(t)$ stands for the measured signal of the system at time t.
with the following boundary condition and initial conditions:

$$\begin{cases} \omega(0,t) = \omega_x(0,t) = \omega_{xx}(L,t) = 0, & t \geq 0, \\ \omega_{xxx}(L,t) - \displaystyle\int_0^t k(t-\tau)\omega_{xxx}(L,\tau)\,d\tau = v(t), & t \geq 0, \\ \omega(x,0) = \omega_0(x), \quad \omega_t(x,0) = y_1(x),\ x \in [0,L], \\ \omega_{\text{out}}(t) = \omega_t(L,t). \end{cases} \tag{2}$$

This System describes the transverse vibration of an extensible beam clamped at $x = 0$ and supported at $x = L$ by a control force.
We propose an adaptive output feedback controller law of the form:

$$\begin{cases} v(t) = f(t)\omega_t(L,t), \\ f_t(t) = r\omega_t^2(L,t),\ f(0) = f_0,\ \ r > 0. \end{cases}$$

This work is divided into three chapters:
Chapter 1: In this chapter, we introduce the fundamental mathematical tools and notations required throughout this work. We begin by reviewing some essential functional spaces, which provide the appropriate framework for analyzing partial differential equations. Next, we recall several important inequalities and theorems which play a crucial role in the derivation of a priori estimates and stability results. Finally, we present the Faedo-Galerkin method, the technique used in proving the existence of weak solutions to PDEs.
Chapter 2: In this chapter, we begin by presenting the precise formulation of the problem under consideration, incorporating a set of well-defined hypotheses to ensure the mathematical rigor of the study. We then proceed to establish the existence and uniqueness of the solution to the problem.
Chapter 3: This chapter is devoted to studying the exponential stability of the solution to the

problem, providing a detailed analysis of how the solution behaves over time under the given conditions.

Chapter 1

Functional Analysis Foundations for PDEs

In this chapter, we review some concepts from functional spaces, along with key inequalities that will be used in subsequent chapters.

1.1 Some Functional Spaces

1.1.1 Banach Space

Definition 1.1.1. *Let X be a vector space. A map $\|\cdot\| : X \to \mathbb{R}_+$ is called a norm if it satisfies the following properties:*

1. $\|x\| = 0 \Leftrightarrow x = 0$ *(definiteness).*
2. $\|x\| \geq 0$, $\forall x \in X$ *(Positivity).*
3. $\|\lambda x\| = |\lambda| \|x\|, \quad \forall x \in X$ *and scalars* $\lambda \in \mathbb{K}$ *(Homogeneity).*
4. $\|x + y\| \leq \|x\| + \|y\|$, $\quad \forall x, y \in X$ *(Triangle inequality).*

The pair $(X, \|.\|)$ is called a ***normed vector space****.*

Definition 1.1.2. *Any complete normed vector space is called a* ***Banach space****. [11]*

1.1.2 Hilbert Space

Definition 1.1.3. *(Inner Product): Let H be a vector space. Inner product on H is a map from $H \times H$ to $\mathbb{K} = \mathbb{R}(or\ \mathbb{C})$, denoted by $\langle \cdot, \cdot \rangle$, which satisfies the following properties.*
For every vectors $x, y, z \in H$ and scalar $\lambda \in \mathbb{K}$:

1. *Positivity and definiteness*

$$\langle x, x \rangle \geq 0 \text{ and } \langle x, x \rangle = 0 \Leftrightarrow x = 0.$$

2. *Bilinearity (or sesquilinearity in the complex case)*

$$\langle x + \lambda z, y\rangle = \langle x, y\rangle + \lambda\langle z, y\rangle,$$

$$\langle x, y + \lambda z\rangle = \langle x, y\rangle + \bar{\lambda}\langle x, z\rangle.$$

3. *Symmetry*

$$\langle x, y\rangle = \overline{\langle y, x\rangle}.$$

where the bar denotes complex conjugate.

Definition 1.1.4. ***(Hilbert space):*** *A Hilbert space is a vector space H equipped with an inner product, and complete with respect to the induced norm by the inner product.*
So Hilbert space is a Banach space [18].

Remark 1.1.1. *The inner product induces a norm defined as:*

$$\|x\| = \sqrt{\langle x, x\rangle}\ , \forall x \in H.$$

Proposition 1.1.1. *Let $x, y \in H$. Then*

$$|\langle x, y\rangle| \leq \|x\|\|y\|,$$

is called Cauchy Schwarz inequality.

Orthogonality

Definition 1.1.5. *Two vectors u and v are said to be orthogonal if $(u, v) = 0$. An element $v \in V$ is said to be orthogonal to a subset $U \subseteq V$ if $(u, v) = 0$ for every $u \in U$.*

Definition 1.1.6. *Let U be a subset of an inner product space V. We define its orthogonal complement to be the set*

$$U^{\perp} = \{v \in V \mid (v, u) = 0 \text{ for all } u \in U\}.$$

The orthogonal complement of any set is a closed subspace.

Definition 1.1.7. *Let V be a finite-dimensional inner product space. A basis $\{v_1, \ldots, v_n\}$ of V is said to be an orthogonal basis if*

$$(v_i, v_j) = 0, \quad \text{for } 1 \leq i \neq j \leq n.$$

1.1.3 Space of Continuous Functions

Let $x = (x_1, x_2, \cdots, x_n)$ denote the generic point of an open set Ω of $\mathbb{R}^n$. Let f be a function defined from Ω to $\mathbb{R}$ we designate by

$$D_i f(x) = f_{x_i}(x) = \frac{\partial}{\partial x_i} f(x)$$

the partial derivative of f with respect to $x_i, (1 \leq i \leq n)$.

Definition 1.1.8. *We denote by $C(\Omega)$ the space of continuous functions defined on the domain Ω. Moreover, the norm on this space is*

$$\|f\|_{C(\Omega)} = \sup_{x \in \Omega} |f(x)|.$$

Definition 1.1.9. *$C^k(\Omega)$ denotes the space of functions that are k-times continuously differentiable on the domain Ω; that is, all $D^\alpha f$ are continuous on Ω.*
The norm on this space is defined by:

$$\|f\|_{C^k(\Omega)} = \sum_{|\alpha| \leq k} \sup_{x \in \Omega} |D^\alpha f(x)|,$$

where α is a multi-index and $D^\alpha f$ denotes the corresponding partial derivative.
The space $C^\infty(\Omega)$ is the space of functions that are infinitely differentiable on Ω; that is,

$$C^\infty(\Omega) = \bigcap_{k=0}^{\infty} C^k(\Omega).$$

*The space $C_0^\infty(\Omega)$ is the set of all infinitely differentiable (smooth) functions with **compact support** contained in Ω. That is,*

$$C_0^\infty(\Omega) = \{\varphi \in C^\infty(\Omega) \mid supp(\varphi) \subset\subset \Omega\}.$$

Functions in $C_0^\infty(\Omega)$ are often called test functions.

1.1.4 Lebesgue Space L^p

We denote by Ω an open domain in $\mathbb{R}^n$, where $n \in \mathbb{N}$ and let $p \in \mathbb{R}$ with $1 \leq p \leq \infty$.

Definition 1.1.10. *Let $p \in \mathbb{R}$ where $1 \leq p < \infty$. We denote by $L^p(\Omega)$ the class of all measurable functions f defined on Ω such that*

$$\int_\Omega |f(x)|^p \, dx < \infty,$$

and we write

$$L^p(\Omega) = \left\{ f : \Omega \to \mathbb{R} \text{ or } \mathbb{C} \mid f \text{ is measurable and } \int_\Omega |f(x)|^p < \infty \right\}.$$

Definition 1.1.11. *Let $p = \infty$. The space $L^\infty(\Omega)$ is defined as the vector space of all measurable functions $f : \Omega \to \mathbb{R}$ or $\mathbb{C}$, which are essentially bounded on Ω, that is:*

$$L^\infty(\Omega) = \Big\{ f : \Omega \to \mathbb{R} \text{ or } \mathbb{C} \;\Big|\; f \text{ is measurable and } \exists k \geq 0 \text{ such that } ; |f(x)| \leq k \quad a.e. on\, \Omega \Big\}.$$

Recall: *A measurable function f is said to be essentially bounded on Ω, if there exists a constant $k > 0$ such that $|f(x)| \leq k$ almost everywhere in Ω.*

Proposition 1.1.2. *The norm in this space is defined by:*

$$\|f\|_{L^p} = \left(\int_\Omega |f(x)|^p dx \right)^{\frac{1}{p}} \qquad \text{for } 1 \leq p < \infty,$$
$$\|f\|_{L^\infty} = \sup_\Omega |f(x)| \qquad\qquad \text{for } p = \infty.$$

Corollary 1.1.1. *$L^2(\Omega)$ is a Hilbert space with respect to the inner product*

$$\langle f, g \rangle = \int_\Omega f(x) g(x)\, dx.$$

Definition 1.1.12. *We denote by $L^p_{loc}(\Omega)$ the space of functions which are L^p on any bounded sub-domain of Ω.*

1.1.5 Vector-Valued Function Spaces

Definition 1.1.13. *Let $a, b \in \mathbb{R}$ with $a < b$, and let X be a Banach space. For $1 \leq p < \infty$, the space $L^p((a,b); X)$ consists of all strongly measurable functions $f : (a,b) \to X$ such that*

$$\|f\|_{L^p((a,b);X)} := \left(\int_a^b \|f(t)\|_X^p \, dt \right)^{1/p} < \infty.$$

For $p = \infty$, the space $L^\infty((a,b); X)$ consists of all essentially bounded measurable functions $f : (a,b) \to X$, with

$$\|f\|_{L^\infty((a,b);X)} := \operatorname*{ess\,sup}_{t \in (a,b)} \|f(t)\|_X < \infty.$$

Equipped with their respective norms, $L^p((a,b); X)$ is a Banach space for all $1 \leq p \leq \infty$.

Lemma 1.1.1. *If $f \in L^p(a,b;X)$, $\dfrac{\partial f}{\partial x} \in L^p(a,b;X)$, then the function f is continuous from $[a,b]$ to X($f \in C(a,b;X)$).*

1.1.6 Sobolev Space

Weak Derivatives

Definition 1.1.14. *[6] We say that a function $f \in L^1_{loc}(\Omega)$ has a weak partial derivative of order α, where α is a multi-index, if there exists a function $g \in L^1_{loc}(\Omega)$ such that*

$$\int_\Omega f(x) D^\alpha \varphi(x)\, dx = (-1)^{|\alpha|} \int_\Omega g(x) \varphi(x)\, dx, \quad \forall \varphi \in \mathcal{D}(\Omega),$$

where $\mathcal{D}(\Omega) = C_0^\infty(\Omega)$ is the space of test functions.
In this case, we write

$$D^\alpha f = g,$$

and we say that g is the α^{th} weak derivative of f.

Sobolev Spaces $\mathcal{W}^{m,p}(\Omega)$

Definition 1.1.15. *Let $m \in \mathbb{N}$ and L^p Lebesgue space where $1 \leq p \leq \infty$, for an open domain $\Omega \subseteq \mathbb{R}^n$. The Sobolev space $\mathcal{W}^{m,p}(\Omega)$ defined as:*

$$\mathcal{W}^{m,p} = \{f \in L^p(\Omega) \mid D^\alpha f \in L^p(\Omega),\ \forall \alpha : |\alpha| \leq m\}$$

where $\alpha \in N^n$, $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_n$ the length of α, and $D^\alpha f$ is the weak (or distributional) partial derivative of f.

The norm in the Sobolev space is described as follows:
For $1 \leq p < \infty$

$$\|f\|_{\mathcal{W}^{m,p}(\Omega)} = \|f\|_{m,p} = \left(\sum_{0 \leq |\alpha| \leq m} \|D^\alpha f\|^p_{L^p(\Omega)} \right)^{\frac{1}{p}}.$$

For $p = \infty$

$$\|f\|_{\mathcal{W}^{m,\infty}(\Omega)} = \|f\|_{m,\infty} = \max_{0 \leq |\alpha| \leq m} \|D^\alpha f\|_{L^\infty(\Omega)}.$$

Theorem 1.1.1. *$\mathcal{W}^{m,p}(\Omega)$ is a Banach space [1].*

Remark 1.1. *In the Sobolev space $\mathcal{W}^{m,p}(\Omega)$, we have the following special cases:*

- *If $m = 0$, then $W^{0,p}(\Omega) = L^p(\Omega)$.*
- *If $p = 2$, then $W^{m,2}(\Omega) = H^m(\Omega)$.*

Definition 1.1.16. *The space $H^m(\Omega)$ is a Hilbert space with the inner product:*

$$\langle f, g \rangle_{H^m(\Omega)} = \sum_{0 \leq |\alpha| \leq m} \langle D^\alpha f, D^\alpha g \rangle.$$

Theorem 1.1.2. *$W^{m,p}(\Omega)$ is separable if $1 \leq p < \infty$, and is uniformly convex and reflexive if $1 < p < \infty$. [1]*

Sobolev Embedding Theorems

Definition 1.1.17. *Let V and W be tow Banach spaces with $V \subseteq W$. We say the space V is continuously embedded in W and write $V \hookrightarrow W$, if*

$$\|v\|_W \leq c\|v\|_V, \quad v \in V. \tag{1.1}$$

We say the space V is compactly embedded in W and write $V \hookrightarrow\hookrightarrow W$, if (1.1) holds and each bounded sequence in V has a convergent subsequence in W .
If $V \hookrightarrow W$, the functions in V are more smooth than the remaining functions in W.

