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MATTER SCIENCES DEPARTMENT

Course and Solved Exercises

Physics I - Mechanics of a Material Point

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This course is written for the students of the 1st year of the LMD Bachelor's program in the fields of ST and SM.

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Chapter 1 : Dimensional analysis.

I.Introduction

The observation of physical phenomena is incomplete if it does not lead to quantitative information, which is the measurement of physical quantities. To study a physical phenomenon, one must examine the important variables; the mathematical relationship between these variables constitutes a physical law.

This is possible in certain cases, but for other cases, it is necessary to use a modeling method such as dimensional analysis.(التحليل البعدي).

II. Definition of Dimensional Analysis تعريف البعذي التحليل

It is a theoretical tool for interpreting problems based on the dimensions of the involved physical quantities: length, time, mass, and so on.

Dimensional analysis allows for:

- Verifying the validity of dimensioned equations.
- Investigating the nature of physical quantities.
- Exploring the homogeneity of physical laws.
- Determining the unit of a physical quantity based on fundamental units (meter, second, kilogram, etc.).

III.Physical Quantity مقدار فيزيائي

A physical quantity is an observable and measurable property through a specifically designed instrument. Mechanics acknowledges seven fundamental physical quantities: length, time, mass, electric current, temperature, quantity of material, and luminous intensity. Other physical quantities, known as derived quantities, are expressed in terms of these three fundamental quantities, such as velocity, acceleration, force, and more…..

Note :

In general, for first-year students in Mathematics and Computer Science (MI), Mathematics (M), and Computer Science (I), the focus is primarily on the first three fundamental quantities: length, time, and mass.

The value of a physical quantity is given in relation to a standard known as a "unit." The first four fundamental units constitute the MKSA International System (Meter, Kilogram, Second, Ampere). Using these fundamental units, derived units can be constructed: area (m^2), velocity (m/s), force (kg m/s^2)…

IV. International System of Units الوحدة في النظام العالمي

Fundamental quantities	Units(in the international system MSKA)	Symbols
Length	Metre	(m)
Mass	Kilogramme	kg
Time	Seconde	(s)
Temperature	Kelvin	(K°)
Current intensity	Ampere	(A)
Light intensity	Candela	(Cd)
Quantity of material	Mole	(mol)

There are specific units such as N (Newton) for force, Hz (Hertz) for frequency, Watt for power, Pascal (Pa) for pressure...

Note: There are two systems of units:

- The International System (SI) known as MKSA (Meter, Kilogram, Second, Ampere), which is the most widely used system.
- The CGS system (Centimeter, Gram, Second), which is less commonly used.

V. Dimensional Equations معادلة الأبعاد

Dimension represents the nature of a physical quantity. A physical quantity has only one possible dimension.

The dimension of a quantity G is denoted by: $[G]=L$**.**

By denoting M, L, and T as the dimensions of the fundamental quantities mass, length, and time, we can express the dimensions of other derived quantities in terms of these three. The resulting equations are the dimension equations for these physical quantities.

Fundamental quantities المقادير الاساسية	Symbols الرمز	Dimensions الأبعاد	Units (الوحدة) in the international system MSKA)
Length	L	[l]=L	Metre
Mass	M	[m]=M	Kilogramme
Time	T	[t]=T	Seconde
Temperature	T	[T]=Ɵ	Kelvin
Current intensity	I	[i]=I	Ampere
Light intensity	J	[j]=J	Candela
Quantity of material	N	[n]=N	Mole

L'équation aux dimensions de toute grandeur G peut se mettre sous la forme :

$$[G] = L^a M^b T^c I^d J^e \theta^f N^g$$

To determine the dimension of a quantity, we need to use known formulas.

Example :

$[\boldsymbol{velocity}] = [\boldsymbol{v}] = \frac{[\boldsymbol{length}]}{[\boldsymbol{time}]} = \frac{\boldsymbol{L}}{\boldsymbol{T}} = \boldsymbol{L}.\boldsymbol{T}^{-1}$ and the unit of speed is (m/s).

$[\boldsymbol{acceleration}] = [\boldsymbol{a}] = \frac{[\boldsymbol{velocity}]}{[\boldsymbol{time}]} = \frac{[\boldsymbol{v}]}{[\boldsymbol{t}]} = \frac{\boldsymbol{L}.\boldsymbol{T}^{-1}}{\boldsymbol{T}} = \boldsymbol{L}.\boldsymbol{T}^{-2}$ and the unit of acceleration (m/s²).

$[\boldsymbol{Force}] = [\boldsymbol{F}] = [\boldsymbol{mass}][\boldsymbol{acceleration}] = [\boldsymbol{m}][\boldsymbol{a}] = \boldsymbol{M}.\boldsymbol{L}.\boldsymbol{T}^{-2}$ and the unit of force is Newton or (kg.m/s²).

Notes :

- The dimension of constants is always equal to 1; we say they are dimensionless.
- Angles and functions like sin, cos, tan, exp, ln, and log are dimensionless functions.

[Numeric value] = 1, [angle] = 1, [cos α] = [sin α] = [tan α] = [cot α] = [ln x] = [e^x] = 1.

VI.Homogeneity of Dimensional Equations معادلة تجانس الأبعاد

The two sides of a dimension equation must have the same dimensions since they represent quantities of the same nature.

G is a physical quantity:

$$G = A \pm B \rightarrow [G] = [G] = [A] = [B]$$

$$G = A * B \rightarrow [G] = [A] * [B]$$

$$G = \frac{A}{B} \rightarrow [G] = \frac{[A]}{[B]}$$

$$G = A^n \rightarrow [G] = [A]^n$$

- A heterogeneous (non-homogeneousغير متجانسة) equation is necessarily False.
- A homogeneous equation is not necessarily true.
- Dimensions cannot be added (or subtracted).

Example 1:

$y = \frac{1}{2}at^2 + v_0 t + y_0$ is the equation of a physical law.

Check that this equation is homogeneous?

This equation is homogeneous if:

$$[y] = \left[\frac{1}{2}at^2\right] = [v_0 t] = [y_0]$$

We have :

$$[y] = [y_0] = L;\ \left[\frac{1}{2}at^2\right] = \left[\frac{1}{2}\right][a][t^2] = 1.L.T^{-2}.T^2 = L;\ [v_0 t] = [v_0][t] = L.T^{-1}.T = L$$

So :

$[y] = \left[\frac{1}{2}at^2\right] = [v_0 t] = [y_0]$ is checked

Hence the equation $y = \frac{1}{2}at^2 + v_0 t + y_0$ is homogeneous.

Notes :

We can use this property of dimension equations to discover physical laws by knowing the variables involved in the given physical phenomenon and the relationship among them.

Example 2:

The period is given in terms of length and severity by the following relationship:

$$T = kl^x g^y$$

Give the physical law of period T ?

For this it is necessary to determine the exponents x and y.

It is assumed that the equation is homogeneous so :

$$[T] = [k][l]^x[g]^y$$

The dimensions of all physical quantities in the study relationship are written.

$[l] = L;\ [k] = 1;\ [g] = L.T^{-2};\ [T] = T$

So :

$[T] = L^{x}(LT^{-2})^{y} = L^{x+y}.T^{-2y} \ldots\ldots\ldots.(1)$

On other hand we have :

$[T] = T = TL^{0} \ldots\ldots\ldots\ldots\ldots..(2)$

By comparing (1) and (2) find :

$$\begin{cases} x + y = 1 \\ -2y = 1 \end{cases} \rightarrow \begin{cases} x = -y = \dfrac{1}{2} \\ y = \dfrac{-1}{2} \end{cases} So = T = kl^{1/2}g^{-1/2}; T = k\sqrt{\frac{l}{g}}$$

it's the law of the period.

Conclusions :

Dimensional analysis serves the following purposes:

- Verification of the homogeneity of physical formulas.
- Determination of the nature of a physical quantity.
- Exploration of the general form of physical laws.

Vector calculation :الحساب الشعاعي

I. Introduction :

Physical quantities are classified into two categories: scalar and vector quantities.

A **scalar quantity** is a real numerical value used to represent certain quantities such as: mass, time, temperature...

A **vector quantity** is a quantity that has a real numerical value and a direction, such as speed, acceleration, force, ...

A vector quantity is represented by what we call a vector. Figure 1

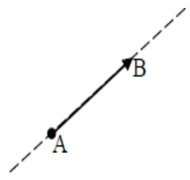


Figure (II.1): Vector $\overrightarrow{AB}$

The vector $\overrightarrow{AB}$ is characterized by :

-The origin (A)

- Support (straight line (AB))

- Vector direction (from A to B)

- The modulus or norm of the vector: real numerical value representing the length of the vector (the distance between A and B).

The modulus of a vector $\overrightarrow{AB}$ is written as follows : $\|\overrightarrow{AB}\| = |\overrightarrow{AB}| = AB$

II. Unit vector :(شعاع الوحدة)

The unit vector is a vector whose modulus is equal to 1. We express a vector $\overrightarrow{AB}$ parallel to the unit vector $\vec{U}$ such that :

$$\overrightarrow{AB} = AB.\vec{U}$$

With $\|\overrightarrow{AB}\| = AB$; $\|\vec{U}\| = 1$

III.Vector components (مركبات الشعاع)

A vector is described by its components, which are determined from a reference frame. This reference system can be linear (a single x component), planar (two components) or spatial (three components).

-Vector coordinates in Cartesian reference system :

The Cartesian reference frame is orthonormal: unit vectors must be orthogonal to each other and normalized to unity.

<u>In the plan $(\boldsymbol{O};\ \vec{\imath};\vec{\jmath})$</u>

The vector $\vec{V}$ is written as :

$\vec{V} = \vec{V_x} + \vec{V_y}$ with $\vec{V_x} = V_x\vec{\imath}$ and $\vec{V_y} = V_y\vec{\jmath}$

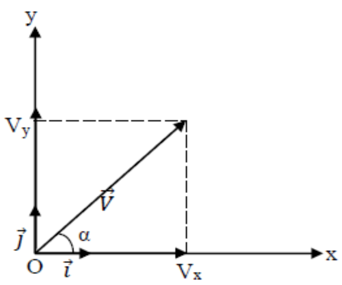


Figure (II .2) : Projection of a vector in the plane $(\boldsymbol{O};\ \vec{\imath};\vec{\jmath})$

$$V_x = V\cos\alpha \;\; ; \;\; V_y = V\sin\alpha$$

The components of vector $\vec{V}$ in the orthonormal plane $O;\ \vec{\imath};\vec{\jmath})$are : V_x and V_y and we write :

$$\vec{V} = \begin{pmatrix} V_x \\ V_y \end{pmatrix} = \begin{pmatrix} V\cos\alpha \\ V\sin\alpha \end{pmatrix}$$

The modulus of vector $\vec{V}$is calculated from its coordinates as follows:

$$V = \sqrt{{V_x}^2 + {V_y}^2}$$

<u>In the plan $(\boldsymbol{O};\ \vec{\boldsymbol{\imath}};\vec{\boldsymbol{\jmath}},\vec{\boldsymbol{k}})$</u>

The vector $\vec{V}$ is written :

$\vec{V} = \vec{V_x} + \vec{V_y} + \vec{V_z}$ with $\vec{V_x} = V_x\vec{\imath}$; $\vec{V_y} = V_y\vec{\jmath}$; $\vec{V_z} = V_z\vec{k}$

$V_x = V\cos\alpha$;

$V_y = V\cos\beta$

$V_y = V\cos\theta$

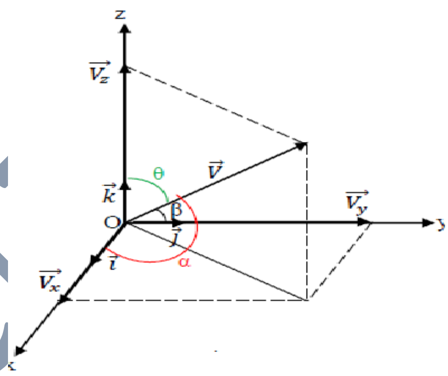


Figure (II. 3) : Projection of a vector in the plane<u>$(\boldsymbol{O};\ \vec{\boldsymbol{\imath}};\vec{\boldsymbol{\jmath}};\vec{\boldsymbol{k}})$</u>

The components of vector $\vec{V}$in space $(O;\vec{\imath};\vec{\jmath};\vec{k})$ are : V_x , V_y and V_z.

The modulus of vector $\vec{V}$ is given by :

$$V = \sqrt{{V_x}^2 + {V_y}^2 + {V_z}^2}$$

IV.Vector operations :

<u>IV.1.The sum of the vectors :</u>

Let two vectors $\vec{V_1}$and $\vec{V_2}$be such that :

$$\vec{V_1} = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix};\ \vec{V_2} = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$$

The sum of two vectors is another vector $\vec{S} = \vec{V_1} + \vec{V_2}$

$\vec{S} = (x_1 + x_2)\vec{\imath} + (y_1 + y_2)\vec{\jmath}$

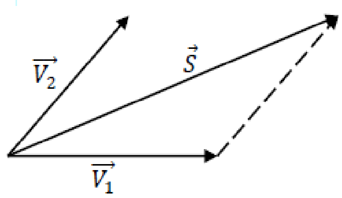


Figure (II .4) : Sum of two vectors

The modulus of $\vec{S}$ is calculated as follows:

$$S = \sqrt{(x_1 + x_2)^2 + (y_1 + y_2)^2}$$

Or by the laws of cosines : $S = \sqrt{{V_1}^2 + {V_2}^2 + 2V_1V_2\cos(\overrightarrow{V_1}, \overrightarrow{V_2})}$

IV.2.__Subtracting vectors__

The subtraction of two vectors is a vector : $\vec{D} = \overrightarrow{V_1} - \overrightarrow{V_2}$ which can be defined as the sum of the vector $\overrightarrow{V_1}$ with the inverse of the vector $\overrightarrow{V_2}$:

$$\vec{D} = \overrightarrow{V_1} + (-\overrightarrow{V_2})$$

$$\vec{D} = ((x_1 - x_2)\vec{\imath} + (y_1 - y_2)\vec{\jmath}$$

We obtain the modulus of $\vec{D}$ as follows:

$$D = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Or : D=$\sqrt{{V_1}^2 + {V_2}^2 - 2V_1V_2\cos(\overrightarrow{V_1}, \overrightarrow{V_2})}$

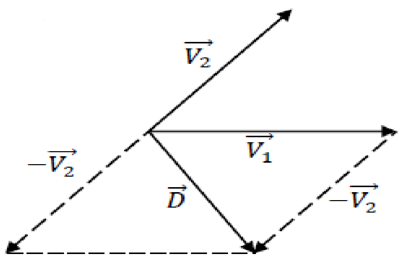


Figure (II. 5) : Subtracting two vectors

IV.3.__The scalar product (or dot product) between two vectors الجداء السلمي لشعاعين__

Scalar product between two vectors $\overrightarrow{V_1} = \binom{x_1}{y_1}$; $\overrightarrow{V_2} = \binom{x_2}{y_2}$ gives a scalar value

$$\overrightarrow{V_1}\overrightarrow{V_2} = V_1V_2\cos(\overrightarrow{V_1}\overrightarrow{V_2})$$

If $\overrightarrow{V_1}$ is parallel to $\overrightarrow{V_2}$, then : $\cos(\overrightarrow{V_1}\overrightarrow{V_2}) = 1$ and the scalar product is $\overrightarrow{V_1}\overrightarrow{V_2} = V_1V_2$

If $\overrightarrow{V_1}$ is perpendicular to $\overrightarrow{V_2}$, then $\cos(\overrightarrow{V_1}\overrightarrow{V_2}) = 0$ and the scalar product is zero $\overrightarrow{V_1}\overrightarrow{V_2} = 0$

On the other:

$$\vec{V}_1.\vec{V_2} = (x_1\vec{\imath} + y_1\vec{\jmath}) * (x_2\vec{\imath} + y_2\vec{\jmath})$$

$$\vec{V}_1.\vec{V_2} = x_1x_1\vec{\imath}\vec{\imath} + x_1y_2\vec{\imath}\vec{\jmath} + y_1x_2\vec{\imath}\vec{\jmath} + y_1y_2\vec{\jmath}\vec{\jmath}$$

We've got : $\vec{\imath}\vec{\jmath} = \vec{\jmath}\vec{\imath} = 0$ *and* $\vec{\imath}\vec{\imath} = \vec{\jmath}\vec{\jmath} = 1$

Hence : $\vec{V}_1.\vec{V_2} = x_1x_2 + y_1y_2$.

Angle Between Two Vectors

The angle θ, ($0 \le \theta \le \pi$), between two vectors can be found using the definition of the dot product:

$$\overrightarrow{V_1}\overrightarrow{V_2} = V_1V_2\cos(\overrightarrow{V_1}\overrightarrow{V_2}) = V_1V_2\cos\theta$$

So : $\cos\theta = \frac{\overrightarrow{V_1}\overrightarrow{V_2}}{V_1V_2} \rightarrow \theta = \cos^{-1}\frac{\overrightarrow{V_1}\overrightarrow{V_2}}{V_1V_2}$

Example1 :

1-If $\overrightarrow{V_1}(2;3;1)$ *and* $\overrightarrow{V_2}(5;-2;2)$ *finf the angle* θ *between* $\overrightarrow{V_1};\overrightarrow{V_2}$

$$\theta = \cos^{-1}\frac{\overrightarrow{V_1}\overrightarrow{V_2}}{V_1V_2}$$

$$\overrightarrow{V_1}\overrightarrow{V_2} = (2,3,1).(5,-2,2) = 6$$

$$V_1 = \sqrt{2^2 + 3^2 + 1^2} = \sqrt{14};\ V_2 = \sqrt{25+4+4} = \sqrt{33}$$

$$\theta = \cos^{-1}\left(\frac{6}{\sqrt{33}\sqrt{14}}\right) = \cos^{-1}(0.2791)$$

$$\theta = 73.8°$$

The angle between$\overrightarrow{V_1}$ and $\overrightarrow{V_2}$ is 73.8◦.

- **Scalar product properties :**

Scalar product is commutative : $\vec{V}_1.\vec{V_2} = \vec{V}_2.\vec{V}_1$

Distributor $\vec{V}_1(\vec{V}_2 + \vec{V}_3) = (\vec{V}_1 + \vec{V}_2) + (\vec{V}_1 + \vec{V}_3)$

$(\alpha\vec{A})\vec{B} = \vec{A}(\alpha\vec{B})$

$$\vec{\imath}\vec{\jmath} = \vec{\jmath}\vec{\imath} = \vec{\imath}\vec{k} = 0 \ and\ \vec{\imath}\vec{\imath} = \vec{\jmath}\vec{\jmath} = \vec{k}\vec{k} = 1$$

Example2 :

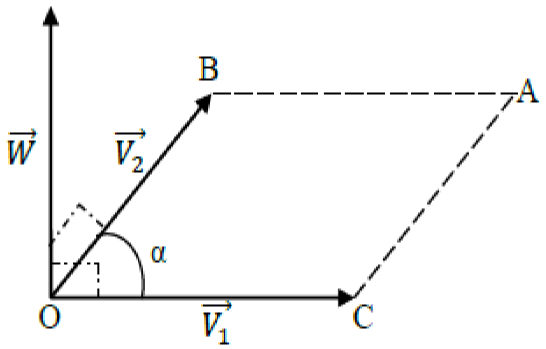
Let two vectors $\overrightarrow{V_1} = 2\vec{i} + 3\vec{j} - \vec{k}$; $\overrightarrow{V_2} = 2\vec{i} + 2\vec{j} - \vec{k}$

The scalar product : $\overrightarrow{V_1}.\overrightarrow{V_2} = 2.2 + 3.2 + (-1).(-1) = 11$

IV.4.<u>The cross product (vector product) between two vectors</u>الجداء الشعاعي لشعاعين

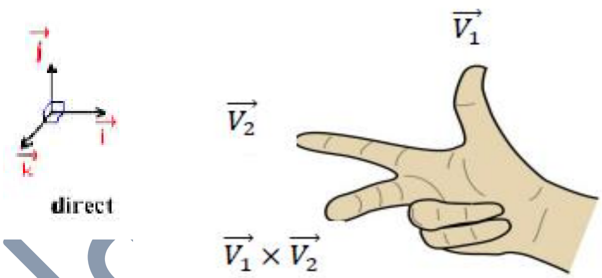
The vector product between two vectors $\vec{V}_1$ and $\vec{V}_2$ is a vector perpendicular to the plane formed by these two vectors.

$\vec{W} = \vec{V}_1 \Delta \vec{V}_2 = \vec{V}_1 \times \vec{V}_2$



Figure(II .6) : vector product $\vec{V}_1 \times \vec{V}_2$

The direction of vector $\vec{W}$ is found using the three-finger rule of the right hand.



$$W = |\vec{V}_1 \times \vec{V}_2| = V_1 * V_2 \sin(\overrightarrow{V_1}, \overrightarrow{V_2})$$

The modulus of the vector W represents the area of the parallelogram (OABC) formed by the two vectors $\vec{V}_1 et\ \vec{V}_2$ (figure 6).

And $W = \frac{|\vec{V}_1 \times \vec{V}_2|}{2}$ represents the area of a triangle.

The vector product can be calculated using the determinant method:

Let two vectors $\vec{V}_1$, $\vec{V}_2$ be such that :

$$\overrightarrow{V_1} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix};\ \overrightarrow{V_2} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}$$

$$\overrightarrow{W} = \vec{V}_1 \Delta \overrightarrow{V_2} = \vec{V}_1 \times \overrightarrow{V_2} = \begin{vmatrix} \vec{\imath} & -\vec{\jmath} & \vec{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} = \begin{vmatrix} y_1 & z_1 \\ y_2 & z_2 \end{vmatrix} \vec{\imath} - \begin{vmatrix} x_1 & z_1 \\ x_2 & z_2 \end{vmatrix} \vec{\jmath} + \begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} \vec{k}$$

$\overrightarrow{W} = (y_1z_2 - z_1y_2)\vec{\imath} - (x_1z_2 - z_1x_2)\vec{\jmath} + (x_1y_2 - y_1x_2)\vec{k}$^ :

From this relationship, we can calculate the modulus of $\overrightarrow{W}$ by :

$W = \sqrt{(y_1z_2 - z_1y_2)^2 + (x_1z_2 - z_1x_2)^2 + (x_1y_2 - y_1x_2)^2}$.

