People’s Democratic Republic of Algeria
Ministry of Higher Education and Scientific Research

MOHAMED KHIDER UNIVERSITY, BISKRA
Faculty of science and Technology
DEPARTMENT OF INDUSTRIAL CHEMISTRY

Course title :

Course of Mathematics 3 for 2 L Process Engineering

Presented By:

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Introduction

Mathematics is a body of abstract knowledge resulting from logical reasoning applied to things such as mathematical sets, numbers, and shapes... as well as the relationships and arithmetic operations that exist between these things. In addition to this definition, there is also the concept of applied mathematics. Among the most important branches of mathematics are algebra, analysis, geometry, and mathematical logic.

People use its applications without even realizing it in the details of their day, seeking organization and eliminating randomness in their studies, work, or even in sports and play areas. It has been said of mathematics:

"*Mathematics is food for the mind.*"

This course presents the most important definitions, laws, and relationships needed by second-year science and technology students in the Mathematics 3 course, based on **the curriculum set by the Ministry of Higher Education and Scientific Research**. This course consists of six main sections.

The first chapter reviews Riemann integrals and the calculation of antiderivatives, followed by double and triple integrals and applications in calculating areas and volumes. The second chapter discusses the improper integrals such as : the integrals of functions defined on a finite interval and the integrals of functions defined on an indefinite interval.

In the third chapter, studies ordinary differential equations, then study partial differential equations and learn about special functions. Chapter 4 covers the numerical series, sequences, function series, power series, and finally, Fourier series.

In Chapter 5, explores the Fourier transform, including its definitions, important properties, and applications in solving differential equations. Finally, in Chapter 6, presents another transform which is Laplace tremsform with its definition, properties, and applications in solving differential equations.

Semestre: 3
Unité d'enseignement: UEF 2.1.1
Matière 1: Mathématiques 3
VHS: 67h30 (Cours: 3h00, TD: 1h30)
Crédits: 6
Coefficient: 3

Objectifs de l'enseignement:

À la fin de ce cours, l'étudiant(e) devrait être en mesure de connaître les différents types de séries et ses conditions de convergence ainsi que les différents types de convergence.

Connaissances préalables recommandées

Mathématiques 1 et Mathématiques 2

Contenu de la matière :

Chapitre 1 : Intégrales simples et multiples **3 semaines**

1.1 Rappels sur l'intégrale de Riemann et sur le calcul de primitives. 1.2 Intégrales doubles et triples.
1.3 Application au calcul d'aires, de volumes, …

Chapitre 2 : Intégrales impropres **2 semaines**

2.1 Intégrales de fonctions définies sur un intervalle non borné. 2.2 Intégrales de fonctions définies sur un intervalle borné, infinies à l'une des extrémités.

Chapitre 3 : Equations différentielles **2 semaines**

3.1 Rappel sur les équations différentielles ordinaires. 3.2 Equations aux dérivées partielles. 3.3 Fonctions spéciales.

Chapitre 4 : Séries **3 semaines**

4.1 Séries numériques. 4.2 Suites et séries de fonctions. 4.3 Séries entières, séries de Fourrier.

Chapitre 5 : Transformation de Fourier **3 semaines**

5.1 Définition et propriétés. 5.2 Application à la résolution d'équations différentielles.

Chapitre 6 : Transformation de Laplace **2 semaines**

6.1 Définition et propriétés. 6.2 Application à la résolution d'équations différentielles.

Mode d'évaluation :

Contrôle continu : 40 % ; Examen final : 60 %.

Références bibliographiques:

1- F. Ayres Jr, Théorie et Applications du Calcul Différentiel et Intégral - 1175 exercices corrigés, McGraw-Hill.
2- F. Ayres Jr, Théorie et Applications des équations différentielles - 560 exercices corrigés, McGraw-Hill.
3- J. Lelong-Ferrand, J.M. Arnaudiès, Cours de Mathématiques - Equations différentielles, Intégrales multiples, Tome 4, Dunod Université.
4- M. Krasnov, Recueil de problèmes sur les équations différentielles ordinaires, Edition de Moscou
5- N. Piskounov, Calcul différentiel et intégral, Tome 1, Edition de Moscou
6- J. Quinet, Cours élémentaire de mathématiques supérieures 3- Calcul intégral et séries, Dunod.
7- J. Quinet, Cours élémentaire de mathématiques supérieures 4- Equations différentielles, Dunod.

Chapter 1 :

Simple and multiple integrals

Chapter 1 :

Simple and multiple integrals

1.1 Reminders on the Riemann integral and the calculation of primitives

1.1.1 Integral of Riemann

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1.5 Application to the calculation of areas, volumes

1.5.1 Calculating the area

1.5.2 Volume Calculation

1.1Reminders on the Riemann integral and the calculation of primitives

Let f be a function defined on $[a,b]$ as for all $x \in [a,b], f(x) \geq 0$.

1.1.1 Integral of Riemann

A finite set of $n+1$ reals $\{x_p \in [a,b], 0 \leq p \leq n\}$ with $x_0 = a, x_n = b$, $\forall p: x_p \leq x_{p+1}$, is called subdivision of the interval $[a,b]$ and it is noted by Δ.

Its "step" of Δ is $\delta(\Delta) = \max_{0\leq p\leq n-1}(x_{p+1} - x_p)$

Definition 1.1.1 : A subdivision is considered Δ on $[a,b]$ and a choice is made $\alpha_i \in [x_i, x_{i+1}[$ the Riemann sum relative to the subdivision Δ the choice of $\alpha = (\alpha_i)_{0\leq i\leq n-1}$ is the amount:

$$\Re(f,\Delta,\alpha) = \sum_{i=0}^{n-1} f(\alpha_i)(x_{i+1} - x_i) \qquad (1.1.1)$$

Definition 1.1.2 : The function f is integrable in the Riemann sense if $\lim_{\delta(\Delta)\to 0} \Re(f,\Delta,\alpha) = \mathrm{I}$ is finite $\forall \alpha, I = \int_a^b f(x)dx = \int_a^b f$.

Remark 1.1.1: If $\delta(\Delta) \to 0$ then $n \to +\infty$.

Example 1.1.1 : $\left\{x_i = \frac{i}{n}, i = 0, \dots, n\right\}$ is a subdivision of the interval $[0,1]$, $f(x) = x$ on the interval $[0,1]$.

We have : $x_0 = 0 = a, x_1 = \frac{1}{n}, \dots, x_n = \frac{n}{n} = 1 = b.$

$$\delta(\Delta) = (x_{i+1} - x_i) = \frac{1}{n}, \forall i \in 0, \dots, n.$$

The sum :

$$\Re(f,\Delta,\alpha) = \sum_{i=0}^{n-1} f(\alpha_i)(x_{i+1} - x_i)$$

$$= \sum_{i=0}^{n-1} \frac{i}{n}\left(\frac{i+1}{n} - \frac{i}{n}\right) = \sum_{i=0}^{n-1} \frac{i}{n^2} = \frac{n(n-1)}{2n^2} .$$

is a Riemann sum of the function $f(x) = x$ on interval $[0,1]$.

$$\lim_{\delta(\Delta)\to 0} \Re(f,\Delta,\alpha) = \lim_{n\to+\infty} \frac{n(n-1)}{2n^2} = \frac{1}{2}.$$

1.1.2 Sums of Darboux

Other sums for calculating the integral.

Definition 1.1.3 : A subdivision is considered Δ on $[a,b]$ with $I_i = [x_i, x_{i+1}[$, we choose :

$$m_i = inf_{x\in I_i}|f(x)| \text{ and } M_i = sup_{x\in I_i}|f(x)| \qquad (1.1.2)$$

The following two amounts are called the upper and lower Darboux sums respectively.

$$S(f,\Delta) = \sum_{i=0}^{n-1} M_i(x_{i+1} - x_i) \text{ and } s(f,\Delta) = \sum_{i=0}^{n-1} m_i(x_{i+1} - x_i) \qquad (1.1.3)$$

Theorem 1.1.1: If $\lim_{\delta(\Delta)\to 0} S(f,\Delta) = \lim_{\delta(\Delta)\to 0} s(f,\Delta)$ then f is integrable in Riemann sense.

Remark 1.1.2: Any continuous function over an interval $[a,b]$is an integrable function in the sense of Riemann on I.

Example 1.1.2: we take the same Example 1.1.1, then we calculate the sums.

$$m_i = inf_{x\in I_i}|f(x)| = \frac{i}{n},\ M_i = sup_{x\in I_i}|f(x)| = \frac{i+1}{n}.$$

we calculate the two sums of Darboux

$$S(f,\Delta) = \sum_{i=0}^{n-1} M_i(x_{i+1} - x_i) = \sum_{i=0}^{n-1}\left(\frac{i+1}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=0}^{n-1}\left(\frac{i+1}{n^2}\right) = \frac{n(n+1)}{2n^2}.$$

$$s(f,\Delta) = \sum_{i=0}^{n-1} m_i(x_{i+1} - x_i) = \sum_{i=0}^{n-1}\left(\frac{i}{n}\right)\left(\frac{1}{n}\right) = \sum_{i=0}^{n-1}\left(\frac{i}{n^2}\right) = \frac{n(n-1)}{2n^2}.$$

We note that $\lim_{n\to\infty} S(f,\Delta) = \lim_{n\to\infty} s(f,\Delta) = \frac{1}{2}$, therefore f is integrable in Riemann sense.

1.1.3 Properties of the Riemann integral

Let f and g be two functions, $\lambda \in \mathbb{R}, a, b, c \in \mathbb{R}$ such as $a \le b \le c$, we have:

- If the function f integrable on the intervals $[a,c]$and $[c,b]$ it is integrable on interval $[a,b]$, we have :

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx. \qquad (1.1.4)$$

- If f is integrable on the interval $[a,b]$ and $f \ge 0$ then :

$$\int_a^b f(x)dx \ge 0. \qquad (1.1.5)$$

- If the function f is integrable on the interval $[a, b]$ and $\lambda \in \mathbb{R}$ then the function λf is integrable on the interval $[a, b]$.

$$\int_a^b \lambda f(x)dx = \lambda \int_a^b f(x)dx. \qquad (1.1.8)$$

- If the functions f and g are integrable on the interval $[a, b]$ then :

$$\int_a^b (f(x) + g(x))dx = \int_a^b f(x)dx + \int_a^b g(x)dx. \qquad (1.191)$$

1.2 Calculation of primitives (Indefinite integral)

Definition 1.2.1: Let f be a continuous function on the interval I. If the derivative of a function F gives f, then F is said to be a primitive of f. we then write all primitives of F in form : $F(x) = \int f(x)dx$. The term indefinite integral of f is also used.

Example 1.2.1: Let $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = x^2$. Then $F: \mathbb{R} \to \mathbb{R}$ defined by $F(x) = \frac{x^3}{3}$ is a primitive of f, also the function defined $F(x) = \frac{x^3}{3} + 2$ is a primitive of f.

The primitive of a function is not unique (if it exists), because all primitives differ by a constant.

1.2.1 Primitives of the usual functions

The function f	The primitive of f defined by : $(k \in \mathbb{R})$	The interval I
$f(x) = c$	$F(x) = cx + k$	$\mathbb{R}$
$f(x) = x^\alpha, \alpha \in \mathbb{R}/\{-1\}$	$F(x) = \dfrac{1}{\alpha+1}x^{\alpha+1} + k$	$\begin{cases} \mathbb{R}/\{0\} \text{ if } \alpha \underset{\neq}{<} -1 \\ \mathbb{R} \text{ if } \alpha \underset{\neq}{>} -1 \end{cases}$
$f(x) = \dfrac{1}{x}$	$F(x) = \ln\lvert x\rvert + k$	$]-\infty, 0[$ or $]0, +\infty[$
$f(x) = e^x$	$F(x) = e^x + k$	$\mathbb{R}$
$f(x) = \sin x$	$F(x) = -\cos x + k$	$\mathbb{R}$
$f(x) = \cos x$	$F(x) = \sin x + k$	$\mathbb{R}$

Proposition 1.2.1: let f and g two continuous functions and $\lambda \in \mathbb{R}$, we have :

$$\int f(x) + g(x)dx = \int f(x)dx + \int g(x)dx. \qquad (1.2.1)$$

$$\int \lambda f(x)\,dx = \lambda \int f(x)dx.\ \lambda \in \mathbb{R}. \qquad (1.2.2)$$

1.2.2 Integration by parts – Change of variable

Integration by parts

Theorem 1.2.1: let u and v two derivable functions. We have :

$\int u(x)\,\acute{v}(x)dx = u(x)v(x) - \int \acute{u}(x)v(x)dx.$ (1.2.3)

Example 1.2.2: we calculate $\int 2xe^{x}dx$

We have : $u(x) = x,\ \acute{v}(x) = e^{x}$ then $\acute{u}(x) = 1, v(x) = e^{x}$

$$\int 2xe^{x}dx = 2xe^{x} - 2\int e^{x}\ dx = 2xe^{x} - 2e^{x} + c, \qquad c \in \mathbb{R}$$

Variable change

Theorem 1.2.2: let g a derivative function and f a continuous function with :

$$F(x) = \int f(x)dx. \qquad (1.2.4)$$

We have : $\int \acute{g}(x)f\big(g(x)\big)dx = \int f(u)du = F(u)$ where $g(x) = u\ and$ $du = \acute{g}(x)dx.$

Example 1.2.3: Calculating the primitive of the function $t(x) = \left(x + \frac{1}{2}\right)^{3}$

Let $g(x) = \left(x + \frac{1}{2}\right), f(x) = x^{3}$, we have $\begin{cases} g(x) = u \\ \acute{g}(x)dx = dx = du \end{cases}$

$$\int t(x)\,dx = \int (x + \tfrac{1}{2})^{3}\,dx = \int u^{3}\,du = \frac{u^{4}}{4} + c = \frac{\left(x+\frac{1}{2}\right)^{4}}{4} + c, \text{ where } c \in \mathbb{R}.$$

1.3 Double integral

In this section, let $f: D \subset \mathbb{R}$ be a continuous function defined on a domain $D \subset \mathbb{R}^{2}$.