Theorem 1.1.3. *Let $\Omega \subset \mathbb{R}^d$ be a non-empty, open and bounded domain with Lipschitz boundary. Then the following Sobolev embedding results hold:*

(a) If $\dfrac{k}{d} < \dfrac{1}{p}$, then

$$\mathcal{W}^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$$

for any $q \leq p^*$, where

$$\frac{1}{p^*} = \frac{1}{p} - \frac{k}{d}.$$

(b) If $\dfrac{k}{d} = \dfrac{1}{p}$, then

$$\mathcal{W}^{k,p}(\Omega) \hookrightarrow L^q(\Omega)$$

for any $q < \infty$.

(c) If $\dfrac{k}{d} > \dfrac{1}{p}$, then

$$\mathcal{W}^{k,p}(\Omega) \hookrightarrow C^{r,\beta}(\Omega),$$

where

$$r = k - \left\lfloor \frac{d}{p} \right\rfloor - 1,$$

and

$$\beta = \begin{cases} \dfrac{d}{p} - \left\lfloor \dfrac{d}{p} \right\rfloor, & \text{if } \dfrac{d}{p} \notin \mathbb{N}, \\ \text{any } \beta \in (0,1), & \text{if } \dfrac{d}{p} \in \mathbb{N}. \end{cases}$$

In the one-dimensional case, with $\Omega = (a,b)$ a bounded interval, we have

$$\mathcal{W}^{k,p}(a,b) \hookrightarrow C[a,b]$$

for any $k \geq 1$ and $p \geq 1$.

Theorem 1.1.4. *Let $\Omega \subset \mathbb{R}^d$ be a non-empty, open, bounded domain with Lipschitz boundary. Then the following Sobolev embedding results hold:*

(a) If $\dfrac{k}{d} < \dfrac{1}{p}$, then

$$\mathcal{W}^{k,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega), \quad \text{for all } q < p^*,$$

where

$$\frac{1}{p^*} = \frac{1}{p} - \frac{k}{d}.$$

(b) *If* $\frac{k}{d} = \frac{1}{p}$, *then*

$$\mathcal{W}^{k,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega), \quad \text{for all } q < \infty.$$

(c) *If* $\frac{k}{d} > \frac{1}{p}$, *then*

$$\mathcal{W}^{k,p}(\Omega) \hookrightarrow\hookrightarrow C^{r,\beta}(\Omega),$$

where

$$r = k - \left\lfloor \frac{d}{p} \right\rfloor - 1,$$

and

$$\beta = \begin{cases} \frac{d}{p} - \left\lfloor \frac{d}{p} \right\rfloor, & \text{if } \frac{d}{p} \notin \mathbb{N}, \\ \text{any } \beta \in (0,1), & \text{if } \frac{d}{p} \in \mathbb{N}. \end{cases}$$

Theorem 1.1.5. *Let k and l be non-negative integers,$k > l$, and $p \in [1, \infty]$. Let $\Omega \subseteq \mathbb{R}^d$ be a non-empty open bounded Lipschitz domain. Then $\mathcal{W}^{k,p}(\Omega) \hookrightarrow\hookrightarrow \mathcal{W}^{l,p}(\Omega)$. [3]*

Aubin-Lions Lemma

Let X_0, X and X_1 be three Banach spaces with $X0 \subseteq X \subseteq X_1$. Assume that X_0 is compactly embedded in X and that X is continuously embedded in X_1, assume also that X_0 and X_1 are reflexive spaces. For $1 < p, q < +\infty$, let

$$W = \{f \in L^p(0, T; X_0); \quad f' \in L^q(0, T; X_1)\}. \tag{1.2}$$

Then the embedding of W into $L^p(0, T; X)$ is also compact.

1.1.7 Types of Convergence

Let X be a normed space, X' its dual space.

Definition 1.1.18. *A sequence $\{x_n\} \subseteq X$ is said to converge strongly to $x \in X$ if*

$$\|u_n - u\|_X \to 0 \quad \text{as } n \to \infty.$$

This is also called convergence in norm and is denoted by

$$u_n \to u \quad \text{in } X.$$

Definition 1.1.19. *A sequence $\{x_n\}$ is said to converge weakly to $x \in X$ if*

$$\langle f, x_n \rangle \to \langle f, x \rangle \quad \text{for all } f \in X',$$

and write $x_n \rightharpoonup x$ as $n \to \infty$.

Definition 1.1.20. *A sequence $\{f_n\} \subset X'$ is said to converge weak-$*$ (or weak-star) to $f \in X'$ if*

$$f_n(x) \to f(x) \quad \textit{for all } x \in X.$$

This is denoted by

$$f_n \overset{*}{\rightharpoonup} f \quad \textit{in } X'.$$

Proposition 1.1.3. *[13] Let $x \in X$, let $\{x_n\} \subset X$. Then:*

(i) If $x_n \to x$ in X, then $x_n \rightharpoonup x$ in X.

(ii) If $x_n \rightharpoonup x$ in X, then the sequence $\{x_n\}$ is bounded in X.

Theorem 1.1.6. *Suppose X is a reflexive Banach space. Then every bounded sequence $\{x_n\} \subset X$ has a weakly convergent subsequence. [3]*

Theorem 1.1.7. *Let X be a separable Banach space (i.e., one which contains a dense countable subset). Then, from any bounded sequence of elements of X', we can extract a subsequence which weakly-$*$ converges in X'.*

1.2 Important Inequalities

Let $1 \leq p \leq \infty$; we denote by q the conjugate exponent

$$\frac{1}{p} + \frac{1}{q} = 1$$

1.2.1 Young's Inequality

Let a and b be real numbers, and $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. Then

$$ab < \frac{a^p}{p} + \frac{b^q}{q}$$

This is known as Young's inequality.

Young's Inequality With η

Let be $a, b > 0$, and let $1 < p, q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. For $\eta > 0$ we have the inequality

$$ab < \eta a^p + \frac{1}{(\eta p q)^{\frac{p}{q}}} b^q$$

This is known as also Young's inequality with η.

Proof. see [6] □

1.2.2 Gronwall's Inequality

Differential Form

Let $\xi(t), \eta(t)$ and $\gamma(t)$ be three continuous functions defined on $[a,b]$ such that ξ is differentiable on $]a,b[$. We assume that

$$\xi'(t) \leq \eta(t)\xi(t) + \gamma(t), \text{ for all } t \in]a,b[,$$

then, we have

$$\xi(t) \leq \exp\Big(\int_a^t \eta(s)\,ds\Big)\xi(a) + \int_a^t \exp\Big(\int_s^t \eta(r)\,dr\Big)\gamma(s)\,ds.$$

Integral Form

Let $\xi(t), \eta(t)$ be continuous in $[0,T]$, with η nondecreasing and γ positive constant. If

$$\xi(t) \leq \eta(t) + \gamma \int_0^t \xi(s)\,ds, \quad \forall t \in [0,T],$$

then

$$\xi(t) \leq \eta(t)\exp(\gamma t), \quad \forall t \in [0,T].$$

1.3 Fubini's Theorem

Let f be summable in $I = I_1 \times I_2 \subset \mathbb{R}^n \times \mathbb{R}^m$. Then

1. $f(x,\cdot) \in L^1(I_2)$ for a.e. $x \in I_1$, and $f(\cdot,y) \in L^1(I_1)$ for a.e. $y \in I_2$.
2. $\displaystyle\int_{I_2} f(\cdot,y)\,dy \in L^1(I_1)$, and $\displaystyle\int_{I_1} f(x,\cdot)\,dx \in L^1(I_2)$
3. the following formulas hold

$$\int_{I_1\times I_2} f(x,y)\,dx\,dy = \int_{I_1}\left(\int_{I_2} f(x,y)\,dy\right)\,dx = \int_{I_2}\left(\int_{I_1} f(x,y)\,dx\right)\,dy.$$

1.4 Leibniz's Rule for Differentiation Under the Integral Sign

Let the integral

$$I(t) = \int_{a(t)}^{b(t)} f(x,t)\,dx$$

where $f(x,t)$ and $\dfrac{\partial f}{\partial t}$ are continuous on the rectangle $[A,B]\times[c,d]$, where $[A,B]$ contains the union of all the intervals $[a(t),b(t)]$, and if $a(t)$ and $b(t)$ are differentiable functions on $[c,d]$, then

$$\frac{d}{dt}I(t) = \int_{a(t)}^{b(t)} \frac{\partial}{\partial t} f(x,t)\,dx + f(b(t),t)b'(t) - f(a(t),t)a'(t).$$

1.5 Faedo-Galerkin method

Definition 1.5.1. *Let V be a separable Hilbert space, and $\{V_n\}_{n\in\mathbb{N}}$ a family of finite–dimensional vector spaces satisffying the axioms*

- $V_n \subset V,\ \dim V_n < \infty,$
- $V_n \to V$ *when* $k \longrightarrow \infty.$

In the following sense: there exists V_n subspace dense in V, such that for all $v \in V$, we can find a sequence $\{v_n\}_{n\in\mathbb{N}} \subset V_n$ satisfying

$$v_n \longrightarrow v \quad \text{in } V \quad \text{when } n \longrightarrow \infty.$$

The space V_n is called a Galerkin approximation of order n.

The Scheme of the Method of Faedo-Galerkin

Let P to be the exact problem for which we want to show the existence of a solution in a function space built on a separable Hilbert space V. Let u to be the unique solution of the problem P.

After having made a choice of a Galerkin approximation V_n of V it is necessary to define an approximate problem P_n in finite-dimensional space V_n having a unique solution u_n. Then, the course of the study is then as follows:

Step 1 : We define the solution u_n of the problem P_n.

Step 2 : We establish estimates on u_n (called a priori estimate) to show that u_n is uniformly bounded.

Step 3 : By using the results that u_n is uniformly bounded, it is possible to extract from $\{u_n\}_{n\in\mathbb{N}}$ a subsequence $\{u'_n\}_{n\in\mathbb{N}}$ which has a limit in the weak topology of the space involved in the estimations of step 2. Let u to be the obtained limit.

Step 4 : We show that u is the solution of the problem P.

Step 5 : Results of strong convergences.

The objective is to build an approximation process which ultimately provides us with a proof of the existence of solution, this process amounts to approaching $u_n(x,t)$ as a linear combination of functions of the bases v_i such that

$$u_n(x,t) = \sum_{i=1}^{n} \phi_i(t) v_i(x), \quad (x,t) \in \Omega \times [0,T],$$

where the $\phi_i(t)$ are then solutions to a system of n linear differential equations.

1.6 Stabilization method (Lyapunov functional)

To establish the desired stability results of the systems, we use the *multiplier method*. The multiplier method is mainly based on the construction of an appropriate Lyapunov function

$L(t)$, which is equivalent to the energy of the solution. By the equivalence $L \sim E$, we mean that there exist positive constants $\alpha, \beta > 0$ such that

$$\alpha E(t) \leq L(t) \leq \beta E(t), \quad \forall t > 0. \tag{1.3}$$

To prove exponential stability, we show that $L(t)$ satisfies the differential inequality

$$L'(t) \leq -\gamma L(t), \quad \forall t > 0, \quad \text{for some } \gamma > 0. \tag{1.4}$$

A simple integration of (1.4) over the interval $(0, t)$, together with the equivalence (1.3), leads to the desired exponential stability result.

1.7 Stabilization types

There are several types of stabilization, classified based on the rate at which the energy of the system decays to zero as time progresses.

- **Strong Stabilization:** This refers to the situation where the energy of the system decays to zero as time tends to infinity. That is

  $$E(t) \to 0 \quad \text{as} \quad t \to \infty.$$

  This type of stabilization does not specify the speed of decay, only that the energy eventually vanishes.

- **Exponential (Uniform) Stabilization:** In this case, the energy decays exponentially fast, which is the fastest type of stabilization. There exist constants $\alpha > 0$ and $C > 0$ such that

  $$E(t) \leq Ce^{-\alpha t}, \quad \forall t > 0.$$

  This implies a uniform and rapid decay of the energy.

- **Polynomial Stabilization:** Here, the energy decays at a polynomial rate, which is slower than exponential decay. There exist constants $\beta > 0$ and $C > 0$ such that

  $$E(t) \leq \frac{C}{t^{\beta}}, \quad \forall t > 0.$$

  This behavior typically arises when exponential decay is not possible due to geometric or damping limitations.