Note:

$$\vec{\imath} \times \vec{\imath} = \vec{\jmath} \times \vec{\jmath} = \vec{k} \times \vec{k} = \vec{o}$$

$$|\vec{\imath} \times \vec{\jmath}| = |\vec{\jmath} \times \vec{k}| = |\vec{\imath} \times \vec{k}| = 1$$

$$\vec{\imath} \times \vec{\jmath} = \vec{k}\ ;\ \vec{\jmath} \times \vec{k} = \vec{\imath}\ ;\ \vec{k} \times \vec{\imath} = \vec{\jmath}$$

- ***Properties of the cross product :***

1-Non commutative $\vec{V}_1 \times \overrightarrow{V_2} = -\overrightarrow{V_2} \times \overrightarrow{V_1}$

2-Non-associative : $\vec{V}_1 \times (\overrightarrow{V_2} \times \overrightarrow{V_3}) \neq (\vec{V}_1 \times \overrightarrow{V_2}) \times \overrightarrow{V_3}$

3-Distributive :$\vec{A} \times (\vec{B} + \vec{C}) = (\vec{A} \times \vec{B}) + (\vec{A} \times \vec{C})$

4-$(\alpha\overrightarrow{A}) \times \vec{B} = \vec{A} \times (\alpha\vec{B})$

5-$\vec{A} \times \vec{B} = 0 \ \rightarrow \vec{A} = 0\ or \vec{B} = 0\ or\ \sin\theta = 0\ \rightarrow \theta = 0\ or\ \theta = k\pi \rightarrow \vec{A} \mathbin{/\!/} \vec{B}$

6-$\vec{\imath} \times \vec{\jmath} = \vec{k}\ and\ \vec{\jmath} \times \vec{k} = \vec{\imath}\ and\ \vec{k} \times \vec{\imath} = \vec{\jmath}\ ;\ \vec{\imath} \times \vec{k} = -\vec{\jmath}$

Example3 : Let be two vectors :

$$\overrightarrow{V_1} = 2\vec{\imath} + 3\vec{\jmath} - 5\overrightarrow{k};\quad \overrightarrow{V_2} = 2\vec{\imath} + 2\vec{\jmath}$$

The vector product(The cross product) :

$$\overrightarrow{W} = \overrightarrow{V_1} \times;\quad \overrightarrow{V_2} = \begin{vmatrix} \vec{\imath} & -\vec{\jmath} & \vec{k} \\ 2 & 3 & -5 \\ 2 & 2 & 0 \end{vmatrix}$$

$$\overrightarrow{W} = (0 + 10)\vec{\imath}$$

V. The mixed product of three vectors:

We define the mixed product between three vectors : $\overrightarrow{V_1} = \begin{pmatrix} x_1 \\ y_1 \\ z_1 \end{pmatrix};\ \overrightarrow{V_2} = \begin{pmatrix} x_2 \\ y_2 \\ z_2 \end{pmatrix}\ and\ \overrightarrow{V_3} = \begin{pmatrix} x_3 \\ y_3 \\ z_3 \end{pmatrix}$

by the scalar $\overrightarrow{V_1}.(\overrightarrow{V_2}\times\overrightarrow{V_3})$ which is calculated by the determinant method such that :

$$\overrightarrow{V_1}.\left(\overrightarrow{V_2}\times\overrightarrow{V_3}\right)=\overrightarrow{V_1}.\begin{vmatrix}\vec{\imath} & \vec{\jmath} & \vec{k}\\ x_1 & y_1 & z_1\\ x_2 & y_2 & z_2\end{vmatrix}=\overrightarrow{V_1}\left[\begin{vmatrix}y_1 & z_1\\ y_2 & z_2\end{vmatrix}\vec{\imath}+\begin{vmatrix}x_1 & z_1\\ x_2 & z_2\end{vmatrix}\vec{\jmath}+\begin{vmatrix}x_1 & y_1\\ x_2 & y_2\end{vmatrix}\vec{k}\right]$$

$$\overrightarrow{V_1}.\left(\overrightarrow{V_2}\times\overrightarrow{V_3}\right)=(y_1z_2-z_1y_2)x_1+(x_1z_2-z_1x_2)y_1+(x_1y_2-y_1x_2)z_1$$

Mixed productrepresent the volume of a parallelogram

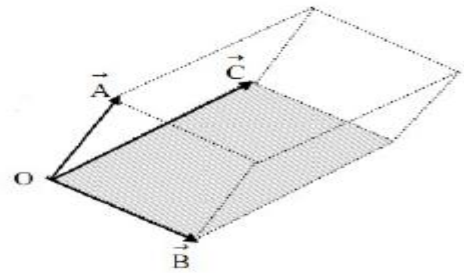


Figure (II .7) : Mixed product

Exercice1 :

Let be the vectos : $\vec{A}=\vec{\imath}+\vec{\jmath}-\vec{k}\,;\ \vec{B}=4\vec{\imath}+3\vec{\jmath}-\vec{k}\ and\ \vec{C}=-\vec{\imath}-2\vec{\jmath}+\vec{k}$

Calculate $\vec{A}.\left(\vec{B}\times\vec{C}\right) and\ \vec{A}\times\left(\vec{B}\times\vec{C}\right)$

Solution :

$$\left(\vec{B}\times\vec{C}\right)=\begin{vmatrix}\vec{\imath} & -\vec{\jmath} & \vec{k}\\ 4 & 3 & -1\\ -1 & -2 & 1\end{vmatrix}=(3.1-2.1)\vec{\imath}-(4.1-1.1)\vec{\jmath}+(4.-2+3.1)\vec{k}$$

$$\left(\vec{B}\times\vec{C}\right)=\vec{\imath}-3\vec{\jmath}-5\vec{k}$$

$$\vec{A}.\left(\vec{B}\times\vec{C}\right)=1.1-3+5=3$$

$$\vec{A}\times\left(\vec{B}\times\vec{C}\right)=\begin{vmatrix}\vec{\imath} & -\vec{\jmath} & \vec{k}\\ 1 & 1 & -1\\ 1 & -3 & -5\end{vmatrix}=(-5-3)\vec{\imath}-(-5+1)\vec{\jmath}+(-3-1)\vec{k}$$

$$\vec{A}\times\left(\vec{B}\times\vec{C}\right)=-7\vec{\imath}+4\vec{\jmath}-4\vec{k}$$

Exercise 02:

Let there be three vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$, such that:

$$\vec{A}=-2\hat{\imath}+\hat{\jmath}+3\hat{k},\quad \vec{B}=2\hat{\imath}-\hat{\jmath}+\hat{k},\quad \vec{C}=x\hat{\imath}+y\hat{\jmath}+z\hat{k}.$$

1. Calculate x and z so that the vector $C^{\rightarrow}$ is:

- (a) Parallel to $A^{\rightarrow}$
.
- (b) Parallel to $B^{\rightarrow}$
.

2. If $\vec{C} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$, calculate x, y, and z so that the vector $\vec{C}$ is **perpendicular** to both $\vec{A}$ and $\vec{B}$ at the same time.

Solution :

VI-Differential operators:

VI.1. Definitions :

A single-variable function is a function that depends on a single variable x : F=f(x). If the function f is derivable at any point x, we define F' the derivative of the function f such that $F' = \frac{df}{dx}$

On the other hand, if the function depends on several variables x, y, z,..., we define what we call a differential.

A dual-variable function is a function that depends on two variables: F=f(x,y).

A three-variable function is a function that depends on three variables x, y and z: F=f(x, y, z).

The total differential of an algebraic function F with three variables x, y, z is written :

$$dF = df(x,y,z) = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy + \frac{\partial f}{\partial z}dz$$

With $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ and $\frac{\partial f}{\partial z}$ are partial differentials.

Example 4 :

$$f(x,y,z) = x^2 - 2y + 4z$$

The total differential is : $df = 2xdx - 2dy + 4dz$

There are multi-variable algebraic functions and multi-variable vector functions.

$$\vec{V} = f(x,y,z)\vec{\imath} + g(x,y,z)\vec{\jmath} + h(x,y,z)\vec{k}$$

$\vec{V}$ is a multivariable vector function.

VI.2. The operators :المؤثرات

VI.2.1.Opérateur nabla :المؤثر نابلا

The nabla $\vec{\nabla}$ operator is a vector that acts on functions as follows:

$$\vec{\nabla} = \frac{\partial}{\partial x}\vec{\imath} + \frac{\partial}{\partial y}\vec{\jmath} + \frac{\partial}{\partial z}\vec{k}$$

VI.2.2.The gradient:التدرج :

The gradient is an operator that acts on algebraic functions, transforming them into vector functions by means of the nabla operator. We define the gradient vector of the algebraic function f as follows :

$$\overrightarrow{grad}f = \vec{\nabla}.f = \frac{\partial f}{\partial x}\vec{\imath} + \frac{\partial f}{\partial y}\vec{\jmath} + \frac{\partial f}{\partial z}\vec{k}$$

Example 5:

Either $f(x,y,z) = xyz^2$

So : $\overrightarrow{grad}f = yz^2\vec{\imath} + xz^2\vec{\jmath} + 2xyz\vec{k}$

VI.2.3.Divergent التباعد:

The divergent operator acts on vector functions, transforming them into algebraic functions using the nabla operator. It is defined as follows:

$$div\vec{V} = \vec{\nabla}.\vec{V} = \frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial Y} + \frac{\partial V_z}{\partial Z}$$

Example 6:

$\vec{V} = 2xy\vec{\imath} + xyz\vec{\jmath} - xyz^2\vec{k}$

$div\vec{V} = 2y + xz - 2xyz$

VI.2.4.The rotational(Curl) : الدوران

Consider a vector .$\vec{V} = V_x\vec{\imath} + V_y\vec{\jmath} + V_z\vec{k}$

The rotational of .$\vec{V}$ is defined as follows:

$$\overrightarrow{rot\vec{V}} = \vec{\nabla} \times \vec{V} = \begin{vmatrix} \vec{\imath} & -\vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

$$\overrightarrow{rot} = \left(\frac{\partial V_z}{\partial y} - \frac{\partial V_y}{\partial z}\right)\vec{\imath} - \left(\frac{\partial V_z}{\partial x} - \frac{\partial V_x}{\partial z}\right)\vec{\jmath} + \left(\frac{\partial V_y}{\partial x} - \frac{\partial V_x}{\partial y}\right)\vec{k}$$

Example7 :

$\vec{\boldsymbol{V}}(x,y,z) = 2xy\vec{\imath} + xyz\vec{\jmath} - xyz^2\vec{k}$

$$\overrightarrow{rot\vec{V}} = \vec{\nabla} \times \vec{V} = \begin{vmatrix} \vec{\imath} & -\vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ V_x & V_y & V_z \end{vmatrix}$$

$$\overrightarrow{rot\vec{V}} = (-xz^2 - xy)\vec{\imath} - (-yz^2 - 0)\vec{\jmath} + (yz - 2x)\vec{k}$$

$$\overrightarrow{rot\vec{V}} = ((-xz^2 - xy)\vec{\imath} + \ yz^2\vec{\jmath} + (yz - 2x)\vec{k}$$

VI.2.5.The laplacian

The laplacian is defined as the divergent of the gradient or the gradient of the divergent.

The Laplacian of an algebraic function is given by the following relation :

$$\vec{\nabla}.\vec{\nabla}(f) = \overrightarrow{\nabla^2}(f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}.$$

Vectorial function Laplacian is given by the following relation

$$\vec{\nabla}.\vec{\nabla}(\vec{V}) = \overrightarrow{\nabla^2}(\vec{V}) = \frac{\partial^2\vec{V}}{\partial x^2} + \frac{\partial^2\vec{V}}{\partial y^2} + \frac{\partial^2\vec{V}}{\partial z^2}$$

Kinematics of material point :حركيةنقطة مادية

I.Introduction :

Material point kinematics is the study of the motion of material bodies as a function of time (position, distance traveled, velocity, acceleration, etc.), without taking into account the causes that cause or modify motion (forces, energy, etc.).

The body under study is assumed to be a material point. The dimensions of the body are assumed to be very small compared with the distance covered.

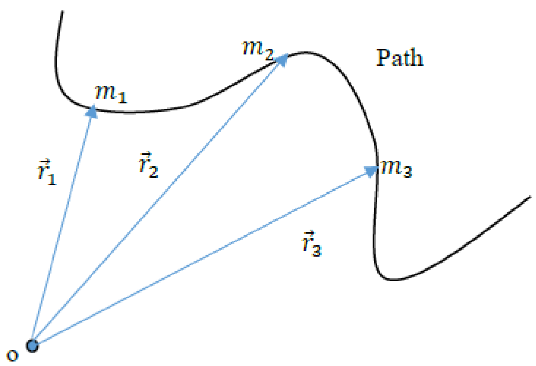
The notion of motion is relative. A body can be, at the same time, in motion relative to one body and at rest relative to another. Consequently, it is necessary to define(a reference frame جملة مرجعية)to determine the position, velocity or acceleration of a moving body at an instant corresponding to the position of the moving body relative to this reference frame.

Several coordinate systems are defined according to the nature of the material point's motion. Cartesian, polar, cylindrical and spherical

II. Descriptive study of the motion of a material point:(دراسة وصفية لحركة نقطة مادية)

II.1.Vector position :شعاع الموضع

Vector position is a vector, which connect the origin of reference to the position of the material point in a given time.



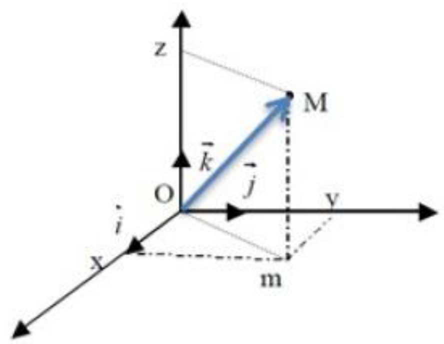


Figure (III.1) : Position vector $\overrightarrow{OM}$ in Cartesian coordinates

And it is written by the following equation:

$$\overrightarrow{OM} = \vec{r} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$$

The x, y and z components of the position vector in the Cartesian coordinate system are the Cartesian coordinates of mobile M. These coordinates change with time as mobile M is in motion: x(t), y(t), z(t).

We will have :

$$\overrightarrow{OM} = \vec{r} = x(t)\vec{\imath} + y(t)\vec{\jmath} + z(t)\vec{k}$$

We call x(t), y(t), and z(t) the time equations of motion.

Note

We have chosen Cartesian coordinates, but we could have chosen other coordinates (polar, cylindrical or spherical coordinates).

II.2. The trajectory :المسار

The trajectory is the collection of successive positions occupied by the moving body over time, relative to a reference frame. It is a continuous line drawn from the point of origin to the point of destination (figure 2). The trajectory defines the nature of the movement. If the trajectory is rectilinear, the motion is rectilinear(فالحركة مستقيمة); if it is curvilinear, the motion is curvilinear(فالحركة منحنية).

Figure(1) shows M's position vector in the Cartesian coordinate system and its trajectory.

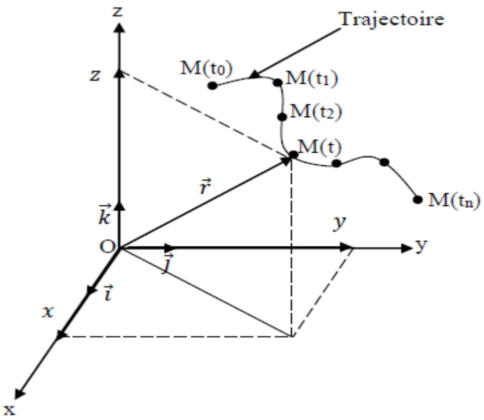


Figure (III.2) : The trajectory of material point.

- ***Equation of the trajectory :***

This is the relationship that links the mobile's coordinates x, y, z to each other, independently of time. To find the equation of the trajectory, we need to eliminate the time between the time equations.

Example1 :

Get the time equations of a moving point in the plane (O, x, y) are :

$$\begin{cases} x = 2t \ldots\ldots\ldots (1) \\ y = 2t + 1 \ldots\ldots\ldots (2) \end{cases}$$

1-Find the equation of the trajectory

2- Find the position vector at time t=2s

From equation (1), we have $t = \frac{x}{2}$. If we replace t in equation (2), we**:**

$$y = 2\left(\frac{x}{2}\right) + 1 \rightarrow y = x + 1$$

This equation of the trajectory is the equation of a straight line of the form y=ax+b, which means that the motion is rectilinear.

$$2\text{-}\overrightarrow{OM} = x\vec{\imath} + y\vec{\jmath}$$

$$\overrightarrow{OM} = 2t\vec{\imath} + (2t+1)\vec{\jmath}$$

$$\overrightarrow{OM}(t2s) = 4\vec{\imath} + 5\vec{\jmath}$$

Example2 :

The motion of a point M is defined in a Cartesian reference frame by :

$$\begin{cases} x(t) = a\sin(\omega t + \varphi) \\ y(t) = a\cos(\omega t + \varphi) \end{cases}$$

What is the trajectory of M.

Solution :

$$x^2 + y^2 = \mathrm{a}^2[\sin^2(\omega t + \varphi) + \cos^2(\omega t + \varphi)]$$

$$x^2 + y^2 = a^2$$

is the equation of a circle with radius a and center O

II.3.The displacement vector :

During motion, the moving body occupies different positions. At time t, it is at point M and at time t'=(t+Δt) , it is at point M' figure(3). The $\overrightarrow{MM'}$ vector is defined as the displacement vector.

$$\overrightarrow{\Delta OM} = \overrightarrow{OM_2}(t') - \overrightarrow{OM_1}(t) = \overrightarrow{M_1M_2}$$

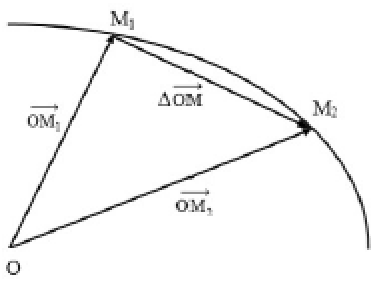


Figure (III.3) : Displacement vector $\overrightarrow{M_1M_2}$ in Cartesian coordinates

II.4.The velocity (speed) vector :

II.4.1.Average velocity :

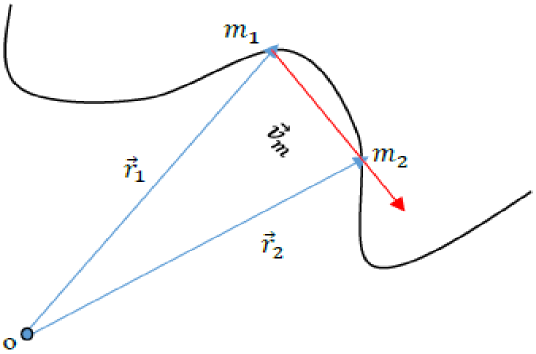
Average velocity is the variation in distance between two positions M_1, M_2 occupied by the moving body, in relation to the time between these two positions.

It is defined as follows :

$$\overrightarrow{v_m} = \frac{\overrightarrow{OM_2}(t') - \overrightarrow{OM_1}(t)}{t' - t} = \frac{\overrightarrow{M_1M_2}}{t' - t} = \frac{\Delta\overrightarrow{OM}}{\Delta t} = \frac{\overrightarrow{\Delta r}}{\Delta t}$$

Note that:

- The direction of average velocity is same as the direction vector displacement.
- The magnitude of the average velocity is called (velocity) **speed**.



II.4.2.Instant velocity :

This is the limit of the average velocity when the time difference is very small, meaning that it tends towards zero.

$$\vec{v} = \lim_{\Delta t \to 0} \overrightarrow{v_m} = \lim_{\Delta t \to 0} \frac{\overrightarrow{M_1M_2}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\overrightarrow{OM_2} - \overrightarrow{OM_1}}{\Delta t} = \frac{d\overrightarrow{OM}}{dt}$$

Let a function y=f(x), its derivative is equal to :

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

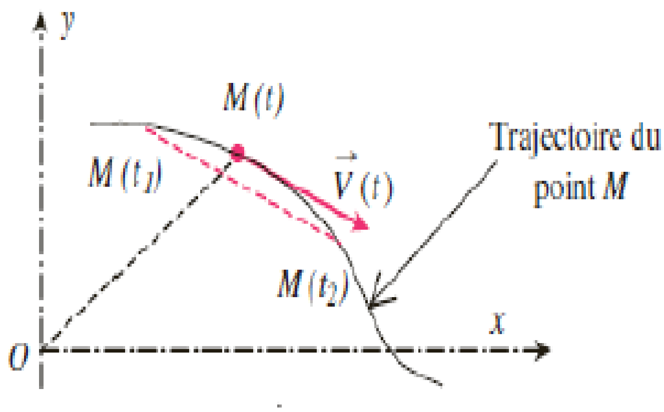
This is simply the definition of the derivative of a function.

"So, instantaneous velocity can be defined as the derivative of the position vector with respect to time."

السرعة اللحظية تعرف على أنها مشتق شعاع الموضع بالنسبة للزمن.

"The instantaneous velocity vector is tangent to the trajectory, and its direction follows the direction of the motion."

شعاع السرعة اللحظية يكون مماسيا للمسار ويكون اتجاهه في نفس جهة الحركة.



The coordinates of the position vector in Cartesian coordinates are :

$$\text{Let } \overrightarrow{OM} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$$

"The velocity vector of point M is obtained by deriving its position vector with respect to time :

$$\vec{v} = \frac{d\overrightarrow{OM}}{dt} = \frac{dx}{dt}\vec{\imath} + \frac{dy}{dt}\vec{\jmath} + \frac{dz}{dt}\vec{k} = v_x\vec{\imath} + v_y\vec{\jmath} + v_z\vec{k}$$

Its modulus is :

$$v=\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2} = \sqrt{(v_x)^2 + \left(v_y\right)^2 + (v_z)^2}$$

II.5.Acceleration vector :

II.5.1.Average Acceleration

We define the average acceleration of material point during the time interval Δt as the difference between the velocity vectors of the particle divided by that time interval,

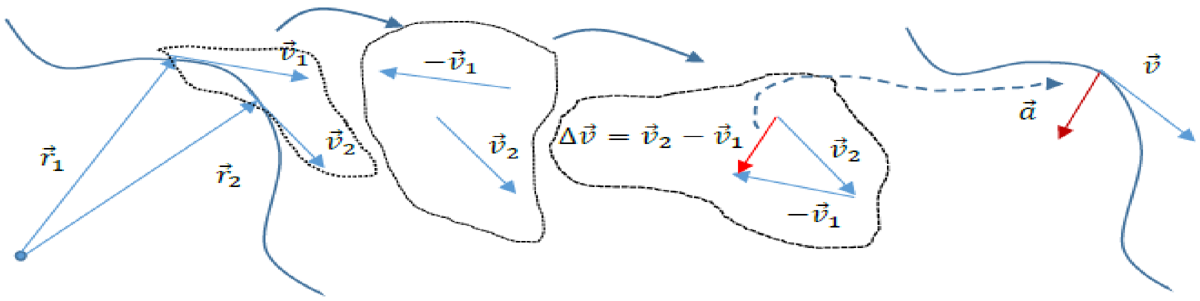
$$\vec{a}_{moy} = \frac{\vec{v}(t+\Delta t) - \vec{v}(t)}{(t+\Delta t) - t} = \frac{\Delta\vec{v}(t)}{\Delta t}$$

Average acceleration has the same direction and sense as $\Delta\vec{v}$

التسارع المتوسط يأخذ نفس جهة ومحنى شعاع تغير السرعة المتوسطة

Vector $\overrightarrow{a_m}$is // at $\overrightarrow{\Delta v}$ and heading towards the concavity of the trajectory.

شعاع التسارع المتوسط يتجه نحو تقعر المسار



II.5.2.The instant acceleration :

As before, we'll go to the $(\Delta t \to 0)$limit to obtain the instantaneous acceleration.

$$\vec{a} = \lim_{\Delta t \to o} \vec{a}_m = \lim_{\Delta t \to 0} \frac{\overrightarrow{\Delta v}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\vec{v}' - \vec{v}}{\Delta t} = \frac{d\vec{v}}{dt} = \frac{d^2\overrightarrow{OM}}{dt^2}$$

The coordinates of the acceleration vector in Cartesian coordinates are:

Let :

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2OM}{dt^2} = \frac{d^2x}{dt^2}\vec{\imath} + \frac{d^2y}{dt^2}\vec{\jmath} + \frac{d^2z}{dt^2}\vec{k}$$

$$\vec{a} = \frac{dv_x}{dt}\vec{\imath} + \frac{dv_y}{dt}\vec{\jmath} + \frac{dv_z}{dt}\vec{k}$$

$$a = \sqrt{(a_x)^2 + (a_y)^2 + (a_z)^2}$$

III. Different types of movement and coordinate systems : أنواع مختلفة من الحركة و أنظمة إحداثيات مختلفة

III.1.Rectilinear motion :الحركة المستقيمة

Rectilinear motion is characterized by a trajectory in the form of a straight line(). The moving body M is identified by Cartesian coordinates along the straight line Ox (if the motion is linear along Ox).

The position vector is :$\overrightarrow{OM} = x(t)\vec{\imath}$

- ✓ **Uniform rectilinear motion :الحركة المستقيمة المنتظمة**

Uniform rectilinear motion is characterized by constant velocity, so acceleration is zero.

$$\vec{a} = \frac{d\vec{v}}{dt} = \vec{o}$$

The velocity vector is : $\vec{v} = \frac{d\overrightarrow{OM}}{dt} = \frac{dx}{dt}\vec{\imath}$

$$dx = vdt$$

Hence :ومن ثم أو بالتالي

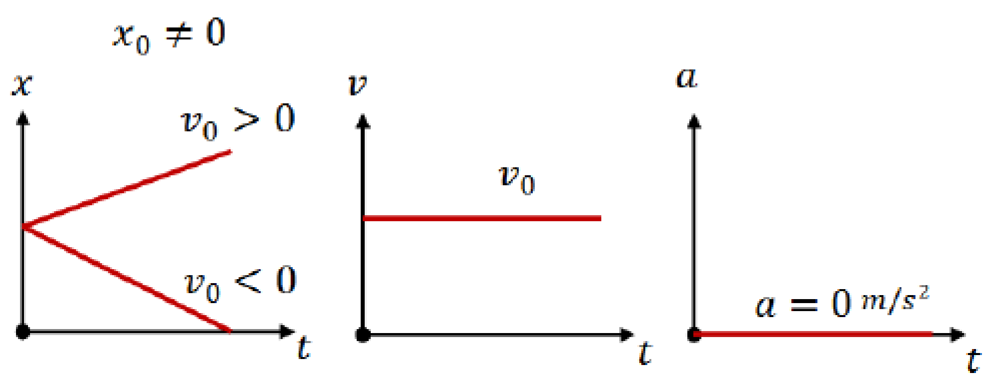
$$\int_{x_0}^{x} dx = \int_{t_0}^{t} v dt$$

Where: x_0 is the position of M at the initial instant t_0.

$$x(t) = v(t - t_0) + x_0$$

This equation is called the time equation of uniform rectilinear motion.