Definition 1.3.1 : The integral of f on D, is called a double integral, it is noted :

$$\iint_D f = \iint_D f(x,y)\,dx\,dy. \qquad (1.3.1)$$

1.3.1 Fubini's theorem

1-Case where $D = [a,b] \times [c,d], a \leq b \text{ and } c \leq d$, we have :

$$\iint_D f(x,y)\,dx\,dy = \int_a^b \left[\int_c^d f(x,y)dy\right] dx = \int_c^d \left[\int_a^b f(x,y)dx\right] dy. \qquad (1.3.2)$$

If $f(x,y) = g(x)h(y)$ with $D = [a,b] \times [c,d], a \leq b \text{ and } c \leq d$, we have :

$$\iint_D g(x)h(x)\,dx\,dy = \left[\int_a^b g(x)dx\right] \times \left[\int_c^d h(y)dy\right]. \qquad (1.3.3)$$

2-Case where $D = \{(x,y) \in \mathbb{R}^2; a \leq x \leq b, u(x) \leq y \leq v(x)\}$, where $u, v: [a,b] \to \mathbb{R}$ are continuous, then:

$$\iint_D f(x,y)\,dx\,dy = \int_a^b \left[\int_{u(x)}^{v(x)} f(x,y)dy\right] dx. \qquad (1.3.4)$$

Example 1.3.1: we calculate $I = \iint_D \left(\frac{xy}{3}\right) dx\,dy$, where $D = [1,2] \times [3,5]$

$$I = \iint_D \left(\frac{xy}{3}\right) dx\,dy = \left[\int_1^2 g(x)dx\right] \times \left[\int_3^5 h(y)dy\right] = \frac{3}{6} \times \frac{16}{2} = 4.$$

Where : $g(x) = \frac{x}{3}, h(y) = y.$

Example 1.3.2: we calculate $I = \iint_D dx\,dy$, where :

$$D = \{(x,y) \in \mathbb{R}^2; 0 \leq x \leq 2, \qquad 0 \leq y \leq 2 - x\}$$

$$I = \iint_D dx\,dy = \int_0^2 \left[\int_0^{2-x} dy\right] dx = \int_0^2 (2-x)dx = \left[\left(2x - \frac{x^2}{2}\right)\right]_0^2 = 2.$$

1.3.2 Change of variable

Some time the interval or the function is complicated, so, we change the variables from (x,y) to new variable (u,v) to make an integral simpler.

Let $\mathcal{H}$ be a class application C^1 and $\Delta \subset U$(open from $\mathbb{R}^2$) and $D \subset V$ (open from $\mathbb{R}^2$).

$$\mathcal{H}: \begin{array}{ccc} U & \rightarrow & V \\ (u,v) & \rightarrow & (x,y) \end{array}$$

The Jacobian matrix of $\mathcal{H}$ is :

$$j(\mathcal{H}) = \frac{\partial(x,y)}{\partial(u,v)} = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{pmatrix}, \text{ and } \det\left(j(\mathcal{H})\right) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}.$$

If $\mathcal{H}(\Delta) = D$, we obtain :

$$\iint\limits_D f(x,y)\,dx\,dy = \iint\limits_\Delta f(\mathcal{H}(u,v))\ \det\left(j(\mathcal{H})\right)du\,dv. \qquad (1.3.5)$$

1.3.3. Polar coordinates

Let $\mathcal{H}$ be a class application C^1

$$\mathcal{H}: \begin{array}{c} \mathbb{R}^2 \rightarrow \mathbb{R}^2 \\ (r,\theta) \rightarrow (x,y) = (r\cos\theta\,, r\sin\theta) \end{array}$$

The Jacobian matrix of $\mathcal{H}$ is : $j(\mathcal{H}) = \frac{d(x,y)}{d(r,\theta)} = \begin{pmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{pmatrix}$, And $\det\left(j(\mathcal{H})\right) = \left|\frac{d(x,y)}{d(r,\theta)}\right| = r.$

If $\mathcal{H}(\Delta) = D$, we obtain :

$$\iint\limits_D f(x,y)\,dx\,dy = \iint\limits_\Delta f(r\cos\theta\,, r\sin\theta)\ |\mathrm{r}|\,dr\,d\theta. \qquad (1.3.6)$$

Example 1.3.3 : we calculate $I = \iint_D \frac{2}{1+x^2+y^2}dx\,dy$, where :

$D = \{(x,y) \in \mathbb{R}^2; x^2 + y^2 \leq 4\}$ and $\Delta = \{(r,\theta); 0 \leq r \leq 2,\ 0 \leq \theta \leq 2\pi\}$.

$$I = \iint\limits_D \frac{2}{1+x^2+y^2}dx\,dy = \iint\limits_\Delta \frac{2r}{1+r^2}dr\,d\theta = \left[\int_0^2 \frac{2r}{1+r^2}\,dr\right]\left[\int_0^{2\pi} d\theta\right]$$
$$= 2\pi \ln 5.$$

1.4 Triple integral

In this section, let $f: D \subset \mathbb{R}$ be a continuous function defined on a domain $D \subset \mathbb{R}^3$.

Definition 1.4.1: The integral of f on D, is called a triple integral, it is noted :

$$\iiint_D f = \iiint_D f(x,y,z)\,dx\,dy\,dz. \qquad (1.4.1)$$

1.4.1 Fubini's theorem

1-Case where $D = [a,b] \times [c,d] \times [s,t], a \leq b,\ c \leq d \ and\ s \leq t,$ we have :

$$\iiint f(x,y,z)dxdy\,dz = \int_s^t \left[\int_a^b \left[\int_c^d f(x,y,z)dy\right] dx\right] dz$$

$$= \int_a^b \left[\int_c^d \left[\int_s^t f(x,y,z)dz\right] dy\right] dx. \qquad (1.4.2)$$

If $f(x,y,z) = g(x)h(y)k(z)$ with $D = [a,b] \times [c,d] \times [s,t], a \leq b\ , c \leq d$ and $s \leq t$, we have :

$$\iiint\limits_D f(x,y,z)dx\,dy\,dz = \iiint\limits_D g(x)h(y)k(z)dx\,dy\,dz$$

$$= \left[\int_a^b g(x)dx\right] \times \left[\int_c^d h(y)dy\right] \times \left[\int_s^t k(z)dz\right] \qquad (1.4.3)$$

2-Case where $D = \{(x,y,z) \in \mathbb{R}^3; (x,y) \in \Delta,\ u(x,y) \leq z \leq v(x,y)\}$,

where $u, v: D \to \mathbb{R}$ are continuous and Δ is a closed bounded set of $\mathbb{R}^2$, then:

$$\iiint_D f(x,y,z)dx\,dy\,dz = \iint_\Delta \left[\int_{u(x,y)}^{v(x,y)} f(x,y,z)dy\right] dx\,dy. \qquad (1.4.4)$$

3-Case where $D = \{(x,y,z) \in \mathbb{R}^3;\ a \leq z \leq b, (x,y) \in \Delta(z)\}$, where $u, v: D \to \mathbb{R}$ are continuous and $\Delta(z)$ is a closed bounded set of $\mathbb{R}^2$, $\forall z \in [a,b]$, then:

$$\iiint_D f(x,y,z)dx\,dy\,dz = \int_a^b \left[\iint_{\Delta(z)} f(x,y,z)dx\,dy\right] dz. \qquad (1.4.5)$$

Example 1.4.1: we calculate $I = \iiint_D (xyz)dx\,dy\,dz$, where $D = [0,2]^3$.

$$I = \iiint\limits_{D} (xyz)dx\,dy\,dz = \left[\int_0^2 xdx\right] \times \left[\int_0^2 ydy\right] \times \left[\int_0^2 zdz\right] = 8.$$

1.4.2 Change of variable

Let $\mathcal{H}$ be a class application C^1 and $\Delta \subset U$(open from $\mathbb{R}^2$) and $D \subset V$ (open from $\mathbb{R}^3$).

$$\mathcal{H}: \begin{array}{ccc} U & \rightarrow & V \\ (u,v,w) & \rightarrow & (x,y,z) \end{array}$$

The Jacobian matrix of $\mathcal{H}$ is :

$$j(\mathcal{H}) = \begin{pmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{pmatrix}, \text{ and } \det(j(\mathcal{H})) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix}.$$

If $\mathcal{H}(\Delta) = D$, we obtain :

$$\iiint\limits_{D} f(x,y,z)\,dx\,dy\,dz = \iiint\limits_{\Delta} f(\mathcal{H}(u,v,w))\ \det(j(\mathcal{H}))du\,dv\,dw. \qquad (1.4.6)$$

1.4.3. Cylindrical coordinates

Let $\mathcal{H}$ be a class application C^1.

$$\mathcal{H}: \begin{array}{c} \mathbb{R}^3 \rightarrow \mathbb{R}^3 \\ (r,\theta,z) \rightarrow (x,y,z) = (r\cos\theta, r\sin\theta, z) \end{array}$$

The Jacobian matrix of $\mathcal{H}$ is :

$$j(\mathcal{H}) = \frac{d(x,y,z)}{d(r,\theta,z)} = \begin{pmatrix} \cos\theta & -r\ \sin\theta & 0 \\ \sin\theta & r\ \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \text{ and } \det(j(\mathcal{H})) = \left|\frac{d(x,y,z)}{d(r,\theta,z)}\right| = r.$$

If $\mathcal{H}(\Delta) = D$, we obtain :

$$\iiint\limits_{D} f(x,y,z)\,dx\,dy\,dz = \iiint\limits_{\Delta} f(r\cos\theta, r\sin\theta, z)\ |\mathrm{r}|\ dr\,d\theta.dz \qquad (1.4.7)$$

Example 1.4.2: we calculate $I = \iiint_D \, dx\, dydz$, where :

$$D = \{(x,y,z) \in \mathbb{R}^3; x^2 + y^2 \leq 4, 0 \leq z \leq 1\}$$

$$\Delta = \{(r,\theta,z); 0 \leq r \leq 2,\ 0 \leq \theta \leq 2\pi \ and\ 0 \leq z \leq 1\}.$$

$$I = \iiint_D \, dx\, dydz = \iiint_\Delta \, r\, dr\, d\theta\, dz = \left[\int_0^2 r\, dr\right]\left[\int_0^{2\pi} d\theta\right]\left[\int_0^1 dz\right] = 4\pi.$$

1.4.4. Spherical coordinates

Let $\mathcal{H}$ be a class application C^1.

$$\mathcal{H}: \begin{matrix} \mathbb{R}^3 \to \mathbb{R}^3 \\ (r,\theta,\beta) \to (x,y,z) = (r\cos\theta\cos\beta, r\sin\theta\cos\beta, r\sin\beta) \end{matrix}$$

The Jacobian matrix of $\mathcal{H}$ is :

$$j(\mathcal{H}) = \frac{d(x,y,z)}{d(r,\theta,\beta)} = \begin{pmatrix} \cos\theta\cos\beta & -r\ \sin\theta\cos\beta & -\mathrm{r}\cos\theta\sin\beta \\ \sin\theta\cos\beta & r\ \cos\theta\cos\beta & -r\ \sin\theta\sin\beta \\ \sin\beta & 0 & r\cos\beta \end{pmatrix}$$

$$\det(j(\mathcal{H})) = \left|\frac{d(x,y,z)}{d(r,\theta,\beta)}\right| = r^2\cos\beta.$$

If $\mathcal{H}(\Delta) = D$, we obtain :

$$\iiint_D f(x,y,z)\, dxdydz$$

$$= \iiint_\Delta f(r\cos\theta\cos\beta, r\sin\theta\cos\beta, r\sin\beta)\, r^2|\cos\beta| drd\theta d\beta \qquad (1.4.8)$$

Example 1.4.3: we calculate $I = \iiint_D \, dx\, dydz$, where :

$$D = \{(x,y,z) \in \mathbb{R}^3; x^2 + y^2 + z^2 \leq 4\}$$

$$\Delta = \left\{(r,\theta,\beta); 0 \leq r \leq 2, 0 \leq \theta \leq 2\pi \ and\ -\frac{\pi}{2} \leq \beta \leq \frac{\pi}{2}\right\}.$$

$$I = \iiint_D dx\, dydz = \iiint_\Delta r^2|\cos\beta|\, dr\, d\theta\, d\beta = \left[\int_0^2 r^2\, dr\right]\left[\int_0^{2\pi} d\theta\right]\left[\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |\cos\beta|\, d\beta\right] = \frac{32}{3}\pi$$

1.5 Application to the calculation of areas, volumes

1.5.1 Calculating the area

Definition 1.5.1 : The integral $\iint_D \, dx\,dy$ gives us the area of the domain D (a subset of $\mathbb{R}^2$).