Chapter 2

Well-Posedness

In this chapter, we will prove the existence and uniqueness of the solutions to the Euler-Bernoulli beam equation by using the Galerkin method.

2.1 Problem Presentation

Let $[0, L]$, with $L > 0$, be an interval in $\mathbb{R}$. Our objective is to investigate the existence and uniqueness of the solution to the Euler-Bernoulli beam equation with a memory term and a boundary output feedback control term.
The problem is mathematically modeled as follows: We seek a real-valued function $\omega(x, t)$, where $x \in [0, L]$ and $t \in \mathbb{R}_+$, that satisfies the following linear partial differential equation

$$\begin{cases} \omega_{tt}(x,t) + \omega_{xxxx}(x,t) - \displaystyle\int_0^t \kappa(t-\tau)\omega_{xxxx}(x,\tau)\, d\tau + g(\omega_t(x,t)) = 0, & x \in [0, L],\ t > 0, \\ \omega(0,t) = \omega_x(0,t) = \omega_{xx}(L,t) = 0, & t \geq 0, \\ \omega_{xxx}(L,t) - \displaystyle\int_0^t \kappa(t-\tau)\omega_{xxx}(L,\tau)\, d\tau = f(t)\omega_t(L,t), & t \geq 0, \\ \omega(x,0) = \omega_0(x), \quad \omega_t(x,0) = \omega_1(x), \quad x \in [0, L] & , \\ f_t(t) = r(\omega_t(L,t))^2, \quad t > 0, \quad f(0) = f_0 > 0, \quad r > 0. & \end{cases} \tag{2.1}$$

The energy of the system (2.1) is given by

$$E(t) = \frac{1}{2}\int_0^L \omega_t^2(x,t) + \omega_{xx}^2(x,t)\, dx. \tag{2.2}$$

In order to demonstrate the existence and uniqueness of the solution to the system (2.1), we first present the essential notation and assumptions required for the proof.

Notation 2.1.1. *Let $L^2(0, L)$ be the usual Hilbert space with the inner product*

$$\langle f, g\rangle = \int_0^L f(x)g(x)\, dx,$$

and the norm

$$\|f\|_{L^2} = \|f\|.$$

• Throughout this paper, we define the space

$$V = \left\{ \omega \in H^2(0, L) \mid \omega(0) = \omega_x(0) = 0 \right\},$$

equipped with norm

$$\|\omega\|_V = \|\omega_{xx}\|.$$

And the space

$$U = \left\{ \omega \in V \cap H^4(0, L) \mid \omega_{xx}(L) = 0 \right\},$$

equipped with norm

$$\|\omega\|_U = \|\omega_{xx}\| + \|\omega_{xxxx}\|.$$

Remark 2.1.1. *According to Poincaré's inequality, $\|\omega\|_U$ and $\|\omega\|_V$ are equivalent to the standard norms of $H^4(0, L)$ and $H^2(0, L)$, respectively.*

• We now state the following hypotheses:

($\boldsymbol{H}_1$) For any $\omega(x, 0) = \omega_0(x) \in U$ and $\omega_t(x, 0) = \omega_1(x) \in L^2(0, L)$, we have

$$\omega(0, 0) = \omega_x(0, 0) = \omega_{xx}(L, 0) = 0, \tag{2.3}$$

$$\omega_{xxx}(L, 0) = f_0 \omega_t(L, 0). \tag{2.4}$$

The functions κ and g are assumed to satisfy the following conditions:

($\boldsymbol{H}_2$) For any $\kappa \in \mathcal{W}^{1,\infty}(0, \infty) \cap \mathcal{W}^{2,1}(0, \infty)$, such that $\kappa(t) \geq 0$ for all $t \geq 0$, and for some $\alpha_1, \alpha_2, \alpha_3, \alpha_4 > 0$, we assume

$$\begin{aligned} -\alpha_1 \kappa(t) \leq \kappa_t(t) \leq -\alpha_2 \kappa(t), \qquad & \forall t \geq t_0, \\ |\kappa_t(t)| \leq \alpha_3 \kappa(t); \qquad & 0 \leq t \leq t_0, \\ 0 \leq \kappa_{tt}(t) \leq \alpha_4 \kappa(t), \qquad & t \geq 0, \end{aligned} \tag{2.5}$$

and

$$\ell = 1 - \int_0^\infty \kappa(\tau)\, d\tau \geq 0. \tag{2.6}$$

($\boldsymbol{H}_3$) Let $g : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function. Assume that there exists a constant $\beta > 0$ such that

$$g(0) = 0 \quad \text{and} \quad (g(u) - g(v))(u - v) \geq \beta |u - v|^2 \quad \text{for all } u, v \in \mathbb{R}. \tag{2.7}$$

• To simplify, we denote by $\diamond$ the operator defined by

$$(\kappa \diamond \omega_{xx})(t) = \int_0^t \kappa(t - \tau) \|\omega_{xx}(\tau) - \omega_{xx}(t)\|^2\, d\tau, \tag{2.8}$$

such that

$$\begin{aligned}\frac{d}{dt}(\kappa\diamond\omega_{xx})(t)=&\int_0^t \kappa'(t-\tau)\|\omega_{xx}(\tau)-\omega_{xx}(t)\|^2\,d\tau+\int_0^t \kappa(t-\tau)\frac{d}{dt}\|\omega_{xx}(\tau)-\omega_{xx}(t)\|^2\,d\tau\\ =&(\kappa'\diamond\omega_{xx})(t)+2\int_0^t \kappa(t-\tau)\langle\omega_{xx}(\tau)-\omega_{xx}(t),-\omega_{xxt}(t)\rangle\,d\tau\\ =&(\kappa'\diamond\omega_{xx})(t)-2\int_0^t \kappa(t-\tau)\langle\omega_{xx}(\tau),\omega_{xxt}(t)\rangle\,d\tau+\frac{d}{dt}\|\omega_{xx}(t)\|^2\int_0^t\kappa(t-\tau)\,d\tau\\ =&(\kappa'\diamond\omega_{xx})(t)+2\int_0^t \kappa(t-\tau)\langle\omega_{xx}(\tau),\omega_{xxt}(t)\rangle\,d\tau-\frac{d}{dt}\left(\|\omega_{xx}(t)\|^2\int_0^t\kappa(\tau)\,d\tau\right)\\ &-\kappa(t)\|\omega_{xx}(t)\|^2.\end{aligned}$$

Thus, we have

$$\begin{aligned}\int_0^t \kappa(t-\tau)\langle\omega_{xx}(\tau),\omega_{xxt}(t)\rangle\,d\tau=&\frac{1}{2}(\kappa'\diamond\omega_{xx})(t)-\frac{1}{2}(\kappa\diamond\omega_{xx})'(t)\\ &+\frac{1}{2}\frac{d}{dt}\left(\|\omega_{xx}(t)\|^2\int_0^t\kappa(\tau)\,d\tau\right)-\frac{1}{2}\kappa(t)\|\omega_{xx}(t)\|^2.\end{aligned}\tag{2.9}$$

•The modified energy is defined by:

$$\begin{aligned}\varphi(t)=&E(t)+\frac{1}{2}\left((\kappa\diamond\omega_{xx})(t)-\|\omega_{xx}(t)\|^2\int_0^t\kappa(\tau)\,d\tau\right)\\ =&\frac{1}{2}\|\omega_t(t)\|^2+\frac{1}{2}(\kappa\diamond\omega_{xx})(t)+\frac{1}{2}\left(1-\int_0^t\kappa(\tau)\,d\tau\right)\|\omega_{xx}(t)\|^2.\end{aligned}\tag{2.10}$$

2.2 Existence of the Solution

Considering the above hypotheses, we have the following theorem.

Theorem 2.2.1. *Let $\omega_0\in U,\omega_1\in L^2(0,L)$. Suppose that assumptions (**H1**), (**H2**), and (**H3**) are satisfied. Then, the problem (2.1) admits a unique solution ω in the following sense: for any $T>0$, we have*

$$\omega\in L^\infty(0,T;U),\quad \omega_t\in L^\infty(0,T;V),$$

$$\omega_{tt}\in L^\infty(0,T;L^2(0,L)),\quad f\in C^1(0,T).$$

By applying the Sobolev embedding theorem, we deduce that the solution

$$\omega\in C\left((0,L)\times[0,T]\right).$$

Next, We will prove the theorem 2.2.1. By using Faedo-Galerkin method.

2.2.1 Variational Form

Let $\xi \in V$. Multiplying the fundamental differential equation of the problem (2.1) by a function ξ, and integrating over the domain $[0, L]$, we obtain

$$\langle \xi, \omega_{tt}(t)\rangle + \langle \xi, \omega_{xxxx}(t)\rangle - \langle \xi, \int_0^t \kappa(t-\tau)\omega_{xxxx}(\tau)\, d\tau\rangle + \langle \xi, g(\omega_t(t))\rangle = 0. \tag{2.11}$$

By applying integration by parts twice, we obtain

$$\begin{aligned}\langle \xi, \omega_{xxxx}(t)\rangle &= \int_0^L \xi(x)\omega_{xxxx}(x,t)\, dx \\ &= \Big[\omega_{xxx}(x,t)\xi(x)\Big]_0^L - \Big[\omega_{xx}(x,t)\xi_x(x)\Big]_0^L + \int_0^L \xi_{xx}(x)\omega_{xx}(x,t)\, dx.\end{aligned}$$

For $\xi \in V$ and the boundary conditions, we get

$$\langle \xi, \omega_{xxxx}(t)\rangle = \xi(L)\omega_{xxx}(L,t) + \langle \xi_{xx}, \omega_{xx}(t)\rangle. \tag{2.12}$$

On the other hand, by Fubini theorem, we have

$$\begin{aligned}\langle \xi, \int_0^t \kappa(t-\tau)\omega_{xxxx}(\tau)\, d\tau\rangle &= \int_0^L \xi(x)\left(\int_0^t \kappa(t-\tau)\omega_{xxxx}(x,\tau)\, d\tau\right)\, dx \\ &= \int_0^L \int_0^t \xi(x)\kappa(t-\tau)\omega_{xxxx}(x,\tau)\, d\tau\, dx \\ &= \int_0^t \int_0^L \xi(x)\kappa(t-\tau)\omega_{xxxx}(x,\tau)\, dx\, d\tau \\ &= \int_0^t \kappa(t-\tau)\left(\int_0^L \xi(x)\omega_{xxxx}(x,\tau)\, dx\right)\, d\tau \\ &= \int_0^t \kappa(t-\tau)\langle \xi, \omega_{xxxx}(\tau)\rangle\, d\tau\langle \omega_t \\ &= \xi(L)\int_0^t \kappa(t-\tau)\omega_{xxx}(L,\tau)\, d\tau + \int_0^t \kappa(t-\tau)\langle \xi_{xx}, \omega_{xx}(\tau)\rangle\, d\tau.\end{aligned} \tag{2.13}$$

By substituting (2.12) and (2.13) in (2.11) then

$$\begin{aligned}\langle \xi, \omega_{tt}(t)\rangle &- \int_0^t \kappa(t-\tau)\langle \xi_{xx}, \omega_{xx}(\tau)\rangle + \xi(L)\left(\omega_{xxx}(L,t) - \int_0^t \kappa(t-\tau)\omega_{xxx}(L,\tau)d\tau\right) \\ &+ \langle \xi, g(\omega_t(t))\rangle + \langle \xi_{xx}, \omega_{xx}(t)\rangle = 0.\end{aligned} \tag{2.14}$$

The problem (2.1) can be formulated as: Found the solution $\omega(t) \in V$ such that

$$\begin{aligned}\langle \xi, \omega_{tt}(t)\rangle + \langle \xi_{xx}, \omega_{xx}(t)\rangle - \int_0^t \kappa(t-\tau)\langle \xi_{xx}, \omega_{xx}(\tau)\rangle + \xi(L)f(t)\omega_t(L,t) \\ + \langle \xi, g(\omega_t(t))\rangle = 0, \quad \forall \xi \in V.\end{aligned} \tag{2.15}$$

Now, we are ready to applying Fadeo-Galerkin approximation.
Let $\{\xi_j\}$ be a complete orthogonal system of V, such that the initial data $\{\omega_0, \omega_1\} \in \text{Span}\{\xi_1, \xi_2\}$.

