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Operational Matrices For Solving Burgers' Equation By Using Clique Polynomials

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Dedication

Praise be to God, who granted me the strength and determination to pursue the path of knowledge and complete this humble work.

To my dear parents, I extend my deepest gratitude and appreciation for the love and support you have given me since my early childhood.

To my beloved family, thank you for your constant encouragement and support.

To my dear siblings, I am truly grateful for your unwavering presence by my side.

And to my friend Amel, heartfelt thanks for your sincere support and companionship throughout this journey.

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I would also like to thank the members of committee, **Dr. Kaci Fatima** and **Dr. Radjeh Fouzia,** and all the professors in the Mathematics Department for their technical support and continuous cooperation.

Finally, I extend my heartfelt thanks and appreciation to my dear parents, my siblings, and all my friends for their unwavering support and constant encouragement.

Thank's

Abstract

This study focuses on the Burgers' equation, which plays an important role in mathematical modeling and describes physical phenomena such as fluid flow and heat transfer. Given the difficulty of solving this equation accurately due to its nonlinear nature, we used a new method based on Clique functions to construct operational matrices that help approximate the solution.

We applied this method to transform the equation into an algebraic form that can be solved numerically, and then compared the results with those of other existing methods. The results showed that the developed method provides accurate solutions and requires less computation time.

In conclusion, this method proved to be efficient and can be applied to similar equations in the future.

Notations and symbols

ODE	:	Ordinary Differential Equation
PDE	:	Partial Differential Equation
I_n	:	The identity matrix of the order $n \times n$
$\det(A)$	:	The determinant of a matrix A
A^{-1}	:	The inverse of a matrix A
$Tr(A)$	:	The trace of a matrix A
A^T	:	The transpose of a matrix A

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Introduction

Partial differential equations (PDEs) are among the most important tools in applied mathematics, given their crucial role in modeling various physical and engineering phenomena such as heat transfer, material diffusion, and fluid flow. Among these equations, the Burgers' equation stands out due to its combination of two essential features: diffusion and nonlinearity. This duality makes it an ideal model for studying complex behaviors in physical systems. In light of the importance of this class of equations, this work focuses on the topic: "**Operational Matrices for Solving Burgers' Equation Using Clique Polynomial**." This choice was motivated by several factors, including a desire to explore numerical solutions for nonlinear equations, the pursuit of accurate and efficient techniques to overcome the absence of analytical solutions in many cases, and the promising capabilities of Clique functions, which are relatively new tools in the field of numerical methods.

Within this context, the central research question guiding this study is: "How can we construct an accurate and efficient numerical method based on operational matrices and Clique functions to solve the Burgers' equation?"

To answer this question, the structure of the memory is organized as follows:

Chapter One provides a theoretical overview of partial differential equations, including their classifications and key properties.

Chapter Two explores the main types of operational matrices, their characteristics, and introduces Clique functions alongside the construction of their associated matrices.

Chapter Three presents a comprehensive analysis of the Burgers' equation, explaining its mathematical formulation, properties, and historical context. We apply the operational matrix method using Clique polynomials to solve this equation, followed by an analysis and comparison of the results with those of other numerical approaches to objectively evaluate the method's performance.

By integrating rigorous theoretical foundations with computer-based numerical applications, we were able to develop an efficient numerical model that yields accurate approximations to the solution of the Burgers' equation. We conducted a thorough review of previous studies, drawing from peer-reviewed articles and research papers that addressed the equation using various approaches, allowing for an insightful evaluation of our proposed method. The references used in this work include recent publications from international scientific journals as well as specialized books in numerical analysis, partial differential equations, and approximation functions. Despite the challenges we encountered particularly the conceptual complexity of Clique functions and the technical difficulties in programming the numerical model the valuable academic guidance of my supervisor enabled us to overcome these obstacles, and we express our deep gratitude for his continuous support.

The memory concludes with a list of the references used, followed by a concise summary of the main results and conclusions drawn from the study.

Chapter 1

Generalities of partial differential equation

In this chapter, we will explore the fundamental concepts, classifications, and various solving techniques of PDEs, with an emphasis on practical applications that highlight their importance in science and engineering.

1.1 Ordinary Differential Equations (ODEs)

Ordinary differential equation is defined as an equation composing of the derivative of the dependent variable having only one independent variable .

Examples include:

1. $\frac{dy}{dx} = 3x$: Dependent variable y, independent variable x.

2. $\frac{d^2y}{d^2x^2} + 2xy = e^x$: Dependent variable y, independent variable x.

3. $\frac{d^2x}{d^2t^2} + 3\frac{dx}{dt} + 2x = sin(2t)$: Dependent variable x, independent variable t.

1.2 Definition of a PDE

A partial differential equation (PDE) are a type of differential equation that contains partial derivatives of a dependent variable (an unknown function) with respect to multiple independent variable. A PDE is generally written as:

$$Au_{xx} + Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = G \tag{1.1}$$

where A, B, C, D, E, F are constants or functions known in terms of X, Y.

Every linear partial differential equation such as (1) represents of the following patterns:

- A parabola
- Hyperbola
- Ellipse

The equation of parabola have the heat flow and diffusion processes and achieve the property

$$B^2 - 4AC = 0$$

Hyperbolic equation describe vibrational and wave motions and satisfy the property

$$B^2 - 4AC > 0$$

The elliptical equation describe steady -state phenomena and satisfy the property

$$B^2 - 4AC < 0$$

Examples of PDEs include:

1. Heat equations:

$u_t = Ku_{xx}(1D); ut = K(u_{xx} + u_{yy})(2D); ut = K(u_{xx} + u_{yy} + u_{zz})(3D);$

describing heat flow in varying dimensions.

2. Wave equations:

$u_{tt} = c^2 u_{xx}(1D); u_{tt} = c^2(u_{xx} + u_{yy})(2D); u_{tt} = c^2(u_{xx} + uyy + u_{zz})(3D);$

describing wave propagation.

3.Laplace equations (time-independent):

$u_{xx} + u_{yy} = 0(2D); u_{xx} + u_{yy} + u_{zz} = 0(3D);$

4. Burgers' equation:

$u_t + uu_x - nuu_{xx} = 0,$

These equations describe phenomena such as heat flow, wave propagation, and fluid dynamics in various dimensions.

1.3 Properties of Partial Differential Equations(PDEs)

Partial Differential Equations($PDEs$) have several important properties that categorize and characterize their behavior. These properties are essential for understanding how to solve $PDEs$ and interpret their solutions. Below are the key properties:

1.3.1 Order of a PDE

The order of a PDE is determined by the highest order of the partial derivative present in the equation. For example**:**

$u_x - u_y = 0$ (**1st order**).

$u_{xx-}u_t = 0$(**2nd order**).

Example : Determining the Order of PDEs

(a) $u_t = u_{xx} + u_{yy}$: The highest derivative is u_{xx} or u_{yy}. **Order: 2.**

(b) $u_x + u_y = 0$: The highest derivative is u_x or u_y. **Order: 1**.

1.3.2 Linear and Nonlinear PDEs

- A partial differential equation is **linear** if:

1. The power of the dependent variable and each partial derivative is one.

2. The coefficients of the dependent variable and its partial derivatives are constants or independent variables.

Examples of linear PDEs:

1. Heat equation: $u_t = Ku_{xx}$.

2. Wave equation:$u_{tt} = c^2 u_{xx}$.

3. Laplace equation: $u_{xx} + u_{yy} = 0$.

4. Linear Schrödinger's equation:$iu_t + u_{xx} = 0$.

- A *partial differential equation is* **non-linear** if it contain nonlinear terms such as: $\sin y, e^y, \sqrt{y}, y^2, yy'$or $\ln y$.

Examples of nonlinear PDEs:

1. Advection equation: $u_t + uu_x = f(x;t)$.

2. Burgers equation: $u_t + uu_x = \alpha u_{xx}$.

3. Sine-Gordon equation: $u_{tt-}u_{xx} = \alpha \sin(u)$.

What is the significance of these equations?

- **Linear** $PDEs$ often describe fundamental processes like diffusion and wave propagation.
- **Nonlinear** $PDEs$ arise in advanced fields like fluid dynamics, plasma physics, and nonlinear optics, often producing solitary wave solutions.

1.3.3 Homogeneous and Inhomogeneous PDEs

One way to classify partial differential equations is based on their structure: they can be **homogeneous** or **non-homogeneous**. This distinction is important in understanding the behavior of solutions and the methods required to solve them.