❖ *Diagram of uniform rectilinear motion:*



✓ **Uniformly variable rectilinear motion :الحركة المستقيمة المتغيرة بانتظام**

Uniformly variable rectilinear motion is characterized by constant acceleration:

$$a = \frac{dv}{dt} = constant$$

$$dv = adt$$

$$\int_{v_0}^{v(t)} dv = a \int_{t_0}^{t} dt$$

Where v_0 is the initial velocity at time t_0.

$$v(t) = a(t - t_0) + v_0$$

On the other hand :

$$v(t) = \frac{dx}{dt} \rightarrow dx = v(t)dt$$

$$\int_{x_0}^{x(t)} dx = \int_{t_0}^{t} v(t)\, dt = \int_{t_0}^{t} [a(t - t_0) + v_0]\, dt$$

$$x(t) = \frac{a}{2}(t^2 - t^2{}_0) + v_0(t - t_0) + x_0$$

If t_0=0, the time equation of uniformly varied rectilinear motion becomes :

$$x(t) = \frac{a}{2}t^2 + v_0 t + x_0$$

If $\vec{a}\vec{v} > 0, motion\ is\ uniformly\ accelerated$

If $\vec{a}\vec{v} < 0, the\ motion\ is\ uniformly\ retarded$

Note: uniformly varied rectilinear motion is characterized by a constant acceleration which can be calculated as follows:

$$a = \frac{dv}{dt} \rightarrow adx = \frac{dv}{dt}dx$$

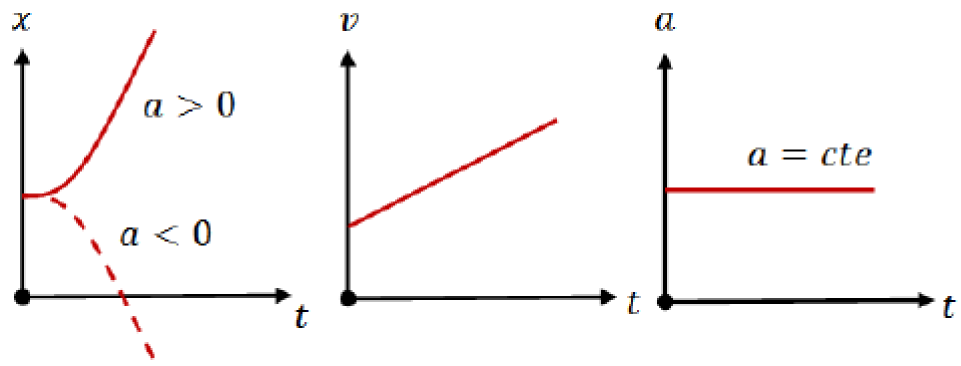
$$\int_{x_0}^{x} adx = \int_{v_0}^{v} vdv$$

$$a(x - x_0) = \frac{1}{2}(v^2 - v_0^2)$$

$$(v^2 - v_0^2) = 2a(x - x_0)$$

x_0 and v_0 are respectively M's position and velocity at time t_0.

❖ *Diagram of uniformly varied rectilinear motion:*



Varied rectilinear motion is characterized by a time-dependent acceleration.

Example : If a moving object is accelerating :$a = 2t - 1$

Determine the particle coordinate x(t) ?

We have : $a = \frac{dv}{dt}$

$$\frac{dv}{dt} = 2t - 1 \rightarrow dv = (2t - 1)dt$$

$$\int_{v_0}^{v} dv = \int_{t_0}^{t} (2t - 1)dt$$

Assume at time t_0=0s, v_0=0m/s and x_0=0m

$$v(t) = t^2 - t$$

$$v = \frac{dx}{dt} \rightarrow dx = vdt$$

$$dx = (t^2 - t)dt$$

$$\int_{0}^{x(t)} dx = \int_{0}^{t} (t^2 - t)dt$$

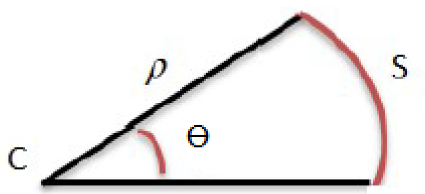
$$x(t) = \frac{t^3}{3} - \frac{t^2}{2}$$

III.2.The study of motion in intrinsic coordinates (Frenet Frame).

دراسة الحركة المنحية في الاحداثيات الذاتية (معلم فرينيه)

We use the Frenet frame in curvilinear motion. The curvature of the trajectory requires knowledge of the radius of curvature and the center of curvature**.**

The trajectory is modeled by a time-varying segment of a circle with radius ρ, center C and angle ϴ.



The length of the arc (S), called the curvilinear abscissa, is given :$S = \rho\theta$

Where : ρ is the radius of curvature, and θ is the angle subtended by the arc in radians.

In some cases, to determine the acceleration at a point M, we use the intrinsic components, which are the algebraic projections of the acceleration.

To determine the expressions of velocity and acceleration in the Frenet frame, we use the **osculating circle**(*الدائرة المتماسكة*) associated with the trajectory of the object M . **The osculating circle** is the circle tangent to the trajectory at a given point, with the same radius of curvature as the trajectory at that moment. Here's how this applies to velocity and acceleration.

One of the vectors of the Frenet frame $\overrightarrow{U_t}$ is tangent to the trajectory at point M and is directed in the positive direction of the movement. The other vector $\overrightarrow{U_n}$ is directed, according to the radius of curvature of the trajectory, towards the center of the osculating circle.

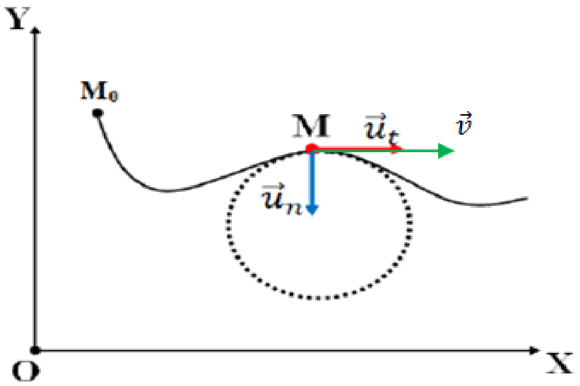


Figure (III.4) : Unit vectors in intrinsic coordinates

We can then define the displacement vector of this moving object between points M and M' :

$$\overrightarrow{MM'} = \overrightarrow{OM} = S\overrightarrow{U_t}$$

Since $\overrightarrow{u_T}$ is a unit vector tangential to the trajectory and in the direction of motion ($\overrightarrow{u_T}$ is parallel to the velocity vector$\vec{v}$), and it is written as : $\vec{v} = v\,\overrightarrow{u_T}$ whereas : $v = \frac{ds}{dt}$

The acceleration vector :

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d(v\overrightarrow{u_t})}{dt} = \frac{dv}{dt}\overrightarrow{u_t} + \frac{d\overrightarrow{u_t}}{dt}v$$

$$\vec{a} = \frac{dv}{dt}\overrightarrow{u_t} + v\frac{d\overrightarrow{u_t}}{dt}$$

When the object moves from point M to point M', the vector $\overrightarrow{u_t}$ rotates by an angle $d\theta$ The time derivative of the vector $\overrightarrow{u_t}$ is therefore.

$$\frac{d\overrightarrow{u_t}}{dt} = \dot{\theta}\overrightarrow{u_n}$$

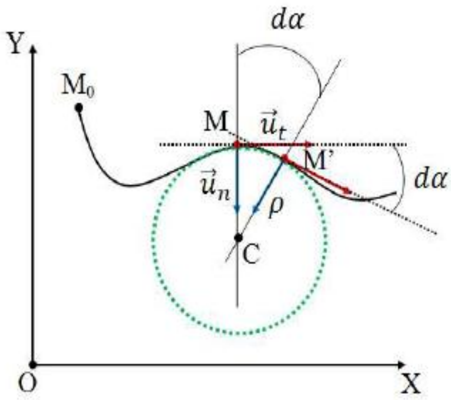


Figure (III.5) : Representation of Curvilinear Motion and Its Elements.

On the other: $\dot{\theta} = \frac{d\theta}{dt} = w = \frac{v}{\rho}\ ; v = \frac{ds}{dt}$ with R is the radius of the curvature of the trajectory.

Hence :

$$\vec{a} = \frac{dv}{dt}\overrightarrow{u_T} + \frac{v^2}{\rho}\overrightarrow{u_N} = \overrightarrow{a_t} + \overrightarrow{a_n}$$

$\overrightarrow{a_n}$ The normal acceleration (التسارع الناظمي) and $\overrightarrow{a_t}$ tangential acceleration (التسارع المماس) are written

By : $\begin{cases} a_t = \frac{d|\vec{v}|}{dt} \\ a_n = \frac{v^2}{R} \end{cases}$ and $a = \sqrt{{a_x}^2 + {a_y}^2} = \sqrt{{a_t}^2 + {a_n}^2}$

$\rho \to \infty$ so the trajectory is a line.

ρ is constant, so the trajectory is circular.

<u>Determining the radius of curvature :</u>

we'll take the vector product of acceleration and velocity.

$$\vec{a} \times \vec{v} = \left(\frac{dv}{dt}\overrightarrow{u_t} + \frac{v^2}{\rho}\overrightarrow{u_n}\right) \times v\overrightarrow{u_t}$$

$$\vec{a} \times \vec{v} = \frac{v^3}{\rho}(\overrightarrow{u_n} \times \overrightarrow{u_t})$$

"Since the cross product is a vector, we will take its magnitude, and thus we can deduce the radius of curvature."

$|\vec{a} \times \vec{v}| = \frac{v^3}{\rho} \to \rho = \frac{v^3}{|\vec{a} \times \vec{v}|}$

Exercice :

Given that its tangential acceleration is β and its normal acceleration is 2β, and let α be the angle between the tangent to the trajectory and the OX axis:

1. If this point is at rest at (t=0), with the curvilinear coordinate (s = 0), deduce the relation s = f(v).
2. Calculate the radius of curvature and show that ρ(t) = s(t).
3. Knowing that (ds = ρdα), determine (s) by setting ($s = s_0$) when (α= 0), and deduce the relation v = f(α).

Solution :

We have :$\frac{dv}{dt} = \beta \quad and\ \frac{v^2}{\rho} = 2\beta$

By integrating the first equation we get :$\int_{v_0}^{v} dv = \beta \int_{t_0}^{t} dt \rightarrow v = \beta t$ because at t=0 speed is zero.

The relationship between the curvilinear coordinate(s) and the modulus of velocity (v) is :

$$v = \frac{ds}{dt} \rightarrow ds = vdt$$

$$\int_{s_0}^{s} ds = \int_{t_0}^{t} vdt \rightarrow s = \int_{t_0}^{t} \beta t\, dt$$

$s = \frac{\beta t^2}{2} + c$ The constant c is zero because s =0 when t=0.

So : $s = \frac{\beta t^2}{2}$

Now, we want to express (s)as a function of (v) instead of (t). From $v = \beta t$, we can express (t)as:

$$t = \frac{v}{\beta}$$

Substituting this into the equation for s(t):

$$s = \frac{\beta t^2}{2} = \frac{v^2}{2\beta}$$

Thus, the relationship between the curvilinear coordinate (s) and the velocity (v) is:

2.Calculate the radius of curvature and show that ρ(t)=s(t) :

We have **:** $\frac{v^2}{\rho} = 2\beta \rightarrow \rho = \frac{v^2}{2\beta}$

3.knowing that ds= ρdα, determine s by posing s=s_0 quant α=0 deduce the relation v=f(α).

The curvilinear coordinate is modelized as an arc, which is the product of the radius and the angle. ds=ρdα and since s=ρ, ds=sdα.

$$\frac{ds}{s} = d\alpha \rightarrow \ln s = \alpha + cste \rightarrow s = Ae^{\alpha}$$

Whereas S(0)=S_0 So : $s = s_0 e^{\alpha}$

From this last equation and the expression for velocity as a function of S, we get :

$\rho = S = \frac{v^2}{2\beta} \rightarrow v = \sqrt{2s\beta}.e^{\alpha/2}$ so velocity is a function of α.

III.3. Circular motion الحركة الدائرية

- **Uniform Circular motion and Angular Velocity**

Let us now consider the special case in which the path is a circle (circular motion). The velocity v, being tangent to the circle and is perpendicular to the radius R = CA. When we measure distances along the circumference of the circle (محيط الدائرة)from the origin 0, we have :

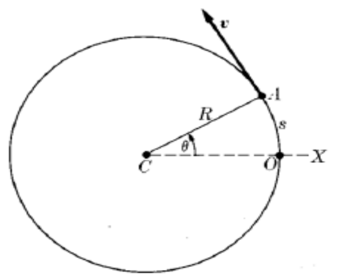


Figure (III.6) : Circular motion.

$$S = R\theta$$

$$v = \frac{dS}{dt} \rightarrow \dot{S} = \frac{d(R\theta)}{dt}$$

$$v = R\frac{d\theta}{dt} \rightarrow v = R\dot{\theta}$$

$\dot{\theta} = \omega$: represent **the angular velocity(السرعة الزاوية)** , which measured in (rad.s^{-1}), the relation between the tangent and the angular velocities is$v = \omega R$.

- ✓ For **uniform circular motion**(الحركة دائرية منتظمة) then $\overrightarrow{a_t} = 0$

The acceleration in this case is : $\vec{a} = \vec{a}_n = \frac{v^2}{\rho}\vec{u}_n$

- ✓ However, if **uniformly variable circular motion(الحركة دائرية متغيرة بانتظام)** in this case the angular velocity ω is not constant and therefore the velocity v is not constant also, then :

$$\vec{a} = \frac{dv}{dt}\overrightarrow{u_t} + \frac{v^2}{\rho}\overrightarrow{u_n}$$

The acceleration in this case is :

$$\vec{a} = \frac{dv}{dt}\overrightarrow{u_t} + \frac{v^2}{\rho}\overrightarrow{u_n} = \rho\frac{d\omega}{dt}\overrightarrow{u_t} + \rho\omega^2\overrightarrow{u_n}$$

$$\begin{cases} a_t = \frac{dv}{dt} = \rho\frac{dw}{dt} \\ a_n = \frac{v^2}{\rho} = \frac{(\rho\omega)^2}{\rho} \end{cases}$$

III.4.Sinusoidal or harmonic motion الحركة الجيبية

The movement is called sinusoidal or harmonic (جيبية)if its evolution over time is written by the equation:

$$x(t) = A\sin(\omega t + \varphi)$$

A: amplitude, ω: angular frequency (التردد الزاوي), and φ: phase.

$$\omega = \frac{2\pi}{T} = 2\pi f$$

T: period and f: frequency

The speed(velocity) :

$$v(t) = A\omega\cos(\omega t + \varphi)$$

The acceleration :

$$a(t) = \frac{dv}{dt} = -A\omega^2 x(t)$$

IV.Expression of velocity and acceleration in different coordinate systems :

IV.1.Polar coordinates system :(الاحداثيات القطبية)

The polar coordinate system is specific to the study of rotationally symmetrical plane motion. A polar axis (Ox) is used, with origin O called the pole. The position of any point M in the plane containing (Ox) can then be determined by the polar radius $\rho(t)$ and the polar angle θ, which can be varied with time using the bases $\overrightarrow{U_\rho}$ which refer to the direction of $\overrightarrow{OM}$ and $\overrightarrow{U_\theta}$ obtained by rotating $\overrightarrow{U_\rho}$by an angle $\frac{\pi}{2}$ in the trigonometric direction.

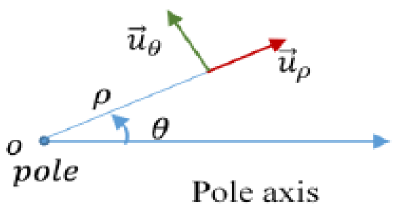


Figure (III.7) : Position vector in polar coordinates

When the motion is in a plane, it's also possible to locate the position of point M using its polar coordinates (ρ, θ).

ρ: polar radius $0 \leq \rho \geq R$

θ: polar angle $0 \leq \theta \geq 2\pi$

The base of this coordinates system is $(\overrightarrow{U_\rho}; \overrightarrow{U_\varphi})$

$\overrightarrow{U_\rho}$ is directed form the origin towards the material point position.

$\overrightarrow{U_\varphi}$is directed in the direction of increase of ɵ.

VI.1.1.Vector position

The vector position is given by $\overrightarrow{OM} = \vec{r} = \rho\overrightarrow{U_\rho}$

Relation between Cartesian and polar coordinates

$\vec{r} = \rho\overrightarrow{U_\rho} = \rho\cos\theta\vec{\imath} + \rho\sin\theta\vec{\jmath}$

$$\begin{cases} x = \rho\sin\theta \\ y = \rho\sin\theta \end{cases} \rightarrow \begin{cases} \rho = \sqrt{x^2 + y^2} \\ \tan\theta = \frac{y}{x} \end{cases}$$

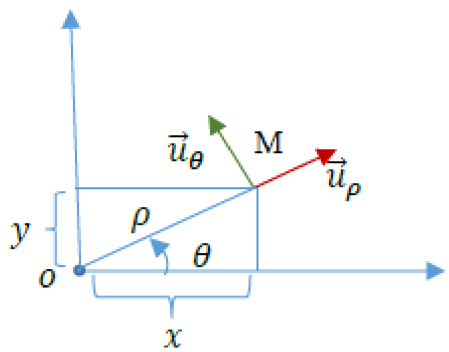
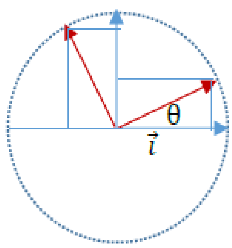


Figure (III.8) : Relationship between polar and Cartesian coordinates

Relation between Cartesian and polar bases :



$\overrightarrow{U_\rho} = \cos\theta\vec{\imath} + \sin\theta\vec{\jmath}$

$\overrightarrow{U_\theta} = \cos\left(\theta + \frac{\pi}{2}\right)\vec{\imath} + \sin\left(\theta + \frac{\pi}{2}\right)\vec{\jmath}$

$\overrightarrow{U_\theta} = -\sin\theta\,\vec{\imath} + \cos\theta\vec{\jmath}$

Also we can write

$\vec{\imath} = \cos\theta\,\overrightarrow{U_\rho} - \sin\theta\,\overrightarrow{U_\theta}$

$\vec{\jmath} = \sin\theta\,\overrightarrow{U_\rho} + \cos\theta\,\overrightarrow{U_\theta}$

4.1.2.Velocity :

$$\vec{v} = \frac{d(\overrightarrow{OM})}{dt} = r = \frac{d\rho}{dt}\overrightarrow{U_\rho} + \rho\frac{d(\overrightarrow{U_\rho})}{dt} = \dot{\rho}\overrightarrow{U_\rho} + \rho\dot{\overrightarrow{U_\rho}} \ldots\ldots..(1)$$

$\overrightarrow{U_\rho}$ Changes its direction with time, so it derivative with respect to time is not equal to zero.

$$\overrightarrow{U_\rho} = \cos\theta\;\vec{\imath} + \sin\theta\,\vec{\jmath} \rightarrow \dot{\vec{U}}_\rho = \dot{\theta}(-\sin\theta\,\vec{\imath} + \cos\theta\,\vec{\jmath}) = \dot{\theta}\,\overrightarrow{U_\theta}$$

$$\overrightarrow{U_\theta} = -\sin\theta\,\vec{i} + \cos\theta\,\vec{j} \rightarrow \dot{\vec{U}}_\theta = \dot{\theta}(-\cos\theta\,\vec{i} - \sin\theta\,\vec{j}) = -\dot{\theta}\overrightarrow{U_\rho}$$

Note: The derivative of a unit vector with respect to an angle is a unit vector perpendicular to the angle in the positive direction.

مشتق شعاع الوحدة بالنسبة إلى الزمن هو شعاع وحدة عمودي على هذا الأخير وفي الاتجاه الموجب

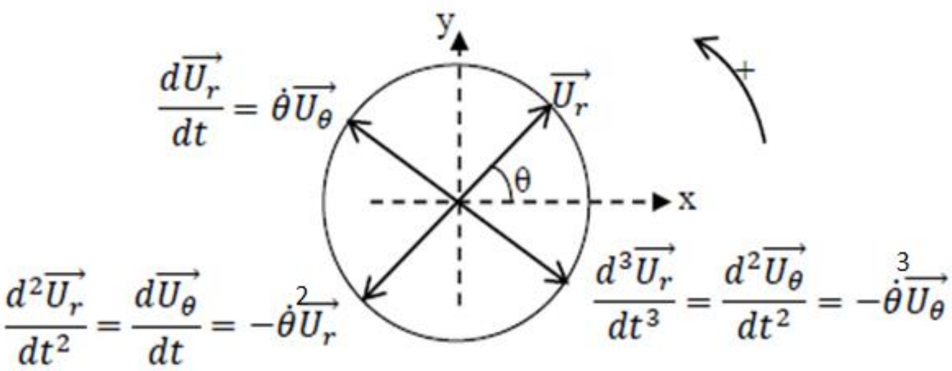


Figure (III.9) : Successive derivation of a unit vector with respect to time

by compensation in relationship (1)

$$\vec{v} = \dot{\rho}\overrightarrow{U_\rho} + \rho\dot{\overrightarrow{U_\rho}} = \dot{\rho}\overrightarrow{U_\rho} + \rho\dot{\theta}\overrightarrow{U_\theta} = v_\rho\overrightarrow{U_\rho} + v_\theta\overrightarrow{U_\theta}$$

$$v = \sqrt{{v_\rho}^2 + {v_\theta}^2}$$

$$\begin{cases} v_\rho : radial\ component \\ v_\theta : transversal\ component \end{cases}$$

VI.1.3.Acceleration

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d\left(\dot{\rho}\overrightarrow{U_\rho} + \rho\,\dot{\theta}\,\dot{\overrightarrow{U_\rho}}\right)}{dt}$$

$$\vec{a} = \ddot{\rho}\overrightarrow{U_\rho} + \dot{\rho}\dot{\overrightarrow{U_\rho}} + \dot{\theta}\dot{\rho}\,\dot{\overrightarrow{U_\rho}} + \rho\ddot{\theta}\,\dot{\vec{U}}_\rho + \rho\,\dot{\theta}\ddot{\overrightarrow{U_\rho}}$$

$$\vec{a} = \ddot{\rho}\overrightarrow{U_\rho} + \dot{\rho}\dot{\theta}\overrightarrow{U_\theta} + \dot{\rho}\dot{\theta}\overrightarrow{U_\theta} + \rho\ddot{\theta}\,\dot{\vec{U}}_\rho + \rho(-\dot{\theta}^2\overrightarrow{U_\rho})$$

$$\vec{a} = (\ddot{\rho} - \rho\dot{\theta}^2)\overrightarrow{U_\rho} + (2\dot{\rho}\dot{\theta} + \rho\ddot{\theta})\overrightarrow{U_\theta}$$

$$\vec{a} = a_\rho\overrightarrow{U_\rho} + a_\theta\overrightarrow{U_\theta}$$

$$\begin{cases} a_\rho : radial\ component \\ a_\theta : transversal\ component \end{cases}$$

$$a = {a_\rho}^2 + {a_\theta}^2$$

Example : Conversion of polar coordinates to Cartesian coordinates

1-The point A with polar coordinates $((1;\pi)$. has Cartesian coordinates $(x$,y) =(-1 ;0)

2-The point B with polar coordinates $\left(2,\frac{\pi}{2}\right)$ has Cartesian coordinates $(x$,y) …..

$$\begin{cases} x = \rho\cos\theta = \cos\pi = -1 \\ y = \rho\sin\theta = \sin\pi = 0 \end{cases} \rightarrow A(-1;0)$$

Example : Conversion of Cartesian coordinates to polar coordinates. (Always choose the angle θ between 0 and 2π).

1- The point A with Cartesian coordinates (−1,−1) has polar coordinates (ρ,θ)…..
2- The point B with Cartesian coordinates (1 2 , √ 3 2) has polar coordinates (ρ,θ) ….

Solution :

$$\begin{cases} \rho = \sqrt{x^2+y^2} = \sqrt{2} \\ \theta = arc\tan\frac{y}{x} = 1 \rightarrow \theta = \frac{\pi}{4} \end{cases} \rightarrow A(\sqrt{2};\frac{\pi}{4})$$

VI.2.Cylindrical coordinates system :

When motion takes place on a cylindrical(أسطواني) or spiral (حلزوني)surface, cylindrical coordinates are often used, defined in relation to the Cartesian system. Mobile M is then represented by :

- the polar coordinates r and θ of its projection “m” on the plane (O, x, y) ;
- its axial coordinate z.