For each $n \in \mathbb{N}$, we define the finite-dimensional subspace $V_n = \text{Span}\{\xi_1, \xi_2, \cdots, \xi_n\}$ of V.
We look for an approximate solution of the form

$$\omega^n(x,t) = \sum_{j=1}^{n} \phi_j(t)\xi_j(x),$$

where $\phi_j(t)$ are time-dependent coefficients to be determined.
The function $\omega^n(x,t)$ is required to satisfy the following weak formulation: for every $\xi \in V_n$, the approximate equation must hold

$$\begin{cases} \langle \xi, \omega^n_{tt}(t)\rangle + & \langle \xi_{xx}, \omega^n_{xx}(t)\rangle - \int_0^t \kappa(t-\tau)\langle \xi_{xx}, \omega^n_{xx}(\tau)\rangle \, d\tau + \xi(L) f^n(t)\omega^n_t(L,t) \\ & +\langle \xi, g(\omega^n_t(t))\rangle = 0, \quad \forall \xi \in V_n, \end{cases} \tag{2.16}$$

with condition

$$\begin{cases} f^n_t(t) = r(\omega^n_t(t))^2 = r\left[\sum_{j=1}^{n} \phi^j_t(t)\xi^j(L)\right]^2, \\ f^n(0) = f^n_0 \geq 0, \\ \omega^n(x,0) = \omega^n_0(x) \to \omega_0(x) \text{ in } U, \quad \omega^n_t(x,0) = \omega^n_1(x) \to \omega_1(x) \text{ in } V. \end{cases} \tag{2.17}$$

Equations (2.16) yield a system of ordinary differential equations with n unknown functions $\phi_j(t)$, where $j = 1, 2, \ldots, n$.
By standard methods in differential equations, we can prove the existence of a solution to equation (2.16) on some interval $[0, t_m)$. Then, using the first estimate below, this solution can be extended to the entire interval $[0, T)$, where $T = \infty$.

2.2.2 Apriori Estimate *I*

Replacing ξ by ω^n_t in (2.16), we get

$$\begin{aligned} &\langle \omega^n_{tt}(t), \omega^n_t(t)\rangle + \langle \omega^n_{xx}(t), \omega^n_{txx}(t)\rangle - \int_0^t \kappa(t-\tau)\langle \omega^n_{xx}(\tau), \omega^n_{txx}(t)\rangle + \langle g(\omega^n_t(t)), \omega^n_t(t)\rangle \\ &\quad + f^n(t)(\omega^n_t(L,t))^2 = 0, \end{aligned}$$

then

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|\omega^n_t(t)\|^2 + \frac{1}{2}\frac{d}{dt}\|\omega^n_{xx}(t)\|^2 + \langle g(\omega^n_t(t)), \omega^n_t(t)\rangle \\ &\qquad = \int_0^t \kappa(t-\tau)\langle \omega^n_{xx}(\tau), \omega^n_{txx}(t)\rangle \, d\tau - f^n(t)(\omega^n_t(L,t))^2. \end{aligned}$$

Consider the function

$$h(t) = \int_0^t \kappa(t-\tau)\langle \omega^n_{xx}(\tau), \omega^n_{xx}(t)\rangle \, d\tau,$$

then

$$\int_0^t \kappa(t-\tau)\langle \omega_{xx}^n(\tau), \omega_{txx}^n(t)\rangle \, d\tau = \frac{d}{dt}\left(\int_0^t \kappa(t-\tau)\langle \omega_{xx}^n(\tau), \omega_{xx}^n(t)\rangle \, d\tau\right) - \int_0^t \kappa_t(t-\tau)\langle \omega_{xx}^n(\tau), \omega_{xx}^n(t)\rangle \, d\tau - \kappa(0)\|\omega_{xx}^n(t)\|^2 \tag{2.18}$$

Hence, we have

$$\frac{1}{2}\frac{d}{dt}\|\omega_t^n(t)\|^2 + \frac{1}{2}\frac{d}{dt}\|\omega_{xx}^n(t)\|^2 + \langle g(\omega_t^n(t)), \omega_t^n(t)\rangle = \frac{d}{dt}\left(\int_0^t \kappa(t-\tau)\langle \omega_{xx}^n(\tau), \omega_{xx}^n(t)\rangle \, d\tau\right) - \int_0^t \kappa_t(t-\tau)\langle \omega_{xx}^n(\tau), \omega_{xx}^n(t)\rangle \, d\tau - \kappa(0)\|\omega_{xx}^m(t)\|^2 - f^n(t)(\omega_t^n(L,t))^2 \tag{2.19}$$

By using Cauchy–Schwarz inequality, then using assumption (2.5), and according to Young's inequality, we have

$$\begin{aligned}\left|\int_0^t \kappa_t(t-\tau)\langle \omega_{xx}^n(\tau), \omega_{xx}^n(t)\rangle d\,\tau\right| &\leq \alpha\|\omega_{xx}^n(t)\| \int_0^t \kappa(t-\tau)\|\omega_{xx}^n(\tau)\| \, d\tau \\ &\leq \frac{\alpha^2}{2}\|\omega_{xx}^n(t)\|^2 + \frac{1}{2}\left(\int_0^t \kappa(t-\tau)\|\omega_{xx}^n(\tau)\|^2 \, d\tau\right)^2 \\ &\leq \frac{\alpha^2}{2}\|\omega_{xx}^n(t)\|^2 + \frac{1}{2}\|\kappa\|_{L^1(0,\infty)} \int_0^t \kappa(t-\tau)\|\omega_{xx}^n(\tau)\|^2 \, d\tau,\end{aligned} \tag{2.20}$$

where $\alpha = \alpha_1 + \alpha_3$.
By assumption (2.7), and inequality (2.20), we get

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\omega_t^n(t)\|^2 + \frac{1}{2}\frac{d}{dt}\|\omega_{xx}^n(t)\|^2 + \beta\|\omega_t^n(t)\|^2 \leq \frac{d}{dt}\left(\int_0^t \kappa(t-\tau)\langle \omega_{xx}^n(\tau), \omega_{xx}^n(t)\rangle \, d\tau\right) \\ + \frac{\alpha^2}{2}\|\omega_{xx}^n(t)\|^2 + \frac{1}{2}\|\kappa\|_{L^1(0,\infty)} \int_0^t \kappa(t-\tau)\|\omega_{xx}^n(\tau)\|^2 \, d\tau - \kappa(0)\|\omega_{xx}^m(t)\|^2 \\ - f^n(t)(\omega_t^n(L,t))^2\end{aligned} \tag{2.21}$$

Integrating (2.21) over $(0,t)$

$$\begin{aligned}&\int_0^t \frac{1}{2}\frac{d}{ds}\|\omega_t^n(s)\|^2 \, ds + \int_0^t \frac{1}{2}\frac{d}{ds}\|\omega_{xx}^n(s)\|^2 \, ds + \int_0^t \beta\|\omega_t^n(\tau)\|^2 \, d\tau \\ &\leq \int_0^t \frac{d}{dt}\left(\int_0^s \kappa(s-\tau)\langle \omega_{xx}^n(\tau), \omega_{xx}^n(s)\rangle \, d\tau\right) ds + \int_0^t \frac{\alpha^2}{2}\|\omega_{xx}^n(\tau)\|^2 \, d\tau \\ &+ \int_0^t \frac{1}{2}\|\kappa\|_{L^1(0,\infty)} \int_0^s \kappa(s-\tau)\|\omega_{xx}^n(\tau)\|^2 \, d\tau \, ds - \int_0^t \kappa(0)\|\omega_{xx}^m(\tau)\|^2 \, d\tau - \int_0^t f^n(\tau)(\omega_t^n(L,\tau))^2 \, d\tau.\end{aligned} \tag{2.22}$$

Then, we obtain

$$\begin{aligned}&\|\omega_t^n(t)\|^2 - \|\omega_t^n(0)\|^2 + \|\omega_{xx}^n(t)\|^2 - \|\omega_{xx}^n(0)\|^2 + 2\beta\int_0^t \|\omega_t^n(\tau)\|^2 \, d\tau \\ &\leq 2\int_0^t \kappa(t-\tau)\langle \omega_{xx}^n(\tau), \omega_{xx}^n(t)\rangle \, d\tau + \left(\alpha^2 + \|\kappa\|_{L^1(0,\infty)}^2 - 2\kappa(0)\right)\int_0^t \|\omega_{xx}^m(\tau)\|^2 \, d\tau - \frac{1}{r}(f^n(t))^2 \\ &+ \frac{1}{r}(f^n(0))^2.\end{aligned} \tag{2.23}$$

By using Schwartz inequality and young's inequality, we have

$$\begin{aligned}|\int_0^t \kappa(t-\tau)\langle \omega_{xx}^n(\tau), \omega_{xx}^n(t)\rangle\, d\tau| &\leq \|\omega_{xx}^n(t)\| \int_0^t \kappa(t-\tau)\|\omega_{xx}^n(\tau)\|\, d\tau \\ &\leq \|\omega_{xx}^n(t)\|\|\kappa\|_{L^1(0,\infty)}^{\frac{1}{2}} \left(\int_0^t \kappa(t-\tau)\|\omega_{xx}^n(\tau)\|^2\, d\tau\right)^{\frac{1}{2}} \\ &\leq \frac{1}{4\eta}\|\omega_{xx}^n(t)\|^2 + \eta\|\kappa\|_{L^1(0,\infty)}\|\kappa\|_{L^\infty(0,\infty)} \int_0^t \|\omega_{xx}^n(\tau)\|^2\, d\tau.\end{aligned} \tag{2.24}$$

Combining the inequalities (2.23)and (2.24), we obtain

$$\begin{aligned}&\|\omega_t^n(t)\|^2 + \|\omega_{xx}^n(t)\|^2 + 2\beta\int_0^t \|\omega_t^n(\tau)\|^2\, d\tau + \frac{1}{r}(f^n(t))^2 \leq \|\omega_{xx}^n(0)\|^2 + \|\omega_t^n(0)\|^2 + \frac{1}{2\eta}\|\omega_{xx}^n(t)\|^2 \\ &+ 2\eta\|\kappa\|_{L^1(0,\infty)}\|\kappa\|_{L^\infty(0,\infty)}\int_0^t \|\omega_{xx}^n(\tau)\|^2\, d\tau + \left(\alpha^2 + \|\kappa\|_{L^1(0,\infty)}^2 - 2\kappa(0)\right)\int_0^t \|\omega_{xx}^m(\tau)\|^2\, d\tau \\ &+ \frac{1}{r}(f^n(0))^2.\end{aligned} \tag{2.25}$$

According to Gronwall's Lemma, we get

$$\|\omega_t^n(t)\|^2 + \|\omega_{xx}^n(t)\|^2 + \frac{1}{r}(f^n(t))^2 \leq Me^{Ct}, \quad \forall t \geq 0,$$

where M is a constant that depends on the initial data ω_0, ω_1, f_0.
Then

$$\|\omega_t^n(t)\|^2 + \|\omega_{xx}^n(t)\|^2 + \frac{1}{r}(f^n(t))^2 \leq C_1. \tag{2.26}$$

This implies that $\omega_t^n(L,t)$ belongs to $L^\infty(0,\infty)$. Consequently, the approximate solution ω^n can be extended to the entire interval $[0,T)$ with $T = \infty$.

2.2.3 Apriori Estimate *II*

First of all, we estimate for $\omega_{tt}^n(0)$ in the L^2-norm.
Let $t = 0$ and $\xi = \omega_{tt}^n(0)$ in (2.16), then we get

$$\langle \omega_{tt}^n(0), \omega_{tt}^n(0)\rangle + \langle \omega_{xxtt}^n(0), \omega_{xx}^n(0)\rangle + \omega_{tt}^n(L,0)f^n(0)\omega_t^n(L,0) + \langle \omega_{tt}^n(0), g(\omega_t^n(0))\rangle = 0.$$

Since $\omega(0,t) = \omega_x(0,t) = \omega_{xx}(L,t) = 0, \omega_0 \in U$, in view of the condition (2.4), we have

$$\begin{aligned}\langle \omega_{xxtt}^n(0), \omega_{xx}^n(0)\rangle &= \int_0^L \omega_{xxtt}^n(x,0)\omega_{xx}^n(x,0)\, dx \\ &= [\omega_{xtt}^n(x,0)\omega_{xx}^n(x,0)]_0^L - [\omega_{tt}^n(x,0)\omega_{xxx}^n(x,0)]_0^L + \int_0^L \omega_{tt}^n(x,0)\omega_{xxxx}^n(x,0)\, dx \\ &= -f_0^n\omega_t^n(L,0)\omega_{tt}^n(L,0) + \langle \omega_{tt}^n(0).\omega_{xxxx}^n(0)\rangle.\end{aligned} \tag{2.27}$$

From the above inequality, we conclude that

$$\|\omega_{tt}^n(0)\|^2 + \langle \omega_{tt}^n(0), \omega_{xxxx}^n(0)\rangle + \langle \omega_{tt}^n(0), g(\omega_t^n(0))\rangle = 0. \tag{2.28}$$

By Cauchy Schwartz, we obtain

$$\|\omega_{tt}^n(0)\|^2 \leq \|\omega_{tt}^n(0)\|\|\omega_{xxxx}^n(0)\| + \|\omega_{tt}^n(0)\|\|g(\omega_t^n(0))\|.$$

As a consequence of equations (2.7) and (2.16-2.17), one can find a constant $C_2 > 0$, depending only on the initial data ω_0 and ω_1, such that

$$\|\omega_{tt}^n(0)\| \leq C_2, \quad \forall n \in \mathbb{N}.$$

Now, we will estimate $\omega_{tt}(t)$, and $\omega_{xxt}(t)$ in L^2-norm.
Differentiating equation (2.16) with respect to time and setting $\xi = \omega_{tt}^n(t)$, we derive the following