- A PDE is **homogeneous** if every term in the equation contains the dependent variable u or one of its derivatives.
- A PDE is **inhomogeneous** if at least one term in the equation does not contain the dependent variable u or any of its derivatives.

Exemples

1. $u_t = 4u_{xx}$: Homogeneous (all terms involve u or its derivatives).

2. $u_t = u_{xx} + x$: Inhomogeneous (term x does not involve u).

3.$u_{xx} + u_{yy} = 0$: Homogeneous.

4. $u_x + u_y = u + 4$: Inhomogeneous (term 4 does not involve u or its derivatives).

This classifcation helps determine whether external sources or independent terms are influencing the system described by the PDE. Homogeneous equations often arise in idealized systems, while inhomogeneous ones account for external forces or inputs.

1.4 Initial conditions

It was indicated before that the $PDEs$ mostly arise to govern physical phenomenon such as heat distribution, wave propagation phenomena and phenomena of quantum mechanics. Most of the $PDEs$, such as the diffusion equation and the wave equation, depend on the time t. Accordingly, the initial values of the dependent variable u at the starting time $t = 0$ should be prescribed. It will be discussed later that for the heat case, the initial value $u(t = 0)$, that defines the temperature at the starting time, should be prescribed. For the wave equation, the initial conditions $u(t = 0)$ and $u_t(t = 0)$ should also be prescribed.

1.5 Boundary Conditions

The general solution of partial differential equations ($PDEs$) is not sufficient; a specific solution must satisfy prescribed conditions.

When a PDE governs the behavior of a physical phenomenon within a bounded domain D, the value of the dependent variable u is specified at the boundaries.

These specified values are known as boundary conditions, which are classified into three types:

1. Dirichlet Boundary Conditions

The function u is explicitly defined at the boundary.

Exemple

For a rod of length L, where $0 < x < L$, the boundary conditions are:

$u(0) = \alpha \ , \quad u(L) = \beta$

For a rectangular plate, values are specified at:

$u(0, y)$, $u(L_1, y)$, $u(x, 0)$, $u(x, L_2)$

2. Neumann Boundary Conditions

The normal derivative $\frac{du}{dn}$of u at the boundary is specified:

$u_x(0, t) = \alpha$, $u_x(L, t) = \beta$

3. Mixed Boundary Conditions

A linear combination of the function u and its normal derivative is specified on the boundary.

Chapter 2

Description of the operational matrix method and application

In this chapter, we will present the definitions of matrices and description of the operational matrix method with some applications on differential equations.

2.1 Matrices

2.1.1 Definition of a square matrix

A square matrix is a matrix in which the number of rows is equal to the number of columns.Its order is represented as $n \times n$, where n is a positive integer.

It can be expressed as follows:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}_{n \times n}$$

where:

- A is the square of order n.
- a_{ij} is the element located at row i and column j.

Exemples

1. **A square matrix of order** 2×2:

$$A = \begin{bmatrix} 3 & 5 \\ 1 & 2 \end{bmatrix}$$

2. **A square matrix of order** 3×3 :

$$B = \begin{bmatrix} 4 & 0 & -2 \\ 1 & 3 & 5 \\ 7 & 8 & 6 \end{bmatrix}$$

2.1.2 Definition of a vectors

A vector is a matrix with only one row or one column, and its values are called components of the vector. Vectors are denoted by lowercase bold letters such as a or b, or using brackets like $a = [a_1]$.

- **Row vector:** It has the form:

$a = \begin{bmatrix} a_1 & a_2 & ... & a_n \end{bmatrix}$ such as

$$a = \begin{bmatrix} -2 & 5 & 0.8 & 0 & 1 \end{bmatrix}$$

- **Column vector:** It has the form:

$$b = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_m \end{bmatrix} \text{ such as } b = \begin{bmatrix} 4 \\ 0 \\ -7 \end{bmatrix}$$

2.2 Properties of square matrices

2.2.1 The matrix trace

Let A be an $n \times n$ square matrix. The trace of A , denoted as $tr\,(A)$, is the sum of the main diagonal elements of the matrix. That is:

$$tr(A) = \sum_{i=1}^{n} a_{ii}$$

Let: $$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad ,B = \begin{bmatrix} 1 & 2 & 0 \\ 3 & 8 & 1 \\ -2 & 7 & -5 \end{bmatrix}$$

To find the trace of A and B, we sum the main diagonal elements

$$tr\,(A) = 1 + 4 = 5$$

$$tr\,(B) = 1 + 8 - 5 = 4$$

Properties of the matrix trace

Let A and B be $n \times n$ matrices. then:

1. $tr\,(A + B) = tr\,(A) + tr\,(B)$

2. $tr\,(A - B) = tr\,(A) - tr\,(B)$

3. $tr\,(kA) = k \cdot tr\,(A)$, where k is a scalar

4. $tr\,(AB) = tr\,(BA)$

5. $tr\left(A^T\right) = tr\,(A)$

2.2.2 Determinants

The determinant of an $n \times n$ matrix A , denoted $det\,(A)$ or $|A|$, is a number given by the following :

- If A is a 1×1 matrix $A = [a]$, then $det(A) = a$.

- If A is a 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

The determinant is calculated using the formula:

$$det(A) = ad - bc.$$

Numerical Example:

If $$A = \begin{bmatrix} 3 & 4 \\ 2 & 5 \end{bmatrix}$$

Then the determinant is:

$$det(A) = (3 \times 5) - (4 \times 2) = 15 - 8 = 7$$

- if A is an $n \times n$ matrix, where $n \geq 2$, then $det\,(A)$ is the number found by taking the cofactor expansion along the first row of A. That is,

$$det\,(A) = a_{1,1}C_{1,1} + a_{1,2}C_{1,2} + \cdots + a_{1,n}C_{1,n}.$$

Determinant Properties

Let A and B be $n \times n$ matrices and let k be a scalar .The following are true :

1. $det\,(kA) = k^n \cdot \det(A)$
2. $det\left(A^T\right) = \det(A)$
3. $det\,(AB) = \det(A)\det(B)$
4. If A is invertible, then

   $\det\left(A^{-1}\right) = \frac{1}{\det(A)}$.
5. A matrix A is invertible if and only if $det\,(A) \neq 0$

2.2.3 Inverse of a matrix:

The inverse of a square matrix A of order $n \times n$ is the matrix A^{-1} that satisfies the equation:

$$AA^{-1} = A^{-1}A = I_n$$

where I_n is the **identity matrix** of the same order.

For a matrix to have an inverse, it must be **non-singular**, meaning that its determinant is not zero:

$$det(A) \neq 0$$

Consider the matrix:

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$$

First, we calculate the determinant:

$$det(A) = (2 \times 4) - (3 \times 1) = 8 - 3 = 5$$

Since the determinant is **not zero**, the matrix is invertible. The inverse is calculated using the formula:

$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Substituting the values:

$A^{-1} = \frac{1}{5} \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$

Thus, the inverse of A is:

$A^{-1} = \begin{bmatrix} \frac{4}{5} & -\frac{3}{5} \\ -\frac{1}{5} & \frac{2}{5} \end{bmatrix}$

Properties of inversible matrices

Let A and B be $n \times n$ inversible matrices. Then:

1. AB is inversible; $(AB)^{-1} = B^{-1}A^{-1}$.

2. A^{-1} is inversible; $(A^{-1})^{-1} = A$.

3. nA is inversible for any nonzero scalar n; $(nA)^{-1} = \frac{1}{n}A^{-1}$.

4. If A is a diagonal matrix, with diagonal entries $d1, d2, \cdots, dn$, where none of the diagonal entries are 0, then A^{-1}exists and is a diagonal matrix. Furthermore, the diagonal entries of $A^{-1}$are

$1/d1, 1/d2, \cdots, 1/dn$.Furthermore,

1. If a product AB is not inversible, then A or B is not inversible.

2. If A or B are not inversible, then AB is not inversible.

2.2.4 transpose of a matrix

The transpose of a square matrix is a new matrix obtained by swapping the rows and columns of the original matrix. If A is a square matrix of order, then its transpose A^T is the matrix where the element in position (i, j) is equal to the element in position (i, j) of the original matrix:

$(A^T)_{ij} = A_{ij}, \forall i, j$

Let $\quad A = \begin{bmatrix} 7 & 8 \\ 9 & 10 \end{bmatrix}$

The transpose of this matrix A^T will be:

$$A^T = \begin{bmatrix} 7 & 9 \\ 8 & 10 \end{bmatrix}$$

Properties of the transpose of a matrix:

Let A and B be matrices where the following operations are defined. Then

1. $(A + B)^T = A^T + B^T$and $(A - B)^T = A^T - B^T$

2. $(kA)^T = kA^T$ for any real number k.