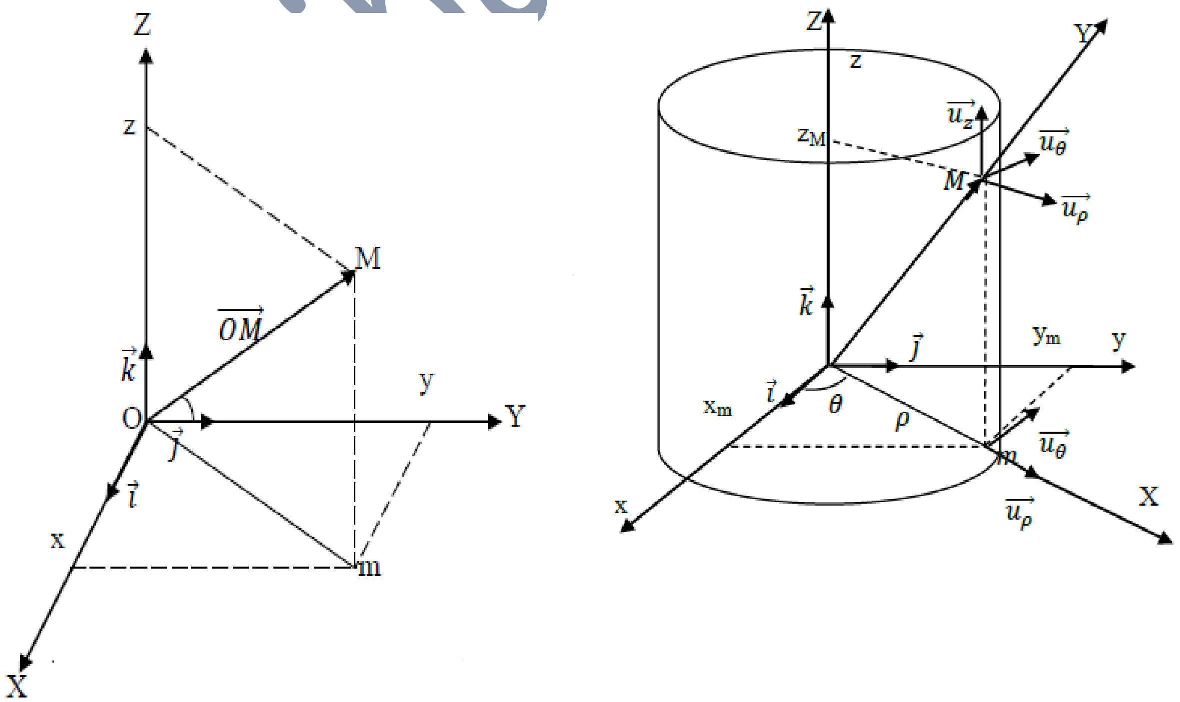


Figure (III.10) : Position vector in cylindrical coordinate$(\overrightarrow{U_\rho};\ \overrightarrow{U_\theta};\ \vec{K})$

The position vector $\overrightarrow{OM}$ is written in cylindrical coordinates as follows:

$$\overrightarrow{OM} = \overrightarrow{Om} + \overrightarrow{mM} = \rho\overrightarrow{U_\rho} + z\vec{k}$$

VI 2.1.*<u>Link between cylindrical and Cartesian coordinates</u>* ***العلاقة** بين الاحداثيات الديكارتية والاحداثيات الأسطوانية*

$$M(x\,;y\,;z) \rightarrow M(\rho;\theta;z)$$

$$\begin{cases} \rho = \left(\sqrt{x^2+y^2}\right) \\ \theta = arc\tan\dfrac{y}{x} \\ z = z \end{cases}$$

$$M(\rho;\theta;z) \rightarrow M(x\,;y\,;z)$$

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \\ z = z \end{cases}$$

The cylindrical base is written in terms of the Cartesian base as follows:

$$\vec{U}_\rho = \cos\theta\,\vec{\imath} + \sin\theta\,\vec{\jmath}$$
$$\vec{U}_\theta = -\sin\theta\,\vec{\imath} + \cos\theta\vec{\jmath}$$
$$\vec{k} = \vec{k}$$

VI.*2.2.<u>Expression of the velocity vector in cylindrical coordinates</u>*

$$\vec{v} = \frac{d(\overrightarrow{OM})}{dt} = \frac{(\rho\overrightarrow{U_\rho} + z\vec{k})}{dt} = r = \frac{d\rho}{dt}\overrightarrow{U_\rho} + \rho\frac{d(\overrightarrow{U_\rho})}{dt} + \frac{dz}{dt}\vec{k} = \dot{\rho}\overrightarrow{U_\rho} + \rho\dot{\overrightarrow{U_\rho}} + \dot{z}\vec{k}$$

We remind you that :

$$\dot{\overrightarrow{U_\rho}} = \dot{\theta}\,\overrightarrow{U_\theta}$$
$$\dot{\overrightarrow{U_\theta}} = -\dot{\theta}\overrightarrow{U_\rho}$$

So the velocity vector is :

$$\vec{v} = \dot{\rho}\overrightarrow{U_\rho} + \rho\dot{\theta}\,\overrightarrow{U_\theta} + \dot{z}\vec{k}$$

VI.*2.3.<u> Expression of the acceleration vector in cylindrical coordinates</u>*

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d\left(\dot{\rho}\overrightarrow{U_\rho} + \rho\,\dot{\theta}\,\dot{\overrightarrow{U_\rho}} + \dot{z}\vec{k}\right)}{dt}$$

$$\vec{a} = \ddot{\rho}\overrightarrow{U_\rho} + \dot{\rho}\dot{\overrightarrow{U_\rho}} + \dot{\theta}\dot{\rho}\,\dot{\overrightarrow{U_\rho}} + \rho\ddot{\theta}\,\dot{\vec{U}}_\rho + \rho\,\dot{\theta}\ddot{\vec{U}}_\rho + \ddot{z}\vec{k}$$

$$\vec{a} = \ddot{\rho}\overrightarrow{U_\rho} + \dot{\rho}\dot{\theta}\overrightarrow{U_\theta} + \dot{\rho}\dot{\theta}\overrightarrow{U_\theta} + \rho\ddot{\theta}\,\dot{\vec{U}}_\rho + \rho\left(-\dot{\theta}^2\overrightarrow{U_\rho}\right) + \ddot{z}\vec{k}$$

$$\vec{a} = \left(\ddot{\rho} - \rho\dot{\theta}^2\right)\overrightarrow{U_\rho} + \left(2\dot{\rho}\dot{\theta} + \rho\ddot{\theta}\right)\overrightarrow{U_\theta} + \ddot{z}\vec{k}$$

Example 2: Express cartesian coordinates (1, -3, 5) as cylindrical coordinates.

$$\begin{cases} \rho = \sqrt{x^2 + y^2} = \sqrt{10} \\ \theta = arctg\dfrac{y}{x} = -3.92 \\ \quad z = z = 5 \end{cases}$$

VI.3.Spherical coordinate system :

The spherical base is determined by the unit vectors$(\overrightarrow{U_r}; \overrightarrow{U_\theta}; \overrightarrow{U_\varphi})$ such that :

φ : is the angle between the position vector $\overrightarrow{OM}$ and the Oz axis.

Θ : the angle between the vector $\overrightarrow{Om}$ and the axis Ox. m is the projection of M in the plane (O,x,y). See figure

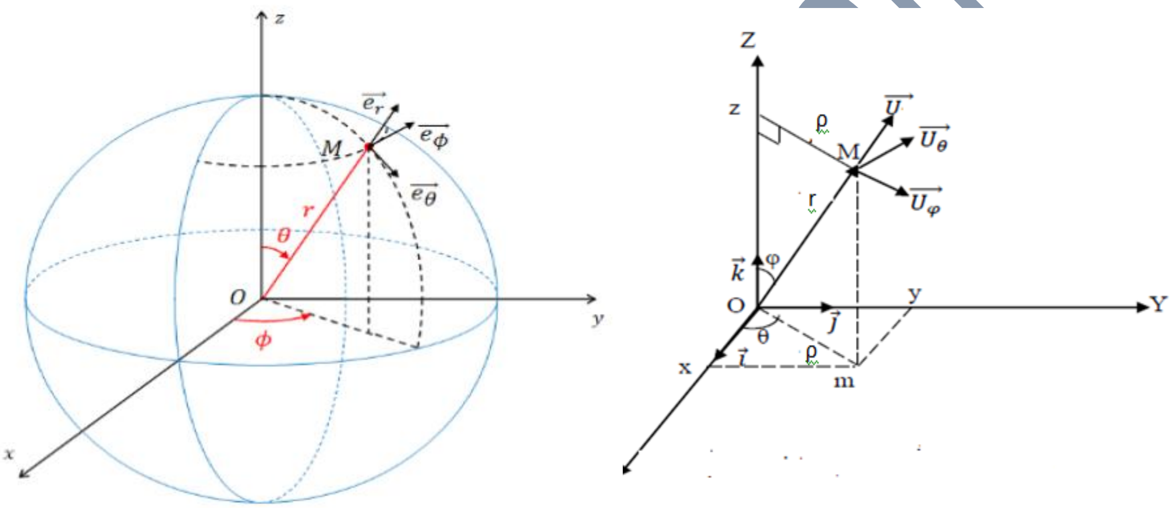


Figure (III.11) : Position vector in spherical coordinates.

A moving point M is given spherical coordinates: (r, θ, φ) and the position vector is written as :

$$\overrightarrow{OM} = r\overrightarrow{U_r}$$

$$\begin{cases} r = OM; \quad 0 < r < R \\ \theta = (ox;\ \overrightarrow{om})\ 0 < \theta < 2\pi \\ \varphi = (oz; \overrightarrow{OM})\ 0 < \varphi < \pi \end{cases}$$

VI.3.1.*__Link between Cartesian and spherical coordinates :__*

$$M(x\,;y\,;z) \rightarrow M(r, \theta, \varphi)$$

$$\begin{cases} r = \sqrt{x^2 + y^2 + z^2} \\ \theta = arc\tan\dfrac{y}{x} \\ \varphi = arc\cos\dfrac{z}{r} \end{cases}$$

$$M(r, \theta, \varphi) \rightarrow M(x\,; y\,; z)$$

$$\begin{cases} x = \rho\cos\theta \\ y = \rho\sin\theta \\ z = r\cos\varphi \end{cases}$$

Knowing that $\rho = r\sin\varphi$ so :

$$\begin{cases} x = r\sin\varphi\cos\theta \\ y = r\sin\varphi\sin\theta \\ z = r\cos\varphi \end{cases}$$

VI.*3.2.*__*Spherical base as a function of Cartesian base*__

$$\overrightarrow{OM} = r\sin\varphi\cos\theta\vec{\imath} + r\sin\varphi\sin\theta\,\vec{\jmath} + r\cos\varphi\,\vec{k} = r\overrightarrow{U_r}$$

$$\overrightarrow{U_r} = \sin\varphi\cos\theta\vec{\imath} + \sin\varphi\sin\theta\,\vec{\jmath} + \cos\varphi\,\vec{k}$$

To find the vector $\overrightarrow{U_\theta}$ as a function of the Cartesian base, we simply plot the unit vectors along the plane (o, x, y). See figure

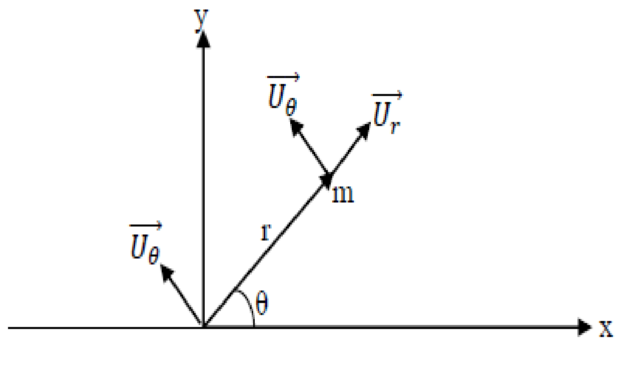


Figure (III.12) : Projection of vector $\overrightarrow{U_\theta}$ along Cartesian coordinates

$$\vec{U}_\theta = -\sin\theta\,\vec{\imath} + \cos\theta\vec{\jmath}$$

To find the vector $\overrightarrow{U_\varphi}$ we have :

$\overrightarrow{U_\theta} \times \overrightarrow{U_r} = \overrightarrow{U_\varphi}$

$$\overrightarrow{U_\varphi} = \begin{vmatrix} \vec{\imath} & -\vec{\jmath} & \vec{k} \\ -\sin\theta & \cos\theta & 0 \\ \sin\varphi\cos\theta & \sin\varphi\sin\theta & \cos\varphi \end{vmatrix} = \cos\theta\cos\varphi\,\vec{\imath} + \sin\theta\cos\varphi\vec{\jmath} - \sin\varphi\vec{k}$$

VI.3.3. Expression of the velocity vector in spherical coordinates

$$\vec{v} = \frac{d(r\overrightarrow{U_r})}{dt} = \dot{r}\overrightarrow{U_r} + r\dot{\vec{U}}_r$$

$$\dot{\vec{U}}_r = \dot{\varphi}\left(\cos\theta\cos\varphi\,\vec{\imath} + \sin\theta\cos\varphi\,\vec{\jmath} - \sin\varphi\,\vec{k}\right) + \dot{\theta}(-\sin\varphi\sin\theta\,\vec{\imath} + \sin\varphi\cos\theta\vec{\jmath})$$

$$\dot{\vec{U}}_r = \dot{\varphi}\overrightarrow{U_\varphi} + \dot{\theta}\sin\varphi\,\overrightarrow{U_\theta}$$

$$\text{So : } \vec{v} = \dot{r}\overrightarrow{U_r} + r\dot{\varphi}\overrightarrow{U_\varphi} + r\dot{\theta}\sin\varphi\,\overrightarrow{U_\theta}$$

VI.3.4. Expression of the acceleration vector in spherical coordinates

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d\left(\dot{r}f + r\dot{\varphi}\overrightarrow{U_\varphi} + r\dot{\theta}\sin\varphi\,\overrightarrow{U_\theta}\right)}{dt}$$

$$\vec{a} = \left(\ddot{r} - r\dot{\varphi}^2 - r\,\dot{\theta}^2\sin^2\varphi\right)\overrightarrow{U_r} + \left(r\ddot{\theta}\sin\varphi + 2\dot{r}\dot{\theta}\sin\varphi + 2r\dot{\theta}\dot{\varphi}\cos\varphi\right)\overrightarrow{U_\theta}$$
$$+ \left(r\ddot{\varphi} + 2\dot{r}\dot{\varphi} - r\dot{\theta}^2\sin\varphi\cos\varphi\right)\overrightarrow{U_\varphi}$$

Example : convert a point from spherical coordinates to Cartesian coordinates $A(2;\frac{2\pi}{4};\frac{2\pi}{4})$.

Exercice(1) :

A material point M is identified by its cartesian coordinates (x, y, z).

1. Write down the relationship between cartesian coordinates and cylindrical coordinates (using a diagram).

2. Write the position vector in cylindrical coordinates and deduce the velocity vector in the same coordinate system.

3. If the position of the point is represented in cylindrical coordinates by :

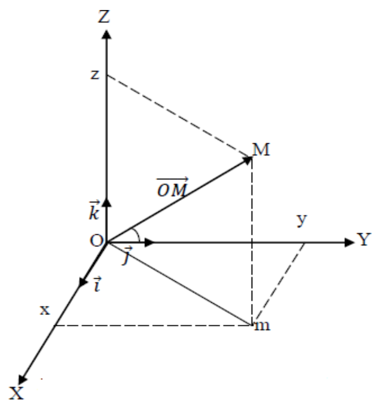
$$:\begin{cases}\rho = 4t^2\\ \theta = \omega t\\ z = \sqrt{t}\end{cases}$$

Find the expression of the velocity vector $\vec{v}$ in cylindrical coordinates

Solution :

1. Writing the relationship between cartesian coordinates and cylindrical coordinates (using a diagram).

$$\begin{cases}x = \rho\cos\theta\\ y = \rho\sin\theta\\ z = z\end{cases}$$



1. ***Write the position vector in cylindrical coordinates***

$$\overrightarrow{OM} = \rho\overrightarrow{U_\rho} + z\vec{k}$$

3. deduce the velocity vector in the same coordinate system.

$$\vec{v} = \frac{d\overrightarrow{OM}}{dt} = \frac{\left(\rho\overrightarrow{U_\rho} + z\vec{k}\right)}{dt} = \dot{\rho}\overrightarrow{U_\rho} + \rho\dot{\theta}\,\overrightarrow{U_\theta} + \dot{z}\vec{k}$$

So : $\begin{cases} \dot{\rho} = 8t \\ \dot{\theta} = \omega \\ \dot{z} = \frac{1}{2\sqrt{t}} \end{cases} \rightarrow \vec{v} = 8t\overrightarrow{U_\rho} + 4t^2\omega\overrightarrow{U_\theta} + \frac{1}{2\sqrt{t}}\vec{k}$

$$\vec{v} = 8t\overrightarrow{U_\rho} + 4t^2\omega\overrightarrow{U_\theta} + \frac{1}{2\sqrt{t}}\vec{k}$$

Exercice(2) : The motion of a particle M moving in the plane (xoy) is described by the following equations: $\begin{cases} x(t) = t\cos t \\ y(t) = t\sin t \end{cases}$

1-Determine the components of the velocity vector and its modulus.

2-Determine the components of the acceleration vector and its modulus.

3-Determine the expressions for the intrinsic components of acceleration as a function of time time t.

4-Deduce the radius of curvature of the trajectory as a function of time.

Solution :

1. ***<u>Determining the components of the velocity vector and its modulus :</u>***

$\vec{v} = \frac{d\overrightarrow{OM}}{dt}$ where : $\overrightarrow{OM} = x\vec{\imath} + y\vec{\jmath} = t\cos t\,\vec{\imath} + t\sin t\,\vec{\jmath}$

$$\vec{v} = \frac{d\overrightarrow{OM}}{dt} = \frac{dx}{dt}\vec{\imath} + \frac{dy}{dt}\vec{\jmath}$$

$$\begin{cases} \dfrac{dx}{dt} = \cos t - t\sin t \\ \dfrac{dy}{dt} = \sin t + t\cos t \end{cases} \rightarrow \vec{v} = (\cos t - t\sin t)\vec{\imath} + (\sin t + t\cos t)\vec{\jmath}$$

So its modulus is : $v = \sqrt{(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2}$

$$v = \sqrt{\cos^2 t - 2t\cos t\sin t + t^2\sin^2 t + \sin^2 t + 2t\sin t\cos t + t^2\cos t^2}$$

$$v = \sqrt{1 + t^2}$$

2.Determine the components of the acceleration vector and its modulus.

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{dv_x}{dt}\vec{\imath} + \frac{dv_y}{dt}\vec{\jmath}$$

$$\begin{cases} \dfrac{dv_x}{dt} = \dfrac{d\,(\cos t - t\sin t}{dt} \\ \dfrac{dv_y}{dt} = \dfrac{d\,(\sin t + t\cos t)}{dt} \end{cases}) \rightarrow$$

$$\begin{cases} a_x = (-\sin t - \sin t - \mathrm{t}\cos t) \\ a_y = (\cos t + \cos t - t\sin t) \end{cases}$$

$$\begin{cases} a_x = (-2\sin t - t\cos t) \\ a_y = (2\cos t - t\sin t) \end{cases}$$

So its modulus is : $a = \sqrt{(-2\sin t - t\cos t)^2 + (2\cos t - t\sin t)^2} = \sqrt{4 + t^2}$

3. ***Determine the expressions for the intrinsic components of acceleration as a function of time t.***

$$a_t = \frac{dv}{dt} = \frac{t}{\sqrt{1+t^2}}$$

$$a_n = \frac{v^2}{\rho}$$

We have : $a = \sqrt{{a_x}^2 + a^2{}_y} = \sqrt{{a_t}^2 + {a_n}^2} = \rightarrow a^2 = {a_t}^2 + {a_n}^2$

$${a_n}^2 = a^2 - {a_t}^2 = (t^2+4) - \frac{t^2}{(1+t^2)} = \frac{(t^2+4)(1+t^2) - t^2}{(1+t^2)}$$

$${a_n}^2 = \frac{t^2 + t^4 + 4 + 4t^2 - t^2}{(1+t^2)} = \frac{t^4 + 4t^2 + 4}{(1+t^2)} = \frac{(t^2+2)^2}{(1+t^2)}$$

4. ***Deduce the radius of curvature of the trajectory as a function of time***

Wehave :

$$a_n = \frac{v^2}{\rho} \rightarrow \rho = \frac{v^2}{a_n} = \frac{\left(\sqrt{1+t^2}\right)^2}{\sqrt{\dfrac{(t^2+2)}{(1+t^2)}}} = \frac{(1+t^2)\sqrt{(1+t^2)}}{\sqrt{(t^2+2)}} = \frac{(1+t^2)^{3/2}}{\sqrt{(t^2+2)}}$$

Chapter VI : Relative motion

I. INTRODUCTION

The motion of a material point can be divided into two distinct motions :

- Motion relative to a fixed reference frame, which we'll call the Absolute reference frame.
  (الحركة بالنسبة إلى إطار مرجعي ثابت، وهو ما سنسميه الإطار المرجعي المطلق).
- A movement relative to a moving frame of reference, which we'll call a relative frame of reference.
  (الحركة بالنسبة إلى إطار مرجعي متحرك، وهو ما سنسميه الإطار المرجعي النسبي.)

All quantities (position, velocity and acceleration) are identified in relation to the appropriate reference frame.

II.Absolute and relative quantities:

A material point M is in motion relative to a moving reference frame R'(o',x',y',z'), itself in motion relative to a fixed reference frame R (O,x,y,z). in motion relative to a fixed reference frame R (O, x, y, z).

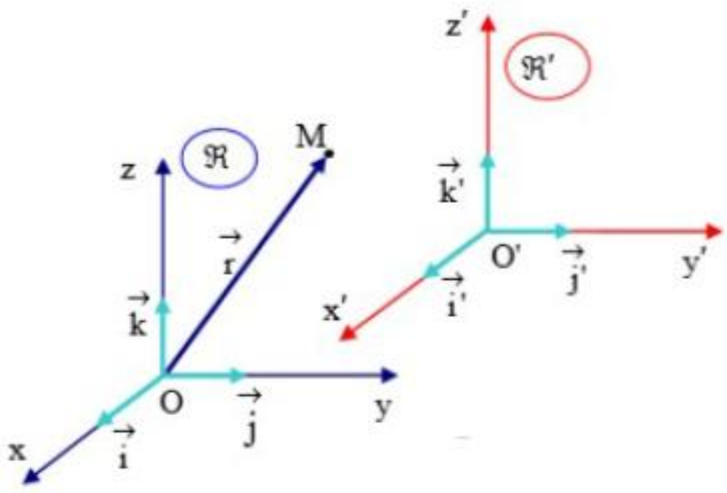


Figure (IV.1) : Presentation of the two reference points

R(O, x, y, z), considered fixed, which is called the absolute reference frame.

R′ (O′, x′, y′, z′), in any motion relative to R, which is the relative reference frame.

The motion of the material point is determined with respect to the absolute reference frame R(O,x,y,z) and derivations are performed in R in which the base $(\vec{\imath},\vec{\jmath},\vec{k})$ is invariant.

II.1.Absolute motion :

Its position vector $\overrightarrow{OM} = \vec{r} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$

Its absolute speed(velocity) vector : $\vec{v}_a(t) = \left.\frac{d\overrightarrow{OM}}{dt}\right|_R = \frac{dx}{dt}\vec{\imath} + \frac{dy}{dt}\vec{\jmath} + \frac{dz}{dt}\vec{k}$

Its absolute acceleration vector : $\vec{a}_a(t) = \left.\frac{d\vec{v}_a}{dt}\right|_R = \frac{d^2x}{dt^2}\vec{\imath} + \frac{d^2y}{dt^2}\vec{\jmath} + \frac{d^2z}{dt^2}\vec{k}$

II.2.Relative motion :

The motion of the material point is determined with respect to the relative reference frame R'(O',x',y',z') and the derivations are performed in R' in which the base (i ' , j ' , k ') is invariant.

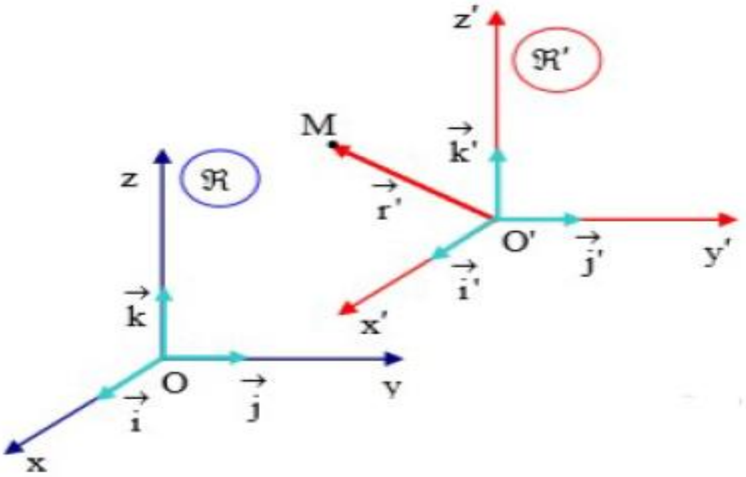


Figure (IV.2) : Relative motion.