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\omega_{tt}^n(t)\|^2 + \frac{1}{2}\frac{d}{dt}\|\omega_{xxt}^n(t)\|^2 - \int_0^t \kappa'(t-\tau)\langle \omega_{xxtt}^n(t), \omega_{xx}^n(\tau)\rangle\, d\tau - \kappa(0)\langle \omega_{xxtt}^n(t), \omega_{xx}^n(t)\rangle \\
&\qquad + \langle \omega_{tt}^n(t) g'(\omega_t^n(t)), \omega_{tt}^n(t)\rangle + r(\omega_t^n(L,t))^3 \omega_{tt}^n(L,t) + f^n(t)(\omega_{tt}^n(L,t))^2 = 0.
\end{aligned}$$

From (2.18), we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\omega_{tt}^n(t)\|^2 + \frac{1}{2}\frac{d}{dt}\|\omega_{xxt}^n(t)\|^2 - \frac{d}{dt}\left(\int_0^t \kappa'(t-\tau)\langle \omega_{xx}^n(\tau), \omega_{txx}^n(t)\rangle\, d\tau\right) + \kappa'(0)\langle \omega_{xx}^n(t), \omega_{xxt}^n(t)\rangle \\
&\quad + \int_0^t \kappa''(t-\tau)\langle \omega_{xx}^n(\tau), \omega_{xxt}^n(t)\rangle\, d\tau - \kappa(0)\frac{d}{dt}\langle \omega_{xx}^n(t), \omega_{xxt}^n(t)\rangle + \kappa(0)\|\omega_{xxt}^n(t)\|^2 \\
&\quad + \langle \omega_{tt}^n(t) g'(\omega_t^n(t)), \omega_{tt}^n(t)\rangle + r(\omega_t^n(L,t))^3 \omega_{tt}^n(L,t) + f^n(t)(\omega_{tt}^n(L,t))^2 = 0.
\end{aligned}$$

Then

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\omega_{tt}^n(t)\|^2 + \frac{1}{2}\frac{d}{dt}\|\omega_{xxt}^n(t)\|^2 + \kappa(0)\|\omega_{xxt}^n(t)\|^2 + \langle \omega_{tt}^n(t) g'(\omega_t^n(t)), \omega_{tt}^n(t)\rangle + r(\omega_t^n(L,t))^3 \omega_{tt}^n(L,t) \\
&\quad + f^n(t)(\omega_{tt}^n(L,t))^2 = \frac{d}{dt}\left(\int_0^t \kappa'(t-\tau)\langle \omega_{xx}^n(\tau), \omega_{txx}^n(t)\rangle\, d\tau\right) - \int_0^t \kappa''(t-\tau)\langle \omega_{xx}^n(\tau), \omega_{xxt}^n(t)\rangle\, d\tau \\
&\quad - \kappa'(0)\langle \omega_{xx}^n(t), \omega_{xxt}^n(t)\rangle + \kappa(0)\frac{d}{dt}\langle \omega_{xx}^n(t), \omega_{xxt}^n(t)\rangle.
\end{aligned} \tag{2.29}$$

According to Cauchy Schwartz and young's inequalities, and from assumption (2.5), we deduce that for all $t \geq 0$

$$\left|\int_0^t \kappa''(t-\tau)\langle \omega_{xx}^n(\tau), \omega_{xxt}^n(t)\rangle\, d\tau\right| \leq \frac{\alpha_4^2}{2}\|\omega_{xxt}(t)\|^2 + \frac{1}{2}\|\kappa\|_{L^1(0,\infty)}\int_0^t \kappa(t-\tau)\|\omega_{xx}^n(\tau)\|^2\, d\tau. \tag{2.30}$$

Since $g \in C^1(\mathbb{R})$ and $(\omega_t^n(t))$ is bounded, there exists $\beta_1 > 0$ depends on the initial data ω_0, ω_1, f_0 such that

$$|\langle \omega_{tt}^n(t) g'(\omega_t^n(t)), \omega_{tt}^n(t)\rangle| \leq \beta_1\|\omega_{tt}^n(t)\|^2. \tag{2.31}$$

Using (2.30) and (2.31), we integrate (2.29) over $(0,t)$, we get

$$\begin{aligned}
&\frac{1}{2}\|\omega_{tt}^n(t)\|^2+\frac{1}{2}\|\omega_{xxt}^n(t)\|^2+\kappa(0)\int_0^t\|\omega_{xxt}^n(\tau)\|^2\,d\tau+\frac{r}{4}(\omega_t^n(L,t))^4\\
&\leq \beta_1\int_0^t\|\omega_{tt}^n(\tau)\|^2\,d\tau+\int_0^t\kappa'(t-\tau)\langle\omega_{xx}^n(\tau),\omega_{xxt}^n(t)\rangle\,d\tau+\frac{\alpha_4^2}{2}\int_0^t\|\omega_{xxt}^n(\tau)\|^2\,d\tau\\
&+\frac{1}{2}\|\kappa\|_{L^1(0,\infty)}^2\int_0^t\|\omega_{xx}^n(\tau)\|^2\,d\tau-\kappa'(0)\int_0^t\langle\omega_{xx}^n(\tau),\omega_{xxt}^n(\tau)\rangle+\kappa(0)\langle\omega_{xx}^n(t),\omega_{xxt}^n(t)\\
&-\int_0^t f^n(\tau)(\omega_{tt}^n(L,\tau))^2\,d\tau+C_1,
\end{aligned}\tag{2.32}$$

where $C_1>0$ depends on the initial data.
By using Cauchy Schwartz and Young's inequalities, we have that

$$\kappa(0)\langle\omega_{xx}^n(t),\omega_{xxt}^n(t)\rangle\leq\frac{\kappa(0)^2}{4\eta}\|\omega_{xx}^n(t)\|^2+\eta\|\omega_{xxt}^n(t)\|^2.\tag{2.33}$$

$$\left|\kappa'(0)\int_0^t\langle\omega_{xx}^n(\tau),\omega_{xxt}^n(\tau)\rangle\right|\leq\frac{(\alpha\kappa(0))^2}{4\eta}\int_0^t\|\omega_{xx}^n(\tau)\|^2\,d\tau+\eta\int_0^t\|\omega_{xxt}^n(\tau)\|^2\,d\tau,\tag{2.34}$$

and

$$\int_0^t\kappa'(t-\tau)\langle\omega_{xx}^n(\tau),\omega_{xxt}^n(t)\rangle\,d\tau\leq\frac{\alpha^2}{4\eta}\|\kappa\|_{L^1(0,\infty)}\|\kappa\|_{L^\infty(0,\infty)}\int_0^t\|\omega_{xx}^n(\tau)\|^2\,d\tau+\eta\|\omega_{xxt}^n(\tau)\|^2\,d\tau,\tag{2.35}$$

such that $\eta>0$.
Thus from (2.32)-(2.35), we obtain

$$\begin{aligned}
&\frac{1}{2}\|\omega_{tt}^n(t)\|^2+\frac{1}{2}\|\omega_{xxt}^n(t)\|^2+\kappa(0)\int_0^t\|\omega_{xxt}^n(\tau)\|^2\,d\tau+\frac{r}{4}(\omega_t^n(L,t))^4\leq\beta_1\int_0^t\|\omega_{tt}^n(\tau)\|^2\,d\tau\\
&+\left(\frac{\alpha^2}{4\eta}\|\kappa\|_{L^1(0,\infty)}\|\kappa\|_{L^\infty(0,\infty)}+\frac{1}{2}\|\kappa\|_{L^1(0,\infty)}^2+\frac{(\alpha\kappa(0))^2}{4\eta}\right)\int_0^t\|\omega_{xx}^n(\tau)\|^2\,d\tau+2\eta\|\omega_{xxt}^n(\tau)\|^2\,d\tau\\
&+\left(\frac{\alpha_4^2}{2}+\eta\right)\int_0^t\|\omega_{xxt}^n(\tau)\|^2\,d\tau+\frac{\kappa(0)^2}{4\eta}\|\omega_{xx}^n(t)\|^2-\int_0^t f^n(\tau)(\omega_{tt}^n(L,\tau))^2\,d\tau+C_1,
\end{aligned}\tag{2.36}$$

Choosing $\eta>0$ sufficiently small and considering the first estimate, then we obtain

$$\begin{aligned}
&\frac{1}{2}\|\omega_{tt}^n(t)\|^2+\left(\frac{1}{2}-2\eta\right)\|\omega_{xxt}^n(t)\|^2+\kappa(0)\int_0^t\|\omega_{xxt}^n(\tau)\|^2\,d\tau+\frac{r}{4}(\omega_t^n(L,t))^4\leq\beta_1\int_0^t\|\omega_{tt}^n(\tau)\|^2\,d\tau\\
&+\left(\frac{\alpha_4^2}{2}+\eta\right)\int_0^t\|\omega_{xxt}^n(\tau)\|^2\,d\tau-\int_0^t f^n(\tau)(\omega_{tt}^n(L,\tau))^2\,d\tau+C,
\end{aligned}\tag{2.37}$$

where $C>0$ depends on ω_0,ω_1,f_0. By applying Gronwall's inequality, we deduce that

$$\|\omega_{tt}^n(t)\|^2+\|\omega_{xxt}^n(t)\|^2+r(f_t^n(L,t))^4\leq M'e^{C't},\quad\forall t\geq 0,$$

where M' is a constant that depends on the initial data ω_0, ω_1, f_0.
Then

$$\|\omega_{tt}^n(t)\|^2 + \|\omega_{xxt}^n(t)\|^2 + r(\omega_t^n(L,t))^4 \leq C_2, \tag{2.38}$$

where $C_2 > 0$ depends on ω_0, ω_1, f_0.

2.2.4 Passage to Limits

By estimates (2.26) and (2.38), we deduce that for all $n \in \mathbb{N}$ and $T > 0$

$$\begin{cases} (\omega^n) \text{ is bounded in } L^\infty(0,T;V) \\ (\omega_t^n) \text{ is bounded in } L^\infty(0,T;V) \\ (\omega_{tt}^n) \text{ is bounded in } L^\infty(0,T;L^2(0,L)) \\ (\omega_t^n(L,t)) \text{ is bounded in } L^\infty(0,T) \\ (f^n) \text{ is bounded in } L^\infty(0,T) \\ (f_t^n) = (r(\omega_t^n(L,t))^2) \text{ is bounded in } L^\infty(0,T) \end{cases} \tag{2.39}$$

Therefore, there exists a subsequence (ω^m) of (ω^n), such that

$$\begin{cases} \omega^m \overset{*}{\rightharpoonup} \omega \quad \text{weak-* in } L^\infty(0,T;V), \\ \omega_t^m \overset{*}{\rightharpoonup} \omega_t \quad \text{weak-* in } L^\infty(0,T;V), \\ \omega_{tt}^m \overset{*}{\rightharpoonup} \omega_{tt} \quad \text{weak-* in } L^\infty(0,T;L^2(0,L)), \\ \omega_t^m(L,t) \overset{*}{\rightharpoonup} \omega_t(L,t) \quad \text{weak-* in } L^\infty(0,T), \\ f^m \overset{*}{\rightharpoonup} f \quad \text{weak-* in } L^\infty(0,T), \\ f_t^m \overset{*}{\rightharpoonup} f \quad \text{weak-* in } L^\infty(0,T). \end{cases} \tag{2.40}$$

Due to the compact embedding $V \hookrightarrow\hookrightarrow L^2(0,L)$, we obtain a subsequence such that

$$\omega_t^m \to \omega_t \quad \text{strongly in } L^2(0,T;L^2(0,L)). \tag{2.41}$$

According to (2.41), it follows that

$$g(\omega_t^m) \to g(\omega_t) \quad \text{a.e. in } x \in (0,L),\ t > 0.$$

From (2.39), (2.40), and (2.41), and using the boundedness of the sequence $(g(\omega_t^m))$ in $L^2(0,T;L^2(0,L))$, we deduce by **Lion's Lemma** that

$$g(\omega_t^m) \rightharpoonup g(\omega_t) \quad \text{weakly in } L^2(0,T;L^2(0,L)).$$

Furthermore, by the **Sobolev embedding theorem** and using (2.40), we obtain

$$f \in C^1[0,T], \quad \text{and} \quad f^m(t)y_t^m(L,t) \rightharpoonup f(t)\omega_t(L,t) \quad \text{weakly in } L^2(0,T).$$

These convergences are sufficient to pass to the limit in the nonlinear terms of equation (2.16). Hence, the existence of global solutions in $[0,T]$ can be established.

