PEOPLE'S DEMOCRATIC REPUBLIC OF ALGERIA

MINISTRY OF HIGHER EDUCATION AND SCIENTIFIC RESEARCH

University Mohamed Khider of Biskra

Faculty of Exact Sciences and Natural and Life Sciences

Department of Mathematics
Course support for the module -Maths 3-
Intended for students in the second year
Sciences & Technologies
Renewable energies

Presented by : Zaamoune Faiza

College year 2024/2025

Semester: 3

Teaching Unit: UEF 2.1.1

Subject: Mathematics 3

Total Hours: 67h30 (Lectures: 3h00, Tutorials: 1h30)

Credits: 6

Coefficient: 3

Objectives

By the conclusion of this course, students will have a clear understanding of the various types of series and the conditions that govern their convergence, along with the different forms of convergence.

Prerequisites

Mathematics 1 and Mathematics 2.

Course Content

Chapter 1: Simple and Multiple Integrals (3 weeks)

1.1 A review of Riemann integrals and basic calculations.

1.2 Multiple Integrals: Concepts and Methods.

1.3 Utilizing for calculating areas, volumes, etc.

Chapter 2: Improper Integrals (2 weeks)

2.1 Integrals of functions defined over unbounded intervals.

2.2 Integrals of functions defined on bounded intervals with infinite limits at one end.

Chapter 3: Differential Equations (2 weeks)

3.1 Examination of ordinary differential equations.

3.2 Exploration of partial differential equations.

3.3 Special functions.

Chapter 4: Series (3 weeks)

4.1 Numerical series.

4.2 Sequences and series of functions.

4.3 Power series and Fourier series.

Chapter 5: Fourier Transform (3 weeks)

5.1 Definition and properties.

5.2 Utilization in the resolution of differential equations.

Chapter 6: Laplace Transform (2 weeks)

6.1 Definition and properties.

6.2 Utilization in the resolution of differential equations.

Approach to Evaluation

Continuous Assessment: 40%

Final Exam: 60%

Contents

Introduction

This document serves as an instructional tool for the "Mathematics 3" module, specifically tailored for second-year undergraduate students in science and technology fields. Its structure is designed to align with the official curriculum while remaining concise enough for the allotted teaching schedule.

The content is organized into six core chapters. The first half establishes a strong foundation in advanced calculus, beginning with a review of **Riemann integrals** before moving to **multiple integrals** and their applications in calculating areas and volumes (Chapter 1). This is followed by a study of **improper integrals**, including the criteria for determining their convergence (Chapter 2), and a comprehensive look at solving **ordinary and partial differential equations** (Chapter 3).

The second half of the course explores infinite series and integral transforms. It covers **numerical and function series**, including Power and Fourier series (Chapter 4). Finally, the manual introduces two powerful techniques for solving differential equations: the **Fourier Transform** (Chapter 5) and the **Laplace Transform** (Chapter 6). Each chapter includes definitions, key theorems, and worked examples to reinforce understanding and build essential problem-solving skills.

Chapter 1

Simple and Multiples Integrals

1.1 A review of Riemann integrals and basic calculations

A Riemann Integral Theoretical Structure

Definition 1.1 (***Riemann integral: constructive view***)

[10] Let $f : [p, q] \to \mathbb{R}$. Consider finite partitions of $[p, q]$, pick one sample in each subinterval, and sum $\sum f(x_i^*)\, \Delta x_i$. If all such sums converge to the same finite value as $\max \Delta x_i \to 0$, we define

$$\int_p^q f(x). \tag{1.1}$$

Theorem 1.1 (***Coincidence of upper/lower limits***)

If the upper and lower sums of f on $[p, q]$ converge to the same limit upon refinement, then f is Riemann integrable, and this limit is equal to $\int_p^q f(x)$.

Proposition 1.2 (***Linearity, positivity, additivity***)

[2] For integrable f, g and scalars α, β:

- ***Additivity**: The integral can be split over subintervals: $\int_p^v f(x)\, dx = \int_p^q f(x)\, dx + \int_q^v f(x)\, dx$.*
- ***Positivity**: If $f(x) \geq 0$, then $\int_p^q f(x)\, dx \geq 0$.*
- ***Linearity**: $\int_p^q (\alpha f(x) + \beta g(x))\, dx = \alpha \int_p^q f(x)\, dx + \beta \int_p^q g(x)\, dx$.*

Proposition 1.3

Riemann integrability over I is true for all continuous functions defined on I.

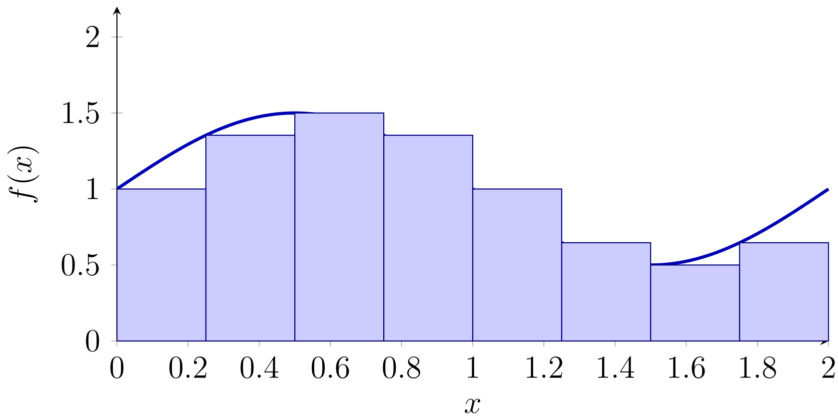


Figure 1.1: Upper rectangles for $f(x) = 1 + \frac{1}{2}\sin(\pi x)$ on $[0, 2]$ with a coarse partition; the sum tightens as the mesh refines.

Definition 1.2 (***Mean value***)

[13] The mean value of f on $[p, q]$ is:

$$M = \frac{1}{q-p}\int_p^q f(x)\,dx. \tag{1.2}$$

Definition 1.3 (***Effective Value***)

The effective value of f on $[p, q]$ is the number d such that:

$$d^2 = \frac{1}{q-p}\int_p^q f^2(x)\,dx. \tag{1.3}$$

1.1.1 Calculation of Primitives

Definition 1.4

An indefinite integral signifies the complete set of functions that yield a specified function $f(x)$ when differentiated. The process, represented by the integral sign $\int$, functions as the reverse of differentiation and aims to identify a function $F(x)$ referred to as the ***antiderivative****. The derivative of any constant equals zero, indicating that the antiderivative lacks uniqueness. Consequently, we incorporate an arbitrary constant of integration, C, to account for all potential solutions.*

$$\int f(x)\,dx = F(x) + C \tag{1.4}$$

Here, $F(x)$ is any function such that $F'(x) = f(x)$, and C can be any real number.

Remark 1.4

The antiderivative of a function, if it exists, is not unique.

Example 1.1

Determine the antiderivative of the function $h(x)$.

1. $\displaystyle\int x^3\,dx = \frac{x^4}{4} + C,$
2. $\displaystyle\int \sin(2x)\,dx = -2\cos(x) + C,$
3. $\displaystyle\int e^{-5x}\,dx = -5e^{-5x} + C,$
4. $\displaystyle\int 4xe^{x^2}dx = 2e^{x^2} + C.$

Definition 1.5

The definite integral represents the signed surface of a line defined by two specific limits. This is depicted as:

$$\int_p^q f(x)\,dx. \tag{1.5}$$

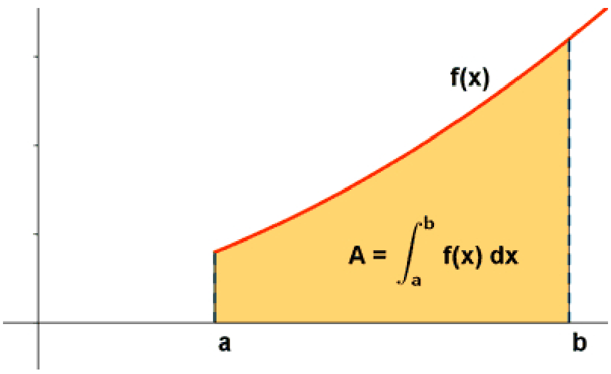


Figure 1.2: Depiction of the region corresponding to a specified integral

Example 1.2

Consider the function $f(x) = 1 + \frac{1}{2}\sin(\pi x)$ on the interval $[0, 2]$. For a uniform partition of $[0, 2]$ with N subintervals, define the upper sums U_N using right endpoints.

We verify that $U_N \to \int_0^2 f(x)\,dx = 2$, *as follows:*

$$\int_0^2 \left(1 + \frac{1}{2}\sin(\pi x)\right)\,dx = \left[x - \frac{1}{2\pi}\cos(\pi x)\right]_0^2 = 2.$$

Integration Techniques

Theorem 1.5 (*Integration by Parts*)

[1] We let $f = u$ *and* $g = dv$, *resulting in:*

$$\int u\,dv = uv - \int v\,du.$$

Example 1.3

Evaluate the integral $\int x\sin(x)\,dx$: *we accept:*

- *Let* $u = x$
- *Let* $dv = \sin(x)\,dx$

At this stage, we determine du *through the differentiation of* u, *and* v *is obtained by integrating* dv:

- $du = dx$
- $v = \int \sin(x)\,dx = -\cos(x)$

So:

$$\begin{aligned}\int x\sin(x)\,dx &= (x)(-\cos(x)) - \int(-\cos(x))\,dx\\ &= -x\cos(x) + \int\cos(x)\,dx\\ &= -x\cos(x) + \sin(x) + C\end{aligned}$$

Theorem 1.6 (*Substitution*)

[10] Let y *denote a differentiable function and* f *represent a continuous function such*

that

$$\int f(x)\,dx = F(x). \tag{1.6}$$

We have:

$$\int y' f(y)\,dx = \int f(y)\,dy = F(y). \tag{1.7}$$

Example 1.4

Evaluate this integral:

$$\int e^{2x}\,dx,$$

we take $y = 2x \Longrightarrow dy = 2dx, dx = \dfrac{dy}{2}$,
so $\int e^{2x}\,dx = \int \dfrac{e^y}{2}, dy = \dfrac{e^y}{2} + C, \quad C \in \mathbb{R}.$

Euclidean Division for Solving Integrals

The Euclidean division approach serves as an effective simplification strategy. It operates by decomposing the integrand into a polynomial and a simpler rational function, so facilitating the integration process.

Definition 1.6 (*Improper vs. proper rational integrand*)

A rational integrand $R(x) = \dfrac{P(x)}{Q(x)}$ *is improper if* $\deg P \geq \deg Q$, *and proper otherwise.*

Proposition 1.7 (*Division to reduce degree*)

If $\deg P \geq \deg Q$, *perform polynomial division to write*

$$\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)}, \qquad \deg R < \deg Q,$$

so that $\int \frac{P}{Q}\,dx = \int S(x)\,dx + \int \frac{R(x)}{Q(x)}\,dx.$

Algorithm (1) Check $\deg P \geq \deg Q$; if yes, divide P by Q to get a polynomial S and a proper remainder R/Q.

(2) Integrate S term-by-term.

(3) Handle the proper part $\int R/Q\,dx$ by partial fractions or a substitution, depending on the factorization of Q.

Example 1.5

Evaluate $\int \frac{x^3+x+1}{x^2+1}\,dx$. *Since* $\deg P = 3 \geq 2 = \deg Q$, *divide to get*

$$\frac{x^3+x+1}{x^2+1} = x + \frac{1}{x^2+1},$$

hence $\int \frac{x^3+x+1}{x^2+1}\,dx = \int x\,dx + \int \frac{1}{x^2+1}\,dx = \frac{x^2}{2} + \arctan x + C.$

Example 1.6

Evaluate $\int \frac{2x^2+3x+5}{x^2-1}\,dx$. *Division yields*

$$\frac{2x^2+3x+5}{x^2-1} = 2 + \frac{3x+7}{x^2-1} = 2 + \frac{A}{x-1} + \frac{B}{x+1},$$

with $3x+7 = A(x+1) + B(x-1) \Rightarrow A = 5,\ B = -2.$ *Therefore*

$$\int \frac{2x^2+3x+5}{x^2-1}\,dx = \int 2\,dx + \int \left(\frac{5}{x-1} - \frac{2}{x+1}\right) dx = 2x + 5\ln|x-1| - 2\ln|x+1| + C.$$

1.2 Multiple Integrals: Concepts and Methods

1.2.1 Double Integrals

Definition 1.7

[7] In the xy-plane, there is a circular and limited region G where the continuous function $f(x,y)$ is defined. On the set of real numbers, the double integral of function f is shown as:

$$\iint_G f(x,y)\,dB \tag{1.8}$$

In this context, dB denotes the differential space element within the region G.

Remark 1.8

The double integral is calculated by dividing the region G into infinitesimally small sub-regions, approximating the volume of each sub-region, and summing up these volumes.

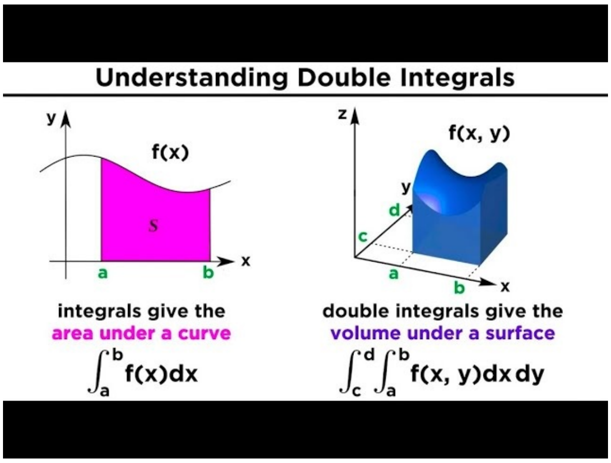


Figure 1.3: A visual representation of the concept of double integrals

Double Integration Properties

Assuming α, β be real numbers and $\Omega \subset \mathbb{R}^2$; we may express f and g as functions of the variables (x, y). Therefore, we can assert:

1. Linear:

$$\iint (\alpha f + \beta g)(x, y)dxdy = \alpha \iint f(x, y)dx + \beta \iint g(x, y)dxdy. \tag{1.9}$$

2. Association: For $\Omega = \Omega_1 \cup \Omega_2$, $\Omega_1 \cap \Omega_2 = \oslash$, so we have :

$$\iint_{\Omega} f(x, y)dxdy = \iint_{\Omega_1} f(x, y)dxdy + \iint_{\Omega_2} f(x, y)dxdy. \tag{1.10}$$

3. A positive sign: for f a positive function in Ω :

$$\iint_{\Omega} f(x, y)dxdy \geq 0. \tag{1.11}$$

4. For the function f, for the variable x and the function g for variable y so the calculation of the double integral $\iint (f.g)dxdy$ in this area $D = [p, q] \times [v, w]$ is equal the multiply the simple integral of each one for it's variable.

$$\int_p^q \int_v^w f(x)g(y)dxdy = (\int_p^q f(x)dx).(\int_v^w g(y)dy). \tag{1.12}$$

Example 1.7

Let $D = [1,2] \times [0,1]$ and $f(x,y) = \ln x \cdot (1+y)$. Since $f(x,y) = \ln x \cdot (1+y)$ is separable, we can write:

$$\iint_D f(x,y)\,dA = \left(\int_1^2 \ln x\,dx\right)\left(\int_0^1 (1+y)\,dy\right).$$

Evaluate the integrals:

$$\int_1^2 \ln x\,dx = [x\ln x - x]_1^2 = (2\ln 2 - 2) - (1\ln 1 - 1) = 2\ln 2 - 1,$$

$$\int_0^1 (1+y)\,dy = \left[y + \frac{1}{2}y^2\right]_0^1 = \left(1 + \frac{1}{2}\cdot 1^2\right) - \left(0 + \frac{1}{2}\cdot 0^2\right) = \frac{3}{2}.$$

So the final result is:

$$(2\ln 2 - 1)\cdot\frac{3}{2}.$$

Techniques of Calculating Double Integrals

1-Fubini's Theorem

Fubini's theorem provides a fundamental method for evaluating double integrals by transforming them into iterated integrals. In a rectangular region G containing a continuous function $f(x,y)$, the theorem permits the computation of the integral by initially integrating with respect to one variable and then the other, confirming that the order of integration does not influence the final outcome.

Theorem 1.9

[13] You may represent the rectangular area $G = \{(x,y)\,|\,p \le x \le q,\ v \le y \le w\}$ as follows: $f(x,y)$ is a continuous function defined on this region.
Then Fubini's theorem states:

$$\iint_R f(x,y)\,dA = \int_p^q \int_v^w f(x,y)\,dy\,dx = \int_c^d \int_p^q f(x,y)\,dx\,dy. \tag{1.13}$$

The double integral of f over G is similar to the iterated integral derived by first integrating with regard to y and subsequently with respect to x, or vice versa.

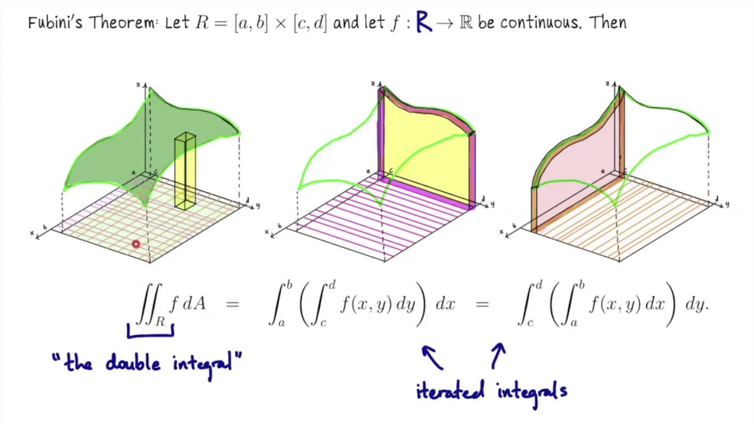


Figure 1.4: Figure for understanding Fubini's theorem

Proposition 1.10

[13] Fubini's Theorem can be applied under the following conditions:

1. *$f(x, y)$ is continuous on the region G.*
2. *The region of integration R is a rectangular region defined by $p \leq x \leq q$ and $v \leq y \leq w$.*

Example 1.8

Using Fubini's theorem, we can interchange the order of integration:

$$\iint_G (x^2 + y^2)\,dA = \int_0^2 \int_1^3 (x^2 + y^2)\,dy\,dx$$

Now, calculate the iterated integrals:

$$\int_0^2 \int_1^3 (x^2 + y^2)\,dy\,dx = \int_0^2 \left[x^2 y + \frac{y^3}{3}\right]_{y=1}^{y=3} dx$$

$$= \int_0^2 \left[3x^2 + 9 - (x^2 + \frac{1}{3})\right] dx$$

$$= \int_0^2 (2x^2 + \frac{26}{3})\,dx$$

$$= \left[\frac{2}{3}x^3 + \frac{26}{3}x\right]_{x=0}^{x=2} = \frac{16}{3} + \frac{52}{3} = \frac{68}{3}.$$

2-Substitution

Substitution simplifies double integrals, particularly in the context of non-rectangular regions. The formula for variable transformation is as follows:

$$\iint_D f(x,y)\,dA = \iint_R f(g(u,v),h(u,v))|J(u,v)|\,du\,dv, \tag{1.14}$$

where R is the transformed region, (u,v) are new variables, and $|J(u,v)|$ is the Jacobian determinant.

Polar Coordinates

$$\begin{cases} x = rcos(\theta) + x_0 \\ y = rsin(\theta) + x_0 \end{cases} \tag{1.15}$$

where:

$dxdy = rdrd\theta$

$$\frac{D(x,y)}{D(r,\theta)} = \begin{pmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{pmatrix} = \begin{pmatrix} cos\theta & -rsin\theta \\ sin\theta & rcos\theta \end{pmatrix} |\frac{D(x,y)}{D(r,\theta)}| = | \begin{matrix} cos\theta & -rsin\theta \\ sin\theta & rcos\theta \end{matrix} | = rcos^2\theta + rsin^2 = r.$$

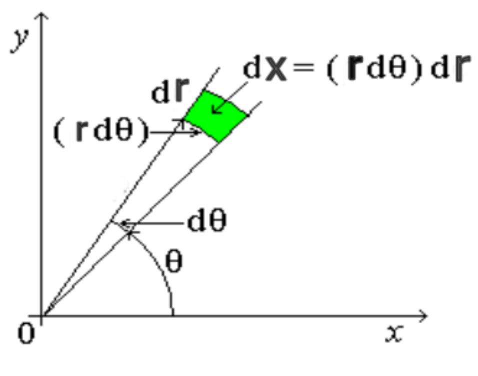


Figure 1.5: demonstration of how to understand polar coordinates

Example 1.9

Evaluate this integral by :

$$\iint_D e^{x^2+y^2}\, dA\; D = \{(x,y) \in \Re^2, x^2+y^2 = 1 \text{ and } x \geq 0\}$$

By used polar coordinates we get :

$$\iint_D e^{x^2+y^2}\, dA = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^1 e^{r^2} \cdot r\, dr\, d\theta = (\int_0^1 re^{r^2})(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\theta) = (\frac{1}{2}e^{r^2})|_0^1)(\theta|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}) =$$

$$(\frac{e}{2}. - \frac{1}{2})(\frac{\pi}{2} + \frac{\pi}{2}) = \frac{(e-1)\pi}{2}.$$

Example 1.10

Evaluate: $\int_D \sqrt{x^2+y^2}\, dxdy$, *where* $D = \{(x,y) \in \mathbb{R}^2 \mid x \geq 0, 1 \leq x^2+y^2 \leq 2y\}$.

