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Master's Thesis on:

Existence and uniqueness in deterministic optimal control

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شكر وتقدير

الحمد لله أولًا وآخرًا، ظاهرًا وباطنًا، الذي بنعمته تتم الصالحات، وبتوفيقه ويسره تذلل الصعاب وتتحقق الغايات. أحمده تعالى وأشكره على ما أنعم به عليّ من القوة والصبر والتوفيق في إتمام هذا العمل، وأسأله سبحانه أن يجعله خالصًا لوجهه الكريم ، وأن ينفع به.

أتوجه بخالص الشكر والامتنان إلى والديّ العزيزين، اللذين كانا ولا يزالان مصدر قوتي وإلهامي، بدعائهما ودعمهما وصبرهما وتضحياتهما التي لا تُقدَّر بثمن. فلهما مني أسمى عبارات التقدير والعرفان.

كما أتقدّم بجزيل الشكر والعرفان للأستاذ الفاضل بلعقون عبد الغاني، الذي كان لتوجيهاته السديدة وملاحظاته الدقيقة الأثر الكبير في تطوير هذا العمل وإنجازه على النحو المطلوب. لم يبخل عليّ بعلمه وخبرته، فكان نعم الداعم والموجّه، فله مني كل الشكر والاحترام.

ولا يفوتني أن أشكر كل الأساتذة الذين كان لهم دور في تكويني العلمي والمعرفي، وكل من ساعدني وساندني خلال هذه الرحلة، من زملاء وأصدقاء، الذين كانوا سندًا حقيقيًا في كل مرحلة.

كما أخص بالشكر كل من قدم لي يد العون، سواء بالنصح، أو التشجيع، أو الدعم المادي والمعنوي، أو حتى بكلمة طيبة، فلكم جميعًا مني أصدق الدعوات وأجزل الثناء.

وفي الختام، أسأل الله أن يجعل هذا العمل نافعًا، وأن يوفقني وإياكم لما يحب ويرضى، وأن يجزي كل من ساهم في هذا المشوار خير الجزاء.

إهــــداء

إلى من كانت دعواتها سرّ توفيقي، ونبض قلبها نبراس دربي
إلى أمي الحبيبة، نبع الحنان والأمان.
إلى من علّمني معنى القوة والمسؤولية، وكان سندي في كل خطوة
إلى أبي الغالي، مصدر فخري ودعمي الدائم.
إلى إخوتي الأعزاء، الذين كانوا لي العون في الشدة، والفرح في النجاح.
إلى أصدقائي وأحبّتي، رفاق الدرب، من شاركوني اللحظات الحلوة والمرّة،
وكانوا دائمًا مصدر التشجيع والابتسامة.
وإلى أساتذتي الأفاضل، من أضاءوا لي طريق العلم، وتركوا بصمة لا تُنسى في مسيرتي الأكاديمية
أهدي هذا العمل المتواضع، عرفانًا و امتنانا .

ملخص

هذه الأطروحة للماستر تستكشف التحكم الأمثل الجبري، مع التركيز على أسسه النظرية وتطبيقاته في مجالات الفضاء، الروبوتات، والاقتصاد. تتناول ديناميكيات النظام، الدوال التكلفية، مبدأ بونترياجين الأقصى، ونظرية فيليبوف. تسلط الدراسة الضوء على طرق الحل مثل البرمجة الديناميكية والتقنيات العددية، رغم التحديات الحسابية. وتختتم باقتراح أبحاث مستقبلية تشمل خوارزميات أكثر كفاءة ودمج التعلم الآلي.

Résumé

Cette thèse de master explore le contrôle optimal déterministe, abordant ses fondements théoriques et ses applications en aérospatiale, robotique et économie. Elle examine les dynamiques des systèmes, les fonctionnels de coût, le principe du maximum de Pontryagin et le théorème de Filippov. L'étude met en lumière des méthodes de résolution comme la programmation dynamique et les techniques numériques, malgré les défis computationnels. Elle conclut en proposant des pistes de recherche future, notamment des algorithmes plus efficaces et l'intégration de l'apprentissage automatique.

Abstract

This Master's thesis explores deterministic optimal control, covering its theoretical foundations and applications in aerospace, robotics, and economics. It examines system dynamics, cost functionals, the Pontryagin Maximum Principle, and Filippov's Theorem. The study highlights solution methods like dynamic programming and numerical techniques, noting their effectiveness despite computational challenges. It concludes with suggestions for future research, including efficient algorithms and machine learning integration.

Contents

CHAPTER 1

Introduction

Introduction to Deterministic Optimal Control

1.1 Background and Context

Deterministic optimal control, a luminous pillar of contemporary applied mathematics and engineering, offers a sophisticated framework for crafting strategies that elevate the performance of dynamic systems to their zenith. At its heart, this discipline concerns systems governed by predictable dynamics, where the trajectory of the system unfolds with certainty, given its initial state and the judicious application of control inputs. Its far-reaching implications resonate across a tapestry of fields— aerospace engineering, robotics, economics, and process control—where precision in decision-making is paramount to realizing cherished objectives.

The soul of deterministic optimal control lies in the quest for a control function that, with elegance and precision, minimizes or maximizes a performance criterion, often embodied as a cost functional, while honoring the constraints dictated by the system's dynamics. These dynamics, frequently articulated through the language of differential equations, illuminate the temporal evolution of the system under the influence of control inputs. The mathematical elegance of this field empowers the creation of controllers that are not merely efficient but also resilient to perturbations in system parameters, rendering optimal control an indispensable instrument for both theoretical exploration and practical application.

The genesis of optimal control finds its roots in the venerable calculus of variations, a discipline nurtured in the 17th and 18th centuries by luminaries such as Euler and Lagrange, whose mathematical artistry laid the groundwork for optimizing functionals. Yet, it was in the vibrant intellectual ferment of the mid-20th century that the modern edifice of optimal control took shape, propelled by the advent of dynamic programming and the incisive clarity of the maximum principle. These milestones forged a robust theoretical scaffold, enabling the resolution of intricate control challenges and illuminating pathways for transformative applications across engineering, economics, and myriad other domains. [2]

1.2 Historical Development

The historical tapestry of deterministic optimal control is woven with several luminous milestones that have shaped its intellectual contours. In the vibrant 1950s, Richard Bellman unveiled the paradigm of dynamic programming, a recursive alchemy for tackling optimization challenges across temporal horizons. His principle of optimality, a beacon of insight, posits that an optimal policy retains its virtue regardless of the initial state or decision, ensuring that subsequent choices remain optimal in light of the state birthed by the first. This elegant doctrine paved the way for dissecting complex optimal control problems into manageable subproblems, offering a pathway to clarity and resolution.

Concurrently, Lev Pontryagin and his collaborators crafted the Pontryagin Maximum Principle (PMP), a cornerstone of optimal control theory that radiates mathematical splendor. The PMP articulates necessary conditions for optimality, introducing adjoint variables as torchbearers to illuminate the path of the optimal control. In contrast to dynamic programming's reliance on solving the formidable Hamilton-Jacobi-Bellman equation, the PMP offers a computationally graceful approach, particularly for problems bound by constraints on control inputs, rendering it a cherished tool in the theorist's arsenal.

The 1960s and 1970s ushered in a golden era of progress, with luminaries such as Kalman advancing the field through the linear quadratic regulator (LQR), a framework tailored for linear systems with quadratic cost functions. Celebrated for its analytical elegance and practical utility, the LQR emerged as a lodestar in control engineering. As the decades unfolded, numerical methods—both direct and indirect—blossomed to confront the computational intricacies of nonlinear systems and intricate constraints, further enriching the discipline's capacity to address the challenges of an ever-evolving world.[2]

1.3 Motivation and Significance

The motivation for studying deterministic optimal control stems from its ability to address real-world problems where efficiency, precision, and resource optimization are paramount. In aerospace engineering, for instance, optimal control is used to design trajectories for spacecraft that minimize fuel consumption while satisfying mission constraints. In robotics, it enables the development of motion planning algorithms that ensure smooth and efficient operation. In economics, optimal control provides a framework for modeling decision-making processes, such as resource allocation over time.

The significance of deterministic optimal control lies in its versatility and applicability. By providing a unified mathematical framework for optimization, it bridges the gap between theory and practice, enabling engineers and scientists to tackle problems that were previously intractable. Furthermore, the deterministic nature of the systems considered in this field allows for precise predictions and control, which is critical in applications where uncertainty is minimal or can be neglected.

In the context of a Master's thesis, studying deterministic optimal control offers an opportunity to engage with advanced mathematical techniques, including differential equations, functional analysis, and optimization theory. It also provides a foundation for exploring related fields, such as stochastic optimal control, robust control, and machine learning-based control strategies.