2.3 Uniqueness of the Solution

Let ω^1, ω^2 be two solutions of problem (2.1), such that

$$\omega^i \in L^\infty(0,T;U), \quad \omega^i_t \in L^\infty(0,T;V), \quad \omega^i_{tt} \in L^\infty(0,T;L^2(0,L)), \quad \text{for } i=1,2, \quad f \in [0,T].$$

We put $y = \omega^1 - \omega^2$, such that

$$\begin{aligned} \langle \xi, y_{tt}(t)\rangle + \langle \xi_{xx}, y_{xx}(t)\rangle - \int_0^t \kappa(t-\tau)\langle \xi_{xx}, y_{xx}(\tau)\rangle \, d\tau \\ + \xi(L,t)f(t)y_t(L,t) + \langle \xi, g(y_t(t))\rangle = 0, \quad \forall \xi \in V. \end{aligned} \tag{2.42}$$

By choosing $\xi = y_t(t)$ as a test function, we obtain:

$$\begin{aligned} \langle y_t, y_{tt}(t)\rangle + \langle y_{txx}, y_{xx}(t)\rangle - \int_0^t \kappa(t-\tau)\langle y_{txx}(t), y_{xx}(\tau)\rangle \, d\tau \\ + f(t)y_t^2(L,t) + \langle y_t, g(y_t(t))\rangle = 0. \end{aligned} \tag{2.43}$$

Using identity (2.9), we get

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\Big[\|y_t(t)\|^2 + (\kappa \diamond y_{xx})(t) + \|y_{xx}(t)\|^2 \Big(1 - \int_0^t \kappa(\tau)\, d\tau\Big)\Big] \\ = -\frac{1}{2}\kappa(t)\|y_{xx}(t)\|^2 - f(t)y_t^2(L,t) - \langle y_t, g(y_t(t))\rangle \\ + \frac{1}{2}(\kappa' \diamond y_{xx})(t), \quad \forall t \geq t_0. \end{aligned} \tag{2.44}$$

We define the modified energy

$$2\varphi(y(t)) = \|y_t(t)\|^2 + (\kappa \diamond y_{xx})(t) + \|y_{xx}(t)\|^2 \Big(1 - \int_0^t \kappa(\tau)\, d\tau\Big). \tag{2.45}$$

According to the hypothesis in (2.5), we obtain

$$\frac{d}{dt}\varphi(y(t)) \leq 0.$$

Integrating over $(0,T)$, we get:

$$\varphi(y(t)) \leq \varphi(y(0)), \quad \text{and} \quad \varphi(y(0)) = 0, \quad \forall t \in [0,T].$$

Thus, $\varphi(y(t)) = 0$ for all $t \in [0,T]$, therefore $y(t) = 0$.
Hence, we obtain

$$\omega^1 = \omega^2.$$

Chapter 3

Exponential Stability Result

In this chapter, we will prove the exponential stability of the solution to the Euler-Bernoulli beam equation with memory and a boundary output feedback control term, using the multiplier technique.

From theorem 2.2.1, we have that the problem (2.1) admits a unique solution satisfying

$$\omega \in L^\infty(0,T;U), \qquad \omega_t \in L^\infty(0,T;V),$$

$$\omega_{tt} \in L^\infty(0,T;L^2(0,L)).$$

The derivative of the energy is given by

$$E'(t) = -f(t)(\omega_t(L,t))^2 + \int_0^t \kappa(t-\tau)\langle \omega_{txx}(t), \omega_{xx}(\tau)\rangle \, d\tau - \langle \omega_t(t), g(\omega_t(t))\rangle. \tag{3.1}$$

Proof. We have

$$E(t) = \frac{1}{2}\|\omega_t(t)\|^2 + \frac{1}{2}\|\omega_{xx}(t)\|^2.$$

The differentiation of the functional energy $E(t)$ yields:

$$\begin{aligned} E'(t) = \frac{d}{dt}E(t) =& \frac{d}{dt}\left(\frac{1}{2}\|\omega_t(t)\|^2 + \|\omega_{xx}(t)\|^2\right) \\ =& \langle \omega_t(t), \omega_{tt}(t)\rangle + \langle \omega_{xx}(t), \omega_{xxt}(t)\rangle. \end{aligned}$$

From the fundamental differential equation of the problem (2.1), we find

$$\omega_{tt} = -\omega_{xxxx} + \int_0^t \kappa(t-\tau)\omega_{xxxx}(\tau)d\tau - g(\omega_t). \tag{3.2}$$

We multiply equation (3.2) by ω_t and integrate over the domain $[0, L]$, we obtain

$$\langle \omega_t(t), \omega_{tt}(t)\rangle = -\langle \omega_t(t), \omega_{xxxx}(t)\rangle + \langle \omega_t(t), \int_0^t \kappa(t-\tau)\omega_{xxxx}(\tau)\, d\tau\rangle - \langle \omega_t(t), g(\omega_t(t))\rangle. \tag{3.3}$$

By applying integration by parts twice, we obtain

$$\begin{aligned}\langle \omega_t(t), \omega_{xxxx}(t)\rangle &= \int_0^L \omega_t(x,t)\omega_{xxxx}(x,t)\,dx \\ &= \Big[\omega_{xxx}(x,t)\omega_t(x,t)\Big]_0^L - \Big[\omega_{xx}(x,t)\omega_{tx}(x,t)\Big]_0^L + \int_0^L \omega_{txx}(x,t)\omega_{xx}(x,t)\,dx.\end{aligned}$$

For

$$\omega \in L^\infty(0,T;U), \qquad \omega_t \in L^\infty(0,T;V),$$

and by the boundary conditions, we get

$$\langle \omega_t(t), \omega_{xxxx}(t)\rangle = f(t)(\omega_t(L,t))^2 + \omega_t(L,t)\int_0^t \kappa(t-\tau)\omega_{xxx}(L,\tau)\,d\tau + \langle \omega_{txx}(t), \omega_{xx}(t)\rangle. \tag{3.4}$$

On the other hand, we have

$$\begin{aligned}\langle \omega_t(t), \int_0^t \kappa(t-\tau)\omega_{xxxx}(\tau)\,d\tau\rangle &= \int_0^L \omega_t(x,t)\left(\int_0^t \kappa(t-\tau)\omega_{xxxx}(x,\tau)\,d\tau\right)dx \\ &= \int_0^L\int_0^t \omega_t(x,t)\kappa(t-\tau)\omega_{xxxx}(x,\tau)\,d\tau\,dx \\ &= \int_0^t\int_0^L \omega_t(x,t)\kappa(t-\tau)\omega_{xxxx}(x,\tau)\,dx\,d\tau \\ &= \int_0^t \kappa(t-\tau)\left(\int_0^L \omega_t(x,t)\omega_{xxxx}(x,\tau)\,dx\right)d\tau \\ &= \int_0^t \kappa(t-\tau)\langle \omega_t(t), \omega_{xxxx}(\tau)\rangle\,d\tau.\end{aligned}$$

From (3.4), we find

$$\begin{aligned}\langle \omega_t(t), \int_0^t \kappa(t-\tau)\omega_{xxxx}(\tau)\,d\tau\rangle = \omega_t(L,t)&\int_0^t \kappa(t-\tau)\omega_{xxx}(L,\tau)\,d\tau \\ &+ \int_0^t \kappa(t-\tau)\langle \omega_{txx}(t), \omega_{xx}(\tau)\rangle\,d\tau.\end{aligned} \tag{3.5}$$

By substituting (3.4) and (3.5) in (3.3), we obtain

$$\begin{aligned}\langle \omega_t(t), \omega_{tt}(t)\rangle = -f(t)(\omega_t(L,t))^2 &- \langle \omega_{txx}(t), \omega_{xx}(t)\rangle + \int_0^t \kappa(t-\tau)\langle \omega_{txx}(t), \omega_{xx}(\tau)\rangle\,d\tau \\ &- \langle \omega_t(t), g(\omega_t(t))\rangle.\end{aligned} \tag{3.6}$$

Substituting (3.6) in $E'(t)$, we get

$$E'(t) = -f(t)(\omega_t(L,t))^2 + \int_0^t \kappa(t-\tau)\langle \omega_{txx}(t), \omega_{xx}(\tau)\rangle\,d\tau - \langle \omega_t(t), g(\omega_t(t))\rangle.$$

Let us consider the derivative of the modified energy, defined by □

$$\varphi_t(t) = -f(t)(\omega_t(L,t))^2 - \langle g(\omega_t(t)), \omega_t(t)\rangle + \frac{1}{2}(\kappa' \diamond \omega_{xx})(t) - \frac{1}{2}\kappa(t)\|\omega_{xx}(t)\|^2. \tag{3.7}$$

Lemma 3.1. *We have*

1. $\varphi(t) \geq 0$, *and* $E(t) \leq \ell^{-1}\varphi(t)$.
2. $\varphi_t \leq 0, \quad \forall t \geq t_0$.

Proof. we have from assumption (2.5) and (2.6)

$$\begin{aligned}\varphi(t) &= \frac{1}{2}\|\omega_t(t)\|^2 + \frac{1}{2}(\kappa \diamond \omega_{xx})(t) + \frac{1}{2}\ell\|\omega_{xx}(t)\|^2 \\ &\geq 0.\end{aligned}$$

Then

$$\begin{aligned}\frac{1}{\ell}\varphi(t) &= \frac{1}{2\ell}\|\omega_t(t)\|^2 + \frac{1}{2\ell}(\kappa \diamond \omega_{xx})(t) + \frac{1}{2}\|\omega_{xx}(t)\|^2 \\ &\geq \frac{1}{2}\|\omega_t(t)\|^2 + \frac{1}{2}\|\omega_{xx}(t)\|^2 + \frac{1}{2\ell}(\kappa \diamond \omega_{xx})(t) \\ &\geq E(t).\end{aligned}$$

By assumptions (2.5) and (2.7) we get $\varphi_t \leq 0, \quad \forall t \geq t_0$. □

Theorem 3.1. *Let ω be the solution given by Theorem 2.2.1 and $\varphi(t)$ be defined by (2.10). Then*

$$\lim_{t\to\infty} \varphi(t) = 0,$$

$$\lim_{t\to\infty} f(t) \leq \sqrt{2r\varphi(t_0) + (f(t_0))^2},$$

$$f(t) \leq \sqrt{2r\varphi(t_0) + (f(t_0))^2}, \qquad \forall t \geq t_0.$$

Now, We define the perturbed energy by

$$\varphi_\epsilon(t) = \varphi(t) + \epsilon\psi(t), \tag{3.8}$$

where $\psi(t) = \delta \int_0^L \omega(x,t)\omega_t(x,t)\,dx$, with $0 < \delta < \frac{1}{2}$.

Proposition 3.1. *There exist a positive constant $m_1 > 0$, such that*

$$|\varphi_\epsilon(t) - \varphi(t)| \leq \epsilon m_1 \varphi(t), \ \ \forall t \geq 0, \ \ \forall \epsilon > 0.$$

Proof. For $\forall t \geq 0$ and $\forall \epsilon > 0$. We have

$$\begin{aligned}|\varphi_\epsilon(t) - \varphi(t)| &= \epsilon|\psi(t)| \\ &= \epsilon\delta \left| \int_0^L \omega(x,t)\omega_t(x,t)\,dx \right|, \quad 0 < \delta < \frac{1}{2}, \\ &\leq \epsilon\delta \int_0^L |\omega(x,t)\omega_t(x,t)|\,dx.\end{aligned}$$

Using Young's inequality, we get

$$|\varphi_\epsilon(t) - \varphi(t)| \leq \epsilon\delta \left(\frac{1}{2}\|\omega_t(t)\|^2 + \frac{1}{2}\|\omega(t)\|^2 \right).$$

We have $V \subseteq H^2(0, L) \subseteq L^2(0, L)$. From sobolev embedding theorem, we obtain

$$\begin{aligned}
|\varphi_\epsilon(t) - \varphi(t)| &\leq \epsilon\delta \left(\frac{1}{2}\|\omega_t(t)\|^2 + \frac{1}{2}m_1'\|\omega_{xx}(t)\|^2 \right) \\
&\leq \epsilon\delta \max(1, m_1') \left(\frac{1}{2}\|\omega_t(t)\|^2 + \frac{1}{2}\|\omega_{xx}(t)\|^2 \right) \\
&\leq \epsilon m_1 \varphi(t).
\end{aligned}$$

□

Proposition 3.2. *There exist constants $m_2, m_3 > 0$ and $\epsilon_1 > 0$ such that*

$$\frac{d}{dt}\varphi_\epsilon(t) \leq -\epsilon m_2 \varphi(t) + \epsilon m_3 f(t)(\omega(L, t))^2,$$

for all $t \geq t_0$ and for all $\epsilon \in (0, \epsilon_1]$.