Use polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$, $dxdy = r\, dr\, d\theta$, *and* $\sqrt{x^2+y^2} = r$. *The region* D *gives* $1 \leq r \leq 2\sin\theta$ *and* $\frac{\pi}{6} \leq \theta \leq \frac{\pi}{2}$ *(first quadrant).*

The integral becomes:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{2}} \int_1^{2\sin\theta} r \cdot r\, dr\, d\theta = \frac{56\sqrt{2}}{36} - \frac{\pi}{9}.$$

1.2.2 Triple Integrals

Definition 1.8

[7] The triple integral extends integration to three-dimensional space. It is used to calculate the volume of a region in 3D. The notation for a triple integral is:

$$\iiint_E f(x,y,z)\, dV, \tag{1.16}$$

where E *denotes the region in three-dimensional space, and* dV *signifies the infinitesimal volume element.*

Remark 1.11

The triple integral is calculated by dividing the region R *in three-dimensional space into infinitesimally small sub-regions. The volume of each sub-region is approximated, and the contributions from all sub-regions are summed up to compute the total value,* *which can represent volume, mass, or other quantities depending on the integrand.*

Figure 1.6: Example for understanding triple integrals

Triple Integration Properties

For any (x, y, z) in $\Omega \subset \mathbb{R}^3$, let f and g be functions of these variables. Then α, β belong to $\mathbb{R}$. Here are the properties that are held:

1. Linearity:

$$\iiint (\alpha f+\beta g)(x,y,z)\,dx\,dy\,dz = \alpha \iiint f(x,y,z)\,dx\,dy\,dz+\beta \iiint g(x,y,z)\,dx\,dy\,dz. \tag{1.17}$$

2. Associativity: For $\Omega = \Omega_1 \cup \Omega_2$ with $\Omega_1 \cap \Omega_2 = \varnothing$, we have:

$$\iiint_{\Omega} f(x,y,z)\,dx\,dy\,dz = \iiint_{\Omega_1} f(x,y,z)\,dx\,dy\,dz + \iiint_{\Omega_2} f(x,y,z)\,dx\,dy\,dz. \tag{1.18}$$

3. Positivity: For f a positive function in Ω:

$$\iiint_{\Omega} f(x,y,z)\,dx\,dy\,dz \geq 0. \tag{1.19}$$

4. Separable Functions: Let $f(x)$ be a function solely dependent on x, $g(y)$ be a function exclusively reliant on y, and $h(z)$ be a function that pertains only to z. For the area $D = [p,q] \times [v,w] \times [e,j]$, the triple integral is valid:

$$\int_p^q \int_v^w \int_e^j f(x)g(y)h(z)\,dx\,dy\,dz = \left(\int_p^q f(x)\,dx\right) \cdot \left(\int_v^w g(y)\,dy\right) \cdot \left(\int_e^j h(z)\,dz\right). \tag{1.20}$$

Example 1.11

Evaluate this integral

$$\iiint_D x^2yz^3\,dx\,dy\,dz,\ D=[0,1]\times[1,2]\times[2,3]$$

Using the separable functions property:

$$\iiint_D x^2yz^3\,dx\,dy\,dz=\int_0^1 x^2\,dx\cdot\int_1^2 y\,dy\cdot\int_2^3 z^3\,dz=\left[\frac{x^3}{3}\right]_0^1\left[\frac{y^2}{2}\right]_1^2\left[\frac{z^4}{4}\right]_2^3=(\frac{1}{3})(\frac{3}{2})(\frac{65}{4})=\frac{65}{8}.$$

Methods of Calculating Triple Integrals

1-Fubini's theorem

Analogous to double integrals, triple integrals can be computed interactively by integrating one variable sequentially.

Theorem 1.12

[7] Consider a continuous function $f(x,y,z)$ based on a rectangular region R in the $x-y-z$-plane, which can be represented as $R=(x,y,z)\,|\,p\le x\le q,\ v\le y\le w\ \ e\le z\le f$. Then Fubini's theorem states:

$$\iiint_E f(x,y,z)dV=\int_p^q(\int_v^w(\int_e^f f(x,y,z)dz)dy)dx=\int_v^w(\int_e^f(\int_p^q f(x,y,z)\,dx)dz)dy.$$

Example 1.12

Determine this integral

$$\iiint_D 2x-y\,dx\,dy\,dz, D=[0,y-z]\times[0,z^2]\times[0,2].$$

$$\int_0^2\int_0^{z^2}\int_0^{y-z}(2x-y)dxdydz=\int_0^2(\int_0^{z^2}(\int_0^{y-z}(2x-y)dx)dy)dz=$$
$$\int_0^2(\int_0^{z^2}(x^2-yx)|_0^{y-z})dy)dz=\int_0^2(\int_0^{z^2}(y-z)^2-y^2+yzdy)dz=$$
$$\int_0^2(\int_0^{z^2}y^2+z^2-2yz-y^2+yzdy)dz=\int_0^2(\int_0^{z^2}z^2-yzdy)dz=\int_0^2(z^2y|_0^{z^2}-\frac{z}{2}y^2|_0^{z^2})dz=$$
$$\int_0^2(z^4-\frac{z^5}{2})dz=\frac{z^5}{5}|_0^2-\frac{z^6}{12}|_0^2=\frac{2^5}{5}-\frac{2^6}{12}.$$

Example 1.13

Determine this integral

$$\iiint_D xe^{-y}\,dx\,dy\,dz, D = [0, 2z] \times [0, \ln x] \times [1, 2].$$

$$\int_1^2 \int_0^{2z} \int_0^{\ln x} xe^{-y} dydxdz = \int_1^2 \int_0^{2z} x \int_0^{\ln x} (e^{-y}) dydxdz =$$
$$\int_1^2 \int_0^{2z} x(-e^{-y})|_0^{\ln x} dxdz = \int_1^2 \int_0^{2z} x(-\frac{1}{x} - 1)dxdz =$$
$$\int_1^2 (\int_0^{2z} -x + 1dx)dz = \int_1^2 (\frac{-x^2}{2}|_0^{2z} + x|_0^{2z})dz =$$
$$\int_1^2 (-2z^2 + 2zdz) = \frac{-2}{3} z^3|_1^2 + z^2|_1^2 = \frac{-14}{3} + 3 = \frac{-5}{3}.$$

2-Substitution

Substitution can simplify triple integrals, particularly in regions with non-standard shapes. The transformation formula for variable changes is expanded to three dimensions:

$$\iiint_E f(x, y, z)\,dV = \iiint_W f(u, v, w)|J(u, v, w)|\,du\,dv\,dw. \tag{1.21}$$

Let W denote the transformed region, with (u, v, w) representing the new variables, and $|J(u, v, w)|$ indicating the Jacobian determinant.

Spherical Coordinates

Spherical coordinates are a system for representing points in three-dimensional space using three parameters: radial distance (r), polar angle (θ), and azimuthal angle (φ), (r, θ, φ).

Spherical coordinates can be converted to Cartesian coordinates as follows:

$$\begin{aligned} x &= r \sin\varphi \cos\theta \\ y &= r \sin\varphi \sin\theta \\ z &= r \cos\varphi \end{aligned} \tag{1.22}$$

The Jacobian matrix of f is

$$J_f = \begin{pmatrix} \cos\theta\cos\varphi & -r\sin\theta\cos\varphi & -r\cos\theta\sin\varphi \\ \sin\theta\cos\varphi & r\cos\theta\cos\varphi & -r\sin\theta\sin\varphi \\ \sin\varphi & 0 & r\cos\varphi \end{pmatrix}$$

and its determinant is

$$\det(J_f) = r^2 \cos\varphi$$

The change of variables formula is then written as:

$$\iiint_D f(x,y,z)\,dx\,dy\,dz = \iiint_\Omega f(r\cos\theta\cos\varphi, r\sin\theta\cos\varphi, r\sin\varphi)\,r^2|\cos\varphi|\,dr\,d\theta\,d\varphi$$

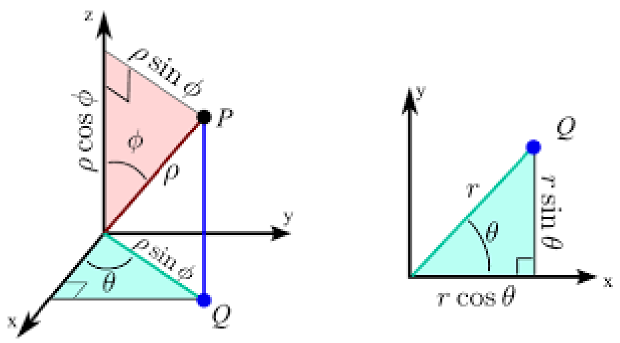


Figure 1.7: Picture to help in comprehending spherical coordinates

Example 1.14

We compute

$$\iiint_D dx\,dy\,dz \quad \textit{where} \quad D = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 \le 1\}$$

with the bounds

$$0 \le r \le 1, \quad 0 \le \theta \le 2\pi, \quad -\frac{\pi}{2} \le \varphi \le \frac{\pi}{2}$$

The formula becomes:

$$\iiint_D f(x,y,z)\,dx\,dy\,dz = \iiint_\Omega r^2|\cos\varphi|\,dr\,d\theta\,d\varphi$$

This can be written as:

$$\left(\int_0^1 r^2\,dr\right)\left(\int_0^{2\pi} d\theta\right)\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\varphi\,d\varphi\right)$$

Which gives:

$$\frac{4}{3}\pi.$$

Example 1.15

Calculate the following integral:

$$I = \iiint\limits_V \sqrt{x^2 + y^2 + z^2}\, dx\, dy\, dz,$$

where V denotes the sphere centered at $(0, 0, 0)$ with radius R.

Switching to spherical coordinates, we obtain:

$$I = \int_0^R \int_0^{2\pi} \int_0^{\pi} \rho^2 |\cos\phi| \sqrt{\rho^2 \cos^2\phi \cos^2\theta + \rho^2 \cos^2\phi \sin^2\theta + \rho^2 \sin^2\phi}\, d\rho\, d\theta\, d\phi.$$

This simplifies to:

$$I = \int_0^R \int_0^{2\pi} \int_0^{\pi} \rho^3 |\cos\phi|\, d\rho\, d\phi\, d\theta.$$

Subsequent integration produces:

$$I = \frac{\pi R^4}{2} \int_{-\pi/2}^{\pi/2} \cos\phi\, d\phi = \frac{\pi R^4}{2} \sin\phi \Big|_{-\pi/2}^{\pi/2} = \pi R^4.$$

Cylindrical Coordinates

Cylindrical coordinates are a system for representing points in three-dimensional space using three parameters: radial distance (r), polar angle (θ), and height (z).

$$f : \Re^3 \to \Re^3, (r, \theta, z) \to (x, y, z) = (r\cos\theta, r\sin\theta, z).$$

Cylindrical coordinates can be converted to Cartesian coordinates as follows:

$$\begin{aligned} x &= r\cos\theta, \\ y &= r\sin\theta, \\ z &= z. \end{aligned} \tag{1.23}$$

The Jacobian matrix of the transformation f is expressed as:

$$J(f) = \begin{bmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The determinant of $J(\theta)$ is:

$$\det(J(f)) = r.$$

If $f(\mathcal{D}) = \mathcal{D}$, the change of variables formula is expressed as:

$$\iiint_{\mathcal{D}} f(x, y, z)\, dx\, dy\, dz = \iiint_{\mathcal{D}} f(r\cos\theta, r\sin\theta, z)\ \det(J(f))\, dr\, d\theta\, dz.$$

with $dxdydz = rdrd\theta dz$.

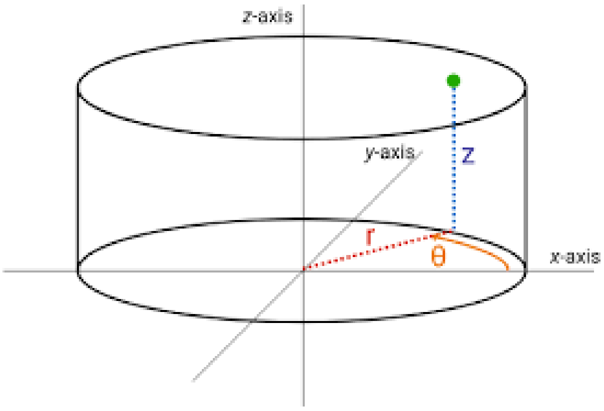


Figure 1.8: Conceptualization of cylindrical coordinates illustrated by graphic

Example 1.16

Calculate the following integral:

$$I = \iiint\limits_{D} \frac{z}{\sqrt{x^2+y^2}}\, dx\, dy\, dz,\ D = (x, y, z) \in \Re^3 : x^2 + y^2 \leq 1, 0 \leq z \leq 1.$$

we used the cylindrical coordinates for solving this integral :

$x = r\cos(\theta)$

$y = r\sin(\theta)$

$z = z$

and $dxdydz = rdrd\theta dz,\ f(x, y, z) = \dfrac{z}{\sqrt{x^2+y^2}} \Longrightarrow_{(r,\theta)} = \dfrac{z}{r}$.

For the r *area we got it from this condition :*

$x^2 + y^2 \leq 1 \Longrightarrow_{(r,\theta)} r^2 \leq 1 \Longrightarrow_{(r,\theta)} 0 \leq r \leq 1$. *we had the* z *area,* $0 \leq z \leq 1$, *and for the* θ *area we always take* $0 \leq \theta \leq 2\pi$.

So we calculate the integral:

$\displaystyle\int_0^{2\Pi}\int_0^1\int_0^1 zdzdrd\theta = (\int_0^{2\Pi} d\theta)(\int_0^1 dr)(\int_0^1 zdz) = (\theta|_0^{2\pi})(r|_0^1)(\frac{z^2}{2}|_0^1) =$
$(2\pi)(1)(\dfrac{1}{2}) = \pi.$

1.3 Utilizing for Calculating Areas and Volumes

Calculating the Area

Example 1.17

Determine the area beneath the curve $y = x^2$ from $x = 0$ to $x = 2$.

The region is denoted by the integral:

$$A = \int_0^2 x^2 \, dx.$$

Evaluating the integral:

$$A = \left[\frac{x^3}{3}\right]_0^2 = \frac{2^3}{3} - \frac{0^3}{3} = \frac{8}{3}.$$

Consequently:

$$A = \frac{8}{3}.$$

Example 1.18

The integral

$$\iint_D dx\, dy,$$

provides the area of the domain D.

We determine the area of the region enclosed by the curves:

$$y + x^2 = 1 \quad \text{and} \quad y = 0.$$

The domain D is defined as:

$$D = \{(x, y) \in \mathbb{R}^2 \mid 0 \le y \le 1 - x^2 \text{ and } -1 \le x \le 1\}.$$

The area is expressed as:

$$A = \int_{-1}^1 \int_0^{1-x^2} 1 \, dy \, dx = \int_{-1}^1 [\int_0^{1-x^2} 1 \, dy] dx = \int_{-1}^1 (y \mid_0^{1-x^2}) dx = \int_{-1}^1 (1 - x^2) dx = (\frac{-x^3}{3} + x) \mid_{-1}^1 = \frac{1}{3} + 1 - (\frac{-1}{3} + 1) = \frac{2}{3}.$$

Calculating the Volumes

Example 1.19

Find the volume of: $\iiint_E (x^2 + y^2 + z^2)\,dV$ *over the region* E *defined by cylindrical coordinates* $r = 1$, $0 \le \theta \le \pi$, *and* $0 \le z \le 2$. *We want to evaluate the triple integral:*

$$\iiint_E (x^2 + y^2 + z^2)\,dV$$

over the region E *defined by cylindrical coordinates* $r = 1$, $0 \le \theta \le \pi$, *and* $0 \le z \le 2$.

In cylindrical coordinates, the volume element dV *is given by:*

$$dV = r\,dr\,d\theta\,dz$$

So, the triple integral becomes:

$$\iiint_E (x^2 + y^2 + z^2)\,dV = \int_0^{\pi}\int_0^{1}\int_0^{2} (r^2 + z^2)\cdot r\,dz\,dr\,d\theta$$

Now, calculate the iterated integrals:

$$\int_0^{\pi}\int_0^{1}\int_0^{2} (r^2 + z^2)\cdot r\,dz\,dr\,d\theta = \int_0^{\pi}\int_0^{1} \left[\frac{r^3}{3}z + \frac{z^3}{3}\right]_{z=0}^{z=2} dr\,d\theta$$

$$= (\int_0^{\pi} d\theta).(\int_0^{1}\left(\frac{2r^3}{3} + \frac{8}{3}\right) dr) = (\frac{r^4}{6}|_0^1)(\theta|_0^{\pi}) = (\frac{\pi}{6}).$$

Example 1.20

Find the volume of:

$$\iiint_E (r^2 z)\,dV$$

across the domain E *delineated by cylindrical coordinates* $0 \le r \le 2$, $0 \le \theta \le \dfrac{\pi}{4}$, *and* $0 \le z \le 3$.

In cylindrical coordinates, the volume element dV *is given by:*

$$dV = r\,dr\,d\theta\,dz$$

So, the triple integral becomes:

$$\iiint_E (r^2 z)\, dV = \int_0^3 \int_0^{\frac{\pi}{4}} \int_0^2 (r^2 z) \cdot r\, dr\, d\theta\, dz$$

$$= \int_0^3 z. \int_0^2 r^3 dr. \int_0^{\frac{\pi}{4}} d\theta = (\frac{z^2}{2}|_0^3).(\frac{r^4}{4}|_0^2).(\theta|_0^{\frac{\pi}{4}}) = (\frac{9}{2})(\frac{16}{4})(\frac{\pi}{4}) = \frac{9\pi}{2}$$

CHAPTER 2

Improper Integral

2.1 Improper Integrals of Functions on an Unbounded Interval

A standard Riemann integral is established for a function that maintains continuity over a closed and finite interval, exemplified by $[p, q]$. An improper integral arises when one or both of these conditions fail to be satisfied. This takes place in two main scenarios:

1. The interval of integration is infinite, for example $[p, \infty)$, $(-\infty, q]$, or $(-\infty, \infty)$.

2. The function $f(x)$ possesses one or more infinite discontinuities within a bounded interval of integration.

Definition 2.1

[10] Define f be an integrable function (in the sense of Riemann) on an unbounded interval $[p, q[\,,]-\infty, q]\,, [a, +\infty[$ or $]p, q]$. We define that f represents improper integration over the interval $[p, +\infty[$ as follows:

$$\lim_{x \to +\infty} \int_p^x f(t)dt, \tag{2.1}$$

existing and finished.

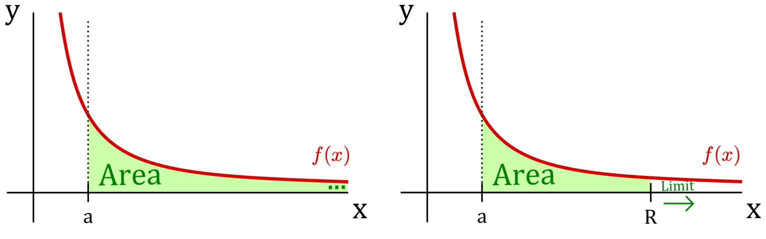


Figure 2.1: Visual aid for comprehending improper integration

Definition 2.2

[4] The following is our definition of the convergent improper integral:

$$\lim_{x \to +\infty} \int_p^x f(t)dt \text{ or } \lim_{x \to a} \int_q^x f(t)dt, \tag{2.2}$$

existing and complete. If the limit is non-existent and infinite, then the improper integral is divergent rather than converging.

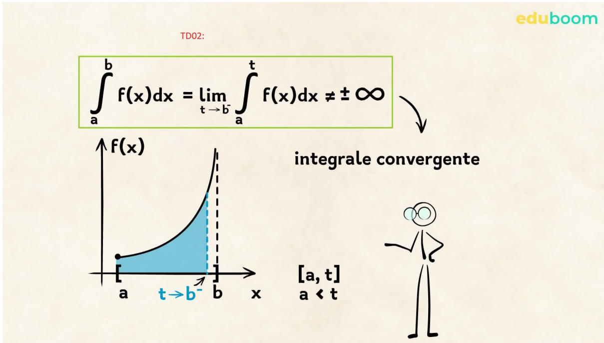


Figure 2.2: Graphical representation of convergent improper integral

Definition 2.3

If the improper integral $\int_p^x f(t)dt$ is evaluated at points p and q, and let v be an element of the interval $]p, q]$. This integral is convergent if the two improper integrals, $\int_p^v f(t)dt$ and $\int_v^q f(t)dt$, are convergent, where:

$$\int_p^q f(t)dt = \int_p^v f(t)dt + \int_v^q f(t)dt. \tag{2.3}$$

Example 2.1

$\int_0^{+\infty} e^t dt = \lim_{x \to +\infty} \int_0^x e^t dt = \lim_{x \to +\infty} (e^t)|_0^x) = \lim_{x \to +\infty} (e^x - 1) = +\infty$. *so we have a divergent integral.*

Example 2.2

$\int_1^2 = \frac{dx}{x-2} = \lim_{x \to 2} \int_1^x \frac{dx}{x-2} = \lim_{x \to 2} \ln|x-2||_1^x = \lim_{x \to 2} (\ln|x-2| - \ln(1)) = -\infty$. *so we have diverged integral.*

Remark 2.1

Let it be $\int_p^q f(t)dt$ *and* $\int_p^q g(t)dt$ *improper integrals at* q *:*

1. *The integral* $\int_p^q f(t)dt$ *is convergent, and the integral* $\int_p^q g(t)dt$ *is also convergent. The integral* $\int_p^q f(t)dt + \int_p^q g(t)dt$ *is convergent.*

2. *The integral* $\int_p^q f(t)dt$ *is divergent, but the integral* $\int_p^q g(t)dt$ *is convergent. The integral* $\int_p^q f(t)dt + \int_p^q g(t)dt$ *is divergent.*

3. *The integrals* $\int_p^q f(t)dt$ *and* $\int_p^q g(t)dt$ *are both divergent, so the nature of integration cannot be assessed if they include different areas of integration.*

Example 2.3

Evaluate this integral:

$$\int_1^2 \frac{dt}{(t-1)(t-2)}$$

$\exists \frac{3}{2} \in]1,2[$ *so*

$$\int_1^2 \frac{dt}{(t-1)(t-2)} = \int_1^{\frac{3}{2}} \frac{dt}{(t-1)(t-2)} + \int_{\frac{3}{2}}^2 \frac{dt}{(t-1)(t-2)} \Rightarrow I = I_1 + I_2$$

$I_1 = \int_1^{\frac{3}{2}} \frac{dt}{(t-1)(t-2)} = \lim_{x\to 1} \int_x^{\frac{3}{2}} \frac{dt}{(t-1)(t-2)} = \lim_{x\to 1} \int_1^{\frac{3}{2}} \frac{-dt}{t-1} + \int_1^{\frac{3}{2}} \frac{dt}{t-2} = \lim_{x\to 1} - \ln|t-1||_x^{\frac{3}{2}} + \ln|t-2||_x^{\frac{3}{2}} = -\infty.$

the same thing for I_2 *we found that* $I_2 = -\infty$ *so the integral* $I = I_1 + I_2$ *divergent integral because the tow integrals* I_1 *and* I_2 *they have difference area integration.*

Riemann's integral[10]

The convergence of integrals of the type $\int \frac{1}{t^\alpha} dt$, commonly referred to as **p-integrals**, is contingent upon the value of α and the limits of integration.

- **Integrals over an Infinite Interval:** For an integral of the form $\int_p^\infty \frac{dt}{t^\alpha}$ with $p > 0$, the integral **converges** if $\alpha > 1$ and **diverges** if $\alpha \leq 1$.
- **Integrals with a Discontinuity at Zero:** For an integral of the form $\int_0^q \frac{dt}{t^\alpha}$ with $q > 0$, where the function is discontinuous at $t = 0$, the integral **converges** if $\alpha < 1$ and **diverges** if $\alpha \geq 1$.

Example 2.4

1. $\int_1^{+\infty} \frac{dt}{t^2}$ *convergent Riemann's integral next to* $+\infty$ *because* $\alpha = 2 > 1$.
2. $\int_3^{+\infty} \frac{dt}{t}$ *divergent Riemann's integral next to* $+\infty$ *because* $\alpha = 1$.
3. $\int_0^1 \frac{dt}{\sqrt{t}}$ *convergent Riemann's integral next to* 0 *because* $\alpha = \frac{1}{2} < 1$.
4. $\int_0^2 \frac{dt}{t^5}$ *divergent Riemann's integral next to* 0 *because* $\alpha = 5 > 1$.

Bertrand integral [10]

1-$(\int_e^{+\infty} \frac{dx}{x^\alpha (\ln x)^\beta})$ convergent $\Rightarrow \alpha > 1$ or $(\alpha = 1 and \beta > 1)$.