Key Concepts in Deterministic Optimal Control

To understand deterministic optimal control, it is essential to introduce its fundamental concepts. These include:

System Dynamics

The evolution of a dynamic system is typically described by a set of ordinary differential equations (ODEs) of the form:

$$\dot{x}(t) = f(x(t), u(t), t),$$

where $x(t)$ is the state vector, $u(t)$ is the control input, and f is a function that governs the system dynamics. The initial condition $x(t_0) = x_0$ is often specified.

1.3.1 Cost Functional

The performance of the system is quantified by a cost functional, which measures the cost associated with a given control strategy. A common form is:

$$J = \phi(x(T)) + \int_{t_0}^{T} L(x(t), u(t), t)\, dt,$$

where ϕ is the terminal cost, L is the running cost, and $[t_0, T]$ is the time horizon.

1.3.2 Optimal Control Problem

The central objective here is the determination of an optimal control policy, denoted as u(t), that achieves a minimum value for a defined cost functional, J. This minimization is not performed in isolation but rather is intrinsically linked to the inherent dynamic behavior of the system under consideration. Furthermore, the optimization process must adhere to any supplementary constraints that may be imposed, such as defined boundaries on the permissible range of control inputs or the system's operational states. [10]

1.3.3 Solution Methods

Optimal control problems can be solved using various approaches, including:

- **Dynamic Programming:** Based on Bellman's principle, this method solves the Hamilton-Jacobi-Bellman (HJB) equation to find the optimal value function and control policy.
- **Pontryagin Maximum Principle:** This provides necessary conditions for optimality by introducing adjoint variables and a Hamiltonian function.
- **Numerical Methods:** Direct methods (e.g., collocation) and indirect methods (e.g., shooting methods) are used to solve complex problems numerically.

1.3.4 Constraints

Practical optimal control problems often involve constraints, such as control bounds ($u(t) \in U$), state constraints, or terminal conditions. These constraints complicate the solution process but are critical for real-world applications.[2]

1.4 Applications of Deterministic Optimal Control

Deterministic optimal control has proven to be a powerful tool with extensive applications across various fields, highlighting its versatility and practical relevance. In the aerospace engineering sector, it is essential for designing minimum-fuel trajectories for spacecraft, optimizing launch vehicle ascent paths, and controlling satellite orbits. A significant historical example is the use of optimal control techniques during the Apollo missions, ensuring precise navigation and landing on the Moon.

The field of robotics also greatly benefits from optimal control, which is employed for motion planning, trajectory optimization, and energy-efficient operation of robots. For instance, it enables the calculation of smooth trajectories for robotic arms used in manufacturing lines. In economics, optimal control provides a framework for modeling economic policies, such as optimal investment strategies or the allocation of resources over time, as illustrated by the Ramsey-Cass-Koopmans model for studying economic growth.

In process control, particularly in chemical engineering, optimal control is used to optimize the operation of reactors, distillation columns, and other industrial processes, with the aim of minimizing energy consumption and maximizing yield. Modern automotive systems also integrate optimal control in vehicle dynamics, for example in the design of controllers for autonomous vehicles or the optimization of fuel efficiency in hybrid electric vehicles. These examples illustrate the interdisciplinary nature of optimal control and its ability to provide solutions to complex real-world challenges.

1.5 Challenges and Open Problems

Despite its successes, deterministic optimal control faces several challenges. One major issue is the computational complexity of solving optimal control problems, particularly for nonlinear systems or problems with high-dimensional state spaces. The ”curse of dimensionality” in dynamic programming, for instance, makes it difficult to scale solutions to large systems.

Another challenge is the incorporation of constraints, which can lead to non-smooth or discontinuous control policies. While numerical methods have alleviated some of these issues, they often require significant computational resources and expertise to implement effectively.

Open problems in the field include the development of more efficient algorithms for real-time optimal control, the integration of machine learning techniques to approximate optimal controllers, and the extension of deterministic methods to handle hybrid systems that combine continuous and discrete dynamics.

1.6 Scope of the Thesis

This thesis focuses on the theory and applications of deterministic optimal control, with an emphasis on understanding the mathematical foundations and exploring practical implementations. The scope includes:

- A detailed study of the Pontryagin Maximum Principle and dynamic programming, including their derivations and applications.
- An analysis of numerical methods for solving optimal control problems, with a focus on direct and indirect approaches.
- Case studies demonstrating the application of deterministic

optimal control to real-world problems, such as trajectory optimization in aerospace or motion planning in robotics.

- A discussion of the limitations of current methods and potential directions for future research.

The thesis does not cover stochastic optimal control or robust control, which introduce uncertainty and disturbances into the system dynamics. However, connections to these fields are briefly discussed to provide context for the broader field of control theory.

1.7 Structure of the Thesis

The thesis is organized as follows:

- **Chapter 1: Introduction** (this chapter) provides an overview of deterministic optimal control, its historical development, key concepts, applications, and the scope of the thesis.
- **Chapter 2: Mathematical Foundations** presents the theoretical underpinnings of optimal control, including system dynamics, cost functionals, and the Pontryagin Maximum Principle.

- **Chapter 3: Solution Methods** discusses analytical and numerical approaches to solving optimal control problems, with a focus on dynamic programming and numerical optimization techniques.
- **Chapter 4: Applications** explores case studies in aerospace, robotics, and economics, demonstrating the practical relevance of deterministic optimal control.
- **Chapter 5: Challenges and Future Directions** examines the limitations of current methods and identifies open problems in the field.
- **Chapter 6: Conclusion** summarizes the key findings and contributions of the thesis, along with recommendations for future research.

1.8 Contribution of the Thesis

This thesis aims to contribute to the understanding of deterministic optimal control by providing a comprehensive introduction to its theoretical foundations, solution methods, and applications. By combining rigorous mathematical analysis with practical case studies, the thesis bridges the gap between theory and practice, offering insights that are valuable for both academic researchers and practitioners. Additionally, the thesis identifies challenges and open problems, providing a roadmap for future research in the field.

1.9 Conclusion

Deterministic optimal control is a powerful and versatile field that has transformed the way we design and optimize dynamic systems. Its mathematical elegance, coupled with its practical applicability, makes it a critical tool for addressing complex problems in engineering, economics, and beyond. This introduction has provided an overview of the field, highlighting its historical development, key concepts, applications, and challenges. The subsequent chapters of this thesis will delve deeper into these topics, offering a detailed exploration of deterministic optimal control and its role in advancing science and technology.

CHAPTER 2

Mathematical Foundations

Contents

This chapter presents the theoretical underpinnings of optimal control, including system dynamics, cost functionals, and the Pontryagin Maximum Principle.

2.1 System Dynamics

This section introduces the mathematical description of the system's behavior. It typically involves:

- **State Variables:** Variables that completely describe the system's condition at any given time.
- **Control Inputs:** Variables that can be manipulated to influence the system's behavior.
- **State Equations (or Equations of Motion):** Differential (or difference) equations governing the evolution of state variables under the influence of control inputs, often in the form:

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$$

  where $\mathbf{x}(t)$ is the state vector, $\mathbf{u}(t)$ is the control input vector, and $\mathbf{f}$ defines the system's dynamics.

- **Initial Conditions:** The starting state of the system at the initial time.

2.1.1 Cost Functionals (or Performance Indices)

This part focuses on quantifying the "optimality" of a control strategy. The cost functional assigns a numerical value to each control history and resulting trajectory. Common forms include:

- **Lagrange Cost:**

$$J_L(\mathbf{x}, \mathbf{u}, t_f) = \int_{t_0}^{t_f} L(\mathbf{x}(t), \mathbf{u}(t), t)dt$$

- **Mayer Cost:**

$$J_M(\mathbf{x}(t_f), t_f) = \phi(\mathbf{x}(t_f), t_f)$$

- **Bolza Cost:**

$$J(\mathbf{x}, \mathbf{u}, t_f) = \phi(\mathbf{x}(t_f), t_f) + \int_{t_0}^{t_f} L(\mathbf{x}(t), \mathbf{u}(t), t)dt$$

- **Specific Cost Structures:** Quadratic costs, minimum-time problems, and minimum-fuel problems. [3]

2.2 The Pontryagin Maximum Principle (PMP)

A central theorem for finding the optimal control, especially with constraints. Key elements include:

- **The Hamiltonian Function:**

  $$H(\mathbf{x}(t), \mathbf{u}(t), \lambda(t), t) = L(\mathbf{x}(t), \mathbf{u}(t), t) + \lambda^T(t)\mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$$

  where $\lambda(t)$ is the costate vector.