3. $(AB)^T = B^T A^T$ for any two matrices that can be multiplied.

4. $(A^{-1})^T = (A^T)^{-1}$

5. $(A^T)^T = A$.

2.3 Types of square matrices

2.3.1 Diagonal matrix

A diagonal matrix is a special type of square matrix in which all the elements outside the main diagonal are zero. It can be represented as follows:

$$D = \begin{bmatrix} d_{11} & 0 & 0 & ... & 0 \\ 0 & d_{22} & 0 & ... & 0 \\ 0 & \cdots & d33 & ... & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & ... & d_{nn} \end{bmatrix}$$

where d_{ii} represents the diagonal elements, which can be real or complex numbers.

Properties of a diagonal matrix:

- **Addition and subtraction:**

If $D_1$and D_2 are diagonal matrices of the same order, their sum and difference will also be diagonal matrices.

$$D_1 + D_2 = diagonal\ matrix$$

- **Miltiplication:**

The product of two diagonal matrices of the same order is also a diagonal matrix, where:

$$(D_1 + D_2) = (D_1)_{ii} \cdot (D_2)_{ii}$$

- **Inverse matrix:**

If all diagonal elements are nonzero, the diagonal matrix is inversible, and its inverse is also a diagonal matrix:

$$(D^{-1})_{ii} = \frac{1}{d_{ii}}$$

- **Matrix Powers:**

If is a diagonal matrix, then:

$$D^k = \begin{bmatrix} d_{11}^k & 0 & 0 & \dots & 0 \\ 0 & d_{22}^k & 0 & \dots & 0 \\ 0 & 0 & d_{33}^k & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & d_{nn}^k \end{bmatrix}$$

- **Eigenvalues and Eigenvectors:**

The eigenvalues of a diagonal matrix are its diagonal elements, and the eigenvectors are the standard basis vectors.

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Properties of this example

- It is a diagonal matrix because all non-diagonal elements are zero.
- Its eigenvalues are $3, -2$ and 5.
- Its inverse, if all diagonal elements are nonzero, is given by:

$$D^{-1} = \begin{bmatrix} \frac{1}{3} & 0 & 0 \\ 0 & \frac{-1}{2} & 0 \\ 0 & 0 & \frac{1}{5} \end{bmatrix}$$

2.3.2 Identity matrix

The **identity matrix** is a square matrix (the number of rows equals the number of columns) that has ones **(1s)** on the main diagonal (extending from the top left to the bottom right) and zeros **(0s)** elsewhere.

It is usually denoted as I_n , where n represents the number of rows (or columns). The identity matrix serves as the **multiplicative identity** in matrix operations.

- **Identity matrix of order $\mathbf{2 \times 2}$** :

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- **Identity matrix of order $\mathbf{3 \times 3}$:**

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Properties of the identity matrix

- **Multiplicative Identity:** When any matrix is multiplied by the identity matrix, the result is the same matrix:

$$A \cdot I_n = I_n \cdot A = A$$

- **Commutativity in Multiplication:** Although matrix multiplication is generally not commutative, multiplying any square matrix by the identity matrix is commutative:

$$AI = IA = A$$

- **Non-Singular Matrix:** The determinant of the identity matrix is always 1, which means it is non-singular and invertible:

$$det(I_n) = 1$$

- **Diagonal Matrix:** The identity matrix is a special case of diagonal matrices, where all the diagonal elements are 1.

- **Symmetric Matrix:** The identity matrix is equal to its transpose:

$$I_n^T = I_n$$

- **Self-Inverse Property:** The inverse of the identity matrix is itself:

$$I_n^{-1} = I_n$$

- **Does Not Affect Eigenvalues:** When a matrix A is multiplied by I_n , its eigenvalues remain unchanged.

2.3.3 Triangular matrix

A triangular matrix is a square matrix (where the number of rows equals the number of columns) in which all elements either above or below the main diagonal are equal to zero. It is classified into two main types:

1. Upper Triangular Matrix

An upper triangular matrix is a square matrix in which all elements below the main diagonal are zero. That is, an element in row i and column j is zero if $i > j$.

A matrix $A = [a_{ij}]$ of order $n \times n$ is an upper triangular matrix if:

$$a_{ij} = 0 \quad \forall i > j$$

Example of an Upper Triangular Matrix:

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 7 \end{bmatrix}$$

2. Lower Triangular Matrix

A lower triangular matrix in which all elements above the main diagonal are zero. That is, an element in row i and column j is zero if $i < j$.

A matrix $B = [b_{ij}]$ of order $n \times n$ is a lower triangular matrix if:

$$b_{ij} = 0 \forall i < j$$

Example of a Lower Triangular Matrix:

$$B = \begin{bmatrix} 5 & 0 & 0 \\ 8 & 6 & 0 \\ 3 & 9 & 4 \end{bmatrix}$$

Properties of triangular matrices

1. The product of two triangular matrices of the same type results in another triangular matrix of the same type.

   - If A and B are both upper triangular matrices, then AB is also an upper triangular matrix.
   - If A and B are both lower triangular matrices, then AB is also a lower triangular matrix

2. The determinant of a triangular matrix is equal to the product of its diagonal elements. If A is a triangular matrix (upper or lower), then:

$$det(A) = a_{11}.a_{22}...a_{nn}$$

3. A triangular matrix is invertible if and only if all its diagonal elements are nonzero.

2.4 Operations on matrices

Matrices are a set of elements arranged in rows and columns, and several mathematical operations can be performed on them. The most important operations include:

2.4.1 Addition of matrices:

If $A[a_{ij}]$ and $B[b_{ij}]$ are two matrices of the same order, then their sum $A+B$ is a new matrix, where each element is the sum of the corresponding elements in A and B. That is:

$$A+B=[a_{ij}+b_{ij}]$$

Consider two matrices A and B of order 2×2. Their sum is given by:

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix}$$

Properties of matrix addition:

If A, B, and C are matrices of the same order, then:

1. **Commutative Property:**

   $A+B=B+A$

2. **Associative Property:**

   $(A+B)+C=A+(B+C)$

3. **Identity Matrix (Additive Identity):**

   $A + O = O + A = A$

   where$O$ is the zero matrix, which does not affect the addition.

4. **Additive Inverse:**

   $A + (-A) = 0 = (-A) + A$

where $(-A)$ is obtained by changing the sign of every element in A, making it the additive inverse of the matrix.

2.4.2 Subtraction of matrices

If A and B are two matrices of the same order, the subtraction is defined as:

$$A - B = A + (-B)$$

For two matrices and of order 2×2 , their difference is given by:

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} - \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_{1-}a_2 & b_1 - b_2 \\ c_1 - c_2 & d_1 - d_2 \end{bmatrix}$$

- Matrices are subtracted by subtracting each element in the first matrix from the corresponding element in the second matrix, $A - B = [a_{ij} - b_{ij}]_{n \times n}$.

2.4.3 Multiplication of matrices

A square matrix is a matrix where the number of rows is equal to the number of columns, meaning its dimensions are $n \times n$. When multiplying two square matrices of the same size, the result is also a square matrix of the same size.

Formula for Multiplying Square Matrices:

If A and B are two square matrices of order $n \times n$, then their product $C = A \times B$ is calculated as follows:

$$C_{i,j} = \sum_{k=1}^{n} A_{i,k} \times B_{k,j}$$

where:

- $C_{i,j}$ is the element at row i and column j in the resulting matrix C .
- $A_{i,k}$ is the element at row i and column k in matrix A .
- $B_{k,j}$ is the element at row k and column j in matrix B .
- n is the number of rows and columns in the matrices.

Example:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 5 & 3 \\ 11 & 5 \end{bmatrix}$$

Properties of square matrices multiplication

1. Non-Commutativity (Non-Swap Property)

In general, matrix multiplication is not commutative:

$$A \times B \neq B \times A$$

except in special cases, such as when both matrices are diagonal or share certain properties.

2. Associativity

Matrix multiplication is associative, meaning:

$$A \times (B \times C) = (A \times B) \times C$$

This means that the order of execution does not affect the result.