2-$(\int_0^{\frac{1}{e}} \frac{dx}{x^\alpha (\ln x)^\beta})$ convergent $\Rightarrow \alpha < 1$ or $(\alpha = 1 and \beta > 1)$.

2.1.1 Improper Integral for Positive Functions

Theorem 2.2 (***Comparison Theorem***)

Let it be f and g positive and continuous functions in the integral $[p, q[$ if exist a $v \in [p, q[$ where:

$$\forall t \in [v, q[\,, 0 \leq f(t) \leq g(t) \tag{2.4}$$

so:

1-$\int_p^q g(t)dt$ convergent $\Rightarrow \int_p^q f(t)dt$ convergent.

2- $\int_p^q f(t)dt$ *divergent* $\Rightarrow \int_p^q g(t)dt$ *divergent.*

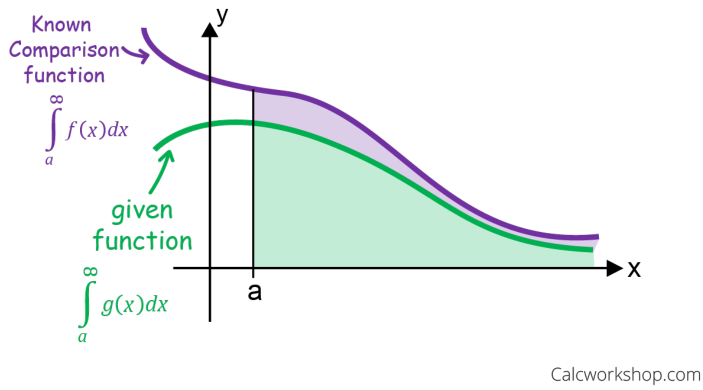


Figure 2.3: Comparison figure for comprehending theorem

Example 2.5

$\int_1^{+\infty} \frac{t}{\sqrt{1+t^6}}dt$, *by the comparison theorem we found:*

$$\frac{t}{\sqrt{1+t^6}} \leq \frac{t}{\sqrt{t^6}} \leq \frac{1}{t^3}$$

the integral $\int_1^{+\infty} \frac{1}{t^3}dt$ *is convergent, since Riemann's criterion indicates that* $\alpha = 3 > 1$.*The integral* $\int_1^{+\infty} \frac{t}{\sqrt{1+t^6}}dt$ *converges using the comparison theorem.*

Example 2.6

$\int_0^1 \frac{e^{-t}}{\sqrt{t}}dt$, *by the comparison theorem we found :*

$$0 < t \leq 1 \rightarrow e^{-t} \leq 1 \rightarrow \frac{e^{-t}}{\sqrt{t}} \leq \frac{1}{\sqrt{t}}$$

the integral $\int_0^1 \frac{1}{\sqrt{t}}dt$, *convergent (Riemann's integral* $\alpha = \frac{1}{2} < 1$)*The integral* $\int_0^1 \frac{e^{-t}}{\sqrt{t}}dt$ *converges using the comparison theorem.*

Proposition 2.3

Let f and g be positive and continuous functions on the interval $[p, q[$. For all $t \in [p, q[$, $g(t) \neq 0$ and $\lim_{t \to q} \frac{f(t)}{g(t)} = m$ therefore:

1. *If it was $0 \leq m < +\infty$ $(m \neq +\infty)$ and $\int_p^q g(t)dt$ convergent $\Rightarrow$ $\int_p^q f(t)dt$ convergent.*
2. *If it was $0 < m \leq +\infty$ $(m \neq 0)$ and $\int_p^q g(t)dt$ diverged $\Rightarrow$ $int_p^q f(t)dt$ divergent.*
3. *If it was $0 < m < +\infty$ $(m \neq 0, m \neq +\infty)$ so we have same nature for $\int_p^q f(t)dt$* and $\int_p^q g(t)dt \to f \sim_q g$ *(f equivalent g behind q).*

Example 2.7

$f(t) = \frac{1}{t\sqrt{1+t^2}}$, $g(t) = \frac{1}{t^2}$ *tow functions define at $[1, +\infty[$ and* $\lim_{x \to +\infty} = \frac{f(x)}{g(x)} = \lim_{x \to +\infty} \frac{x^2}{x\sqrt{1+x^2}} = 1$ *given that the integral* $\int_1^{+\infty} g(x)dx = \int_1^{+\infty} \frac{1}{t^2}$ *convergent because Riemann's integral $\alpha = 2 > 1$ so the integral* $\int_1^{+\infty} f(x)dx$ *convergent.*

Remark 2.4

1. *If $\lim_{t \to b} f(t)g(t)$ exist and finished so :* $\int_a^b g(t)f^{'}(t)dt$ *and* $\int_a^b f(t)g^{'}(t)dt$ *they have same nature.*
   $\int_a^b f^{'}(t)g(t)dt = F(t)g(t)|_a^b - \int_a^b f(t)g^{'}(t)dt$ *(integration by parts way).*
2. *The integrals* $\int_a^b f(t)dt$ *and* $\int_\alpha^\beta f(\varepsilon(x))\varepsilon'(x)dx$ *they have same nature and we have :*
   $\int_a^b f(t)dt = \int_\alpha^\beta f(\varepsilon(x))\varepsilon'(x)dx$ *(substitution way).*

Example 2.8

Calculate the following integral:

$\int_0^1 \frac{\ln(t)}{\sqrt{t}} dt$, *the function is improper in* $x = 0$ *so :*

$\lim_{x\to 0} \int_x^1 \ln(t) t^{\frac{-1}{2}}$, *we use integral by parts method:*

$$\begin{aligned} u &= \ln t \to u' = \frac{1}{t}, \\ v' &= t^{\frac{-1}{2}} \to v = 2\sqrt{t}. \end{aligned}$$

$\lim_{x\to 0} \int_x^1 \ln(t) t^{\frac{-1}{2}} = \lim_{x\to 0} (2\ln t\sqrt{t}\Big|_x^1 - 2\int_x^1 \frac{1}{\sqrt{t}} dt) = \lim_{x\to 0}(-2\ln x\sqrt{x} - 4\sqrt{t}\Big|_x^1) = \lim_{x\to 0}(-2\ln x\sqrt{x} - 4 + \sqrt{x}) = -4.$

So the integral $\int_0^1 \frac{\ln(t)}{\sqrt{t}} dt$ *is convergent.*

Example 2.9

$\int_{\frac{2}{\Pi}}^{+\infty} \ln(\sin(\frac{1}{t}))dt$, *by the comparison theorem we found :*

$$\sin(\frac{1}{t}) \sim_{+\infty} \frac{1}{t} \to \ln(\sin(\frac{1}{t})) \sim_{+\infty} \ln(\frac{1}{t}) = -\ln(t)$$

so the integral $\int_{\frac{2}{\Pi}}^{+\infty} -\ln(t)dt = \lim_{x\to+\infty} \int_{\frac{2}{\Pi}}^{x} -\ln(t)dt = \lim_{x\to+\infty} (t\ln(t) - t)|_{\frac{2}{\Pi}}^{x} = \lim_{x\to+\infty}(x\ln(x) - x - \frac{2}{\Pi}\ln(\frac{2}{\Pi}) + \frac{2}{\Pi}) = +\infty.$

so the integral $\int_{\frac{2}{\Pi}}^{+\infty} -\ln(t)dt$ *divergent* $\Rightarrow$ $\int_{\frac{2}{\Pi}}^{+\infty} \ln(\sin(\frac{1}{t}))dt$ *is divergent by the comparison theorem.*

Riemann's rule

Proposition 2.5

[18] Consider f *be a positive function defined on the interval* $[p, q[$. *and*

$$\lim_{t\to q} (q - t)^{\alpha} f(t) = m, \tag{2.5}$$

where $\alpha \in \Re$ *and* m *finished and existing so:*

1/If $\alpha < 1 \Rightarrow \int_p^q f(t)dt$ *convergent.*

2/If $\alpha \geq 1, m \neq 0 \Rightarrow \int_p^q f(t)dt$ *divergent.*

Proposition 2.6

If $\int_p^q f(t)dt$ *improper integral at* p *when we apply Riemann's rule for know this integral nature we calculate :* $\lim_{t\to p}(t-p)^{\alpha} f(t) = m$.
Let it be f *a positive function integration in* $[a, +\infty[\,, p > 0$ *and*

$$\lim_{t\to+\infty} (t)^{\alpha} f(t) = m, \tag{2.6}$$

where $\alpha \in \Re$ *and* m *finished and existing.*
1/If $\alpha > 1 \Rightarrow \int_p^{+\infty} f(t)dt$ *convergent.*
2/If $\alpha \leq 1, m \neq 0 \Rightarrow \int_p^{+\infty} f(t)dt$ *divergent.*

Theorem 2.7

[15] Let it be f *a positive function and continuous by piece at* $[p, +\infty[$ *if found tow real numbers* α *and* $\lambda \neq 0$ *where* $f(t) \sim_{+\infty} \frac{\lambda}{t^{\alpha}}$ *so the integral* $\int_p^{+\infty} f(t)dt$ *convergent if* $\alpha > 1$ *and divergent for* $\alpha \leq 1$. *The same thing for interval* $]p, q]\,, q > 0$ *where* $f(t) \sim_0 \frac{\lambda}{t^{\alpha}}$ *so the integral* $\int_p^0 f(t)dt$ *convergent if* $\alpha < 1$ *and divergent for* $\alpha \geq 1$.

Example 2.10

$\int_1^2 \frac{dt}{\sqrt{t^3-1}}$ *improper integral at 1, by Riemann's rule application we have :*
$\lim_{t\to1}(t-1)^{\alpha} \frac{1}{\sqrt{t^3-1}}$
$\exists\alpha = \frac{1}{2}, \lim_{t\to1} \frac{\sqrt{t-1}}{\sqrt{t^3-1}}$ *we have* $t^3 - 1 = (t-1)(t^2+t+1)$ *so*
$k = \lim_{t\to1} \frac{\sqrt{t-1}}{\sqrt{t-1}\sqrt{(t^2+t+1)}} = \frac{1}{\sqrt{3}}$ *finished and exist.*
$\alpha = \frac{1}{2} < 1$ *and* k *finished and existing* $\Rightarrow \int_1^2 \frac{dt}{\sqrt{t^3-1}}$ *convergent.*

Remark 2.8

1. $\int_p^q f(t)dt = \int_p^v f(t)dt + \int_v^q f(t)dt$, $\int_p^v f(t)dt$ *limited integral so,* $\int_p^q f(t)dt$ *and* $\int_v^q f(t)dt$ *they have same nature.*

2. *Let it be* f *negative, for improper integral studied at* q $g(t) = -f(t)$ *is a positive*

function then in order to study the nature of integration $\int_p^q g(t)dt$ *we apply the studied methods,* $\int_p^q f(t)dt$ *and* $\int_p^q g(t)dt$ *they have same nature.*

3. $\int_p^q f(t)dt$ *and* $\int_p^q \lambda f(t)dt$ *tow improper integrals at* q *and* $\lambda \in \Re$ *they have same nature.*

4. *Some equivalent functions :*

   $*\ln(1+h(x)) \sim_{+\infty} \pm h(x)$

   $h(x) \longrightarrow 0\ ,\ x \longrightarrow +\infty.$

   $*\sin(h(x)) \sim_{+\infty} h(x)$

   $h(x) \longrightarrow 0\ ,\ x \longrightarrow +\infty.$

   $*\cos(h(x)) \sim_{+\infty} 1 - \frac{h(x)^2}{2}.$

   $h(x) \longrightarrow 0\ ,\ x \longrightarrow +\infty.$

   $*\cosh(h(x)) \sim_{+\infty} \frac{h'(x)^2}{2} + 1$

   $h(x) \longrightarrow 0\ ,\ x \longrightarrow +\infty.$

   $*\arctan(h(x)) \sim_{+\infty} h(x)$

   $h(x) \longrightarrow 0\ ,\ x \longrightarrow +\infty.$

   $*e^{h(x)} \sim_{+\infty} h(x) + 1$

   $h(x) \longrightarrow 0\ ,\ x \longrightarrow +\infty.$

   $*\sinh(h(x)) \sim_{+\infty} h(x)$

   $h(x) \longrightarrow 0\ ,\ x \longrightarrow +\infty.$

   $*\frac{a_\alpha x^\alpha + a_{\alpha-1}x^\alpha - 1 + ... + a_0}{b_\beta x^\beta + b_{\beta-1} + ... + b_0} \sim_{+\infty} \frac{a_\alpha x^\alpha}{b_\beta x^b}.$

Example 2.11

$\int_0^{+\infty} \frac{dt}{(t+1)(t+2)}$, *the function is positive and integration and by the comparison theorem we found :*

$$\frac{1}{(t+1)(t+2)} \sim_{+\infty} \frac{1}{t^2}$$

and the integral $\int_1^{+\infty} \frac{1}{t^2}dt$ *convergent (Riemann's integral* $\alpha = 2 > 1$*) so the integral* $\int_0^{+\infty} \frac{dt}{(t+1)(t+2)}$ *convergent.*

Example 2.12

$\int_4^{+\infty} \ln(1+\frac{1}{x})dx$, the function is positive and integration and by the comparison theorem we found :

$$\ln(1+\frac{1}{x}) \sim_{+\infty} \frac{1}{x}$$

and the integral $\int_4^{+\infty} \frac{1}{x}dt$ divergent (Riemann's integral $\alpha = 1$) so the integral $\int_4^{+\infty} \ln(1+\frac{1}{x})$ divergent.

Improper Integrals of Functions Defined on a Bounded Interval

Importants Definitions

Let $f : [p, q[\to \mathbb{R}$ be a function integrable (in the Riemann sense) on $[p, x]$ for all $x \in [p, q[$. Define

$$F(x) = \int_a^x f(t)\, dt. \tag{2.7}$$

Definition 2.4

[13] We say that f admits a convergent improper integral on $[p, q[$ if $F(x)$ has a limit as x tends to b. In this case, we write:

$$\lim_{x\to q} \int_p^x f(t)\, dt = \int_p^q f(t)\, dt. \tag{2.8}$$

If this condition is met, f is considered semi-integrable on the interval $[p, q[$.

Example 2.13

Consider the function $f(x) = \frac{1}{\sqrt{-x}}$ defined on $[-1, 0[$. We observe that this function is integrable in the Riemann sense on $[-1, \beta]$ for all $\beta \in [-1, 0[$. Moreover:

$$\int_{-1}^x f(t)\, dt = -2\sqrt{-x} + 2.$$

Thus,

$$\lim_{x\to 0} \int_{-1}^x f(t)\, dt = 2.$$

Therefore, f admits a convergent improper integral on $[-1, 0[$.

Remark 2.9

1. *If a function* $f\ :]p,q]\ \rightarrow \mathbb{R}$ *is integrable (in the Riemann sense) on* $[x,b]$ *for all* $x\ \in]p,q]$*, we say that it admits a convergent improper integral (or is semi-integrable) on* $)]p,q]$ *if the function*

$$F(x) = \int_{x}^{q} f(t)\,dt, \tag{2.9}$$

has a limit as x *tends to* p*. In this case, we write:*

$$\lim_{x\to a} \int_{x}^{q} f(t)\,dt = \int_{p}^{q} f(t)\,dt. \tag{2.10}$$

2. *For a function defined on an open interval* $]p,q[$ *and integrable on any interval* $[x,y]$ *with* $p < x < y < q$*, we say that its improper integral on* $]p,q[$ *converges if, for* $p < v < q$*, its improper integrals on* $]p,v]$ *and* $[v,q[$ *exist.*

Example 2.14

The improper integral

$$\int_{-\infty}^{\infty} \frac{1}{1-t^2}\,dt,$$

is divergent because the integral

$$\int_{-1}^{0} \frac{1}{1-t^2}\,dt,$$

is divergent.

Definition 2.5 (*Absolute Value Convergent*)

[10] Let $f : [p,q[\rightarrow \mathbb{R}$ *be a function defined for integration over any interval of the form* $[p,q[$ *for any* $\alpha < q$*. The improper integral of f is considered absolutely convergent on* $[p,q[$ *if the improper integral of* $|f|$ *converges on* $[p,q[$*.*

Remark 2.10

1. *If* $\int_{p}^{q} f(t)dt$ *absolute convergent* $\Rightarrow \int_{p}^{q} f(t)dt$ *convergent.*

2. *If* $\int_{p}^{q} f(t)dt$ *convergent* $\nRightarrow \int_{p}^{q} f(t)dt$ *absolute convergent.*

3. *If* $\int_p^q f(t)dt$ *divergent* $\Leftrightarrow \int_p^q f(t)dt$ *non absolute convergent.*

Example 2.15

The improper integral on $]2, +\infty]$ *of the function* $f(x) = \dfrac{\cos x}{x^2}$ *is absolutely convergent.*

Because :

$\int_2^{+\infty} \frac{|\cos x|}{x^2} dx \leq \int_2^{+\infty} \frac{dx}{x^2} dx$

$\int_2^{+\infty} \frac{dx}{x^2} dx$ *convergent because* $\alpha = 2 > 1$ *Riemann's integral.*

Chapter 3

Differential Equations

3.1 Examination of Ordinary Differential Equations

3.1.1 First-Order Ordinary Differential Equations

Consider φ denote a continuous function. A first-order ordinary differential equation is defined as one that can be stated in the form

$$[11] y' = \varphi(t, \gamma), \tag{3.1}$$

We denote the differential function as $\gamma_0 : J$ is a subset of the real numbers, mapping to the real numbers, and represents a solution to the first-order differential equation. Let $\gamma' = \varphi(t, \gamma)$ be defined on the interval J if it satisfies the condition $\gamma_0' = \varphi(t, \gamma_0)$ for all $t \in J$.

Example 3.1

The function $x : \mathbb{R} \longrightarrow \mathbb{R}$, defined by $x(t) = e^t$, serves as the answer to the subsequent equation:

$\gamma' = \gamma$ on $\Re$, since $x'(t) = e^t = x(t)$, for all $t \in \Re$.

Differential Equations of Separate Variables

A first-order differential equation is said to have separable variables if it can be represented in the following form:

$$\varphi(\gamma)\gamma' = h(x), \tag{3.2}$$

Thus, $\Phi(\gamma) = H(x) + m$, where $m \in \mathbb{R}$. Let Φ be an antiderivative of γ, and let H be an antiderivative of h. So

$$\gamma = \Phi^{-1}(H(x) + c). \tag{3.3}$$

Example 3.2

Solve on $J = [2, +\infty[$ *differential equation*

$$x\gamma' \ln x = (lnx + 1)y$$

We may isolate the variables x *and* γ *by dividing by* $\gamma x \ln x$,
$\frac{\gamma'}{\gamma} = \frac{(\ln x + 1)}{x \ln x}$ *so*
$\ln \gamma = \int \frac{\ln x + 1}{x \ln x} dx + c$, $c \in \Re$,
that's mean :
$\gamma = e^{\int \frac{\ln x+1}{x \ln x} dx + c}$, $c \in \Re$. $\gamma = e^{\ln |x \ln x + c|} = m(x \ln x)$, $m = e^c \in \Re$.

Example 3.3

Solve on $J = \Re$ *differential equation* $\gamma' = \gamma \cos(x)$

Separate variables:

$$\frac{d\gamma}{\gamma} = \cos(x)\, dx$$

Integrate both sides:

$$\ln |\gamma| = \sin(x) + C$$

Thus,

$$\gamma = me^{\sin(x)}\, m \in \Re.$$

Linear Ordinary Differential Equation

Definition 3.1

[17] A linear differential equation is expressed in the following form:

$$s_0(x)\gamma + s_1(x)\gamma' + ... + s_n(x)\gamma^{(n)} = \varphi(x), \tag{3.4}$$

here is the homogeneous equation that corresponds:

$$s_0(x)\gamma + s_1(x)\gamma' + ... + s_n(x)\gamma^{(n)} = 0. \tag{3.5}$$

Proposition 3.1

1. *If* γ_1 *and* γ_2 *are two solutions of Eq.*(3.4)*. Thus,* $\gamma_1+\gamma_2$ *and* $\alpha\gamma_1$ *are also solutions to Eq.* (3.4)*.*

2. *If* S_0 *is the solution set of Eq.*(3.4) *and* γ_* *is a particular solution of Eq.*(3.5)*, so the solution set of Eq.*(3.4) *is given by* $S = (\gamma + \gamma_*,\, y \in S_0)$*.*

First Order Linear Differential Equation

Definition 3.2

[11] A first-order linear differential equation is a first-order differential equation characterized by:

$$\gamma' = \varphi(t)\gamma + h(t), \tag{3.6}$$

with φ*,* h *continuous functions on* $\Re$*. The associated homogeneous differential equation is :*

$$\gamma' = \varphi(t)\gamma. \tag{3.7}$$

Example 3.4

The differential equation $\gamma' = t^2\gamma + t$ *is a first-order linear differential equation. The corresponding homogeneous differential equation is* $\gamma' = t^2\gamma$*.*

Example 3.5

The differential equation $\gamma' = e^{2t+8}y + 5t^3$ *is a linear differential equation of order 1. The associated homogeneous differential equation is,* $\gamma' = e^{2t+8}\gamma$*.*

Resolution of a Homogeneous Linear Differential Equation

We initiate the process in this manner: The equation $\gamma' = \varphi(t)\gamma$ represents a relationship where the rate of change of the function γ is directly proportional to the function itself and a function dependent on the variable t. $\dfrac{\gamma'}{\gamma} = \varphi(t)$ We will analyze the equation $\ln|\gamma| = \displaystyle\int \varphi(t)dt + c$, where $c \in \Re$. The solution is represented as $\gamma = me^{\int \varphi(t)dt}$, where $m \in \Re$.

The Solution of a Non-Homogeneous Linear Differential Equation

The process of solving a non-homogeneous linear differential equation begins with the identification of the general solution, denoted as γ_1, for the corresponding homogeneous equation. Subsequently, a specific solution, denoted as γ_*, is pursued for the non-homogeneous equation. The general solution, denoted as γ_g, is derived by combining the homogeneous solution and the particular solution, expressed as $y_g = \gamma_1 + \gamma_*$.

Example 3.6

We consider the equation: $2x\gamma' - \gamma = x$.

We start with the relevant homogeneous equation: The generic solution of the homogeneous equation $2x\gamma' - \gamma = 0$ *is*

$\gamma' = \dfrac{1}{2x}\gamma \Rightarrow \gamma_1 = me^{\frac{1}{2}\int \frac{1}{x}} \Rightarrow \gamma_1 = me^{\frac{1}{2}\ln x} \Rightarrow \gamma_1 = me^{\ln x^{\frac{1}{2}}} \Rightarrow \gamma_1 = m\sqrt{x}$.

The specific solution of the non-homogeneous equation is provided by: $\gamma_* = m(x)\sqrt{x}$.

So $\gamma_*'(x) = m'(x)\sqrt{x} + \dfrac{m(x)}{2\sqrt{x}}$ *also :* $2x\gamma_*' - \gamma_* = x$ *we find*

$2xm'(x)\sqrt{x} + \dfrac{2xm(x)}{2\sqrt{x}} - m(x)\sqrt{x} = x \Rightarrow 2xm'(x)\sqrt{x} + m(x)\sqrt{x} - m(x)\sqrt{x} = x \Rightarrow$

$2xm'(x)\sqrt{x} = x \Rightarrow m'(x) = \dfrac{1}{2\sqrt{x}}$.