Proof. Using (3.2), we deduce that

$$\begin{aligned}
\frac{d}{dt}\psi(t) =& \delta \frac{d}{dt} \left(\int_0^L \omega(x, t)\omega_t(x, t)\, dx \right) \\
=& \delta \int_0^L (\omega_t(x, t))^2\, dx + \delta \int_0^L \omega(x, t)\omega_{tt}(x, t)\, dx \\
=& \delta \int_0^L (\omega_t(x, t))^2\, dx - \delta \int_0^L \omega(x, t)\omega_{xxxx}(x, t)\, dx \\
&+ \delta \int_0^L \omega(x, t) \int_0^t \kappa(t - \tau)\omega_{xxxx}(x, \tau)\, d\tau\, dx - \delta \int_0^L \omega(x, t) g(\omega_t(x, t))\, dx.
\end{aligned} \tag{3.9}$$

By using integration by parts twice, and from $\omega(0, t) = \omega_x(0, t) = \omega_{xx}(L, t) = 0,$ and $\omega \in L^\infty(0, T; U)$, we get

$$\begin{aligned}
\int_0^L \omega(x, t)\omega_{xxxx}(x, t)\, dx &= [\omega(x, t)\omega_{xxx}(x, t)]_0^L + [\omega_x(x, t)\omega_{xx}(x, t)]_0^L + \int_0^L (\omega_{xx}(x, t))^2\, dx \\
&= \omega(L, t)\omega_{xxx}(L, t) + \int_0^L (\omega_{xx}(x, t))^2\, dx.
\end{aligned}$$

And by Fubini's theorem, we obtain

$$\begin{aligned}
&\int_0^L \omega(x, t) \left(\int_0^t \kappa(t - \tau)\omega_{xxxx}(x, \tau)\, d\tau \right) dx = \int_0^t \kappa(t - \tau) \left(\int_0^L \omega(x, t)\omega_{xxxx}(x, \tau)\, dx \right) d\tau \\
&= \int_0^t \kappa(t - \tau) \left([\omega(x, t)\omega_{xxx}(x, \tau)]_0^L + \int_0^L \omega_x(x, t)\omega_{xxx}(x, \tau)\, dx \right) d\tau \\
&= \int_0^t \kappa(t - \tau) \left(\omega(L, t)\omega_{xxx}(L, \tau) + [\omega_x(x, t)\omega_{xx}(x, \tau)]_0^L + \int_0^L \omega_{xx}(x, t)\omega_{xx}(x, \tau)\, dx \right) d\tau
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^t \kappa(t-\tau)\left(\omega(L,t)\omega_{xxx}(L,\tau) + \int_0^L \omega_{xx}(x,t)\omega_{xx}(x,\tau)\,dx\right)d\tau \\
&= \int_0^t \kappa(t-\tau)\omega(L,t)\omega_{xxx}(L,\tau)\,d\tau + \int_0^t \kappa(t-\tau)\left(\int_0^L \omega_{xx}(x,t)\omega_{xx}(x,\tau)\,dx\right)d\tau \\
&= \omega(L,t)\left(\int_0^t \kappa(t-\tau)\omega_{xxx}(L,\tau)\,d\tau\right) + \int_0^L \omega_{xx}(x,t)\left(\int_0^t \kappa(t-\tau)\omega_{xx}(x,\tau)\,d\tau\right)dx.
\end{aligned}
$$

By compensating in (3.9), we get

$$
\begin{aligned}
\psi_t(t) =&\delta\int_0^L(\omega_t(x,t))^2\,dx - \delta\omega(L,t)\left[\omega_{xxx}(L,t) - \int_0^t \kappa(t-\tau)\omega_{xxx}(L,\tau)\,d\tau\right] \\
&- \delta\int_0^L(\omega_{xx}(x,t))^2\,dx + \delta\int_0^L \omega_{xx}(x,t)\int_0^t \kappa(t-\tau)\omega_{xx}(x,\tau)\,d\tau \\
&- \delta\int_0^L \omega(x,t)g(\omega_t(x,t))\,dx.
\end{aligned}
\tag{3.10}
$$

Let us estimate the terms of the equation (3.13), we start with the integral

$$
J = \int_0^L \omega_{xx}(t)\left(\int_0^t \kappa(t-\tau)\omega_{xx}(\tau)\,d\tau\right)dx.
$$

From Fubini's theorem, we have

$$
\begin{aligned}
J &= \int_0^t \kappa(t-\tau)\left(\int_0^L \omega_{xx}(t)\omega_{xx}(\tau)\,dx\right)d\tau \\
&= \int_0^t \kappa(t-\tau)\left(\int_0^L(\omega_{xx}(\tau)-\omega_{xx}(t))\omega_{xx}(t)\,dx\right)d\tau + \int_0^t \kappa(t-\tau)\left(\int_0^L(\omega_{xx}(t))^2\,dx\right)d\tau,
\end{aligned}
$$

and

$$
I = \int_0^t \kappa(t-\tau)\left(\int_0^L(\omega_{xx}(\tau)-\omega_{xx}(t))\omega_{xx}(t)\,dx\right)d\tau.
$$

Using Young's inequality with η and $p = q = 2$, we get

$$
\int_0^L(\omega_{xx}(\tau)-\omega_{xx}(t))\omega_{xx}(t)\,dx \le \int_0^L \frac{1}{4\eta}(\omega_{xx}(\tau)-\omega_{xx}(t))^2 + \eta(\omega_{xx}(t))^2\,dx.
$$

From the assumtions (2.5), multiplying the both sides by $\kappa(t-\tau)$, and integrating over $(0,t)$, we obtain

$$
\begin{aligned}
I &\le \int_0^t \kappa(t-\tau)\left(\int_0^L \frac{1}{4\eta}(\omega_{xx}(\tau)-\omega_{xx}(t))^2 + \eta(\omega_{xx}(t))^2\,dx\right)d\tau \\
&\le \int_0^t \kappa(t-\tau)\left(\int_0^L \frac{1}{4\eta}(\omega_{xx}(\tau)-\omega_{xx}(t))^2\,dx\right)d\tau + \int_0^t \kappa(t-\tau)\left(\int_0^L \eta(\omega_{xx}(t))^2\,dx\right)d\tau.
\end{aligned}
$$

From the assumtions (2.6), we have $\int_0^\infty \kappa(\tau) \leq 1$. Then

$$\begin{aligned} I \leq & \frac{1}{4\eta}\int_0^t (\kappa(t-\tau))^2 \left(\int_0^L (\omega_{xx}(\tau)-\omega_{xx}(t))^2\,dx\right) d\tau + \eta\int_0^L (\omega_{xx}(t))^2\,dx \\ \leq & \frac{1}{4\eta}\|\kappa\|_{L^\infty(0,\infty)}\int_0^t \kappa(t-\tau)\|\omega_{xx}(\tau)-\omega_{xx}(t)\|_2^2\,d\tau + \eta\|\omega_{xx}(t)\|_2^2. \end{aligned}$$

Hence

$$J \leq \frac{1}{4\eta}\|\kappa\|_{L^\infty(0,\infty)}(\kappa\diamond\omega_{xx})(t) + \eta\|\omega_{xx}(t)\|^2 + \int_0^t \kappa(t-\tau)\,d\tau\|\omega_{xx}(t)\|^2. \tag{3.11}$$

By Young's inequality, we have

$$\begin{aligned} \delta\left|\int_0^L \omega(x,t)g(\omega_t(x,t))\,dx\right| &\leq \delta\int_0^L |\omega(x,t)|\,|g(\omega_t(x,t))|\,dx \\ &\leq \delta\frac{1}{2}\left(\|\omega(t)\|^2 + \|g(\omega_t(t))\|^2\right). \end{aligned}$$

From the assumptions on g, and by Sobolev Embedding theorem, we get

$$\delta\left|\int_0^L \omega(x,t)g(\omega_t(x,t))\,dx\right| \leq \delta\frac{1}{2}\left(c_1\|\omega(t)\|^2 + c_2\|\omega_t(t)\|^2\right).$$

then

$$\delta\left|\int_0^L \omega(x,t)g(\omega_t(x,t))\,dx\right| \leq M_1\|\omega_t(t)\|^2 + M_2\|\omega_{xx}(t)\|^2.$$

By combining (3.13) and (3.11), we get

$$\begin{aligned} \psi_t(t) \leq & \delta\|\omega_t(t)\|^2 - \delta\omega(L,t)f(t)\omega_t(L,t) - \delta\|\omega_{xx}(t)\|^2 + \frac{\delta}{4\eta}\|\kappa\|_{L^\infty(0,\infty)}(\kappa\diamond\omega_{xx})(t) \\ & + \delta\eta\|\omega_{xx}(t)\|^2 + \delta\int_0^t \kappa(\tau)\,d\tau\|\omega_{xx}(t)\|^2 + M_1\|\omega_t(t)\|^2 + M_2\|\omega_{xx}(t)\|^2 \\ \leq & (\delta + M_1)\|\omega_t(t)\|^2 + \left(\delta\eta - \delta + \delta\int_0^t \kappa(\tau)\,d\tau + M_2\right)\|\omega_{xx}(t)\|^2 \\ & - \delta\omega(L,t)f(t)\omega_t(L,t) + \frac{\delta}{4\eta}\|\kappa\|_{L^\infty(0,\infty)}(\kappa\diamond\omega_{xx})(t) \\ \leq & -\delta\varphi(t) + \left(\frac{3\delta}{2} + M_1\right)\|\omega_t(t)\|^2 + \left(\delta\eta - \frac{\delta}{2} + \frac{\delta}{2}\int_0^t \kappa(\tau)\,d\tau + M_2\right)\|\omega_{xx}(t)\|^2 \\ & - \delta\omega(L,t)f(t)\omega_t(L,t) + \left(\frac{\delta}{4\eta}\|\kappa\|_{L^\infty(0,\infty)} + \frac{\delta}{2}\right)(\kappa\diamond\omega_{xx})(t). \end{aligned} \tag{3.12}$$

Hence, from (3.7) and (3.12), we deduce that

$$
\begin{aligned}
\frac{d}{dt}\varphi_\epsilon(t) &= \frac{d}{dt}\varphi(t) + \epsilon\frac{d}{dt}\psi(t) \\
&\le -f(t)(\omega_t(L,t))^2 - \langle g(\omega_t(t)), \omega_t(t)\rangle + \frac{1}{2}(\kappa' \diamond \omega_{xx})(t) - \frac{1}{2}\kappa(t)\|\omega_{xx}(t)\|^2 \\
&\quad - \epsilon\delta\varphi(t) + \epsilon\left(\frac{3\delta}{2} + M_1\right)\|\omega_t(t)\|^2 + \epsilon\left(\delta\eta - \frac{\delta}{2} + \frac{\delta}{2}\int_0^t \kappa(\tau)\,d\tau + M_2\right)\|\omega_{xx}(t)\|^2 \\
&\quad - \epsilon\delta\omega(L,t)f(t)\omega_t(L,t) + \epsilon\left(\frac{\delta}{4\eta}\|\kappa\|_{L^\infty(0,\infty)} + \frac{\delta}{2}\right)(\kappa \diamond \omega_{xx})(t),
\end{aligned}
$$

and from (2.5)and (2.7)

$$
\begin{aligned}
\frac{d}{dt}\varphi_\epsilon(t) &\le -f(t)(\omega_t(L,t))^2 - \beta\|\omega_t(t)\|^2 - \frac{\alpha_2}{2}(\kappa \diamond \omega_{xx})(t) - \frac{1}{2}\kappa(t)\|\omega_{xx}(t)\|^2 - \epsilon\delta\varphi(t) \\
&\quad + \epsilon\left(\frac{3\delta}{2} + M_1\right)\|\omega_t(t)\|^2 + \epsilon\left(\delta\eta - \frac{\delta}{2} + \frac{\delta}{2}\int_0^t \kappa(\tau)\,d\tau + M_2\right)\|\omega_{xx}(t)\|^2 \\
&\quad - \epsilon\delta\omega(L,t)f(t)\omega_t(L,t) + \epsilon\left(\frac{\delta}{4\eta}\|\kappa\|_{L^\infty(0,\infty)} + \frac{\delta}{2}\right)(\kappa \diamond \omega_{xx})(t) \\
&\le -\epsilon\delta\varphi(t) - \left(\beta - \epsilon\left(\frac{3\delta}{2} + M_1\right)\right)\|\omega_t(t)\|^2 - \left(\frac{\alpha_2}{2} - \epsilon\left(\frac{\delta}{4\eta}\|\kappa\|_{L^\infty(0,\infty)} + \frac{\delta}{2}\right)\right)(\kappa \diamond \omega_{xx})(t) \\
&\quad - \left(\frac{1}{2}\kappa(t) - \epsilon\left(\delta\eta - \frac{\delta}{2} + \frac{\delta}{2}\int_0^t \kappa(\tau)\,d\tau + M_2\right)\right)\|\omega_{xx}(t)\|^2 - f(t)\left(1 - \frac{\delta\epsilon}{2}\right)(\omega_t(L,t))^2 \\
&\quad + \frac{\delta\epsilon}{2}f(t)(\omega(L,t))^2, \qquad \forall t \ge t_0.
\end{aligned} \tag{3.13}
$$