3. Distributive Property Over Addition

Matrix multiplication satisfies the distributive property over addition:

$$A \times (B + C) = A \times B + A \times C$$

$$(B + C) \times A = B \times A + C \times A$$

4. Zero Matrix (Multiplicative Zero Property)

If any matrix is multiplied by the zero matrix O, the result is also a zero matrix:

$$A \times O = O \times A = O$$

5. Transposition Property

When multiplying two square matrices and then transposing the result, we get the same result if we transpose each matrix first and reverse the order of multiplication:

$$(A \times B)^T = B^T \times A^T.$$

2.5 Description of the clique polynomial operational matrix method

Graph theory is one of the gifted subjects in applied mathematics. A graph G is contained with a nonempty finite set of n vertices called the vertex set $V(G)$, along with a prescribed set of m unordered pairs of members of $V(G)$ called edge set $E(G)$. These unordered pairs are joined by a line called an edge. Whenever two vertices share a common edge, then those two edges are coined to be adjacent. If all the vertices and edges present in a graph $G'$are from another graph G then G' is said to be a subgraph of G. A graph in which all pair of vertices are adjacent is called a complete graph and K_n is the notion for the complete graph on n vertices. A complete subgraph with k vertices of a graph G is called as $k-$clique of G. For graph-theoretic definitions, symbols, and related works we refer $[3, 8]$. Hoede et al.[5] defined clique polynomial of a graph G, denoted by $C(G; x)$, is defined by

$$C(G; x) = \sum_{k=0}^{n} a_k x^k$$

where a_k represent the total distinct $k-$cliques in graph of size k, with $a_0 = 1$. In general, the clique polynomial of a complete graph K_n with $n-$vertices is given by

$$C\left(K_n; x\right) = (1 + x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$$

where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$

In particular

$$\begin{aligned} C(K_0; x) &= 1 \\ C(K_1; x) &= 1 + x \\ C(K_2; x) &= 1 + 2x + x^2 \\ C(K_3; x) &= 1 + 3x + 3x^2 + x^3 \end{aligned}$$

Theorem 1. Let $f(y)$ be the bounded function in $L^2(R)$ defined on $[0, 1]$, then the clique polynomial expansion of $f(y)$ converges to it.

Proof. See Ref $[3, 8]$.

Let $B = \{C_n(x) = C(K_n, x), n \in N\}$. Clearly B is Banach space on closed subset

A of R with norm given by

$$\|C_n\| = \sup_{\forall x \in A} |C_n(x)| \qquad \forall C_n \in B(A)$$

We can approximate any function $f(x)$ in $L^2[0,1]$ in terms of the clique polynomial

$$f(x) \approx \widetilde{f}(x) = \sum_{i=0}^{n-1} a_i C(K_i; x) = \sum_{i=0}^{n-1} a_i \left(\sum_{k=0}^{i} \binom{i}{k} x^k\right) = A^T P X(x)$$

where $A^T = [a_0, a_1, \ldots, a_{n-1}]$, $X(x) = [1, x, \ldots, x^{n-1}]^T$ and P is the lower triangular $n \times n$ matrices defined by

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 2 & 1 & \ddots & \cdots & 0 \\ 1 & 3 & 3 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 & 0 \\ 1 & n-1 & \frac{(n-1)(n-2)}{2!} & \cdots & n-1 & 1 \end{bmatrix}$$

where

$$p_{ij} = \begin{cases} 0 \quad j > i, i, j = 1, 2, ..., n \\ \frac{(i-1)!}{(i-j)!(j-1)} \quad i \geq j, i, j = 1, 2, ..., n \end{cases}$$

2.6 Numerical solution of differential equations

2.6.1 Operational matrix method

We consider the clique polynomial operational matrix method along with collocation points to solve the following fourth order of differential equations

$$y^{(4)} = f(x, y, y', y'', y'''), 0 \leq x \leq 1 \tag{1}$$

with the initial conditions

$$y(0) = b_1, y'(0) = b_2, y''(0) = b_3, y'''(0) = b_4 \tag{2}$$

where b_1, b_2, b_3, b_4 are real constants and f is a given continuous on $[0, 1]$, nonlinear function. We assume that

$$y^{(4)}(x) = A^T P X(x) \tag{3}$$

Where A is an unknown vector to be determined $A^T = [a_0, a_1, \dots, a_{n-1}]$, $X(x)$ is the known vector defined above and

$$P = \begin{bmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 2 & 1 & \ddots & \cdots & 0 \\ 1 & 3 & 3 & \ddots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 & 0 \\ 1 & n-1 & \frac{(n-1)(n-2)}{2!} & \cdots & n-1 & 1 \end{bmatrix}$$

For solving the Equation (1), we calcul the derivatives $y^{(k)}(x)$ where $k = 0, 1, 2, 3, x \in$

$[0,1]$ and with the initial conditions (2). It is easy to prove that this identity

$$\underbrace{\int_0^x \int_0^x \dots \int_0^x}_{k \text{ times}} A^T P X(t)\, dt = A^T P M_k x^k X(x)$$

where M_k is the $n \times n$ matrices

$$M_k = \begin{bmatrix} \frac{1}{k!} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{2\times 3\times \dots (k+1)} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{3\times 4\times \dots (k+2)} & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \frac{1}{n(n+1)\dots(n+k-1)} \end{bmatrix}$$

Integrating Equation (3) fourth times on bothside with respect to x limit between 0 and x, we obtain

$$y(x) = b_1 + b_2 x + \frac{b_3}{2}x^2 + \frac{b_4}{6}x^3 + \int_0^x \int_0^x \int_0^x \int_0^x A^T P X(t) dt$$

After integration yields

$$y(x) = b_1 + b_2 x + \frac{b_3}{2}x^2 + \frac{b_4}{6}x^3 + A^T P M_4 x^4 X(x)$$

where

$$M_4 = \begin{bmatrix} \frac{1}{4!} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{5!} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{3\times 4\times 5\times 6} & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \frac{1}{n(n+1)(n+2)(n+3)} \end{bmatrix}$$

Now by substituting $y, y', y'', y''', y^{(4)}$ into Equation (1) and collocate this equation by the following collocation points $x_i = \frac{2i-1}{2n}, i = 1, ..., n$, we get a system of n non linear equations with n unknowns $(a_0, a_1, ..., a_{n-1})$. This system can be solved by using the Newton method.

Example 1 Consider the following Lane-Emden equation

$$y'' + \frac{2}{x}y' + 1 = 0 \tag{4}$$

with initial conditions

$$y\,(0) = 1, y'\,(0) = 0 \tag{5}$$

The exact solution of the above problem is

$$y = 1 - \frac{x^2}{6} \tag{6}$$

By solving the Equation (4) with conditions (5) we obtain the vector A for $n = 10$

$$A = \begin{bmatrix} -0.211563 \\ -0.755242 \\ 2.064859 \\ -3.265769 \\ 3.292303 \\ -2.193601 \\ 0.965798 \\ -0.270905 \\ 0.043921 \\ -0.003135 \end{bmatrix}$$

Table 1 shows that the numerical solutions and the errors obtained for Lane-

Emden equation of problem (4) (Example **1**) by using the present method and compared with the exact solution (6) for $n = 10$. Figure **2.1** shows the numerical results for Example **1.**

x	Exact solution	Numerical solution	Errors
0.1	0.998333333333333	0.998333333424079	$-9.07457442522741E-11$
0.2	0.993333333333333	0.993333333363829	$-3.04952729734964E-11$
0.3	0.985	0.985000000121763	$-1.21762711025042E-10$
0.4	0.973333333333333	0.973333333624439	$-2.91105806127234E-10$
0.5	0.958333333333333	0.958333333868015	$-5.34681854347241E-10$
0.6	0.94	0.940000000313705	$-3.13704728860387E-10$
0.7	0.918333333333333	0.918333332677134	$6.56199095061538E-10$
0.8	0.893333333333333	0.893333331842293	$1.49104029123492E-09$
0.9	0.865	0.864999998067961	$1.93203875120673E-09$
1	0.833333333333333	0.833333330673594	$2.65973987279011E-09$

Table 1 Numerical results for Example **1**

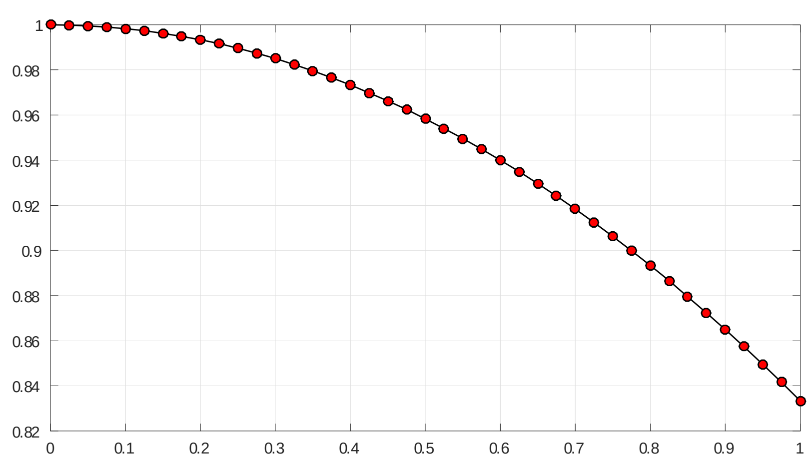

Figure 2.1: Comparison of the numerical solution (○) with the exact solution (—) for Example 1.