So $m(x) = \sqrt{x} \Rightarrow y_p = m(x)\sqrt{x} \Rightarrow \gamma_* = x$.

The definitive answer is:

$$\gamma_g = \gamma_1 + \gamma_* = m\sqrt{x} + x,\ m \in \Re.$$

3.1.2 Linear Differential Equations of Order 2 (LDE of Order 2)

Definition 3.3

[11] A second-order linear differential equation with constant coefficients is expressed in the following form:

$$s_1\gamma'' + s_2\gamma' + s_3\gamma = \varphi(x), \tag{3.8}$$

where $s_1, s_2, s_3 \in \Re(a \neq 0)$ *and* $\varphi \in C^{0(J)}(J \subset \Re)$.

The associated homogeneous equation (or without a second member) is:

$$s_1\gamma'' + s_2\gamma' + s_3\gamma = 0. \tag{3.9}$$

Proposal if γ_1 is a general solution of (3.1.2) and γ_*represents a specific solution of (3.3). So $\gamma_g = \gamma_1 + \gamma_*$ is a general solution of (3.3).

Addressing the corresponding homogeneous equation (HE)

$$s_1\gamma'' + s_2\gamma' + s_3\gamma = 0$$

We seek the solution in the form $\gamma(x) = e^{rx}, r \in \Re$. So $\gamma'(x) = r\gamma(x)$ and $\gamma''(x) = r^2\gamma(x)$, so, (3.1.2) became :

$$\gamma(s_1r^2 + s_2r + s_3) = 0. \tag{3.10}$$

Definition 3.4

The equation $s_1r^2 + s_2r + s_3 = 0$ is referred to as the characteristic equation of (3.1.2).

Proposition 3.2

Following the sign of $\Delta = s_2^2 - 4s_2s_3$, we have the following results:

1) $\Delta > 0$: Eq. (3.1.2) accept tow distinct real roots $r_1 \neq r_2$, and

$$\gamma(x) = c_1e^{r_1x} + c_2e^{r_2x},\ c_1, c_2 \in \Re.$$

is a general solution of Eq. (3.1.2).

2)$\Delta = 0$: Eq. (3.1.2) accept a double root $r \in \Re$,

$$\gamma(x) = (c_1x + c_2)e^{rx},\ c_1, c_2 \in \Re.$$

3)$\Delta < 0$: Eq. (3.1.2) accept a two complex roots conjugated $r_{1,2} = a \pm ib$, $(a, b \in \Re, b \neq 0)$ and

$$\gamma(x) = e^{ax}(c_1\cos(bx) + c_2\sin(bx)),\ c_1, c_2 \in \Re.$$

Example 3.7

1. ***Case 1: Two Distinct Real Roots*** $(\Delta > 0)$

   Consider the equation: $\gamma'' - 6\gamma' + 8\gamma = 0$.

   The characteristic equation is $r^2 - 6r + 8 = 0$, which factors to $(r-2)(r-4) = 0$.

   The roots are distinct and real: $r_1 = 2$ and $r_2 = 4$. The solution is:

$$\gamma(x) = c_1e^{2x} + c_2e^{4x}, \quad c_1, c_2 \in \mathbb{R}$$

2. ***Case 2: One Repeated Real Root*** ($\Delta = 0$)

   Consider the equation: $\gamma'' - 6\gamma' + 9\gamma = 0$.

   The characteristic equation is $r^2 - 6r + 9 = 0$, *which factors to* $(r-3)^2 = 0$. *This gives a repeated real root* $r = -3$. *The solution is:*

$$\gamma(x) = (c_1 x + c_2)e^{3x}, \quad c_1, c_2 \in \mathbb{R}$$

3. ***Case 3: Two Complex Conjugate Roots*** ($\Delta < 0$)

   Consider the equation: $\gamma'' + 4\gamma' + 13\gamma = 0$.

   The characteristic equation is $r^2 + 4r + 13 = 0$. *The discriminant is* $\Delta = 4^2 - 4(1)(13) = 16 - 52 = -36$. *The complex roots are* $r = \frac{-4 \pm \sqrt{-36}}{2} = -2 \pm 3i$. *The solution is:*

$$\gamma(x) = e^{-2x}(c_1 \cos(3x) + c_2 \sin(3x)), \quad c_1, c_2 \in \mathbb{R}$$

Determining the Specific Solution

To determine a specific solution γ_* for the equation $s_1\gamma'' + s_2\gamma' + s_3\gamma = \varphi(x)$, the Method of Undetermined Coefficients can be employed. This method is applicable when the function $\varphi(x)$ is a polynomial, an exponential, a sine or cosine function, or a product of these types.

Step 1: Determine the Initial Guess for γ_* The initial step involves formulating a preliminary assumption for γ_* that mirrors the structure of $\varphi(x)$. The table below presents the accurate predictions for the most prevalent cases.

Table 3.1: Initial Guess Forms for the Particular Solution (γ_*)

If the term $\varphi(x)$ has the form...	**The corresponding guess for $\gamma_*(x)$ is...**
$P_n(x)$ (A polynomial of degree n)	$A_n x^n + \cdots + A_1 x + A_0$
$P_n(x)e^{\alpha x}$	$(A_n x^n + \cdots + A_0)e^{\alpha x}$
$P_n(x)e^{\alpha x}\cos(\beta x)$ or $P_n(x)e^{\alpha x}\sin(\beta x)$	$(A_n x^n + \ldots)e^{\alpha x}\cos(\beta x) +$ $(B_n x^n + \ldots)e^{\alpha x}\sin(\beta x)$

Step 2: Apply the Modification Rule This is the most important step and replaces the repetitive conditions in your original text. You must compare your initial guess for γ_* with the terms in your homogeneous solution, γ_1.

- **Case A (No Overlap):** If no term in your proposed solution for γ_* corresponds to a solution of the homogeneous equation (γ_1), then your initial assumption is valid. This applies when the components of $\varphi(x)$ (such as a or $a \pm ib$ do not constitute roots of the characteristic equation.

- **Case B (Overlap):**If any component of your proposed solution for γ_* is already included in γ_1, it is necessary to multiply your entire proposal by x. This is the criterion for a clear root of the characteristic equation.

- **Case C (Repeated Overlap):** If, after multiplying by x, the new guess still includes a term from γ_1, it is necessary to multiply by x once more (i.e., multiply the original guess by x^2). This addresses the scenario of a double root of the characteristic equation.

Upon applying this rule to obtain the final form of γ_*, one can substitute it into the differential equation to determine the unknown coefficients (A, B, etc.).

Example 3.8

$\gamma'' - 4\gamma' + 4\gamma = (x^2+1)e^x$ the homogeneous equation is $\gamma'' - 4\gamma' + 4\gamma = 0$ so the characteristic equation is: $r^2 - 4r + 4 = 0$ we have $\Delta = 0, r = 2$ double root so the solution is

$$\gamma_1 = (c_1 x + c_2)e^{2x},\ c_1, c_2 \in \Re.$$

2)We seek a specific solution to the non-homogeneous equation. as the $m = 1$ a not root of the characteristic equation so the $\gamma_*(x) = Q(x)e^{2x}$ such as $degQ = 2$,

$$\gamma_*(x) = (ax^2 + bx + c)e^x \text{ so}$$
$$\gamma_*'(x) = (2ax + b)e^x + (ax^2 + bx + c)e^x$$
$$\gamma_*''(x) = 2ae^x + 2(2ax + b)e^x + (ax^2 + bx + c)e^x\ .$$

By replacing in (2.1); $\gamma_*'' - 4\gamma_*' + 4\gamma_* = (x^2+1)e^x$ *we find*

$((a-1)x^2 + (b-4a)x + c + b + 2a - 1)e^x = 0$

$$\begin{cases} a-1=1 \\ b-4a=0 \\ c+2a-2b-1=1 \Rightarrow \end{cases} \quad \begin{cases} a=2 \\ b-4a=8 \\ c=15 \end{cases}$$

the specific solution $\gamma_* = (2x^2+8x+15)e^x$

finally the overall answer is

$$\gamma_g = \gamma_1 + \gamma_* = (c_1x + c_2)e^{2x} + (2x^2+8x+15)e^x,\ c_1, c_2 \in \Re.$$

Example 3.9

Resolve these differential equations: $\gamma'' - 4\gamma' + 3\gamma = 3x + 2$

The homogeneous equation is:

$$\gamma'' - 4\gamma' + 3\gamma = 0$$

The characteristic equation is:

$$r^2 - 4r + 3 = 0$$

Factoring:

$$(r-1)(r-3) = 0$$

Thus, the roots are $r = 1$ *and* $r = 3$. *The alternate solution is as follows:*

$$\gamma_1 = C_1e^{r_1x} + C_2e^{r_2x} = C_1e^x + C_2e^{3x}$$

The non-homogeneous term is $3x+2$. *Consider a specific solution represented as follows:*

$$\gamma_* = Ax + B$$

Compute derivatives:

$$\gamma_*' = A, \quad \gamma_*'' = 0$$

Substitute into the original equation:

$$0 - 4A + 3(Ax + B) = 3x + 2$$

Simplify:

$$3Ax + (3B - 4A) = 3x + 2$$

Equating coefficients:

$$3A = 3 \quad \Rightarrow \quad A = 1$$

$$3B - 4A = 2 \quad \Rightarrow \quad 3B - 4(1) = 2 \quad \Rightarrow \quad 3B = 6 \quad \Rightarrow \quad B = 2$$

Thus:

$$\gamma_* = x + 2$$

The comprehensive solution is as follows:

$$\gamma_g = \gamma_1 + \gamma_* = C_1 e^x + C_2 e^{3x} + x + 2$$

Solving the associated non-homogeneous equation with an arbitrary function

Technique of Variation of Constants

The equation is:

$s_1\gamma'' + s_2\gamma' + s_3\gamma = \rho(x)$ where $\varphi(x)$ is arbitrary function.

When the function $\rho(x)$ is arbitrary, the **variation of parameters** method is effective. First, solve the associated homogeneous equation to find the basis solutions, let's say $\gamma_a(x)$ and $\gamma_b(x)$, giving $\gamma_h = c_1\gamma_a(x) + c_2\gamma_b(x)$. To determine the specific solution, we replace the constants c_1 and c_2 with functions $F(x)$ and $E(x)$, such that $\gamma_* = F(x)\gamma_a(x)(x) + E(x)\gamma_b(x)(x)$. These functions are then found by solving a specific system of equations. We modify the constants c_1 and c_2, and denote a variable by x, thus we express:

$$\gamma_* = c_1(x)\varphi(x) + c_2(x)h(x)$$
$$\gamma'_* = c'_1(x)\varphi'(x) + c'_2(x)h'(x)$$

where the functions φ, h are the basis of the solution γ_1.

so γ_1 is solution if only if $c'_1(x)$ and $c'_2(x)$ they investigate this conditions:

$$c'_1(x)\varphi(x) + c'_2(x)h(x) = 0$$
$$c'_1(x)\varphi'(x) + c'_2(x)h'(x) = \rho(x)$$

Example 3.10

Examine this equation:

$$\gamma''(x) + 2\gamma'(x) + \gamma(x) = \tan(x)$$

The corresponding (HE)is:

$$\gamma''(x) + 2\gamma'(x) + \gamma(x) = 0$$

Assume two linearly independent solutions for (HE): $\gamma_1(x) = e^{-x}(c_1 + c_2 x)$. *We will now identify the specific solution using the method of variation of parameters. The parameters* $c_1(x)$ *and* $c_2(x)$ *are established by resolving the system of equations:*

$$\begin{cases} c_1'(x)e^{-x} + c_2' x e^{-x} = 0 \\ -c_2'(x)e^{-x} + c_2'(1-x)e^{-x} = \tan(x) \end{cases}$$

Simplify:

$$\begin{cases} c_1'(x) = -xe^{x}\tan(x) \\ c_2'(x) = \tan(x) \end{cases}$$

Integrate to find $c_1(x)$ *and* $c_2(x)$*:*

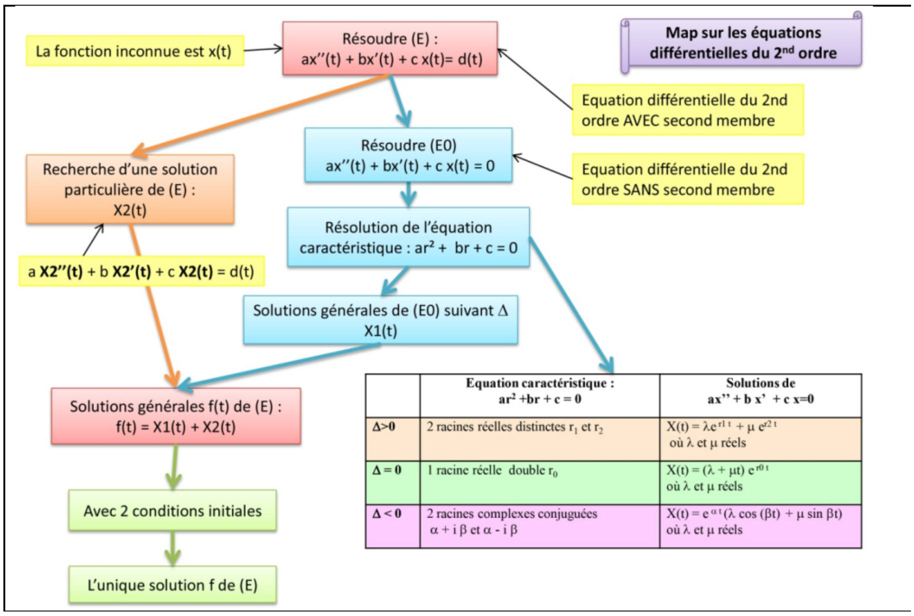
$$\begin{cases} c_1(x) = \displaystyle\int -xe^{x}\tan(x)\,dx \\ c_2(x) = \displaystyle\int \tan(x)\,dx \end{cases}$$

The specific solution is provided by:

$$\gamma_1(x) = (-xe^{x}\tan(x))e^{-x} + (-\ln|\cos(x)|)xe^{-x}$$

.

This is the specific solution to the provided differential equation.



	Equation caractéristique : $ar^2 + br + c = 0$	Solutions de $ax'' + b x' + c x = 0$
$\Delta > 0$	2 racines réelles distinctes r_1 et r_2	$X(t) = \lambda e^{r1\,t} + \mu e^{r2\,t}$ où λ et μ réels
$\Delta = 0$	1 racine réelle double r_0	$X(t) = (\lambda + \mu t) e^{r0\,t}$ où λ et μ réels
$\Delta < 0$	2 racines complexes conjuguées $\alpha + i\beta$ et $\alpha - i\beta$	$X(t) = e^{\alpha t}(\lambda \cos(\beta t) + \mu \sin \beta t)$ où λ et μ réels

Figure 3.1: Diagram for the synthesis of second-order differential equations with a secondary term.

3.2 Exploration of Partial Differential Equations

A function $\varphi : \Re \times \Re \to \Re$ of two real variables defined in the vicinity of the point $A(p, q)$. If the function $x \to \varphi(x, q)$ possesses a derivative with regard to x, we denote it as $\varphi'_x(p, q)$.

The partial derivative with regard to x can be defined as

$$f'_x : \Re \times \Re \to \Re, (x, y) \longrightarrow f'_x(x, y). \quad (3.11)$$

The partial derivative with respect to γ is defined using the same method, we note

$$\varphi'_x = \frac{\partial \varphi}{\partial x}, \varphi'_\gamma = \frac{\partial \varphi}{\partial \gamma}. \quad (3.12)$$

Successive derivatives

In the same way, we can define the derivative of the function φ'_x with respect to x, we note it φ''_{x^2} or $\dfrac{\partial^2 \varphi}{\partial x^2}$.

also the derivative of the function φ'_x with respect to γ, we note $\varphi''_{x\gamma}$ or $\dfrac{\partial^2 \varphi}{\partial \gamma \partial x}$.

The derivative of the function $\varphi'_{y\gamma}$ with respect to γ, we note φ''_{γ^2} or $\dfrac{\partial^2 \varphi}{\partial \gamma^2}$.

The derivative of the function φ'_γ with respect to x, we note $\varphi''_{\gamma x}$ or $\dfrac{\partial^2\varphi}{\partial x\partial\gamma}$.

Theorem 3.3

[5] If the partial derivatives $\varphi''_{\gamma x}$ and $\varphi''_{x\gamma}$ are continuous, then these derivatives are equal:

$$\varphi''_{x\gamma} = \varphi''_{\gamma x}. \tag{3.13}$$

Example 3.11

Determine $\dfrac{\partial^2\varphi}{\partial\gamma\partial x}, \dfrac{\partial^2\varphi}{\partial\gamma^2}, \dfrac{\partial^2\varphi}{\partial x^2}, \dfrac{\partial^2\varphi}{\partial x\partial\gamma}$, *for the function* $\varphi(x,\gamma) = x^2 + x\gamma^2 - \gamma$

$$\frac{\partial\varphi}{\partial x}(x,\gamma) = 2x + \gamma^2,\ \frac{\partial^2\varphi}{\partial x^2}(x,\gamma) = 2,\ \frac{\partial^2\varphi}{\partial\gamma\partial x}(x,\gamma) = 2\gamma.$$
$$\frac{\partial\varphi}{\partial\gamma}(x,\gamma) = 2x\gamma,\ \frac{\partial^2\varphi}{\partial\gamma^2}(x,\gamma) = 2x,\ \frac{\partial^2\varphi}{\partial x\partial\gamma}(x,\gamma) = 2\gamma.$$

Definition 3.5

[5] An equation involving a function φ with multiple independent variables $x_1, ..., x_n$ and its partial derivatives relative to these variables takes the form:

$$F(x_1, ..., x_n, \varphi, \frac{\partial\varphi}{\partial x_1}, ..., \frac{\partial^2\varphi}{\partial x_1^2}, \frac{\partial^2\varphi}{\partial x_2^2}, ...)\partial x_1\partial x_2, ..., \frac{\partial^m\varphi}{\partial^m x_n}) = 0, \tag{3.14}$$

is a partial differential equation.

Example 3.12

1-$(\dfrac{\partial u}{\partial x})^2 - u\dfrac{\partial u}{\partial\gamma} = 0.$

2-$x^2 + \gamma^2(\dfrac{\partial u}{\partial x} + 2\dfrac{\partial u}{\partial\gamma}) = x + \gamma + u.$

3.2.1 Partial Differential Equations of the 1^{st} Order

Definition 3.6

[17] An equation involving a function f of multiple independent variables $x_1, ..., x_n$ and the first-order partial derivatives of φ with respect to these variables, specifically

in the form:

$$F(x_1, ..., x_n, \varphi, \frac{\partial \varphi}{\partial x_1}, ..., \frac{\partial \varphi}{\partial x_n}) = 0, \tag{3.15}$$

It is classified as a first-order partial differential equation (PDE).

Proposition 3.4

Any function $\varphi(x_1, ..., x_n)$ which identically satisfies this equation is a solution of it.

Remark 3.5

In the following, we often use the notations u or z instead of φ. In the case of two variables x, γ, we have

$$F(x, \gamma, \varphi, \frac{\partial f}{\partial x}, \frac{\partial \varphi}{\partial \gamma}) = 0.$$

Example 3.13

1-$\frac{\partial \varphi}{\partial x} = 0 \Leftrightarrow \varphi(x, y) = \varphi(\gamma)$.
2-$\frac{\partial \varphi}{\partial \gamma} = 0 \Leftrightarrow \varphi(x, \gamma) = \varphi(x)$.
3-$\frac{\partial \varphi}{\partial x} = g(x)$. *If g is integral and G is one of its primitives, then* $\frac{\partial}{\partial x}(\varphi(x, \gamma) - G(x)) = 0$, *from where* $\varphi(x, \gamma) = G(x) + \varphi(\gamma)$.

Remark 3.6

The generic solution of a first-order partial differential equation relies on an arbitrary function.

Method of of Characteristic System for Solving the PDE

Definition 3.7

A prevalent method for addressing linear first-order partial differential equations is the method of characteristics. This method is applicable to equations expressed in standard form:

$$Z(x, \gamma, z)\frac{\partial \varphi}{\partial x} + K(x, \gamma, z)\frac{\partial f}{\partial y} = N(x, \gamma, z), \tag{3.16}$$

where Z, K, and N are functions of x, γ, and z defined on an open subset of $\mathbb{R}^3$. A

function $\Phi(x,\gamma)$ serves as a first integral of a differential system and simultaneously constitutes a solution to the corresponding partial differential equation.
the characteristic system associated with the PDE president of the form is

$$\frac{dx}{Z(x,\gamma,z)} = \frac{dy}{K(x,\gamma,z)} = \frac{dz}{N(x,\gamma,z)}. \tag{3.17}$$

Remark 3.7

The PDE solution is reduced to the search for two prime integrals of a system such that:

$$\Phi_1(x,\gamma) = c_1,\ \Phi_2(x,\gamma) = c_2$$

where the general solution is $\Phi(c_1,c_2) = 0$ with $c_2 = \Phi(c_1)$.

Example 3.14

Ascertain the comprehensive solution of the equation:

$$\gamma z\frac{\partial\varphi}{\partial x} + xz\frac{\partial\varphi}{\partial\gamma} = -x\gamma.$$

Let $\varphi(x,\gamma,z)$ be the function to solve. Rewrite the equation:

$$\gamma z\varphi_x + xz\varphi_\gamma = -x\gamma.$$

we use this equality:

$$\frac{dx}{\gamma z} = \frac{d\gamma}{xz} = \frac{dz}{-x\gamma}$$

we choose the first equality:

This is a first-order linear PDE. To solve, introduce the characteristic equations:

$$\int xdx = \int \gamma d\gamma$$

$\implies C_1 = y^2 - x^2,$

where C_1 is a constant.
2. From $\frac{dx}{\gamma z} = \frac{dz}{-x\gamma}$, solve:

$$\implies -xdx = zdz \implies x^2 + z^2 = C_2,$$

where C_2 is another constant.

Thus, the general solution is:

$$g(x^2 - \gamma^2, z^2 + x^2) = 0, g(x^2 - \gamma^2) = z^2 + x^2$$

where g and φ are an arbitrary functions.

Example 3.15

Solve the first-order (PDE):

$$\gamma^2 \frac{\partial u}{\partial x} + x^2 \frac{\partial u}{\partial \gamma} = 0.$$

Rewrite the equation in shorthand notation:

$$\gamma^2 u_x + x^2 u_\gamma = 0.$$

We apply the method of characteristics. The characteristic equations are:

$$\frac{dx}{\gamma^2} = \frac{d\gamma}{x^2} = \frac{du}{0}.$$

First, solve the characteristic equation $\frac{dx}{\gamma^2} = \frac{d\gamma}{x^2}$:

$$\int x^2\, dx = \int \gamma^2\, d\gamma \implies \frac{x^3}{3} = \frac{\gamma^3}{3} + c.$$

This yields the first characteristic:

$$C_1 = \gamma^3 - x^3.$$

Next, solve $\frac{du}{0} = \frac{d\gamma}{x^2}$. Since $\frac{du}{0}$ implies $du = 0$, we have:

$$u = C_2.$$

In its most basic form, the answer is a functional link among the following characteristics:

$$\phi(\gamma^3 - x^3, u) = 0, \quad \text{or equivalently,} \quad u = \varphi(\gamma^3 - x^3),$$

where φ is an arbitrary differentiable function.