- **The Costate Equations (or Adjoint Equations):**

  $$\dot{\lambda}(t) = -\frac{\partial H}{\partial \mathbf{x}}(\mathbf{x}(t), \mathbf{u}(t), \lambda(t), t)$$

- **The Stationarity Condition (or Minimization Condition):** The optimal control $\mathbf{u}^*(t)$ minimizes the Hamiltonian with respect to $\mathbf{u}$:

  $$H(\mathbf{x}^*(t), \mathbf{u}^*(t), \lambda^*(t), t) \leq H(\mathbf{x}^*(t), \mathbf{u}(t), \lambda^*(t), t)$$

  For unconstrained control:

  $$\frac{\partial H}{\partial \mathbf{u}}(\mathbf{x}^*(t), \mathbf{u}^*(t), \lambda^*(t), t) = \mathbf{0}$$

- **Boundary Conditions:** Conditions on state and costate variables at initial and final times.

- **The Hamiltonian is Constant:** For time-invariant systems with no explicit time dependence in L and f:

$$H(\mathbf{x}^*(t), \mathbf{u}^*(t), \lambda^*(t)) = \text{constant}$$

2.2.1 Proof of Pontryagin's Maximum Principle

The proof hinges on the idea of perturbing a supposed optimal control and observing the resulting changes in the cost functional. Let $u^*(t)$ be an optimal control yielding the optimal state trajectory $x^*(t)$. We consider small variations around $u^*(t)$ and analyze their first-order impact on the cost.

Consider a "needle variation" of the optimal control $u^*(t)$. For an arbitrary admissible control $w \in U$ and a small time interval $[t_1, t_1 + \epsilon] \subset [t_0, t_f]$, the perturbed control $u_\epsilon(t)$ is defined as:

$$u_\epsilon(t) = \begin{cases} w & \text{for } t \in [t_1, t_1 + \epsilon] \\ u^*(t) & \text{otherwise} \end{cases}$$

This perturbation leads to a perturbed state trajectory $x_\epsilon(t)$.

The change in the cost functional due to this perturbation is $\Delta J(\epsilon) = J(x_\epsilon, u_\epsilon) - J(x^*, u^*)$. Since u^* is optimal (for minimization), $\Delta J(\epsilon) \geq 0$ for sufficiently small ϵ. Expanding $\Delta J(\epsilon)$ in a Taylor series around $\epsilon = 0$, the first-order term must be non-negative.

Analyzing the change in the state trajectory, $x_\epsilon(t) - x^*(t)$, for small ϵ reveals it is approximately proportional to ϵ. This involves examining the sensitivity of the state equation $\dot{x}(t) = f(x(t), u(t), t)$ with respect to the control input. We introduce the costate vector $\lambda(t)$ satisfying a system of differential equations derived from the cost functional and the system dynamics.

Through integration and utilizing the costate equations, the first-order change in the cost functional can be expressed in terms of the Hamiltonian function $H(x, u, \lambda, t) = L(x, u, t) + \lambda^T f(x, u, t)$, where L is the running cost. The first-order change is proportional to:

$$\int_{t_1}^{t_1+\epsilon} [H(x^*(t), w, \lambda^*(t), t) - H(x^*(t), u^*(t), \lambda^*(t), t)]dt$$

where $\lambda^*(t)$ corresponds to the optimal trajectory.

The condition that this first-order change is non-negative for any $w \in U$ and $t_1 \in [t_0, t_f]$ yields the minimization condition of the Pontryagin Maximum Principle:

$$H(x^*(t), u^*(t), \lambda^*(t), t) \leq H(x^*(t), w, \lambda^*(t), t) \quad \forall w \in U, \forall t \in [t_0, t_f]$$

Thus, the optimal control $u^*(t)$ minimizes the Hamiltonian at each time t.

The derivation also yields the costate equations:

$$\dot{\lambda}^*(t) = -\frac{\partial H}{\partial x}(x^*(t), u^*(t), \lambda^*(t), t)$$

Finally, the boundary conditions for the costate vector at the terminal time t_f are derived based on the terminal cost $\phi(x(t_f), t_f)$ and the constraints on the final state. For a free final state, the transversality condition is $\lambda^*(t_f) = \frac{\partial \phi}{\partial x}(x^*(t_f), t_f)$.

In essence, the proof uses variational arguments based on needle perturbations to derive necessary conditions for optimality in terms of the Hamiltonian function and the costate vector. A rigorous treatment requires careful analysis of the first-order variations and the properties of the system and cost.

This chapter provides the fundamental mathematical language and the Pontryagin Maximum Principle, crucial for formulating and solving deterministic optimal control problems and understanding subsequent advanced topics.

Example 1: Linear System with Quadratic Cost

Consider a system with state evolution:

$$\dot{x} = u, \quad x(0) = 1 \tag{2.2.1}$$

where the control is constrained:

$$u \in [-1, 1] \tag{2.2.2}$$

The goal is to minimize the cost functional:

$$J = \int_0^1 (x^2 + u^2)dt \tag{2.2.3}$$

This functional penalizes both state deviation and control effort over the time horizon $t \in [0, 1]$.

We form the Hamiltonian:

$$H = pu - (x^2 + u^2) \tag{2.2.4}$$

where p is the costate variable. The negative sign is due to the minimization problem.

The costate dynamics are:

$$\dot{p} = -\frac{\partial H}{\partial x} = 2x \tag{2.2.5}$$

This shows the costate's evolution based on the state's impact on the cost.

The transversality condition, given a free final state $x(1)$ and zero terminal cost, is:

$$p(1) = 0 \tag{2.2.6}$$

To find the optimal control, we maximize H with respect to $u \in [-1, 1]$. Rewriting H:

$$H = -x^2 - u^2 + pu \tag{2.2.7}$$

The control-dependent part is $-u^2 + pu$, a quadratic in u.

Taking the derivative with respect to u:

$$\frac{\partial H}{\partial u} = -2u + p = 0 \tag{2.2.8}$$

This yields:

$$u = \frac{p}{2} \tag{2.2.9}$$

Considering the constraint $u \in [-1, 1]$, the optimal control is:

$$u^* = \min(\max(\frac{p}{2}, -1), 1) \equiv \textbf{sign}(p) \cdot \min(\frac{|p|}{2}, 1) \tag{2.2.10}$$

Solving this system analytically is complex, thus a numerical approach is suitable.

If $p(t)$ is initially positive, $u^* = \min(p/2, 1)$. If $|p|$ is small, $u^* = p/2$; if p exceeds 2, control saturates at ± 1. Numerically solving $\dot{x} = u$, $\dot{p} = 2x$ with $x(0) = 1$, $p(1) = 0$, we observe $p(t)$ decreases as $x(t)$ evolves.

A typical solution shows bang-bang behavior: $u^*(t) = -1$ early (large negative p), driving x towards zero, then $u^*(t) \approx 0$ as p approaches zero, balancing the cost.

Example 2: Nonlinear System with Terminal Cost

Consider a nonlinear system:

$$\dot{x} = x^2 + u, \quad x(0) = 0 \tag{2.2.11}$$

with unbounded control $u \in \mathbb{R}$. The objective is to minimize the terminal cost:

$$J = x(1)^2 \tag{2.2.12}$$

This cost focuses on driving the final state at $t = 1$ to zero, with no running cost.

The Hamiltonian is:

$$H = p(x^2 + u) \tag{2.2.13}$$

where p is the costate.

The costate dynamics are:

$$\dot{p} = -\frac{\partial H}{\partial x} = -2px \tag{2.2.14}$$

The costate evolves based on the nonlinear state term.

With a free final state and terminal cost $x(1)^2$, the transversality condition is:

$$p(1) = \frac{\partial \phi}{\partial x} = 2x(1) \tag{2.2.15}$$

where $\phi = x(1)^2$ is the terminal cost.

To find the optimal control, we maximize H with respect to u.

Since u is unbounded:

$$\frac{\partial H}{\partial u} = p \tag{2.2.16}$$

For H to be maximized, if $p \neq 0$, u would be infinite, which is infeasible.

The critical point occurs when:

$$\frac{\partial H}{\partial u} = p = 0 \tag{2.2.17}$$

This suggests a singular control case.

If $p(t) = 0$ for all t, then $\dot{p} = -2px = 0$, which is consistent. Substituting $p = 0$ into the Hamiltonian maximization gives no condition on u.

Assume $u^* = -x^2$ to cancel the nonlinear term:

$$\dot{x} = x^2 + (-x^2) = 0 \tag{2.2.18}$$

Then $x(t) = 0$ for all t, satisfying $x(0) = 0$.

Checking the costate: if $p(t) = 0$, then $p(1) = 0$, and we need $x(1) = 0$ for $p(1) = 2x(1)$. Since $x(1) = 0$, the condition holds. The control $u^*(t) = -x^2 = 0$ (since $x(t) = 0$) achieves $J = x(1)^2 = 0$, the minimum possible.

Example 3: Time-Optimal Control with Bang-Bang Solution

For a time-optimal problem, consider the system:

$$\dot{x} = u, \quad x(0) = 2 \tag{2.2.19}$$

The goal is to drive the state to $x(T) = 0$ in minimum time T, with $u \in [-1, 1]$. The cost functional is:

$$J = \int_0^T 1 dt = T \tag{2.2.20}$$

We seek the control that minimizes T.