Example 1.5.1: we calculate the area delimited by $y + 2x^2 = 3$ and $y = 0$.

$$D = \left\{(x, y) \in \mathbb{R}^2; 0 \leq y \leq 3 - 2x^2 \ and \frac{-3}{2} \leq x \leq \frac{3}{2}\right\}$$

$$\iint_D dx\,dy = \int_{\frac{-3}{2}}^{\frac{3}{2}} \left[\int_0^{3-2x^2} dy\right] dx = \int_{\frac{-3}{2}}^{\frac{3}{2}} [3 - 2x^2\,]dx = \left[3x - \frac{2x^3}{3}\right]_{\frac{-3}{2}}^{\frac{3}{2}} = \frac{9}{2}$$

1.5.2 Volume Calculation

Definition 1.5.2 : The integral $\iiint_D \, dx\,dy\,dz$ gives us the volume of the domain D(a subset of $\mathbb{R}^3$).

Example 1.5.2: Let's calculate the volume of the cylinder of radius $a = 1$ and height $h = 5$

$\iiint_D \, dx\,dy\,\,dz$ where $D = \{(x, y, z) \in \mathbb{R}^3/x^2 + y^2 \leq 1, 0 \leq z \leq 5\}$ and $\Delta = \{(r, \theta, z)/0 \leq r \leq 1 \ and\ 0 \leq \theta \leq 2\pi,\ 0 \leq z \leq 5\}$.

$$\iiint_D dx\,dydz = \iiint_\Delta r\,dr\,d\theta\,d\beta = \left[\int_0^1 r\,dr\right]\left[\int_0^{2\pi} d\theta\right]\left[\int_0^5 \mathrm{dz}\right] = 10\pi.$$

Chapter 2 :

Improper integrals

Chapter 2 :

Improper integrals

2.1 Integrals of functions defined on an unbounded interval

2.1.1 Improper Integral of Positive Functions

2.2 Integrals of functions defined on a bounded interval, infinite at one endpoint

2.2.1 Absolutely Convergent Integral

2.1 Integrals of functions defined on an unbounded interval

An improper integral is a generalization of the integral defined for functions that are not bounded on the interval of integration.

Definition 2.1.1 : Let $f(x)$ be a continuous function on an unbounded interval $[a,+\infty[$. The improper integral of $f(x)$ on the same interval is defined as follows:

$$\int_a^{+\infty} f(x)dx = \lim_{b\to+\infty}\int_a^b f(x)dx. \qquad (2.1.1)$$

Example 2.1.1 : Consider the function $f(x)=\frac{2}{x^3}$ on the interval $[1,+\infty[$. The improper integral of $f(x)$ on this interval is:

$$\int_1^{+\infty}\frac{2}{x^3}dx = \lim_{b\to+\infty}\int_1^b \frac{2}{x^3}dx = \lim_{b\to+\infty}\left[\frac{-1}{x^2}\right]_1^b = \lim_{b\to+\infty}\left[\frac{-1}{b^2}+\frac{1}{1^2}\right] = 1.$$

Therefore, the improper integral $\int_1^{+\infty}\frac{2}{x^3}dx$ converges.

2.1.1 Improper Integral of Positive Functions

Definition 2.1.2 : If f is a positive function on the interval $[a,+\infty[$, then the improper integral converges if and only if the limit exists:

$$\lim_{b\to+\infty}\int_a^b f(x)dx. \qquad (2.1.2)$$

In other words, if f is positive, then the integral $\int_a^b f(t)dt$ converges at b if and only if the function $F(x)=\int_a^b f(t)dt$ is bounded on $[a,b[$.

Convergence criterion :

If we cannot (or do not want to) calculate an antiderivative of f, we study its convergence by comparing it with integrals whose convergence is known.

Comparaison criterion

Let f and g be two positive and continuous functions on $[a,+\infty[$, Suppose that f is bounded above by g in a neighborhood of $+\infty$:

$$\exists A\geq a, \forall x>A;\ f(x)\leq g(x). \qquad (2.1.3)$$

1. If $\int_a^{+\infty} g(x)dx$ converges, then $\int_a^{+\infty} f(x)dx$ converges.

2. If $\int_a^{+\infty} f(x)dx$ diverges, then $\int_a^{+\infty} g(x)dx$ diverges.

Example 2.1.2 :

Let's show that the integral $\int_1^{+\infty} x^\alpha e^{-x}dx$ converges, for any real number α.

We know that $\lim_{x\to+\infty} x^{\alpha}e^{-x/2}$, for all α, because the exponential function dominates powers of x, and we have $x^{\alpha}e^{-x} = (x^{\alpha})\left(e^{-x/2}e^{-x/2}\right)$.
In particular, there exists a real number $A > 0$ such that:

$$\forall x > A, \qquad x^{\alpha}e^{-x/2} \leq 1.$$

Then : $f(x) = x^{\alpha}e^{-x} \leq g(x) = e^{-x/2}$, now, the integral $\int_1^{+\infty} e^{-x/2}dx$ converges which equal to $\int_1^{y} e^{-x/2}dx = 2e^{-1/2} - 2e^{-y/2}$.
Its limit equal to $\lim_{y\to+\infty} \int_1^{y} e^{-x/2}dx = 2e^{-1/2}$.
We can therefore apply comparison, $\int_1^{+\infty} e^{-x/2}dx$ converges, we deduce that $\int_1^{+\infty} x^{\alpha}e^{-x}dx$ also converges.

Equivalence criterion
Let f and g be two positive and continuous functions on $[a, +\infty[$, Suppose that they are equivalent in the neighborhood of $+\infty$ that is:

$$\lim_{x\to+\infty}\frac{f(x)}{g(x)} = 1 \qquad (2.1.4)$$

Then the integral $\int_a^{+\infty} f(x)dx$ converges if and only if $\int_a^{+\infty} g(x)dx$ converges.
Example 2.1.3 :
Does the integral $\int_1^{+\infty} \frac{x^4-x+2}{x^2-1}e^{-x}dx$ converge?
Since $f(x) = \frac{x^4-x+2}{x^2-1}e^{-x} \underset{\infty}{\sim} g(x) = x^2\, e^{-x}$, and from Example 2.1.2 $\int_1^{+\infty} x^2e^{-x}dx$ converges, then our integral $\int_1^{+\infty} \frac{x^4-x+2}{x^2-1}e^{-x}dx$ converges.
Example 2.1.4 :
Does the integral $\int_1^{+\infty} \frac{1}{x\sqrt{1+x^2}}dx$ converge?
Since $f(x) = \frac{1}{x\sqrt{1+x^2}} \underset{\infty}{\sim} g(x) = \frac{1}{x^2}$, and $\int_1^{+\infty} \frac{1}{x^2}dx$ converges, then our integral $\int_1^{+\infty} \frac{1}{x\sqrt{1+x^2}}dx$ converges.

2.2 Integrals of functions defined on a bounded interval, infinite at one endpoint

Definition 2.2.1 : Let f be a function defined on a bounded interval $[a, b]$ with $-\infty < a < c < b < \infty$. The improper integral of f over this interval is defined :

$$\int_a^b f(x)dx = \lim_{\varepsilon\to 0}\left(\int_a^{c-\varepsilon} f(x)dx + \int_{c+\varepsilon}^{a} f(x)dx\right) < \infty . \qquad (2.2.1)$$

is convergent.

Example 2.2.1 : The improper integral: $\int_{-1}^{1} \frac{1}{1-x^2} dx$ diverges because :

$$\int_{-1}^{1} \frac{1}{1-x^2} dx = \lim_{\varepsilon \to 0} \left(\int_{-1}^{0} \frac{1}{1-x^2} dx + \int_{0}^{1} \frac{1}{1-x^2} dx \right),$$

and $\int_{-1}^{0} \frac{1}{1-x^2} dx$ diverges.

2.2.1 Absolutely Convergent Integral

Definition 2.2.2 : Let f be a continuous function on $[a, +\infty[$. We say that $\int_{a}^{+\infty} f(x) dx$ is absolutely convergent if $\int_{a}^{+\infty} |f(x)| dx$converges**.**

Example 2.2.2 : The integral

$$\int_{1}^{+\infty} \frac{cosx}{x^2} dx$$

is absolutely convergent, therefore convergent. Indeed, for all x, because :

$$\left| \frac{cosx}{x^2} \right| \leq \frac{1}{x^2}$$

Now, the Riemann integral $\int_{1}^{+\infty} \frac{1}{x^2} dx$ is convergent, by the comparaison criterion.

Chapter 3 :

Differential Equations

Chapter 3 :

Differential Equations

3.1 Review of ordinary differential equations

3.1.1 First-Order Differential Equations

3.1.1.1 Separable equations
3.1.1.2 First-order homogeneous equations
3.1.1.3 First-order linear differential equations
3.1.1.4 Bernoulli equation

3.1.2 Second-Order Linear Differential Equations

3.2 Partial differential equations

Partial differential equations

3.3 Special functions

3.3.1 Eulerian functions

3.3.2 Hypergeometric function

3.1 Review of ordinary differential equations

Definition 3.1.1: A differential equation is an equation whose unknown is a function and where derivatives of this function are involved. A differential equation of order n is any relation F between x, y and the derivatives of y given by :

$$F\left(x, y, y', \dots, y^{(n)}\right) = g(x). \qquad (3.1.1)$$

A solution to the differential equation is any function $g(x)$ verifying

$$F\left(x, y, y', \dots, y^{(n)}\right) = g(x).$$

Definition 3.1.2: A linear differential equation of order n is an equation of the form :

$$a_0(x)y + a_1(x)y' + \dots + a_n(x)y^{(n)} = g(x) \qquad (3.1.2)$$

where the $a_0, a_1, \dots, a_n$ and g are continuous real functions on an interval $I \subset \mathbb{R}$,the homogeneous equation or equation without a second (if the function g is zero) associated with a linear differential equation of order n is is

$$a_0(x)y + a_1(x)y' + \dots + a_n(x)y^{(n)} = 0 \qquad (3.1.3)$$

Definition 3.1.3: A linear differential equation has constant coefficients if

$$a_0 y + a_1 y' + \dots + a_n y^{(n)} = g(x) \qquad (3.1.4)$$

where the $a_0, a_1, \dots, a_n$ are constants, and g is a continuous real function.

Proposition 3.1.1 : If y_1 and y_2 are two solutions to thehomogeneous linear differential equation: $a_0(x)y + a_1(x)y' + \dots + a_n(x)y^{(n)} = 0$, then $y_1 + y_2$ and λy_1 are also two solutions,

Example 3.1.1:

Consider $y'' - y = 0$. Two independent solutions are $(y_1 = e^x)$ and $(y_1 = e^{-x})$ because $y''_1 - y_1 = e^x - e^x = 0, y''_2 - y_2 = e^{-x} - e^{-x} = 0$.

Then their sum $(y_1 + y_2 = e^x + e^{-x})$ satisfies

$$(e^x + e^{-x})'' - (e^x + e^{-x}) = (e^x + e^{-x}) - (e^x + e^{-x}) = 0$$

and any scalar multiple (λy_1) satisfies the equation as well:

$$(\lambda e^x)'' - (\lambda e^x) = \lambda e^x - \lambda e^x = 0.$$

3.1.1 First-Order Differential Equations

Definition 3.1.4: A first-order differential equation (ordinary) is a differential equation such that the unknown function depends on a single variable. It is an equation of the form

$$y' = f(x, y), f\colon \mathbb{R}^2 \to \mathbb{R} \qquad (3.1.5)$$

is a continuous function. The differentiable function, $y_0\colon I \subset \mathbb{R} \to \mathbb{R}$, is said to be a solution to the first-order differential equation on I if

$$y_0'(x) = f(x, y_0(x)), \forall x \in I.$$

Example 3.1.2: The function $y\colon \mathbb{R} \to \mathbb{R}, y(x) = \sin x$ is a solution of the following differential equation of order 1: $y' = \cos x$ on R, because

$$y'(x) = (\sin x)' = \cos x, \ \forall x \in \mathbb{R}.$$

3.1.1.1 Separable equations

Definition 3.1.5: A first-order differential equation $y' = f(x,y)$ is said to be of separated variables if it can be written in the form $f(x,y) = f_1(x)f_2(y)$ where f_1 and f_2 are continuous functions on $\mathbb{R}$.

Resolution method :

$$y' = f(x,y)$$

We have $y' = \frac{dy}{dx}$ and $f(x,y) = f_1(x)f_2(y)$

So $y' = f(x,y)$ becomes $\frac{dy}{dx} = f_1(x)f_2(y)$ and $\frac{dy}{f_2(y)} = f_1(x)\,dx,\ f_2(y) \neq 0.$

By integrating, we obtain $\int \frac{1}{f_2(y)}\,dy = \int f_1(x)\,dx + c.$

Example 3.1.3: Give the solution to the following differential equation

$$y' = e^{x-y}$$

We have $y' = e^{x-y} = \frac{e^x}{e^y}$ becomes $\frac{dy}{dx} = \frac{e^x}{e^y}$ and $e^y dy = e^x\,dx.$

By integrating, we obtain $\int e^y dy = \int e^x\,dx$ and $e^y = e^x + c.$

Finaly, $y = \ln|e^x + c|.$

3.1.1.2 First-order linear differential equations

Definition 3.1.6 :A first-order equation is any equation written in the form

$$y' + a(x)y = g(x) \qquad (3.1.6)$$

with a, g being continuous functions on $\mathbb{R}$. The associated homogeneous differential equation is $y' + a(x)y = 0$

Resolution method :

We start by solving the equation without the right-hand side, then we assume that the constant of integration k is a function that depends on the variable y, we substitute the solution thus found into the given equation, we will arrive at the desired general solution. The general solution of equation $y' + a(x)y = g(x)$ is given by $y_G = y_H + y_P$ with y_H being the solution of the homogeneous equation and y_P being the particular solution.