Example 3.16

Resolve the first-order PDE:

$$\frac{1}{x}\frac{\partial u}{\partial x} + \frac{1}{\gamma}\frac{\partial u}{\partial \gamma} = 2.$$

The characteristic equations are:

$$\frac{dx}{\frac{1}{x}} = \frac{d\gamma}{\frac{1}{\gamma}} = \frac{du}{2}.$$

Simplify equations:

$$x\,dx = \gamma\,d\gamma = \frac{du}{2}.$$

First, solve the equation $\frac{dx}{\frac{1}{x}} = \frac{d\gamma}{\frac{1}{\gamma}}$*, or equivalently,* $x\,dx = \gamma\,d\gamma$*:*

$$\int x\,dx = \int \gamma\,d\gamma \implies \frac{x^2}{2} = \frac{\gamma^2}{2} + c.$$

This yields to:

$$C_1 = \gamma^2 - x^2.$$

Next, solve the equation $\frac{d\gamma}{\frac{1}{\gamma}} = \frac{du}{2}$*, or equivalently,* $\gamma\,d\gamma = \frac{du}{2}$*:*

$$\int \gamma\,d\gamma = \int \frac{du}{2} \implies \frac{\gamma^2}{2} = \frac{u}{2} + c \implies u - \gamma^2 = C_2.$$

Thus, the second characteristic is:

$$C_2 = u - \gamma^2.$$

The overall solution is represented as a functional relationship among the characteristics.

$$\phi(\gamma^2 - x^2, u - \gamma^2) = 0, \quad \text{or equivalently,} \quad u = \varphi(\gamma^2 - x^2) + \gamma^2,$$

where ϕ *and* φ *are arbitrary differentiable functions.*

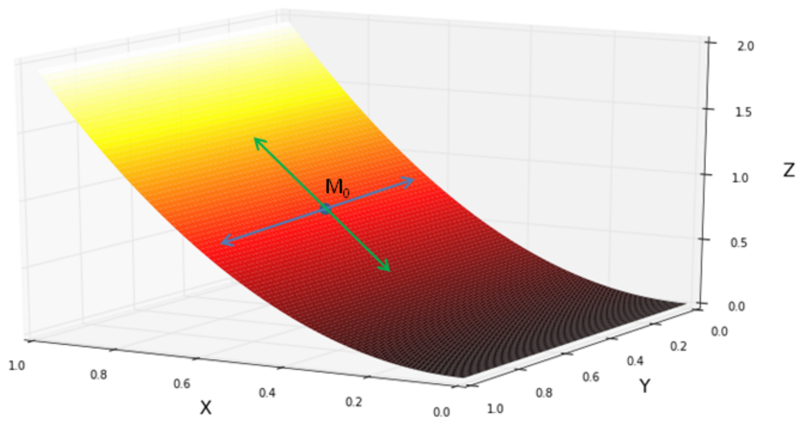


Figure 3.2: Illustration of first-order 1^{st} partial differential equations for synthesis

3.2.2 Partial Differential Equations of the 2^{st} Order

Definition 3.8

[5] Let φ be a function of two variables, x and γ. A second-order partial differential equation is defined as a relation of the form

$$F\left(x,\gamma,\varphi,\frac{\partial\varphi}{\partial x},\frac{\partial\varphi}{\partial\gamma},\frac{\partial^2\varphi}{\partial x^2},\frac{\partial^2\varphi}{\partial x\partial\gamma},\frac{\partial^2\varphi}{\partial\gamma^2}\right)=0, \tag{3.18}$$

involving the function φ and its partial derivatives up to the second order.

Example 3.17

1- $\dfrac{\partial^2\varphi}{\partial x^2}=0,$

2- $\dfrac{\partial^2\varphi}{\partial x\partial\gamma}=0.$

Remark 3.8

The solution to a second-order partial differential equation is contingent upon two arbitrary functions.

Proposition 3.9

Examine a second order partial differential equation represented in the following man-

ner:

$$s_1(x,\gamma)\frac{\partial^2 z}{\partial x^2} + 2s_2(x,\gamma)\frac{\partial^2 z}{\partial x \partial \gamma} + s_3(x,\gamma)\frac{\partial^2 z}{\partial \gamma^2} = F(x,\gamma,z,\frac{\partial z}{\partial x},\frac{\partial z}{\partial \gamma}), \quad (3.19)$$

in the context where $z = \varphi(x,\gamma)$ *represents an unknown function, and* s_1, s_2, s_3, F *are specified functions defined within a domain* $D \subset \Re^2$.

Definition 3.9

A characteristic for the above equation is a curve in D *satisfying the differential equation:*

$$s_1(\frac{\partial \gamma}{\partial x})^2 - 2s_2(\frac{\partial \gamma}{\partial x}) + s_3 = 0. \quad (3.20)$$

Classification of PDE of the 2^{nd} order

The type of equations depends on their discriminant $\Delta = s_2^2 - s_1 s_2$.

Definition 3.10

Only when $\Delta > 0$ *can the equation be classified as hyperbolic.*

Example 3.18

The equation for vibrating strings:

$$\frac{\partial^2 z}{\partial x^2} = k^2 \frac{\partial^2 z}{\partial t^2}$$

where $z(x,t)$ *represents the motion of the point at abscissa* x *at time* t. *This is a hyperbolic equation represented as* $(s_1 = 1, s_2 = 0, s_3 = -k^2)$.

Definition 3.11

Only when $\Delta = 0$ *can the equation be said to be of the parabolic form.*

Example 3.19

Chaleur equation:

$$\frac{\partial z}{\partial t} = \alpha \frac{\partial^2 z}{\partial x^2},$$

where t *is the time,* z *is the temperature of a body and* α *is a constant. It is a parabolic equation*$(s_1 = \alpha^2, s_2 = s_3 = 0)$.

Definition 3.12

The equation is of the elliptic type if and only if $\Delta < 0$.

Example 3.20

The equation of harmonic functions (or Laplace equation with two variables):

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial \gamma^2} = 0,$$

is elliptic $(s_1 = s_3 = 1, s_2 = 0)$.

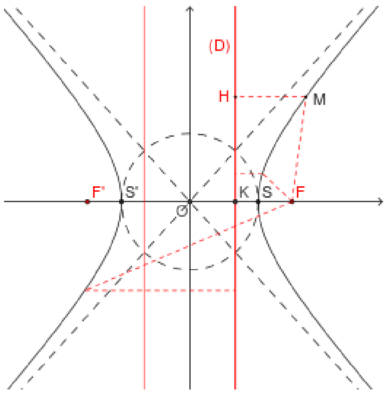


Figure 3.3: Define hyperbolic trajectory

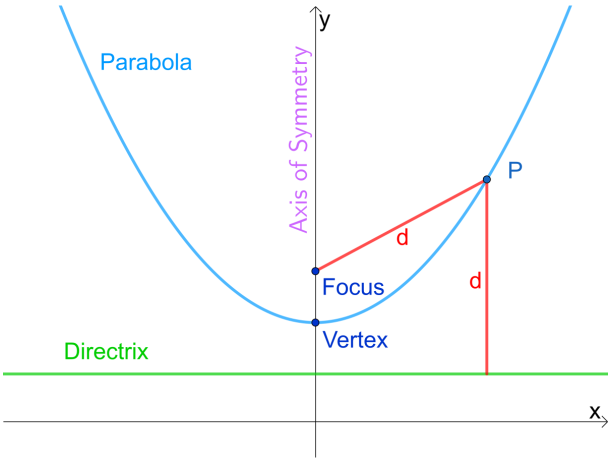


Figure 3.4: Define parabolic trajectory

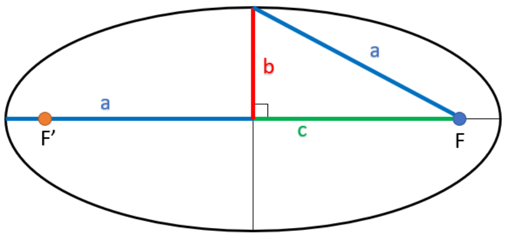


Figure 3.5: Define elliptic trajectory

Analysis of the Equation for Vibrating Strings

Let it be

$$\frac{\partial^2 z}{\partial x^2} = k^2 \frac{\partial^2 z}{\partial t^2},\ x \in \Re,\ t \geq 0 z(x,0) = \varphi(x),\ \frac{\partial z}{\partial t}(x,0) = \Psi(x). \tag{3.21}$$

in which the functions $\varphi(x)$ and $\Psi(x)$ are defined $]-\infty, +\infty[$.

Alembert solution

$\frac{\partial^2 z}{\partial x^2} - k^2 \frac{\partial^2 z}{\partial t^2} = 0$

$a = 1, b = 0, c = -k^2 \rightarrow \Delta = k^2$

$\frac{dx}{dt} = \pm \frac{k}{2} = \pm k \Rightarrow dx = \pm dt \Rightarrow$

$\begin{cases} x = kt + c_1 \\ x = -kt + c_2 \Rightarrow \end{cases} \begin{cases} \zeta = x - kt \\ \eta = x + kt \end{cases}$

Any class solution C^2 from the equation of vibrating strings is of the form $g(x - kt) + h(x + kt)$ where g and h are arbitrary class functions to define them we use the conditions:

$\begin{cases} g(x) + h(x) = \varphi(x) \\ g(x) - h(x) = -\frac{1}{k} \int_0^x \Psi(\tau) d\tau \end{cases}$

so the classical solution of the equation 3.2.2 is :

$$z(x,t) = \frac{1}{2}(\varphi(x - kt) + \varphi(x + kt)) + \frac{1}{2k} \int_{x-kt}^{x+at} \Psi(\tau) d\tau.$$

Remark 3.10

Consider an equation that contains only partial derivatives with respect to a single variable. The equation can be analyzed as an ordinary differential equation, with the integration constants treated as functions of the variable that serves as a parameter. In a similar manner, if an equation contains only partial derivatives of a specific partial derivative concerning one of the variables, it is possible to analyze the equation by *treating the latter partial derivative as an intermediate unknown.*

Example 3.21

1-$\dfrac{\partial^2 f}{\partial x^2} = 0 \Rightarrow \dfrac{\partial}{\partial x}(\dfrac{\partial f}{\partial x}) \Rightarrow \dfrac{\partial f}{\partial x} = g(\gamma)$ *so* $\varphi(x,y) = xg(\gamma)+h(y)$. *where* g, h *are arbitrary functions.*
2-$\dfrac{\partial^2 \varphi}{\partial x \partial \gamma} = 0 \Rightarrow \dfrac{\partial}{\partial y}(\dfrac{\partial \varphi}{\partial x}) \Rightarrow \dfrac{\partial \varphi}{\partial x} = g(x)$ *so* $\varphi(x,y) = h(x) + \varphi(\gamma)$.

3.3 Special functions

Eulerian functions

Definition 3.13

We call the Eulerian integral of the first kind or beta function, the integral depending on two parameters x *and* γ *defined as follows*

$$\beta(x,\gamma) = \int_0^1 t^{x-1}(1-t)^{\gamma-1}dt. \text{ or}$$
$$\beta(x,\gamma) = 2\int_0^{\frac{\Pi}{2}} \sin^{2x-1} t \cos^{2\gamma-1} t dt.$$

Definition 3.14

We call the Eulerian integral of the second kind or gamma function, the integral depending on the parameter x, *defined as follows*

$$\Gamma(x,\gamma) = \int_0^{+\infty} t^{x-1}e^{-t}dt, \text{ or}$$
$$\Gamma(x,\gamma) = 2\int_0^{+\infty} t^{2x-1}e^{-t^2}dt.$$

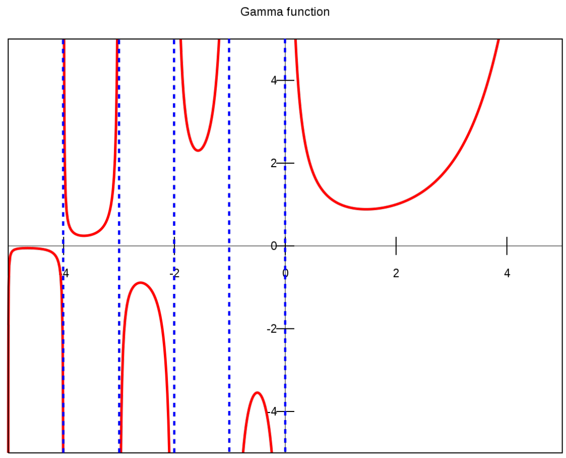


Figure 3.6: Define Gamma function

Hypergeometric function

Definition 3.15

Definition

The hypergeometric function is defined by the series

$$F_1(a,b,c,z) = \sum_{k=0}^{+\infty} \frac{\Gamma(a+k+1)\Gamma(b+k+1)\Gamma(1+c)z^k}{\Gamma(1+a)\Gamma(1+b)\Gamma(c+1+k)k!}.$$

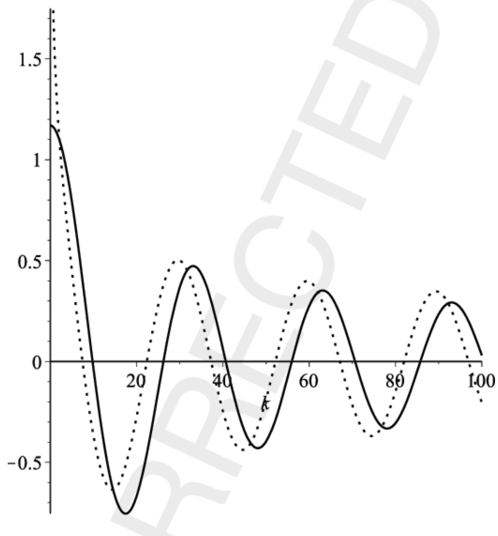


Figure 3.7: Define hypergeometric function

Bessel function of the first kind

Definition 3.16

We call the Bessel function the function $J_\lambda(x)$, defined by

$$J_\lambda(x) = \frac{1}{2\Pi}\int_{-\Pi}^{\Pi} e^{i(x\sin\theta-\lambda\theta)}d\theta = \frac{1}{\pi}\int_{-\Pi}^{\Pi}\cos(x\sin\theta-\lambda\theta)d\theta.$$

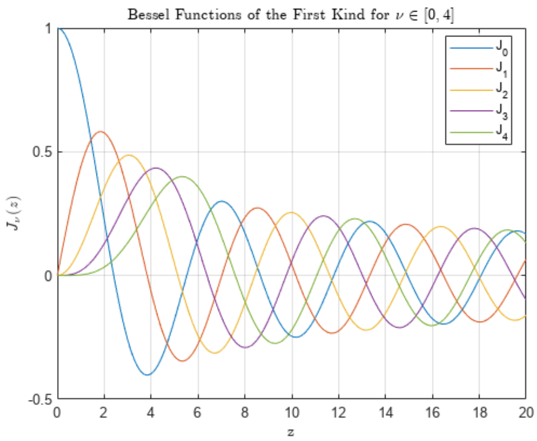


Figure 3.8: Define the Bessel function

Chapter 4

Series

4.1 Numerical Series

4.1.1 Series with Positive Terms

Let be $(u_n)_{n\in N}$ a real numerical sequence.

Definition 4.1

[10] We refer to a real numerical series by its general name u_n

$$\sum_{k=0}^{+\infty} u_k = u_0 + u_1 + ... + u_n + ... \tag{4.1}$$

We denote by s_n *the partial sum of the first* n *terms and we have*

$$s_n = u_0 + u_1 + ... + u_{n-1}. \tag{4.2}$$

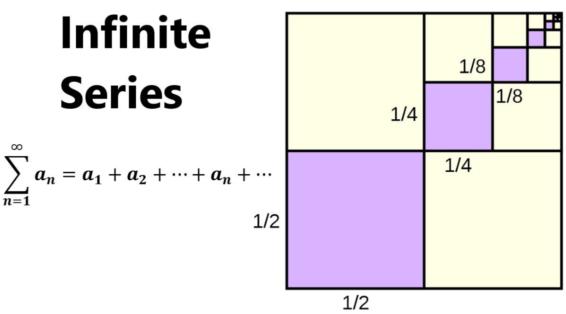


Figure 4.1: Define numerical series

Proposition 4.1

The numerical series $\sum_{k=0}^{+\infty} u_k$ *is considered convergent if and only if its sequence of* *partial sums* $(s_n)_{n\in\mathbb{N}}$ *converges, meaning that* $\lim_{n\to+\infty} s_n$ *exists and is finite; otherwise, it is classified as divergent.*

Example 4.1

Examine the convergence properties of the series characterized by the general term $a_n = \dfrac{1}{n^2+3n+2}$ *for* $n \geq 1$.

First, we simplify the general term using partial fraction decomposition. The denominator factors as $(n+1)(n+2)$, *so we can write:*

$$a_n = \frac{1}{(n+1)(n+2)} = \frac{1}{n+1} - \frac{1}{n+2}$$

This form suggests that the series is a ***telescoping series****. Let's examine the sequence of partial sums,* S_N:

$$\begin{aligned} S_N &= \sum_{n=1}^{N} \left(\frac{1}{n+1} - \frac{1}{n+2}\right) \\ &= \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \left(\frac{1}{4} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{N+1} - \frac{1}{N+2}\right) \end{aligned}$$

In this sum, the second part of each term cancels the first part of the next term. The only terms that do not cancel are the first term, $\frac{1}{2}$, *and the last term,* $-\frac{1}{N+2}$.

$$S_N = \frac{1}{2} - \frac{1}{N+2}$$

To assess the convergence of the series, we evaluate the limit of the partial sums as $N \to \infty$:

$$S = \lim_{N\to\infty} S_N = \lim_{N\to\infty} \left(\frac{1}{2} - \frac{1}{N+2}\right) = \frac{1}{2} - 0 = \frac{1}{2}$$

Since the limit of the partial sums is a finite number, the series ***converges*** *to* $\frac{1}{2}$.

Theorem 4.2

The convergence characteristics of a series remain unchanged when a finite number of terms are removed.

Proposition 4.3 (*Series Properties*)

Let it be $\sum_{n=0}^{+\infty} u_n$ *and* $\sum_{n=0}^{+\infty} v_n$ *tow numerical series* λ *an arbitrary number we have*

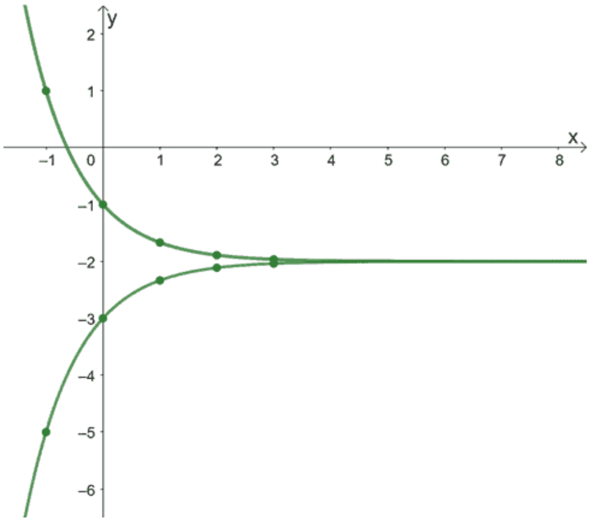


Figure 4.2: Graph of simple convergence the series.

$$1\text{-}\sum_{n=0}^{+\infty} u_n \text{ convergent} \Rightarrow \sum_{n=0}^{+\infty} (\lambda u_n) \text{ convergent.}$$

$$2\text{-}\sum_{n=0}^{+\infty} u_n \text{ and } \sum_{n=0}^{+\infty} v_n \text{ convergent} \Rightarrow \sum_{n=0}^{+\infty} (u_n \pm v_n) \text{ convergent.}$$

Necessary condition for the convergence of series

Theorem 4.4

[1] If $\sum_{n=0}^{+\infty} u_n$ *convergent so* $\lim_{n\to+\infty} u_n = 0$.

Corollary 4.1

If $\lim_{n\to+\infty} u_n \neq 0$ *so* $\sum_{n=0}^{+\infty} u_n$ *is divergent.*
With regard to the investigation of the character of a series, it is not always possible to compute its sum. Nevertheless, we are able to ascertain the nature of the series by employing other methods; in order to do so, we want additional tools.

Example 4.2

Study the nature of this series:

$U_n = \sqrt{n^2+1} - \sqrt{n}$

Simplify U_n and calculated the limite:

$$\lim_{n\to+\infty} U_n = \lim_{n\to+\infty} n(\sqrt{1+\frac{1}{n^2}} + \sqrt{\frac{1}{n}}) = +\infty.$$

The series diverges due to the essential condition for convergence.

Example 4.3

Study the nature of this series:
$U_n = n\sin\left(\frac{1}{n}\right)$
Simplify U_n and calculated the limit:

$$U_n = \frac{\sin\frac{1}{n}}{\frac{1}{n}} \Rightarrow \lim_{n\to+\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}}$$

by use this change $\frac{1}{n} = y$ so $\Rightarrow \lim_{y\to+\infty} \frac{\sin y}{y} = 1$ The series diverges due to the essential condition for convergence.

Criteria for Series with Positive Terms

Theorem 4.5 (*Comparison criteria*)

Let it be $\sum_{n=0}^{+\infty} u_n$ and $\sum_{n=0}^{+\infty} v_n$ two numerical series with positive terms verifying

$$\forall n \in N\ u_n \leq v_n. \tag{4.3}$$

or 4.5 is true from a certain rank.

Theorem 4.6

1-If $\sum_{n=0}^{+\infty} v_n$ is convergent so $\sum_{n=0}^{+\infty} u_n$ is convergent.
2-If $\sum_{n=0}^{+\infty} u_n$ is divergent so $\sum_{n=0}^{+\infty} v_n$ is divergent.

Remark 4.7

1/The series with the sum of a geometric progression which converges if the ratio q is between 1 and -1 ($|q| < 1$).
2/Riemann's series $\sum_{n=1}^{+\infty} \frac{1}{n^\alpha}$ is convergent when $\alpha > 1$ and divergent when $\alpha \leq 1$.

3/Bertrand series $\sum_{n=1}^{+\infty} \frac{1}{n^{\alpha}(\ln n)^{\beta}}$ *is convergent when* $\alpha > 1$ *or* $\alpha = 1$ *and* $\beta > 1$.

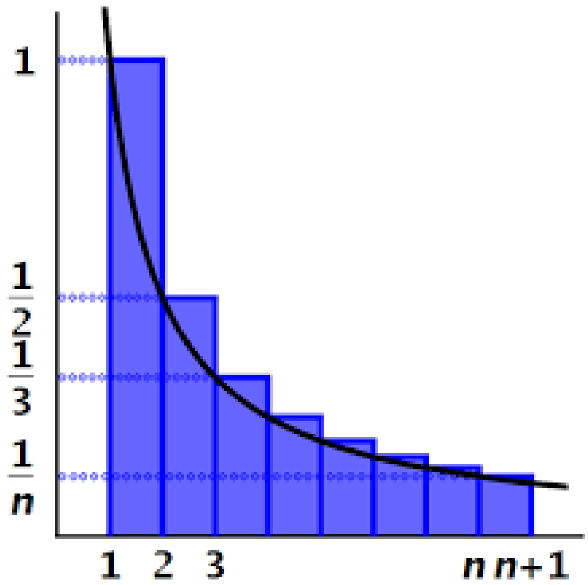


Figure 4.3: Define Riemann series

Theorem 4.8 (*Equivalence Criteria*)

Let it be $\sum_{n=0}^{+\infty} u_n$ *and* $\sum_{n=0}^{+\infty} v_n$ *two numerical series with positive terms where* $u_n \sim_{+\infty} v_n$ *then the two series are of the same nature.*

Remark 4.9

The study of the nature of a series amounts to the study of its development limited to the neighborhood of infinity, because the latter two are equivalent.