By setting

$$
\sigma = \min\left\{\frac{2\beta}{2M_1 + 3\delta}, \frac{2\alpha_2\eta}{\delta(2\eta + \|\kappa\|_{L^\infty})}, \frac{\kappa(t)}{(2M_2 + 2\eta\delta + \delta\int_0^t \kappa(\tau)\,d\tau - \delta)}, \frac{2}{\delta}\right\}, \quad t \ge t_0, \tag{3.14}
$$

where, $\epsilon \in [0, \sigma]$ and $(2\eta - \ell)\delta + 2M_2 > 0$, from (3.13) and (3.14), we obtain

$$
\frac{d}{dt}\varphi_\epsilon(t) \le -\epsilon\delta\varphi(t) + \frac{\delta\epsilon}{2}f(t)(\omega(L,t))^2, \qquad \forall t \ge t_0. \tag{3.15}
$$

□

Proof of theorem 3.1 . From proposition (3.1), we have

$$
|\varphi_\epsilon(t) - \varphi(t)| \le \epsilon m_1\varphi(t),
$$

then

$$
(1 - \epsilon m_1)\varphi(t) \le \varphi_\epsilon(t) \le (\epsilon m_1 + 1)\varphi(t), \tag{3.16}
$$

thus, hence

$$
\frac{\varphi_\epsilon(t)}{\epsilon m_1 + 1} \le \varphi(t), \tag{3.17}
$$

by (3.15) and (3.17) ,we have

$$\frac{d}{dt}\varphi_\epsilon(t) \leq \frac{-\epsilon\delta}{\epsilon m_1+1}\varphi_\epsilon(t) + \frac{\delta\epsilon}{2}f(t)(\omega(L,t))^2, \qquad \forall t \geq t_0. \tag{3.18}$$

we put $k = \dfrac{-\epsilon\delta}{\epsilon m_1+1}$, from Gronwall's theorem, we get for all $t \geq t_0$

$$\begin{aligned}\varphi_\epsilon(t) &\leq \exp\Big(-\int_{t_0}^t k\,d\tau\Big)\varphi_\epsilon(t_0) + \int_{t_0}^t \exp\Big(-\int_\tau^t k\,ds\Big)\frac{\delta\epsilon}{2}f(\tau)(\omega(L,\tau))^2\,d\tau \\ &\leq \exp(-k(t-t_0))\varphi_\epsilon(t_0) + \frac{\delta\epsilon}{2}\int_{t_0}^t \exp(-k(t-\tau))f(\tau)(\omega(L,\tau))^2\,d\tau \\ &\leq \exp(-kt)\exp(kt_0)\varphi_\epsilon(t_0) + \frac{\delta\epsilon}{2}\int_{t_0}^t \exp(-k(t-\tau))f(\tau)(\omega(L,\tau))^2\,d\tau \\ &\leq \exp(-kt)\varphi_\epsilon(t_0) + \frac{\delta\epsilon}{2}\sup_{t\geq 0}|f(t)|\int_0^t \exp(-k(t-\tau))(\omega(L,\tau))^2\,d\tau. \end{aligned} \tag{3.19}$$

As in [8], we deduce that

$$\begin{aligned}\int_0^t \exp(-k(t-\tau))(\omega(L,\tau))^2\,d\tau &= \int_0^{\frac{t}{2}} \exp(-k(t-\tau))(\omega(L,\tau))^2\,d\tau + \int_{\frac{t}{2}}^t \exp(-k(t-\tau))(\omega(L,\tau))^2\,d\tau \\ &\leq \max_{0\leq\tau\leq\frac{t}{2}}\exp(-k(t-\tau))\int_0^{\frac{t}{2}}(\omega(L,\tau))^2\,d\tau + \max_{\frac{t}{2}\leq\tau\leq t}\exp(-k(t-\tau))\int_{\frac{t}{2}}^t(\omega(L,\tau))^2\,d\tau \\ &\leq \exp\Big(-\frac{kt}{2}\Big)\int_0^{\frac{t}{2}}(\omega(L,\tau))^2\,d\tau + \int_{\frac{t}{2}}^t(\omega(L,\tau))^2\,d\tau \\ &\leq \exp\Big(-\frac{kt}{2}\Big)\Big(\int_0^{\frac{t}{2}}(\omega(L,\tau))^2\,d\tau + \int_{\frac{t}{2}}^t(\omega(L,\tau))^2\,d\tau\Big) \\ &\leq \exp\Big(-\frac{kt}{2}\Big)\Big(\int_0^t(\omega(L,\tau))^2\,d\tau\Big) \\ &\leq \exp\Big(-\frac{kt}{2}\Big)\int_0^\infty(\omega(L,\tau))^2\,d\tau. \end{aligned}$$

Therefore, we obtain

$$\varphi_\epsilon(t) \leq \exp(-kt)\varphi_\epsilon(t_0) + \frac{\delta\epsilon}{2}\sup_{t\geq 0}|f(t)|\exp\Big(-\frac{kt}{2}\Big)\int_0^\infty(\omega(L,\tau))^2\,d\tau,$$

By Theorem 2.2.1, together with the embedding $U \hookleftarrow L^2(0,T;L^2(0,T))$ and the fact that $\omega(L,t) \in L^2(0,\infty)$, we can pass to the limit as $t \to \infty$ to obtain the desired result

$$\lim_{t\to\infty}\varphi_\epsilon(t) = 0.$$

Let $\varepsilon \in (0,\varepsilon_0]$ where $\varepsilon_0 = \min\Big\{\varepsilon_1, \dfrac{1}{2m_1}\Big\}$, so we have $\varepsilon \leq \dfrac{1}{2m_1}$.
From (3.16), we obtain

$$\frac{1}{2}\varphi(t) \leq \varphi_\varepsilon(t) \leq \frac{3}{2}\varphi(t), \quad \forall t \geq t_0.$$

Therefore, we have

$$\lim_{t\to\infty} \varphi(t) = 0. \tag{3.20}$$

Now, we consider

$$v(t) = \varphi(t) + \frac{f^2(t)}{2r}. \tag{3.21}$$

Differentiating the above expression yields

$$\begin{aligned}
\frac{d}{dt}v(t) &= \frac{d}{dt}\varphi(t) + \frac{d}{dt}\left(\frac{f^2(t)}{2r}\right) \\
&= -f(t)(\omega_t(L,t))^2 - \langle g(\omega_t(t)), \omega_t(t)\rangle + \frac{1}{2}(\kappa' \diamond \omega_{xx})(t) - \frac{1}{2}\kappa(t)\|\omega_{xx}(t)\|^2 + \frac{1}{r}f(t)f_t(t) \\
&= -f(t)(\omega_t(L,t))^2 - \langle g(\omega_t(t)), \omega_t(t)\rangle + \frac{1}{2}(\kappa' \diamond \omega_{xx})(t) - \frac{1}{2}\kappa(t)\|\omega_{xx}(t)\|^2 + \frac{1}{r}f(t)r(\omega_t(L,t))^2 \\
&= -\langle g(\omega_t(t)), \omega_t(t)\rangle + \frac{1}{2}(\kappa' \diamond \omega_{xx})(t) - \frac{1}{2}\kappa(t)\|\omega_{xx}(t)\|^2 \\
&\le 0.
\end{aligned} \tag{3.22}$$

From (3.22), we have

$$\sup_{t\ge 0} v(t) \le M_3',$$

where $M_3' \ge 0$ is a constant depending on the initial data, and for $t \ge t_0$ we have

$$\varphi(t) + \frac{1}{2r}f^2(t) \le \varphi(t_0) + \frac{1}{2r}f^2(t_0).$$

Thus when $t \to \infty$, we have

$$\varphi(\infty) + \frac{1}{2r}f^2(\infty) \le \varphi(t_0) + \frac{1}{2r}f^2(t_0).$$

From (3.20)

$$f(\infty) \le \sqrt{2r\varphi(t_0) + f^2(t_0)}.$$

Since $f(t)$ is nondecreasing, we obtain

$$f(t) \le \sqrt{2r\varphi(t_0) + f^2(t_0)}, \quad \forall t \ge t_0. \tag{3.23}$$

□

Theorem 3.2. *Let ω be the solution of problem (2.1), and let $\varphi(t)$ be the modified energy defined by (2.10). Then, there exist constants $K > 0$ and $\lambda > 0$ such that*

$$E(t) \le K\exp(-\lambda t), \quad \forall t \ge t_0.$$

Proof. From (3.18), we see that

$$\frac{d}{dt}\varphi_\epsilon(t) \le \frac{-\epsilon\delta}{\epsilon m_1 + 1}\varphi_\epsilon(t) + \frac{\delta\epsilon}{2}f(t)(\omega(L,t))^2, \qquad \forall t \ge t_0.$$

By integrating over (t_0, t) and using (3.23), we get

$$\begin{aligned}\varphi_\epsilon(t) &\le \varphi_\epsilon(t_0) - k\int_{t_0}^{t} \varphi_\epsilon(\tau)\,d\tau + \frac{\delta\epsilon}{2}\int_{t_0}^{t} f(\tau)(\omega(L,\tau))^2\,d\tau \\ &\le \varphi_\epsilon(t_0) - k\int_{t_0}^{t} \varphi_\epsilon(\tau)\,d\tau + \frac{\delta\epsilon}{2}\|f\|_{L^\infty}\int_{0}^{t} (\omega(L,\tau))^2\,d\tau. \qquad (3.24)\end{aligned}$$

Since $\int_0^\infty (\omega(L,\tau))^2\,d\tau \le k'$, we have

$$\varphi_\epsilon(t) \le \varphi_\epsilon(t_0) - k\int_{t_0}^{t} \varphi_\epsilon(\tau)\,d\tau + \frac{\delta\epsilon}{2}k'\|f\|_{L^\infty}.$$

By Gronwall's inequality, we get

$$\varphi_\epsilon(t) \le (K_1 + \varphi_\epsilon(t_0))\exp(-kt), \quad \forall t \ge t_0.$$

For sufficiently small $\epsilon \in (0, \frac{1}{m_1})$, using the proposition (3.1), we get

$$E(t) \le \ell^{-1}\varphi(t) \le K\exp(-\lambda t), \quad \forall t \ge t_0,$$

where $K = \ell^{-1}\dfrac{K_1 + \varphi_\epsilon(t_0)}{1-\epsilon m_1}$ and $\lambda = k$. □

Conclusion

The aim of this work is to conduct a qualitative analysis of the Euler–Bernoulli beam equation, which is a partial differential equation that describes the bending of elastic beams under various loadings. In this study, the model is extended by incorporating a viscoelastic memory term, reflecting the material's dependence on its deformation history, along with a boundary output feedback control law.

This work is structured into three main chapters: The first chapter presents some fundamental functional spaces and mathematical inequalities that will be used in the subsequent analysis. In the second chapter, we introduce the mathematical formulation of the problem and prove the existence of a solution using the Galerkin method. The third and final chapter is devoted to studying the exponential stability of the solution through the construction of a suitable Lyapunov functional and the application of the multiplier technique.

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الملخص

في هذه الدراسة، نتناول معادلة عارضة أويلر-بيرنولي مع تضمين تأثير الذاكرة اللزجة المرنة و التحكم في التغذية الراجعة للمخرجات الحدودية. يبدأ العمل بمراجعة لمفاهيم التحليل الدالي الأساسية وبعض المتراجحات، والتي تُعد أدوات أساسية في المعالجة الرياضية للمشكلة. بعد ذلك، نثبت وجود الحل ووحدانيته باستخدام طريقة التقريب غالركين. وأخيرًا، من خلال تطبيق تقنية المضاعف، نُظهر أن طاقة النظام تتناقص أُسّيًا مع مرور الوقت.

كلمات مفتاحية : معادلة شعاع أويلر-بيرنولي، طريقة غالركين، حد التحكم في التغذية الراجعة، حد الذاكرة، الإستقرار الأسي .

Abstract

In this study, we address the Euler–Bernoulli beam equation incorporating a viscoelastic memory effect and a boundary output feedback control term. The work begins with a review of essential functional analysis concepts and fundamental inequalities, which serve as foundational tools for the mathematical treatment of the problem. We then establish the existence and uniqueness of the solution using the Galerkin approximation scheme. Finally, by applying the multiplier technique, we prove that the energy of the system decays exponentially over time.

Key words: Euler-Bernoulli Beam equation, Galerkin method, Output feedback control term, Memory term, exponentiel stability.

Résumé

Dans cette étude, nous abordons l'équation de poutre d'Euler–Bernoulli en incorporant un effet de mémoire viscoélastique et un terme de contrôle en rétroaction basé sur la sortie aux frontières. Le travail commence par une révision des concepts fondamentaux de l'analyse fonctionnelle et des inégalités de base, qui constituent des outils essentiels pour le traitement mathématique du problème. Ensuite, nous établissons l'existence et l'unicité de la solution à l'aide de la méthode d'approximation de Galerkin. Enfin, en appliquant la technique du multiplicateur, nous démontrons que l'énergie du système décroît de façon exponentielle au cours du temps.

Mots clés: Équation de la poutre d'Euler-Bernoulli, méthode de Galerkin, terme de contrôle de rétroaction de sortie, terme de mémoire, stabilité exponentielle.