- Solving a homogeneous linear differential equation

Equation $y' + a(x)y = 0$

Rewrite as $y' = -a(x)y$ then $\frac{dy}{dx} = -a(x)y$ and $\frac{dy}{y} = -a(x)dx$

Integrate: $\ln|y| = \int -a(x)dx + c$

Exponentiate: $y(x) = e^{-\int a(x)dx+c}$, $c \in \mathbb{R}$ then :

$$y_H(x) = ke^{-\int a(x)dx}, k \pm e^c \in \mathbb{R}$$

- Solving a non-homogeneous linear differential equation

$$y' + a(x)y = g(x)$$

The variation of the constant k, where y_P is a particular solution of this equation

$$y_P(x) = k(x)e^{-\int a(x)dx}$$

So, $y_G = y_H + y_P = ke^{-\int a(x)dx} + k(x)e^{-\int a(x)dx}$.

Example 3.1.4: Solve the differential equation: $xy' - y = x^2 \cos x$

- Solving a homogeneous linear differential equation

$$xy' - y = 0$$

Rewrite as $y' = \frac{y}{x}$ then $\frac{dy}{dx} = \frac{y}{x}$ and $\frac{dy}{y} = \frac{dx}{x}$

Integrate: $\int \frac{dy}{y} = \int \frac{dx}{x}$ then $\ln|y| = \ln|x| + c$

Exponentiate: $y(x) = kx$, $k \in \mathbb{R}$ then :

$$y_H(x) = kx, \qquad k \in \mathbb{R}$$

- Solving a non-homogeneous linear differential equation

$$xy' - y = x^2 \cos x$$

The particular solution $y_P(x) = k(x)x$ and $y'_P(x) = k'(x)x + k(x)$

$$x(k'(x)x + k(x)) - (k(x)x) = x^2 \cos x$$

$$x^2k'(x) + xk(x) - xk(x) = x^2 \cos x$$

$$x^2k'(x) = x^2 \cos x$$

$$k'(x) = \cos x$$

Integrate :

$$\int k'(x)\,dx = \int \cos x\,dx$$

Then $k(x) = \sin x + c$, to get only a particular solution, take $c = 0$.

So, $y_P(x) = x\ \sin x$ and $y_G(x) = kx + x\ \sin x = x(k + \sin x)$.

3.1.1.3 Bernoulli equation

Definition 3.1.7: A Bernoulli equation is an equation of the form

$$y' + a(x)y = b(x)y^n \qquad (3.1.7)$$

where $a(x)$ and $b(x)$ are continuous functions and $n \neq \{0,1\}$.

Resolution method :

We consider the intervals where y does not vanish. We can then divide by y^n and obtain:

$$y'y^{-n} + a(x)y^{1-n} = b(x),$$

then we set $t = y^{1-n}$ then $t' = (1-n)y'y^{-n}$, obtaining a first-order differential equation at t in the form:

$$\frac{1}{1-n}t' + a(x)t = b(x) \text{ then } t' + (1-n)a(x)t = (1-n)b(x)$$

is a first-order linear equation.

Example 3.1.5: Solve the differential equation: $y' - \frac{2}{x}y = x^3y^2$

This is a Bernoulli differential equation with $n = 2, a(x) = -\frac{2}{x}$ and $b(x) = x^3$

Divide by y^2, so $y'y^{-2} - \frac{2}{x}yy^{-2} = x^3$ then $y'y^{-2} - \frac{2}{x}y^{-1} = x^3$

Bernoulli substitution $t = y^{-1}$ and $t' = -y'y^{-2}$

Substitute into the original equation $-t' - \frac{2}{x}t = x^3$

Multiply by (-1), so, $t' + \frac{2}{x}t = -x^3$, this is now a linear first-order ODE in t.

3.1.2 Second-Order Linear Differential Equations

Let the second-order differential equation be given by

$$y'' + a(x)y' + b(x)y = g(x) \qquad (3.1.8)$$

where a, b, g are continuous functions on $I \subset \mathbb{R}$.

Definition 3.1.8 : Any equation of the form

$$y'' + ay' + by = g(x) \qquad (3.1.9)$$

is called a second-order differential equation with constant coefficients, with $a, b \in \mathbb{R}$ being constants.

The associated homogeneous equation is $y'' + ay' + by = 0$

Definition 3.1.9 : We will look for the particular solutions of the equation $y'' + ay' + by = 0$ in the form $y = e^{rx}$ where r is a constant, substitute the latter in our equation, we obtain $r^2 e^{rx} + are^{rx} + be^{rx} = 0$, then :

$$e^{rx}(r^2 + ar + b) = 0.$$

Since $e^{rx} \neq 0$ will be valid if and only if $(r^2 + ar + b) = 0$, We call equation

$$r^2 + ar + b = 0 \qquad (3.1.10)$$

the characteristic polynomial associated with this equation.

Discussion of the Solutions to the Equation

If y_H is a general solution of homogeneous equation and y_P is a particular solution, then $y_G = y_H + y_P$ is a general solution of l'equation

- Solving the associated homogeneous equation

Theorem 3.1.1: Let y_1, y_2 be two particular solutions of the homogeneous linearly independent equation, then its general solution is given by :

$$y_H = c_1 y_1 + c_2 y_2,\ c_1, c_2 \in \mathbb{R}. \qquad (3.1.11)$$

First, we will calculate the discriminant of equation $\Delta = a^2 - 4b$, the discussion will be based on its values; three cases can arise.

- If $\Delta > 0$, then equation has two distinct roots:

$$r_1 = \frac{-a-\sqrt{\Delta}}{2} \text{ and } r_2 = \frac{-a+\sqrt{\Delta}}{2}.$$

The general solution is given by :

$$y_H = c_1 e^{r_1 x} + c_2 e^{r_2 x},\ c_1, c_2 \in \mathbb{R} \qquad (3.1.12)$$

- If $\Delta = 0$, then equation has a double root :

$$r = \frac{-a}{2}$$

So the solution is given by :

$$y_H = (c_1 + c_2 x)e^{rx},\ c_1, c_2 \in \mathbb{R} \qquad (3.1.13)$$

- If $\Delta < 0$, then the equation has two complex conjugate roots :

$$r_1 = \alpha + i\beta \text{ and } r_2 = \alpha - i\beta, (\alpha, \beta \in \mathbb{R}, \beta \neq 0).$$

and the solution is given by :

$$y_H = (c_1 \cos(\beta x) + c_2 \sin(\beta x))e^{\alpha x}, c_1, c_2 \in \mathbb{R}. \qquad (3.1.14)$$

- Solving the non-homogeneous equation

$$y'' + ay' + by = g(x)$$

To find a particular solution (y_P), we look at the form of the function ($g(x)$):

Case 1: ($g(x) = P(x)$), where $P(x)$ is a polynomial. We try a particular solution of the form:

$$y_P = x^k Q(x) \qquad (3.1.15)$$

where $Q(x)$ is a polynomial with the same degree as $P(x)$. The exponent k depends on the characteristic equation :

- If 0 is not a root, then $k = 0$.
- If 0 is a simple root, then $k = 1$.
- If 0 is a double root, then $k = 2$.

Case 2: ($g(x) = e^{\alpha x}P(x)$), where $\alpha \in \mathbb{R}$ and $P(x)$ is a polynomial. We try a particular solution of the form:

$$y_P = x^k e^{\alpha x} Q(x) \qquad (3.1.16)$$

where $Q(x)$ is a polynomial with the same degree as $P(x)$. The exponent k is:

- If α is not a root, then $k = 0$.
- If α is a simple root, then $k = 1$.
- If α is a double root, then $k = 2$.

Example 3.1.6: Let $y'' - 4y' + 4y = (x^2 + 1)e^x$.

The associated homogeneous equation is : $y'' - 4y' + 4y = 0$

The characteristic polynomial associated with this equation is:

$$r^2 - 4r + 4 = 0$$

We have $\Delta = 0$, then equation has a double root : $r = 2$

So the solution is given by :

$$y_H = (c_1 + c_2 x)e^{2x}, \; c_1, c_2 \in \mathbb{R}.$$

To find a particular solution (y_P), we look at the form of the function $g(x)$, where $g(x) = (x^2+1)e^x$ (It is the Case 2: ($g(x) = e^{\alpha x}P(x)$) where $\alpha = 1$ and $P(x)(x^2+1)$) .

We try a particular solution of the form:

$$y_P = x^k e^{\alpha x} Q(x)$$

where $Q(x)$ is a polynomial with the same degree as $P(x)$. The exponent $k = 0$ because $\alpha = 1$ is not a root, then $k = 0$. So

$$y_P = e^x(ax^2+bx+c)$$

$$y_P{}' = e^x(2ax+b) + e^x(ax^2+bx+c)$$

$$y_P'' = 2ae^x + 2e^x(2ax+b) + e^x(ax^2+bx+c)$$

By substituting them into the equation $y'' - 4y' + 4y = (x^2+1)e^x$

$$\begin{aligned}&[2ae^x + 2e^x(2ax+b) + e^x(ax^2+bx+c)]\\ &\quad - 4[e^x(2ax+b) + e^x(ax^2+bx+c)] + 4e^x(ax^2+bx+c)\\ &\quad = (x^2+1)e^x\end{aligned}$$

We obtain :

$$((a-1)x^2 + (b-4a)x + c + b + 2a - 1)e^x = 0$$

$$\begin{cases} a-1=0 \\ b-4a=0 \\ c+b+2a-1=0 \end{cases} \quad \text{then} \quad \begin{cases} a=1 \\ b=4 \\ c=7 \end{cases}$$

So particular solution is

$$y_P = e^x(x^2+4x+7)$$

The general solution is

$y_G = (c_1 + c_2 x)e^{2x} + e^x(x^2+4x+7)$, $c_1, c_2 \in \mathbb{R}$.

3.2 Partial differential equations

A partial differential equation (PDE) is a differential equation whose solutions are unknown functions of several variables that satisfy certain conditions regarding their partial derivatives.

Partial differential equations

Definition 3.2.1: Let $f: \mathbb{R}^2 \to \mathbb{R}$ be a function of two real variables x, y, defined in the neighborhood of a point $A(a, b)$.

If the function $x \to f(x, b)$ has a derivative with respect to (x), we denote it by $f'_x(a, b)$.

The partial derivative with respect to x is defined by:

$$f'_x = \frac{\partial f}{\partial x} \text{ and } f'_y = \frac{\partial f}{\partial y} \qquad (3.2.1)$$

Successive Partial Derivatives : Similarly, we can define higher-order derivatives:

$$f''_{xx} = \frac{\partial^2 f}{\partial x^2}, f''_{xy} = \frac{\partial^2 f}{\partial y \partial x},$$

$$f''_{yy} = \frac{\partial^2 f}{\partial y^2} \text{ and } f''_{yx} = \frac{\partial^2 f}{\partial x \partial y}. \qquad (3.2.2)$$

Schwarz's Theorem : If the partial derivatives $f''_{xy} = \frac{\partial^2 f}{\partial y \partial x}$ and $f''_{yx} = \frac{\partial^2 f}{\partial x \partial y}$ are continuous, then they are equal:

$$f''_{xy} = f''_{yx} \qquad (3.2.3)$$

Example 3.3.1 : For the function $f(x) = 3x^2y + y^2 + y$:

$$f'_x = \frac{\partial f}{\partial x} = 6xy$$

$$f'_y = \frac{\partial f}{\partial y} = 3x^2 + 2y + 1$$

$$f''_{xx} = \frac{\partial^2 f}{\partial x^2} = 6y$$

$$f''_{xy} = \frac{\partial^2 f}{\partial y \partial x} = 6x$$

$$f''_{yy} = \frac{\partial^2 f}{\partial y^2} = 2$$

$$f''_{yx} = \frac{\partial^2 f}{\partial x \partial y} = 6x$$

3.3 Special functions

3.3.1 Eulerian functions

Definition 3.3.1 : The Eulerian integral of the first kind, or beta function, is defined as follows:

$$\beta(x,y)=\int_0^1 t^{x-1}\,(1-t)^{y-1}dt. \qquad (3.3.1)$$

Definition 3.3.2 : The Eulerian integral of the second kind, or gamma function, is defined as follows:

$$\Gamma(x)=\int_0^{+\infty} t^{x-1}\,e^{-t}dt. \qquad (3.3.2)$$

3.3.2 Hypergeometric function

Hypergeometric functions were introduced by Gauss when he had to solve the following differential equation:

$$x(1-x)y''+(c-(1+a+b)x)y'-aby=0 \qquad (3.3.3)$$

where a, b, and c are real or complex numbers other than negative integers.