Example 4.4

Study the nature of this series: $U_n = \frac{\cos(n\pi)}{n^2}$

we use comparison theorem $\mid \frac{\cos(n\pi)}{n^2} \mid < \frac{1}{n^2}$ *since* $\Rightarrow \sum \frac{1}{n^2}$ *Riemann serie converge* $\alpha = 2 > 1$ *so* $\sum \frac{\cos(n\pi)}{n^2}$ *the series converges absolutely so is converge by comparison.*

Example 4.5

Study the nature of this series: $U_n = \dfrac{1}{\ln(1+\frac{1}{n^2})}$

For large n, $\ln(1+\dfrac{1}{n^2}) \sum \dfrac{1}{n^2}$, *so:*

$$U_n \sum \frac{1}{\frac{1}{n^2}} = n^2 \Rightarrow \lim_{n\to+\infty} n^2 = +\infty.$$

equivalency causes the series to diverge, since $\sum n^2$ *diverges.*

Theorem 4.10 (*Alembert Criteria*)

[13] If in a series with positive terms $\sum_{n=0}^{+\infty} u_n$ *the limit of the report* $\dfrac{u_{n+1}}{u_n}$ *is finite and is equal* l, *then:*

1-The series convergent for $l < 1$.

2-The series divergent for $l > 1$.

3-For $l = 1$ *we can not conclude.*

Example 4.6

Study the nature of this series: $\sum_{n=1}^{+\infty} \dfrac{\ln n}{n!}$:

we applied Alembert criteria:

$\lim_{n\to+\infty} \dfrac{u_{n+1}}{u_n} = \lim_{n\to+\infty} \dfrac{\ln n+1}{n+1!}\dfrac{n!}{\ln n} = \lim_{n\to+\infty} \dfrac{\ln n+1}{(n+1)n!}\dfrac{n!}{\ln n} = \lim_{n\to+\infty} \dfrac{\ln n+1}{n+1}\dfrac{1}{\ln n} = 0 =$

$l < 1 \Rightarrow \sum_{n=1}^{+\infty} \dfrac{\ln n}{n!}$ *is convergent.*

Example 4.7

Study the nature of this series: $U_n = \dfrac{n!}{n}$

we applied Alembert criteria:

$$\lim_{n\to+\infty} \frac{U_{n+1}}{U_n} = \lim_{n\to+\infty} \frac{(n+1)!}{n+1}\frac{n!}{n} = \lim_{n\to+\infty} n = +\infty > 0.$$

The series diverges by Alembert.

Theorem 4.11 (*Cauchy Criteria*)

[10] If in a series with positive terms $\sum_{n=0}^{+\infty} u_n$ *the limit of the* $\lim_{n\to+\infty} \sqrt[n]{u_n} = l$:

1-If $l > 1$ so $\sum_{n=0}^{+\infty} u_n$ is divergent.

2-If $l < 1$ so $\sum_{n=0}^{+\infty} u_n$ is convergent.

3-If $l = 1$ so we can not judge.

Example 4.8

Study the nature of this series: $\sum_{n=1}^{+\infty} 2^{n^2}$

we applied Cauchy criteria:

$$\lim_{n\to+\infty} \sqrt[n]{u_n} = \lim_{n\to+\infty} \sqrt[n]{2^{n^2}} = \lim_{n\to+\infty} 2^n = +\infty > 1 \text{ so } \sum_{n=1}^{+\infty} 2^{n^2} \text{ is divergent.}$$

Example 4.9

Study the nature of this series: $\sum_{n=1}^{+\infty} \frac{1}{n^n}$

we applied Cauchy criteria:

$$\lim_{n\to+\infty} \sqrt[n]{u_n} = \lim_{n\to+\infty} \sqrt[n]{\frac{1}{n}} = \lim_{n\to+\infty} (\frac{1}{n})^{\frac{n}{n}} == \lim_{n\to+\infty} \frac{1}{n} = 0 < 1 \text{ so } \sum_{n=1}^{+\infty} \frac{1}{n^n} \text{ is convergent.}$$

Theorem 4.12 (*Riemann criteria*)

[10] Let it be $\sum_{n=0}^{+\infty} u_n$ series with positive terms and $\exists\alpha \in \Re$, $\lim_{n\to+\infty} n^\alpha u_n = l$

1-If $\alpha > 1$ and $0 \leq l < +\infty \Rightarrow \sum_{n=0}^{+\infty} u_n$ is convergent.

2-If $\alpha \leq 1$ and $0 < l \leq +\infty \Rightarrow \sum_{n=0}^{+\infty} u_n$ is divergent.

Example 4.10

Study the nature of this series: $\sum_{n=1}^{+\infty} \frac{1}{\sqrt{n(n+1)}}$

we applied Riemann criteria:

$\lim_{n\to+\infty} n^\alpha u_n, -\alpha = 1),$

$\lim_{n\to+\infty} \frac{n}{n(\sqrt{1+\frac{1}{n}})} = 1$, so: $l \neq 0$ and $\alpha = 1 \Rightarrow$ the series is diverge.

Theorem 4.13 (*Cauchy integral criteria*)

[10] Let it be $\sum_{n=0}^{+\infty} u_n$ a numerical series with positive terms and decreasing (starts to decrease from a certain rank n_0), we define the application $f(x)$ as follows $f(n) = u_n$ so $\sum_{n=0}^{+\infty} u_n$ and $\int_{n_0}^{+\infty} f(x)dx$ are the same nature.

Example 4.11

Let it be $\sum_{n=3}^{+\infty} \frac{\ln n}{n}$ we applied Cauchy integral criteria: the series with positive terms and decreasing from $n_0 = 3$, we define the function $\frac{\ln x}{x}$, $\forall x \in [3, +\infty[$ so $\sum_{n=3}^{+\infty} \frac{\ln n}{n}$ and $\int_{3}^{+\infty} \frac{\ln x}{x}dx$ they have same nature we calculate this integral so :
$\int_{3}^{+\infty} \frac{\ln x}{x}dx = \frac{1}{2}(\ln x)^2|_{x=3}^{+\infty} = +\infty$ so the integral is divergent and the same for the series.

4.1.2 Series Represented by Various Symbols

Definition 4.2

A series is considered to have mixed signs if it contains both positive and negative terms.

Remark 4.14

Alternating series represent a specific category of series characterized by terms that can have varying signs.

Definition 4.3

- *A series is termed **absolutely convergent** if the series of its absolute values, $\sum |a_n|$, converges. A fundamental theorem asserts that every completely convergent series is convergent.*
- *A series is defined as conditionally convergent (or semi-convergent) when the original series $\sum a_n$ converges, while the series of its absolute values, $\sum |a_n|$, diverges.*

Example 4.12

$\sum_{n=1}^{+\infty} \frac{\sin n}{n\sqrt{n}}$ *we study the absolutely convergent that mean:*
$\sum_{n=1}^{+\infty} |\frac{\sin n}{n\sqrt{n}}|$ *by the comparison we have:*
$|\sin n| \leq 1 \Rightarrow |\frac{\sin n}{n^{\frac{3}{2}}}| \leq \frac{1}{n^{\frac{3}{2}}}$
$\sum_{n=1}^{+\infty} \frac{1}{n^{\frac{3}{2}}}$ *is convergent because it's a Riemann's series* $\alpha = \frac{3}{2} > 1$ *so by comparison the series* $\sum_{n=1}^{+\infty} |\frac{\sin n}{n\sqrt{n}}|$ *is convergent* $\Rightarrow \sum_{n=1}^{+\infty} \frac{\sin n}{n\sqrt{n}}$ *is absolutely convergent* $\Rightarrow \sum_{n=1}^{+\infty} \frac{\sin n}{n\sqrt{n}}$ *is convergent*

Theorem 4.15

All totally convergent series are convergent; however, the reverse is not necessarily true.

Theorem 4.16 (*Abel*)

Suppose that $u_n = w_n.v_n$ *where* w_n *and* v_n *check the following conditions:*

1-$(v_n)_{n\ inN}$ *is decreasing and* $\lim_{n\to+\infty} v_n = 0$.

2- s_n *is bounded where en is defined as in (B) so the* $\sum_{n=0}^{+\infty} u_n$ *is convergent.*

Theorem 4.17

Suppose that $u_n = w_n.v_n$ *where* w_n *and* v_n *check the following conditions:*

1-$(v_n)_{n\ inN}$ *is a decreasing and* $\lim_{n\to+\infty} v_n = l$, $(l \in \Re)$.

2-$\sum_{n=0}^{+\infty} w_n$ *is convergent. So* $\sum_{n=0}^{+\infty} u_n$ *is convergent.*

Example 4.13

Study the nature of this series : $U_n = a^n e^{-n}$, $|a| < 1$

Rewrite $U_n = v_n.w_n = a^n e^{-n}$ *we use Abel theorem. Since* $\sum a^n$, *the series converges geometrically (* $|a| < 1$*).*

And $\lim_{n\to+\infty} e^{-n} = 0$ *so by Abel the serie is converge.*

Definition 4.4 (***The Alternating Series Test***)

An alternating series is characterized by a specific structure that follows this form:

$$u_1 - u_2 + u_3 + ... + (-1)^{n+1}u_n + ..., \tag{4.4}$$

where $u_n \geq 0$ *for all* $n \in N^*$.

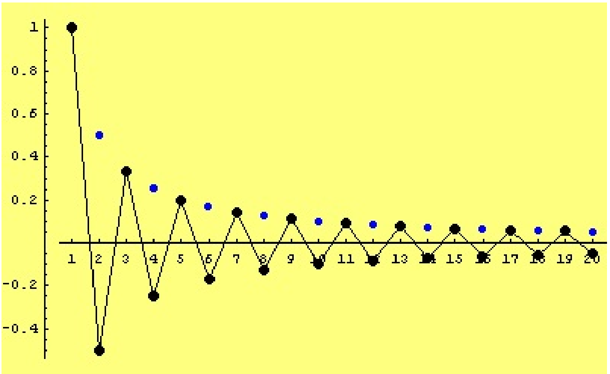


Figure 4.4: Define alternating series

Theorem 4.18 (***Leibniz***)

[13] An alternating series consists of terms that change in sign from one to the next. A series of the form $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ *is assured to converge if it satisfies all three of the following criteria:*

1. *The values of* b_n *are required to be positive for every* n *(i.e.,* $b_n > 0$*).*
2. *The sequence of terms* $\{b_n\}$ *is required to be decreasing; specifically,* $b_{n+1} \leq b_n$ *for all* n *after a certain threshold.*
3. *The terms must converge to zero. The limit of* b_n *as* n *approaches infinity is 0.*

Example 4.14

$\sum_{n=1}^{+\infty} \frac{(-1)^n}{n^2}$ *for study the convergent we use theorem Leibniz:*

$v_n = \frac{1}{n^2}$ *is decrease sequence and* $\lim_{n \to +\infty} \frac{1}{n^2} = 0$ *so* $\sum_{n=1}^{+\infty} \frac{(-1)^n}{n^2}$ *is convergent.*

4.2 Sequences and Series of Functions

4.2.1 Sequences of Functions

Definition 4.5

A sequence of functions is defined as a sequence in which each term is a function that depends on a parameter n, denoted as follows:

$$(f_n)_{n\in N}. \tag{4.5}$$

Categories of Convergences

Simple convergence

Definition 4.6

A sequence of functions (f_n) demonstrates simple convergence to a function f on an interval I if, for every distinct point $x \in I$, the sequence of values $f_n(x)$ converges to $f(x)$.

Definition 4.7

Let $f : I \subset \mathbb{R} \longrightarrow \mathbb{R}$ be a sequence of functions. We state that the sequence (f_n) converges simply to x_0 on I if and only if the sequence $f_n(x_0)$ is convergent.

Remark 4.19

The sequence of functions f_n converges pointwise to f if the following conditions hold:

1. *For all x, the limit of the sequence satisfies:*

$$\lim_{n\to+\infty} f_n(x) = f(x).$$

2. *For all x, the absolute difference converges to zero:*

$$\lim_{n\to+\infty} |f_n(x) - f(x)| = 0.$$

Example 4.15

We study this sequence $f_n(x) = \dfrac{\sqrt{n}x}{1+\sqrt{n}x^2}$.

1/ $\lim_{n\to+\infty} f_n(x) = \lim_{n\to+\infty} \dfrac{\sqrt{n}x}{1+\sqrt{n}x^2} = \lim_{n\to+\infty} (\dfrac{x}{\frac{1}{\sqrt{n}}+x^2}) = \dfrac{1}{x} = f(x)$ *where* $x \in \Re^{*+}$ *so*

$$f(x) = \begin{cases} 0 & x = 0 \\ \dfrac{1}{x} & x \in]0,+\infty[\end{cases}$$

1-For $x = 0$*:*

$\lim_{n\to+\infty} |f_n(x) - f(x)| = \lim_{n\to+\infty} |f_n(0) - f(0)| = 0$ *so* f_n *is simple convergent for the point* $x_0 = 0$.

2-For $x \in]0,+\infty[$*:*

$\lim_{n\to+\infty} |f_n(x) - f(x)| = \lim_{n\to+\infty} |\dfrac{\sqrt{n}x}{1+\sqrt{n}x^2} - \dfrac{1}{x}| = \lim_{n\to+\infty} \dfrac{1}{x(1+\sqrt{n}x^2)} = 0$ *so* $f_n(x) \longrightarrow f$ *simple convergent in* $]0,+\infty[$ *so* $f_n(x) \longrightarrow f$ *simple convergent in* $]0,+\infty[$.

Uniform Convergence

Definition 4.8

Uniform convergence*, however, is a more stringent condition. It requires that the convergence occurs at the same rate across the entire interval. Formally, a sequence* (f_n) *converges uniformly to* f *on* I *if the supremum of the absolute difference between the functions approaches zero:*

$$\lim_{n\to\infty} \sup_{x\in I} |f_n(x) - f(x)| = 0$$

Remark 4.20

Uniform convergence necessarily leads to simple convergence.

Theorem 4.21

Let it be $(f_n)_{n\in N}$ *a sequence of functions is simply convergent to* f *if* $\dfrac{\partial f_n}{\partial x}$ *is bounded then the convergence is uniform.*

Example 4.16

We study uniform convergent the sequence $f_n(x) = \dfrac{\sqrt{n}x}{1+\sqrt{n}x^2}$.

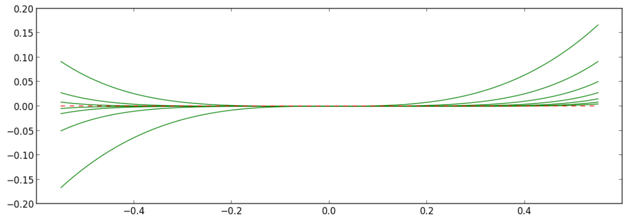


Figure 4.5: Graph of defining the uniform convergence

For the simple convergent we have:

$$f(x) = \begin{cases} 0 \ x = 0 \\ \frac{1}{x} \ x \in]0, +\infty[\end{cases}$$

For the uniform convergent, we have:

$$\lim_{n \to +\infty} sup|f_n(x) - f(x)| = 0$$

1/$g_n(x) = |f_n(x) - f(x)|$

2/$sup g_n(x)/x \in I$

3/$\lim_{n \to +\infty} g_n(x)$

1/For $x \in]0, +\infty[$: $f(x) = \frac{1}{x}$ *and* $g_n(x) = (\frac{1}{x(1+\sqrt{n}x^2})$ *so we seek for* $sup g_n(x)$ *we find the derivative of* $g_n(x)$:

$g'_n(x) = \frac{-(1+3\sqrt{n}x^2)}{(1+\sqrt{n}x^2)^2}$ *so the* $sup_{x \in \Re^{*+}} g_n(x) = \lim_{x \to 0} g_n(x) = \lim_{x \to 0} \frac{1}{(1+\sqrt{n}x^2)^2} = +\infty$ *so* $f_n(x)$ *not uniformly convergent to* f.

Properties of Functions Sequences

Proposition 4.22 (*Continuity Theory*)

Let it be $f : I \subset \Re \longrightarrow \Re$ *a sequence of functions continuous uniformly converges to* f *on* I. *Then* f *is continuous.*

Proposition 4.23 (*Derivation Theory*)

Let $(f_n)_{n \in \mathbb{R}}$ *be a sequence of functions that converges uniformly to the function* f. *The sequence of functions* $(f'_n)_{n \in \Re}$ *converges uniformly to* f'.

Proposition 4.24 (*Integral Theory*)

Let it be $(f_n)_{n \in \Re}$ *a sequence of Riemann-integral functions converges uniformly to a* *function f on $[a,b]$, so*

$$\lim_{n\to+\infty}\int_a^b f_n(x)dx = \int_a^b \lim_{n\to+\infty} f_n(x)dx = \int_a^b f(x)dx.$$

4.2.2 series of functions

Definition 4.9

Consider a sequence of functions denoted as $f_n : I \subset \Re \longrightarrow \Re$. A series of functions is defined as the infinite sum of the terms of f_n, which we denote as follows:

$$\sum_{n=0}^{+\infty} f_n(x) = f_1 + f_2 + f_3 + ... + f_n + f_{n+1} + ... \tag{4.6}$$

Definition 4.10

We denote $(s_n)_{n\in\Re}$ as the partial sums for a sequence of functions f_n:

$$s_n = \sum_{n=0}^{n} f_k = f_0 + f_2 + f_3 + ... + f_n. \tag{4.7}$$

Simple Convergent

Definition 4.11

Consider the series of functions $\sum_{n=0}^{+\infty} f_n(x)$ defined on the interval $I \in \Re$. We characterize it as simply convergent if the series of partial sums $(s_n)_{n\in\Re}$ converges simply on I:

$$\lim_{n\to+\infty} s_n(x) = s(x) \Rightarrow \lim_{n\to+\infty} |s_n(x) - s(x)| = 0. \tag{4.8}$$

Uniform Convergence

Definition 4.12

We state that the series $\sum_{n=0}^{+\infty} f_n(x)$ converges uniformly on the interval I.

$$\Leftrightarrow \lim_{n\to+\infty} sup|s_n(x) - s(x)| = 0. \tag{4.9}$$

Remark 4.25

To analyze the convergence of the series $\sum_{n=0}^{+\infty} f_n(x)$, *one can employ various criteria such as comparison criteria, equivalent criteria, Riemann's criteria, the Alembert criterion, and Cauchy's criterion. It is important to note that if* f_n *is either positive or negative on the interval* I, *where* x *is treated as a constant variable, then* $\sum_{n=0}^{+\infty} f_n(x)$ *can be regarded as a numerical series with a positive general term.*

Normally Convergent

Definition 4.13

A series of functions $\sum f_n(x)$ *is said to be* ***normally convergent*** *on an interval* I *if the series formed by the supremum of the absolute values of each function is a convergent numerical series. That is, if:*

$$\sum_{n=0}^{\infty} \left(\sup_{x \in I} |f_n(x)| \right) < \infty$$

Normal convergence implies uniform convergence, making it a very strong condition.

Theorem 4.26 (*Weirtrass*)

[13] Let it be $\sum_{n=0}^{+\infty} f_n(x)$ *a series of functions define in* $I \in \Re$ *if we find a numerical sequence* $(C_n)_{n \in N}$ *where:*

$$\forall x \in \Re,\ |f_n(x)| \leq C_n, \tag{4.10}$$

so if it was $\sum_{n=0}^{+\infty} C_n$ *convergent so* $\sum_{n=0}^{+\infty} f_n(x)$ *normally convergent in* I.

Example 4.17

For previously example: $\sum_{n=0}^{+\infty} f_n = \sum_{n=0}^{+\infty} \frac{x^n}{1+n^2}$.

1/Simple convergent:

$\forall x \in [0, +\infty[,\ f_n(x) = \frac{x^n}{1+n^2} \geq 0.$

For fixed x *we applique role Riemann:*

$\exists \alpha \in \Re,\ \lim_{n \to +\infty} n^{\alpha} f_n(x),$

$\exists \alpha = 2,$ *where,* $\lim_{n \to +\infty} n^{\alpha} \frac{x^n}{1+n^2} = \lim_{n \to +\infty} \frac{n^2 x^n}{1+n^2} \lim_{n \to +\infty} x^n = \lim_{n \to +\infty} x^n$

$$\lim_{n\to+\infty} x^n = \begin{cases} 0\ if\ x \in [0,1[\\ 1\ if\ x = 1 \\ +\infty\ if x \in]1,+\infty[\end{cases}$$

$\exists\alpha = 2 > 1$:

$$\lim_{n\to+\infty} n^\alpha f_n(x) = \begin{cases} 0\ if\ x \in [0,1[\\ 1\ if\ x = 1 \\ +\infty\ if x \in]1,+\infty[\end{cases}$$

According to $\forall x \in [0,1] \Rightarrow \sum_{n=0}^{+\infty} f_n(x)$ *simply convergent.*

For $x \in]1,+\infty[$ *we applique Alembert method:*

$\lim_{n\to+\infty} \frac{f_{n+1}(x)}{f_n(x)} = \lim_{n\to+\infty} \frac{x^{n+1}}{1+(n+1)^2}\frac{1+n^2}{x^n} = \frac{n^2+1}{1+(n+1)^2}x = x$, *so,* $\lim_{n\to+\infty} \frac{f_{n+1}(x)}{f_n(x)} = x > 1$ *according to Alembert* $\Rightarrow \sum_{n=0}^{+\infty} f_n(x)$ *not simply convergent in* $]1,+\infty[\Rightarrow \sum_{n=0}^{+\infty} f_n(x)$ *not simply convergent in* $[1,+\infty[$.