Example 2 Consider the linear third order initial value problem

$$y''' = sin(x), 0 \leq x \leq 1 \tag{7}$$

with initial conditions

$$y(0) = -1, y'(0) = 0, y''(0) = 1 \tag{8}$$

The analytic solution of the above problem is

$$y = cos(x) + x^2 - 2 \tag{9}$$

We have

$$y(x) = -1 + \frac{1}{2}x^2 + A^T P M_3 x^3 X(x) \tag{10}$$

Substituting equation (10) into (7) yields

$$A^T P X(x) = sin(x)$$

By using the conditions (8), the obtained system is solved. **Table 2** shows that the numerical solutions and the errors obtained for linear third order initial value problem (7) (Examples **2**) by using the present method and compared with the exact solution (9) for $n = 10$. In Figure. **2.2,** numerical results are shown for Example **2**.

x	Exact solution	Numerical solution for $n = 10$	Errors
0	-1	-1	0
0.1	-0.994995834721974	-0.994995834723177	$1.20281562487889E-12$
0.2	-0.979933422158758	-0.979933422162680	$3.92197385679083E-12$
0.3	-0.954663510874394	-0.954663510881710	$7.31581462076747E-12$
0.4	-0.918939005997115	-0.918939006007955	$1.08402176124400E-11$
0.5	-0.872417438109627	-0.872417438122700	$1.30726540703563E-11$
0.6	-0.814664385090322	-0.814664385113334	$2.30125918321278E-11$
0.7	-0.745157812715512	-0.745157812779116	$6.36040109469604E-11$
0.8	-0.663293290652835	-0.663293290813850	$1.61015867305991E-10$
0.9	-0.568390031729336	-0.568390032047044	$3.17708304109487E-10$
1	-0.459697694131860	-0.459697694888935	$7.57074403168190E-10$

Table 2 Numerical results for Example **2**

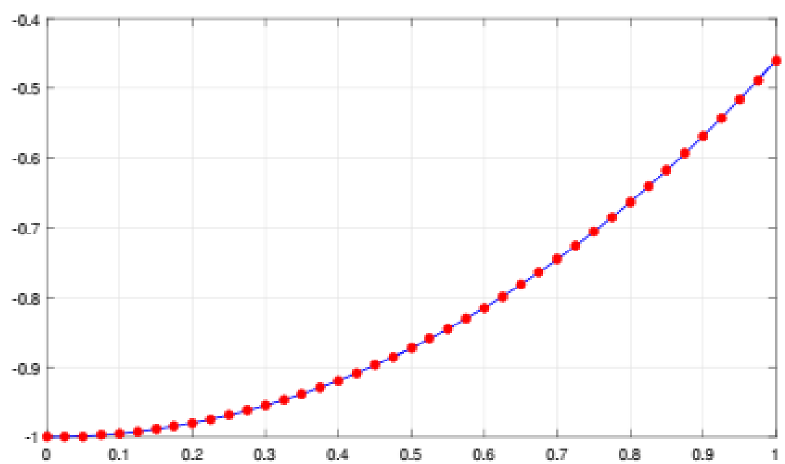

Figure 2.2: Comparison of the numerical solution (○) with the exact solution (—) for Example 2.

Example.3 Consider the non-linear fourth boundary value problem

$$y^{(4)} = \sin x + (\sin x)^2 - (y'')^2, 0 \leq x \leq 1 \tag{11}$$

with boundary conditions

$$y(0) = 0, y'(0) = 1, y(1) = \sin 1, y'(1) = \cos 1 \tag{12}$$

The exact solution of this problem is

$$y(x) = \sin x \tag{13}$$

Table 3 and 4 show that the numerical solutions and the errors obtained for the non-linear fourth boundary value problem (11) (Example **3**) by using the present method and compared with the exact solution (13) for $n = 10$. Figure **2.3**. shows the numerical results which compared with the exact solution (13) for Example **3**.

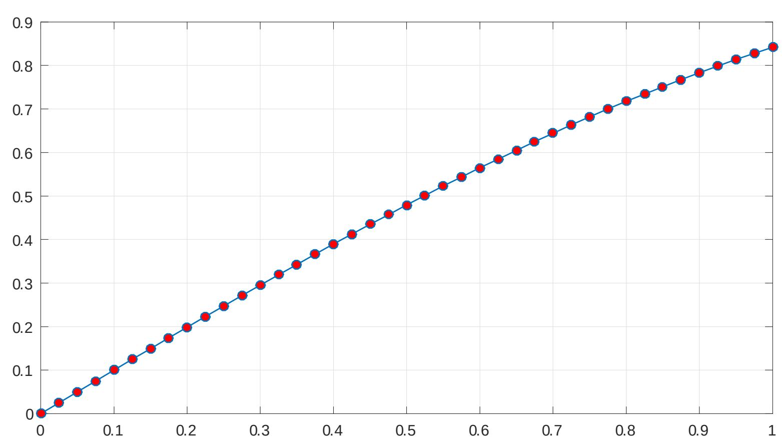

Figure 2.3: Comparison of the numerical solution (○) with the exact solution (—) for Example 3.

x	Exact solution	Numerical solution at $n = 10$	Method in [1, 2]
0	0	0	$9.5923E - 14$
0.1	0.0998334166	0.0998334165	0.0998334945
0.2	0.1986693307	0.1986693304	0.1986696031
0.3	0.2955202066	0.2955202061	0.2955207315
0.4	0.3894183423	0.3894183416	0.3894191196
0.5	0.4794255386	0.4794255378	0.4794265100
0.6	0.5646424733	0.5646424727	0.5646435236
0.7	0.6442176872	0.6442176867	0.6442186501
0.8	0.7173560908	0.7173560905	0.7173567749
0.9	0.7833269096	0.7833269095	0.7833271803
1	0.8414709848	0.8414709848	0.8414709848

Table 3 Numerical results for example 3

x	Errors ((CP) method)	Errors in [1, 2]
0	0	$9.5923E - 14$
0.1	$1.0752E - 10$	$7.7856E - 08$
0.2	$3.2231E - 10$	$2.7231E - 07$
0.3	$5.3800E - 10$	$5.2489E - 07$
0.4	$6.9509E - 10$	$7.7730E - 07$
0.5	$7.5288E - 10$	$9.7145E - 07$
0.6	$6.9431E - 10$	$1.0502E - 06$
0.7	$5.3058E - 10$	$9.6286E - 07$
0.8	$3.0506E - 10$	$6.8407E - 07$
0.9	$9.4464E - 11$	$2.7069E - 07$
1	$3.3306E - 16$	$1.5676E - 13$

Table 4 Absolute errors obtained for Example **3**.

Chapter 3

Numerical solution of Burgers' equation

3.1 Burgers' equation

Burgers' equation is a nonlinear parabolic partial differential equation arises in the theory of shock waves ,in turbulence problems and in continuous stochastic processes. It has a large variety of applications in modeling of water in unsaturated soil, gas dynamics, heat conduction, elasticity, statics of flow problems, mixing and turbulent diffusion, cosmology, seismology, are the popular ones (Burger,1948)

Burgers' equation is an important and simple model in understanding the physical flows. It describes various kinds of phenomena such as mathematical model of turbulence and the approximate theory of flow through a shock wave travelling in a viscous fluid. This equation provides the simplest nonlinear models of turbulence in the phenomena process.