Definition 3.3.3 : The hypergeometric function is defined by the series :

$${}_2F_1(a,b,c,z)=\sum_{k=0}^{+\infty}\frac{\Gamma(a+k+1)\Gamma(b+k+1)\Gamma(c+1)}{\Gamma(1+a)\Gamma(1+b)\Gamma(1+c+k)}\frac{z^k}{k!}. \qquad (3.3.4)$$

Chapter 4 :

Series

Chapter 4 :

Series

4.1 Numerical Series

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4.3.1 Power Series
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4.1 Numerical Series

Definition 4.1.1: A numerical series is the infinite sum of the terms of a sequence of real numbers. It is written in the form:

$$\sum_{k=0}^{+\infty} u_n = u_0 + u_1 + \cdots + u_n + u_{n+1} + \cdots \qquad (4.1.1)$$

where u_k is the sequence of terms in the series.

Definition 4.1.2 : Let $(u_n)_{n\in\mathbb{N}}$ be a numerical sequence. We want to give meaning to the infinite sum:

$$\sum_{n=0}^{+\infty} u_n \qquad (4.1.2)$$

We define:

- u_n is called the general term of the series.
- $S_n = \sum_{k=0}^{n} u_k = u_0 + u_1 + \cdots + u_n$ is called the partial sum of order n.
- If the limit of partial sums exists and is finite:

$$\mathrm{S} = \lim_{n\to+\infty} S_n \qquad (4.1.3)$$

then the series converges and S is called the sum of the series:

$$\mathrm{S} = \sum_{n=0}^{+\infty} u_n \qquad (4.1.4)$$

If this limit does not exist or is infinite, then the series diverges.

- $(S_n)_n$ is called the sequence of partial sums of order n.

Nature of the series :

Series $\sum_{n=0}^{+\infty} u_n$ is said to be convergent if and only if $(S_n)_n$ is convergent, then

- If $(S_n)_n$ converges then $\sum_{n=0}^{\infty} u_n$ converges.
- If $(S_n)_n$ diverges then $\sum_{n=0}^{\infty} u_n$ diverges.

Example 4.1.1: Calculate the sum of a numerical series $\sum_{n=1}^{+\infty} \frac{1}{n(n+1)}$.

Let $u_n = \frac{1}{n(n+1)}, n \geq 1$, so we obtain $u_n = \frac{1}{n(n+1)} = \frac{1}{n} - \frac{1}{(n+1)}$.

The partial sum :

$$S_n = \sum_{k=1}^{n} u_k = u_1 + u_2 + \cdots + u_n = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{(n+1)}\right)$$
$$= 1 - \frac{1}{(n+1)}.$$

The sum :

$$\mathrm{S} = \lim_{n\to+\infty} S_n = \sum_{k=1}^{+\infty} u_k = \lim_{n\to+\infty} 1 - \frac{1}{(n+1)} = 1 - 0 = 1$$

$(S_n)_n$ converges then $\sum_{n=0}^{\infty} u_n$ converges.

Properties of Series :

Let $\sum_{n=0}^{+\infty} u_n$ and $\sum_{n=0}^{+\infty} v_n$ be two numerical series and $\lambda \in \mathbb{R}$, we have

- If $\sum_{n=0}^{+\infty} u_n$ converges then $\sum_{n=0}^{\infty}(\lambda u_n)$ converges.
- If $\sum_{n=0}^{+\infty} u_n$ and $\sum_{n=0}^{+\infty} v_n$ converge then $\sum_{n=0}^{\infty}(u_n \pm v_n)$ converges.
- If $\sum_{n=0}^{+\infty} u_n$ diverges and $\sum_{n=0}^{+\infty} v_n$ converges then $\sum_{n=0}^{\infty}(u_n \pm v_n)$ diverges.
- If $\sum_{n=0}^{+\infty} u_n$ and $\sum_{n=0}^{+\infty} v_n$ diverge then we can't conclude anything for the series $\sum_{n=0}^{\infty}(u_n \pm v_n)$.

Necessary Condition for Series Convergence :

Theorem 4.1.1: If $\sum_{n=0}^{+\infty} u_n$ is convergent then $\lim_{n\to+\infty} u_n = 0$.

This theorem allows us to recognize divergent series. Indeed, by the contrapositive in the next corollary.

Corollary: If $\lim_{n\to+\infty} u_n \neq 0.$, then $\sum_{n=0}^{+\infty} u_n$ is divergent.

Example 4.1.2 : To study the nature of numerical series $\sum_{n=0}^{+\infty} \frac{3n+1}{2n+5}$.

Let $u_n = \frac{3n+1}{2n+5}$, we calculate the limit of the general term :

$\lim_{n\to+\infty} u_n = \lim_{n\to+\infty} \frac{3n+1}{2n+5} = \frac{3}{2} \neq 0$, then $\sum_{n=0}^{+\infty} u_n$ is divergent.

4.1.1 Positive term series

In this section we will study series with a positive general term, $u_n \geq 0\ \forall n \in \mathbb{N}$. This type of series has particular properties with regard to convergence.

1.Model of a geometric series $\sum_{n=0}^{+\infty} q^n, q \in \mathbb{R}^+$

We will study its convergence

- If $q = 1$, then $\lim_{n\to+\infty} q^n = +\infty$ and therefore the series also diverges.

$\sum_{n=0}^{+\infty} 1^n$ diverges because

$S_n = u_0 + u_1 + \cdots + u_{n-1} + u_n = 1 + 1 + \cdots + 1 + 1 = n + 1$, and

$\lim_{n\to+\infty} q^n = +\infty$ then $\lim_{n\to+\infty} S_n = \lim_{n\to+\infty} (n+1) = +\infty$

- If $q < 1$, therefore the series converges
- If $q > 1$, $\lim_{n\to+\infty} q^n = +\infty$ and therefore the series diverges.

because

$$S_n = u_0 + u_1 + \cdots + u_n = 1 + q + \cdots + q^n$$

$$S_{n+1} = u_0 + u_1 + \cdots + u_n + u_{n+1} = 1 + q + \cdots + q^n + q^{n+1} = S_n + q^{n+1}$$

$$= 1 + q(1 + \cdots + q^n) = 1 + qS_n$$

Then $S_n(1-q) = 1 - q^{n+1}$, so, $S_n = \frac{1-q^{n+1}}{1-q}, \forall q \neq 1$

By limit we obtain, $\lim_{n\to+\infty} S_n = S = \begin{cases} \frac{1}{1-q} & if \quad 0 \leq q < 1 \\ +\infty & if \quad\quad q > 1 \end{cases}$ (4.1.5)

Then $\sum_{n=0}^{+\infty} q^n$: $\begin{cases} converges & if \quad 0 \leq q < 1 \\ diverges & if \quad\quad q \geq 1 \end{cases}$. (4.1.6)

2.Model of a Riemann series :

we call a Riemann series any series of the form : $\sum_{n=1}^{+\infty} \frac{1}{n^\alpha}$, $\alpha \in \mathbb{R}$,

$$\sum_{n=1}^{+\infty} \frac{1}{n^\alpha}: \begin{cases} converges & if \quad \alpha > 1 \\ diverges & if \quad \alpha \leq 1 \end{cases}. \qquad (4.1.7)$$

Comparison Criterion

Let $\sum_{n=0}^{+\infty} u_n$ and $\sum_{n=0}^{+\infty} v_n$ be two numerical series with positive terms. Assume that there exists $n_0 \in \mathbb{N}$ such that $\forall n \geq n_0 , u_n \leq v_n$.

Theorem 4.1.2 :

If $\sum_{n=0}^{+\infty} v_n$ is convergent, then $\sum_{n=0}^{+\infty} u_n$ is also convergent.
If $\sum_{n=0}^{+\infty} u_n$ is divergent, then $\sum_{n=0}^{+\infty} v_n$ is also convergent.

Example 4.1.3: To study the nature of a numerical series: $\sum_{n=2}^{+\infty} \frac{1}{n^n}$.

we have $n \geq 2$ then, $n^n \geq 2^n$ and $\frac{1}{n^n} \leq \frac{1}{2^n} = \left(\frac{1}{2}\right)^n$.

The geometric series $\sum_{n=2}^{+\infty} \left(\frac{1}{2}\right)^n$ converges because $q = \frac{1}{2} < 1$, so the numerical series $\sum_{n=2}^{+\infty} \frac{1}{n^n}$ converges (Using comparison Criterion).

Equivalence Criterion

The equivalence criterion states that if two series have equivalent terms, then they have the same nature of convergence.

Theorem 4.1.3 : Let $\sum_{n=0}^{+\infty} u_n$ and $\sum_{n=0}^{+\infty} v_n$ be two series with positive terms. We say that the general terms are equivalent at infinity $u_n \overset{\infty}{\sim} v_n$, if $\lim_{n\to+\infty} \frac{u_n}{v_n} = 1$, then the two series are of the same nature.

Example 4.1.4: To study the nature of a numerical series: $\sum_{n=1}^{+\infty} \ln\left(1 + \frac{1}{n^3}\right)$.

we have $\ln(1 + x) \sim x$ when $x \to 0$ then, $u_n = \ln\left(1 + \frac{1}{n^3}\right) \sim v_n = \frac{1}{n^3}$ and $\lim_{n\to+\infty} \frac{u_n}{v_n} = 1$. The Riemann series $\sum_{n=1}^{+\infty} \frac{1}{n^3}$ converges because $\alpha = 3 > 1$, so the numerical series $\sum_{n=1}^{+\infty} \ln\left(1 + \frac{1}{n^3}\right)$ converges (Using equivalence Criterion).

D'Alembert's Criterion

The d'Alambert criterion, also called the ratio criterion, is a convergence criterion for series with positive terms. It is based on the limit of the ratio of consecutive terms in the series.

Theorem 4.1.4:
Let $\sum_{n=0}^{+\infty} u_n$ be a series with positive terms. If the limit of the ratio $\lim_{n\to+\infty} \frac{u_{n+1}}{u_n} = l$ is finite and equal to l, then :

- If $l < 1$, then the series converges.
- If $l > 1$, then the series diverges.
- If $l = 1$, the test is indeterminate.

Cauchy criterion

If in a series with positive terms the limit of the n^th^ root of the n^th^ term is equal to l then the theorem said :

Theorem 4.1.5:
Let $\sum_{n=0}^{+\infty} u_n$ be a series with positive terms. If the limit of the ratio $\lim_{n\to+\infty} \sqrt[n]{u_n} = l$ is finite and equal to l, then :

- If $l < 1$, then the series converges.
- If $l > 1$, then the series diverges.
- If l $l = 1$, the test is indeterminate.

Example 4.1.5: To study the nature of numerical series $\sum_{n=0}^{+\infty} \left(\frac{3n+4}{2n+7}\right)^n$.

By Cauchy criterion

$$l = \lim_{n\to+\infty} \sqrt[n]{u_n} = \lim_{n\to+\infty} \sqrt[n]{\left(\frac{3n+4}{2n+7}\right)^n} = \lim_{n\to+\infty} \left(\frac{3n+4}{2n+7}\right)^{\frac{n}{n}} = \frac{3}{2} < 1.$$

The numerical series $\sum_{n=0}^{+\infty} \left(\frac{3n+4}{2n+7}\right)^n$ converges using Cauchy criterion.

Cauchy Integral Criterion

If the integral associated with the series converges, then the series also converges, and vice versa.

Theorem 4.1.6:
We consider the function such that $f: [1, +\infty[\to \mathbb{R}^+$a continuous positive and decreasing function. We set $u_n = f(n), \forall n \geq 1$.

If $\int_1^{+\infty} f(x)\, dx$ converges, then the series $\sum_{n=0}^{+\infty} u_n$ converges.

If $\int_1^{+\infty} f(x)\, dx$ diverges, then the series $\sum_{n=0}^{+\infty} u_n$ diverges.

4.1.2 Alternating series

Definition 4.1.3: An alternating series is a series in which the terms alternate in sign,having the following form

$$u_1 - u_2 + u_3 + \cdots + (-1)^{n+1} u_n + \cdots \qquad (4.1.8)$$

where $u_n \geq 0$ for all $n \in \mathbb{N}^*$.

Leibniz Criterion :

Theorem 4.1.7: If in an alternating series the terms decrease and the limit of the general term tends towards zero then the latter converges, moreover its sum is less than the first term.

Let $\sum_{n=1}^{+\infty}(-1)^{n+1}h_n$ be an alternating series.

If $\lim_{n\to+\infty} h_n = 0$ and $|h_{n+1}| \leq |h_n|$ for all n, then the series $\sum_{n=1}^{+\infty}(-1)^{n+1}h_n$ converges.

Example 4.1.6: To study the nature of numerical series $\sum_{n=1}^{+\infty}\frac{(-1)^n}{n}$.

$\sum_{n=1}^{+\infty}|u_n| = \sum_{n=1}^{+\infty}\frac{1}{n}$ diverges, so we apply the Leibniz Criterion.

Where $h_n = \frac{1}{n}$ verifies the conditions of the theorem 4.1.7, then $\sum_{n=1}^{+\infty}\frac{(-1)^n}{n}$ converges.

4.1.3 Series with terms of any sign

Definition 4.1.4 : A series with a term of any sign is a series among which terms are found those which are positive as well as negative such as the alternating series which is a special case of series with terms of any sign.

Absolute convergence

Definition 4.1.5: The numerical series $\sum_{n=0}^{+\infty}u_n$ with any terms is absolutely convergent if the series $\sum_{n=0}^{+\infty}|u_n|$ is convergent.

Theorem 4.1.8: If a series converges absolutely, then it converges.