2/Normally convergent in $[0,1]$*:*

$\sum_{n=0}^{+\infty} f_n(x)$ *normally convergent in* $[0,1] \Leftrightarrow \sum_{n=0}^{+\infty} sup|f_n(x)|$ *convergent.*

1-$|f_n(x)| = |\frac{x^n}{1+n^2}$

2-$sup_{x\in[0,1]}|f_n(x)| = sup_{x\in[0,1]}\frac{x^n}{1+n^2}$

we calculate the derivative of $f_n(x)$ *so:*

$(\frac{x^n}{1+n^2})' = \frac{nx^{n-1}}{1+n^2} > 0,\ \forall x \in]0,1]$ *so the sup of the function be at value* $x = 1 \Rightarrow \sum_{n=0}^{+\infty} sup|f_n(x)| = \sum_{n=0}^{+\infty}\frac{1}{1+n^2}$

we study the nature of this series by using the equivalent method:

$\frac{1}{1+n^2} \sim_{+\infty} \frac{1}{n^2}$, *is convergent because it's Riemann series* $\alpha = 2 > 1 \Rightarrow \sum_{n=1}^{+\infty}\frac{1}{1+n^2}$ *is convergent* $\Rightarrow \sum_{n=0}^{+\infty}\frac{1}{1+n^2}$ *is convergent so,*

$\sum_{n=0}^{+\infty} sup_{x\in]0,1]}|f_n(x)|$*is convergent* $\Leftrightarrow \sum_{n=0}^{+\infty} f_n(x)$ *normally convergent in* $]0,1]$

For $x = 0$*:*

$\sum_{n=0}^{+\infty} f_n(x) = \sum_{n=0}^{+\infty} f_n(0) = 0$ *is normally convergent in* $x = 0$ *so* $\forall x \in [0,1] \sum_{n=0}^{+\infty} f_n(x)$ *is normally convergent.*

Absolutely Convergent

Definition 4.14

Let it be $\sum_{n=0}^{+\infty} f_n(x)$ *series of functions define in* $I \in \Re$ *we say that* $\sum_{n=0}^{+\infty} f_n(x)$ *absolutely convergent if* $\sum_{n=0}^{+\infty} |f_n(x)|$ *simple convergent in* $I \in \Re$.

Example 4.18

We study the convergence of the series:

$$\sum_{n=0}^{\infty} \frac{\sin(nx)}{n^2}, \quad x \in \mathbb{R}.$$

To determine its convergence, we consider the absolute convergence by examining:

$$\sum_{n=0}^{\infty} \left|\frac{\sin(nx)}{n^2}\right| = \sum_{n=0}^{\infty} \frac{|\sin(nx)|}{n^2}.$$

For all $x \in \mathbb{R}$, *we have* $|\sin(nx)| \geq 0$. *To apply the comparison test, note that:*

$$|\sin(nx)| \leq 1 \quad \Rightarrow \quad \frac{|\sin(nx)|}{n^2} \leq \frac{1}{n^2}.$$

The series $\sum_{n=0}^{\infty} \frac{1}{n^2}$ *is a convergent p-series (Riemann zeta function at* $p = 2$*), since* $p = 2 > 1$. *By the comparison test, since* $\frac{|\sin(nx)|}{n^2} \leq \frac{1}{n^2}$, *the series* $\sum_{n=0}^{\infty} \frac{|\sin(nx)|}{n^2}$ *converges. Therefore, the original series* $\sum_{n=0}^{\infty} \frac{\sin(nx)}{n^2}$ *is absolutely convergent for all* $x \in \mathbb{R}$.

Properties of Function Series

Continuity theory

Proposition 4.27 (*Continuity Theory*)

Suppose we have $\sum_{n=0}^{+\infty} f_n(x)$ *continuous series of functions and uniformly convergent in* $I \in \Re$ *so:* $s(x) = \sum_{n=0}^{+\infty} f_n(x)$ *is a continuous function in* $I \in \Re$.

Proposition 4.28 (*Derivation Theory*)

Suppose we have $\sum_{n=0}^{+\infty} f_n(x)$ *a series of functions uniformly convergent. Then the series of functions* $\sum_{n=0}^{+\infty} f'_n$ *uniformly convergent in* $I \in \Re$.

Proposition 4.29 (*Integral theory*)

Suppose we have $\sum_{n=0}^{+\infty} f_n(x)$ *a series of Riemann-integral functions uniformly convergent on* $[a, b] \in \Re$, *so*

$$\sum_{n=0}^{+\infty}\left(\int_a^b f_n(x)dx\right) = \int_a^b \left(\sum_{n=0}^{+\infty} f_n(x)\right)dx = \int_a^b s(x).$$

4.3 Power Series and Fourier Series

Power Series and Radius of Convergence

A **power series** centered at x_0 is an infinite series of the form:

$$\sum_{n=0}^{\infty} a_n(x - x_0)^n. \tag{4.11}$$

A key feature of any power series is its **domain of convergence**, which is always an interval centered at x_0. The size of this interval is determined by the **radius of convergence**, denoted by R.

The series is guaranteed to converge absolutely for all x within this radius (i.e., for $|x - x_0| < R$) and diverge for all x outside of it (i.e., for $|x - x_0| > R$). The convergence at the endpoints ($x = x_0 \pm R$) must be checked separately. The value of R can be found using the Ratio Test:

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| \tag{4.12}$$

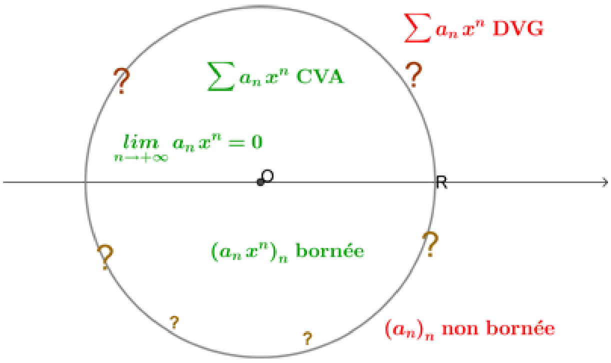


Figure 4.6: Define the power series

Determination of the Radius of Convergence

Theorem 4.30 (***he Ratio Test (Alembert's Criterion)***)

$$R = \lim_{n \to +\infty} |\frac{a_n}{a_{n+1}}|. \quad (4.13)$$

Example 4.19

$R = \lim_{n \to +\infty} |\frac{a_{3n+1}}{a_{3n+2}}|$ *or,* $R = \lim_{n \to +\infty} |\frac{a_{n+17}}{a_{n+18}}|$.

Theorem 4.31 (***Cauchy Criteria***)

$R = \lim_{n \to +\infty} \frac{1}{\sqrt[n]{|a_n|}}$.

Example 4.20

$R = \lim_{n \to +\infty} \frac{1}{\sqrt[n^3]{|a_{n^3}|}}$ *or,* $R = \lim_{n \to +\infty} \frac{1}{\sqrt[2n+3]{|a_{2n+3}|}}$.

4.3.1 Some properties of Power Series

Proposition 4.32 (***Continuity***)

The sum of an integer series constitutes a continuous function across any sub-domain within its convergence domain.

Theorem 4.33 (***Derivation***)

Let it be $\sum_{n=0}^{+\infty} a_n x^n$ *power series whose domain of convergence is* $]-R, R[$ *and the sum*

$s(x)$, either $\sum_{n=0}^{+\infty} n a_n x^{n-1}$ deduced from the first by term-to-term derivation, the latter has the same domain of convergence as the first, moreover its sum $l = s'(x)$.

Remark 4.34

The above theorem can be generalized to any order of differentiation; specifically, every series obtained from a given series through n differentiations possesses the same domain of convergence, and its sum corresponds to the n^{th} derivative of the original series.

Theorem 4.35 (*Integral*)

Let it be $\sum_{n=0}^{+\infty} a_n x^n$ power series whose domain of convergence is $]-R, R[$ and the sum $s(x)$, either $c + \sum_{n=0}^{+\infty} \frac{a_n}{n+1} x^{n+1}$ deduced from the first by term-to-term integral, the latter has the same domain of convergence as the first, moreover its sum $m(x) = \int s(x)dx$.

4.3.2 Power Series (Séries de puissances)

Definition 4.15

A power series is defined as any series expressed in the following form:

$$\sum_{n=0}^{+\infty} a_n (x - x_0)^n, \tag{4.14}$$

where a_n is a numerical series.

Remark 4.36

One can consider the change as $X = x - x_0$, allowing this expression to transform into a complete series.

Remark 4.37

The convergence interval of the power series is centered at x_0.

Remark 4.38

The Mac Laurent series development represents an entire series, on the other hand that of Taylor represents a power series in the vicinity of the point x_0.

Applications of Power Series

Limits Calculating

Corollary 4.2

f we are unable to determine the limit, we can compute the limit of the function's development in within close proximity of the desired point.

Example 4.21

we calculated this limit: $\lim\limits_{x\to 0} \dfrac{e^x - 1}{\sin x}$.
We have; $\lim\limits_{x\to 0} \dfrac{e^x - 1}{\sin x} = \dfrac{0}{0}$. *We replace* e^x *and* $\sin x$ *by their Mac Laurent developments, we obtain:*
$\lim\limits_{x\to 0} \dfrac{e^x - 1}{\sin x} = \lim\limits_{x\to 0} \dfrac{1 + x - 1}{x} = 1.$

Extension by Continuity

Corollary 4.3

To determine whether a function can be extended by continuity at a point where it is not defined, it suffices to express it as an entire series. If this series is continuously differentiable, the function can be extended at that point.

Example 4.22

Can we extend the function $f(x) = \dfrac{\sin x}{x}$ *in a continuous function on* $\Re$. *We have:*
$\sin x = \sum\limits_{n=0}^{+\infty} \dfrac{x^{2n+1}}{(2n+1)!} \Rightarrow \dfrac{\sin x}{x} = \sum\limits_{n=0}^{+\infty} \dfrac{x^{2n}}{(2n+1)!}$.
The resulting series is defined and belongs to the class C^∞ *in* $\Re$, *so the function* $f(x)$ *permits a continuous extension.*

Integral Calculation

Corollary 4.4

Every continuous function possesses an anti-derivative; however, many functions have anti-derivatives that cannot be directly defined, albeit they can be computed via series.

Example 4.23

We calculate this integral: $\int_0^1 e^{x^2}\,dx$, *we have:*

$$e^x = \sum_{n=0}^{+\infty} \frac{x^n}{n!} \Rightarrow e^{-x^2} = \sum_{n=0}^{+\infty} \frac{(-1)^n x^{2n}}{n!} \Rightarrow \int_0^1 e^{x^2} dx = \sum_{n=0}^{+\infty} \frac{(-1)^n}{n!(2n+1)}.$$

Integration of Ordinary Differential Equations

Corollary 4.5

When tasked with integrating a specific ordinary differential equation (O.D.E) whose resolution cannot be reduced to a quadrature, we can consistently derive and approximation solution by assuming that the answer can be expressed as a power series expansion.

Example 4.24

We solve the differential equation using a power series expansion of $y(x)$

$$\begin{cases} (x+1)y' - xy = -1 \\ y(0) = 2 \qquad\qquad x \in [-1, +\infty[\end{cases}$$

Let

$$y(x) = \sum_{n=0}^{\infty} a_n x^n.$$

Then,

$$y'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n,$$

where we relabeled the index of the second sum by setting $n = n - 1$.

Inserting $y(x)$ *and* $y'(x)$ *into the differential equation:*

$$(x+1) \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - x \sum_{n=0}^{\infty} a_n x^n = -1.$$

Expanding $(x+1)y'$ *and collecting terms:*

$$\sum_{n=0}^{\infty} (n+1) a_{n+1} x^{n+1} + \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=0}^{\infty} a_n x^{n+1} = -1.$$

Shift the indices of the sums to align powers of x^n*:*

$$\sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} (n+1) a_{n+1} x^n - \sum_{n=1}^{\infty} a_{n-1} x^n = -1.$$

Combine terms for x^n *and write the equation as:*

$$(n+1) a_{n+1} + n a_n - a_{n-1} = 0 \quad \text{for } n \geq 1,$$

and the constant term a_0 satisfies:

$$a_1 + a_0 = -1.$$

Using the initial condition $y(0) = 2$, we have:

$$a_0 = 2.$$

From the recurrence relation, compute coefficients a_n:

$$a_1 = -1 - a_0 = -1 - 2 = -3,$$

$$a_2 = \frac{a_1}{2} = \frac{-3}{2},$$
$$a_3 = \frac{-a_2}{3} = \frac{3}{2 \cdot 3} = \frac{1}{2},$$

and so on.

Thus, the solution is:

$$y(x) = 2 - 3x - \frac{3}{2}x^2 + \frac{1}{2}x^3 + \cdots .$$

4.3.3 Fourier Series

Definition 4.16

We define a function f as periodic with period p if it meets the following condition:

$$\forall x \in D, \quad x + p \in D : f(x + p) = f(x), \tag{4.15}$$

where D denotes the domain of definition of f and p represents the smallest positive integer satisfying (4.16).

Definition 4.17

We define a function f as monotonic by slice on the interval $[a, b]$ if it can be partitioned into subintervals where f is monotonic on each subinterval.

Remark 4.39

Let f be a slice-monotonic function that is limited on $[a, b]$; if it exhibits discontinuities, *they can only be of the first kind.*

Definition 4.18

A trigonometric series is characterized as any series represented in the following format:

$$f(x) = \frac{a_0}{2} + \sum_{n=0}^{+\infty}(a_n \cos(nx) + b_n \sin(nx)), \tag{4.16}$$

where a_0, a_i, b_i are real numbers called coefficients of the series.

Determination of Fourier Coefficients

Consider a $2\Pi-$periodic function that can be represented by a trigonometric series of the form (4.18), suppose that the series $\frac{a_0}{2} + \sum_{n=0}^{+\infty}(a_n + b_n)$ is absolutely convergent, therefore (4.18) is uniformly convergent, we can therefore integrate it term by term. Knowing that:

$$\int_{-\Pi}^{+\Pi} \cos(nx)\cos(kx)dx = \int_{-\Pi}^{+\Pi} \sin(nx)\sin(kx)dx = \begin{cases} \Pi \; if \; n = k \\ 0 \; if \; n \neq k \end{cases} \tag{4.17}$$

and

$$\int_{-\Pi}^{+\Pi} \cos(nx)dx = \int_{-\Pi}^{+\Pi} \sin(nx) = \int_{-\Pi}^{+\Pi} \cos(nx)\sin(kx)dx = 0, \forall n, k. \tag{4.18}$$

The integration of the two members of (4.18) gives:

$$a_0 = \frac{1}{\Pi}\int_{-\Pi}^{+\Pi} f(x)dx. \tag{4.19}$$

We will now multiply both sides of (4.18) by $\cos(kx)$ and thereafter integrate the result from $-\Pi$ to Π, yielding:

$$a_n = \frac{1}{\Pi}\int_{-\Pi}^{+\Pi} \cos(nx)f(x)dx. \tag{4.20}$$

Let us multiply both sides of (4.18) by $\sin(kx)$ and integrate the result between $-\Pi$ and Π, we obtain:

$$b_n = \frac{1}{\Pi}\int_{-\Pi}^{+\Pi} \sin(nx)f(x)dx. \tag{4.21}$$

Thus we found the coefficients of the series. These are called Fourier coefficients.

Theorem 4.40

Any periodic, limited, and monotonous function can be expressed as a Fourier series. Its sum $s(x) = f(x)$ at points of continuity equals the function value, while at points of discontinuity, it corresponds to the arithmetic mean of the right-hand and left-hand limits.

Remark 4.41

The Fourier series expansion of even functions does not contain the $b_n (b_n = 0)$.

Remark 4.42

The Fourier series expansion of odd functions does not contain the an $(a_0 = a_n = 0)$.

4.3.4 Fourier Series of Periodic Period Functions $\neq 2\Pi$

Suppose that the function $f(x)$ is $2l$-periodic $(l \neq 0, l \neq \Pi)$, if moreover f is monotonic by slice and bounded then it is developable in Fourier series, whose Fourier coefficients are worth

$$a_0 = \frac{1}{l}\int_{-l}^{+l} f(x)dx, a_n = \frac{1}{l}\int_{-l}^{+l} f(x)\cos(\frac{n\Pi x}{l}), b_n = \frac{1}{l}\int_{-l}^{+l} f(x)\sin(\frac{n\Pi x}{l}). \tag{4.22}$$

In addition we have:

$$f(x) = \frac{a_0}{2} + \sum_{n=0}^{+\infty}(a_n\cos(\frac{n\Pi x}{l}) + b_n\sin(\frac{n\Pi x}{l})). \tag{4.23}$$

Parseval Equality

Theorem 4.43

Let f be a function developable in Fourier series of period $2l$ we have:

$$\frac{1}{l}\int_{-l}^{+l} f^2(x)dx = \frac{a_0^2}{2} + \sum_{n=0}^{+\infty}(a_n^2 + b_n^2). \tag{4.24}$$

Fourier Series in Complex Form

As previously stated, if f can be represented as a Fourier series, we designate it as follows:

$$\frac{a_0}{2} + \sum_{n=0}^{+\infty}(a_n\cos(\frac{n\Pi x}{l}) + b_n\sin(\frac{n\Pi x}{l})), \tag{4.25}$$

where:

$$\cos(\frac{n\Pi x}{l}) = \frac{e^{i\frac{n\Pi x}{l}} + e^{-i\frac{n\Pi x}{l}}}{2}, \tag{4.26}$$

and:

$$\sin(\frac{n\Pi x}{l}) = \frac{e^{i\frac{n\Pi x}{l}} - e^{-i\frac{n\Pi x}{l}}}{2i}, \tag{4.27}$$

so we found:

$$f(x) = \sum_{n=-\infty}^{+\infty} c_n e^{i\frac{n\Pi x}{l}}, \tag{4.28}$$

where:

$$c_0 = \frac{a_0}{2},\ c_n = \frac{a_n - ib_n}{2}\ \textit{and}\ c_{-n} = \frac{a_n + ib_n}{2}, \tag{4.29}$$

or simply:

$$c_n = \frac{1}{2\Pi}\int_{-\Pi}^{+\Pi} f(x)e^{-i\frac{n\Pi x}{l}}. \tag{4.30}$$

Example 4.25

We determine the Fourier series of this function $f(x) = |sin(x)|$ we have

1)The function f being periodic, bounded and monotonic per slice therefore can be developed in Fourier series.

2)$a_0 = \frac{1}{\Pi}\int_{-\Pi}^{+\Pi} f(x)dx = \frac{4}{\Pi}$.

$a_n = \frac{1}{\Pi}\int_{-\Pi}^{+\Pi} f(x)\cos nx dx \Rightarrow a_n \begin{cases} 0\ if\ n\ is\ odd \\ \frac{4}{\Pi(1-n^2)}\ n\ is\ even \end{cases}$

and we have the function f is even, then $b_n = 0$.

We can therefore write:

$a_{2n} = \frac{4}{\Pi(1-n^2)}$ *and:*

$f(x) = \frac{2}{\Pi} + \sum_{1}^{+\infty} \frac{4}{\Pi(1-n^2)}\cos(2nx)$.

3)Deduction of $\sum_{1}^{+\infty} \frac{4}{\Pi(1-n^2)}$, *$f(x)$ is is continuous in $x_0 = 0 \Rightarrow f(0) = 0$ so:*

$\sum_{1}^{+\infty} \frac{4}{\Pi(1-n^2)} = \frac{1}{2}$.

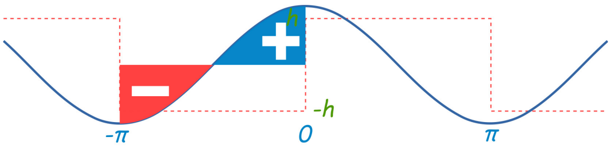


Figure 4.7: Graph of a_n coefficient.

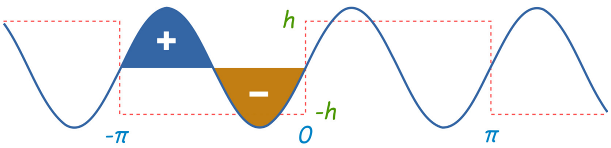


Figure 4.8: Graph of b_n coefficient.

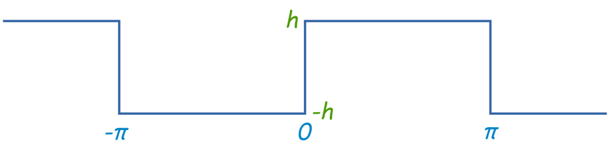


Figure 4.9: Graph of design Fourier series.

Fourier Transformation

The Fourier transform (FT) serves as a significant mathematical instrument that transitions a function from its time or spatial domain into the frequency domain, broadening the application of Fourier series to encompass non-periodic functions.

Definition 5.1

[12] L^1 is the set of summable functions.

Definition 5.2

[12] L^2 denotes the collection of functions whose squares are summable.

Definition 5.3

The set $\mathcal{L}^1$ consists of functions specified on $\Re$, which may be discontinuous at a finite number of points, but are continuous elsewhere and exhibit absolute integrability over $\Re$.

Definition 5.4

Let it be $f : \Re \longrightarrow C$, f is said to be rapidly decreasing, and we note $f \in S$ if $f \in C^\infty$ and if,

$$\forall s, m \in N,\ \exists G_{s,m} : |x^m f^{(s)}(x)| \leq G_{s,m},\ \forall x \in \Re. \tag{5.1}$$

Definition 5.5

Let it be $f : \Re \longrightarrow C$, f is a test function if it is indefinitely differentiable with compact support, and we note $f \in D$, in other words

$$f \in C^\infty \text{ and } \exists p, q \in \Re : p \leq q,\ \forall x \in \Re/[p, q],\ f(x) = 0. \tag{5.2}$$

Definition 5.6 (*Fourier Transformation*)

[12] Let it be $f : \Re \longrightarrow \Re$, a locally and absolutely integrable function on $\Re$. The

function denoted $F : \Re \longrightarrow C$, as a result,

$$F(f)(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t)e^{-i\omega t}\, dt. \quad (5.3)$$

Definition 5.7 (*Inverse Fourier Transformation*)

[8] Let $C \longrightarrow \Re$ retrieves the original function from its frequency representation:

$$f(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(f)(\omega)e^{i\omega t}\, d\omega. \quad (5.4)$$

Remark 5.1

The variable t is referred to as a temporal variable as it often represents time. The variable x is referred to as a frequency variable since it often represents frequency or pulsation. The expression $f(t)$ is referred to as a signal.

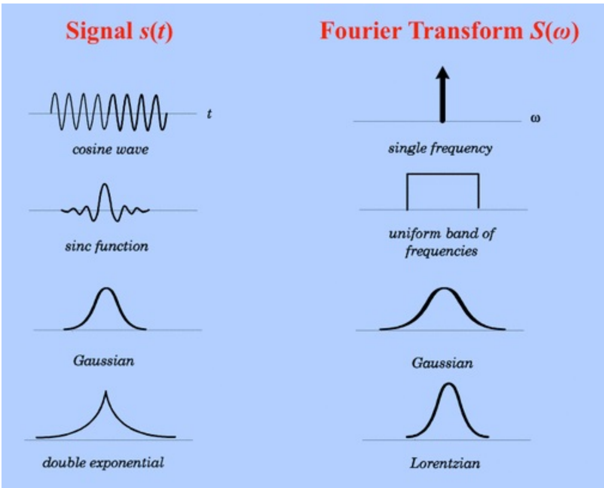


Figure 5.1: Design a Fourier transformation

Example 5.1

Analyze the Heaviside function:

$$H(t) = \begin{cases} 0 \; if \;\; t < 0 \\ 1 \; if \;\; t \geq 0. \end{cases}$$

The (FT) of the Heaviside function is computed as follows:

$$F(H(t)) = \frac{1}{2\pi} \int_{-\infty}^{\infty} H(t)e^{-i\omega t}\, dt = \frac{1}{2\pi} \int_{0}^{\infty} e^{-i\omega t}\, dt \frac{-i}{-iw\sqrt{2\pi}} e^{-i\omega t}|_{t=0}^{t=+\infty} = \frac{-i}{\omega\sqrt{2\pi}}.$$

5.1 Properties of Fourier Transformation

5.1.1 Linearity

For constants a and b, and functions $f_1(t)$ and $f_2(t)$, the (FT) is characterized as follows:

$$F[af_1(t) + bf_2(t)] = aF[f_1(t)] + bF[f_2(t)] \tag{5.5}$$

Linearity of the Inverse Fourier Transform

Let α and β denote two arbitrary complex numbers, and let $\hat{f}(\omega)$ and $\hat{g}(\omega)$ represent two functions that are locally integrable and absolutely integrable over $\mathbb{C}$. Then:

$$\mathscr{F}^{-1}\left[\alpha\hat{f} + \beta\hat{g}\right](t) = \alpha f(t) + \beta g(t). \tag{5.6}$$

Differentiation in the Time Domain

The (FT) of the n-th derivative of $f(t)$ is expressed as:

$$\mathscr{F}\left(\frac{d^n f}{dt^n}\right)(\omega) = (i\omega)^n \hat{f}(\omega). \tag{5.7}$$

Differentiation in the Frequency Domain

For the n-th derivative of $\hat{f}(\omega)$, the inverse Fourier transform is given by:

$$\frac{d^n \hat{f}}{d\omega^n}(\omega) = \mathscr{F}\left((it)^n f(t)\right). \tag{5.8}$$

Translation in t

For a time shift t_0, the Fourier transform satisfies:

$$\mathscr{F}(f(t - t_0)) = e^{-i\omega t_0}\hat{f}(\omega). \tag{5.9}$$

Remark 5.2

A translation in the time domain results in a linear phase shift in the frequency domain, represented by $e^{-i\omega t_0}$. This operation does not influence the magnitude of the (FT).