The one -dimensional form of burgers equation is:

Where:

$$\frac{\partial y(x,t)}{\partial t} + y(x,t)\frac{\partial y(x,t)}{\partial x} = v\frac{\partial^2 y(x,t)}{\partial x^2}, \qquad 0 < x < 1, \quad 0 < t < T. \qquad \text{(I)}$$

- $y(x,t)$ represents the fluid velocity.
- v is the velocity coefficient.
- The term $\frac{\partial y}{\partial t}$ represesents temprral variation of velocity.
- The term $v\frac{\partial y}{\partial x}$ represesents nonlinear convection.
- The term $v\frac{\partial^2 y}{\partial x^2}$ accounts for diffusion due to viscosity.

Subject to initial condition

$$y(x,0) = g(x), 0 \leq x \leq 1$$

and boundary conditions

$$y(0,t) = h_1(t) \text{ and } y(1,t) = h_2(t), 0 \leq t \leq T.$$

Where $v = \frac{1}{\mathrm{Re}}$ (Re is Reynolds number) is the positive coefficient of kinematic viscosity and g, h_1 and h_2 are the sufficiently smooth given functions.

One of the methods for solving this equation is :

1. Simplification Using the Cole-Hopf Transformation

Due to the nonlinear nature of the equation, the Cole-Hopf transformation is introduced by defining a new variable ϕ such that:

$$y = -2v\frac{\partial}{\partial x}\ln\phi$$

Substituting this transformation into Burgers' equation converts it into the **linear heat equation**:

$$\frac{\partial \phi}{\partial t} = v\frac{\partial^2 \phi}{\partial x^2}$$

2. Applying the Separation of Variables Method

To solve the transformed equation, we assume a separable solution of the form:

$$\phi(x,t) = X(x)\,T(t)$$

Where X is a function of x only and T is a function of t only. It other words, the solution of the given PDE, $\phi(x,t)$, is the product of two functions that depend only on x and t .

Substituting this into the Burgers equation yields:

$$X(x)\frac{dT}{dt} = vT(t)\frac{d^2X}{dx^2}$$

Dividing both sides by $X(x)\,T(t)$, we obtain two independent equations:

$$\frac{1}{T}\frac{dT}{dt} = v\frac{1}{X}\frac{d^2X}{dx^2} = -\lambda$$

where λ is a separation constant.

It this way, two distinct ODE_S are derived from equation

$$\frac{dT}{dt} = -v\lambda T(t)\,,$$

$$\frac{d^2X}{dx^2} = -\lambda X(x)\,.$$

As a result, two ordinary differential equations that are simples to solve are generated by the method of separation of variable.

3.2 History of Burgers' equation

The origins of **Burgers' equation** date back to 1915 when Bateman first derived it in a physical context. In 1923, Fay rederived it within the framework of acoustics. Later, in 1940, Burgers highlighted the significance of this equation, emphasizing its role in describing turbulence phenomena in fluid mechanics.

It was discovered that Burgers' equation could be transformed into the linear heat equation, a transformation published by Cole and known as **the Cole-Hopf** transformation. Independently, Blackstock and Hopf rediscovered this transformation in 1950. In the field of aerodynamics, the Fay series was introduced as an approximate solution to Burgers' equation for a sinusoidal initial condition.

Burgers' equation has been employed in studying the propagation of one-dimensional finite-amplitude acoustic signals, with Blackstock and Lighthill utilizing it for this purpose. In 1969, Lagerstrom applied it to analyze shock structures in the Navier-Stokes equations. Due to its characteristics, Burgers' equation is considered a mathematical approximation of the Navier-Stokes equations, making it a simplified model for them.

Burgers' equation consists of two primary terms: **the diffusion term**, representing viscosity effects, and **the convection term**, which accounts for the nonlinear transport of momentum.

3.3 Numerical results of Burgers' equation

3.3.1 Description of the operational matrix method

In this section, clique polynomials of complete graphs with the collocation method are used to solve the nonlinear Burgers equation defined in Eq. (I) with different initial-boundary conditions.

Assume that

$$\frac{\delta^3 y\left(x,t\right)}{\delta x^2 \delta t} = X^T\left(t\right) A^T P X\left(x\right) \tag{1}$$

Where A is an unknown vector to be determined $A^T = [a_0, a_1, \ldots, a_{n-1}]$ and $X\left(x\right)$ is the known vector.

Now, integrate Eq. (1) concerning t from 0 to t, we get

$$\frac{\delta^2 y(x,t)}{\delta x^2} = \frac{\delta^2 y(x,0)}{\delta x^2} + \int_0^t X^T(t)\, A^T P X(x)\, dt \tag{2}$$

After integration yields

$$\frac{\delta^2 y(x,t)}{\delta x^2} = \frac{\delta^2 y(x,0)}{\delta x^2} + M_1 t X(t)\, A^T P X(x) \tag{3}$$

Integrating Eq. (3) concerning x from 0 to x

$$\begin{aligned}\frac{\delta y(x,t)}{\delta x} &= \frac{\delta y(0,t)}{\delta x} + \frac{\delta y(x,0)}{\delta x} - \frac{\delta y(0,0)}{\delta x} + \int_0^x M_1 t X(t)\, A^T P X(x)\, dx \\ &= \frac{\delta y(0,t)}{\delta x} + \frac{\delta y(x,0)}{\delta x} - \frac{\delta y(0,0)}{\delta x} + M_1 t X(t)\, A^T P M_1 x X(x)\end{aligned} \tag{4}$$

Integrating Eq. (4) concerning x from 0 to x

$$\begin{aligned}y(x,t) &= y(0,x) + y(t,0) - y(0,0) + x\left[\frac{\delta y(0,t)}{\delta x} - \frac{\delta y(0,0)}{\delta x}\right] + \int_0^x M_1 t X(t)\, A^T P M_1 x X(x)\, dx \\ &= y(0,x) + y(t,0) - y(0,0) + x\left[\frac{\delta y(0,t)}{\delta x} - \frac{\delta y(0,0)}{\delta x}\right] + M_1 t X(t)\, A^T P M_2 x^2 X(x)\end{aligned} \tag{5}$$

Put $x = 1$ in the equation (5)

$$y(1,t) = y(0,1) + y(t,0) - y(0,0) + \left[\frac{\delta y(0,t)}{\delta x} - \frac{\delta y(0,0)}{\delta x}\right] + M_1 t X(t)\, A^T P M_2 X(1)$$

Hence

$$\left[\frac{\delta y(0,t)}{\delta x} - \frac{\delta y(0,0)}{\delta x}\right] = y(0,1) + y(t,0) - y(0,0) - y(1,t) + M_1 t X(t)\, A^T P M_2 X(1) \tag{6}$$

Substitute Eq. (6) in (5) we get

$$\begin{aligned} y(x,t) = y(0,x) + y(t,0) - y(0,0) + +M_1 t X(t) A^T P M_2 x^2 X(x) + \\ x\left[y(0,1) + y(t,0) - y(0,0) - y(1,t) + M_1 t X(t) A^T P M_2 X(1)\right] \end{aligned} \tag{7}$$

where

$$M_1 = \begin{bmatrix} \frac{1}{1!} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{2} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{3} & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \frac{1}{n} \end{bmatrix}$$

and

$$M_2 = \begin{bmatrix} \frac{1}{2!} & 0 & 0 & \cdots & 0 \\ 0 & \frac{1}{2\times 3} & 0 & \cdots & 0 \\ 0 & 0 & \frac{1}{3\times 4} & \ddots & 0 \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & 0 & \frac{1}{n(n+1)} \end{bmatrix}$$

Differentiating Eq. (7) concerning t , we get

$$\begin{aligned} \frac{\delta y(x,t)}{\delta t} = \frac{\delta y(t,0)}{\delta t} + \frac{\delta\left(M_1 t X(t) A^T P M_2 x^2 X(x)\right)}{\delta t} + \\ x\frac{d}{dt}\left[y(0,1) + y(t,0) - y(0,0) - y(1,t) + M_1 t X(t) A^T P M_2 X(1)\right] \end{aligned}$$