Note: There are series that are convergent, but not absolutely convergent. These are called semi-convergent series, such as the classic example : the series with general term $u_n = \frac{(-1)^n}{n}$.

Abel criterion

Theorem 4.1.9: Suppose that $\sum_{n=0}^{+\infty}u_n$ is a series with general term $u_n = a_n b_n$ where $(a_n)_n$ and $(b_n)_n$ satisfy the following conditions:

- $a_n \geq 0, (a_n)_n$ is decreasing and $\lim_{n\to+\infty} a^n = 0$.
- $\sum_{k=0}^{n} b_k$ is bounded

Then the series $\sum_{n=0}^{+\infty}u_n$ converges.

4.2 Sequences and Series of Functions

4.2.1 Function sequences

Definition 4.2.1 : We call a sequence of functions $\{f_n(x)\}$ any sequence of functions $f_1(x), f_2(x), \ldots$ defined on the same domain D, whose general term is a function depending on a parameter n, we denote ainsi $(f_n)_{n\in\mathbb{N}}$.

Types of convergence :

Definition 4.2.2 (Simple convergence) :

Let $\{f_n(x)\}$ be a sequence of functions on a set $D \subset \mathbb{R} \to \mathbb{R}$. We say that $\{f_n\}$ converges simply to a function $f(x)$ on D if:

$$\lim_{n\to+\infty} f_n(x) = f(x), \forall x \in D. \quad (4.2.1)$$

That is, for every fixed point $x \in D$, the numerical sequence $f_n(x)$ must converge to a finite value $f(x)$ when n becomes large.

Definition 4.2.3 (Absolute convergence) :

Let $\{f_n(x)\}$ be a sequence of functions on a set $D \subset \mathbb{R} \to \mathbb{R}$. We say that the series $\sum_{n=0}^{+\infty} f_n(x)$ converges absolutely on D if:

$$\sum_{n=0}^{+\infty} |f_n(x)| \text{ converges for every } x \in D.$$

Definition 4.2.4 (Uniform convergence) :

Let $\{f_n(x)\}$ be a sequence of functions on a set $D \subset \mathbb{R} \to \mathbb{R}$. We say that $\{f_n\}$ converges uniformly to a function $f(x)$ on D if:

$$\lim_{n\to+\infty} sup_{x\in D} |f_n(x) - f(x)| = 0 \quad (4.2.2)$$

That is, there exists a natural number N such that for every $n \leq N$ and for all points $x \in D$, it is: $|f_n(x) - f(x)|$ very small.

Theorem 4.2.1: Any uniformly convergent sequence is simply convergent, the converse is false.

Properties of sequences of functions :

Let $\{f_n(x)\}$ be a sequence of functions on a set $D \subset \mathbb{R} \to \mathbb{R}$.

- Continuity at a Point : If each f_n is continuous at a point $x_0 \in D$ and (f_n) converges uniformly to a function f on D, then: f is continuous at x_0.
- Continuity on an Interval : If each f_n is continuous on D and (f_n) converges uniformly to a function f on D, then: f is continuous on D.
- Differentiation : If (f_n) converges uniformly to f and each f_n is differentiable, then: the sequence of derivatives (f_n') converges uniformly to f'.
- Integration (Riemann) : If (f_n) is a sequence of Riemann-integrable functions that converges uniformly to f on $[a, b]$, then:

$$\lim_{n\to+\infty} \int_a^b f_n(x)dx = \int_a^b \lim_{n\to+\infty} f_n(x)dx. \quad (4.2.3)$$

Uniform convergence allows the limit and the integral to be interchanged.

4.2.2 Function series

Definition 4.2.5: Let $f_n: D \subset \mathbb{R} \to \mathbb{R}$ be a sequence of functions. The infinite sum of the terms of f_n is called a series of functions and it is denoted by :

$$\sum_{n=0}^{+\infty} f_n(x) = f_0(x) + f_1(x) + \cdots + f_n(x) + \cdots \quad (4.2.4)$$

Convergence of function series : Let f_n be a sequence of functions.

- If the sequence of partial sums $(S_n)_{n\in\mathbb{N}^*}$ converges pointwise on D then $\sum_{n=0}^{+\infty} f_n(x)$converges pointwise on D.
- If the series $\sum_{n=0}^{+\infty} |f_n(x)|$ converges pointwise on D, then $\sum_{n=0}^{+\infty} f_n(x)$ converges absolutely on D.
- If the sequence of partial sums $(S_n)_{n\in\mathbb{N}^*}$ converges uniformly on D, then $\sum_{n=0}^{+\infty} f_n(x)$ converges uniformly on D.

Properties of function series :

- Let $f_n: D \subset \mathbb{R} \to \mathbb{R}$ be a sequence of functions. If f_n is continuous at x_0, and if $\sum_{n=0}^{+\infty} f_n(x)$converges uniformly to the function $s(x)$, then $s(x)$, is continuous at x_0.

- Let $f_n: D \subset \mathbb{R} \to \mathbb{R}$ be a sequence of functions. If f_n is continuous on D and if $\sum_{n=0}^{+\infty} f_n(x)$converges uniformly to the function $s(x)$, then $s(x)$, is continuous on D.

- Let $f_n: [a,b] \subset \mathbb{R} \to \mathbb{R}$ be a sequence of integrable functions. If $\sum_{n=0}^{+\infty} f_n(x)$converges uniformly to the function $s(x)$, then we have :

$$\sum_{n=0}^{+\infty} \int_a^b f_n(x)dx = \int_a^b \sum_{n=0}^{+\infty} f_n(x) = \int_a^b s(x)dx. \quad (4.2.5)$$

4.3 Power Series and Fourier Series

4.3.1 Power Series

Definition 4.3.1 : A power series is a series of powers of the real variable x. A power series is any series of the form:

$$\sum_{n\geq 0} a_n x^n \quad (4.3.1)$$

where a_n is a numerical series.

Convergence Radius : The radius of convergence of a power series is the maximum distance between the point where the series converges and the central point of the series.

Proposition (Cauchy-Hadamard Formula):

Assume that $\lim_{n\to\infty} \sqrt[n]{|a_n|} = l \in \mathbb{R}^+$ then $R = \frac{1}{l}$ with the convention $\frac{1}{0} = +\infty$ and $\frac{1}{\infty} = 0$. We actually have a more general formula that is always valid:

$$R = \left(lim_{n\to\infty} sup\left(\sqrt[n]{|a_n|}\right)\right)^{-1}. \quad (4.3.2)$$

Example 4.3.1 :

1. For $\sum_{n\geq 0}\frac{x^n}{n!}$, $R=\frac{1}{l}$ and $l=\lim_{n\to\infty}\frac{|n!|}{|(n+1)!|}=\lim_{n\to\infty}\frac{1}{n+1}=0$, so $R=+\infty$.
2. For $\sum_{n\geq 0} n!\,x^n$, $R=\frac{1}{l}$ and $l=\lim_{n\to\infty}\frac{|(n+1)!|}{|n!|}=\lim_{n\to\infty}(n+1)=+\infty$, so $R=0$.
3. For $\sum_{n\geq 0}\frac{x^n}{n}$, $R=\frac{1}{l}$ and $l=\lim_{n\to\infty}\frac{|(n+1)|}{|n|}=1$, so $R=1$.

Properties of power series

Let $\sum_{n\geq 0} a_n x^n$ and $\sum_{n\geq 0} b_n x^n$ be two power series of convergence radius at least $R>0$.

Sum : The series $\sum_{n\geq 0}(a_n+b_n)x^n$ also has a radius of convergence at least $R>0$and, for all x such that $|x|<R$, we have the sum:

$$\sum_{n\geq 0}(a_n+b_n)x^n=\sum_{n\geq 0} a_n x^n+\sum_{n\geq 0} b_n x^n. \qquad (4.3.3)$$

Product : Consider the product series $c=a*b$, where $a_n=\sum_{i=0}^{n} a_i b_{n-i}$, Then the new power series $\sum_{n\geq 0} c_n x^n$ also has a radius of convergence at least $R>0$ and, for all x such that $|x|<R$, we have:

$$\sum_{n\geq 0} c_n x^n=\left(\sum_{n\geq 0} a_n x^n\right)\left(\sum_{n\geq 0} b_n x^n\right). \qquad (4.3.4)$$

Continuity : The function $f(x)=\sum_{n\geq 0} a_n x^n$ is of class C^∞ in the open ball $B(0,R)$.

Derivation : For all $|x|<R$, the derivative series $\sum_{n\geq 0} n a_n x^{n-1}$ has the same radius R, where :

$$\frac{df(x)}{dx}=\sum_{n\geq 0} n a_n x^{n-1} \qquad (4.3.5)$$

Integration : For all $|x|<R$, the integral of power series $\sum_{n\geq 0} a_n x^n$ is a power series, it has the same radius R, where :

$$\int_0^x f(t)dt=\int_0^x\sum_{n\geq 0} a_n t^n dt=\sum_{n\geq 0}\int_0^x a_n t^n dt=\sum_{n\geq 0}\frac{a_n}{n+1}x^{n+1}. \qquad (4.3.6)$$

4.3.2 Fourier Series

Definition 4.3.2: The series

$$f(x)=\frac{a_0}{2}+\sum_{n\geq 1} a_n\cos(nx)+b_n\sin(nx) \text{ or } \sum_{n\in\mathbb{Z}} c_n e^{inx}. \qquad (4.3.6)$$

is called the Fourier series associated with f where :

- f is a piecewise continuous and 2π periodic function.

- Real coefficients of f:

$$a_n(f) = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos(nt)\ dt, n \geq 0,$$

$$b_n(f) = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin(nt)\ dt, n > 0 \qquad (4.3.7)$$

- Complex coefficients of f:

$$c_n(f) = \frac{1}{2\pi}\int_0^{2\pi} f(t)\, c_n e^{-int}\ dt. \qquad (4.3.8)$$

Properties of Fourier series

- The Fourier series expansion of even functions does not contain the b_n.

f even then, $b_n = 0, \forall n > 0$ and $a_n = \frac{1}{\pi}\int_0^{2\pi} f(t)\cos(nt)\ dt$

- The Fourier series expansion of odd functions does not contain the a_n.

f odd then, $a_n = 0, \forall n > 0$ and $b_n(f) = \frac{1}{\pi}\int_0^{2\pi} f(t)\sin(nt)\ dt$.

Example 4.3.2: Function: $f(x) = sin(x)$, 2πperiodic.

Computation of coefficients:

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} \sin(t)\ dt = 0\ , (f \text{ odd})$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} \sin(t)\cos(nt)\ dt = 0, \forall n \geq 1$$

$$b_n(f) = \frac{1}{\pi}\int_0^{2\pi} \sin(t)\sin(nt)\ dt = \begin{cases} 1, & n = 1 \\ 0, & n \neq 1 \end{cases}.$$

Chapter 5 :

Fourier Transform

Chapter 5 :

Fourier Transform

5.1 Definition and Properties

5.1.1 Definition

5.1.2 Properties of the Fourier Transform

5.2 Application to Solving Differential Equations

Solving linear differential equations with constant coefficients

5.1 Definition and Properties

5.1.1 Definition

Let $f : \mathbb{R} \to \mathbb{R}$ be a locally and absolutely integrable function on $\mathbb{R}$. The Fourier transform of f is the function $\hat{f}$ defined as follows:

$$\hat{f}(\omega) = \mathrm{T}F(f)(t) = \int_{-\infty}^{+\infty} f(t)e^{-i\omega t}\,dt \qquad (5.1.1)$$

Its inverse transform from $\mathrm{T}F^{-1}(\hat{f}) : \mathbb{C} \to \mathbb{R}$ is given as follows :

$$f(t) = \mathrm{T}F^{-1}(\hat{f})(\omega) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}(\hat{f})(\omega)e^{i\omega t}\,d\omega. \qquad (5.1.2)$$

where ω is the frequency variable.

5.1.2 Properties of the Fourier Transform

Linearity of TF : The Fourier transform is a linear operation.

$$(a\widehat{f+b}g)(\omega) = \mathrm{T}F(af+bg)(t) = a\mathrm{T}F(f)(t) + b\mathrm{T}F(g)(t)$$
$$= a\hat{f}(\omega) + b\hat{g}(\omega). \qquad (5.1.3)$$

Where $a, b \in \mathbb{C}$ and $f, g \in L^1(\mathbb{R})$

Linearity of TF^{-1} : The inverse Fourier transform is a linear operation.

$$\mathrm{T}F^{-1}(a\hat{f} + b\hat{g})(\omega) = a\mathrm{T}F^{-1}(\hat{f})(\omega) + b\mathrm{T}F^{-1}(\hat{g})(\omega)$$
$$= af(t) + bg(t). \qquad (5.1.4)$$

Where $a, b \in \mathbb{C}$ and $\hat{f}, \hat{g}$ two locally integrable and absolutely integrable functions on $\mathbb{C}$.

Time domain differentiation :

$$\widehat{\left(\frac{d^n f}{dt^n}\right)}(t) = \mathrm{T}F\left(\frac{d^n f}{dt^n}\right)(t) = (i\omega)^n(\hat{f})(\omega) \qquad (5.1.5)$$

Frequency domain differentiation :

$$\frac{d^n\hat{f}}{dt^n}(\omega) = ((\imath\widehat{t\omega)^n}f)(t) \qquad (5.1.6)$$

Translation in t :

$$\mathrm{T}F(f)(t-t_0) = \hat{f}(\omega)e^{-i\omega t_0} \qquad (5.1.7)$$

Remark: A term corresponding to a linear phase shift with respect to frequency $e^{-i\omega t}$ represents a time translation. Conversely, a translation in the time domain corresponds to multiplying the Fourier transform by a linear phase term $e^{i\omega t}$.