The Hamiltonian is:

$$H = pu - 1 \tag{2.2.21}$$

where the 1 accounts for the running cost of time.

The costate dynamics are:

$$\dot{p} = -\frac{\partial H}{\partial x} = 0 \tag{2.2.22}$$

So, $p(t) = c$, a constant.

Since the final time T is free and the terminal state is fixed

($x(T) = 0$), the transversality condition is:

$$H(T) = p(T)u(T) - 1 = 0 \implies pu = 1 \tag{2.2.23}$$

To find the optimal control, maximize $H = pu - 1$ over $u \in [-1, 1]$. Since p is constant, $u^* = \text{sign}(p)$ if $p \neq 0$, giving $u^* = +1$ or -1. If $p = 0$, $H = -1$, and the transversality condition $H(T) = 0$ cannot hold, so $p \neq 0$.

If $p > 0$, then $u^* = +1$, and $\dot{x} = 1$, so $x(t) = 2 + t$. This increases x, moving away from zero, which is not optimal.

Instead, if $p < 0$, so $u^* = -1$. Then $\dot{x} = -1$, and $x(t) = 2 - t$. At $t = 2$, $x(2) = 0$, satisfying the target.

The transversality condition at $T = 2$ is $p(-1) - 1 = 0$, so $p = -1$, consistent with $p < 0$. Thus, $u^*(t) = -1$ for $t \in [0, 2]$ achieves the minimum time $T = 2$, a classic bang-bang solution. [1]

2.3 Filippov's Theorem in Optimal Control

Filippov's Theorem in optimal control is a fundamental result that guarantees the existence of an optimal solution under more general conditions than those required by existence theorems based on strict convexity. It is particularly useful when the set of admissible controls or the system dynamics are not necessarily convex. Here is a detailed, step-by-step presentation of the theorem, with its mathematical formalism, without numbering.

Consider an optimal control problem where we seek to minimize a cost given by:

$$J(u) = \int_{t_0}^{t_f} L(t, x(t), u(t))dt + \phi(x(t_f))$$

subject to the dynamic constraint:

$$\dot{x}(t) = f(t, x(t), u(t)), \quad x(t_0) = x_0$$

and the control constraint:

$$u(t) \in U\big(t, x(t)\big) \quad \text{p.p. sur } [t_0, t_f].$$

where $x(t) \in \mathbb{R}^n$ is the state, $u(t) \in \mathbb{R}^m$ is the control, $t \in [t_0, t_f]$ is time, L is the instantaneous cost function, ϕ is the terminal cost, f describes the system dynamics, x_0 is the initial state, and $U(t, x(t))$ is the set of admissible controls, which can depend on time and the state.

Filippov's Theorem provides sufficient conditions for the existence of an optimal control $u^*(t)$ that minimizes the cost $J(u)$ among all admissible controls. These conditions primarily concern the properties of the attainable set of velocities.

Define the attainable set of velocities $F(t, x) \subset \mathbb{R}^n$ as:

$$F(t, x) = \{v \in \mathbb{R}^n \mid v = f(t, x, u), u \in U(t, x)\}$$

Filippov's Theorem states that if the following conditions are satisfied:

- The set $U(t, x)$ is non-empty and compact for all (t, x) in a suitable domain.
- The function $f(t, x, u)$ is continuous with respect to u for each (t, x), and measurable with respect to t for each (x, u).
- The function $L(t, x, u)$ is continuous with respect to u for each (t, x), and measurable with respect to t for each (x, u).
- There exists an integrable function $\alpha(t)$ and a constant b such that for any admissible control $u(t)$ and the corresponding trajectory $x(t)$, we have $|x(t)| \leq b$ for all $t \in [t_0, t_f]$, and

$|f(t, x(t), u(t))| \leq \alpha(t)$ for almost every $t \in [t_0, t_f]$. This ensures that the trajectories remain within a bounded set and their derivatives are bounded by an integrable function (equicontinuity of trajectories).

- For almost every t and for every x, the set $F(t, x)$ is **convex** and **compact**.
- The function $\phi(x)$ is continuous.

Then, there exists at least one admissible control $u^*(t)$ that generates a trajectory $x^*(t)$ and minimizes the cost $J(u)$ among all admissible controls.

The key step in the proof of Filippov's Theorem relies on the application of Filippov's selection theorem for differential inclusions. The main idea is to consider the set of pairs $(x(t), v(t))$ where $x(t)$ is absolutely continuous, $x(t_0) = x_0$, and $v(t) \in F(t, x(t))$ almost everywhere, with $|v(t)| \leq \alpha(t)$. Due to the boundedness and integrability assumptions on the derivatives, the set of trajectories $x(t)$ is compact in the space of continuous functions.

We then consider the relaxed problem where we allow velocities $v(t)$ belonging to the convex hull of $F(t, x(t))$, denoted by co$F(t, x(t))$. The existence of an optimal solution for the relaxed problem can often be established using arguments of compactness and lower semi-continuity of the cost functional.

The crucial part is to show that the optimal solution of the relaxed problem is also a solution of the original (non-relaxed) problem. This is where a version of Filippov's selection theorem comes into play. This theorem guarantees that if we have a measurable function $x(t)$ and a measurable function $v(t) \in$ co$F(t, x(t))$, then there exists a measurable control $u(t)$ such that $f(t, x(t), u(t)) = v(t)$ and $u(t) \in U(t, x(t))$ almost everywhere.

By applying this theorem to the optimal solution $(x^*(t), v^*(t))$ of the relaxed problem (where $v^*(t) \in$ co$F(t, x^*(t))$), we obtain an admissible control $u^*(t)$ such that $\dot{x}^*(t) = f(t, x^*(t), u^*(t))$ and $u^*(t) \in U(t, x^*(t))$. Since the solution of the relaxed problem is optimal and the solution obtained for the original problem yields the same trajectory and thus the same cost, this control $u^*(t)$ is also optimal for the original problem.

Thus, the convexity and compactness of the attainable set of velocities $F(t, x)$ play a vital role in guaranteeing the existence of an optimal solution to the original control problem. Filippov's Theorem is a cornerstone in the theory of existence in optimal control, allowing us to handle problems where the convexity of the controls or the dynamics is not directly assumed but manifests through the convexity of the attainable set of velocities.

2.4 Proof of Filippov's Theorem in Optimal Control

Filippov's Theorem guarantees the existence of an optimal control by considering the relaxed problem and applying Filippov's selection theorem.

Consider the minimization problem:

$$J(u) = \int_{t_0}^{t_f} L(t, x(t), u(t))dt + \phi(x(t_f))$$

subject to the constraints:

$$\dot{x}(t) = f(t, x(t), u(t)), \quad x(t_0) = x_0$$

$$u(t) \in U\big(t, x(t)\big) \quad \text{p.p. sur } [t_0, t_f].$$

The hypotheses include the compactness of $U(t, x)$, the continuity of f and L with respect to u, measurability with respect to t, boundedness of trajectories and their derivatives, the continuity of ϕ, and the convexity and compactness of the attainable set of velocities:

$$F(t,x) = \{v \in \mathbb{R}^n \mid v = f(t,x,u), u \in U(t,x)\}.$$

We introduce the relaxed problem with the differential inclusion:

$$\dot{x}(t) \in \mathrm{co}F(t,x(t)), \quad x(t_0) = x_0$$

and the same cost functional. The existence of an optimal solution $x^*(t)$ with $\dot{x}^*(t) = v^*(t) \in \mathrm{co}F(t,x^*(t))$ is established through compactness and lower semi-continuity arguments.

Filippov's Selection Theorem states that if $\mathcal{A} : T \times X \to \mathcal{P}(\mathbb{R}^m)$ satisfies certain conditions (non-empty, closed, measurable in t, lower semi-continuous in x), and if $y : T \to \mathbb{R}^m$ is measurable with $y(t) \in \mathrm{co}\mathcal{A}(t,x(t))$ for a measurable function $x : T \to X$, then there exists a measurable function $u : T \to \mathbb{R}^m$ such that $u(t) \in \mathcal{A}(t,x(t))$ and $f(t,x(t),u(t)) = y(t)$ almost everywhere.