Example: On consider the Burgers equation

$$\frac{\delta y}{\delta t}+y\frac{\delta y}{\delta x}=v\frac{\delta^2 y}{\delta^2 x},(x,t)\in[0,1]\times[0,T]$$

with intial condition

$$y\left(x,0\right)=\sin\left(\pi x\right),t\in[0,T]$$

and boundary conditions

$$y\left(0,t\right)=0,y\left(1,t\right)=0,x\in[0,1]$$

The exact solution of this problem by using the Cole-Hopf transformation is

$$y\left(x,t\right)=2\pi v\frac{\sum_{n=1}^{\infty}c_n\exp\left(-n^2\pi^2vt\right)n\sin\left(n\pi x\right)}{c_0+\sum_{n=1}^{\infty}c_n\exp\left(-n^2\pi^2vt\right)n\cos\left(n\pi x\right)}$$

where

$$\begin{aligned}c_0&=\int_0^1\exp\left(-\frac{1}{2\pi x}\left(1-\cos\left(\pi x\right)\right)\right)dx\\c_n&=2\int_0^1\exp\left(-\frac{1}{2\pi x}\left(1-\cos\left(\pi x\right)\right)\right)\cos\left(n\pi x\right)dx\end{aligned}$$

The numerical computations were done by using the uniform mesh. For the comparison we compute the analytical and numerical solution at some mesh points for the given time step, $Dt=0.01$. Tables **1** and **2** give the numerical and exact values of the solution y for $v=1$ and 0.1. The results by the proposed method are in good agreement with exact solution. In Figure. 3.1 numerical results with uniform mesh are shown for Example of Burgers equation at different times for $t=0.01,0.1,0.2$ and $v=1$. Figure **3.3**. shows the numerical results for Example at different times for $t = 0.01, 0.05, 0.1$ and $v = 0.1$. Figures **3.2** and **3.4** show the numerical solution of the Burgers equation for $v = 1$ and $v = 0.1$ respectively. These numerical predictions exhibit good physical behaviour.

x	Exact solution	Present method	Method [13]
0	0.000000	0.000000	0.000000
0.125	0.135829	0.131176	0.128578
0.250	0.253638	0.251968	0.239809
0.375	0.336742	0.337973	0.317851
0.5	0.371577	0.372804	0.350090
0.625	0.350123	0.348647	0.329320
0.75	0.272582	0.266194	0.256060
0.875	0.149239	0.137828	0.140092
1	0.000000	0.000000	0.000000

Table 1. Comparison numerical solution with exact solution for Example of Burgers equation with $v = 1$, $t = 0.1$.

x	Errors	Errors in [13]
0	0	0
0.125	0.004653	0.007251
0.250	0.00167	0.013829
0.375	0.001231	0.01889
0.5	0.001227	0.021487
0.625	0.001476	0.020803
0.75	0.006388	0.016522
0.875	0.011411	0.009147
1	0	0

Table 2. The errors for the solution of Burgers equation for $v = 1$ and $t = 0.1$.

x	Exact solution	present method	method [13]
0	0	0	0
0.125	0.278023	0.278617	0.271187
0.25	0.534143	0.531141	0.519585
0.375	0.743852	0.729710	0.721350
0.5	0.877280	0.843188	0.848611
0.625	0.897099	0.838978	0.867467
0.75	0.761797	0.691667	0.739558
0.875	0.447836	0.398221	0.438783
1	0	0	0

Table 3. Comparison numerical solution with exact solution for Example of Burgers equation with $v = 0.1$, $t = 0.1$.

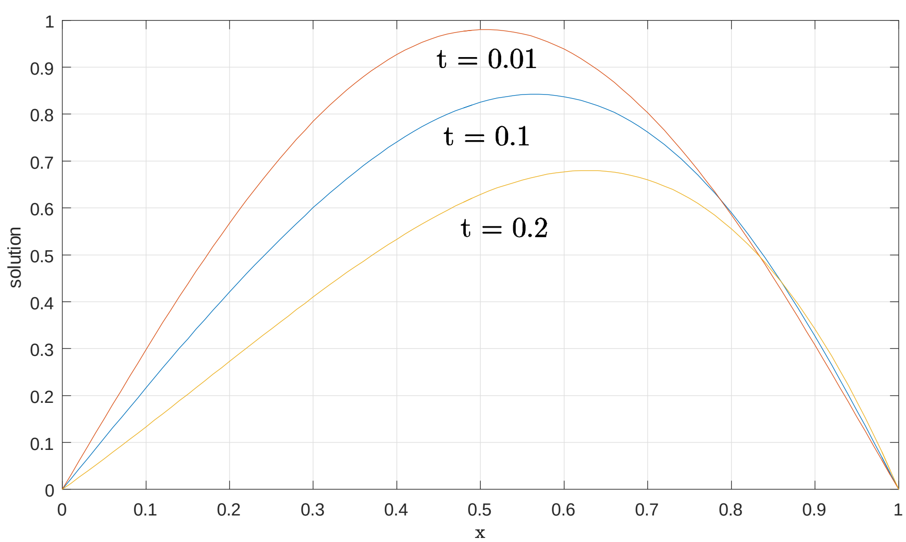

Figure 3.1: Numerical results at different times for Example of Burgers equation for $v = 1.0$ and $t = 0.1$.

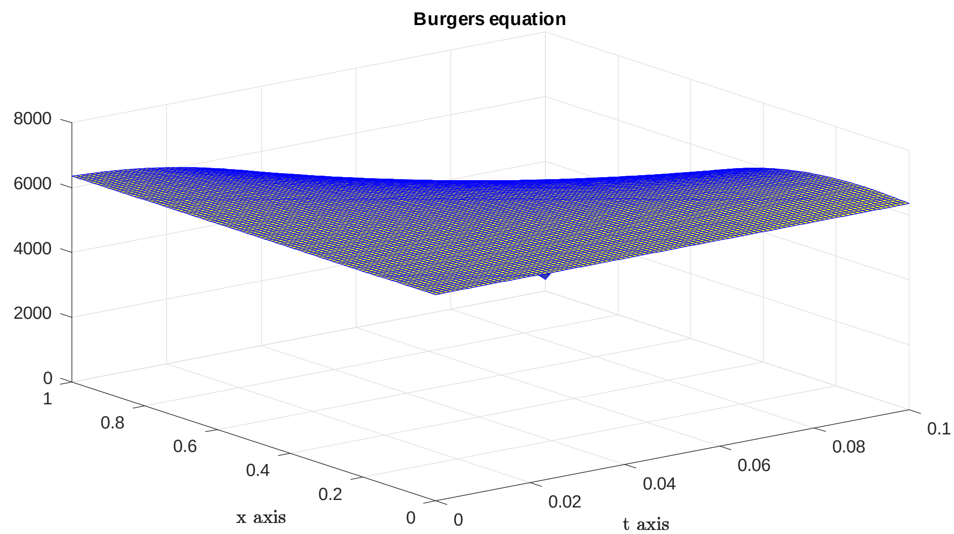

Figure 3.2: Numerical solution of the problem produced for the parameter $v = 1.0$.

The figure illustrates the numerical solution of the Burgers' equation for the viscosity coefficient $v = 1.0$. The resulting surface shows the behavior of the function $y(x,t)$ with respect to both spatial and temporal variables. The solution is characterized by smoothness and regularity, indicating the efficiency and accuracy of the adopted numerical method based on Clique functions and the operational matrix approach. It is also observed that the values decrease in the middle and increase toward the boundaries, reflecting the expected physical behavior of the Burgers' equation under diffusion effects.

It is observed that the solution rapidly drops to large negative values over time, indicating a potential numerical instability in the applied method, especially for lower viscosity values. In contrast, the solution appears more stable in the higher viscosity case.

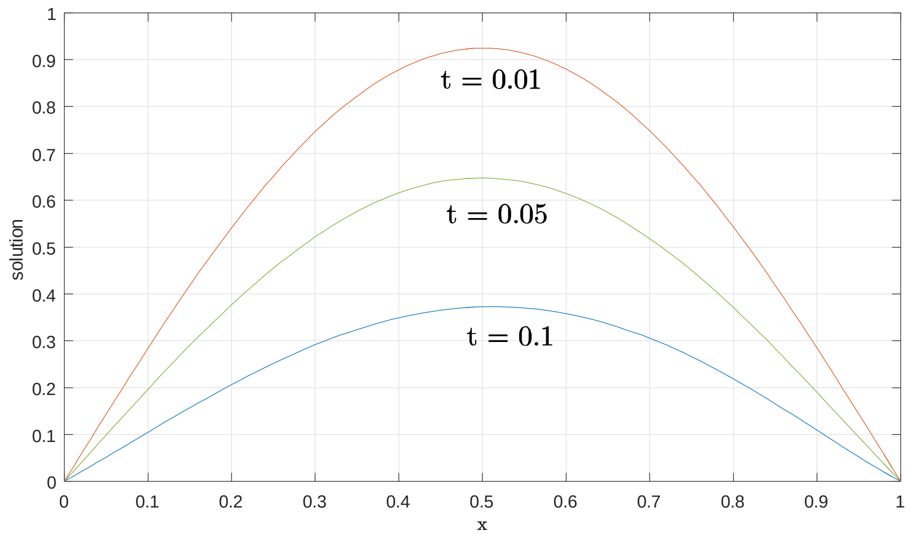

Figure 3.3: Numerical results at different times for Example of Burgers equation for $v = 0.1$ and $t = 0.1$.