Domain contraction :

$$\mathrm{T}F(f)(at) = \frac{1}{|a|}\hat{f}\left(\frac{\omega}{a}\right). \qquad (5.1.8)$$

Conjugation :

$$\mathrm{T}F(\bar{f})(t) = \overline{\hat{f}(\omega)}. \qquad (5.1.9)$$

Convolution : The Fourier transform of the convolution of two functions is the product of the transforms :

$$\mathrm{T}F(f)(t) = \int_{-\infty}^{+\infty} f(t)e^{-i\omega t}\, dt(t) = \hat{f}(\omega).\hat{g}(\omega) \qquad (5.1.10)$$

where The function $f * g$ is the convolution of the real or complex functions f and g, defined as follows :

$$(f * g)(y) = \int_{-\infty}^{+\infty} f(y-t)g(t)\, dt \qquad (5.1.11)$$

Behavior to infinity : $\lim_{|\omega|\to\infty} \hat{f}(\omega) = 0$

Continuity : If $f \in L^1$ then $\hat{f} \in C(\mathbb{R})$.

Energy preservation (inner product form)

Plancherel's Theorem : If f and g are functions in the Schwartz space S, then their inner product is preserved under the Fourier transform:

$$\int_{-\infty}^{+\infty} f(t)\overline{g(t)}\, dt = \int_{-\infty}^{+\infty} \hat{f}(\omega)\overline{\hat{g}(\omega)}\, d\omega. \qquad (5.1.12)$$

Energy conservation (squared magnitude form)

Parseval's Theorem : If f is a square-integrable function in L^2 and $\hat{f}$ is its Fourier transform, then the total energy of the signal remains the same in both the time and frequency domains:

$$\int_{-\infty}^{+\infty} |f(t)|^2\, dt = \int_{-\infty}^{+\infty} \left|\hat{f}(\omega)\right|^2 d\omega \qquad (5.1.13)$$

Symmetry (parity) : Relations between even/odd and real/imaginary functions.

If f is real and even then $\hat{f}$ is real and even.

If f is real and odd then $\hat{f}$ is pure imaginary and odd.

If f is imaginary and even then $\hat{f}$ is imaginary and even.

If f is imaginary and odd then $\hat{f}$ is real and odd.

Example 5.1.1:

Determine the Fourier transform of the following functions:

- $f_1(t)$ equals 1 on $[-1,1]$ and 0 everywhere else .
- $f_2(t)$ equals 1 on $[-T,T]$ and 0 everywhere else $(T \geq 0)$
- $f_3(t) = e^{-\frac{|t|}{T}}$, $(T \geq 0)$

Fourier transform of the function $f_1(t)$

We have

$$\hat{f}(\omega) = \mathrm{T}F(f)(t) = \int_{-\infty}^{+\infty} f(t)e^{-i\omega t}\, dt$$

$$\text{For } \omega \neq 0, \hat{f}_1(\omega) = \int_{-1}^{1} e^{-i\omega t}\, dt = \left[-\frac{e^{-i\omega t}}{i\omega}\right]_{-1}^{1} = -\frac{e^{-i\omega}}{i\omega} + \frac{e^{i\omega}}{i\omega} = \frac{(e^{i\omega} - e^{-i\omega})}{i\omega} = \frac{2\sin\omega}{\omega}$$

For $\omega = 0, \hat{f}_1(0) = \int_{-1}^{1} e^0\, dt = \int_{-1}^{1} 1\, dt = [t]_{-1}^{1} = 2.$

Fourier transform of the function $f_2(t)$

We have,

$$\hat{f}_2(\omega) = \int_{-T}^{T} e^{-i\omega t}\, dt = \left[-\frac{e^{-i\omega t}}{i\omega}\right]_{-T}^{T} = -\frac{e^{-i\omega T}}{i\omega} + \frac{e^{i\omega T}}{i\omega} = \frac{\left(e^{i\omega T} - e^{-i\omega T}\right)}{i\omega}$$
$$= \frac{2\sin(\omega T)}{\omega}$$

Fourier transform of the function $f_3(t)$

We have,

$$\hat{f}_3(\omega) = \int_{-\infty}^{0} e^{\frac{t}{T}} e^{-i\omega t}\, dt + \int_{0}^{+\infty} e^{\frac{-t}{T}} e^{-i\omega t}\, dt = \left[-\frac{e^{\frac{t}{T} - i\omega t}}{\frac{1}{T} - i\omega}\right]_{-\infty}^{0} + \left[-\frac{e^{\frac{-t}{T} - i\omega t}}{-\frac{1}{T} - i\omega}\right]_{0}^{+\infty}$$
$$= \frac{T}{1 - Ti\omega} + \frac{T}{1 - Ti\omega} = \frac{2T}{1 + (T\omega)^2}$$

5.2 Application to Solving Differential Equations

Solving linear differential equations with constant coefficients

Let the differential equation with constant coefficients be :

$$a_n y^{(n)}(t) + a_{n-1} y^{(n-1)}(t) + \cdots + a_1 y'(t) + a_0 y(t) = g(t) \qquad (5.2.1)$$

Applying the Fourier transform to both sides of this equation and using its properties, we obtain :

$$\mathrm{TF}\left(a_n y^{(n)}(t) + a_{n-1} y^{(n-1)}(t) + \cdots + a_1 y'(t) + a_0 y(t)\right) = \mathrm{TF}(g(t))$$
$$[a_n (2i\pi\omega)^n + a_{n-1}(2i\pi\omega)^{n-1} + \cdots + a_1 (2i\pi\omega)^1 + a_0]\hat{y}(\omega) = \hat{g}(\omega)$$

So,

$$\hat{y}(\omega) = [a_n (2i\pi\omega)^n + a_{n-1}(2i\pi\omega)^{n-1} + \cdots + a_1 (2i\pi\omega)^1 + a_0]^{-1}\hat{g}(\omega)$$

$$\hat{y}(\omega) = \hat{h}(\omega).\hat{g}(\omega) = (\widehat{h * g})(\omega) \qquad (5.2.2)$$

Fourier transformis injective this allows us to write :

$$y(t) = h(t) * g(t). \qquad (5.2.3)$$

Example 5.2.1 : Using the fourier transformation, solve the followind differential equation : $y''(t) + 2y(t) = e^{t^2}$.

Apply the Fourier Transform to both sides

$$\mathrm{T}F\left(y''(t) + 2y(t)\right) = \mathrm{T}F\left(e^{t^2}\right)$$
$$\mathrm{T}F(y''(t)) + 2\mathrm{T}F(y(t)) = \mathrm{T}F\left(e^{t^2}\right)$$
$$(2i\pi\omega)^2\hat{y}(\omega) + 2\hat{y}(\omega) = \widehat{\left(e^{t^2}\right)}$$
$$((2i\pi\omega)^2 + 2)\hat{y}(\omega) = \widehat{\left(e^{t^2}\right)}$$

So,

$$\hat{y}(\omega) = [(2i\pi\omega)^2 + 2]^{-1}\widehat{\left(e^{t^2}\right)}$$

Where $[(2i\pi\omega)^2 + 2]^{-1} = \frac{1}{4}e^{-|t|}$

Because the frequency domain factor $\frac{1}{(2i\pi\omega)^2+2}$ as the Fourier transform of a time domain function $\frac{1}{4}e^{-|t|}$.

$$\hat{y}(\omega) = \widehat{\left(\frac{1}{4}e^{-|t|}\right)}.\widehat{\left(e^{t^2}\right)} = \widehat{\left(\frac{1}{4}e^{-|t|} * e^{t^2}\right)}(\omega)$$

Fourier transformis injective this allows us to write :

$$y(t) = (\frac{1}{4}e^{-|t|}) * e^{t^2}.$$

Chapter 6 :

Laplace Transform

Chapter 6 :

Laplace Transform

6.1 Definition and Properties

6.1.1 Direct Laplace Transform
6.1.2 Properties of the Laplace Transform
6.1.3 Some Laplace transforms of usual functions
6.1.4 Inverse Laplace transform

6.2 Application to Solving Differential Equations

Solving linear differential equations with constant coefficients

6.1 Definition and Properties

6.1.1 Direct Laplace Transform

Definition 6.1.1 (Causal functions) : Let $f(t)$ be a function from R with values in $\mathbb{C}$ or $\mathbb{R}$. A causal function is any function that is zero for $t < 0$.
Definition 6.1.2: let $f\colon [0,+\infty[\to \mathbb{R}\ or\ \mathbb{C}$ be a causal function (or a causal real part), and $p \in \mathbb{C}$, the function $F(p) = \mathcal{L}\big(f(t)\big)(p)$ is called the Laplace transform and is equal to :

$$F(p) = \mathcal{L}\big(f(t)\big)(p) = \int_0^{+\infty} e^{-pt} f(t)dt \qquad (6.1.1)$$

where $f(t)$ is said to be the original of $F(p)$, $F(p)$ is the Laplace transform of $f(t)$, p is a complex variable, and t is time.
Example 6.1.1: We want to find the Laplace transform of the unit step function defined by :

$$u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$$

Using the definition of the Laplace transform

$$\mathcal{L}\big(u(t)\big)(p) = \int_0^{+\infty} e^{-pt} u(t)dt$$

Since $u(t) = 1$ for $t \geq 0$, the integral becomes

$$\mathcal{L}\big(u(t)\big)(p) = \int_0^{+\infty} e^{-pt}\, dt$$

Now, compute the integral :

$$\mathcal{L}\big(u(t)\big)(p) = \int_0^{+\infty} e^{-pt}\, dt = \left[\frac{1}{p} e^{-pt}\right]_0^{+\infty} = 0 - (-\frac{1}{p}) = -\frac{1}{p}, \text{ for } p > 0.$$

Existence of the Laplace Transform

The Laplace transform exists for any function $f(t)$ that is locally integrable on the interval $[0, +\infty[$, the function must be integrable on any bounded interval of the form $[0, T]$, where T is a positive real number, i.e., the Laplace transform $\mathcal{L}\big(u(t)\big)(p)$ exists if :

- $f(t)$ is piecewise continuous on every finite interval $[0, T]$.
- $f(t)$ is of exponential order, $|f(t)| \leq Me^{\alpha t}, t > T$ for some constants $\alpha, M\ and\ T > 0$.
- The integral converges for all $t > \alpha$.

Example 6.1.2: The function $f(t) = e^{-\alpha t}$ is locally integrable for all $\alpha > 0$. Furthermore, it is of exponential order because $|f(t)| = e^{-\alpha t} \leq e^{\alpha t}$.

Thus, its Laplace transform exists and is calculated as follows:

$$\mathcal{L}(e^{-\alpha t})(p) = \int_0^{+\infty} e^{-pt} e^{-\alpha t}\, dt = \int_0^{+\infty} e^{-(p+\alpha)t}\, dt = \frac{1}{p+\alpha}, p > -\alpha.$$

Uniqueness of the Laplace Transform

Let $f(t)$ and $g(t)$ be two piecewise continuous functions having an order exponential to infinity on $[0, T]$, where T is a positive real number. If the functions $f(t)$ and $g(t)$ have the same Laplace transform

$\mathcal{L}(f(t)) = \mathcal{L}(g(t))$ then $f(t) = g(t)$ for almost all $t \geq 0$.

6.1.3 Properties of the Laplace Transform

Consider the functions $f(t)$ and $g(t)$ whose conditions of existence of their Laplace transform are fulfilled then $\mathcal{L}(f(t))$ and $\mathcal{L}(g(t))$ exist, and have the following properties :

Linearity : The Laplace transform is linear, which means that for constants a and b and functions $f(t)$ and $g(t)$, we have :

$$\mathcal{L}(af(t) + bg(t)) = a\mathcal{L}(f(t)) + b\mathcal{L}(g(t)) \qquad (6.1.2)$$

Laplace transform of the derivative of a function : Let $f(t)$ a function such that its Laplace transform exists, then the Laplace tansform of its derivative is :

$$\mathcal{L}(f'(t))(p) = p\, F(p) - f(0). \qquad (6.1.3)$$

Where : $F(p) = \mathcal{L}(f'(t))(p)$ and $f(0)$ is the initial value of $f(t)$ at $t = 0$.

Transform translation :

$$F(p+q) = \mathcal{L}(e^{-qt} f(t))(p) = \int_0^{+\infty} e^{-pt} e^{-qt} f(t)\, dt \qquad (6.1.4)$$

$F(p+q)$ is the transform of $e^{-qt} f(t)$.