Scaling in t (Time Domain Contraction)

For scaling t by a factor a, the (FT) is given by:

$$\mathscr{F}(f(at)) = \frac{1}{|a|}\hat{f}\left(\frac{\omega}{a}\right). \tag{5.10}$$

5.1.2 Shift Theorem

A temporal shift in the original function is equivalent to a phase shift in the frequency domain.

$$F[f(t-\tau)] = e^{-i\omega\tau}F[f(t)]0. \tag{5.11}$$

The (FT) of $f(t) = e^{-at^2}$ is expressed as:

$$F(\omega) = \sqrt{\frac{\pi}{a}}e^{-\frac{\omega^2}{4a}}. \tag{5.12}$$

5.1.3 Modulation

$$F(e^{i\omega_0 t}f(t) = F(\omega - \omega_0. \tag{5.13}$$

5.1.4 Conjugation

$$F(\overline{f}) = \overline{F}(\omega). \tag{5.14}$$

5.1.5 Convolution

The convolution product of the real or complex functions f and g is the function $f * g$ defined as follows:

$$(f * g)(x) = \int_{-\infty}^{+\infty} f(x-t)g(t)dt. \tag{5.15}$$

The convolution product of the real or complex functions f and g is defined as the function $f * g$.

$$F(f * g)(t) = F(f)(\omega).F(g)(\omega). \tag{5.16}$$

5.1.6 Continuity

$$f \in \mathcal{L}^1 \Rightarrow F(f) \in C(\Re). \tag{5.17}$$

5.1.7 Behavior at infinity

$$\lim_{|\omega| \to +\infty} F(f)(\omega) = 0. \tag{5.18}$$

Remark 5.3

The Fourier transform $\mathscr{F}(f)$ has the following properties depending on the symmetry and nature of $f(t)$:

- *If $f(t)$ is a real and even function, then its Fourier transform $\mathscr{F}(f)$ is also real and even.*
- *If $f(t)$ is a real and odd function, then its Fourier transform $\mathscr{F}(f)$ is wholly imaginary and odd.*
- *If $f(t)$ is imaginary and even, then $\mathscr{F}(f)$ is imaginary and even.*
- *If $f(t)$ is an imaginary and even function, then its Fourier transform $\mathscr{F}(f)$ is also imaginary and even.*

5.1.8 Theorem of Plancherel

Let it be the functions $f, g \in S$ we have:

$$\int_{-\infty}^{+\infty} f(t)\overline{g}(t)dt = \int_{-\infty}^{+\infty} F(f)(\omega)\overline{F}(g)(\omega)d\omega. \tag{5.19}$$

5.1.9 Theorem Parseval

Let $f(t)$ be a finite energy function ($f \in L^2$) and $F(f)(\omega)$ denote its (FT). These are the assignments $f(t)$ and $F(f)(\omega)$ possess equivalent energy. That is to say:

$$\int_{-\infty}^{+\infty} |f(t)|^2 dt = \int_{-\infty}^{+\infty} |F(f)(\omega)|^2 d\omega. \tag{5.20}$$

Remark 5.4

The selection of parameters in the Fourier Transform can profoundly influence the outcome. Comprehending the characteristics and dynamics of functions in both temporal and spectral domains is essential for precise analysis.

5.2 Utilization in the Resolution of Differential Equations

The Fourier transform enables the explicit resolution of a linear differential equation by converting it into a more manageable form.

Results for Constant-Coefficient Linear Differential Equations

Proposition 5.5

Let f be an integrable function on $\mathbb{R}$ taking values in $\mathbb{C}$, such that $\mathscr{F}(f) = F$. Then we have:

$$\mathscr{F}\left(\frac{d^n f}{dt^n}(t)\right) = (2i\pi\omega)^n F(\omega), \quad n \in \mathbb{N}. \tag{5.21}$$

In general:

$$\mathscr{F}\left(\frac{df}{dt}(t)\right) = (2i\pi\omega)F(\omega), \quad \mathscr{F}\left(\frac{d^2 f}{dt^2}(t)\right) = (2i\pi\omega)^2 F(\omega). \tag{5.22}$$

Consider the differential equation:

$$\sum_{j=0}^{n} a_{n-j} y^{(n-j)}(t) = g(t), \quad y^{(0)}(t) = y(t). \tag{5.23}$$

The (FT) is used $\mathscr{F}$ to both sides:

$$\sum_{j=0}^{n} a_{n-j} \mathscr{F}\left(y^{(n-j)}(t)\right) = \mathscr{F}(g(t)). \tag{5.24}$$

With $y^{(0)}(t) = y(t)$. Making use of the (FT) to both sides of (5.2) and using its properties, we obtain:

$$\sum_{j=0}^{n} a_{n-j} \mathcal{F}\left(y^{(n-j)}(t)\right) = \mathcal{F}(g(t)). \tag{5.25}$$

Using Proposition 5.5, (5.2) this becomes:

$$\left[\sum_{j=0}^{n} a_{n-j}(2i\pi\omega)^{n-j}\right] F(\omega) = \mathscr{F}(g(t)). \tag{5.26}$$

Permitting $\mathscr{F}(y(t)) = F(\omega)$, currently, we possess:

$$F(\omega) = \left[\sum_{j=0}^{n} a_{n-j}(2i\pi\omega)^{n-j}\right]^{-1} \mathscr{F}(g(t)). \tag{5.27}$$

This can be written as:

$$F(\omega) = \mathscr{F}(h(t))\mathscr{F}(g(t)) = \mathscr{F}(h(t) * g(t)), \tag{5.28}$$

where $*$ denotes convolution. Using the inverse (FT) instead:

$$y(t) = h(t) * g(t).$$

the function f should be defined on and be integrable. $\mathbb{R}$ that takes values in $\mathbb{C}$, and let its Fourier transform be denoted as $\mathscr{F}(f) = F$. Subsequently, we present:

$$\mathscr{F}(tf(t)) = \frac{i}{2\pi}\frac{dF}{d\omega}(\omega), \quad \mathscr{F}(t^2 f(t)) = \left(\frac{i}{2\pi}\right)^2 \frac{d^2F}{d\omega^2}(\omega),$$

and in general:

$$\mathscr{F}(t^n f(t)) = \left(\frac{i}{2\pi}\right)^n \frac{d^nF}{d\omega^n}(\omega).$$

Solving a Differential Equation Using the Fourier Transform

Example 5.2

Examine the subsequent ordinary differential equation:

$$\frac{du}{dt} + au = 0, \quad u(0) = u_0.$$

Applying the Fourier Transform, we get:

$$(i\omega + a)U(\omega) = u_0.$$

Solving for $U(\omega)$, we find:

$$U(\omega) = \frac{u_0}{i\omega + a}.$$

Example 5.3

Solve the following starting point problem using the (FT):

$$u''(t) - +4u(t) = \cos(3t),$$

1.On both sides of the ODE, apply the (FT).

$$F\{u''(t) + 4u(t)\} = F\{\cos(3t)\}$$

2.Solve for the transformed variable $U(\omega)$ using the properties of the Fourier Transform.

$$(i\omega)^2 U(\omega) + 4U(\omega) = \frac{s}{s^2 + 9}$$

3.Determine using the Inverse (FT) method: $u(t)$.

$$u(t) = F^{-1}\{U(\omega)\}$$

Solving Partial Differential Equations (PDEs)

Example 5.4

Examine the one-dimensional heat equation:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad u(x,0) = f(x), \quad u(0,t) = u(L,t) = 0.$$

What we get when we use the (FT) on the x and t variables is:

$$i\omega U(\omega,t) = -k\omega^2 U(\omega,t) \quad \text{(after inverse transform).}$$

Solving for $U(\omega,t)$ and then applying the inverse Fourier Transform gives the solution.

1. Apply Fourier Transform in x and t:

$$i\omega U(\omega,t) = -k\omega^2 U(\omega,t)$$

2. Solve for $U(\omega,t)$:

$$U(\omega,t) = c(\omega)e^{-k\omega^2 t}$$

3. Use Initial Condition to Find $c(\omega)$:

$$U(\omega,0) = F\{f(x)\}$$

$$c(\omega) = \frac{1}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)\,dx$$

4. *Substitute Back to Find $u(x,t)$:*

$$u(x,t) = F^{-1}\{U(\omega,t)\}$$

$$u(x,t) = \sum_{n=1}^{\infty} c(\omega)\sin\left(\frac{n\pi x}{L}\right)e^{-k\omega^2 t}$$

Example 5.5

Determine the solution to the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}, \quad u(x,0) = \sin(x), \quad \frac{\partial u}{\partial t}(x,0) = 0 \tag{5.29}$$

where $0 < x < L$ and $t > 0$.

1. *Apply Fourier transform in x and t:*

$$i\omega U(\omega,t) = -k\omega^2 U(\omega,t)$$

$$U(\omega,t) = c(\omega)e^{-k\omega^2 t}$$

2. *Use initial condition to find $c(\omega)$:*

$$U(\omega,0) = F\{\sin(x)\}$$

$$c(\omega) = \frac{2}{\pi}\frac{1}{\omega^2 + k^2}$$

3. *Substitute back to find $u(x,t)$:*

$$u(x,t) = F^{-1}\{U(\omega,t)\}$$

$$u(x,t) = \frac{2}{\pi}\int_0^{\infty}\frac{1}{\omega^2 + k^2}\sin(x)e^{-k\omega^2 t}\,d\omega$$

Remark 5.6

By transforming temporal or geographical concerns to frequency domain issues, the Fourier transform solves differential equations. This method is widely used in science and engineering.

Chapter 6

Laplace Transformation

The **Laplace Transform** (LT) is an integral transform that plays a crucial role in various fields of science and engineering. The main function is to transform linear ordinary differential equations, which may pose challenges when addressed in their original form (the "time domain"), into straightforward algebraic problems (in the "frequency domain"). Upon resolving the algebraic problem, the application of the inverse Laplace transform facilitates the determination of the solution to the original equation. This approach offers a structured framework for examining and addressing issues associated with dynamic systems.

Definition 6.1 (*Laplace transform*)

[9] Let $f : [0, +\infty[\longrightarrow \Re$ or C and $s \in C$, where the Laplace transform of a function $f(t)$ is defined as:

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\, dt. \tag{6.1}$$

Definition 6.2

Any zero function is defined as a causative function for $t < 0$.

Definition 6.3 (*Existence of the Laplace transform*)

[9] Let $f(t)$ be a piecewise continuous function defined on the closed interval $[0, p]$, $(p > 0)$. We define that a function f exhibits an exponential α order at infinity if:

$$\exists A > 0,\ \exists \alpha > 0,\ \forall s > a,\ \ wehave\ |f(t)| = Ae^{\alpha t}. \tag{6.2}$$

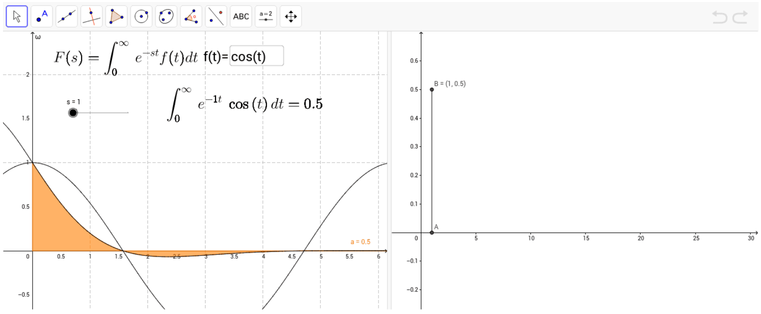


Figure 6.1: Design of the Laplace transformation

Theorem 6.1

Any function continues piecewise, checking $f \in D$ and,

$$\exists \beta,\ 0 < \beta < 1\ :\ \lim_{t \to 0} t^{\beta} |f(t)| = 0, \tag{6.3}$$

admits a Laplace transform (it exists).

Remark 6.2

The equality (6.1) is equivalent to

$$\lim_{t \to +\infty} \left| e^{-bt} f(t) \right| = 0. \tag{6.4}$$

Definition 6.4 (*Singularityof the Laplace transformation*)

Let it be $f(t)$, $g(t)$ two piecewise continuous functions having an order exponential to infinity on $[0, p]$, $(a > 0)$ if:

$$\mathcal{L}\{f(t)\} = \mathcal{L}\{g(t)\} \Rightarrow f(t) = g(t),\ \forall t \in [0, p]. \tag{6.5}$$

Definition 6.5 (*Inverse Laplace Transformation*)

$$f(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st} F(s) ds. \tag{6.6}$$

6.1 Properties of the Laplace Transform

6.1.1 Linearity

The Laplace transform is a linear operation, which indicates that it adheres to the following property:

$$\mathcal{L}\{pf(t) + qg(t)\} = p\mathcal{L}\{f(t)\} + q\mathcal{L}\{g(t)\}. \tag{6.7}$$

Example 6.1

Calculate the Laplace transform of the function $f(t) = \cos(kt)$. We possess:

$$\cos(kt) = \frac{e^{ikt} + e^{-ikt}}{2}, \text{ so}$$
$$\mathcal{L}(\cos(kt)) = \mathcal{L}(\frac{e^{ikt} + e^{-ikt}}{2})$$
$$= \frac{1}{2}\mathcal{L}(e^{ikt}) + \frac{1}{2}\mathcal{L}(e^{-ikt})$$
$$= \frac{1}{2}[\int_0^\infty e^{-st}e^{ikt}dt + \int_0^\infty e^{-st}e^{-ikt}dt]$$
$$\frac{1}{2}[\int_0^\infty e^{(-s+ik)t}dt + \int_0^\infty e^{(-s-ik)t}dt] = \frac{s}{s^2 + k^2}.$$

6.1.2 Derivative

The (LT) of a function's derivative is expressed as:

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0). \tag{6.8}$$

6.1.3 Integration

The (LT) of a function's integral is expressed as:

$$\mathcal{L}\{\int_0^t f(\tau)\, d\tau\} = \frac{1}{s}\mathcal{L}\{f(t)\}. \tag{6.9}$$

6.1.4 The Transformation's Translation

Let $f(t)$ be a function that meets the criteria outlined in Theorem 6.1; consequently, we obtain

$$\mathcal{L}\left(e^{-wt}f(t)\right) = F(p + w). \tag{6.10}$$

6.1.5 Time Delay

Let $f(t)$ be a function that meets the criteria of Theorem 6.1; thus, we obtain

$$\mathcal{L}\left(f(t-\tau)\right) = e^{-p\tau}F(p). \tag{6.11}$$

6.1.6 Convolution

The (LT) of convolution as follows:

$$\mathcal{L}\{f(t) * g(t)\} = F(s).G(s). \tag{6.12}$$

Remark 6.3

if the limits exist so:

1-$\lim\limits_{t\to 0} f(t) = \lim\limits_{s\to+\infty} sF(s)$.

2- $\lim\limits_{t\to+\infty} f(t) = \lim\limits_{s\to 0} sF(s)$.

6.2 Utilization in the Resolution of Differential Equations

This study will analyze the differential equation:

$$a_0x^{(n)}(t) + a_1x^{(n-1)}(t) + \cdots + a_nx(t) = f(t), \tag{6.13}$$

under the beginning conditions:

$$x^{(i)}(0) = x_i \quad \text{for all } i \quad 0 \leq i \leq n-1, \tag{6.14}$$

where the zeroth-order derivative is equal to the function itself ($x^{(0)}(t) = x(t)$).

Let $a_0 \neq 0$, and consider the functions $f(t)$ and $x(t)$, along with their derivatives up to order n, to be original. Applying the (LT) to both sides of Equation (6.13), and using the differentiation rule and linearity of the (LT), (6.13) becomes:

$$\left(a_0p^n + a_1p^{n-1} + \cdots + a_n\right) X(p) = F(p) + B(p), \tag{6.15}$$

where:

$$X(p) = \mathcal{L}\{x(t)\}, \quad F(p) = \mathcal{L}\{f(t)\},$$

and:

$$B(p) = x_0(a_0p^{n-1}+a_1p^{n-2}+\cdots+a_{n-1})+x_1(a_0p^{n-2}+a_1p^{n-3}+\cdots+a_{n-2})+\cdots+x_{n-1}(a_0).$$

The solution to (6.15) is given by:

$$X(p) = \frac{F(p)+B(p)}{a_0p^n + a_1p^{n-1} + \cdots + a_n}. \tag{6.16}$$

Thus, the solution to (6.13) is:

$$x(t) = \mathcal{L}^{-1}\{X(p)\}. \tag{6.17}$$

Solving Ordinary Differential Equations (ODEs)

Example 6.2

Consider the first-order ODE:

$$\frac{dy}{dt} + 3y = e^{-2t}, \quad y(0) = 1$$

Compute the (LT) of both sides:

$$\mathcal{L}\left\{\frac{dy}{dt}\right\} + 3\mathcal{L}\{y(t)\} = \mathcal{L}\{e^{-2t}\}$$

$$sY(s) - y(0) + 3Y(s) = \frac{1}{s+2}$$

Substitute $y(0) = 1$*:*

$$sY(s) - 1 + 3Y(s) = \frac{1}{s+2}$$

2. Solve for $Y(s)$*:*

$$(s+3)Y(s) = 1 + \frac{1}{s+2}$$

$$Y(s) = \frac{1}{s+3} + \frac{1}{(s+3)(s+2)}$$

3. Find $y(t)$ *by taking the inverse (LT): Break* $Y(s)$ *into partial fractions:*

$$\frac{1}{(s+3)(s+2)} = \frac{A}{s+3} + \frac{B}{s+2}$$

Solve for A and B: $A = 1, B = -1$:

$$Y(s) = \frac{1}{s+3} + \frac{1}{s+2} - \frac{1}{s+3}$$

$$Y(s) = \frac{1}{s+2}$$

Taking the inverse (LT) so the soution is:

$$y(t) = e^{-2t}$$

Example 6.3

Examine the second-order ODE:

$$y''(t) + 2y'(t) + y(t) = 0, \quad y(0) = 1, \quad y'(0) = -2$$

1. *Take the (LT) of both sides:*

$$\mathcal{L}\{y''(t)\} + 2\mathcal{L}\{y'(t)\} + \mathcal{L}\{y(t)\} = \mathcal{L}\{0\}$$

$$s^2Y(s) - sy(0) - y'(0) + 2(sY(s) - y(0)) + Y(s) = 0$$

2. *Substitute initial conditions:*

$$s^2Y(s) - s(1) - (-2) + 2(sY(s) - 1) + Y(s) = 0$$

$$s^2Y(s) - s + 2 + 2sY(s) - 2 + Y(s) = 0$$

$$(s^2 + 2s + 1)Y(s) = s - 4$$

3. *Solve for $Y(s)$:*

$$Y(s) = \frac{s-4}{(s+1)^2}$$

4. *Determine $y(t)$ through the utilization of the inverse (LT): Decompose $Y(s)$:*

$$Y(s) = \frac{s}{(s+1)^2} - \frac{4}{(s+1)^2}$$

Taking the inverse (LT) so the soution is:

$$y(t) = e^{-t}(1 - 4t)$$

Example 6.4

Let the second-order ODE:

$$y''(t) + 4y'(t) + 4y(t) = 2u(t), \quad y(0) = 0, \quad y'(0) = 1$$

1. *Submit the (LT) on both sides:*

$$\mathcal{L}\{y''(t)\} + 4\mathcal{L}\{y'(t)\} + 4\mathcal{L}\{y(t)\} = \mathcal{L}\{2u(t)\}$$

$$s^2Y(s) - sy(0) - y'(0) + 4(sY(s) - y(0)) + 4Y(s) = \frac{2}{s}$$

Substitute $y(0) = 0$ and $y'(0) = 1$:

$$s^2Y(s) - 1 + 4sY(s) + 4Y(s) = \frac{2}{s}$$

$$(s^2 + 4s + 4)Y(s) = \frac{2}{s} + 1$$

2. *Solve for $Y(s)$:*

$$Y(s) = \frac{\frac{2}{s} + 1}{(s+2)^2}$$

$$Y(s) = \frac{2s + 3}{s(s+2)^2}$$

3. *Determine $y(t)$ through the utilization of the inverse (LT): Decompose $Y(s)$:*

$$Y(s) = \frac{2}{s} + \frac{3}{(s+2)^2}$$

The answer can be obtained by using the inverse (LT) function:

$$y(t) = 2 + 3te^{-2t}$$

Remark 6.4

The Laplace Transform is an effective instrument for resolving linear differential equations and streamlining the mathematical analysis of dynamic systems. Its linearity and many attributes render it a versatile method in engineering and applied mathematics.

Fonction	transformée
δ	1
H	$\frac{1}{p}$
$t^a \ (a > -1)$	$\frac{\Gamma(a+1)}{p^{a+1}}$
$e^{-\lambda t}$	$\frac{1}{p+\lambda}$
$t^a e^{-\lambda t}$	$\frac{\Gamma(a+1)}{(p+\lambda)^{a+1}}$
$\cos(wt)$	$\frac{p}{p^2+w^2}$
$\sin(wt)$	$\frac{w}{p^2+w^2}$
$e^{-\lambda t}\sin(wt+\alpha)$	$\frac{w\cos\alpha+(p+\lambda)\sin\alpha}{(p+\lambda)^2+w^2}$
$e^{-\lambda t}\cos(wt+\alpha)$	$\frac{(p+\lambda)\cos\alpha+w\sin\alpha}{(p+\lambda)^2+w^2}$
$t^n\sin(wt)$	$n!\frac{\mathrm{Im}(p+iw)^{n+1}}{(p^2+w^2)^{n+1}}$
$t^n\cos(wt)$	$n!\frac{\mathrm{Re}(p+iw)^{n+1}}{(p^2+w^2)^{n+1}}$

Figure 6.2: Table of some transforms

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