Its position vector $\overrightarrow{O'M} = \vec{r'} = x'\vec{\imath} + y'\vec{\jmath} + z'\vec{k}$

Relative velocity vector : $\vec{v_r}(t) = \left.\frac{d\overrightarrow{O'M}}{dt}\right|_{R'} = \frac{dx'}{dt}\vec{\imath} + \frac{dy'\vec{\jmath}}{dt} + \frac{dz'}{dt}\vec{k}$

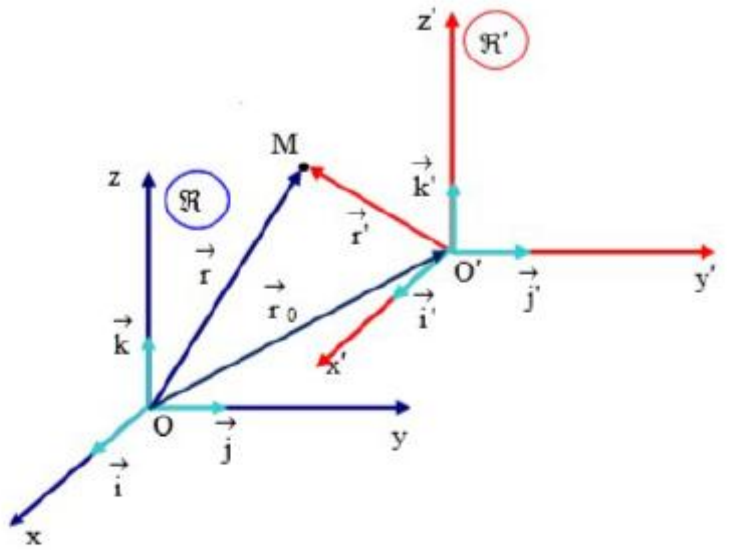
Relative acceleration vector : $\vec{a}_r = \left.\frac{d\vec{v_r}}{dt}\right|_{R'} = \frac{d^2x'}{dt^2}\vec{\imath} + \frac{d^2y'\vec{\jmath}}{dt^2} + \frac{d^2z'}{dt^2}\vec{k}$

II.3.Composition of velocity vectors :

As defined previously, the absolute velocity of point M is given by :

$$\vec{v}_a(t) = \left.\frac{d\overrightarrow{OM}}{dt}\right|_R = \frac{dx}{dt}\vec{\imath} + \frac{dy}{dt}\vec{\jmath} + \frac{dz}{dt}\vec{k}$$

Using the Chasles relationship



Using the Chasles(شال) relationship : $\overrightarrow{OM} = \overrightarrow{OO'} + \overrightarrow{O'M}$

The derivative of $\overrightarrow{OM}$ with respect to time, "given that the base $(\vec{\imath}', \vec{\jmath}', \vec{k}')$ can vary in time with respect to R', gives :'

$$\vec{v}_a(t) = \left.\frac{d\overrightarrow{OM}}{dt}\right|_R = \frac{d\left(\overrightarrow{OO'} + \overrightarrow{O'M}\right)}{dt} = \frac{d\overrightarrow{OO'}}{dt} + \frac{dx'}{dt}\vec{\imath}' + \frac{dy'}{dt}\vec{\jmath}' + \frac{dz'}{dt}\vec{k}' + x'\frac{d\vec{\imath}'}{dt} + y'\frac{d\vec{\jmath}'}{dt} + z'\frac{d\vec{k}'}{dt}$$

Rearranging the terms gives :

$$\vec{v}_a(t) = \left[\frac{d\overrightarrow{OO'}}{dt} + x'\frac{d\vec{\imath}'}{dt} + y'\frac{d\vec{\jmath}'}{dt} + z'\frac{d\vec{k}'}{dt}\right] + \left[\frac{dx'}{dt}\vec{\imath}' + \frac{dy'}{dt}\vec{\jmath}' + \frac{dz'}{dt}\vec{k}'\right]$$

$$\vec{v}_a(t) = \vec{v}_e + \vec{v}_r(t)$$

We write :

$$\vec{v}_e = \left[\frac{d\overrightarrow{OO'}}{dt} + x'\frac{d\vec{\imath}'}{dt} + y'\frac{d\vec{\jmath}'}{dt} + z'\frac{d\vec{k}'}{dt}\right]$$

$$\vec{v}_r(t) = \left[\frac{dx'}{dt}\vec{\imath}' + \frac{dy'}{dt}\vec{\jmath}' + \frac{dz'}{dt}\vec{k}'\right]$$

$\vec{v}_e$:The speed (velocity) of movement of the mobile reference frame R' relative to the fixed reference frame R.

$\vec{v}_r(t)$:The relative velocity is the velocity of M with respect to the mobile reference frame R'.

II.4.Composition of acceleration vectors :

The derivative of the absolute velocity vector with respect to time gives the absolute acceleration vector defined in the reference frame R.

$$\overrightarrow{a_a} = \frac{\vec{v}_a(t)}{dt} = \frac{\mathrm{d}}{\mathrm{dt}}\left[\frac{d\overrightarrow{OO'}}{dt} + x'\frac{d\vec{i'}}{dt} + y'\frac{d\vec{j'}}{dt} + z'\frac{d\vec{k'}}{dt}\right] + \left[\frac{dx'}{dt}\vec{i'} + \frac{dy'}{dt}\vec{j'} + \frac{dz'}{dt}\vec{k'}\right]$$

$$\overrightarrow{a_a} = \frac{d^2OO'}{dt} + \left(\frac{dx'}{dt}\frac{d\vec{i'}}{dt} + \frac{dy'}{dt}\frac{d\vec{j'}}{dt} + \frac{dz'}{dt}\frac{d\vec{k'}}{dt}\right) + \left[x'\frac{d^2\vec{i'}}{dt} + y'\frac{d^2\vec{j'}}{dt} + z'\frac{d^2\vec{k'}}{dt}\right]$$
$$+ \left(\frac{d^2x'}{dt}\vec{i'} + \frac{d^2y'}{dt}\vec{j'} + \frac{d^2z'}{dt}\vec{k'}\right) + \left(\frac{dx'}{dt}\frac{d\vec{i'}}{dt} + \frac{dy'}{dt}\frac{d\vec{j'}}{dt} + \frac{dz'}{dt}\frac{d\vec{k'}}{dt}\right)$$

$$\overrightarrow{a_a} = \left(\frac{d^2OO'}{dt} + x'\frac{d^2\vec{i'}}{dt} + y'\frac{d^2\vec{j'}}{dt} + z'\frac{d^2\vec{k'}}{dt}\right) + 2\left(\frac{dx'}{dt}\frac{d\vec{i'}}{dt} + \frac{dy'}{dt}\frac{d\vec{j'}}{dt} + \frac{dz'}{dt}\frac{d\vec{k'}}{dt}\right)$$
$$+ \left(\frac{dx'}{dt}\frac{d\vec{i'}}{dt} + \frac{dy'}{dt}\frac{d\vec{j'}}{dt} + \frac{dz'}{dt}\frac{d\vec{k'}}{dt}\right)$$

$\overrightarrow{a_a} = \overrightarrow{a_e} + \overrightarrow{a_c} + \overrightarrow{a_r}$ such that :

$$\overrightarrow{a_e} = \left(\frac{d^2\overrightarrow{OO'}}{dt} + x'\frac{d^2\vec{i'}}{dt} + y'\frac{d^2\vec{j'}}{dt} + z'\frac{d^2\vec{k'}}{dt}\right)$$ The acceleration of entrainment.

$$\overrightarrow{a_c} = 2\left(\frac{dx'}{dt}\frac{d\vec{i'}}{dt} + \frac{dy'}{dt}\frac{d\vec{j'}}{dt} + \frac{dz'}{dt}\frac{d\vec{k'}}{dt}\right)$$ The Coriolis acceleration.

$$\overrightarrow{a_r} = \left(\frac{dx'}{dt}\frac{d\vec{i'}}{dt} + \frac{dy'}{dt}\frac{d\vec{j'}}{dt} + \frac{dz'}{dt}\frac{d\vec{k'}}{dt}\right)$$ Relative acceleration

- The motion of the reference frame R' can be translational or rotational with respect to the fixed reference frame R. The relative velocity does not change, but the entrainment velocity changes, and consequently, the entrainment acceleration changes, as does the Coriolis acceleration.

➢ ***Case of a reference frame in translational motion :***

The reference frame R' is in translational motion(حركة انسحابية) with respect to R when the unit vectors of reference frame R' do not change over time and maintain the same orientation and direction as those of reference frame R : $\vec{i} = \vec{i'}; \vec{j} = \vec{j'}\,; \vec{z} = \vec{z'}$

$$\frac{d\vec{i'}}{dt} = \frac{d\vec{j'}}{dt} = \frac{d\vec{k'}}{dt} = 0$$

The entrainment velocity becomes:

$$\vec{v_e} = \frac{d\overrightarrow{OO'}}{dt}$$

The training acceleration becomes :

$$\vec{a_e} = \frac{d^2\overrightarrow{OO'}}{dt}$$

And the Coriolis acceleration cancels out : $\vec{a_c} = 0$

Note:

The Coriolis acceleration cancels out if the mobile reference frame R' is in translational motion with respect to the fixed reference frame R or if the mobile M is stationary with respect to the mobile reference frame R'.

يحذف تسارع كوريوليس إذا كان الإطار المرجعي المتحرك R' في حركة انتقالية بالنسبة إلى الإطار المرجعي الثابت R أو إذا كان المتحرك M ثابتًا بالنسبة إلى الإطار المرجعي المتحرك R’

➢ ***Case of a reference frame in rotational motion without translation.***

We assume that the reference frame R' is rotating about the z-axis with an angular velocity $\vec{\omega}_{R'/R} = \omega\vec{k}$ and we consider that O≡O′ .Any vector in rotation about a perpendicular axis has its time derivative as the vector product of its angular velocity $\vec{\omega}$ and the rotating vector.

$$\frac{d\vec{i'}}{dt} = \vec{\omega} \times \vec{i'} \ ; \ \frac{d\vec{j'}}{dt} = \vec{\omega} \times \vec{j'} \ , \frac{d\vec{k'}}{dt} = \vec{\omega} \times \vec{k'}$$

The entrainment velocity becomes:

$$\vec{v}_e = \frac{d\overrightarrow{OO'}}{dt} + \left[x'\frac{d\vec{i'}}{dt} + y'\frac{d\vec{j'}}{dt} + z'\frac{d\vec{k'}}{dt}\right]$$

$$\vec{v}_e = \frac{d\overrightarrow{OO'}}{dt} + x'\left(\vec{\omega} \times \vec{i'}\right) + y'\left(\vec{\omega} \times \vec{j'}\right) + z'(\vec{\omega} \times \vec{k'})$$

$$\vec{v}_e = \frac{d\overrightarrow{OO'}}{dt} + \left(\vec{\omega} \times x'\vec{i'}\right) + \left(\vec{\omega} \times y'\vec{j'}\right) + (\vec{\omega} \times z'\vec{k'})$$

$$\vec{v}_e = \frac{d\overrightarrow{OO'}}{dt} + (\vec{\omega} \times \overrightarrow{O'M})$$

For training acceleration:

We have :

$$\frac{d\vec{\imath}'}{dt} = \vec{\omega} \times \vec{\imath}'$$

Hence :

$$\frac{d^2\vec{\imath}'}{dt^2} = \left(\frac{d\vec{\omega}}{dt} \times \vec{\imath}'\right) + \left(\vec{\omega} \times \frac{d\vec{\imath}'}{dt}\right)$$

$$\vec{a}_e = \frac{d^2\overrightarrow{OO'}}{dt} + x'\frac{d^2\vec{\imath}'}{dt} + y'\frac{d^2\vec{\jmath}'}{dt} + z'\frac{d^2\vec{k}'}{dt}$$

So :

$$\vec{a}_e = \frac{d^2\overrightarrow{OO'}}{dt} + x'\left[\left(\frac{d\vec{\omega}}{dt} \times \vec{\imath}'\right) + \left(\vec{\omega} \times \frac{d\vec{\imath}'}{dt}\right)\right] + y'\left[\left(\frac{d\vec{\omega}}{dt} \times \vec{\jmath}'\right) + \left(\vec{\omega} \times \frac{d\vec{\jmath}'}{dt}\right)\right] + z'\left[\left(\frac{d\vec{\omega}}{dt} \times \vec{k}'\right) + \left(\vec{\omega} \times \frac{d\vec{k}'}{dt}\right)\right]$$

$$\vec{a}_e = \frac{d^2\overrightarrow{OO'}}{dt} + \left[\left(\frac{d\vec{\omega}}{dt} \times x'\vec{\imath}'\right) + \left(\vec{\omega} \times x'\frac{d\vec{\imath}'}{dt}\right)\right] + \left[\left(\frac{d\vec{\omega}}{dt} \times y'\vec{\jmath}'\right) + \left(\vec{\omega} \times y'\frac{d\vec{\jmath}'}{dt}\right)\right] + \left[\left(\frac{d\vec{\omega}}{dt} \times z'\vec{k}'\right) + \left(\vec{\omega} \times z'\frac{d\vec{k}'}{dt}\right)\right]$$

$$\vec{a}_e = \frac{d^2\overrightarrow{OO'}}{dt} + \left(\frac{d\vec{\omega}}{dt} \times \overrightarrow{O'M}\right) + (\vec{\omega} \times (\vec{\omega} \times \overrightarrow{O'M}))$$

$\frac{d^2\overrightarrow{OO'}}{dt}$: Represents the translational motion of R'/R.

$(\vec{\omega} \times (\vec{\omega} \times \overrightarrow{O'M}))$:Represents the rotational movement of R'/R.

$\left(\frac{d\vec{\omega}}{dt} \times \overrightarrow{O'M}\right)$:Involved only if rotation is non-uniform.

For Coriolis acceleration :

$$\vec{a_c} = 2\left(\frac{dx'}{dt}\frac{d\vec{\imath}'}{dt} + \frac{dy'}{dt}\frac{d\vec{\jmath}'}{dt} + \frac{dz'}{dt}\frac{d\vec{k}'}{dt}\right)$$

$$\vec{a_c} = 2\,\frac{dx'}{dt}(\vec{\omega} \times \vec{\imath}') + \frac{dy'}{dt}\,(\vec{\omega} \times \vec{\jmath}') + \frac{dz'}{dt}(\vec{\omega} \times \vec{k}')$$

$$\vec{a}_c = 2\vec{\omega} \times \frac{dx'}{dt}\vec{\imath}' + \frac{dy'}{dt}\vec{\jmath}' + \frac{dz'}{dt}\vec{k}'$$

$$a_c = 2(\omega \times \vec{v}_r)$$

Exercice01 :

Let be a moving reference frame R'(o',x',y',z') in motion relative to a fixed one R(o,x,y,z) with speed Ve (1,0,0). Assume that x',y' and z' are the coordinates of a material point M in the frame R' such that :

$$(x' = 6t^2 3t \quad ; \quad y' = -3t^2 \quad ; z' = 3)$$

we assume that at time t=0 this point is at position O(0,0,0) in the fixed reference frame R.

1. Give the relative speed of this point and its absolute speed.
2. Deduce the coordinates of point M in the fixed reference frame R.
3. Determine the expression for relative and absolute acceleration.

Solution :

1. **Relative and Absolute Speeds**

Relative Velocity ($\vec{V}'_{rel}$) in R':

The velocity of M in R' is the time derivative of its coordinates in R':

$$\vec{V}'_{rel} = \frac{d}{dt}(x', y', z').$$

Compute each component:

$$\frac{dx'}{dt} = \frac{d}{dt}(6t^2 + 3t) = 12t + 3,$$

$$\frac{dy'}{dt} = \frac{d}{dt}(-3t^2) = -6t,$$

$$\frac{dz'}{dt} = \frac{d}{dt}(3) = 0.$$

Thus:

$$\vec{V}'_{rel} = (12t + 3)\hat{\imath} + (-6t)\hat{\jmath} + 0\hat{k}.$$

Absolute Velocity ($\vec{V}_{abs}$) in R:

The absolute velocity of M in R is the sum of its velocity relative to R' and the velocity of R' relative to R:

$$\vec{V}_{abs} = \vec{V}'_{rel} + \vec{V}_e.$$

Substitute:

$$\vec{V}_e = (1,0,0), \quad \vec{V}'_{rel} = \big((12t + 3), -6t, 0\big).$$

$$\vec{V}_{abs} = (12t + 3 + 1)\hat{\imath} + (-6t)\hat{\jmath} + 0\hat{k}.$$

$$\vec{V}_{abs} = (12t + 4)\hat{\imath} + (-6t)\hat{\jmath} + 0\hat{k}.$$

2. Coordinates of M in$R$:

The coordinates of M in R are found by adding the displacement due to the velocity of R' to the coordinates of M in R'. At time t, the displacement of R' in R is given by $x_{R'} = V_e t$ along the x-axis.

Thus, the coordinates of M in R are:

$$x = x' + V_e t, \quad y = y', \quad z = z'.$$

Substitute $x' = 6t^2 + 3t$, $y' = -3t^2$, $z' = 3$, and $V_e = 1$:

$$x = (6t^2 + 3t) + t = 6t^2 + 4t,$$

$$y = -3t^2,$$

$$z = 3.$$

Thus, the coordinates of M in R are:

$$(x, y, z) = (6t^2 + 4t, -3t^2\,; 3).$$

3. **Relative and Absolute Accelerations**

Relative Acceleration ($\vec{a}'_{rel}$) in R':

The relative acceleration is the time derivative of the relative velocity:

$$\vec{a}'_{rel} = \frac{d}{dt}\vec{V}'_{rel}.$$

Compute each component:

$$\frac{d}{dt}(12t + 3) = 12, \quad \frac{d}{dt}(-6t) = -6, \quad \frac{d}{dt}(0) = 0.$$

$$\vec{a}'_{rel} = 12\hat{\imath} - 6\hat{\jmath} + 0\hat{k}.$$

Absolute Acceleration ($\vec{a}_{abs}$) in R:

The absolute acceleration is the same as the relative acceleration because the velocity of R' is constant (no acceleration of R'):

$$\vec{a}_{abs} = \vec{a}'_{rel}.$$

$$\vec{a}_{abs} = 12\hat{\imath} - 6\hat{\jmath} + 0\hat{k}.$$

Exercice02 :

A moving body is described by the position vector in a moving frame of reference R' by :

$\overrightarrow{O'M} = 5t\overrightarrow{i'} + (2t^2 - t)\overrightarrow{j'} - 2t\overrightarrow{k'}$ .This frame of reference is in rectilinear translational motion relative to a fixed frame of reference R, with velocity vector : $\overrightarrow{v_e} = 2t\vec{i} + \vec{j} + \vec{k}$

1.Find the expression for the absolute velocity of M with respect to the reference frame R. 2.Then, deduce the position vector of M in the fixed frame R, knowing that at time t=0, M is at the point (0,1,−2) in frame R.

3. Calculate the relative $\overrightarrow{a_r}$ and absolute acceleration $\overrightarrow{a_a}$ of M.

Chapter V : Dynamics of the material point

I.Definition

Dynamics is the study of motion in relation to the causes that produce it.

This chapter is devoted to dynamics, the relationships that exist between a movement and the forces that cause it.

II.Notion of force :

A material point is in motion because of interactions between the particle and its environment, which is subject to them. These interactions are called forces. These forces depend on the nature of the particle and the nature of its environment.

-Its origin is the point of contact (force/body).

- Its direction: is the direction of the movement supported on the wire, in the case of the force: wire tension, for example.

- Its modulus: is the value of the force in Newton (N)

The force exerted on any body is the vector sum of all the forces applied to it.

III. Inertia principle :(مبدا العطالة)

Galileo was the first to establish the principle of inertia. This principle states, "If a body is not subjected to any force, then it either continues its motion in a straight line at the same speed or remains at rest if it was already at rest."

The law of inertia states, "Any free or isolated body moves in a straight line at a constant speed." This is the law of inertia, or Newton's first law (1678).

The principle of inertia is valid for a Galilean reference frame (system) in which any free body moves at a constant speed.

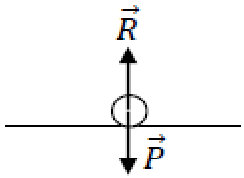
The terrestrial reference frame is suitable for studying a body as long as accelerations are negligible compared to terrestrial accelerations.

Example:

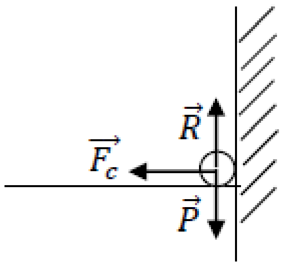
We throw a ball on the ground and study its motion without taking frictional forces into account. The ball is in uniform rectilinear motion.

If we study the forces exerted on the ball, we have : $\vec{P} + \vec{R} = \vec{0}$

$$\vec{P} + \vec{R} = \vec{0}$$



If the ball hits a wall (receives an obstacle), the ball is subjected to another force because its movement changes. In this case, we have : $\vec{P} + \vec{R} + \vec{F_c} \neq \vec{0}$



The ball changes direction due to the force of contact of the ball with the wall $\overrightarrow{Fc}$.
On the other hand, if the ball is at rest, it remains at rest if $\Sigma\vec{F} = \vec{0}$.

IV.Mass concept (مفهوم الكتلة)

We all know that the greater the mass of a body, the harder it is to change its velocity vector or alter its movement (its direction). "It is easier for a person to move a table than to move a wardrobe."
Mass is a scalar physical quantity that represents the amount of matter that makes up a particle and reflects the inertia of the body.

V.Amount of motion (كمية الحركة)

The momentum is defined as the product of the mass of a body and its velocity $\vec{p} = m\vec{v}$, is a vector quantity that has the same direction as velocity. It's an important physical concept because it unites two elements describing motion: mass and velocity.
Note that the direction of the momentum vector is the same as the direction of the velocity vector.
For a system of N particles, the total momentum is : $\vec{p} = \sum_{i=1}^{n} \vec{p}_i$

VI.Principle of conservation of momentum

To verify this principle, we conduct a simple experiment. A body M_1 interacts with another body M_2, and we neglect any interaction with the external environment on both bodies (isolated system).

Experience :

We have two carts sliding on a smooth track. The masses of the two carts are m_1=2kg and m_2=3 kg. The first cart is pushed with an initial velocity v_1=4 m/s, while the second cart is stationary with v_2=0 m/s.

- **Momentum Before Collision**

To calculate momentum, we use the formula: $\overrightarrow{p_1} = m_1\overrightarrow{v_1} = 8kg.m.s^{-1}$; $\overrightarrow{p_2} = m_2\overrightarrow{v_2} = 0kg.m.s^{-1}$ so the **total momentum before collision:**

$$p_{total\ befor} = p_1 + p_2 = 8kg.m.s^{-1}$$

After the collision, the velocity of the first cart became 0.5m/s, while the velocity of the second cart became 2.33 m/s.

- **Momentum after collision**

$$p_{total\ after} = p'_1 + p'_2 = 2.05 + 3.2.338 = 8kg.m.s^{-1}$$

Before, during, and after the interaction, we observe that the momentum of the system remains constant.

$$\overrightarrow{p_1} + \overrightarrow{p_2} = \overrightarrow{p'_1} + \overrightarrow{p'_2}\ of\ the\ \overrightarrow{p_1} - \overrightarrow{p'_1} = \overrightarrow{p'_2} - \overrightarrow{p_2} = -(\overrightarrow{p_2} - \overrightarrow{p'_2})$$

This leads to :$\Delta\vec{p}_1 = -\Delta\vec{p}_2$

They represent the changes in the momentum of the two bodies.

The momentum lost by one body is gained by the other.

Note: This principle of conservation of momentum applies only to an isolated system..

VII.Dynamics of a body (حركية الجسم)

VII.1.Strength (force) :

Schematically, a force is a factor that can change the speed of an object. Changing the speed can mean giving it a speed if it was initially zero or altering either its magnitude, its direction, or both simultaneously.

Regardless of their nature, and regardless of how they manifest (at a distance or through contact between two bodies), forces (such as the weight of a body) are vector quantities. Therefore, each time we consider a force, we must determine its direction.

From the previous equation:$\Delta\vec{p}_1 = -\Delta\vec{p}_2$ dividing the two terms by the interval of the time Δt we find :$\frac{\Delta\vec{p}_1}{\Delta t} = -\frac{\Delta\vec{p}_2}{\Delta t}$ the average variations of the momentum vectors during the interval are equal and opposite in direction. Quantity $\frac{\Delta\vec{p}_1}{\Delta t}$ is called the average force acting on the body over the time interval Δt , we write :$\vec{F}_m = \frac{\Delta\vec{p}_1}{\Delta t}$ if the time interval $\Delta t$is very small $\Delta t \to 0$we define the instantaneous force.

$$\vec{F}_i = \lim_{\Delta t \to 0} \vec{F}_m = \lim_{\Delta t \to 0} \frac{\Delta\vec{p}}{\Delta t} = \frac{d\vec{p}}{dt}$$

VII.2.Newton's laws :

VII.2.1.Fundamental relationship of dynamics (Newton's 2nd law) :المبدأ الأساسي للتحريك

The force applied to a body :

$$\vec{F}_i = \frac{d\vec{p}}{dt} = \frac{d(m\vec{v})}{dt}$$

If mass does not vary with velocity, then :

$$\vec{F}_i = \frac{d\vec{p}}{dt} = \frac{d(m\vec{v})}{dt} = m\frac{d\vec{v}}{dt} = m\vec{a}$$

The force applied to a body is the resultant of the reactions due to all the forces. $\vec{F}_1$; $\vec{F}_2$; $\vec{F}_3$; ... $\vec{F}_n$

$$\vec{F} = \sum \vec{F}_i = m\vec{a}$$

Relation Fondamentale de la Dynamique (RFD ou PFD)

Note:

Newton's law defines the differential equations of motion.

$$\vec{F} = \sum \vec{f}_i = m\vec{a} = m\frac{d\vec{v}}{dt} = m\frac{d^2\vec{r}}{dt^2}$$

$$\vec{F} = \sum \vec{f}_i = m\vec{a} = \begin{cases} F_X \\ F_Y \\ F_Z \end{cases} = m\begin{cases} \frac{dv_x}{dt} \\ \frac{dv_y}{dt} \\ \frac{dv_z}{dt} \end{cases} = \begin{cases} \frac{d^2x}{dt^2} \\ \frac{d^2y}{dt^2} \\ \frac{d^2z}{dt^2} \end{cases}$$

VII.2.2.Newton's first law: the law of inertia(القانون الأول لنيوتن-مبدأ العطالة)

If a body is free, isolated or moving at constant speed : $a = 0 \rightarrow \vec{F} = m\vec{a} = 0$

If the force applied to a body is zero $\vec{F} = \vec{0} \rightarrow \vec{a} = \vec{0}$ the body has uniform motion (constant speed) is the law of inertia (Newton's 1st law).