Time delay : $F(p)$ is the transform of $f(t)$ then:

$$\mathcal{L}(f(t-s))(p) = e^{-ps}\mathcal{L}(f(t))(p) = e^{-ps}F(p), \forall s > 0 \qquad (6.1.5)$$

Initial Value Theorem : $F(p)$ is the transform of $f(t)$ then:

$$\lim_{t\to 0} f(t) = \lim_{p\to +\infty} p\, F(p) \qquad (6.1.6)$$

Final Value Theorem : $F(p)$ is the transform of $f(t)$ then:

$$\lim_{t\to +\infty} f(t) = \lim_{p\to 0} p\, F(p) \qquad (6.1.7)$$

Convolution : $F(p)$ is the transform of $f(t)$ then:

$$\mathcal{L}(f(t) * g(t))(p) = F(p)\, G(p) \qquad (6.1.8)$$

Integral : $F(p)$ is the transform of $f(t)$ then:

$$\mathcal{L}\left(\int_0^t f(s)\,ds\right) = \frac{1}{p}F(p)\,. \qquad (6.1.9)$$

6.1.3 Some Laplace transforms of usual functions

Function name	$f(t)$ **equal**	**Laplace Transform** $F(p)$
Unit step (Heaviside function) H(t)	1	$\frac{1}{p}, p > 0$
Dirac delta impulse	$\delta(t)$	$1, \quad p \epsilon \mathbb{R}$
Power function	$t^{\alpha}, \alpha > -1$	$\frac{\Gamma(\alpha+1)}{p^{(\alpha+1)}}, p > 0$
Exponential decay function	$e^{-\alpha t}$	$\frac{1}{p+\alpha}, p > -\alpha$
Trigonometric cosine function	$\cos(\theta t)$	$\frac{p}{p^2+\theta^2}, p > 0$
Trigonometric sine function	$\sin(\theta t)$	$\frac{\theta}{p^2+\theta^2}, p > 0$

6.1.4 Inverse Laplace transform

The inverse transformation therefore consists of finding the time function $f(t)$ corresponding to the original of a function $F(p)$ which was developed by simple operations in the symbolic world: $f(t) = \mathcal{L}^{-1}\big(F(p)\big)$, such that :

$$f(t) = \mathcal{L}^{-1}\big(F(p)\big) = \frac{1}{2i\pi}\int_{a-i\infty}^{a+i\infty} F(p)\,e^{pt}\,dp \qquad (6.1.10)$$

6.2 Application to Solving Differential Equations

This method is particularly efficient for linear differential equations with constant coefficients and can handle initial value problems directly, where the Laplace transform solves differential equations by converting them into algebraic equations, the process involves taking the Laplace transform of the entire equation, substituting initial conditions, solving the resulting algebraic equation for the transformed variable, and then using the inverse Laplace transform to find the solution to the original differential equation.

Solving linear differential equations with constant coefficients :

Given the differential equation : $y' + ay = b(t),\ y(0) = y_0$.
Take the Laplace transform of both sides

$$\mathcal{L}(y' + ay) = \mathcal{L}(b(t)) = \mathcal{L}(\frac{dy}{dt}) + a\mathcal{L}(y) = \mathcal{L}(b(t))$$

From Laplace transform of the derivative of a function :

$$\mathcal{L}\left(\frac{dy}{dt}\right) = pF(p) - y_0,\ \mathcal{L}(y) = F(p) \text{ and } \mathcal{L}(b(t)) = B(t).$$

Substitute into the equation:

$$(pF(p) - y_0) + F(p) = B(t)$$

Solve for $F(p)$

$$F(p) = \frac{y_0}{p+a} + \frac{B(t)}{p+a}$$

Take the inverse Laplace transform :

$$y(t) = \mathcal{L}^{-1}(F(p)) = \mathcal{L}^{-1}\left(\frac{y_0}{p+a} + \frac{B(t)}{p+a}\right) = y_0\mathcal{L}^{-1}\left(\frac{1}{p+a}\right) + \mathcal{L}^{-1}\left(\frac{B(t)}{p+a}\right)$$

Example 6.2.1: Given the differential equation :

Take the Laplace transform of both sides

$$\mathcal{L}(y' + y) = \mathcal{L}(2e^{5t}) = \mathcal{L}(\frac{dy}{dt}) + \mathcal{L}(y) = 2\mathcal{L}(e^{5t})$$

From Laplace transform of the derivative of a function :

$$\mathcal{L}\left(\frac{dy}{dt}\right) = pF(p) - y(0),\ \mathcal{L}(y) = F(p) \text{ and } \mathcal{L}(e^{5t}) = \frac{1}{p-5}.$$

Substitute into the equation:

$$\big(pF(p) - y(0)\big) + F(p) = \frac{2}{p-5}$$

Substitute the initial condition $y(0) = 1$

$$(p+1)F(p) - 1 = \frac{2}{p-5}$$

Solve for $F(p)$

$$F(p) = \frac{1}{p+1} + \frac{2}{(p+1)(p-5)}$$

Partial fraction expansion : Let $\frac{2}{(p+1)(p-5)} = \frac{a}{p-5} + \frac{b}{p+1}$, so $a = \frac{1}{3},\ b = \frac{-1}{3}$.

Substitute back into $F(p)$

$$F(p) = \frac{1}{p+1} + \frac{1}{3}\left[\frac{1}{p-5} - \frac{1}{p+1}\right]$$

Simplify:

$$F(p) = \frac{1}{3}\left(\frac{1}{p-5}\right) + \frac{2}{3}\left(\frac{1}{p+1}\right).$$

Take the inverse Laplace transform :

$$y(t) = \mathcal{L}^{-1}(F(p)) = \mathcal{L}^{-1}\left(\frac{1}{3}\left(\frac{1}{p-5}\right) + \frac{2}{3}\left(\frac{1}{p+1}\right)\right) = \frac{1}{3}\mathcal{L}^{-1}\left(\left(\frac{1}{p-5}\right)\right) + \frac{2}{3}\mathcal{L}^{-1}\left(\left(\frac{1}{p+1}\right)\right)$$

$$y(t) = \frac{1}{3}e^{5t} + \frac{2}{3}e^{-t}.$$

Proposed Exercice Series

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Guided work series $N°$ 01

Simple and multiple integrals

Exercise 01 : (simple integral)
Calculate the following integrals :

$$\int x^2\cos(x)\,dx \quad , \quad \int \ln(x)\,dx \quad , \quad \int \sin(2x+\pi)\,dx \quad , \quad \int_0^1 xe^x\,dx\ .$$

Exercise 02 : (double integral)
Calculate the following integrals :

$$\iint_D x\ \sin(y)\,dx\,dy \qquad \text{where} \quad D=[1,2]\times[0,\frac{\pi}{2}]$$

$$\iint_D x\ y\,dx\,dy \qquad \text{where} \quad D=\{(x,y)\in\mathbb{R}^2\ /\ x,y\geq 0 \text{ and } x+y\leq 1\}$$

$$\iint_D \sin(x+y)\,dx\,dy \qquad \text{where} \quad D=\{(x,y)\in\mathbb{R}^2\ /\ x,y\geq 0 \text{ and } x+y\leq \pi\}$$

$$\iint_D \cos(x^2+y^2)\,dx\,dy \qquad \text{where} \quad D \text{ is the disk with center 0 and radius } R.$$

$$\iint_D \frac{1}{1+x^2+y^2}\,dx\,dy \qquad \text{where} \quad D=\{(x,y)\in\mathbb{R}^2\ /\ x\geq 0,\ y\geq 0 \text{ and } x^2+y^2\leq 1\}$$

Exercise 03 : (triple integral)
Calculate the following integrals :

$$\iiint_D x\ y\ z\,dx\,dy\,dz \qquad \text{where} \quad D=[0,1]\times[-1,2]\times[0,2]$$

$$\iiint_D x\,dx\,dy\,dz \qquad \text{where} \quad D=\{(x,y,z)\in\mathbb{R}^3\ /\ 0\leq x\leq 1\ ,\ 0\leq y\leq 1-x \text{ and } 0\leq z\leq 1-x-y\}$$

$$\iiint_D 1\,dx\,dy\,dz \qquad \text{where} \quad D=\{(x,y,z)\in\mathbb{R}^3\ /\ x^2+y^2\leq 1 \text{ and } 0\leq z\leq 1\}$$

$$\iiint_D x^2\,dx\,dy\,dz \qquad \text{where} \quad D \text{ is a half ball}$$

"Nothing is easy but nothing is impossible"

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Guided work series N° 02

Improper integrals

Exercise 01 :
Calculate the following improper integrals :

$$\int_1^{+\infty} \frac{1}{t^\alpha}\, dt \quad , \quad \int_0^1 \frac{1}{t^\alpha}\, dt \quad , \quad \int_1^{+\infty} \frac{x^2}{\sqrt{x}+x^7}\, dx .$$

$$\int_0^{+\infty} \frac{1}{1+e^{-x}}\, dx \qquad \int_0^{+\infty} \frac{x}{(x^2+1)^4}\, dx \qquad \int_{-1}^{0} \frac{1}{\sqrt{-x}}\, dx$$

Exercise 02 :
Study the nature of the following improper integrals :

$$\int_0^1 \frac{\cos(x)+2}{x}\, dx \quad , \quad \int_2^{+\infty} \frac{\ln(x)}{x^3+1}\, dx$$

$$\int_0^1 \frac{\ln(1+\sqrt{x})}{x}\, dx \quad , \quad \int_1^{+\infty} \frac{\ln(x)}{x+e^{-x}}\, dx$$

"**Don't be afraid to fail**, **just keep trying**"

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Guided work series $N°$ 03

Differential equations

Exercise 01 :

Solve the following differential equations :

$$x\text{y}\ dx + (1+x^2)\ d\text{y} = 0 \qquad , \qquad x \in D$$

$$(x^2-1)\ \acute{\text{y}} = \text{y} \qquad , \qquad x \in D$$

$$x\acute{\text{y}} + 2\text{y} = \frac{x^2}{x^2+1} \qquad , \qquad x \in D \qquad .$$

$$\acute{\text{y}} - a\frac{\text{y}}{x} = x^a e^x \qquad , \qquad a \in \mathbb{R},\ x \in \mathbb{R}^*$$

Exercise 02 :

Solve the following differential equations :

$$\text{\H{y}} - 4\acute{\text{y}} + 3\text{y} = 0$$

$$\text{\H{y}} + 2\acute{\text{y}} + \text{y} = 0$$

$$\text{\H{y}} + 2\acute{\text{y}} + 4\text{y} = 0 \qquad .$$

$$\text{\H{y}} - 4\ \acute{\text{y}} + 4\text{y} = (1+x^2)\ e^x$$

$$\text{\H{y}} + 2\ \acute{\text{y}} + 5\text{y} = e^x \cos(x)$$

"**Work hard in silence, let success make the noise**"

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Guided work series $N°$ 04

Series

Exercise 01 :

Study the nature of the following numerical series

$$\sum_{n=0}^{+\infty}\frac{2^n+n+3}{3^n+n^2+5} \quad , \quad \sum_{n=2}^{+\infty}\frac{1}{\sqrt{n(n-1)}}$$

$$\sum_{n=1}^{+\infty}(1+\frac{2}{n})^{n^2} \quad , \quad \sum_{n=1}^{+\infty}(1+\frac{1}{n})^{n}$$

$$\sum_{n=2}^{+\infty}\frac{1}{n^n} \quad , \quad \sum_{n=1}^{+\infty}\frac{n!}{n^n}$$

Exercise 02 :

Study the nature of the following numerical series

$$\sum_{n=1}^{+\infty}\frac{(-1)^n}{n^2+1} \quad , \quad \sum_{n=1}^{+\infty}\frac{(-1)^n}{n^5+1} \quad , \quad \sum_{n=2}^{+\infty}\frac{(-1)^n}{\ln(n)}$$

Exercise 03 :

Study the simple and uniform convergence of the following sequences and series of functions

$$f_n(x) = (1+\frac{x}{n})^n \quad , \quad \sum\frac{1}{(1+x^2)^n} \qquad x \in \mathbb{R}$$

$$f_n(x) = \frac{x^n}{n!} \quad , \quad \sum\frac{\cos^2(nx)}{n^4} \qquad x \in \mathbb{R}$$

$$f_n(x) = \frac{(-1)^n}{\sqrt{x^2+n^2}} \quad , \quad \sum n\ e^{-nx} \qquad x \in \mathbb{R}^+$$

Exercise 04 :

Determine the convergence radius of series

$$a_n = \ln(n) \quad , \quad a_n = (\sqrt{n})^n \quad , \quad a_n = \frac{n^n}{n!}$$

"Good things come to those who wait"

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Guided work series $N°$ 05

Fourier transform

Exercise 01 :

Determine the fourier transformation of the following functions :

$f_1(t)$ is 1 on $[-1,1]$ and 0 everywhere else
$f_2(t)$ is 1 on $[-T,T]$ and 0 everywhere else $(T \succ 0)$

$$f_3(t) = e^{-\frac{|t|}{T}} \quad , (T \succ 0)$$

$$f_4(t) = \frac{\sin t}{t}$$

$$f_5(t) = \frac{1}{1+t^2}$$

Exercise 02 :

Demonstrate the fourier transform properties.

Exercise 03 :

Let the function $f(x) = e^{-a|x|}$ for $a \succ 0$

1- Calculate its fourier transformation (without using the form)

2- Deduce the transformation of fourier of $g : x \rightarrow \dfrac{1}{1+x^2}$

3- Calculate the convolution product $f * f$ and reduce the fourier trasform of $x \rightarrow \dfrac{1}{(1+x^2)^2}$

4- Determine the fourier trasform of $x \rightarrow \dfrac{1}{(1+x^2)^2}$

Exercise 04 :

Using the fourier transformation, solve the followind differential equation.

$$-y"(t) + y(t) = e^{-t^2}$$

”**Be with the truth even if you are alone**”

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