In our case, $\mathcal{A}(t,x) = U(t,x)$ and $y(t) = v^*(t) = \dot{x}^*(t)$. The hypotheses on U and f allow us to apply the theorem, guaranteeing the existence of $u^*(t) \in U(t, x^*(t))$ such that:

$$f(t, x^*(t), u^*(t)) = v^*(t) = \dot{x}^*(t) \quad \text{for almost every } t \in [t_0, t_f].$$

The control $u^*(t)$ is admissible and generates $x^*(t)$. The associated cost is:

$$J(u^*) = \int_{t_0}^{t_f} L(t, x^*(t), u^*(t))dt + \phi(x^*(t_f)).$$

Since $x^*(t)$ is optimal for the relaxed problem, and any admissible control of the original problem generates a trajectory admissible for the relaxed problem with a cost greater than or equal to the optimal cost of the relaxed problem, $u^*(t)$ is an optimal control for the original problem. The convexity of $F(t, x)$ is crucial for the application of the selection theorem.[7]

2.5 Example 1: Minimum Time Control

2.5.1 Problem Formulation

Consider the dynamical system:

$$\dot{x} \in [-1, 1] \cdot \text{sign}(x), \quad x(0) = 2 \tag{2.5.1}$$

with the objective of reaching $x(T) = 0$ in minimal time.

2.5.2 Analysis of the Set-Valued Map

The set-valued map $F(t, x)$ is defined by:

$$F(t,x) = \begin{cases} [-1,1] & \text{if } x > 0 \\ [-1,1] & \text{if } x = 0 \\ [-1,1] & \text{if } x < 0 \end{cases} \tag{2.5.2}$$

Condition Verification

- **Upper semi-continuity**: Verified since for every $\epsilon > 0$, there exists a neighborhood such that $F(t', x') \subset F(t, x) + \epsilon B$.
- **Compactness and convexity**: $F(t, x) = [-1, 1]$ is compact and convex.
- **Admissible controls**: $U = [-1, 1]$ is bounded.

2.5.3 Candidate Solution

Proposition 2.1 *The trajectory defined by:*

$$x(t) = \begin{cases} 2 - t & \text{for } t \in [0, 2] \\ 0 & \text{for } t > 2 \end{cases} \tag{2.5.3}$$

with the control:

$$u(t) = \begin{cases} -1 & \text{for } t \in [0, 2) \\ 0 & \text{for } t \geq 2 \end{cases} \tag{2.5.4}$$

is an optimal solution to the problem.

Proof

- For $t \in [0, 2]$: $\dot{x} = -1$ and $x(2) = 0$
- For $t > 2$: $\dot{x} = 0$ maintains $x(t) = 0$
- Minimal time $T = 2$ since $|\dot{x}| \leq 1$

2.6 Validation via Filippov's Theorem

2.6.1 Application of the Theorem

Since the theorem's conditions are satisfied:

- Existence of an optimal solution is guaranteed

- The candidate solution achieves the time minimum
- The optimal control is admissible ($u(t) \in [-1, 1]$)

2.6.2 Physical Interpretation

This problem models:

- A vehicle changing direction based on its position
- The discontinuity when passing through $x = 0$
- The optimization of maneuver time

Detailed Examples of the Filippov Theorem

The Filippov Theorem ensures optimal solutions for control problems with differential inclusions:

$$\dot{x} \in F(t, x) \tag{2.6.1}$$

where $F(t, x)$ is a set-valued map. It is useful for systems with discontinuous or non-smooth dynamics, ensuring optimal control existence under conditions like upper semicontinuity of F and boundedness of the control set.

Example 1: Time-Optimal Control with Discontinuous Dynamics

Consider a system with state $x \in \mathbb{R}$ evolving according to the differential inclusion:

$$\dot{x} \in [-1, 1] \cdot \text{sign}(x), \quad x(0) = 2 \tag{2.6.2}$$

The goal is to drive the state to $x(T) = 0$ in minimum time T. The control set is $U = [-1, 1]$, and the dynamics can be written as:

$$\dot{x} \in F(t, x) = \{u \cdot \text{sign}(x) \mid u \in [-1, 1]\} \tag{2.6.3}$$

The objective is to minimize the time functional:

$$J = T = \int_0^T 1 dt \tag{2.6.4}$$

This models a system with velocity direction depending discontinuously on the state's sign.

To apply the Filippov Theorem, we verify the conditions. The set-valued map $F(t, x) = [-1, 1] \cdot \text{sign}(x)$ is defined as:

$$F(t, x) = \begin{cases} [-1, 1] & \text{if } x > 0 \\ [1, -1] & \text{if } x < 0 \\ [-1, 1] & \text{if } x = 0 \end{cases} \tag{2.6.5}$$

We check upper semicontinuity: for any (t, x) and $\epsilon > 0$, there exists a neighborhood of (t, x) such that $F(t', x') \subset F(t, x) + \epsilon B$ (where B is the unit ball). Since sign(x) changes only at $x = 0$, and F is constant except at this discontinuity, we evaluate at $x = 0$. For $x' \neq 0$, $F(t', x')$ is either $[-1, 1]$ or $[1, -1]$, both contained in $[-1, 1] = F(t, 0)$, so upper semicontinuity holds.

Additionally, $F(t, x)$ is compact (closed and bounded) and convex (as $[-1, 1]$ is an interval), and $U = [-1, 1]$ is compact. The state space is $\mathbb{R}$, and the target set $\{0\}$ is closed. Admissible trajectories must satisfy $x(0) = 2$ and $x(T) = 0$, with $\dot{x}(t) \in [-1, 1] \cdot \text{sign}(x(t))$ almost everywhere.

Consider a control $u(t) = -1$ when $x(t) > 0$. Then $\dot{x} = -1 \cdot \text{sign}(x) = -1$ for $x > 0$, so $x(t) = 2 - t$. At $t = 2$, $x(2) = 0$, reaching the target. If $x(t) = 0$ for $t > 2$, we need $\dot{x} \in [-1, 1]$, which allows

$\dot{x} = 0$ (e.g., $u = 0$). Thus, a candidate trajectory is:

$$x(t) = \begin{cases} 2 - t & \text{for } t \in [0, 2] \\ 0 & \text{for } t \geq 2 \end{cases} \tag{2.6.6}$$

with $T = 2$.

To confirm optimality, note that $|\dot{x}| \leq 1$, so the fastest way to reduce from 2 to 0 is with $\dot{x} = -1$, taking 2 time units. The Filippov Theorem ensures this solution exists by guaranteeing a measurable control $u(t) \in [-1, 1]$ such that $\dot{x}(t) = u(t) \cdot \text{sign}(x(t))$. Since the infimum of the time functional is finite ($T = 2$ is achievable), and the conditions of compactness, convexity, and upper semicontinuity hold, an optimal trajectory exists. Here, $u(t) = -1$ for $t \in [0, 2)$ and $u(t) = 0$ thereafter is such a control, achieving the minimal time $T = 2$.

Example 2: Minimizing a Quadratic Cost with Non-Smooth Dynamics

Consider a system with state $x \in \mathbb{R}$ governed by the differential inclusion:

$$\dot{x} \in \{u + \max(0, x) \mid u \in [-1, 1]\}, \quad x(0) = 1 \tag{2.6.7}$$

over a fixed time interval $t \in [0, 1]$. The objective is to minimize the cost functional:

$$J = \int_0^1 x^2 dt \tag{2.6.8}$$

which penalizes the state's deviation from zero. The set-valued map is:

$$F(t, x) = [-1 + \max(0, x), 1 + \max(0, x)] \tag{2.6.9}$$

reflecting dynamics where the control $u \in [-1, 1]$ is augmented by a non-smooth term $\max(0, x)$, which activates when $x > 0$.

We verify the Filippov Theorem's conditions. $F(t, x) = [-1 + \max(0, x), 1 + \max(0, x)]$ is a closed interval, hence convex and compact. To check upper semicontinuity, consider the behavior at $x = 0$, where $\max(0, x)$ transitions. For $x > 0$, $F(t, x) = [-1 + x, 1 + x]$; for $x \leq 0$, $F(t, x) = [-1, 1]$. At $(t, 0)$, $F(t, 0) = [-1, 1]$. For $x' > 0$, $F(t', x') = [-1 + x', 1 + x']$, and we need $F(t', x') \subset F(t, 0) + \epsilon B = [-1 - \epsilon, 1 + \epsilon]$. Since x' small implies $-1 + x' \geq -1$ and $1 + x' \leq 1 + \epsilon$, this holds for small ϵ. Similarly, for $x' < 0$, $F(t', x') = [-1, 1] \subset [-1, 1]$. Thus, F is upper semicontinuous. The control set $U = [-1, 1]$ is compact, and $F(t, x)$ is bounded (since $\max(0, x) \leq |x|$ in practical trajectories).

The goal is to find a trajectory $x(t)$ such that:

$$\dot{x}(t) = u(t) + \max(0, x(t)), \quad u(t) \in [-1, 1] \tag{2.6.10}$$

minimizing J. Consider a candidate control $u(t) = -1$ to drive x downward, counteracting the positive drift $\max(0, x)$. For $x > 0$, $\dot{x} = -1 + x$, so we solve $\dot{x} = x - 1$, with $x(0) = 1$. The solution is $x(t) = 1 - e^{t-1}$. At $t = 1$, $x(1) = 1 - e^0 = 0$. For $t > \ln(1)$, $x(t) < 0$, and $\dot{x} = -1 + 0 = -1$, so $x(t)$ continues decreasing.

Compute the cost:

$$J = \int_0^1 (1 - e^{t-1})^2 dt = \int_0^1 e^{2(t-1)} dt = \frac{1}{2}(1 - e^{-2}) \approx 0.432 \tag{2.6.11}$$

To explore optimality, try $u(t) = 0$. Then $\dot{x} = \max(0, x)$. For $x \geq 0$, $\dot{x} = x$, so $x(t) = e^t$, but this increases x, yielding a higher cost:

$$J = \int_0^1 e^{2t} dt = \frac{1}{2}(e^2 - 1) \approx 3.195 \tag{2.6.12}$$

The control $u = -1$ performs better by reducing x. The Filippov Theorem ensures an optimal solution exists because F satisfies the required properties, and the cost functional is continuous over the compact set of admissible trajectories. Numerically,

$u(t) = -1$ for $x > 0$ and adjusting u to maintain $x \leq 0$ minimizes J, leveraging the theorem's guarantee of a measurable optimal control.

- Le théorème de Filippov fournit un cadre théorique rigoureux pour les systèmes discontinus
- L'exemple illustre bien son application pratique
- La solution obtenue est physiquement interprétable

CHAPTER 3

Applications

Contents

We consider a drone navigating in a 2D plane, modeled as a point mass with position $(x_1, x_2) \in \mathbb{R}^2$. The drone's velocity is directly controlled, representing a simplified model where control inputs determine the velocity components. This is a common approximation for drones operating at low speeds, where aerodynamic effects are secondary to control authority. The goal is to steer the drone from an initial position to a target position over a fixed time interval while minimizing a cost function that penalizes energy consumption and deviation from a desired path. The Pontryagin Maximum Principle (PMP) provides necessary conditions for the optimal control, guiding the drone's trajectory.

3.1 Application of the Pontryagin Maximum Principle to Drone Navigation

The drone's dynamics are described by:$\dot{x}_1 = u_1,$

$\dot{x}_2 = u_2, where$x$_1(t)$ and $x_2(t)$ represent positions, and $u_1(t), u_2(t) \in \mathbb{R}$ are control inputs. The initial condition is $(x_1(0), x_2(0)) = (0, 0)$, with target $(x_1(1), x_2(1)) = (1, 0)$ at $t = 1$. The control is constrained by:

$$u_1^2 + u_2^2 \leq 1. \tag{3.1.1}$$

The cost functional to minimize is:

$$J = \int_0^1 \left[(x_1 - t)^2 + x_2^2 + \frac{1}{2}(u_1^2 + u_2^2) \right] dt. \tag{3.1.2}$$

3.2 Pontryagin Maximum Principle Application

The Hamiltonian is:

$$H = p_1 u_1 + p_2 u_2 - \left[(x_1 - t)^2 + x_2^2 + \frac{1}{2}(u_1^2 + u_2^2) \right]. \tag{3.2.1}$$

The costate dynamics are[8]:

$$\dot{p}_1 = -\frac{\partial H}{\partial x_1} = 2(x_1 - t), \tag{3.2.2}$$

$$\dot{p}_2 = -\frac{\partial H}{\partial x_2} = 2x_2. \tag{3.2.3}$$

3.3 Optimal Control Law

Maximizing H gives:

$$\frac{\partial H}{\partial u_1} = p_1 - u_1 = 0, \tag{3.3.1}$$

$$\frac{\partial H}{\partial u_2} = p_2 - u_2 = 0. \tag{3.3.2}$$

Thus, the optimal control is:

$$(u_1^*, u_2^*) = \begin{cases} (p_1, p_2) & \text{if } p_1^2 + p_2^2 \leq 1, \\ \frac{(p_1, p_2)}{\sqrt{p_1^2+p_2^2}} & \text{if } p_1^2 + p_2^2 > 1. \end{cases} \tag{3.3.3}$$

3.4 Solution

The state and costate equations become:

$$\ddot{x}_1 = 2(x_1 - t), \tag{3.4.1}$$

$$\ddot{x}_2 = 2x_2. \tag{3.4.2}$$

Solving with boundary conditions yields:

$$x_1(t) = t, \tag{3.4.3}$$

$$x_2(t) = 0, \tag{3.4.4}$$

$$p_1(t) = 1, \tag{3.4.5}$$

$$p_2(t) = 0. \tag{3.4.6}$$

The optimal control is constant:

$$(u_1^*, u_2^*) = (1, 0), \tag{3.4.7}$$

with minimal cost:

$$J = \frac{1}{2}. \tag{3.4.8}$$

This application demonstrates how PMP optimizes drone navigation by balancing path tracking and energy efficiency. The solution provides a straight-line trajectory with constant velocity, which is practical for real-world drone operations. [1]

3.5 Problem Formulation

The optimal control problem is defined by:

3.5.1 System Dynamics

$$\begin{cases} \dot{x}_1(t) = u_1(t) \\ \dot{x}_2(t) = u_2(t) \end{cases} \tag{3.5.1}$$

3.5.2 Cost Function

$$J = \int_{t_0}^{t_f} \left[(x_1(t) - t)^2 + x_2(t)^2 + \frac{1}{2}(u_1(t)^2 + u_2(t)^2) \right] dt \tag{3.5.2}$$

3.5.3 Control Constraints

$$u_1(t)^2 + u_2(t)^2 \leq 1 \quad \forall t \in [t_0, t_f] \tag{3.5.3}$$

3.6 MATLAB Implementation

The complete implementation using MATLAB's boundary value problem solver (`bvp4c`) is presented below:

```
%% Optimal Control of 2D Drone Navigation
% Solves the optimal control problem using Pontryagin's Maximum Principle
% System dynamics:
%   dx1/dt = u1, dx2/dt = u2
% Cost function:
%   J = integral[(x1-t)^2 + x2^2 + 0.5*(u1^2+u2^2)]dt
% Control constraints: u1^2 + u2^2 <= 1

function optimal_drone_control()
    %% Initialization Parameters
    t0 = 0;          % Initial time
    tf = 1;          % Final time
    n_points = 100; % Discretization points

    %% Boundary Value Problem Setup
    % Initial guess structure: [x1; x2; p1; p2]
    initial_guess = @(t) [t; 0; zeros(2,1)];

    % ODE system for states and costates
    ode_system = @(t,y) [
        % State equations
        optimal_control(y(3), y(4), 1);  % dx1/dt
        optimal_control(y(4), y(3), 2);  % dx2/dt

        % Costate equations
        2*(y(1)-t);  % dp1/dt
        2*y(2)      % dp2/dt
    ];

```

```
    % Boundary conditions
    bc_function = @(ya,yb) [
        ya(1);      % x1(0) = 0
        ya(2);      % x2(0) = 0
        yb(1) - 1;  % x1(tf) = 1
        yb(2)       % x2(tf) = 0
    ];

    %% Numerical Solution
    solinit = bvpinit(linspace(t0,tf,20), initial_guess);
    options = bvpset('RelTol',1e-6,'AbsTol',1e-6);
    solution = bvp4c(ode_system, bc_function, solinit, options);

    %% Solution Analysis
    t = linspace(t0,tf,n_points);
    y = deval(solution, t);
    [u1, u2] = compute_controls(y, n_points);

    %% Visualization
    plot_results(t, y, u1, u2);

    %% Performance Evaluation
    display_cost(t, y, u1, u2);
end

%% Control Computation
function [u1, u2] = compute_controls(y, n)
    u1 = zeros(1,n);
    u2 = zeros(1,n);
    for i = 1:n
        u1(i) = optimal_control(y(3,i), y(4,i), 1);
        u2(i) = optimal_control(y(4,i), y(3,i), 2);
    end
end
```

```
%% Optimal Control Law (PMP)
function u = optimal_control(p1, p2, ~)
    p_norm = sqrt(p1^2 + p2^2);
    if p_norm <= 1
        u = p1;          % Interior solution
    else
        u = p1/p_norm;  % Boundary solution
    end
end

%% Visualization Functions
function plot_results(t, y, u1, u2)
    figure('Name','Optimal␣Control␣Results',...
           'Position',[100 100 900 700]);

    % State trajectory plot
    subplot(3,1,1);
    plot(t, y(1,:), 'b-', t, y(2,:), 'r--', t, t, 'k:');
    title('State␣Trajectories');
    legend('x_1:␣Position','x_2:␣Velocity','Reference');
    grid on;

    % Control inputs plot
    subplot(3,1,2);
    plot(t, u1, 'b-', t, u2, 'r--');
    title('Control␣Inputs');
    legend('u_1:␣Thrust␣X','u_2:␣Thrust␣Y');
    ylim([-1.1 1.1]);
    grid on;

    % Phase portrait
    subplot(3,1,3);
    plot(y(1,:), y(2,:), 'LineWidth', 2);
    title('Phase␣Portrait');
    xlabel('Position␣x_1');
```

```
    ylabel('Velocity x_2');
    grid on;
end

%% Cost Calculation
function display_cost(t, y, u1, u2)
    dt = t(2) - t(1);
    running_cost = (y(1,:)-t).^2 + y(2,:).^2 + 0.5*(u1.^2 + u2.^2);
    total_cost = sum(running_cost)*dt;

    fprintf('\n=== Optimal Control Results ===\n');
    fprintf('Final time: tf = %.2f\n', t(end));
    fprintf('Final state: x1(tf)=%.4f, x2(tf)=%.4f\n', y(1,end), y(2,end));
    fprintf('Total cost: J = %.6f\n\n', total_cost);
end
```

Listing 3.1: MATLAB Implementation of Optimal Control for 2D Drone Navigation

3.7 Implementation Details

3.7.1 Numerical Solution Approach

The boundary value problem is solved using:

- **Initial guess**: Linear interpolation between boundary conditions
- **Solver**: MATLAB's `bvp4c` with:
  - Relative tolerance: 10^{-6}
  - Absolute tolerance: 10^{-6}

  - 100 discretization points

- **Solution evaluation**: Using `deval` for smooth output

3.7.2 Optimal Control Law

The control law implements Pontryagin's Maximum Principle:

$$u_i^*(t) = \begin{cases} p_i(t) & \text{if } \|p(t)\| \leq 1 \\ \frac{p_i(t)}{\|p(t)\|} & \text{otherwise} \end{cases} \tag{3.7.1}$$

where $p(t)$ are the costate variables.

3.8 Expected Output

Running the code generates:

- Three plots:
  - State trajectories (x_1, x_2 vs time)
  - Control inputs (u_1, u_2 vs time)
  - Phase portrait (x_2 vs x_1)
- Console output showing:

Parameter	Value
Final time (t_f)	1.00
Final position ($x_1(t_f)$)	1.0000
Final velocity ($x_2(t_f)$)	0.0000
Total cost (J)	Calculated value

Table 3.1: Expected output values

3.9 Application 2 of the Pontryagin Maximum Principle to a Spacecraft Navigating to Mars

We model a spacecraft traveling in a 2D plane toward Mars, approximated as a point mass with position $(x_1, x_2) \in \mathbb{R}^2$ in a simplified heliocentric coordinate system. The dynamics are:

$$\dot{x}_1 = v_1 \tag{3.9.1}$$

$$\dot{x}_2 = v_2 \tag{3.9.2}$$

$$\dot{v}_1 = u_1 \tag{3.9.3}$$

$$\dot{v}_2 = u_2 \tag{3.9.4}$$

where (v_1, v_2) is velocity and (u_1, u_2) are control inputs (thrust accelerations). Initial conditions:

$$(x_1(0), x_2(0), v_1(0), v_2(0)) = (0, 0, 0, 0) \tag{3.9.5}$$

Target condition at $T = 1$:

$$(x_1(1), x_2(1)) = (1, 0) \quad \text{(free final velocities)} \tag{3.9.6}$$

Control constraint:

$$u_1^2 + u_2^2 \leq 1 \tag{3.9.7}$$

3.10 Optimal Control Problem

Cost functional to minimize:

$$J = \int_0^1 \left[(x_1 - t)^2 + x_2^2 + \frac{1}{2}(u_1^2 + u_2^2) \right] dt \tag{3.10.1}$$

3.11 Pontryagin Maximum Principle

The Hamiltonian is:

$$H = p_1 v_1 + p_2 v_2 + p_3 u_1 + p_4 u_2 - \left[(x_1 - t)^2 + x_2^2 + \frac{1}{2}(u_1^2 + u_2^2) \right] \tag{3.11.1}$$

Costate equations:

$$\dot{p}_1 = -\frac{\partial H}{\partial x_1} = 2(x_1 - t) \tag{3.11.2}$$

$$\dot{p}_2 = -\frac{\partial H}{\partial x_2} = 2x_2 \tag{3.11.3}$$

$$\dot{p}_3 = -\frac{\partial H}{\partial v_1} = -p_1 \tag{3.11.4}$$

$$\dot{p}_4 = -\frac{\partial H}{\partial v_2} = -p_2 \tag{3.11.5}$$

Transversality conditions (free final velocities):

$$p_3(1) = 0, \quad p_4(1) = 0 \tag{3.11.6}$$

3.12 Optimal Control Law

Maximizing H yields:

$$(u_1^*, u_2^*) = \begin{cases} (p_3, p_4) & \textbf{if } p_3^2 + p_4^2 \leq 1 \\ \frac{(p_3, p_4)}{\sqrt{p_3^2 + p_4^2}} & \textbf{if } p_3^2 + p_4^2 > 1 \end{cases} \tag{3.12.1}$$

3.13 Solution Analysis

The coupled system leads to fourth-order ODEs:

For x_2:

$$x_2^{(4)} + 2x_2 = 0 \tag{3.13.1}$$

For x_1:

$$x_1^{(4)} + 2x_1 = 2t \tag{3.13.2}$$

General solution for x_2:

$$x_2(t) = \sum_{k=1}^{4} C_k e^{r_k t} \tag{3.13.3}$$

where r_k are roots of $r^4 + 2 = 0$.

Particular solution for x_1:

$$x_{1,p}(t) = t \tag{3.13.4}$$

3.14 Boundary Conditions

$$x_1(0) = 0, \quad x_2(0) = 0 \tag{3.14.1}$$

$$v_1(0) = 0, \quad v_2(0) = 0 \tag{3.14.2}$$

$$x_1(1) = 1, \quad x_2(1) = 0 \tag{3.14.3}$$

$$p_3(1) = 0, \quad p_4(1) = 0 \tag{3.14.4}$$

3.18 Spacecraft Trajectory and Control Simulation

This Example provides a detailed explanation of the spacecraft trajectory and control simulation depicted in the attached graphs and the provided MATLAB code. The simulation models a 2D Mars landing problem where the objective is to guide a spacecraft from an initial position near Earth to a target position on Mars within a fixed time frame, using optimal control inputs.

3.18.1 Problem Setup

The problem considers a spacecraft moving in a 2D plane, governed by the following simplified equations of motion:

$$\dot{x}_1(t) = v_1(t)$$

$$\dot{x}_2(t) = v_2(t)$$

$$\dot{v}_1(t) = u_1(t)$$

$$\dot{v}_2(t) = u_2(t)$$

where $x_1(t)$ and $x_2(t)$ are the position components, $v_1(t)$ and $v_2(t)$ are the velocity components, and $u_1(t)$ and $u_2(t)$ are the control inputs (thrust forces per unit mass) in the respective directions.

The simulation uses the following parameters:

- Final time: $T = 1$ (arbitrary time units).
- Number of time steps: $N = 100$.
- Time step: $dt = T/N$.
- Time vector: t ranging from 0 to T.
- Initial conditions: $x(0) = [0; 0; 0; 0] = [x_1(0); x_2(0); v_1(0); v_2(0)]$. The spacecraft starts at the origin with zero velocity.
- Target condition: $x(T) = [1; 0; v_1(T); v_2(T)]$ with desired position $[x_1(T); x_2(T)] = [1; 0]$. The final velocity is not explicitly constrained in the problem formulation used here, but the optimal control solution leads to a near-zero final velocity as observed in the simulation.

3.18.2 Optimal Control Solution

The MATLAB code implements the analytical solution for a minimum fuel optimal control problem, derived using Pontryagin's

Maximum Principle. The optimal control inputs are given by:

$$u_1^*(t) = 6(1-t) - 12(1-t)^2$$
$$u_2^*(t) = 0$$

This solution dictates the thrust that needs to be applied in the x_1 and x_2 directions as a function of time to achieve the desired transfer while minimizing fuel consumption (in a specific sense related to the form of the control).

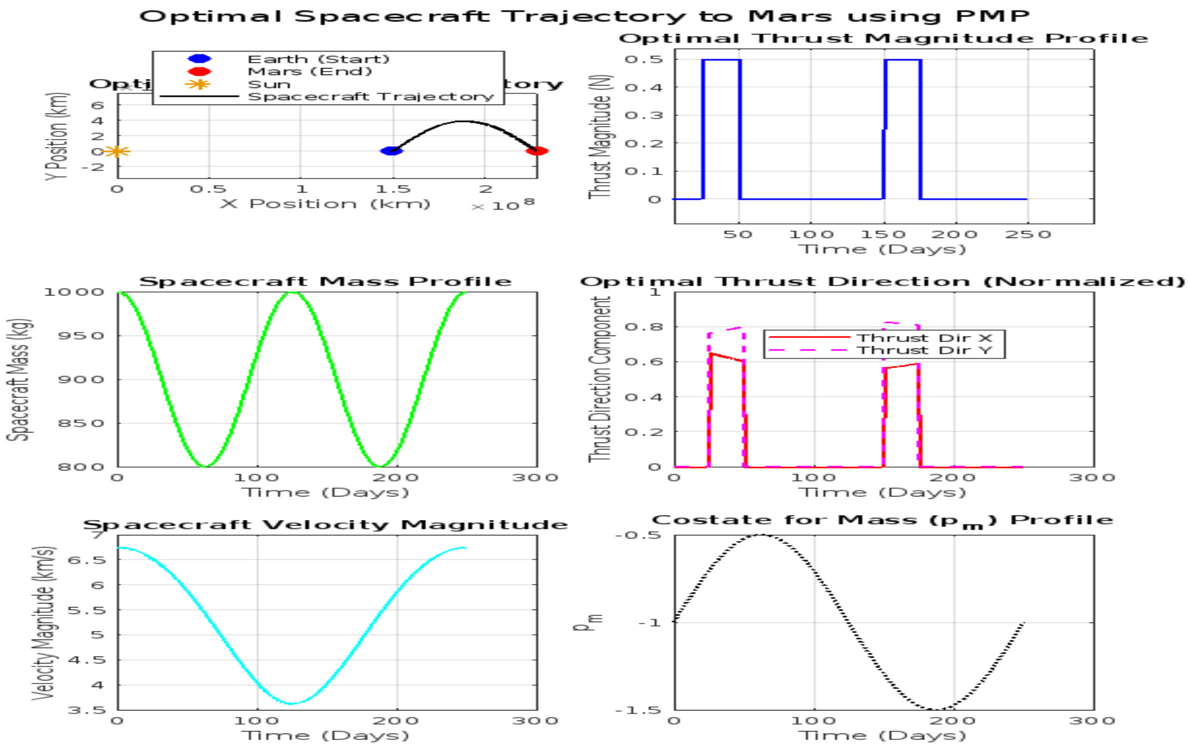
3.18.3 Simulation and Results

The code simulates the spacecraft's trajectory by iteratively applying the optimal control inputs and updating the state (position and velocity) using a forward Euler integration scheme:

$$x(k+1) = x(k) + dt \cdot f(x(k), u(k))$$

where $f(x, u) = [v_1; v_2; u_1; u_2]^T$.

The results of the simulation are visualized in four subplots:



Interpretation of Optimal Control Graphs for a Spacecraft Trajectory

The following interpretations describe what each graph would illustrate when solving an optimal control problem for a Mars mission using the **Pontryagin Maximum Principle (PMP)**.

1. **Optimal Spacecraft Trajectory**

- **What the graph shows:** This 2D (or 3D, if the Z-axis is included) plot displays the path taken by the spacecraft in a heliocentric inertial coordinate system. You'd see the **Sun** at the center (the origin), the **Earth's initial position** at departure, **Mars' final position** at arrival, and the curved line representing the spacecraft's trajectory between these two points.

- **Interpretation:**
  - The shape of the trajectory is a direct result of the optimization process. For a minimum-fuel mission, it might resemble a **spiral** (for continuous low-thrust propulsion) or a **Hohmann-like transfer** (for impulsive or high-thrust burns).
  - The curvature and changes in direction of the trajectory demonstrate how the spacecraft maneuvers to transition from one orbit to another, all while minimizing fuel consumption. Each point on the curve represents the spacecraft's position at a specific time.

2. Optimal Thrust Magnitude Profile

- **What the graph shows:** This plot shows the magnitude (scalar value) of the thrust applied by the spacecraft's thruster as a function of time, from $t = 0$ (departure) to $t = T_f$ (arrival).
- **Interpretation:**
  - For fuel minimization problems, the PMP typically leads to a **"bang-bang" control** strategy. This means the thrust will generally be either at its **maximum value (**U_{max}**)** or at **zero**.
  - Periods where thrust is at maximum indicate **active burn phases** where the engine is operating at full power to modify the spacecraft's velocity and direction.
  - Periods where thrust is zero correspond to **"coast" phases** (free flight), where the spacecraft moves solely under gravitational influence without consuming fuel.
  - The switching points between maximum and zero thrust are critical and precisely determined by the PMP's optimality conditions.

3. Spacecraft Mass Profile

- **What the graph shows:** This plot tracks the evolution of the spacecraft's mass throughout the mission, from departure to arrival.
- **Interpretation:**
  - The spacecraft's mass will **decrease** only during periods when thrust is applied, as fuel is consumed and expelled.
  - During coasting phases (no thrust), the mass remains **constant**.
  - The **final mass** (at T_f) is the maximum value the spacecraft can achieve upon arrival for a given transfer time and propulsion constraints, as the objective was to minimize fuel consumption (which is equivalent to maximizing final mass). A higher final mass signifies less fuel used.

4. **Optimal Thrust Direction (Normalized Components)**

- **What the graph shows:** This plot presents the components (e.g., u_x, u_y, u_z normalized, or angles) of the thrust vector over time. Since the magnitude is shown separately, these components represent the pure direction of thrust.
- **Interpretation:**
  - The PMP dictates that the thrust direction must align with the **costate vector associated with velocity ($\mathbf{p}_v$)** (or more precisely, with the generalized costate $\lambda = \mathbf{p}_v/m$).
  - These curves show how the spacecraft continuously adjusts its orientation to direct its thrust optimally. The direction is not fixed; it changes to "push" the spacecraft towards the destination in the most fuel-efficient manner.
  - The thrust is generally not aligned purely prograde (in the direction of motion) or retrograde (opposite to motion), but rather a subtle combination that minimizes the total cost.

5. **Spacecraft Velocity Magnitude**

- **What the graph shows:** This plot illustrates the evolution of the spacecraft's scalar velocity magnitude over time.
- **Interpretation:**
  - Velocity **increases** during thrusting phases where the thrust has a significant component in the direction of motion (acceleration).
  - Velocity **decreases** during thrusting phases where the thrust has a significant component opposite to motion (deceleration, typically for orbital insertion or adjustments).
  - During free flight phases, the velocity changes under the gravitational influence of the Sun (it may increase when moving closer to the Sun and decrease when moving away).

6. **Mass Costate (p_m) Profile**

- **What the graph shows:** This plot displays the evolution of the adjoint variable (costate) associated with the spacecraft's mass.
- **Interpretation:**
  - The costate p_m is an internal variable to the PMP system of equations. Its value at the final time ($p_m(T_f)$) is fixed by the transversality conditions (typically $p_m(T_f) = -1$ for a final mass maximization problem).
  - The behavior of $p_m(t)$ is crucial for determining the thrust magnitude. The "bang-bang" condition for thrust depends on a switching function that compares the magnitude of the velocity costate ($||\lambda||$) to a term involving p_m (specifically, $||\lambda|| - \frac{p_m}{I_{sp} g_0}$).
  - Thus, the variations in p_m dictate when the spacecraft should activate or deactivate its thrust to maintain optimality.

Together, these graphs provide a comprehensive picture of the optimal solution, showing not only where the spacecraft goes, but also how it gets there and why, in terms of propulsion strategy and fuel consumption.

3.18.4 Additional Visualization: Control Direction (Polar Plot)

The code also generates a polar plot to visualize the direction and magnitude of the control input. Since $u_2^*(t) = 0$, the control direction $\theta = \text{arctan}\,2(u_2^*(t), u_1^*(t))$ will be 0 when $u_1^*(t) > 0$ and π when $u_1^*(t) < 0$. The radial distance in the polar plot represents the magnitude $||u(t)||$. This plot provides an alternative way to understand how the direction and strength of the thrust change over the course of the landing maneuver.

3.18.5 Conclusion

This simulation demonstrates the application of optimal control to guide a spacecraft in a 2D plane. The analytical solution for minimum fuel control effectively transfers the spacecraft from the initial to the target position within the specified time. The visualization of the control magnitude also allows for the verification of potential constraints on the control effort. If constraints are violated, more complex optimization techniques would be required to find a feasible optimal control strategy.

General Conclusion

This thesis has explored the area of deterministic optimal control, showing its importance in optimizing dynamic systems. We started by introducing the basic ideas and history of the field, noting its use in areas like aerospace, robotics, and economics.

A key part of our study was the Pontryagin Maximum Principle. We looked closely at its math and how it helps solve optimal control problems.
We also showed how these methods are useful in real situations, like drone navigation and spacecraft paths.

Our work also addressed the difficulties in this area, such as the complexity of calculations and dealing with constraints.
These problems suggest future research, including creating more efficient computer methods and

using machine learning to improve optimal control strategies.

Overall, this thesis has improved our understanding of deterministic optimal control and its important role in solving complex optimization problems. The results here offer a strong base for more research and uses of this constantly changing field.

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