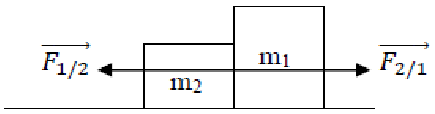
VII.2.3.Newton's third law (action and reaction)مبدأ الفعل ورد الفعل:

When two bodies interact, based on the principle of conservation of momentum

$$\frac{\Delta\vec{p}_1}{\Delta t} = -\frac{\Delta\vec{p}_2}{\Delta t} \rightarrow \frac{d\vec{p}_1}{dt} = -\frac{d\vec{p}_2}{dt} \rightarrow \vec{F}_{12} = -\vec{F}_{21}$$

$\vec{F}_{12}$: Force applied by body 1 to body 2**.**

$\vec{F}_{21}$:Force applied by body 2 to body 1.



$$\overrightarrow{F_{1/2}} = -\overrightarrow{F_{2/1}}$$

V.III.Fundamental interactions (التفاعلات الاساسية)

Different types of forces act on bodies.

The main forces result from fundamental interactions.

V.III.1.Electromagnetic interaction :

Resultant force of a magnetic field and a moving charge "Laplace-Lorentz force

$$\vec{F} = q\vec{E} + q\vec{v} \times \vec{B}$$

V.III.2.Strong nuclear interaction :

The feeble interaction manifests itself in certain types of nuclear reaction, such as radioactivity.

There are two types of forces:

V.III.3.Forces of distance interaction (the actor and the receiver are not in contact): examples include gravitational forces, electromagnetic forces, and nuclear cohesion forces.

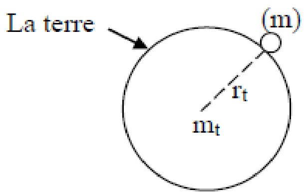
V.III.3.1.Gravitational interaction force " Newton in 1650 " : It's a force of attraction between all particles. Like bodies attracted to the Earth in its vicinity.

It is an interaction force between two masses. Two masses m and M attract each other, separated by a distance r.

The force of attraction is given by :

$$F = G\frac{mM}{r^2}$$

- ***On the earth's surface :***



m_t and r_t are the mass and radius of the earth respectively, such that :

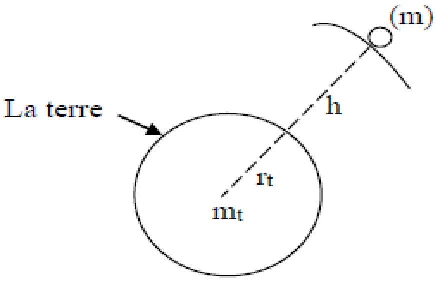
$$m_t = 5.98.10^{24}kg \ ; r_t = 6.37.10^6 m$$

$$F_g = mg = G\frac{mm_t}{r^2}$$

Gravitation g is : $g = G\frac{m_t}{r^2}$

Numerical application: g=9.8N/kg

- ***At a height h from the earth***



$$F_{g'} = mg' = G\frac{mm_t}{r^2}$$

Such as : $r = r_t + h$

Where : $g' = g\frac{r_t^2}{r^2}$

Mass weight (ثقل كتلة)

The weight of a mass represents its gravitational interaction with the earth.$\vec{p} = m\vec{g}$

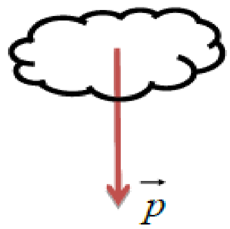


Figure (V.1) : le poids d'une masse.

V.III.4.Linking or contact forces :(قوى الترابط أو التلامس)

A contact force is any force that occurs when two plane surfaces come into contact.

V.III.4.1. Normal Force ($\vec{N}$)

If an object placed on a table, the table exerts an upward action force $\vec{N}$ (normal force) on the object. The reaction of the table ($\vec{N}$) on the object m is distributed over the entire table-object contact surface and represents the resultant of all the actions exerted on the contact surface. The normal force $\vec{N}$ is the force that prevents the object from falling through the table, and can have any value up to the point of breaking the table.

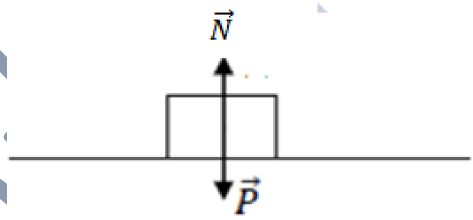


Figure (V.2) : normal Force.

$$\vec{P} = m\vec{g}$$

The object is in equilibrium: $\vec{p} + \vec{N} = \vec{0} \rightarrow \vec{p} = -\vec{N}$

V.III.4.2.Forces of friction :

The forces of friction are forces that occur when an object moves. Friction opposes the movement of moving objects. There are two types of friction:

- solid friction (solid-solid contact).

- viscous friction (solid-fluid contact).

V.III.4.2.1.Friction forces (قوى الاحتكاك)-(solid-solid contact)

The solid object, placed on a solid support, is in motion under the action of the driving force $\vec{F_e}$. A force opposite to the driving force $\vec{F_e}$. will appear and will slow down the motion; this force called the force of friction (frictional force) or simply friction$\overrightarrow{f_f}$. It is always in the opposite direction to the motion (velocity). **When the friction is ignored, the surface is said to be *frictionless***

A distinction is made between solid and fluid friction forces and between static and dynamic friction forces:

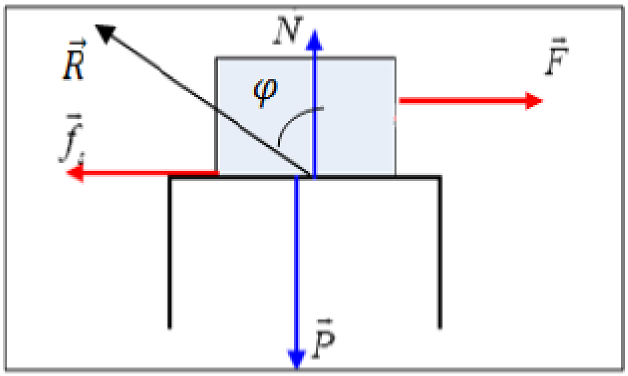
a-Static friction forces :(قوى الاحتكاك في حالة السكون)

Example: A body resting on a horizontal plane :

Consider the body shown in the figure below. It is subjected to four forces.

Let f_s , be the static friction force and $\vec{p}$ and $\vec{N}$ be the weight and normal reaction force of the support respectively.

For the body on the table to move, a minimum force $\vec{F}$ must be applied.



The mass remains stationary as long as F<fs, there is resistance to movement.

In this case the reaction of the support is the resultant force given by :$\vec{R} = \vec{N} + \overrightarrow{f_s}$

At equilibrium:$\sum \vec{F} = \vec{0} \rightarrow \vec{N} + \vec{f_s} + \vec{F} + \vec{P} = \vec{0} \rightarrow \vec{R} + \vec{F} + \vec{P} = \vec{0}$

By projecting onto the two axes Ox and Oy :

On Oy : P=N et sur Ox : F=f_s

The mass starts moving when F>f_s

Experience shows that the ratio (f_s/N) is constant.

$$\tan\varphi = \frac{f}{N} = k = \mu$$

μ: is the coefficient of friction and φ is the angle of friction.

The coefficient of friction is called static when the body is stationary. The coefficient of static friction is a ratio between the static frictional force of an object and the normal force, and is written as follows :

$$\mu_s = \frac{f_s}{N} = \tan\varphi$$

- **Definition of the Static Friction :مفهوم الاحتكاك السكوني**

The static friction coefficient is a physical property associated with two contacting surfaces, and it determines the amount of force required to overcome static friction and initiate relative motion between the surfaces. In other words, it is a measure of how difficult it is to start moving an object on a given surface.

معامل الاحتكاك السكوني هو خاصية فيزيائية ترتبط بسطحين متلامسين، وتحدد مقدار القوة اللازمة للتغلب على الاحتكاك السكوني وبدء الحركة النسبية بين السطحين. بعبارة أخرى، هو مقياس لمدى صعوبة بدء حركة جسم على سطح ما.ويعطى بالعلاقة التالية:

$$\mu_s = \frac{f_s}{N} = \tan\varphi$$

f_s :Static friction force.

μ_s: is the coefficient of static friction.

N : is the normal force between the surfaces, and it is often equal to the weight of the object on a horizontal surface.

b-Dynamic friction forces (قوى الاحتكاك في حالة الحركة)

Kinetic or dynamic friction is the frictional force present when an object is in motion on another object.

The dynamic friction coefficient is a ratio between the dynamic friction force of an object and the normal force.

Mass starts moving when $F>f_d$.

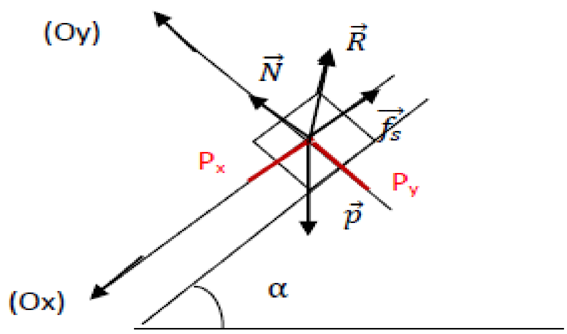
The coefficient of dynamic friction is written as :

$$\tan\varphi = \mu_k = \frac{f_k}{N}$$

Application exercise:

Consider a small block(كتلة صغيرة) of mass m abandoned without initial velocity at point **A** of an inclined plane(مستو مائل) at an angle α=30° to the horizontal. Point **A** is at height h.

1- What is the value of the coefficient of static friction μs that keeps the mass in equilibrium at point **A**.



Corrected:

At equilibrium: $\sum \vec{F}_{ext} = \vec{0} \Rightarrow \vec{N} + \vec{P} + \vec{f_s} = \vec{R} + \vec{P}$

Following (Ox) : $-f_s + p_x = 0 \Rightarrow f_s = p_x = p \sin\alpha$

Following (Oy) : $-N + p_y = 0 \Rightarrow N = p_y = p \cos\alpha$

In order for the body to remain stationary on the plane, the following conditions must be met

$$f_s > p_x$$

We have :

$$\mu_s = \frac{f_s}{N} = \tan\varphi = \frac{mg \sin\alpha}{mg \cos\alpha}$$

The maximum value that the coefficient of static friction μs can take is tg α.

Note: experience shows that $\mu_s \geq \mu_d$

The values of the dimensionless coefficients μ_s s and μ_k c depend on the nature of the surfaces, not on their areas.

When we apply a force $\overrightarrow{F_e}$ on the object to move it to the right, the object will remain immobile if $\overrightarrow{F_e}$ is not large enough. The frictional force $\overrightarrow{f_f}$ acts to the left and **keeps the object immobile**. We call this frictional force the **force of static friction**. If we increase $\overrightarrow{F_e}$, the static frictional force increases $\overrightarrow{f_s}$, while the object remains at rest. When the applied force $\overrightarrow{F_e}$, **reaches a certain value**, the object will be on the verge of slipping and the frictional force will be maximum, $\overrightarrow{f_{fmax}}$. When $\overrightarrow{F_e}$ **exceeds f_{smax}, the object moves to the right**. When the object is in motion, the frictional force becomes less than $f_{s,max}$ and is called the **force of kinetic friction** f_k.

Exercise 1

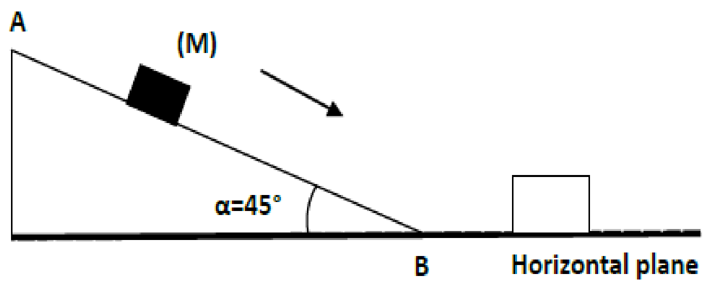
A block (M) of mass m is thrown from the top of an inclined plane AB=1m at an angle α=45° to the horizontal, with initial velocity v_A=1m/s.

1- Knowing that the coefficient of friction μ=0.5 on AB.

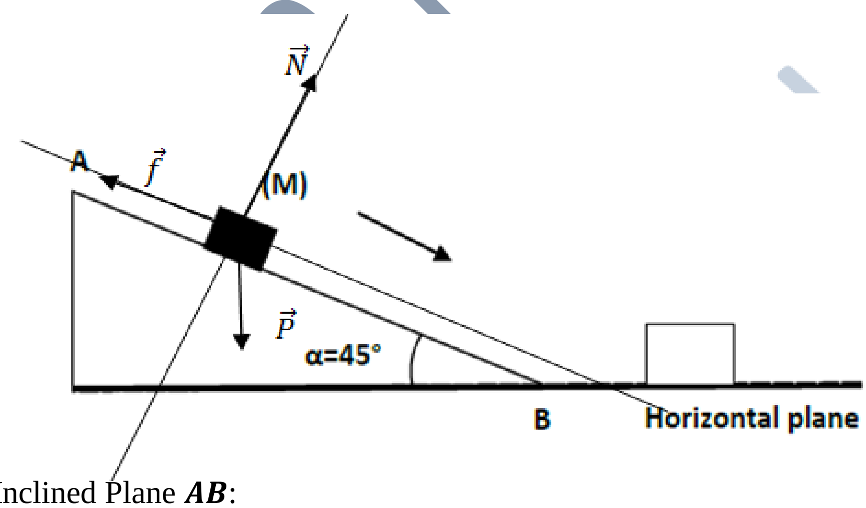
- Demonstrate, what is the nature of the motion on AB?
- Calculate the speed of (M) when it reaches point B.

2- Friction forces are considered negligible on the horizontal plane:

- Demonstrate the nature of the motion on the horizontal plane.
- Will the block (M) stop? Justify your answer.



<u>Solution :</u>



1. Motion on the Inclined Plane $\boldsymbol{AB}$:

Known data:

- Length of the inclined plane: $AB = 1\,\text{m}$
- Angle of inclination: $\alpha = 45°$
- Initial velocity: $v_A = 1\,\text{m/s}$
- Coefficient of friction: $\mu = 0.5$
- Acceleration due to gravity: $g = 9.8\,\text{m/s}^2$

(a) Nature of the Motion on AB:

To determine the nature of motion, calculate the net force acting on the block along the inclined plane.

3. **Gravitational force along the plane (F_{gravity}):**

$$F_{\text{gravity}} = mg\sin\alpha$$

Substituting $g = 9.8\,\text{m/s}^2$ and $\sin 45° = \frac{\sqrt{2}}{2}$:

$$F_{\text{gravity}} = m \cdot 9.8 \cdot \frac{\sqrt{2}}{2} = 6.93\,m\,\text{N}$$

4. **Frictional force (F_{friction}):**

$$F_{\text{friction}} = \mu \cdot F_{\text{normal}}$$

The normal force on an inclined plane is $F_{\text{normal}} = mg\cos\alpha$. Substituting $\cos 45° = \frac{\sqrt{2}}{2}$:

$$F_{\text{friction}} = 0.5 \cdot m \cdot 9.8 \cdot \frac{\sqrt{2}}{2} = 3.47\,m\,\text{N}$$

5. **Net force (F_{net}) along the incline:**

$$F_{\text{net}} = F_{\text{gravity}} - F_{\text{friction}} = 6.93\,m - 3.47\,m = 3.46\,m\,\text{N}$$

6. **Acceleration (a) along the incline:** Using Newton's second law ($F = ma$):

$$a = \frac{F_{\text{net}}}{m} = \frac{3.46\,m}{m} = 3.46\,\text{m/s}^2$$

Since the acceleration is positive, the block moves down the incline with increasing velocity.

Thus, the motion is uniformly **accelerated**.

(b) Speed of the Block at Point B:

The block starts at $v_A = 1\,\text{m/s}$ and travels a distance $AB = 1\,\text{m}$ with acceleration $a = 3.46\,\text{m/s}^2$.

Using the kinematic equation:

$$v_B^2 = v_A^2 + 2ad$$

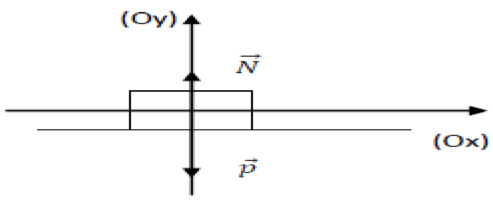
Substitute the values:

$$v_B^2 = (1)^2 + 2 \cdot 3.46 \cdot 1 = 1 + 6.92 = 7.92$$

$$v_B = \sqrt{7.92} \approx 2.81\,\text{m/s}$$

2. Motion on the Horizontal Plane:

(a) Nature of Motion:



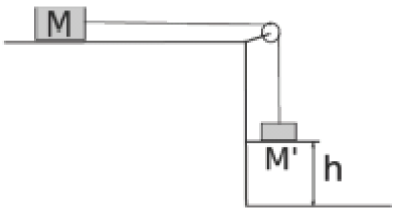
On the horizontal plane, friction forces are negligible, and no other horizontal forces act on the block. Hence, there is **no net force** along the horizontal plane. According to Newton's first law, the block will continue moving with a **constant velocity**.

(b) Will the Block Stop?

Since there is no opposing force to decelerate the block, it will **not stop** and will continue moving indefinitely with the velocity it had at point B, which is $v_B = 2.81\,\mathrm{m/s}$.

Exercise 2

Two bodies M and M', with masses m and m' respectively, are connected by an inextensible string passing over a pulley of negligible mass. Initially, the body M' is at a height h above the ground and is released without any initial velocity. The contact between the body M and the horizontal plane is characterized by static and kinetic friction coefficients μ_s and μ_k, respectively. $\mu_s = 0.6$, $\mu_k = 0.4$, $m = 6\,\mathrm{kg}$, $h = 1.5\,\mathrm{m}$, and $g = 10\,\mathrm{m/s^2}$.



1- Give the expression for the mass m'min for the system to start moving, as a function of m and µs.

2) We now take a mass $m' = 4\,\mathrm{kg}$, and the system starts moving. Considering the two phases of the movement of mass M until it comes to a stop:

a) What is the nature of the movement of mass M? Justify.

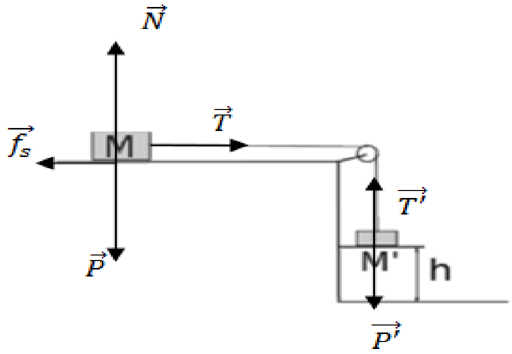
b) Calculate the acceleration in the first phase.

c) Deduce the velocity at the end of this phase.

d) Calculate the acceleration in the second phase.

e) Deduce the total distance D traveled by mass M. Provide its value.

Solution:



Step 1: Expression for $\boldsymbol{m'_{\min}}$ to start moving

To determine the minimum mass $m'_{\min}$ for the system to start moving, we need to analyze the forces acting on M and M'.

For block $\boldsymbol{M}$*:*

- Weight: $W_M = mg$
- Normal force: $N = mg$
- Static friction: $f_s = \mu_s N = \mu_s mg$

For block $\boldsymbol{M'}$*:*

- Weight: $W_{M'} = m'g$

For the system to start moving, the force exerted by $m'g$ must overcome the maximum static friction acting on M. This gives:

$$m'g > \mu_s mg$$

Cancel g from both sides:

$$m' > \mu_s m$$

Thus, the minimum mass is:

$$m'_{\min} = \mu_s m$$

Substitute values:

$$m'_{\min} = 0.6 \cdot 6 = 3.6\,\text{kg}$$

So, $m'_{\min} = 3.6\,\text{kg}$

Step 2: When $\boldsymbol{m'} = \boldsymbol{4}\,\text{kg}$

Since $m' = 4\,\text{kg} > m'_{\min} = 3.6\,\text{kg}$, the system starts moving. We now analyze the two phases of motion.

Phase 1: $\boldsymbol{M'}$ descends to the ground

(a) Nature of the motion:

- The motion is uniformly accelerated because:
  - The force due to $m'g$ exceeds the kinetic friction acting on M.

- Once the system starts moving, the frictional force transitions to kinetic friction ($f_k = \mu_k mg$), which is constant.

- The net force on the system remains constant, leading to a constant acceleration.

(b) Calculate acceleration in the first phase

We analyze the net force on the system to calculate acceleration. Let the tension in the string be T.

For M:

$$T - f_k = ma$$

$$T - \mu_k mg = ma$$

For M':

$$m'g - T = m'a$$

From these two equations, solve for a by eliminating T:

$$T = \mu_k mg + ma$$

$$m'g - (\mu_k mg + ma) = m'a$$

$$m'g - \mu_k mg = (m + m')a$$

Solve for a:

$$a = \frac{m'g - \mu_k mg}{m + m'}$$

Substitute values:

$$a = \frac{4 \cdot 10 - 0.4 \cdot 6 \cdot 10}{6 + 4}$$

$$a = \frac{40 - 24}{10} = \frac{16}{10} = 1.6\,\mathrm{m/s^2}$$

So, the acceleration is:

$$a = 1.6\,\mathrm{m/s^2}$$

(c) Velocity at the end of this phase

We use the kinematic equation:

$$v^2 = u^2 + 2as$$

Where:

- $u = 0$ (initial velocity),
- $a = 1.6\,\mathrm{m/s^2}$,
- $s = h = 1.5\,\mathrm{m}$.

Substitute:

$$v^2 = 0 + 2 \cdot 1.6 \cdot 1.5$$

$$v^2 = 4.8$$

$$v = \sqrt{4.8} \approx 2.19 \, \text{m/s}$$

The velocity at the end of this phase is:

$$v \approx 2.19 \, \text{m/s}$$

Phase 2: After $\boldsymbol{M'}$ hits the ground

When M' hits the ground, the tension in the string disappears, and M continues moving under the effect of kinetic friction until it comes to a stop.

(d) Calculate acceleration in the second phase

The only force acting on M is the kinetic friction, which opposes its motion. Thus, the acceleration is:

$$a = -\frac{f_k}{m}$$

Substitute $f_k = \mu_k mg$:

$$a = -\frac{\mu_k mg}{m}$$

Simplify:

$$a = -\mu_k g$$

Substitute values:

$$a = -0.4 \cdot 10 = -4 \, \text{m/s}^2$$

The acceleration is:

$$a = -4 \, \text{m/s}^2$$

(e) Total distance traveled by $\boldsymbol{M}$

To find the total distance D, we calculate the distance traveled during the second phase after M stops.

Velocity at the start of the second phase:

From phase 1, $v = 2.19 \, \text{m/s}$

Distance in the second phase:

Use the kinematic equation:

$$v^2 = u^2 + 2as$$

Where:

- $v = 0$ (final velocity),

- $u = 2.19\ \text{"}m/s\text{"}$,
- $a = -4$ 〖"m/s" 〗^2.

Solve for s:

$$0 = (2.19)^2 + 2 \cdot (-4) \cdot s$$

$$(2.19)^2 = 8s$$

$$s = \frac{(2.19)^2}{8} = \frac{4.8}{8} = 0.6\ \text{m}$$

The distance in the second phase is:

$$s = 0.6\ \text{m}$$

Total distance D:

The total distance traveled by M is:

$$D = \text{distance in phase 1} + \text{distance in phase 2}$$

$$D = h + s = 1.5 + 0.6 = 2.1\ \text{m}$$

V.III.4.2.2.Tension forces :قوة التوترأو القوة الارجاعية

When a cord (or spring) is attached to a body and pulled(سحب.), the cord is said to be in **tension**. The tension in the cord is defined as the force that the cord exerts on the body. This force is denoted usually by the symbol $\vec{T}$. A cord is considered to be *massless* (i.e., its mass is negligible compared to the body's mass) and *non-stretchable(غير قابل للتمدد).*

Example: Spring

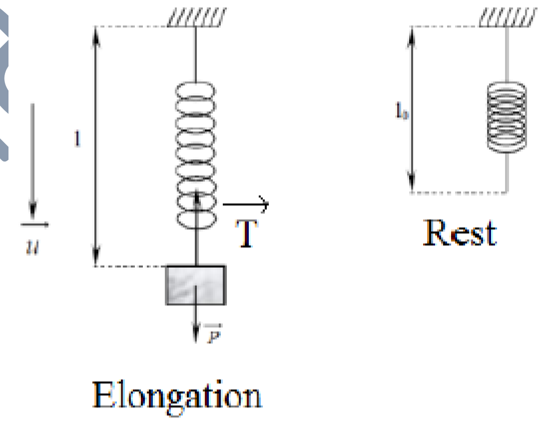


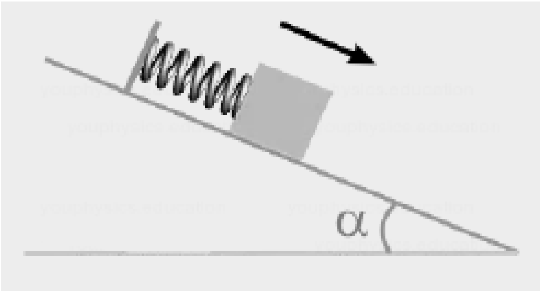
Figure (V.3) : Elastic force: Spring tension.

$$\vec{T} = -k(l - l_0)\vec{u} = -kx\vec{u}$$

k: coefficient of elongation (spring stiffness coefficient).