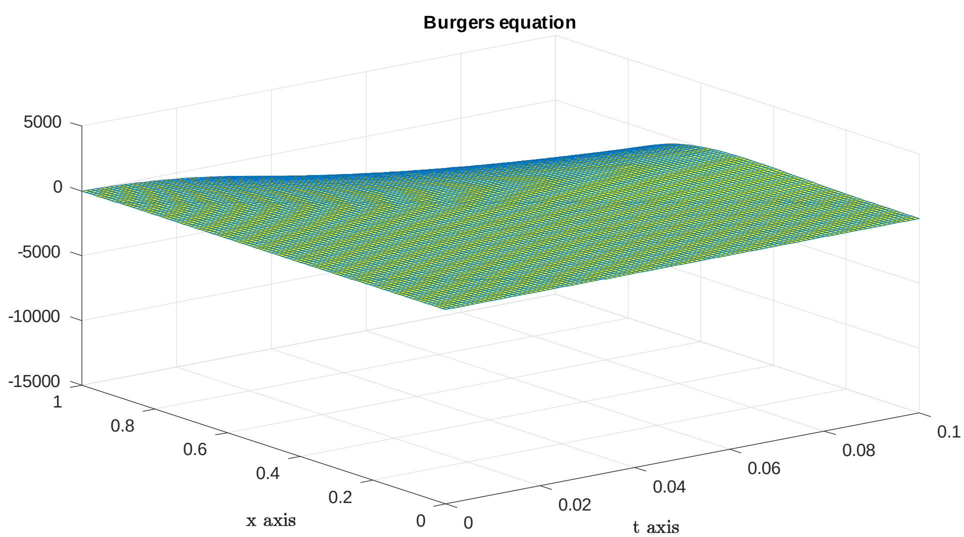

Figure 3.4: Numerical solution of the problem produced for the parameter $v = 0.1$.

Conclusion

In this work, Burgers' equation, a mathematical model that combines diffusion and nonlinearity and is used to represent a variety of physical events, was the subject of our numerical investigation. We used a numerical approach based on operational matrices and Clique functions to solve this equation because of their precision and simplicity of use. The chosen approach demonstrated efficacy in approximating answers through theoretical analysis and numerical experimentation, producing precise findings with a significant decrease in numerical error. The creation of appropriate matrices to manage the equation in an orderly fashion was also made easier by the application of Clique functions.

In conclusion, we believe that this work represents a foundational step that can be further developed to address more complex differential equations or to integrate this methodology with other numerical tools to enhance efficiency and accuracy.

Bibliography

[1] Agarwal. R. P, Boundary value problems for higher order differential equations, World Scientific, Singapore, 1986.

[2] Asadpour Sasan , Hosseinzadeh Hassan and Yazdani AllahBakhsh , Numerical solution of the Lane-Emden equations with moving least squares method, Applications and Applied Mathematics: An International Journal (AAM), Vol. 14 (2), (2019) 762 – 776.

[3] Butcher J.C., Numerical methods for ordinary differential equations, John Wiley Sons, Chichester, 2008.

[4] Gregory Hartman, Fundamentals matrix algebra, Third Edition, APEX Calculus, 2011.

[5] Ganji R. M., Jafari H., Kgarose M., Mohammadi A., Numerical solutions of time-fractional Klein-Gordon equations by clique polynomials, Alexandria Engineering Journal, Vol. 60, (2021) 4563-4571.

[6] Hoede C., Li X., Clique polynomials and independent set polynomials of graphs, Discr. Math, Vol. 125, (1994) 219-228.

[7] Kumbinarasaiah S., A new approach for the numerical solution for nonlinear Klein-Gordon equation, SeMA Journal, Vol. 77, (2020) 435-456.

[8] Kumbinarasaiah S., Ramane H. S., Pise K. S. and Hariharan G., Numerical solution for non-linear Klein Gordon equation via operational matrix by clique polynomial of complete graphs, Int. J. Appl. Comput. Math, Vol. 7(1), (2021) 1-19.

[9] Khataybeh S. N., Hashim I., Alshbool M., Solving directly third-order ODEs using operational matrices of Bernstein polynomials method with applications to fluid flow equations, Journal of King Saud University Science, Vol. 31, (2019) 822–826.

[10] Martin Hermann and Saravi Masoud , Nonlinear ordinary differential equations (Analytical Approximation and Numerical Methods), Springer India, Garmany, 2016.

[11] Mehrkanoon S., A direct variable step block multistep method for solving general third-order ODEs, Numer. Algorithms, Vol. 57, (2011) 53–66.

[12] Mohan K. Kadalbajoo, Sharma K.K. , Awasthi A., A parameter-uniform implicit difference scheme for solving time-dependent Burgers' equations, Applied Mathematics and Computation, Vol. 170, (2005) 1365–1393.

[13] Momani Shaher and Moadi Khaled , A reliable algorithm for solving fourth-order boundary value problems, J. Appl. Math. Computing, Vol. 22(3), (2006) 185-197.

[14] Noor Muhammad Aslam , Syed Tauseef Mohyud-Din, An efficient method for fourth-order boundary value problems, Computers and Mathematics with Applications, Vol. 54, (2007) 1101-1111.

[15] Wazwaz Abdul-Majid , Partial Differential Equations and Solitary Waves Theory, Spingerlink, Garmany, 2009.

Abstract:

This study focuses on the Burgers' equation, which plays an important role in mathematical modeling and describes physical phenomena such as fluid flow and heat transfer. Given the difficulty of solving this equation accurately due to its nonlinear nature, we used a new method based on Clique functions to construct operational matrices that help approximate the solution.

We applied this method to transform the equation into an algebraic form that can be solved numerically, and then compared the results with those of other existing methods. The results showed that the developed method provides accurate solutions and requires less computation time.

In conclusion, this method proved to be efficient and can be applied to similar equations in the future.

Keywords: Burgers' equation, operational matrices, Clique functions, numerical methods, approximate solutions.

Résumé :

Cette étude porte sur l'équation de Burgers, qui joue un rôle important dans la modélisation mathématique et décrit des phénomènes physiques tels que l'écoulement des fluides et le transfert de chaleur. En raison de la difficulté à résoudre cette équation avec précision en raison de sa nature non linéaire, nous avons utilisé une nouvelle méthode basée sur les fonctions de clique pour construire des matrices opérationnelles permettant d'approximer la solution.

Nous avons appliqué cette méthode pour transformer l'équation en une forme algébrique pouvant être résolue numériquement, puis nous avons comparé les résultats à ceux obtenus par d'autres méthodes connues. Les résultats ont montré que la méthode développée fournit des solutions précises et nécessite moins de temps de calcul.

En conclusion, cette méthode s'est révélée efficace et peut être appliquée à des équations similaires à l'avenir.

Mots-clés : Équation de Burgers, matrices opérationnelles, fonctions de clique, méthodes numériques, solutions approchées.

الملخص:

تركز هذه الدراسة على معادلة برغر، التي تلعب دورًا مهمًا في النمذجة الرياضية وتصف ظواهر فيزيائية مثل جريان الموائع وانتقال الحرارة. ونظرًا لصعوبة حل هذه المعادلة بدقة بسبب طبيعتها غير الخطية، فقد استخدمنا طريقة جديدة تعتمد على الدوال المضغوطة لبناء مصفوفات تشغيلية تساعد في تقريب الحل.

قمنا بتطبيق هذه الطريقة لتحويل المعادلة إلى صيغة جبرية يمكن حلّها عددياً، ثم قارنا النتائج مع تلك الناتجة عن طرق أخرى معروفة. وقد أظهرت النتائج أن الطريقة المطورة توفّر حلولاً دقيقة وتتطلب وقتًا حسابيًا أقل.

وفي الختام، أثبتت هذه الطريقة كفاءتها، ويمكن تطبيقها على معادلات مماثلة في المستقبل.

الكلمات المفتاحية: معادلة برغر، المصفوفات التشغيلية، الدوال المضغوطة ، الطرق العددية، الحلول التقريبية.