Exercise :

A mass m = 15 kg suspended from a spring of stiffness K = 100N / m descends along an inclined plane which makes an angle α = 30º with the horizontal. Assuming there is no friction, determine the normal reaction of the support and the acceleration of the mass when the spring is stretched by a length x = 0.02m.



Given:

7. Mass $m = 15$ kg.
8. Spring stiffness $K = 100\ \mathrm{N/m}$.
9. Inclined plane angle $\alpha = 30°$.
10. Spring elongation $x = 0.02$ m.
11. Frictionless surface.

Solution :

Force Analysis:

1. Forces Acting on the Mass:

- **Weight (W):**

$$W = mg$$

where $g = 10\ \mathrm{m/s^2}$ (gravitational acceleration). Thus:

$$W = 15 \times 10 = 150\ \mathrm{N}.$$

- **Component of Weight Parallel to the Inclined Plane ($W_{/\!/}$):**

$$W_{/\!/} = W\sin\alpha = 147 \sin 30° = 150 \times 0.5 = 75\mathrm{N}.$$

- **Component of Weight Perpendicular to the Inclined Plane ($W_{\perp}$):**

$$W_{\perp} = W\cos\alpha = 150 \cos 30° = 150 \times \frac{\sqrt{3}}{2} = 127.1\ \mathrm{N}.$$

- **Spring Force (F_s):**

$$F_s = Kx = 100 \times 0.02 = 2\ \mathrm{N}.$$

2. Normal Reaction (N):

- The normal reaction is equal to the perpendicular component of the weight since there is no vertical motion:

$$N = W_{\perp} = 127.1\ \mathrm{N}.$$

*3. **Acceleration (a):***

- The net force along the inclined plane is the difference between the parallel component of the weight and the spring force:

$$F_{\text{ext}} = W_{\parallel} - F_s = 73.5 - 2 = 71.5\text{ N}.$$

- Using Newton's Second Law ($F_{\text{ext}} = ma$):

$$a = \frac{F_{\text{ext}}}{m}.$$

Substituting the values:

$$a = \frac{71.5}{15} = 4.77\text{ m/s}^2.$$

Final Results:

12. **Normal Reaction:**

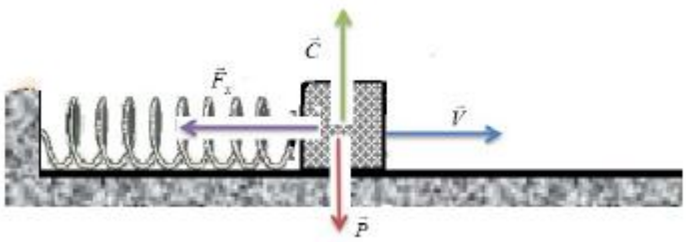
$$N = 127.1\text{ N}.$$

13. **Acceleration of the Mass:**

$$a = 4.77\text{ m/s}^2.$$

Application:

A body is placed horizontally with a spring(نابض) attached to its end.



Figure(V.4) : a body submitted to the stiffening force of a spring.

$$\sum F_{ext} = m\vec{a} \Rightarrow \vec{P} + \vec{C} + \vec{F_x} = m\vec{a}$$

By projection on ox : $-F_x = ma$

By projection on oy :$P - C = 0 \Rightarrow P = C = mg$

So :$-kx = ma \Rightarrow kx + ma = 0$

$\frac{d^2x}{dt^2} + \frac{k}{m}x = 0 \Rightarrow \ddot{x} + \frac{k}{m}x = 0 \Leftrightarrow \ddot{x} + \omega^2{}_0 x = 0$

Where : $\omega^2{}_0 = \frac{k}{m}$ Which is the differential equation of the second order .

The Solution of this equation is:

$x = A\cos(\omega t + \varphi) \rightarrow$ it's a harmonic motion.

With ω the pulsation ; A and φ are calculated from the initial conditions.

V.III.4.3.Viscous friction (in liquids) (الاحتكاك اللزج)

When an object moves through a fluid (liquid or gas) at a relatively low velocity, the force of friction is proportional to the velocity, and in the opposite direction:

$$\overrightarrow{f_f} = -\alpha\vec{v} = -k\eta\vec{v} \quad \text{(Formula valid only for low velocities)}$$

k :positive coefficient linked to the shape of the object.

η :is the coefficient of viscosity ($Kg.m^{-1}.s^{-1}$) of the fluid.

$\vec{v}$:object velocity vector

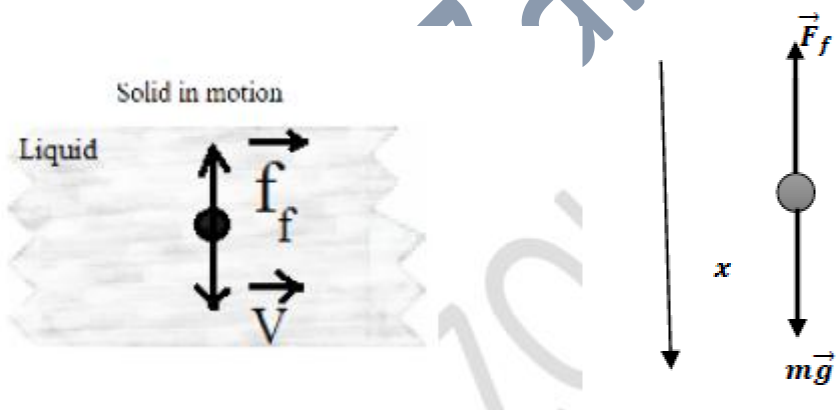


Figure (V.5) : Representation of the forces acting on a solid body moving in a liquid.

For high velocities : $f_f = kv^n$ $(n \geq 2)$

Examples

01: What is the terminal (limit) speed of a spherical ball in free fall.

$$k = 6\pi R$$

$$\sum F_{ext} = m\vec{a} \Rightarrow \vec{p} + \overrightarrow{F_f} = m\vec{a}$$

Projection on $\overrightarrow{ox}$gives:

$$mg - \alpha v = ma \Rightarrow g - \frac{\alpha}{m}v = a$$

$$g - \frac{\alpha}{m}v = \ddot{x} \Rightarrow g - \frac{\alpha}{m}v = \frac{dv}{dt}$$

Solving the Differential Equation :

$$dt = \frac{dv}{g - \frac{\alpha}{m}v} \Rightarrow dt = \frac{dv}{u}$$

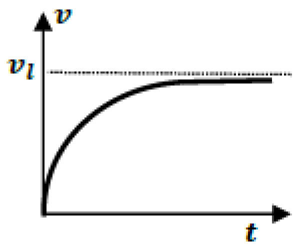
We put:

$g - \frac{\alpha}{m}v = u \Longrightarrow -\frac{\alpha}{m}dv = du \Rightarrow \frac{du}{u} = -\frac{\alpha}{m}dt$

$\Rightarrow \int \frac{du}{u} = -\frac{\alpha}{m}\int dt \Rightarrow \mathrm{Ln}(u) = -\frac{\alpha}{m}t \Longrightarrow u = Ce^{-\frac{\alpha}{m}t}$

$\Rightarrow g - \frac{\alpha}{m}v = Ce^{-\frac{\alpha}{m}t}$ If at $t = 0, v_0 = 0 \Rightarrow C = g$

$\Rightarrow v = \frac{mg}{\alpha}\left(1 - e^{-\frac{\alpha}{m}t}\right)$



IX. Fundamental principle of dynamics (FPD) in a non-galilean reference frame :

Considering a non-Galilean reference frame R' in motion relative to a Galilean reference frame R:

The equation of motion (Newton's second law)PFD in the Galilean reference frame R is written as:

$$\sum \vec{F}_{\text{ext}} = m\overrightarrow{a_{\left(\frac{M}{R}\right)}}$$

Where R the absolute reference is frame and R' is the relative reference frame. The law of acceleration composition is expressed as:

$$\vec{a}\left(\frac{M}{R}\right) = \vec{a}_a(M) = \vec{a}_r(M) + \vec{a}_e(M) + \vec{a}_c(M)$$

The equation of motion in R then becomes:

$$\sum \vec{F}_{\text{ext}} = m\vec{a}_r(M) + m\vec{a}_e(M) + m\vec{a}_c(M)$$

This allows us to write the equation of motion in the non-Galilean (relative) reference frame R' as:

$$m\vec{a}\left(\frac{M}{R'}\right) = m\vec{a}_r(M)$$

$$= \sum \vec{F}_{\text{ext}} - m\vec{a}_e(M) - m\vec{a}_c(M)$$

$$= \sum \vec{F}_{\text{ext}} + \vec{F}_e + \vec{F}_c$$

where the terms $-m\vec{a}_e(M) - m\vec{a}_c(M)$ the inertial forces. In particular, we have:

$$\vec{F}_{ie} = -m\vec{a}_e(M)\text{: is the driving inertia force}$$

$\vec{F}_{ic} = -m\vec{a}_c(M)$: is the Coriolis force of inertia.

In a non-Galilean reference frame, it is necessary, in addition to the external forces acting on the material point, to take into account the inertial forces. However, it is important to note that the inertial forces are not due to any specific interaction. Therefore, they are not considered as real forces in the same sense as other forces, even though their physical effects are real.

Remarks:

- If (R') is a Galilean reference frame, the inertial forces are zero, and the equation of motion (PFD) applies without modifications.

The equation of motion in a non-Galilean reference frame is also expressed using the momentum.

$$\left.\frac{d\vec{p}^{\,(M/R')}}{dt}\right|_{R'} = \sum \vec{F}_{ext} + \vec{F}_{ie} + \vec{F}_{ic}$$

If (R') is not Galilean.

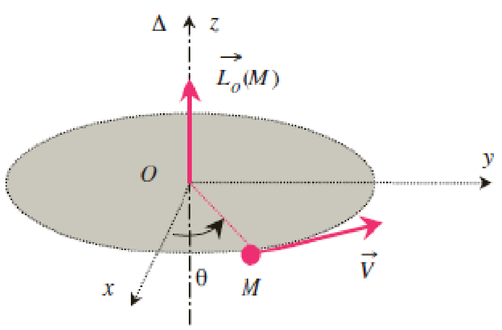
X.Angular momentum☹(العزم الحركي)

X.1.Definition :

The angular momentum at a point "O" of a material point M, of mass (m) and velocity $\vec{v}$ is the following vector product:

$$\vec{L}_{M/O} = \overrightarrow{OM} \times \vec{P}$$

العزم الحركي، أو كمية الحركة الزاوية، هو كمية فيزيائية تصف حركة جسم يدور حول نقطة معينة أو محور معين. يشبه العزم الحركي كمية الحركة الخطية(Linear Momentum) ، لكنه خاص بالحركة الدورانية.



Figure(V.6) Three-dimensional representation of the angular momentum of a body at a specific point.

(the unit of angular momentum is kg.m^2 .s^{-1}).

The angular momentum is a vector perpendicular to the plane formed by $\overrightarrow{OM}$ *and* $\vec{v}$

X.2.Curvilinear movement in the plane : (الحركة المنحنية في المستوي)

$$\overrightarrow{OM} = \rho\overrightarrow{u_\rho}$$

$$\vec{v} = \dot{\rho}\overrightarrow{u_\rho} + \rho\dot{\theta}\overrightarrow{u_\theta}$$

$$\vec{L}_{M/O} = \overrightarrow{OM} \times \vec{P} = \rho\overrightarrow{u_\rho} \times m(\dot{\rho}\overrightarrow{u_\rho} + \rho\dot{\theta}\overrightarrow{u_\theta})$$

$$(\rho\overrightarrow{u_\rho} \times m\dot{\rho}\overrightarrow{u_\rho}) + (\rho\overrightarrow{u_\rho} \times \rho\dot{\theta}\overrightarrow{u_\theta})$$

$$\overrightarrow{u_\rho}\ is\ parallels\ with\ m\dot{\rho}\overrightarrow{u_\rho}\ which\ gives(\rho\overrightarrow{u_\rho} \times m\dot{\rho}\overrightarrow{u_\rho}) = \vec{0}$$

$$\vec{L}_{M/O} = m\rho^2\dot{\theta}\vec{k}$$

If the motion is uniformly circular (ρ= R et = w= constant) :

$$\vec{L}_{M/O} = mR^2\omega\vec{k}$$

XI.The angular momentum theorem : TMC(نظرية العزم الحركي)

The derivation of angular momentum with respect to time gives :

$$\frac{\vec{L}_{M/O}}{dt} = \left(\frac{\overrightarrow{OM}}{dt} \times \vec{P}\right) + \left(\overrightarrow{OM} \times \frac{\vec{P}}{dt}\right)$$

$$\frac{d\vec{L}_{M/O}}{dt} = \left(\frac{d\overrightarrow{OM}}{dt} \times m\vec{v}\right) + \left(\overrightarrow{OM} \times \frac{md\vec{v}}{dt}\right)$$

$$\frac{d\vec{L}_{M/O}}{dt} = (\vec{v} \times m\vec{v}) + \left(\overrightarrow{OM} \times m\frac{d\vec{v}}{dt}\right)$$

$\vec{v}\ is\ collinear\ with\ m\vec{v}\ which\ gives\ \vec{v} \times m\vec{v} = \vec{0}$

The previous equation becomes :

$$\frac{d\vec{L}_{M/O}}{dt} = \overrightarrow{OM} \times m\frac{d\vec{v}}{dt} = \overrightarrow{OM} \times \vec{F} = \vec{\mathcal{M}}_{/O}(\vec{F})$$

The derivative of angular momentum with respect to time gives the moment of force F with respect to point “O”.

So the moment of force with respect to « O » is :

$$\frac{d\overrightarrow{L_{M/O}}}{dt} = \overrightarrow{\mathcal{M}}_{/O}(\overrightarrow{F}) = \overrightarrow{OM} \times \vec{F}$$

If $\overrightarrow{\mathcal{M}}_{/O}(\overrightarrow{F}) = 0 \Rightarrow L_{/O} = C^{ste}$

$\overrightarrow{OM}$: is the position vector from the point O (origin or reference point) to the point M where the force is applied.

$\vec{F}$: is the force vector applied at point M.

$\overrightarrow{\mathcal{M}}_{/O}(\overrightarrow{F})$: represent the torque vector (sometimes denote as $\vec{\tau}_{/O}$) about the point O caused by the force.

The torque vector points perpendicular to the plane formed by $\overrightarrow{OM}$ and $\vec{F}$ with its direction determined by **the right-hand rule.**

The angular momentum theorem TMC is similar to the fundamental relation of the dynamics PFD :

The PFD relates the resultant of the forces applied at point M to the variation in $\vec{P}$.

The TMC relates the sum of the moments of these forces relative to O and the variation of $\vec{L}$ relative to this point.

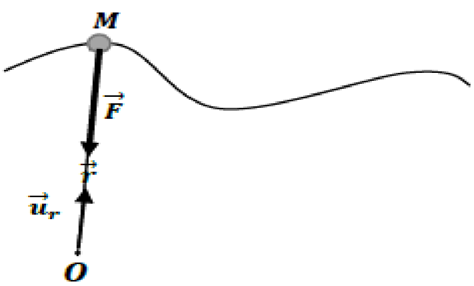
Note:

Angular moment plays a role in rotation $\frac{d\overrightarrow{L_{M/O}}}{dt}$ a role similar to that of force in translation $\frac{d\vec{P}}{dt} = \vec{F}$

XII.Central Forces(القوى المركزية)

Central force is the force that acts on an object moving in a circular path, directed toward the center of the circle. This force is responsible for keeping the object in circular motion by constantly changing the direction of its velocity without altering its magnitude.

القوة المركزية هي القوة التي تعمل على جذب جسم نحو مركز معين أو دفعه بعيدًا عنه، وهي المسؤولة عن إبقاء الجسم في **مسار دائري**. عادةً، تعمل هذه القوة بشكل عمودي على سرعة الجسم، مما يجعلها تُغيّر اتجاه الحركة دون تغيير مقدار السرعة.

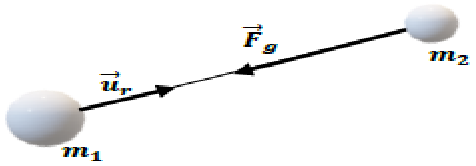


$$\vec{\boldsymbol{F}} = -\boldsymbol{f}(\boldsymbol{r})\overrightarrow{\boldsymbol{u_r}}$$

Example:

Gravitational Forces between two masses $\boldsymbol{m_1}$ and $\boldsymbol{m_2}$:

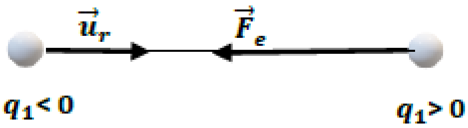
$$\vec{F_g} = -G\frac{m_1 m_2}{r^2}\overrightarrow{U_r}$$



$m_1; m_2$**:** masses of the 2 bodies.

Electrostatic Forces between two chargeses q_1 and q_2:

$$\vec{F_e} = k\frac{q_1 q_2}{r^2}\overrightarrow{U_r}$$



$q_1; q_2$**:** Charges of the 2 bodies.

A system subjected to central forces has a constant angular momentum $\vec{L} = cste$

$$\frac{d\vec{L}}{dt} = \vec{r} \times \vec{F} = r\vec{u_r} \times f(r)\overrightarrow{U_r} = \vec{0}$$

If force $\vec{F}$is central: $\vec{F}$ is parallel to $\vec{r}$. So the angular momentum with respect to the center of forces is constant. The opposite is true, i.e. if angular momentum is constant, then the force is central.

إذا كانت القوة $\longrightarrow$ مركزية، فإن $\longrightarrow$ تكون موازية لـ $\vec{r}$ وبالتالي، فإن العزم الزاوي بالنسبة إلى مركز القوة يكون ثابتًا. والعكس صحيح، أي إذا كان العزم الزاوي ثابتًا، فإن القوة تكون مركزية.

<u>Exercise</u>

The position vector of a body with a mass of 6 kg is given as:

$$\vec{r} = \hat{\imath}(3t^2 - 6t) + \hat{\jmath}(-4t^3) + \hat{k}(3t + 2)\text{ m.}$$

Find:

14. The force $\vec{F}$ acting on the body.
15. Its angular momentum $\vec{L}$ with respect to the origin.
16. The linear momentum $\vec{p}$ of the body and its angular momentum $\vec{L}$ with respect to time.
17. Verify that $\frac{d\vec{L}}{dt} = \vec{\tau}$ and $\vec{F} = \frac{d\vec{p}}{dt}$.

Solution :

Force acting on the body:

$$\vec{F} = m\vec{a} = 6(6\hat{\imath} - 24t\hat{\jmath}),$$

$$\vec{F} = 36\hat{\imath} - 144t\hat{\jmath}.$$

2. **Torque with respect to the origin**:

$$\vec{\tau} = \vec{r} \times \vec{F}.$$

Substituting the values of $\vec{r}$ and $\vec{F}$:

$$\vec{\tau} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3t^2 - 6t & -4t^3 & 3t + 2 \\ 36 & -144t & 0 \end{vmatrix}.$$

Expanding:

$$\vec{\tau} = (432t^2 + 288t)\hat{\imath} + (108t + 72)\hat{\jmath} + (-288t^3 + 864t^2)\hat{k}.$$

3. **Linear momentum of the body**:

$$\vec{p} = m\vec{v} = 6\left[(6t - 6)\hat{\imath} + (-12t^2)\hat{\jmath} + 3\hat{k}\right],$$

$$\vec{p} = (36t - 36)\hat{\imath} - 72t^2\hat{\jmath} + 18\hat{k}.$$

Angular momentum with respect to the origin:

$$\vec{L} = \vec{r} \times \vec{p}.$$

Substituting:

$$\vec{L} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3t^2 - 6t & -4t^3 & 3t + 2 \\ 36t - 36 & -72t^2 & 18 \end{vmatrix}.$$

Expanding:

$$\vec{L} = (144t^3 + 144t^2)\hat{\imath} + (54t^2 + 72t + 72)\hat{\jmath} + (72t^4 + 288t^3)\hat{k}.$$

4. **Verification**:

$$\vec{F} = \frac{d\vec{p}}{dt}.$$

This confirms the relationship between force and momentum.

$$\vec{p} = (36t - 36)\hat{\imath} - 72t^2\hat{\jmath} + 18\hat{k}.$$

$$\frac{\overrightarrow{dp}}{dt} = 36\vec{\imath} - 144t\vec{\jmath}$$

$$\vec{L} = (144t^3 + 144t^2)\hat{\imath} + (54t^2 + 72t + 72)\hat{\jmath} + (72t^4 + 288t^3)\hat{k}.$$

$$\frac{\overrightarrow{dL}}{\mathrm{dt}} = (432\mathrm{t}^2 + 288\mathrm{t})\vec{\mathrm{i}} + (108\mathrm{t} + 72)\vec{\mathrm{j}} + (-288\mathrm{t}^3 + 864\mathrm{t}^2)\vec{\mathrm{k}}$$

Exercise :

We displace a point mass (m) from its equilibrium position, which is suspended by an inextensible string of length(l). The position of the mass (m) is determined by the angle (theta) between the vertical and the direction of the string.

Task: Derive the differential equation of motion using:

1) The fundamental principle of dynamics (using the polar coordinate system).

2) The theorem of angular momentum.

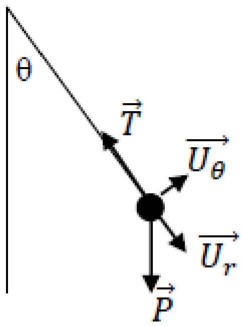
Solution:

The forces applied are: wire tension and weight.

1)- Applying the fundamental principle of dynamics :

$$\sum \vec{F} = m\vec{a}$$

$$\vec{P} + \vec{T} = m\vec{a}$$



The acceleration in polar coordinates is :

$$(\ddot{\rho} - \rho\dot{\theta})\overrightarrow{U_\rho} + (2\dot{\rho}\dot{\theta} + \rho\ddot{\theta})\overrightarrow{U_\theta}$$

In this case, we have :ρ=l

So : $\vec{a} = l\dot{\theta}^2\overrightarrow{U_\rho} + l\ddot{\theta}\overrightarrow{U_\theta}$

Projection on $\overrightarrow{U_\rho}$ and $\overrightarrow{U_\theta}$ gives us :

$$-T + mg\cos\theta = -ml\dot{\theta}^2$$

$$-mgl\sin\theta = ml\ddot{\theta}$$

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

Applying the angular momentum theorem **:**

$$\frac{d\vec{L}}{dt} = \sum \vec{\tau}(F)$$

$$where: \vec{L} = \overrightarrow{OM} \times m\vec{v}$$

The velocity in polar coordinates is :

$$\vec{v} = l\dot{\theta}\overrightarrow{U_\theta}$$

$$\vec{L} = \overrightarrow{OM} \times m\vec{v}$$

$$\vec{L} = l\overrightarrow{u_r} \times ml\dot{\theta}\overrightarrow{U_\theta} = ml^2\dot{\theta}\vec{k}$$

$$\frac{d\vec{L}}{dt} = ml^2\ddot{\theta}\vec{k}$$

The moment of the forces is :

$$\vec{\tau}(\vec{P}) = \overrightarrow{OM} \times \vec{P} = -mlg\sin\theta\vec{k}$$

$$\vec{\tau}(\vec{T}) = \overrightarrow{OM} \times \vec{T} = \vec{0}$$

$$\frac{d\vec{L}}{dt} = \vec{\tau}(\vec{P}) + \vec{\tau}(\vec{T})$$

$$ml^2\ddot{\theta} = -mlg\sin\theta$$

$$\ddot{\theta} + \frac{g}{l}\sin\theta = 0$$

Work and Energy

I.Introduction

The aim of this chapter is to present the energy tools used in mechanics to solve problems. Indeed, sometimes the fundamental principle of dynamics is not enough to solve a problem. Newton's laws can be used to solve all the problems of classical mechanics. If we know the position and initial velocity of the particles in a system, as well as all the forces acting on them. But in practice, we don't always know all the forces at play, and even if we do, the equations to be solved are too complex. In this case, other concepts such as work and energy must be used. Before describing the different types of energy (kinetic, potential and mechanical) and using them in energy theorems, we'll introduce the notions of power and work of a force.

II.The work :(العمل)

All motion under the action of external forces $\vec{}$, implies work by these forces. In other words; work supplied by a force moves a body in its own direction and creates motion.

كل حركة تحت تأثير القوى الخارجية (F) تتضمن شغلًا من قبل هذه القوى. بمعنى آخر، الشغل الناتج عن قوة يؤدي إلى تحريك جسم في اتجاهها الخاص ويُحدث حركة.

II.1.Work performed by a constant force: (العمل المنجز بواسطة قوة ثابتة)

Let (M) be a material point moving along a straight line (AB) and subjected to a force (F). The work of the force is defined as :

$$w = \vec{F}.\overrightarrow{AB} = F.AB.\cos\alpha$$

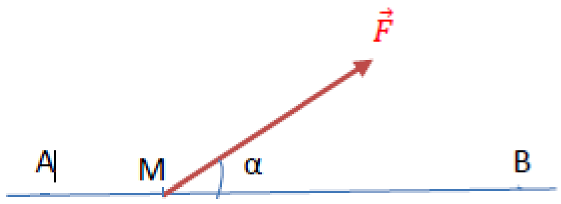


Figure (..) : Work of a constant force over a rectilinear displacement.

This work can be positive, negative or zero, depending on the sign of cos α :

- If: $0 < \alpha < \frac{\pi}{2} \Rightarrow \cos\alpha > 0$ "$it's\ motor\ work$".
- If: $\frac{\pi}{2} < \alpha < \pi \Rightarrow \cos\alpha < 0$ "$Resistive\ work$"
- If : $\alpha = \frac{\pi}{2} \Rightarrow \cos\alpha = 0$ "no work"

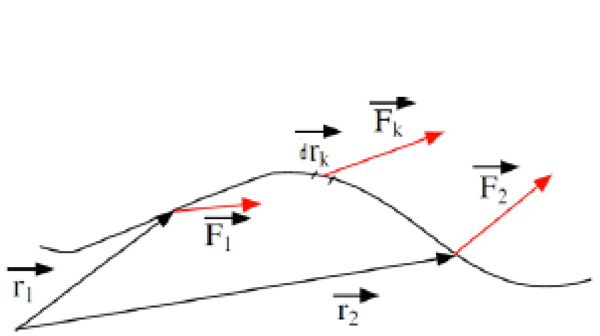
Unity of work in the system MKSA is « **Joule** ».

Note :

Note that work is a scalar quantity, unlike force and displacement, which are vectors.

II.2.The work performed by a variable force: (العمل المنجز بواسطة قوة متغيرة)

In the case where the force (F) is variable and the displacement is arbitrary, the work of this force is calculated for an infinitesimal rectilinear displacement (dr). This is referred to as the elementary work (dW), defined by: $dW = \vec{F}.\vec{dr}$



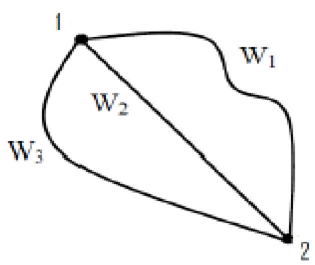
To find the total work between the first point and the second, we integrate this last equation.

$$W_{1\to 2} = \int_{r_1}^{r_2} \vec{F}.\vec{dr}$$

If we want to use Cartesian coordinates, we express the force and the displacement in this system. For this, we will have:

$$\vec{F} = F_x\vec{\imath} + F_y\vec{\jmath} + F_z\vec{k} \quad ; \quad \vec{dr} = dx\vec{\imath} + dy\vec{\jmath} + dz\vec{k}$$

$$W_{1\to 2} = \int_1^2 F_x\ dx + F_y dy + F_z dz$$



The calculation of work requires knowing the path taken between the two points. For each path, a specific work is obtained. In general, the work depends on the path taken.

يتطلب حساب العمل معرفة المسار المتبع بين النقطتين. لكل مسار يتم الحصول على شغل معين. وبشكل عام، يعتمد الشغل على المسار المتبع.

Example :

A particle is subjected to the force $\vec{F} = 2xy\vec{\imath} + x^2\vec{\jmath}$ Calculate the work done by the force $\vec{F}$ when the particle moves from point (0,0) to point (2,0) along the OX axis.

Solution

1. Work Formula:

The work done by a force is given by the equation:

$$W = \int \vec{F} \cdot d\vec{r}$$

Where:

- $\vec{F} = 2xy\hat{\imath} + x^2\hat{\jmath}$ is the force.
- $d\vec{r} = dx\hat{\imath} + dy\hat{\jmath}$ is the infinitesimal displacement.

2. Expressing Force and Displacement Along the Path:

Since the particle moves along the OX-axis:

- $y = 0$ Throughout the motion.
- $dy = 0$ Because the displacement in the y-direction is zero.
- The displacement is $d\vec{r} = dx\hat{\imath}$ (only in the x-direction).

Now, the force becomes:

$$\vec{F} = 2x(0)\hat{\imath} + x^2\hat{\jmath} = 0\hat{\imath} + x^2\hat{\jmath} = x^2\hat{\jmath}.$$

3. Dot Product of Force and Displacement:

To calculate the work, we need to take the dot product of the force $\vec{F}$ and the displacement $d\vec{r}$:

$$\vec{F} \cdot d\vec{r} = (x^2\hat{\jmath}) \cdot (dx\hat{\imath}).$$

Since $\hat{\jmath}$ is perpendicular to $\hat{\imath}$, their dot product is zero:

$$\vec{F} \cdot d\vec{r} = 0.$$

4. Total Work Done:

Since the dot product is zero:

$$W = \int_0^2 0\ dx = 0.$$

The work done by the force $\vec{F}$ as the particle moves from $(0,0)$ to $(2,0)$ along the OX-axis is:

$$\boxed{W = 0}.$$

III.Power :الاستطاعة

The average power (P_m) is defined as the work done per unit of time.

$$P_m = \frac{\Delta w}{\Delta t}$$

Instant power :$P_i = \lim_{\Delta t \to 0} \frac{\Delta w}{\Delta t} = \frac{dw}{dt} = \frac{\vec{F}\overrightarrow{dr}}{dt} = \vec{F}.\vec{v}$

$$P = \vec{F}.\vec{v} = F.v\cos\theta$$

Note :

- ✓ The unit of power is the « Watt ».
  - ✓ This force can be classified into three types:

- It is driving, if its power is positive which corresponds to an angle $\alpha<\pi/2$.
- It is resistive, if its power is negative which corresponds to an angle $\alpha>\pi/2$.
- Finally, it can be of zero power, in which case $\alpha=\pi/2$.

IV.Energy (الطاقة)

In physics, energy is defined as the capacity of a system to produce work. Energy is not a material substance: it is a physical quantity that characterizes the state of a system; it can be stored and exists in many forms.

في الفيزياء، تُعرَّف الطاقة على أنها قدرة النظام على إنجاز الشغل. الطاقة ليست مادة ملموسة؛ بل هي كمية فيزيائية تُميّز حالة النظام. يمكن تخزينها وتوجد في أشكال عديدة.

IV.1.Kinetic energ(الطاقة الحركية).

Kinetic energy, denoted as E_c, is the energy a body possesses due to its motion relative to a given reference. Kinetic energy is equal to the work done by the applied forces.

$$dw = \vec{F}\overrightarrow{dr} = m\vec{a}\overrightarrow{dr}$$

$$dw = m\frac{d\vec{v}}{dt}\overrightarrow{dr} = mvdv$$

$W = \int mvdv$ if the body moves from A-B we'll get.

$$W = \int_A^B mvdv = m\int_A^B vdv = m\frac{v^2}{2}\Big|_A^B$$

$$W = \frac{1}{2}m({v^2}_A - {v^2}_B) = \frac{1}{2}m{v^2}_A - \frac{1}{2}m{v^2}_B$$

$\sum W(F)_{A\to B} = E_{c(B)} - E_{c(A)} = \Delta E_c$ The work of the applied force is equal to the variation in kinetic energy of the material point.

Remark :

When $E_{cB} > E_{cA}$ the work is motor work and $v_B > v_A$

A force perpendicular to the displacement (W=0) implies $E_{cB} = E_{cA}$ and the velocity of the body remains constant.

IV.1.1.The theorem of change in kinetic energy (نظرية التغير في الطاقة الحركية)

The variation in kinetic energy of a material point subjected to a set of external forces between two positions A and B is equal to the sum of the work of these forces between these two points.

$$\sum W(F)_{A\to B} = E_{c(B)} - E_{c(A)} = \Delta E_c$$

IV.2.Potential Energy- Forces conservatrices:(الطاقة الكامنة أو القوى المحافظة)

IV.2.1.Definition

A force is said to be conservative if its work between two points, M_1 and M_2, depends only on the initial position and the final position. In other words, the work is independent of the path taken to move from M_1 to M_2.

IV.2.2.Equivalent definition :

Force is a gradient, and it is also said that force derives from potential energy according to the relation: $\vec{F} = -\overrightarrow{grad}E_p$

القوة هي تدرج، ويُقال أيضًا إن القوة مشتقة من طاقة كامنة وفقًا للعلاقة

This relation is important because it allows us to determine the force from the potential energy from which it derives.

Conversely, it also allows us to determine the potential energy if the force is a gradient.

هذه العلاقة مهمة لأنها تتيح لنا تحديد القوة من طاقة الوضع التي تشتق منها .وعلى العكس، تتيح لنا أيضًا تحديد طاقة الوضع إذا كانت القوة تمثل تدرجًا.

$$dE_p = -\vec{F}\overrightarrow{dr}$$

$$dw = \vec{F}\overrightarrow{dr}$$

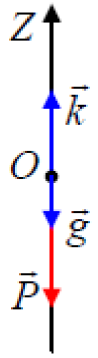
$$dw = -dE_p \Rightarrow w = -\Delta E_p = E_p(A) - E_p(B)$$

Example 1: The potential energy of a spring(الطاقة الكامنة في نابض)

Return force is : $\vec{F} = -kx\vec{\imath}$ where x is the elongation of the spring.

$$dE_p = -\vec{F}\overrightarrow{dr} = kxdx \Rightarrow E_p = \frac{1}{2}kx^2$$

Example2 : The force of gravity (قوة الجابية الارضية او الطاقة الكامنة الثقالية)



In the Cartesian coordinate system, where OZ is the vertical axis oriented upwards:

$$\vec{P} = \vec{F} = -mg\vec{k}$$

Using the vectorial displacement expression in Cartesian coordinates:

$$d\vec{r} = dx\vec{\imath} + dy\vec{\jmath} + dz\vec{k}$$

We can deduce:

$$dW = \vec{F} \cdot d\vec{r} = -mg\,dz$$

By integrating this equation, we see that the work for the displacement between two points A and B depends only on their vertical positions (heights) and not on the trajectory taken between them.

$$w = -\int_{z1}^{z2} mgdz = -mg(z_2 - z_1) = -mg\Delta h.$$

If the two points are at the same level, then the work done by the gravitational force is zero, which indicates that the gravitational force is conservative.

Thus, if ($z_1 = z_2$), we have (W = 0), proving that gravity is a conservative force.

IV.3.The conservative force- The force derived from potential. (القوة المحافظة او القوة المشتقة من كمون)

A force is said to be conservative, or to derive from a potential, if its work is independent of the path taken, whatever the probable displacement between the starting point and the end point.

Conservative forces include the force of gravity, spring return force and the tension force of a wire.

تشمل القوى المحافظة قوة الجاذبية، وقوة إرجاع النابض، وقوة الشد في السلك

Remark:

If the force $\vec{F}$ is conservative, we say that the path is closed, and we write:

إذا كانت القوة (F) متحفظة(محافظة)، فإننا نقول إن المسار مغلق، ونكتب:

$$w = \oint \vec{F} \cdot d\vec{r} = 0$$

A force is conservative if it satisfies one of the three conditions:

- Its work does not depend on the path taken.
- It is derived from a potential $\vec{F} = -\overrightarrow{grad}E_p(x, y, z)$
- $\overrightarrow{rot}\vec{F} = \vec{0}$

Exercise1:

The force $\vec{\mathbf{F}} = (x^2 - y^2)\vec{\mathbf{i}} + 3xy\vec{\mathbf{j}}$ moves from point $O(0,0)$ to point $B(2,4)$ via two paths: $y = x^2$ and $y = 2x$.

Is this force conservative?

<u>Solution:</u>

Along the first path $(y = 2x)$

$$y = 2x \Rightarrow \vec{F} = -3x^2\vec{\imath} + 6x^2\vec{\jmath}$$

$$dy = 2dx\,;\, d\vec{r} = dx\vec{\imath} + dy\vec{\jmath} \Rightarrow d\vec{r} = dx\vec{\imath} + 2dx\vec{\jmath}$$

$$w = \int \vec{F}\, d\vec{r} = \int \left(F_x dx + F_y dy\right) = \int -3x^2dx + 12x^2dx = \int 9x^2dx$$

$$3x^3\big|_0^2 = 24joule$$

Along the second path $(y = x^2)$

$$y = x^2 \Rightarrow \vec{F} = (x^2 - x^4)i + 3x^3j$$

$$dy = 2xdx\,;\, d\vec{r} = dx\vec{\imath} + dy\vec{\jmath} \Rightarrow d\vec{r} = dx\vec{\imath} + 2xdx\vec{\jmath}$$

$$w = \int F.dr = \int \left(F_x dx + F_y dy\right) = \int (x^2 - x^4)dx + 6x^4\,dx$$

$$\int (x^2 + 5x^4)\,dx = x^5 + \frac{1}{3}x^3\bigg|_0^2 \Rightarrow w = 34.6joule$$

The two works are not equal, and thus the force in this case is non-conservative.

Exercise2:

Find the work done by the force $\vec{\mathbf{F}} = xy^3\vec{\mathbf{i}} + xy\vec{\mathbf{j}}$ along three different paths from the point $(0,0)$ to the point $(1,1)$.

c)Determine the work done by $\vec{\mathbf{F}}$ for the displacement of M along a circular trajectory of radius R centered at $O(0,0)$.

b) Is this force conservative?

c) If yes, find the potential function $U(x, y)$ from which the force is derived. To solve the exercise:

a) **Find the work done by the force along three different paths from (0,0) to (1,1):**

The work done by a force along a path C is given by:

$$W = \int_C \mathbf{F} \cdot d\mathbf{r},$$

Where: $\vec{\mathbf{F}} = xy^3\vec{\mathbf{i}} + xy\vec{\mathbf{j}}$ and $d\vec{\mathbf{r}} = dx\vec{\mathbf{i}} + dy\vec{\mathbf{j}}$. The dot product becomes:

$$\vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = xy^3\, dx + xy\, dy.$$

We compute W for three different paths:

1. Path 1: Straight Line $\boldsymbol{y = x}$:

Along this path, $y = x$ and $dy = dx$. Substitute $y = x$ into the integral:

$$W = \int_0^1 (x(x)^3\, dx + x(x)\, dx) = \int_0^1 (x^4 + x^2)\, dx.$$

Compute the integral:

$$W = \int_0^1 x^4\, dx + \int_0^1 x^2\, dx = \left[\frac{x^5}{5}\right]_0^1 + \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{5} + \frac{1}{3}.$$

$$W = \frac{3}{15} + \frac{5}{15} = \frac{8}{15}.$$

2. Path 2: Along $\boldsymbol{x}$-axis, then $\boldsymbol{y}$-axis:

- **Step 1: Move along the x-axis ($y = 0, dy = 0$)**

$$W_1 = \int_0^1 x\,(0)^3\, dx + x(0)\, dy = 0.$$

- **Step 2: Move along the y-axis ($x = 1, dx = 0$)**

$$W_2 = \int_0^1 (1y^3\, dx + 1y\, dy) = \int_0^1 y\, dy.$$

Compute the integral:

$$W_2 = \int_0^1 y\, dy = \left[\frac{y^2}{2}\right]_0^1 = \frac{1}{2}.$$

Total work along this path:

$$W = W_1 + W_2 = 0 + \frac{1}{2} = \frac{1}{2}.$$

3. Path 3: Along $\boldsymbol{y}$-axis, then $\boldsymbol{x}$-axis:

- **Step 1: Move along the y-axis ($x = 0, dx = 0$)**

$$W_1 = \int_0^1 0\,(y)^3\, dx + 0(y)\, dy = 0.$$

- **Step 2: Move along the $\boldsymbol{x}$-axis ($y = 1, dy = 0$)**

$$W_2 = \int_0^1 \mathrm{x}(1)^3\, dx + 0\, dy) = \int_0^1 x\; dx.$$

Compute the integral:

$$W_2 = \int_0^1 x\; dx = \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}.$$

Total work along this path:

$$W = W_1 + W_2 = 0 + \frac{1}{2} = \frac{1}{2}.$$

b) **Is the force conservative?**

To determine if the force is conservative, compute the curl:

$$\vec{\nabla} \times \vec{\mathbf{F}} = \frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}(xy^3).$$

Compute each partial derivative:

$$\frac{\partial}{\partial x}(xy) = y, \quad \frac{\partial}{\partial y}(xy^3) = 3xy^2.$$

Subtract:

$$\vec{\nabla} \times \vec{\mathbf{F}} = y - 3xy^2.$$

Since the curl is not zero ($\vec{\nabla} \times \vec{\mathbf{F}} \neq 0$), the force is **not conservative**.

Since the path is closed, the work will be zero (The starting point is the same as the ending point.).

$$w = \oint \vec{F} \cdot d\vec{r} = 0$$

c)Since the force not conservative, there is no potential function U(x,y) for this force.

Exercise3:

$$\vec{F} = 3x^2\,\vec{\imath} + (4y + z)\,\vec{\jmath} + y\,\vec{k}$$

Is this a gradient field, and what is the potential E_p?

Solution:

Check if the force field is a gradient field

We compute the curl of $\vec{F}$:

$$\mathrm{Rot}\,\vec{F} = \vec{\nabla} \times \vec{F}$$

Using the determinant method:

$$\text{Rot}\,\vec{F}=\begin{vmatrix}\vec{\imath} & -\vec{\jmath} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 3x^2 & 4y+z & y\end{vmatrix}$$

$$\text{Rot}\,\vec{F}=\left(\frac{\partial(y)}{\partial y}-\frac{\partial(4y+z)}{\partial z}\right)\vec{\imath}-\left(\frac{\partial(y)}{\partial x}-\frac{\partial(3x^2)}{\partial z}\right)\vec{\jmath}+\left(\frac{\partial(4y+z)}{\partial x}-\frac{\partial(3x^2)}{\partial y}\right)\vec{k}$$

$$\text{Rot}\,\vec{F}=(1-1)\,\vec{\imath}-(0-0)\,\vec{\jmath}+(0-0)\,\vec{k}=\vec{0}$$

Since "Rot" $\vec{F}=\overrightarrow{0}$, $\overrightarrow{F}$ is indeed(بالفعل) a gradient field.

Determine the potential E_p:

We have:

$$\vec{F}=-\vec{\nabla}E_p$$

$$\vec{F}=-\left(\frac{\partial E_p}{\partial x}\,\vec{\imath}+\frac{\partial E_p}{\partial y}\,\vec{\jmath}+\frac{\partial E_p}{\partial z}\,\vec{k}\right)$$

From $\vec{F}=3x^2\,\vec{\imath}+(4y+z)\,\vec{\jmath}+y\,\vec{k}$, we compare components:

18. $\frac{\partial E_p}{\partial x}=-3x^2$

$$E_p=-x^3+g(y,z)$$

3. $\frac{\partial E_p}{\partial y}=-(4y+z)$

$$\frac{\partial g(y,z)}{\partial y}=-4y-z$$

$$g(y,z)=-2y^2-yz+l(z)$$

4. $\frac{\partial E_p}{\partial z}=-y$

$$\frac{\partial l(z)}{\partial z}=0 \quad\Rightarrow\quad l(z)=\text{constant}$$

Final Result:

$$E_p=-x^3-2y^2-yz+C$$

Hence finally:

$$E_p(\mathrm{x},\mathrm{y},\mathrm{z})=-x^3-2y^2-yz+C$$

We have:

$$E_p(0{,}0{,}0)=0\Rightarrow \mathrm{C}=0\ \ \text{So: } E_p(\mathrm{x},\mathrm{y},\mathrm{z})=-x^3-2y^2-yz$$

IV.4.Mechanical Energy(Total energy): Law of Conservation

Let us consider a material point subjected to a single force, which is derived from potential energy (this force $\vec{F}$ may be the resultant of several applied forces).
For a displacement AB of the material point, we can write at the same time:

لنعتبر نقطة مادية تخضع لقوة واحدة مستمدة من طاقة الوضع (قد تكون هذه القوة $\vec{F}$ محصلة عدة قوى مطبقة). بالنسبة للإزاحة AB للنقطة المادية، يمكننا كتابة ما يلي في نفس الوقت:

$$w_{A\to B} = E_{p(A)} - E_{p(B)} \; and \; w_{A\to B} = E_{c(B)} - E_{c(A)}$$

$$E_{c(A)} + E_{p(A)} = E_{c(B)} + E_{p(B)} = cste \Rightarrow \left(E_c + E_p\right)_A = \left(E_c + E_p\right)_B$$

The mechanical energy, or total energy, of the material point is defined as:

$$\left(E_c + E_p\right) = E_m$$

<u>Statement:</u>

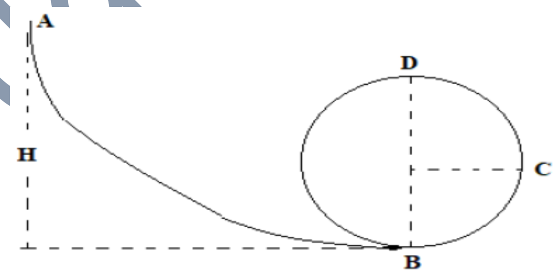
If a material point is subjected only to a force derived from potential energy, its mechanical (total) energy remains conserved.

النص:

إذا كانت النقطة المادية خاضعة فقط لقوة مشتقة من طاقة كامنة، فإن طاقتها الميكانيكية (الإجمالية) تبقى محفوظة.

Exercise4:

A body of mass M is released from rest at the top of a mountain of height H, as shown in the figure below.



- Find the velocities of the body M at points B, C, and D.
- Find the reaction forces of the surface on the body M at points C and D.

<u>Solution</u>

1. Find the velocities of body M at points B, C, and D.

1. Find the reaction forces of the surface on body M at points C and D.

- In the absence of friction, the total mechanical energy is conserved:

$$(E_T)_{(A)} = (E_T)_{(B)} \quad \rightarrow \quad (E_C)_{(A)} + (E_P)_{(A)} = (E_C)_{(B)} + (E_P)_{(B)}$$

Taking level (B) as the reference level for the potential energy:

$$(E_P)_{(B)} = 0$$

Since $v_A = 0$:

$$(E_C)_{(A)} = 0$$

$$MgH = \frac{1}{2}Mv_B^2 \quad \rightarrow \quad v_B = \sqrt{2gH}$$

- At point C:

$$(E_T)_{(A)} = (E_T)_{(C)} \quad \rightarrow \quad (E_C)_{(A)} + (E_P)_{(A)} = (E_C)_{(C)} + (E_P)_{(C)}$$

$$MgH = \frac{1}{2}Mv_C^2 + MgR \quad \rightarrow \quad v_C = \sqrt{2g(H-R)}$$

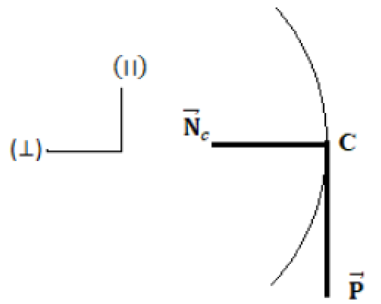
- At point D:

$$(E_T)_{(A)} = (E_T)_{(D)} \quad \rightarrow \quad (E_C)_{(A)} + (E_P)_{(A)} = (E_C)_{(D)} + (E_P)_{(D)}$$

$$MgH = \frac{1}{2}Mv_D^2 + 2MgR \quad \rightarrow \quad v_D = \sqrt{2g(H-2R)}$$

- By applying the fundamental principle of dynamics for the body at point C:

$$\vec{\mathbf{P}} + \vec{\mathbf{N}}_{\mathbf{C}} = M\vec{\mathbf{a}}_{\mathbf{C}}$$



By projecting onto the two axes (∥) and (⊥):

- Along ∥:

$$Ma_{\parallel} = -Mg$$

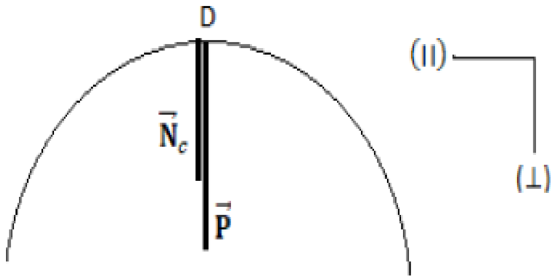
- Along (⊥):

$$Ma_{\perp} = N_C$$

$$N_C = M\frac{v_C^2}{R} = 2\frac{(H-R)}{R}Mg$$

By applying the fundamental principle of dynamics for the body at point D:

$$\vec{\mathbf{P}} + \vec{\mathbf{N}}_{\mathbf{D}} = M\vec{\mathbf{a}}_{\mathbf{D}}$$



By projecting onto the two axes (∥) and (⊥):

- Along (//):

$$Ma_{\parallel} = 0$$

- Along (⊥):

$$Ma_{\perp} = Mg + N_D \quad \rightarrow \quad N_D = Ma_{\perp} - Mg$$

$$N_D = M\frac{v_D^2}{R} - Mg = \frac{(2H - 5R)}{R}Mg$$

Bibliography: