Mohamed Khider University of Biskra
Exact sciences Faculty
Material sciences department

Lecture notes :

Mechanical Element

Intended for:

first-year students in the LMD system,

Sciences and Water Engineering (FRL/FRR)

Faculty of Urban Planning, Architecture

Civil and Irrigation Engineering

Department of Urban and Irrigation Engineering

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2025/2026

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Chapter I :Reminder

I.1. Introduction

The term "physics" comes from the Ancient Greek word *physikḗ*, which means "understanding of nature." Physics is the natural science of matter, focusing on the examination of substances, their fundamental components, their motion and behavior through space and time, and the associated concepts of energy and force. Physics is among the most essential scientific fields, with its primary objective being to comprehend how the universe operates.

Physics can be categorized into two principal branches: classical physics and modern physics:

- Classical physics primarily deals with matter and energy at the everyday scale of observation.
- Modern physics studies the properties and behavior of matter and energy under extreme circumstances or at very large or very small scales.

In physics, we describe quantities in two main forms: scalar quantities and vector quantities:

- Scalar quantities are completely characterized by a numerical value and a unit; for example, the mass m of an object, the length l of an item, and so on.
- Vector quantities are defined by a number, a unit, and a direction (often represented geometrically by an arrow); for example, velocity, force, and weight.

I.2. Physical quantities and units

I.2.1. International System of Units (SI)

Internationally recognized by the abbreviation SI (from the French term *Système International*, commonly referred to as the metric system), it is the most widely adopted system of measurement in the world. The SI system is organized and regulated by the General Conference on Weights and Measures (CGPM).

I.2.1.1 Base Units

The definitions of the base units of the International System (SI) are based on reproducible physical phenomena.

• The meter (m):

The meter is the length of the path traveled by light in vacuum during a time interval of $\frac{1}{299\,792\,458}$ of a second.

Historically, the first official and practical definition of the meter (1791) was based on the Earth's circumference and was equal to $\frac{1}{40\,000\,000}$ of a meridian.

Previously, the meter, as a proposed universal decimal unit of measurement, was defined as the length of a pendulum that oscillates with a half-period of one second.

• The kilogram (kg):

The kilogram is the mass of the international prototype of the kilogram. This prototype, made of a platinum–iridium alloy (90%–10%), is kept at the International Bureau of Weights and Measures in Sèvres, France.

Historically, the kilogram was defined as the mass of one cubic decimeter of water (one liter).

• The second (s):

The second is the duration of 9,192,631,770 periods of the radiation corresponding to the transition between the two levels of the ground state of the cesium-133 atom at a temperature of 0 kelvin.

Originally, the second was based on the duration of the Earth's day, divided into 24 hours of 60 minutes, each minute lasting 60 seconds (that is, 86,400 seconds per day).

• The ampere (A):

The ampere is the intensity of a constant electric current which, if maintained in two straight, parallel conductors of infinite length, of negligible circular cross-section, and placed one meter apart in vacuum, would produce between these conductors a force equal to

2×10^{-7}newton per meter of length.

• **The kelvin (K):**

The kelvin, the unit of thermodynamic temperature, is equal to the fraction $1/273.16$of the thermodynamic temperature of the triple point of water.

• **The mole (mol):**

The mole is the amount of substance of a system containing as many elementary entities as there are atoms in 0.012 kilogram of carbon-12.

This number of elementary entities is called Avogadro's number. When the mole is used, the elementary entities must be specified and may be atoms, molecules, ions, electrons, other particles, or specified groups of such particles.

• **The candela (cd):**

The candela is the luminous intensity, in a given direction, of a source that emits monochromatic radiation of frequency

540×10^{12} hertz and that has a radiant intensity in that direction of $\frac{1}{683}$watt per steradian.

Table.I.1. SI base Units.

Quantity	**Name**	**Symbol**
Time	second	**s**
Length	meter	**m**
Mass	kilogram	**kg**
Electric current	ampere	**A**
Temperature	kelvin	**K**
Amount of substance	mole	**mol**
Luminous intensity	candela	**cd**

I.2.2. MKSA system

The MKS system of units is a system of physical measurement that uses the meter, kilogram, and second (MKS) as its fundamental units. When the ampere is included as a fourth base unit, the system is sometimes called the MKSA system. In 1960, the system was expanded by adding the Kelvin and candela as base units, creating the International System of Units (SI). Later, in 1971, the mole was introduced as the seventh base unit.

I.2.3. CGS system of units

The centimeter–gram–second system of units (abbreviated CGS or cgs) is a version of the metric system that uses the centimeter as the unit of length, the gram as the unit of mass, and the second as the unit of time.

I.3. Derived units

Derived units are associated with derived quantities; for example, velocity is a quantity that is derived from the base quantities of time and length, and thus the SI derived unit is meter per second (symbol m/s).

Table. I. 2. SI derived units with special names and symbols.

Name	Symbol	Quantity	In SI base units	In other SI units
radian	rad	plane angle	m/m	1
hertz	Hz	frequency	s^{-1}	
newton	N	force, weight	$kg \cdot m \cdot s^{-2}$	
pascal	Pa	pressure, stress	$kg \cdot m^{-1} \cdot s^{-2}$	$N/m^2 = J/m^3$
joule	J	energy, work, heat	$kg \cdot m^2 \cdot s^{-2}$	$N \cdot m = Pa \cdot m^3$
watt	W	power, radiant flux	$kg \cdot m^2 \cdot s^{-3}$	J/s
coulomb	C	electric charge	$s \cdot A$	
volt	V	electric potential, voltage, emf	$kg \cdot m^2 \cdot s^{-3} \cdot A^{-1}$	W/A = J/C
farad	F	capacitance	$kg^{-1} \cdot m^{-2} \cdot s^4 \cdot A^2$	$C/V = C^2/J$
ohm	Ω	resistance, impedance, reactance	$kg \cdot m^2 \cdot s^{-3} \cdot A^{-2}$	$V/A = J \cdot s/C^2$
siemens	S	electrical conductance	$kg^{-1} \cdot m^{-2} \cdot s^3 \cdot A^2$	Ω^{-1}
weber	Wb	magnetic flux	$kg \cdot m^2 \cdot s^{-2} \cdot A^{-1}$	$V \cdot s$
tesla	T	magnetic flux density	$kg \cdot s^{-2} \cdot A^{-1}$	Wb/m^2
henry	H	inductance	$kg \cdot m^2 \cdot s^{-2} \cdot A^{-2}$	Wb/A
degree Celsius	°C	temperature relative to 273.15 K	K	

I.4. Dimensional analysis

In engineering and science, dimensional analysis is the study of the relationships between various physical quantities by identifying their fundamental dimensions (such as length, mass, time, and electric current) and their units of measurement (such as meters and grams) and keeping track of these dimensions during calculations or comparisons. It can also be used to verify the consistency of scientific formulas. The units included in the system are presented in the following table:

Table. I. 3. Dimensional Analysis of Physical Quantities.

Physical Quantity	Symbol	SI Unit	Dimension
Length	(L)	meter (m)	([L])
Mass	(M)	kilogram (kg)	([M])
Time	(T)	second (s)	([T])
Electric Current	(I)	ampere (A)	([I])
Temperature	(\Theta)	kelvin (K)	([\Theta])
Luminous Intensity	(J)	candela (cd)	([J])
Amount of Substance	(N)	mole (mol)	([N])

Any derived quantity G can be expressed as a combination of the fundamental quantities (Mass M, Length L, Time T, Electric Current I, etc.) raised to certain powers:

$$G = k\, M^a L^b T^c I^d \Theta^e J^f N^g \tag{I.1}$$

Where:

k: dimensionless constant

a, b, c, d, e, f, g: dimensional exponents corresponding to each fundamental quantity

M, L, T, I, Θ, J, N : fundamental quantities for mass, length, time, electric current, temperature, luminous intensity, and amount of substance, respectively.

Simplified version (most common form)

If only M, L, T are considered:

$$G = k\, M^a L^b T^c \tag{I.2}$$

Where a, b, c are the exponents for mass, length, and time.

Example:

- **Velocity:**

$$v = \frac{\text{length}}{\text{time}} \Rightarrow [v] = [L][T]^{-1}$$

- **Force:**

$$F = \text{mass} \times \text{acceleration} = \text{mass} \times \frac{\text{length}}{\text{time}^2} \Rightarrow [F] = [M][L][T]^{-2}$$

- **Energy:**

$$E = \text{Force} \times \text{displacement} = \text{mass} \times \frac{\text{length}}{\text{time}^2} \times \text{length} \Rightarrow [E] = [M][L]^2[T]^{-2}$$

I.5 Reference System

A reference frame (or coordinate system) is essentially a "point of view" from which we observe and describe motion. It consists of:

1. An origin – a fixed point that acts as a reference for position.
2. Axes – directions along which we measure positions (usually 3 axes: X, Y, Z in 3D space).
3. A clock – to measure time for moving objects.

Motion is not absolute. The speed, direction, and even path of an object depend on the reference frame from which it is observed. For example, a passenger walking inside a moving train has a speed relative to the train but a different speed relative to the ground.

I.5.1 Copernican Reference Frame

The Copernican reference frame is centered at the barycenter (center of mass) of the solar system. Its axes point toward very distant stars, which appear fixed in the sky (the "fixed stars"). This frame is almost ideal for describing motion of planets, satellites, and celestial bodies because it is effectively non-accelerating on the scale of the solar system.

Example:

The motion of Earth around the Sun is described more simply using the Copernican frame rather than using a point on Earth as reference.

I.5.2 Galilean Reference Frame

A Galilean reference frame moves with uniform straight-line motion relative to the Copernican frame. It is often used for everyday physics problems because most objects on Earth move relatively slowly compared to the speed of light, and gravity dominates over other effects. Objects obey Newton's laws in a Galilean frame. Any frame moving at constant velocity relative to another Galilean frame is also a Galilean frame (principle of relativity in classical mechanics).

Cartesian Reference Frame (within a Galilean Frame), this is the most common way to define coordinates (see Fig.I.1). It consists of:

1. Origin O– the reference point.
2. Three orthonormal vectors $(\vec{\imath}, \vec{\jmath}, \vec{k})$ – mutually perpendicular unit vectors that define the directions OX, OY, and OZ.

Why orthonormal vectors?

- They simplify calculations, because the axes are independent and distances can be measured directly using Pythagoras' theorem.

Example in practice:

If an object has position $\vec{r} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$, we can easily find its displacement, velocity, or acceleration along each axis.

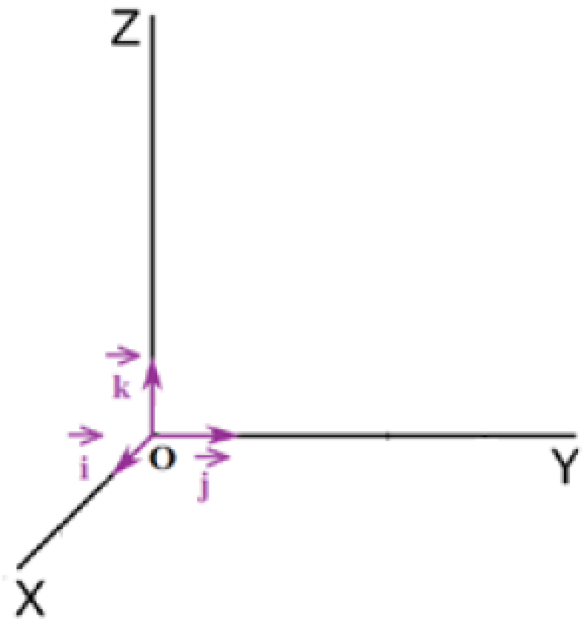


Fig.I.1. Cartesian reference frame.

I.6. Material point

In physics, a point generally refers to an object that is considered to have negligible size and no shape, and it is represented by a single position or coordinate in space. This concept is fundamental in geometry and is commonly used to describe the position of an object in three-dimensional space using three coordinates (x, y, z) (see Fig.I.2)

- x: abscissa (coordinate along the X-axis)
- y: ordinate (coordinate along the Y-axis)
- z: applicate (coordinate along the Z-axis)

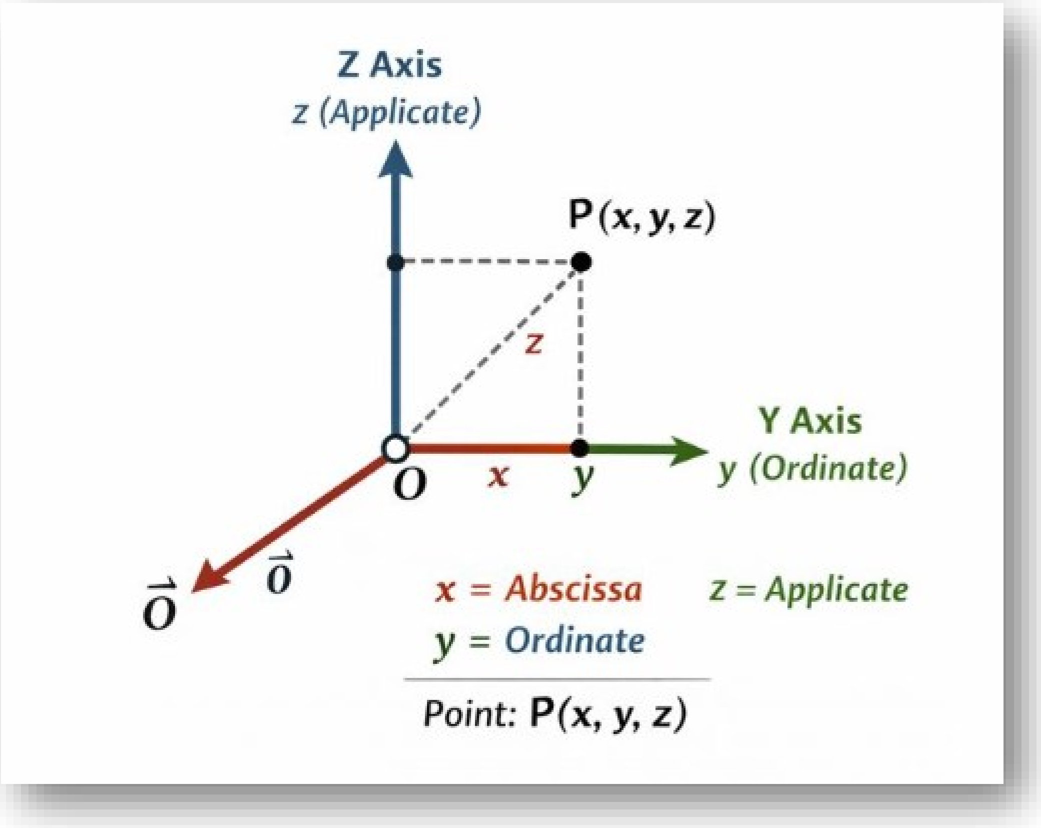


Fig.I.2. Three coordinates (x, y, z).

I.7. Vector

A vector ($\overrightarrow{AB}$) is represented by a directed segment (an arrow) with a starting point (tail) (A) and an end point (head or tip) (B).

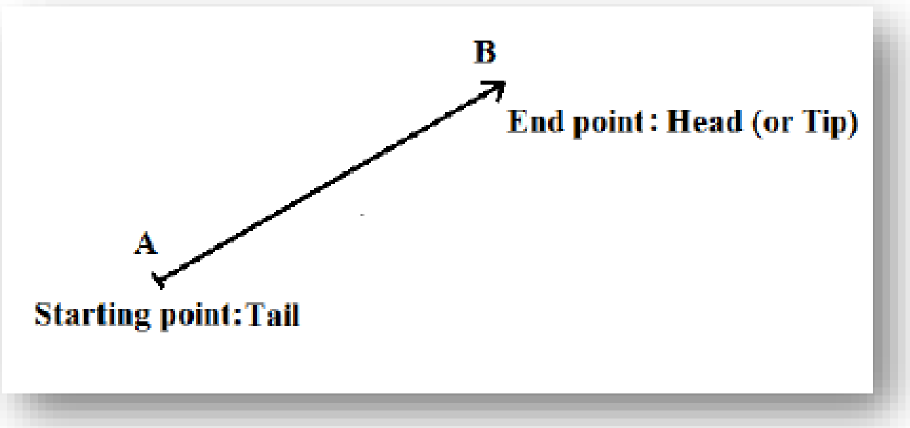


Fig.I.3. vector $\overrightarrow{AB}$

A vector is defined by:

- a length (a magnitude): $\|\overrightarrow{AB}\|$,

- a support: the straight line (AB),

- a direction (orientation): from A to B.

We can define a vector as follows:

$$\overrightarrow{AB} = \|\overrightarrow{AB}\|\vec{u}$$

is the unit vector with norm 1 ($\|\vec{u}\|$ =1) and the same direction as $\overrightarrow{AB}$

Noted that:

$$\overrightarrow{AB} = (x_b - x_a)\vec{\imath}+(y_b - y_a)\vec{\jmath} + (z_b - z_a)\vec{k} \tag{I.3}$$

$$(\vec{A}=\begin{pmatrix} x_a \\ y_a \\ z_a \end{pmatrix} , \vec{B}=\begin{pmatrix} x_B \\ y_B \\ z_B \end{pmatrix})$$

$$\|\overrightarrow{AB}\| = AB = \left(\sqrt{(x_b - x_a)^2 + (y_b - y_a)^2 + (z_b - z_a)^2}\right) \tag{I.4}$$

Remarks:

1- The vector $\overrightarrow{BA}$ is the opposite vector to the $\overrightarrow{AB}$ (same length, same support, but opposite direction)

2- The vector $\overrightarrow{AA}$ and $\overrightarrow{BB}$ is the zero vectors.

3- The vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are equal or equipollent if they have the same length, the same direction, the same support or parallel supports.

4- Colinear (or linearly dependent) vectors are vectors carried by parallel lines.

I.7.1 Types of vectors

I.7.1.1 Free vector

It is a vector with a non-specific support. In physics and mathematics, a free vector is a vector that is defined only by its magnitude and direction, but its position in space is not fixed.

In other words, a free vector can be slid anywhere in space without changing its properties.

Free vectors:

- Represented by an arrow showing direction and length proportional to its magnitude.
- Only the magnitude and direction matter; the initial point is not fixed.
- Useful for describing quantities like force, velocity, and displacement, where the exact position of the vector is not important.

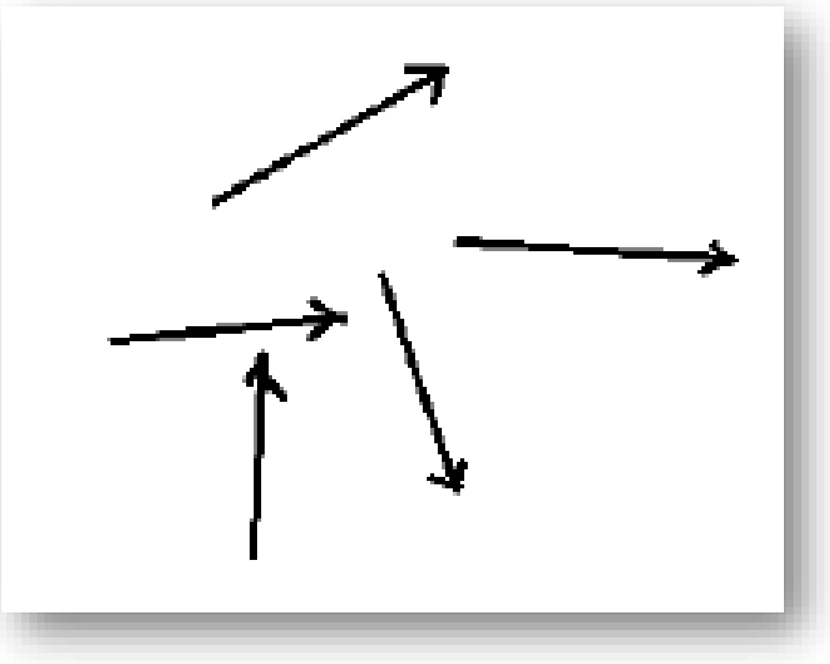


Fig.I.3. Free vector.

I.7.2 Sliding vector

A vector is called a "sliding vector" if we impose its support (Δ). Example: The vectors $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are representatives of the sliding vector $\vec{V}$.

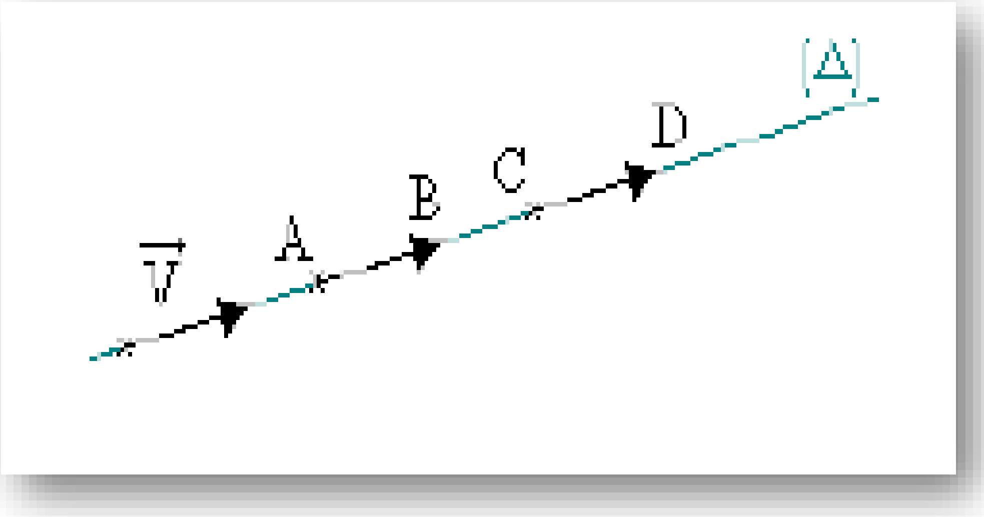


Fig.I.4. Sliding vector.

I.7.3 Linked vector

A vector is called a "linked vector" if we fix the point of application A. The position of the vector is completely defined on the support (Δ).

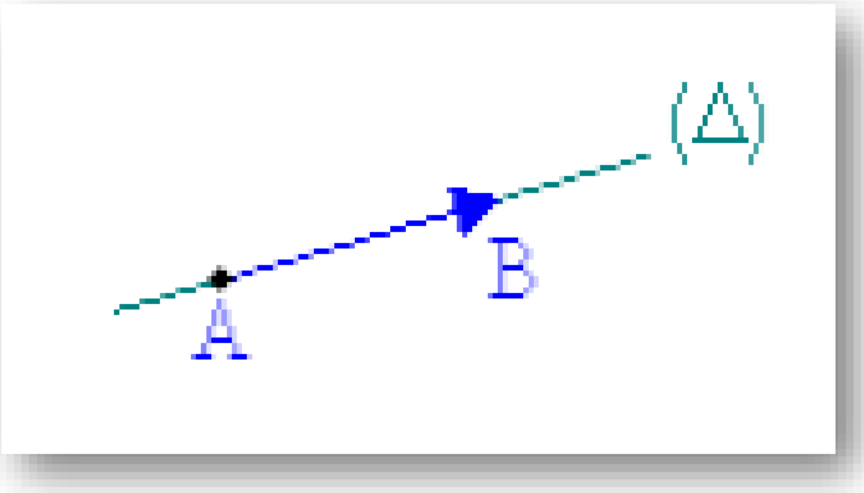


Fig.I.5. Linked vector.

I.7.4. Vector operations

I.7.4.1 Addition of two vectors

Vector addition is the process of combining two or more vectors to produce a resultant vector. The resultant vector represents the combined effect of the original vectors.

Methods of Vector Addition

- Graphical Method (Tip-to-Tail Rule)
  - Place the tail of the second vector at the tip of the first vector.
  - The resultant vector is drawn from the tail of the first vector to the tip of the second vector.
  - Works for any two vectors in 2D or 3D space.

Example:

If $\vec{A}$and $\vec{B}$are two vectors:

$$\vec{R} = \vec{A} + \vec{B} \tag{I.5}$$

where $\vec{R}$ is the resultant vector.

- Parallelogram Method
  - Place the tails of both vectors at the same point.
  - Complete the parallelogram using the two vectors as adjacent sides.
  - The diagonal of the parallelogram from the common tail point is the resultant vector.

- Analytical (Component) Method
  - Resolve each vector into its components along the X, Y, and Z axes:

$$\vec{A} = A_x\vec{\imath} + A_y\vec{\jmath} + A_z\vec{k}, \quad \vec{B} = B_x\vec{\imath} + B_y\vec{\jmath} + B_z\overrightarrow{k}$$

  - Add the corresponding components:

$$\vec{R} = (A_x + B_x)\vec{\imath} + (A_y + B_y)\vec{\jmath} + (A_z + B_z)\vec{k} \tag{I.6}$$

- This gives the resultant vector in component form.

Example (2D):

- $\vec{A} = 3\hat{\imath} + 4\hat{\jmath}$
- $\vec{B} = 1\hat{\imath} + 2\hat{\jmath}$

Resultant:

$$\vec{R} = (3+1)\hat{\imath} + (4+2)\hat{\jmath} = 4\hat{\imath} + 6\hat{\jmath}$$

Magnitude of Resultant:

$$|\vec{R}| = \sqrt{4^2 + 6^2} = \sqrt{16 + 36} = \sqrt{52} \approx 7.21$$

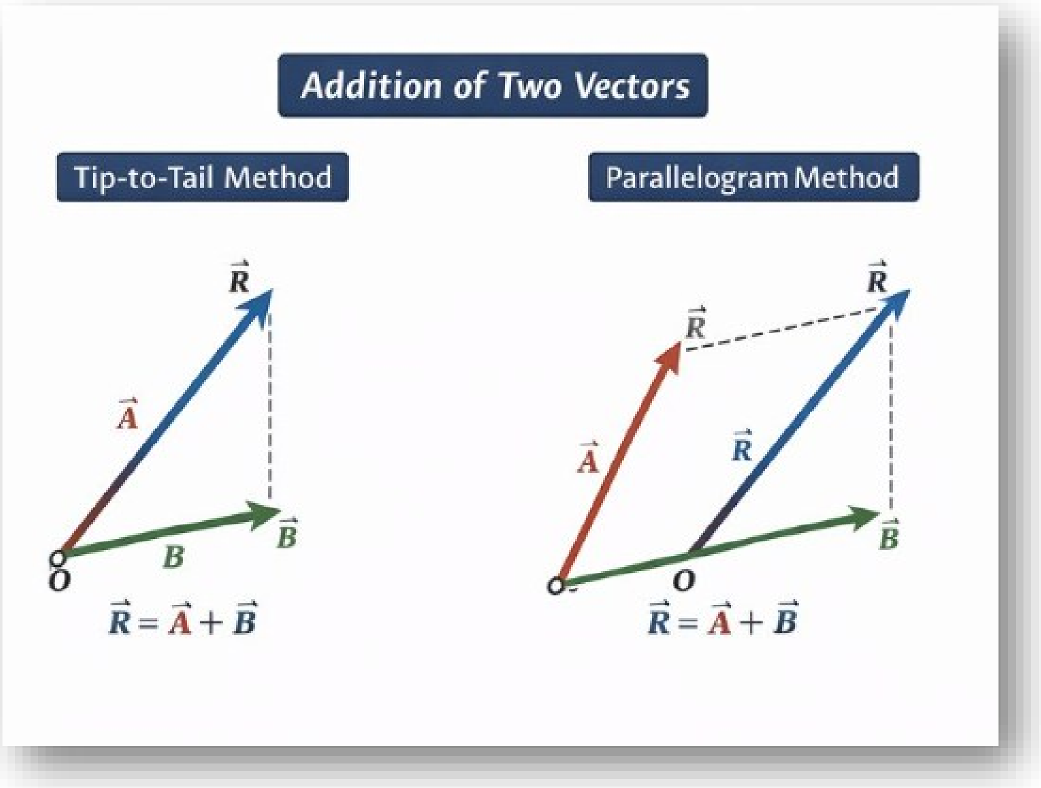


Fig.I.6. Addition of two vectors.

Commutative law:

$$\vec{A} + \vec{B} = \vec{B} + \vec{A} \tag{I.6}$$

Associative law:

$$\vec{A} + (\vec{B} + \vec{C}) = (\vec{A} + \vec{B}) + \vec{C} \tag{I.7}$$

Distributive law:

$$\lambda(\vec{A} + \vec{B}) = \lambda\vec{A} + \lambda\vec{B} \tag{I.8}$$

I.7.4.2 Subtraction of Two Vectors

Vector subtraction is essentially adding a negative vector. To subtract a vector $\vec{B}$ from a vector $\vec{A}$:

$$\vec{R} = \vec{A} - \vec{B} = \vec{A} + (-\vec{B}) \tag{I.9}$$

Where $-\vec{B}$ is a vector with the same magnitude as $\vec{B}$ but opposite direction.

Methods of Vector Subtraction

- Graphical Method (Tip-to-Tail)
  - Reverse the direction of $\vec{B}$ to get $-\vec{B}$.
  - Add $\vec{A}$ and $-\vec{B}$ using the tip-to-tail method.
  - The vector from the tail of $\vec{A}$ to the tip of $-\vec{B}$ is the resultant $\vec{R}$.

- Analytical (Component) Method
  - Resolve vectors into components:

$$\vec{A} = A_x\vec{\imath} + A_y\vec{\jmath}, \quad \vec{B} = B_x\vec{\imath} + B_y\vec{\jmath}$$

  - Subtract corresponding components:

$$\vec{R} = (A_x - B_x)\vec{\imath} + (A_y - B_y)\vec{\jmath} \tag{I.10}$$

Example 1:

$\vec{A} = 5\vec{\imath} + 3\vec{\jmath}$

$\vec{B} = 2\vec{\imath} + 1\vec{\jmath}$

$\vec{R} = \vec{A} - \vec{B} = (5-2)\vec{\imath} + (3-1)\vec{\jmath} = 3\vec{\imath} + 2\vec{\jmath}$

Example 2:

$\vec{A} = \vec{\imath} + 4\vec{\jmath}$

$\vec{B} = 3\vec{\imath} - 2\vec{\jmath}$

So:

$\vec{C} = \vec{A} - \vec{B} = -2\vec{\imath} + 6\vec{\jmath}$

Subtraction is not commutative:

$$\vec{A} - \vec{B} \neq \vec{B} - \vec{A} \tag{I.11}$$

Distributive law:

$$\lambda(\vec{A} - \vec{B}) = \lambda\vec{A} - \lambda\vec{B} \tag{I.12}$$

I.7.4.3 Scalar product (or the Dot Product)

The scalar product of two vectors $\vec{A}$ and $\vec{B}$ (denoted $\vec{A} \cdot \vec{B}$ ($\vec{A}\, scalar\, \vec{B}$)) is defined as the product of the magnitude of $\vec{A}$ and $\vec{B}$ by the cosine of the angle between the two vectors :

$$\vec{A} \cdot \vec{B} = A \cdot B \cdot cos\theta \tag{I.13}$$

Remarks

1- The product $\vec{A} \cdot \vec{B}$ is a scalar and not a vector.

2- The scalar product satisfies the following laws:

2.a-) $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A}$ (the scalar product is commutative).

2. b-) $\vec{A}\cdot(\vec{B}+\vec{C}) = \vec{A}\cdot\vec{B}+\vec{A}\cdot\vec{C}$ (the scalar product is distributive).

2.c-) $\lambda(\vec{A}\cdot\vec{B}) = (\lambda\vec{A})\cdot\vec{B} = \vec{A}\cdot(\lambda\vec{B})$, λ is a scalar value.

3- If $\vec{A}\cdot\vec{B}=0$ and if $\vec{A}$ and $\vec{B}$ are zero vectors, then are perpendicular.

4- The orthogonal unit vectors $(\vec{\imath},\vec{\jmath},\vec{k})$ which form the Cartesian basis satisfy :

$\vec{\imath}.\vec{\jmath} = \vec{\imath}.\vec{k} = \vec{k}.\vec{\jmath} = 0$ and $\vec{\imath}.\vec{\imath} = \vec{k}.\vec{k} = \vec{\jmath}.\vec{\jmath} = 1$

5- The analytical expression for the scalar product is:

$$\vec{A}\cdot\vec{B} = (x_A\vec{\imath} + y_A\vec{\jmath} + z_A\vec{k})\cdot(x_B\vec{\imath} + y_B\vec{\jmath} + z_B\vec{k}) = x_Ax_B + y_Ay_B + z_Az_B$$

I.7.4.4 The vector product (or the Cross Product)

The vector product of two vectors $\vec{A}$ and $\vec{B}$ (denoted ($\vec{A}\wedge\vec{B}$) $\vec{A}$ vectorial $\vec{B}$) is a vector . It is defined as the product of the moduli of $\vec{A}$ and $\vec{B}$ by the sine of the angle between the two vectors :

$$\vec{A}\wedge\vec{B} = (A.B.\sin\theta).\vec{u} = \vec{C} \tag{I.14}$$

Where $\vec{u}$ is a unit vector indicating the direction of $\vec{A}\wedge\vec{B}$ which is perpendicular to the plane formed by $\vec{A}$ and $\vec{B}$ Noted that $\vec{A},\vec{B}$ and $\vec{u}$ (or $\vec{C}$)form a direct trihedron (i.e., a corkscrew that which turns from $\vec{A}$ to $\vec{B}$ advances in the direction of $\vec{u}$ (or $\vec{C}$).

<u>Note that:</u>

1- If $\vec{A}$ and $\vec{B}$ are parallel, $\vec{A}\wedge\vec{B} = 0$.

2- The orthogonal unit vectors $(\vec{\imath}, \vec{\jmath}, \vec{k})$ (form a direct trihedron, they satisfy:

$\vec{\imath} \wedge \vec{\imath} = \vec{k} \wedge \vec{k} = \vec{\jmath} \wedge \vec{\jmath} = 0$

$\vec{\imath} \wedge \vec{\jmath} = \vec{k}$, $\vec{\jmath} \wedge \vec{k} = \vec{\imath}$, $\vec{k} \wedge \vec{\imath} = \vec{\jmath}$ (circular permutation)

3- The analytical expression of the vector product:

$$\vec{A} \wedge \vec{B} = \left(x_A\vec{\imath} + y_A\vec{\jmath} + z_A\vec{k}\right) \wedge \left(x_B\vec{\imath} + y_B\vec{\jmath} + z_B\vec{k}\right)$$

$$\vec{A} \wedge \vec{B} = (y_A z_B - z_A y_B)\vec{\imath} - (z_A x_B - z_B x_A)\vec{\jmath} + (x_A y_B - x_B y_A)\vec{k} \qquad (I.15)$$

Or, by using the determinant:

$$\vec{A} \wedge \vec{B} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ x_A & y_A & z_A \\ x_B & y_B & z_B \end{vmatrix}$$

$$\vec{A} \wedge \vec{B} = \begin{vmatrix} y_A & z_A \\ y_B & z_B \end{vmatrix}\vec{\imath} - \begin{vmatrix} x_A & z_A \\ x_B & z_B \end{vmatrix}\vec{\jmath} + \begin{vmatrix} x_A & y_A \\ x_B & y_B \end{vmatrix}\vec{k}$$

$$\vec{A} \wedge \vec{B} = (y_A z_B - z_A y_B)\vec{\imath} - (z_A x_B - z_B x_A)\vec{\jmath} + (x_A y_B - x_B y_A)\vec{k}$$

4-The vector product is not commutative:

$$\vec{A} \wedge \vec{B} = -\vec{B} \wedge \vec{A}. \qquad (I.16)$$

5-The vector product is distributive:

$$\vec{A} \wedge \overline{(\vec{B}+\vec{C})} = \vec{A} \wedge \vec{B} + \vec{A} \wedge \vec{C} \qquad (I.17)$$

6- The double vector product:

$$\vec{A} \wedge \overline{(\vec{B} \wedge \vec{C})} = \vec{B}.(\vec{A}\ .\vec{C}) - \vec{C}.(\vec{A}\ .\vec{B}) \qquad (I.18)$$

7- $\left|\vec{A} \wedge \vec{B}\right|$ is the area of a parallelogram with sides $\vec{A}$ and $\vec{B}$.

I.7.4.5 The mixed product

A mixed product of three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$ is defined by:

$$\vec{A}\cdot(\vec{B}\wedge\vec{C}) = \begin{vmatrix} x_A & y_A & z_A \\ x_B & y_B & z_B \\ x_c & y_c & z_c \end{vmatrix} \qquad \text{(I.19)}$$

This mixed product represents the volume of a parallelepiped with sides $\vec{A}$, $\vec{B}$ and $\vec{C}$.

We also have $\vec{A}\cdot(\vec{B}\wedge\vec{C}) = \vec{B}\cdot(\vec{C}\wedge\vec{A}) = \vec{C}\cdot(\vec{A}\wedge\vec{B})$ (circular permutation).

I.7.4.6 Derivatives of vectors

Let be $\vec{A}(t)$ the vector function in terms of time (t):

$\vec{A}(t) = x(t)\vec{\imath} + y(t)\vec{\jmath} + z(t)\vec{k}$

The first derivative of with respect to t is defined by:

$$\frac{d\vec{A}(t)}{dt} = \frac{dx(t)}{dt}\vec{\imath} + \frac{dy(t)}{dt}\vec{\jmath} + \frac{dz(t)}{dt}\vec{k} \qquad \text{(I.20)}$$

The second derivative is :

$$\frac{d^2\vec{A}(t)}{d^2t} = \frac{d}{dt}\left(\frac{d\vec{A}(t)}{dt}\right) = \frac{d^2x(t)}{d^2t}\vec{\imath} + \frac{d^2y(t)}{d^2t}\vec{\jmath} + \frac{d^2z(t)}{d^2t}\vec{k} \qquad \text{(I.21)}$$

So if (t) is a scalar function and if and are vectors

$$\frac{d(\lambda\vec{A})}{dt} = \frac{d\lambda}{dt}\vec{A} + \lambda\frac{d\vec{A}}{dt} \qquad \text{(I.22)}$$

$$\frac{d}{dt}(\vec{A}\cdot\vec{B}) = \frac{d\vec{A}}{dt}\vec{B} + \vec{A}\frac{d\vec{B}}{dt} \qquad \text{(I.23)}$$

$$\frac{d}{dt}(\vec{A}\wedge \vec{B}) = \frac{d\vec{A}}{dt}\wedge\vec{B} + \vec{A}\wedge\frac{d\vec{B}}{dt} \qquad (I.24)$$

I.7.4.7 Vector integrals21

Let the vector $\vec{A}(t) = x(t)\vec{\imath} + y(t)\vec{\jmath} + z(t)\vec{k}$ which is a vector function of t. We define an integral of $\vec{A}(t)$ by:

$$\int\vec{A}(t)dt = \vec{\imath}\int x(t)dt + \vec{\jmath}\int y(t)dt + \vec{k}\int z(t)dt \qquad (I.25)$$

<u>Exercise 1</u>

Using the expressions of different forces, find the dimensions and SI units of the following quantities:

1. Viscosity coefficient $\eta$of a fluid in which a sphere of radius Rmoves with velocity v, subject to a frictional force given by:

$F = 6\pi\eta R v$

2. Elastic friction coefficient λ, given that the elastic friction force is:

$F = \lambda v$

3. Universal gravitational constant G, given that:

$F = G\frac{m_1 m_2}{r^2}$

4. Permittivity of free space ε_0, given that:

$F = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}$

5. Magnetic permeability of free space μ_0, given that:

$F = \frac{\mu_0}{2\pi}\frac{LI^2}{r}$

Then, find the expression for the speed of light in vacuum $c$in terms of the above quantities:

$c = k\, G^{\alpha}\, \varepsilon_0^{\beta}\, \mu_0^{\gamma}$

where $k$is a dimensionless constant.

Note:

- $m_1$and $m_2$are masses,
- $q_1$and $q_2$are electric charges,
- $L$and $r$are lengths,
- $I$is electric current.

Solution:

Given expressions:

1. Viscosity:

$F = 6\pi\eta R v$

2. Elastic friction coefficient:

$F = \lambda v$

3. Gravitational constant:

$F = G\frac{m_1 m_2}{r^2}$

4. Permittivity of free space:

$F = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2}$

5. Magnetic permeability:

$F = \frac{\mu_0}{2\pi}\frac{LI^2}{r}$

6. Speed of light relation:

$c = k\, G^{\alpha}\, \varepsilon_0^{\beta}\, \mu_0^{\gamma}$

We are to find the dimensions and SI units of $\eta, \lambda, G, \varepsilon_0, \mu_0$and then the exponents $\alpha, \beta, \gamma$for c.

- Find dimensions of η(viscosity)

From:

$F = 6\pi\eta R v$

- Fhas dimension: $[F] = MLT^{-2}$
- Rhas dimension: $[R] = L$
- $v$has dimension: $[v] = LT^{-1}$

$[F] = [\eta][R][v] \Rightarrow [\eta] = \frac{[F]}{[R][v]}$

$[\eta] = \frac{MLT^{-2}}{L \cdot LT^{-1}} = ML^{-1}T^{-1}$

Dimensions: $[ML^{-1}T^{-1}]$

SI unit: $kg \cdot m^{-1} \cdot s^{-1}$

- Elastic friction coefficient λ

From:

$F = \lambda v$

$[F] = MLT^{-2},\ [v] = LT^{-1}$

$[\lambda] = \frac{[F]}{[v]} = \frac{MLT^{-2}}{LT^{-1}} = MT^{-1}$

Dimensions: $[MT^{-1}]$

SI unit: $kg \cdot s^{-1}$

- Gravitational constant G

From:

$F = G\frac{m_1 m_2}{r^2} \Rightarrow [G] = \frac{[F][r^2]}{[m]^2}$

$[F] = MLT^{-2}, [r] = L, [m] = M$

$[G] = \frac{MLT^{-2} \cdot L^2}{M^2} = M^{-1}L^3T^{-2}$

Dimensions: $[M^{-1}L^3T^{-2}]$

SI unit: $m^3 \cdot kg^{-1} \cdot s^{-2}$

-: Permittivity of free space ε_0

From:

$F = \frac{1}{4\pi\varepsilon_0}\frac{q_1 q_2}{r^2} \Rightarrow [\varepsilon_0] = \frac{[q]^2}{[F][r^2]}$

- Electric charge: $[q] = IT$
- $F = MLT^{-2},\ r = L$

$[\varepsilon_0] = \frac{(IT)^2}{MLT^{-2} \cdot L^2} = M^{-1}L^{-3}T^4I^2$

Dimensions: $[M^{-1}L^{-3}T^4I^2]$

SI unit: $A^2 \cdot s^4 \cdot kg^{-1} \cdot m^{-3}$

- Magnetic permeability μ_0

From:

$F = \frac{\mu_0}{2\pi} \frac{LI^2}{r} \Rightarrow [\mu_0] = \frac{[F][r]}{[L][I^2]}$

- L= length, I= current
- $F = MLT^{-2}, r = L$

$[\mu_0] = \frac{MLT^{-2} \cdot L}{L \cdot I^2} = MLT^{-2}I^{-2}$

Dimensions: $[MLT^{-2}I^{-2}]$

SI unit: $kg \cdot m \cdot s^{-2} \cdot A^{-2}$

Step 6: Find exponents for speed of light

We want:

$c = kG^{\alpha}\varepsilon_0^{\beta}\mu_0^{\gamma}$

- $c$has dimension $[c] = LT^{-1}$
- Substituting dimensions:

$[LT^{-1}] = (M^{-1}L^3T^{-2})^{\alpha}(M^{-1}L^{-3}T^4I^2)^{\beta}(MLT^{-2}I^{-2})^{\gamma}$

Combine exponents for M, L, T, I:

1. Mass (M):

$$-\alpha - \beta + \gamma = 0$$

2. Length (L):

$$3\alpha - 3\beta + \gamma = 1$$

3. Time (T):

$$-2\alpha + 4\beta - 2\gamma = -1$$

4. Current (I):

$$2\beta - 2\gamma = 0 \Rightarrow \beta = \gamma$$

Solve the system:

From I: $\beta = \gamma$

- M: $-\alpha - \beta + \gamma = -\alpha - \beta + \beta = -\alpha = 0 \Rightarrow \alpha = 0$
- L: $3\alpha - 3\beta + \gamma = 0 - 3\beta + \beta = -2\beta = 1 \Rightarrow \beta = -\frac{1}{2}, \gamma = -\frac{1}{2}$
- T: check: $-2\alpha + 4\beta - 2\gamma = 0 + 4(-1/2) - 2(-1/2) = -2 + 1 = -1$

Final exponents for speed of light

$$\boxed{\alpha = 0, \beta = -\frac{1}{2}, \gamma = -\frac{1}{2}}$$

So the expression for c is: $\boxed{c = k\frac{1}{\sqrt{\varepsilon_0 \mu_0}}}$

Exercise 2

Let $(\vec{\imath}, \vec{\jmath}, \vec{k})$ be the unit vectors of a right-handed, orthonormal coordinate system $OXYZ$. Consider:

$$\vec{A} = 2\vec{\imath} + 3\vec{\jmath} - \vec{k}, \vec{B} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}, (x, y, z) \in \mathbb{R}$$

1. Find the relation between the variables x, y, z so that:
   - $\vec{A}$ is perpendicular to $\vec{B}$
   - $\vec{A}$ is parallel to $\vec{B}$
2. Find the values of x, y, z so that $\vec{B}$ is a unit vector in the direction of $\vec{A}$.

Solution:

We are given:

$$\vec{A} = 2\vec{\imath} + 3\vec{\jmath} - \vec{k}, \vec{B} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$$

Step 1: Condition for perpendicular vectors

Two vectors are perpendicular if their dot product is zero:

$$\vec{A} \cdot \vec{B} = 0$$

Compute the dot product:

$$\vec{A}\cdot\vec{B} = (2)(x) + (3)(y) + (-1)(z) = 0$$

$$\boxed{2x + 3y - z = 0}$$

This is the relation between x, y, z for $\vec{A} \perp \vec{B}$.

Step 2: Condition for parallel vectors

Two vectors are parallel if one is a scalar multiple of the other:

$$\vec{B} = k\vec{A} \text{for some } k \in \mathbb{R}$$

This gives component-wise equations:

$$x = 2k, y = 3k, z = -1 \cdot k = -k$$

So the relation for $\vec{A} \parallel \vec{B}$is: $\boxed{x : y : z = 2 : 3 : -1}$

Step 3: Make $\vec{B}$a unit vector in the direction of $\vec{A}$

A unit vector has magnitude 1:

$$|\vec{B}| = 1$$

If $\vec{B}$is along $\vec{A}$, we can write:

$$\vec{B} = k\vec{A} = k(2\vec{\imath} + 3\vec{\jmath} - \vec{k})$$

Magnitude:

$$|\vec{B}| = |k| \quad |\vec{A}| = 1 \Rightarrow |k| = \frac{1}{|\vec{A}|}$$

Compute $|\vec{A}|$:

$$|\vec{A}| = \sqrt{2^2 + 3^2 + (-1)^2} = \sqrt{4 + 9 + 1} = \sqrt{14}$$

$$k = \pm\frac{1}{\sqrt{14}}$$

So the unit vector $\vec{B}$ is:

$$\vec{B} = \pm\frac{1}{\sqrt{14}}(2\vec{\imath} + 3\vec{\jmath} - \vec{k})$$

Exercise 3

Calculate the first and second derivatives with respect to time for the following vectors:

$$\vec{A} = \sin^2(t)\,\vec{\imath} - \cos(2t)\,\vec{\jmath}, \vec{B} = 3e^{-2t}\,\vec{\imath} + 5t\,\vec{\jmath}$$

Then, using the properties of vector differentiation, calculate:

1. $\frac{d}{dt}(\vec{A} \cdot \vec{B})$(time derivative of the dot product)
2. $\frac{d}{dt}(\vec{A} \wedge \vec{B})$(time derivative of the cross product)

Solution:

We are given:

$$\vec{A}(t) = \sin^2(t)\,\vec{\imath} - \cos(2t)\,\vec{\jmath}, \vec{B}(t) = 3e^{-2t}\,\vec{\imath} + 5t\,\vec{\jmath}$$

Step 1: First derivatives of $\vec{A}$and $\vec{B}$

Derivative of $\vec{A}$:

$$\vec{A} = \sin^2(t)\,\vec{\imath} - \cos(2t)\,\vec{\jmath}$$

- $\frac{d}{dt}[\sin^2(t)] = 2\sin(t)\cos(t) = \sin(2t)$
- $\frac{d}{dt}[-\cos(2t)] = 2\sin(2t)$

$$\boxed{\frac{d\vec{A}}{dt} = \sin(2t)\,\vec{\imath} + 2\sin(2t)\,\vec{\jmath}}$$

Derivative of $\vec{B}$:

$$\vec{B} = 3e^{-2t}\,\vec{\imath} + 5t\,\vec{\jmath}$$

- $\frac{d}{dt}[3e^{-2t}] = -6e^{-2t}$
- $\frac{d}{dt}[5t] = 5$

$$\boxed{\frac{d\vec{B}}{dt} = -6e^{-2t}\,\vec{\imath} + 5\,\vec{\jmath}}$$

Step 2: Second derivatives

Second derivative of $\vec{A}$:

$$\frac{d^2\vec{A}}{dt^2} = \frac{d}{dt}(\sin(2t)\,\vec{\imath} + 2\sin(2t)\,\vec{\jmath})$$

- $d/dt[\sin(2t)] = 2\cos(2t)$

$$\frac{d^2\vec{A}}{dt^2} = 2\cos(2t)\,\vec{\imath} + 4\cos(2t)\,\vec{\jmath}$$

Second derivative of $\vec{B}$:

$$\frac{d^2\vec{B}}{dt^2} = \frac{d}{dt}(-6e^{-2t}\,\vec{\imath} + 5\,\vec{\jmath}) = 12e^{-2t}\,\vec{\imath} + 0\,\vec{\jmath} = 12e^{-2t}\,\vec{\imath}$$

Step 3: Time derivative of the dot product

Property:

$$\frac{d}{dt}(\vec{A}\cdot\vec{B}) = \frac{d\vec{A}}{dt}\cdot\vec{B} + \vec{A}\cdot\frac{d\vec{B}}{dt}$$

Compute $\frac{d\vec{A}}{dt}\cdot\vec{B}$:

$$\frac{d\vec{A}}{dt} = \sin(2t)\vec{\imath} + 2\sin(2t)\vec{\jmath}, \vec{B} = 3e^{-2t}\vec{\imath} + 5t\vec{\jmath}$$

Dot product:

$$(\sin(2t))(3e^{-2t}) + (2\sin(2t))(5t) = 3\sin(2t)e^{-2t} + 10t\sin(2t)$$

Compute $\vec{A}\cdot\frac{d\vec{B}}{dt}$:

$$\vec{A} = \sin^2(t)\vec{\imath} - \cos(2t)\vec{\jmath}, \frac{d\vec{B}}{dt} = -6e^{-2t}\vec{\imath} + 5\vec{\jmath}$$

Dot product:

$$(\sin^2(t))(-6e^{-2t}) + (-\cos(2t))(5) = -6\sin^2(t)e^{-2t} - 5\cos(2t)$$

Combine terms:

$$\frac{d}{dt}(\vec{A}\cdot\vec{B}) = (3\sin(2t)e^{-2t} + 10t\sin(2t)) + (-6\sin^2(t)e^{-2t} - 5\cos(2t))$$

$$\boxed{\frac{d}{dt}(\vec{A}\cdot\vec{B}) = -6\sin^2(t)e^{-2t} + 3\sin(2t)e^{-2t} + 10t\sin(2t) - 5\cos(2t)}$$

Step 4: Time derivative of the cross product

Property:

$$\frac{d}{dt}(\vec{A}\wedge\vec{B})=\frac{d\vec{A}}{dt}\wedge\vec{B}+\vec{A}\wedge\frac{d\vec{B}}{dt}$$

- Use determinants to compute cross products if needed.

$\vec{A}$and $\vec{B}$are in the i,jplane ($k=0$), so their cross product will point along k:

$$\vec{A}\wedge\vec{B}=(A_xB_y-A_yB_x)\vec{k}$$

- $A_x=\sin^2(t), A_y=-\cos(2t)$
- $B_x=3e^{-2t}, B_y=5t$

$$\vec{A}\wedge\vec{B}=(\sin^2(t)\cdot 5t-(-\cos(2t)\cdot 3e^{-2t}))\vec{k}=(5t\sin^2(t)+3\cos(2t)e^{-2t})\vec{k}$$

Derivative with respect to t:

$$\frac{d}{dt}(\vec{A}\wedge\vec{B})=\left[\frac{d}{dt}(5t\sin^2(t)+3\cos(2t)e^{-2t})\right]\vec{k}$$

- $d/dt[5t\sin^2(t)]=5\sin^2(t)+10t\sin(t)\cos(t)=5\sin^2(t)+5t\sin(2t)$
- $d/dt[3\cos(2t)e^{-2t}]=3[-2\sin(2t)e^{-2t}+\cos(2t)(-2e^{-2t})]=$ $-6e^{-2t}(\sin(2t)+\cos(2t))$

$$\frac{d}{dt}(\vec{A}\wedge\vec{B})=[5\sin^2(t)+5t\sin(2t)-6e^{-2t}(\sin(2t)+\cos(2t))]\vec{k}$$

Final Answers:

First derivatives:

$$\frac{d\vec{A}}{dt}=\sin(2t)\vec{\imath}+2\sin(2t)\vec{\jmath},\frac{d\vec{B}}{dt}=-6e^{-2t}\vec{\imath}+5\vec{\jmath}$$

Second derivatives:

$$\frac{d^2\vec{A}}{dt^2}=2\cos(2t)\vec{\imath}+4\cos(2t)\vec{\jmath},\frac{d^2\vec{B}}{dt^2}=12e^{-2t}\vec{\imath}$$

Dot product derivative:

$$\frac{d}{dt}(\vec{A}\cdot\vec{B})=-6\sin^2(t)e^{-2t}+3\sin(2t)e^{-2t}+10t\sin(2t)-5\cos(2t)$$

Cross product derivative:

$$\frac{d}{dt}(\vec{A}\wedge\vec{B})=[5\sin^2(t)+5t\sin(2t)-6e^{-2t}(\sin(2t)+\cos(2t))]\vec{k}$$

Chapter II :Kinematics

II.1 Introduction

Kinematics is a subfield of physics, developed in classical mechanics. Kinematics is the study of the motion of objects without reference to the forces that caused the motion.

II.2 characteristics of the motion

In kinematics, the two fundamental concepts are space and time, because the motion takes place in space as a function of time. Mathematically solving kinematics problems in physics will involve understanding, calculating, and measuring several physical quantities:

- **Position vector** ($\overrightarrow{\boldsymbol{OM}}$)**:** determines the object's physical location in space relative to an origin in a defined coordinate system.

- **Velocity vector** ($\vec{\boldsymbol{V}}$)**:** which determines the variation in magnitude and position of the position vector.

- **Acceleration vector** ($\vec{\boldsymbol{a}}$)**:** which determines the variation in magnitude and position of the velocity vector.

II.2.1. Position vector

The position of an object $\overrightarrow{\boldsymbol{OM}}$ is given by its displacement relative to O. It changes with time (Fig.II.1).

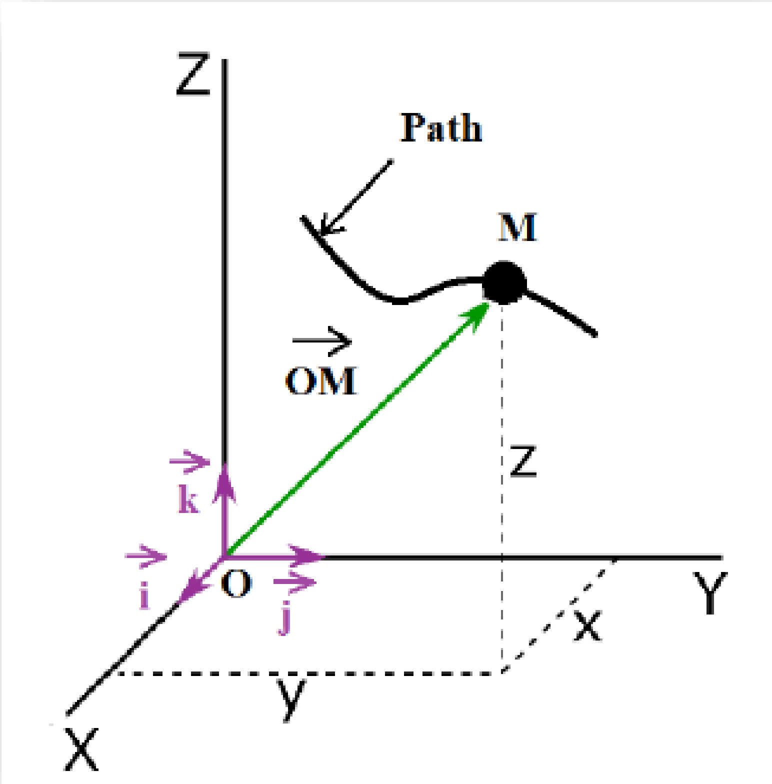


Fig.II.1. The position of an object $\overrightarrow{OM}$.

$\vec{\imath}, \vec{\jmath}$ *and* $\vec{k}$: unit vectors.

x, y and z: point coordinates.

Path of motion

The path followed by the object is the set of successive positions or line along which point P moves in space.

Parametric equation of the path: After removing time, we get the relations between the x, y, z coordinates.

II.2.2. Velocity vector

Velocity vector is vector quantity that characterizes the rate of change in the position of a body in space ([V] =m/s). The direction of velocity is the same as the direction of motion.

II.2.2.1. Average velocity vector

The average velocity vector $\vec{V}$ between M_1 and M_2 (or between two times t_1 and t_2) is defined as the ratio of the displacement $\Delta\overrightarrow{OM} = \overrightarrow{OM_2} - \overrightarrow{OM_1}$ to the time interval $\Delta t = t_2 - t_1$. That is:

$$\overrightarrow{V_{avg}} = \frac{\overrightarrow{OM_2} - \overrightarrow{OM_1}}{t_2 - t_1} = \frac{\overrightarrow{M_1M_2}}{t_2 - t_1} = \frac{\Delta\overrightarrow{OM}}{\Delta t} \quad \text{(II.1)}$$

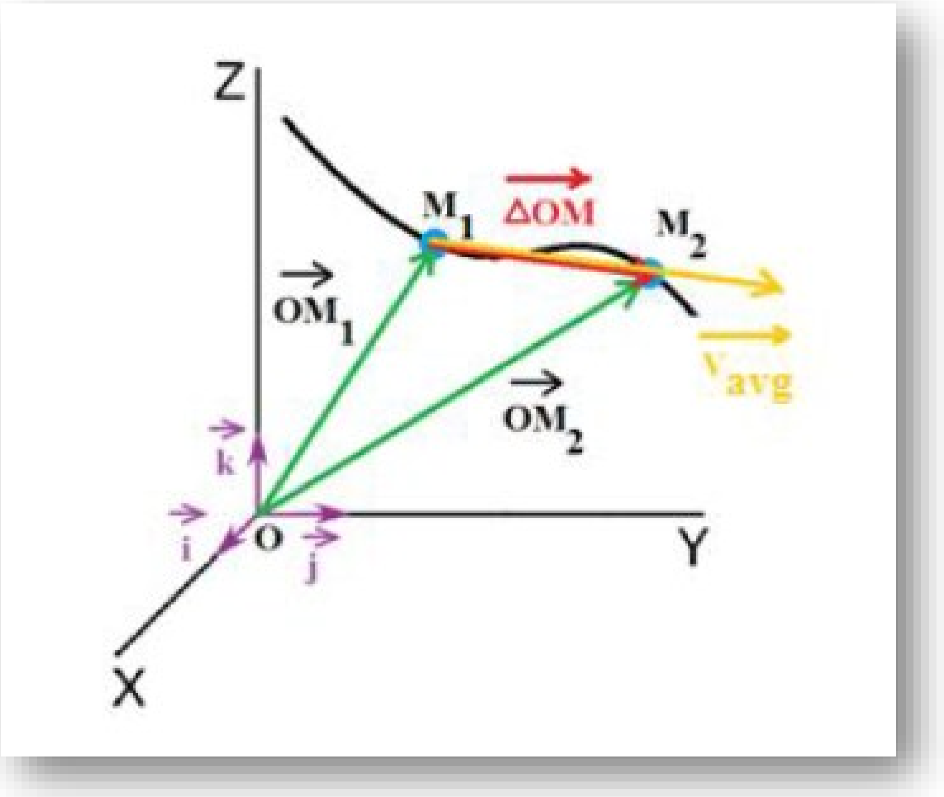


Fig.II.2. The average velocity vector.

II.2.2.2. Instantaneous velocity vector

The instantaneous velocity $\vec{V}$ is defined as the limiting value of the ratio $\frac{\Delta\overrightarrow{OM}}{\Delta t}$ as Δt approaches zero. Mathematically, $\vec{V}$ can be expressed as:

$$\vec{V} = \lim_{\Delta t \to 0} \frac{\Delta \overrightarrow{OM}}{\Delta t} = \frac{d\overrightarrow{OM}}{dt} \tag{II.2}$$

II.2.3 Acceleration vector

The acceleration vector is vector quantity that characterizes the variation of the velocity vector with respect to time

II.2.3.1. Average acceleration vector

The average acceleration $\overrightarrow{a_{avg}}$ is defined as the ratio of the change in velocity $\Delta\vec{V} = \overrightarrow{V_2} - \overrightarrow{V_1}$ to the time interval $\Delta t = t_2 - t_1$.

That is:

$$\overrightarrow{a_{avg}} = \frac{\overrightarrow{V_2} - \overrightarrow{V_1}}{t_2 - t_1} = \frac{\Delta\vec{V}}{\Delta t} \tag{II.3}$$

II.2.3.2. Instantaneous acceleration vector

Instantaneous acceleration is defined as the limiting value of the ratio $\frac{\Delta\vec{V}}{\Delta t}$ when Δt approaches zero. It is defined as follows:

$$\vec{a} = \lim_{\Delta t \to 0} \frac{\Delta\vec{V}}{\Delta t} = \frac{d\vec{V}}{dt} = \frac{d^2\overrightarrow{OM}}{dt^2} \tag{II.4}$$

III.3. Motions in various coordinate systems and bases

In mechanics, before studying the motion of a system, it is necessary to indicate the coordinate system in which the motion will be describe. We will explain the motion in different coordinate systems and bases, i.e. the set of three vectors on which we will give the expressions of the position vector, velocity vector and acceleration vector. The elementary surface area and volume will also be given**.**

III.3.1. Cartesian coordinate system

The Cartesian coordinates system is orthonormal and it consists of three axes (OX, OY, OZ). The directions of (OX, OY, OZ) are determined by three unit vectors $(\vec{\imath}, \vec{\jmath}, \vec{k})$ which are fixed in the observation frame of reference (neither the norm, nor the support, nor the direction of these vectors change with time).

Each point M is marked by its coordinates (x, y, z) in the base $(\vec{\imath}, \vec{\jmath}, \vec{k})$.

<u>The position vector</u>: is defined by the origin point O and the coordinates (x,y,z):

$$\overrightarrow{OM} = \vec{r}(t) = x\vec{\imath} + y\vec{\jmath} + z\vec{\jmath}$$

$$\overrightarrow{OM} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$\| \overrightarrow{OM} \| = \sqrt{x^2 + y^2 + z^2}$$

$\vec{u}$ is the unit vector $\vec{u} = \frac{\overrightarrow{OM}}{\|\overrightarrow{OM}\|}$

- The elementary displacement of M:

$$d\overrightarrow{OM} = dx\vec{\imath} + dy\vec{\jmath} + dz\vec{k} \qquad \text{(II.6)}$$

- The elementary surface:

$$ds = dxdy \ or: ds = dydz, or: ds = dxdz \qquad \text{(II.7)}$$

- The elementary volume:

$$dV = dxdydz. \qquad \text{(II.8)}$$

<u>The velocity vector:</u>

$$\vec{v} = \frac{d\overrightarrow{OM}}{dt} = \frac{dx}{dt}\vec{\imath} + \frac{dy}{dt}\vec{\jmath} + \frac{dz}{dt}\vec{k}$$

$$\vec{v} = \dot{x}\vec{\imath} + \dot{y}\vec{\jmath} + \dot{z}\vec{k} \qquad \text{(II.9)}$$

$$\| \vec{v} \| = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}$$

<u>The acceleration vector:</u>

$$\vec{a} = \frac{d^2\overrightarrow{OM}}{dt^2} = \frac{d^2x}{dt^2}\vec{\imath} + \frac{d^2y}{dt^2}\vec{\jmath} + \frac{d^2z}{dt^2}\vec{k}$$

$\vec{a} = \ddot{x}\vec{\imath} + \ddot{y}\vec{\jmath} + \ddot{z}\vec{k}$ (II.10)

$$\| \vec{a} \| = \sqrt{\ddot{x}^2 + \ddot{y}^2 + \ddot{z}^2}$$

- The elementary displacement of M:

$d\overrightarrow{OM} = dx\vec{\imath} + dy\vec{\jmath} + dz\vec{k}$ (II.11)

- The elementary surface:

$ds = dxdy \; or: ds = dydz, or: ds = dxdz$ (II.12)

- The elementary volume:

$dV = dxdydz.$ (II.13)

II.3.2. Polar coordinates system

The polar reference frame is orthonormal. It consists of two unit vectors $(\overrightarrow{u_\rho}, \overrightarrow{u_\theta})$ which move with time.

Each point M is identified by its coordinates (ρ, θ) in the base $(\overrightarrow{u_\rho}, \overrightarrow{u_\theta})$,((Fig.II. 3).

θ is the angle between $\overrightarrow{OM}$ and $\vec{\imath}$.

$\overrightarrow{u_\rho}$ is perpendicular to $\overrightarrow{u_\theta}$.

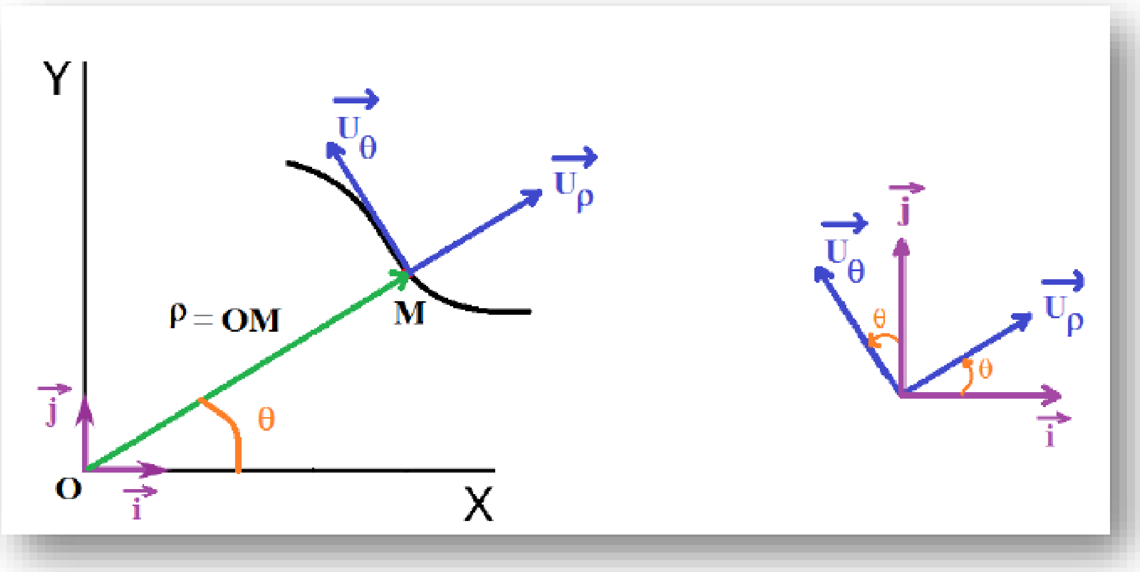


Fig.II.3. The polar reference frame.

Each point M is identified by its coordinates (ρ, θ) in the base $(\overrightarrow{u_\rho}\,,\ \overrightarrow{u_\theta})$

$$\overrightarrow{OM} = \vec{r} = \rho\overrightarrow{u_\rho}$$

$$\overrightarrow{OM} = \binom{\rho}{\theta}$$

$$\| \overrightarrow{OM} \| = \rho$$

In the Cartesian reference frame:

The position vector:

$$\overrightarrow{OM} = \vec{r} = x\vec{\imath} + y\vec{\jmath}$$

So we have:

$$\begin{cases} x = \rho cos\theta \\ y = \rho sin\theta \end{cases}$$

$$\rho = \sqrt{x^2 + y^2}$$

$$arctg\theta = \frac{y}{x}$$

$$\begin{cases} \overrightarrow{u_\rho} = (\cos\theta\,\vec{\imath} + \sin\theta\,\vec{\jmath}\,) \\ \overrightarrow{u_\theta} = (-\sin\theta\ \vec{\imath} + \cos\theta\,\vec{\jmath}) \end{cases}$$

Noted that: $\frac{d\overrightarrow{u_\rho}}{dt} = \dot{\theta}\overrightarrow{u_\theta}$

The velocity vector:

$$\vec{v} = \frac{d\overrightarrow{OM}}{dt} = \frac{d(\rho\overrightarrow{u_\rho})}{dt} = \frac{d\rho}{dt}\overrightarrow{u_\rho} + \rho\frac{d\theta}{dt}\overrightarrow{u_\theta}$$

$$\Rightarrow \vec{v} = \dot{\rho}\overrightarrow{u_\rho} + \rho\dot{\theta}\overrightarrow{u_\theta} \qquad \text{(II.14)}$$

$$\| \vec{v} \| = \sqrt{\dot{\rho}^2 + (\rho\dot{\theta})^2}$$

- The acceleration vector:

$$\vec{a} = \frac{d^2\overrightarrow{OM}}{dt^2} = \frac{dv}{dt} = \frac{d(\dot{\rho}\overrightarrow{u_\rho} + \rho\dot{\theta}\overrightarrow{u_\theta})}{dt}$$

$$\vec{a} = \frac{d\dot{\rho}}{dt}\overrightarrow{u_\rho} + \dot{\rho}d\overrightarrow{u_\rho} + \left(\frac{d\rho}{dt}\right)\dot{\theta}\overrightarrow{u_\theta} + \rho\frac{d\dot{\theta}}{dt}\overrightarrow{u_\theta} + \rho\dot{\theta}\frac{d(\overrightarrow{u_\theta})}{dt}$$

$$\vec{a} = \ddot{\rho}\overrightarrow{u_\rho} + \dot{\rho}\dot{\theta}\overrightarrow{u_\theta} + \dot{\rho}\dot{\theta}\overrightarrow{u_\theta} + \rho\ddot{\theta}\overrightarrow{u_\theta} - \rho\dot{\theta}^2\overrightarrow{u_\rho}$$

$$\vec{a} = (\rho^{\cdot\cdot} - \rho\theta^{\cdot 2})\overrightarrow{u_\rho} + \left(2\dot{\rho}\dot{\theta} + \rho\ddot{\theta}\right)\overrightarrow{u_\theta} \quad \text{(II.15)}$$

$$\| \vec{a} \| = \sqrt{(\rho^{\cdot\cdot} - \rho\theta^{\cdot 2})^2 + \left(2\dot{\rho}\dot{\theta} + \rho\ddot{\theta}\right)^2}$$

The elementary displacement of M:

$$\mathrm{d}\overrightarrow{OM} = d\rho\overrightarrow{u_\rho} + \rho d\theta\overrightarrow{u_\theta} \quad \text{(II.16)}$$

The elementary surface:

$$ds = \rho d\rho d\theta \quad \text{(II.17)}$$

II.3.3. Cylindrical coordinates system

The cylindrical reference frame is orthonormal. It consists of three unit vectors $(\overrightarrow{u_\rho},\ \overrightarrow{u_\theta},\vec{k})$ which the two vectors $(\overrightarrow{u_\rho},\ \overrightarrow{u_\theta})$ varies with time while $\vec{k}$ is invariable (Fig.II.4)

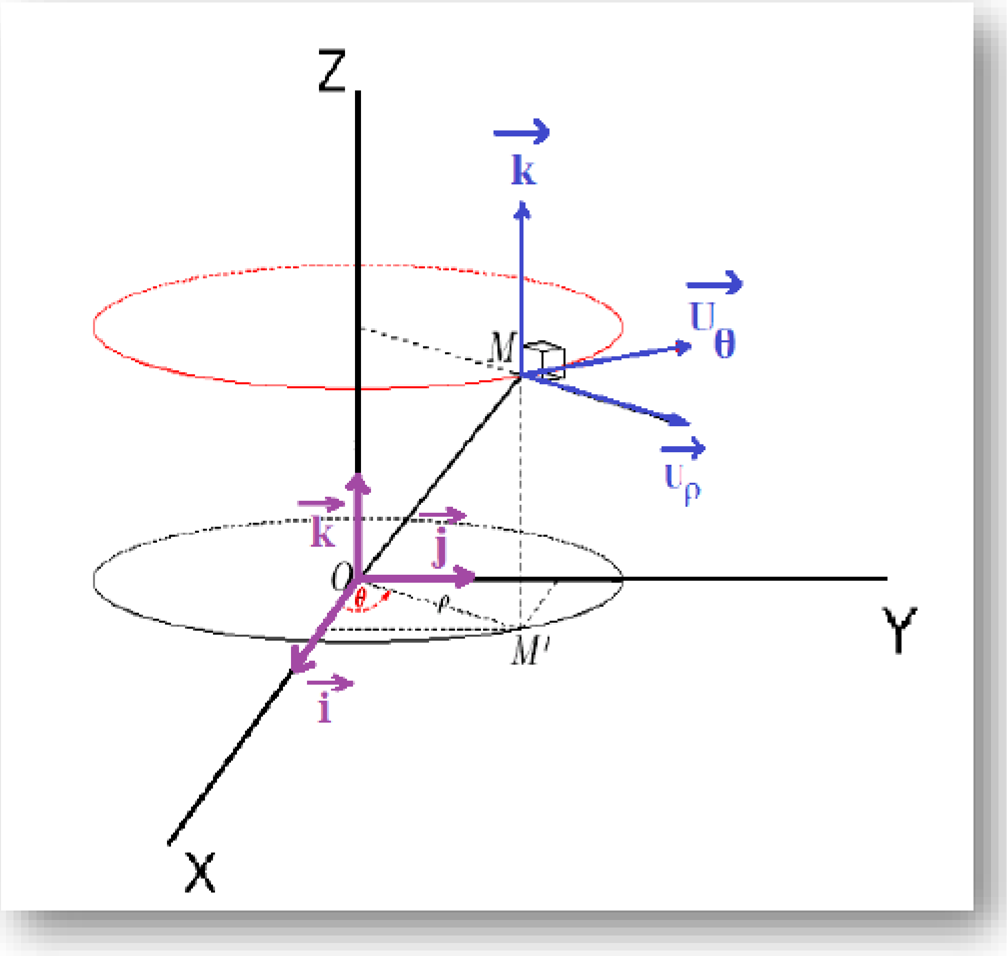


Fig.II.4. Cylindrical coordinates system.

Each point M is identified by its coordinates (ρ, θ) in the base $(\overrightarrow{u_\rho}\,,\ \overrightarrow{u_\theta},\vec{k})$ (Fig. 4).

θ is the angle between $\overrightarrow{OM'}$ and $\vec{\imath}$, M' is the projection of M in the plane (xoy).

<u>The position vector</u>:

The position vector is defined from the origin point O and the coordinates (ρ, θ, z):

$$\overrightarrow{OM} = \overrightarrow{OM'} + \overrightarrow{M'M} = \rho\overrightarrow{u_\rho} + z\vec{k}$$

$$\overrightarrow{OM} = \begin{pmatrix} \rho \\ \theta \\ z \end{pmatrix}$$

$$\| \overrightarrow{OM} \| = \sqrt{\rho^2 + z^2}$$

In the Cartesian reference frame:

$+\overrightarrow{OM} = \vec{r} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$

So we have:

$$\begin{cases} x = \rho cos\theta \\ y = \rho sin\theta \\ z = z \end{cases}$$

$$\| \overrightarrow{OM} \| = \sqrt{\rho^2 + z^2}$$

$$\rho = \sqrt{x^2 + y^2}$$

$arctg\theta = \frac{y}{x}$

$$\begin{cases} \overrightarrow{u_\rho} = (\cos\theta\,\vec{\imath} + \sin\theta\,\vec{\jmath}\,) \\ \overrightarrow{u_\theta} = (-\sin\theta\ \vec{\imath} + \cos\theta\,\vec{\jmath}) \\ \vec{k} = \vec{k} \end{cases}$$

The velocity vector:

$\vec{v} = \frac{d\overrightarrow{OM}}{dt} = \frac{d(\rho\overrightarrow{u_\rho} + z\vec{k})}{dt} = \frac{d\rho}{dt}\overrightarrow{u_\rho} + \rho\frac{d\theta}{dt}\overrightarrow{u_\theta} + \frac{dz}{dt}\vec{k}$

$\Rightarrow \vec{v} = \dot{\rho}\overrightarrow{u_\rho} + \rho\dot{\theta}\overrightarrow{u_\theta} + \dot{z}\vec{k}$ (II.18)

$$\| \vec{v} \| = \sqrt{\dot{\rho}^2 + (\rho\dot{\theta})^2 + \dot{z}^2}$$

The acceleration vector:

$\vec{a} = \frac{d^2\overrightarrow{OM}}{dt^2} = \frac{dv}{dt} = \frac{d(\dot{\rho}\overrightarrow{u_\rho} + \rho\dot{\theta}\overrightarrow{u_\theta} + \dot{z}\vec{k})}{dt}$

$\vec{a} = \frac{d\dot{\rho}}{dt}\overrightarrow{u_\rho} + \dot{\rho}d\overrightarrow{u_\rho} + \left(\frac{d\rho}{dt}\right)\dot{\theta}\overrightarrow{u_\theta} + \rho\frac{d\dot{\theta}}{dt}\overrightarrow{u_\theta} + \rho\dot{\theta}\frac{d(\overrightarrow{u_\theta})}{dt} + \ddot{z}\vec{k}$

$\vec{a} = \ddot{\rho}\overrightarrow{u_\rho} + \dot{\rho}\dot{\theta}\overrightarrow{u_\theta} + \dot{\rho}\dot{\theta}\overrightarrow{u_\theta} + \rho\ddot{\theta}\overrightarrow{u_\theta} - \rho\dot{\theta}^2\overrightarrow{u_\rho} + \ddot{z}\vec{k}$

$\vec{a} = (\rho^{\cdot\cdot} - \rho\theta^{\cdot 2})\overrightarrow{u_\rho} + (2\dot{\rho}\dot{\theta} + \rho\ddot{\theta})\overrightarrow{u_\theta} + \ddot{z}\vec{k}$ (II.19)

$$\| \vec{a} \| = \sqrt{(\ddot{\rho} - \rho\dot{\theta}^2)^2 + (2\dot{\rho}\dot{\theta} + \rho\ddot{\theta})^2 + \ddot{z}^2}$$

-The elementary displacement of M:

$$d\overrightarrow{OM} = d\rho\overrightarrow{u_\rho} + \rho d\theta\overrightarrow{u_\theta} + dz\vec{k} \quad \text{(II.20)}$$

-The elementary surface:

$$ds = \rho d\theta dz \quad \text{(II.21)}$$

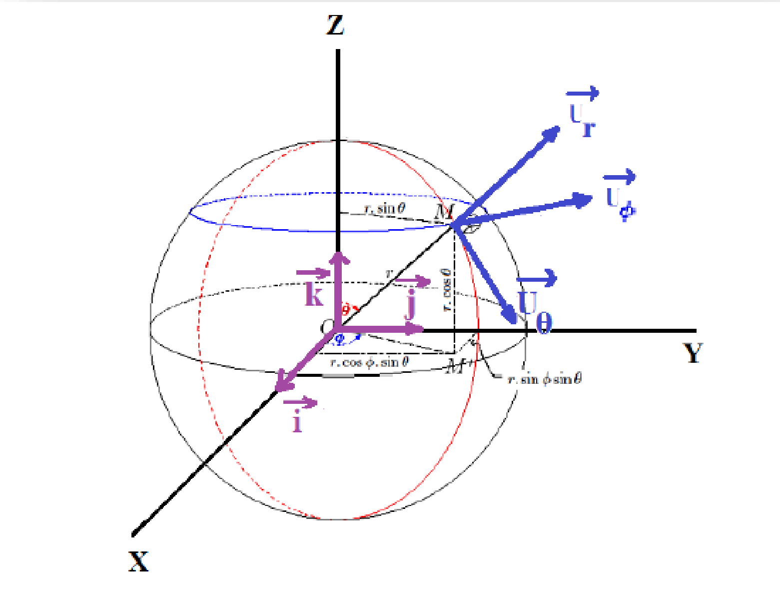
- The elementary volume:

$$dV = \rho d\rho d\theta dz \quad \text{(II.22)}$$

II.4. Spherical coordinates system

The spherical reference frame is orthonormal. It consists of three unit vectors $(\overrightarrow{u_r},\ \overrightarrow{u_\theta},\overrightarrow{u_\varphi})$ which vary with time.

Each point M is identified by its coordinates (r, θ,φ) in the base $(\overrightarrow{u_r},\ \overrightarrow{u_\theta},\overrightarrow{u_\varphi})$ (Fig.II. 5).



$\theta = (\widehat{oz, oM}),\ \Phi = (\widehat{ox, oM'})$

Fig.II.5. Spherical coordinates system.

The position vector:

The position vector is defined from the origin point O and the coordinates (r, θ,φ):

$$\overrightarrow{OM} = r\overrightarrow{u_r}$$

$$\overrightarrow{OM} = \begin{pmatrix} r \\ \theta \\ \varphi \end{pmatrix}$$

$$\| \overrightarrow{OM} \| = r$$

$$\overrightarrow{OM} = \overrightarrow{OM'} + \overrightarrow{M'M}$$

$$\overrightarrow{OM'} = r'\,(cos\varphi\vec{\imath} + sin\varphi\vec{\jmath})$$

$$r' = rsin\theta$$

$$\overrightarrow{OM'} = rsin\theta(cos\varphi\vec{\imath} + sin\varphi\vec{\jmath}) = rsin\theta cos\varphi\vec{\imath} + rsin\theta sin\varphi\vec{\jmath}$$

$$\overrightarrow{M'M} = rcos\theta\vec{k}$$

So:

$$\overrightarrow{OM} = rsin\theta cos\varphi\vec{\imath} + rsin\theta sin\varphi\vec{\jmath} + rcos\theta\vec{k} \qquad \text{(II.23)}$$

In the Cartesian reference frame:

$+\overrightarrow{OM} = \vec{r} = x\vec{\imath} + y\vec{\jmath} + z\vec{k}$

So we have:

$$\begin{cases} x = rsin\theta cos\varphi \\ y = rsin\theta sin\varphi \\ z = rcos\theta \end{cases}$$

$$\overrightarrow{OM} = r\left(sin\theta cos\varphi\vec{\imath} + sin\theta sin\varphi\vec{\jmath} + cos\theta\vec{k}\right)$$

$$\overrightarrow{OM} = r\overrightarrow{u_r}$$

$$r = \sqrt{x^2 + y^2 + z^2}$$

$\overrightarrow{u_r} = sin\theta cos\varphi\vec{\imath} + sin\theta sin\varphi\vec{\jmath} + cos\theta\vec{k}$, $\overrightarrow{u_r}$ is the radial unit vector

$$\frac{d\overrightarrow{u_r}}{d\theta} = \overrightarrow{u_\theta} = cos\theta cos\varphi\vec{\imath} + cos\theta sin\varphi\vec{\jmath} - sin\theta\vec{k} \qquad \text{(II.24)}$$

$\overrightarrow{u_\theta}$ *is* the ortho-radial vector

$$\overrightarrow{u_\varphi} = -sin\varphi\vec{\imath} + cos\varphi\vec{\jmath} \tag{II.25}$$

- The velocity vector :

$$\vec{v} = \frac{d\overrightarrow{OM}}{dt} = \frac{d(r\overrightarrow{u_r})}{dt} = \frac{dr}{dt}\overrightarrow{u_r} + r\frac{d\overrightarrow{u_r}}{dt} \quad \Rightarrow$$

$$\vec{v} = \frac{dr}{dt}\overrightarrow{u_r} + r\left(\frac{d\overrightarrow{u_r}}{d\theta}\frac{d\theta}{dt} + \frac{d\overrightarrow{u_r}}{d\varphi}\frac{d\varphi}{dt}\right)$$

$$\vec{v} = \dot{r}\overrightarrow{u_r} + r\left(\dot{\theta}\overrightarrow{u_\theta} + \dot{\varphi}sin\theta\overrightarrow{u_\varphi}\right) \tag{II.26}$$

$$\parallel \vec{v} \parallel = \sqrt{\dot{r}^2 + \left(r\dot{\theta}\right)^2 + (\dot{\varphi}sin\theta)^2}$$

- The acceleration vector:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\dot{r}\overrightarrow{u_r} + r\left(\dot{\theta}\overrightarrow{u_\theta} + \dot{\varphi}sin\theta\overrightarrow{u_\varphi}\right)\right)$$

$$\vec{a} = \begin{cases} a_r = \ddot{r} - r\dot{\varphi}^2(sin\theta)^2 - r\dot{\theta}^2 \\ a_\theta = 2r\dot{\theta} + r\ddot{\theta} - r\dot{\varphi}sin\theta cos\theta \\ a_\varphi = 2\dot{r}\dot{\varphi}sin\theta + r\ddot{\varphi}sin\theta + 2r\dot{\theta}\dot{\varphi}cos\theta \end{cases} \tag{II.27}$$

$$\parallel \vec{a} \parallel = \sqrt{{a_r}^2 + {a_\theta}^2 + {a_\varphi}^2}$$

- The elementary displacement of M:

$$\mathrm{d}\overrightarrow{OM} = dr\overrightarrow{u_r} + rd\overrightarrow{u_r}$$

$$\mathrm{d}\overrightarrow{OM} = dr\overrightarrow{u_r} + r\left(d\theta\overrightarrow{u_\theta} + sin\theta d\varphi\overrightarrow{u_\varphi}\right) \tag{II.28}$$

-The elementary surface:

$$ds = r^2 sin\theta d\theta d\varphi \tag{II.29}$$

- The elementary volume:

$$dV = r^2 dr sin\theta d\theta \tag{II.30}$$

III.4. Intrinsic coordinate system (Frenet system)42

The intrinsic coordinate system for each point of the trajectory is defined as a system of reference formed by two axes $(\overrightarrow{u_T}, \overrightarrow{u_N})$ (Fig.II.6):

- ***Tangent axis***: its direction is *tangent* to the trajectory and is positive in the same direction than the velocity at that point. It is defined by the unit vector $\overrightarrow{u_T}$
- ***Normal axis***: it is *perpendicular* to the trajectory and is positive toward the center of curvature of the trajectory. It is defined by the unit vector $\overrightarrow{u_N}$

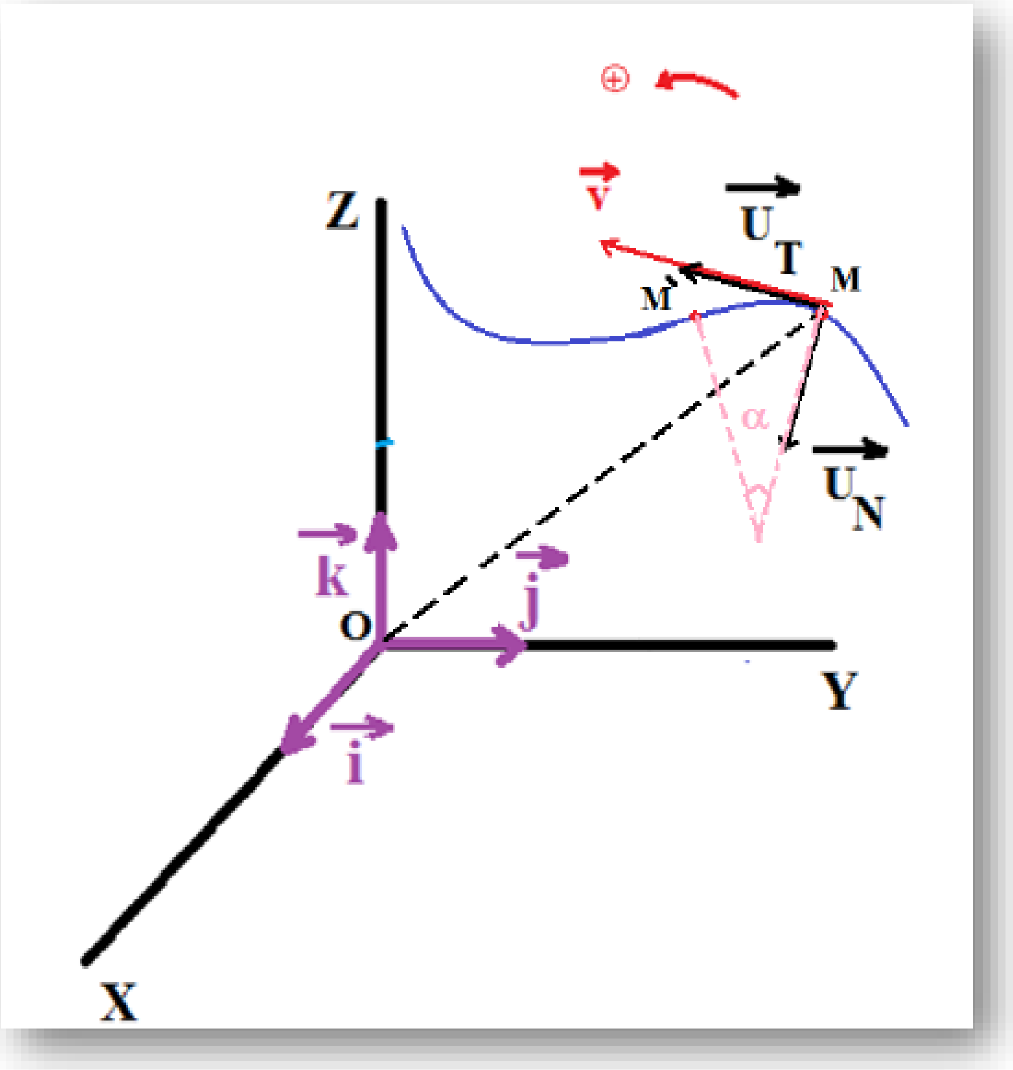


Fig.II.6. Intrinsic coordinate system (Frenet system).

Curvilinear abscissa

In this frame of reference, we define the curvilinear abscissa ***S*** of the point M along the trajectory as being equal to the length of the arc $\widehat{MM'}$. Noting that:

$\overrightarrow{u_N}$ =R$\frac{d\overrightarrow{u_T}}{ds}$ is the radius of curvature.

The velocity vector:

$$\vec{V} = V\overrightarrow{u_T} = \frac{ds}{dt} \tag{II.31}$$

The acceleration vector:

$$\vec{a} = \frac{dV}{dt} = \frac{d(V\overrightarrow{u_T})}{dt} = \frac{dV}{dt}\overrightarrow{u_T} + V\frac{d\overrightarrow{u_T}}{dt}$$

$$ds = Rd\alpha \tag{II.32}$$

$$d\overrightarrow{u_T} = \overrightarrow{u_N}\frac{ds}{R}$$

$$\overrightarrow{u_N} = R\frac{d\overrightarrow{u_T}}{ds}$$

$$\frac{d\overrightarrow{u_T}}{dt} = \frac{d\overrightarrow{u_T}}{ds}\frac{ds}{dt}$$

$$\frac{d\overrightarrow{u_T}}{dt} = \frac{d\overrightarrow{u_T}}{ds}V$$

$$\frac{d\overrightarrow{u_T}}{dt} = \frac{V}{R}\overrightarrow{u_N}$$

$$\vec{a} = \frac{dV}{dt}\overrightarrow{u_T} + V\frac{d\overrightarrow{u_T}}{dt}$$

$$\vec{a} = \frac{dV}{dt}\overrightarrow{u_T} + V\frac{V}{R}\overrightarrow{u_N}$$

$$\vec{a} = \frac{dV}{dt}\overrightarrow{u_T} + \frac{V^2}{R}\overrightarrow{u_N} \tag{II.33}$$

$$\vec{a} = a_T\overrightarrow{u_T} + a_N\overrightarrow{u_N}$$

$a_T = \frac{dV}{dt}$, *tangential acceleration*

$a_N = \frac{V^2}{R}$, normal *acceleration*

$$\| \vec{a} \| = \sqrt{a_T{}^2 + a_N{}^2}$$

II.5. Study of motions

II.5.1. Rectilinear motion

A rectilinear (Linear) motion is one-dimensional motion along a straight line. It can be described mathematically using only one spatial dimension.

II.5.1.1. Uniform rectilinear motion

It is characterized by constant velocity (zero acceleration) :

$a = 0$ and $V = V_0 = C^{te}$

$V = V_0 = \frac{dx}{dt} \Rightarrow dx = V_0 dt \Rightarrow \int_{x_0}^{x} dx = \int_{t_0}^{t} V_0 dt \Rightarrow x - x_0 = V_0 t$

$$x = x_0 + V_0 t \tag{II.34}$$

If $V = 0$, the object is stationary.

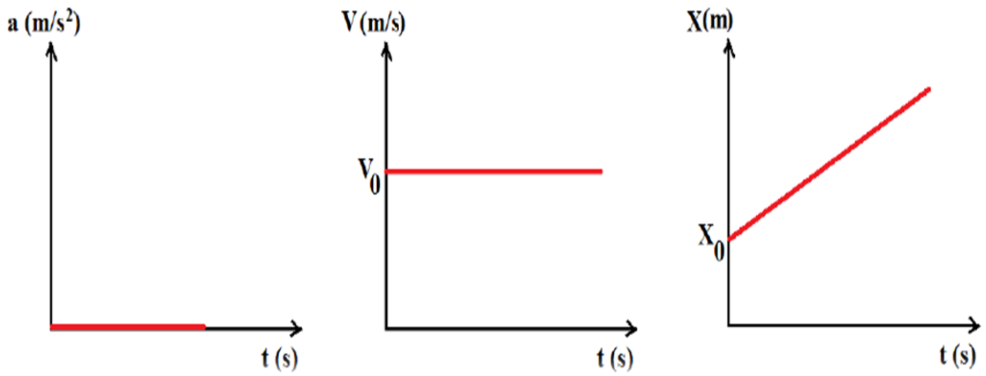


Fig.II.7. Uniform rectilinear motion

II.5.1.2. Non-uniform linear motion45

It can be uniformly accelerated (or retarded) rectilinear motion. It is characterised by variable velocity (non-zero acceleration).

$$a = a_0 = \frac{dV}{dt} \Rightarrow dV = a_0 dt \Rightarrow \int_{V_0}^{V} dV = \int_0^t a_0\, dt \Rightarrow V - V_0 = a_0 t$$

$$V = a_0 t + V_0 \tag{II.35}$$

$$V = \frac{dx}{dt} \Rightarrow dx = V dt \Rightarrow \int_{x_0}^{x} dx = \int_0^t (V_0 + a_0 t)\, dt \Rightarrow x - x_0 = V_0 t + \frac{1}{2} a_0 t^2$$

$$x = \frac{1}{2} a_0 t^2 + V_0 t + x_0 \tag{II.36}$$

(equation of a parabola)

- If acceleration and velocity are in the same direction ($\vec{V} \cdot \vec{a} > 0$) the movement is accelerated.
- If acceleration and speed are in opposite directions ($\vec{V} \cdot \vec{a} < 0$), the movement is retarded.

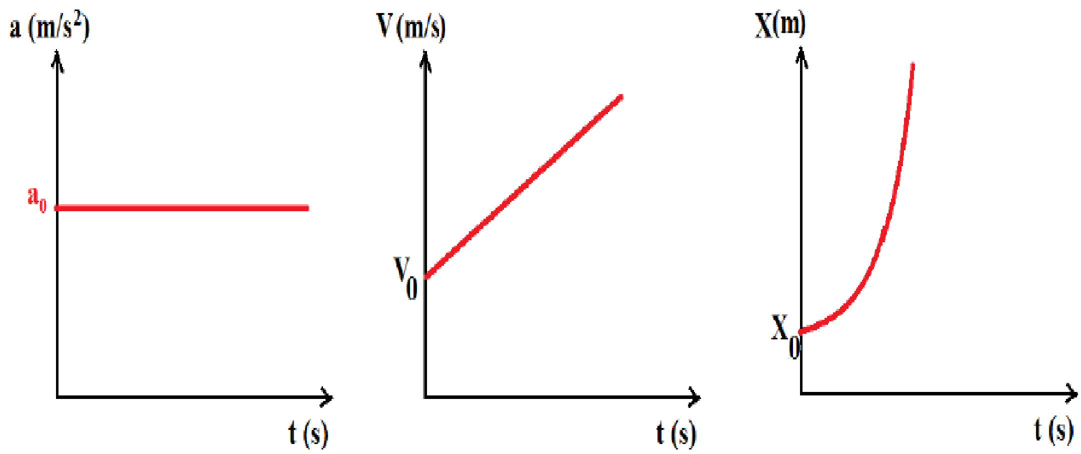


Fig.II.8. Non-uniform linear motion.

II.5.2. Projectile (2 dimensions)

Any object that is thrown into the air is called a *projectile*.

Let us assume that at $t = 0$ the projectile leaves the origin (i.e. $x_0 = y_0 = 0$) with initial velocity $\vec{V}$ that makes an angle θ_0 with the positive x direction as in Fig.II.9

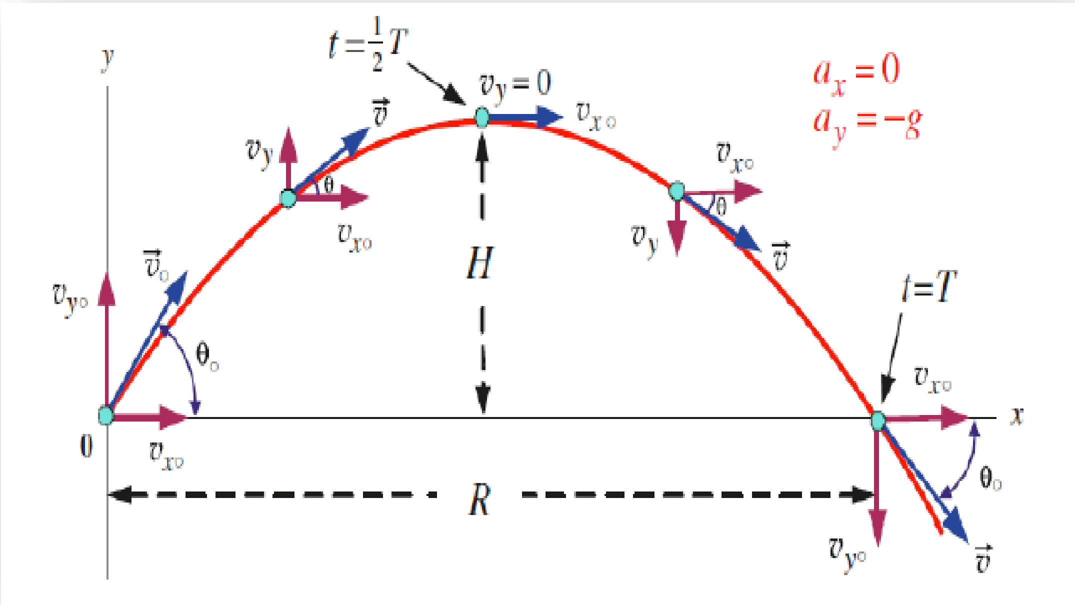


Fig.II.9. Projectile.

$a_x = 0$ and $a_y = -g$

$\vec{V_0} = V_{x_0}\vec{\imath} + V_{y_0}\vec{\jmath}$, $V_{x_0} = V_0 cos\theta_0$, $V_{y_0} = V_0 sin\theta_0$

We decompose the horizontal motion and vertical motion as described below:

$a_x = 0 \Rightarrow \quad V_x = V_{x_0} = V_0 cos\theta_0 \quad \Rightarrow x = V_{x_0} t$

$$\Rightarrow x = (V_0 cos\theta_0)t \tag{II.37}$$

$a_y = -g \Rightarrow V_y = V_{y_0} - gt \quad \Rightarrow \quad V_y = V_0 sin\theta_0 - gt$

$$\Rightarrow \quad y = (V_0 sin\theta_0)t - \frac{1}{2}gt^2 \qquad \text{(II.38)}$$

- Calculus of the horizontal range R (the distance traveled by the projectile when it returns to *y* =0) after time *t* =*T*

Set *x* = *R* at time *t* =*T and y* =0

$R = (V_0 cos\theta_0)T$

$y = (V_0 sin\theta_0)T - \frac{1}{2}gT^2$

$$T = \frac{2V_0 sin\theta_0}{g} \qquad \text{(II.39)}$$

$R = (V_0 cos\theta_0)T \quad \Rightarrow \quad R = (V_0 cos\theta_0)\frac{2V_0 sin\theta_0}{g} \quad , \ 2sin\theta_0 cos\theta_0 = sin2\theta_0$

$$R = \frac{{V_0}^2 sin2\theta_0}{g} \qquad \text{(II.40)}$$

- Calculus of maximum height H:

we set $V_y = 0$

$V_y = V_0 sin\theta_0 - gt = 0 \quad \Rightarrow \quad t = \frac{V_0 sin\theta_0}{g}$

$H = (V_0 sin\theta_0)\frac{V_0 sin\theta_0}{g} - \frac{1}{2}g(\frac{V_0 sin\theta_0}{g})^2$

$$H = \frac{{V_0}^2 sin^2\theta_0}{2g} \qquad \text{(II.41)}$$

- Equation of the Trajectory :

$x = (V_0 cos\theta_0)t \ \Rightarrow \ t = \frac{x}{V_0 cos\theta_0}$

$y = (V_0 sin\theta_0)\frac{x}{V_0 cos\theta_0} - \frac{1}{2}g(\frac{x}{V_0 cos\theta_0})^2$

$$\Rightarrow \; y = -\left(\frac{g}{2V_0{}^2 cos^2\theta_0}\right)x^2 + (tg\theta_0)x \qquad \text{(II.42)}$$

This can be written in the form

$y = -ax^2 + bx$, which is the equation of a *parabola* that passes through the origin.

II.5.3. Curvilinear movement

II.5.3.1. Circular movement

In this case, the trajectory is not a straight line, but a circle of radius R (R is constant) (Fig.II.10).

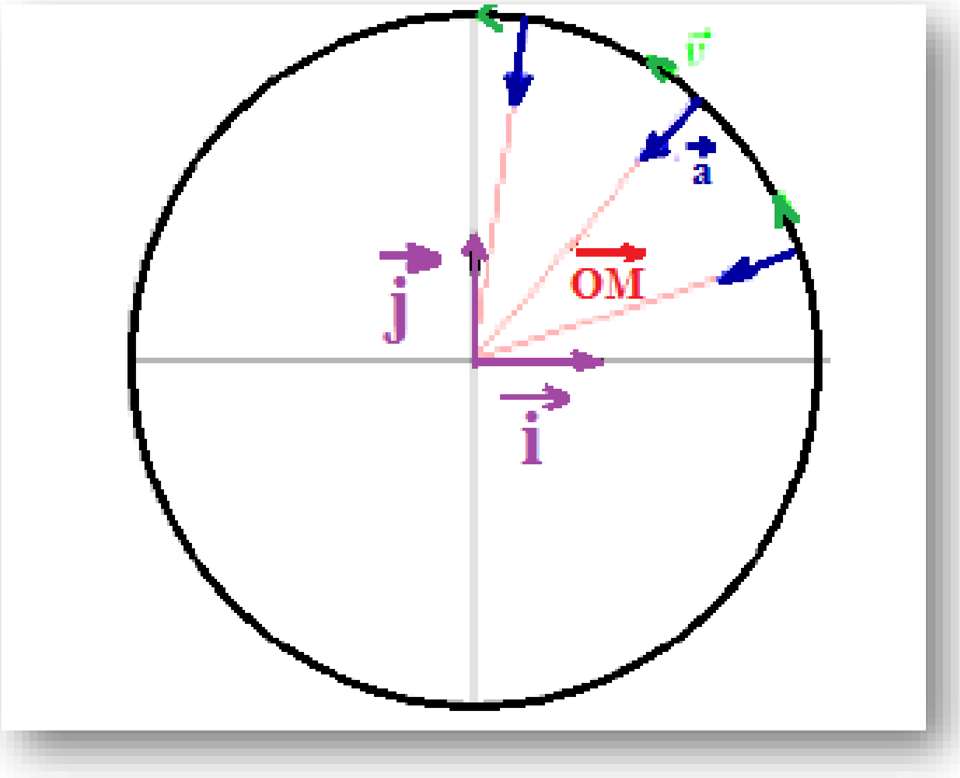


Fig.II.10. Circular movement.

Using polar coordinates:

$$\overrightarrow{OM} = \vec{r} = \rho\overrightarrow{u_\rho}$$

The velocity vector:

$$\vec{v} = \frac{d\overrightarrow{OM}}{dt} = \frac{d(R\overrightarrow{u_\rho})}{dt} = \frac{dR}{dt}\overrightarrow{u_\rho} + R\frac{d\theta}{dt}\overrightarrow{u_\theta} \quad \Rightarrow$$

$$\vec{v} = R\dot{\theta}\overrightarrow{u_\theta} \qquad \text{(II.43)}$$

$$\| \ \vec{v} \ \| = R\dot{\theta} = Rw$$

$\dot{\theta} = w$ is the angular velocity

- The acceleration vector:

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{d(R\dot{\theta}\overrightarrow{u_\theta})}{dt}$$

$$\vec{a} = R\frac{d\dot{\theta}}{dt}\overrightarrow{u_\theta} + R\dot{\theta}\frac{d(\overrightarrow{u_\theta})}{dt}$$

$$\vec{a} = R\ddot{\theta}\overrightarrow{u_\theta} - R\dot{\theta}^2\overrightarrow{u_\rho}$$

$$\vec{a} = R\left(-\theta^{\cdot 2}\overrightarrow{u_\rho} + \ddot{\theta}\overrightarrow{u_\theta}\right) \qquad \text{(II.44)}$$

$\ddot{\theta} = \alpha$ angular acceleration.

$$\| \ \vec{a} \ \| = R\sqrt{(\theta^{\cdot 2})^2 + (\alpha)^2}$$

- Using intrinsic coordinates:

$$\overrightarrow{u_\rho} = -\overrightarrow{u_N}$$

$$\overrightarrow{u_\theta} = \overrightarrow{u_T}$$

$$\vec{v} = R\dot{\theta}\overrightarrow{u_T}$$

The acceleration vector:

$$\vec{a} = R(\dot{\theta}^2\overrightarrow{u_N} + \ddot{\theta}\overrightarrow{u_T}) \qquad \text{(II.45)}$$

$$a_T = \frac{dV}{dt} = R\ddot{\theta} \quad , \textit{tangential acceleration}$$

$a_N = \frac{V^2}{R} = R\dot{\theta}^2$, normal *acceleration*

$\frac{V^2}{R} = R\dot{\theta}^2 \quad \Rightarrow \frac{(R\dot{\theta}^2)^2}{R} = R\dot{\theta}^2$ ⇒ R=R (radius of curvature = radius of the circle)

The curvilinear abscissa:

$$V = \frac{ds}{dt} = R\frac{d\theta}{dt} \Rightarrow ds = Rd\theta \quad \Rightarrow \quad \int_{s_0}^{s} ds = \int_{\theta_0}^{\theta} Rd\theta \quad \Rightarrow \quad s - s_0 = R(\theta - \theta_0)$$

$$\Rightarrow \quad s = R(\theta - \theta_0) + s_0 \qquad \text{(II.46)}$$

II.5.3.2. Uniform circular motion

$\dot{\theta} = w_0 = C^{te}$, $\ddot{\theta} = 0$

$$\dot{\theta} = \frac{d\theta}{dt} = w_0 \Rightarrow \int_{\theta_0}^{\theta} d\theta = \int_0^t w_0 dt \quad \Rightarrow \theta - \theta_0 = w_0 t \quad \Rightarrow$$

$$\theta = w_0 t + \theta_0 \qquad \text{(II.47)}$$

In polar coordinates :

$$\vec{v} = Rw_0\overrightarrow{u_\theta} \qquad \text{(II.48)}$$

$$\vec{a} = R\left(-\dot{\theta}^2\overrightarrow{u_\rho} + \ddot{\theta}\overrightarrow{u_\theta}\right)$$

$$= -Rw_0{}^2\overrightarrow{u_\rho} \qquad \text{(II.49)}$$

In intrinsic coordinates:

$$\vec{v} = Rw_0\overrightarrow{u_T} \qquad \text{(II.50)}$$

$$\vec{a} = Rw_0{}^2\overrightarrow{u_N} \qquad \text{(II.51)}$$

The curvilinear abscissa:

$$s = R(\theta - \theta_0) + s_0$$

$$= Rw_0 t + s_0 \qquad \text{(II.52)}$$

IV.2.1.2- Accelerated uniform circular motion51

$\ddot{\theta} = \alpha_0 = \dot{w} = C^{te} \Rightarrow \frac{dw}{dt}\alpha_0 \Rightarrow \int_{w_0}^{w} w_0 dt = \int_0^t \alpha_0 dt \Rightarrow w - w_0 = \alpha_0 t$

$$\Rightarrow \quad w = \alpha_0 t + w_0 \qquad \text{(II.53)}$$

$w = \frac{d\theta}{dt} \quad \Rightarrow \int_{\theta_0}^{\theta} \theta d\theta = \int_0^t w dt \Rightarrow \int_{\theta_0}^{\theta} \theta d\theta = \int_0^t (\alpha_0 t + w_0) dt$

$\Rightarrow \quad \theta - \theta_0 = -\frac{1}{2}\alpha_0 t^2 + w_0 t$

$$\Rightarrow \theta = -\frac{1}{2}\alpha_0 t^2 + w_0 t + \theta_0 \qquad \text{(II.54)}$$

$\vec{v} = R\dot{\theta}\overrightarrow{u_T} = Rw_0\overrightarrow{u_T}$

$$\Rightarrow \vec{v} = R(\alpha_0 t + w_0)\overrightarrow{u_T} \qquad \text{(II.55)}$$

$$a_T = \frac{dV}{dt} = R\alpha_0 \qquad \text{(II.56)}$$

$$a_N = \frac{V^2}{R} = R(\alpha_0 t + w_0)^2 \qquad \text{(II.57)}$$

$s = R(\theta - \theta_0) + s_0$

$$\Rightarrow \quad s = R(\frac{1}{2}\alpha_0 t^2 + w_0 t) + s_0 \qquad \text{(II.58)}$$

II.6. Angular rotation velocity vector

Let the plane of motion be the (xoy) plane and (oz) the axis of rotation.

The angular velocity vector of rotation, or simply vector rotation is $\vec{w}$:

$\vec{w} = w\vec{k}$

$\vec{v} = Rw_0\overrightarrow{u_\theta} = Rw_0(\vec{k} \wedge \overrightarrow{u_\rho}) = w\vec{k} \wedge R\overrightarrow{u_\rho}$

$\overrightarrow{OM} = R\overrightarrow{u_\rho}$

$\vec{v} = \overrightarrow{w} \wedge \overrightarrow{OM}$ (II.59)

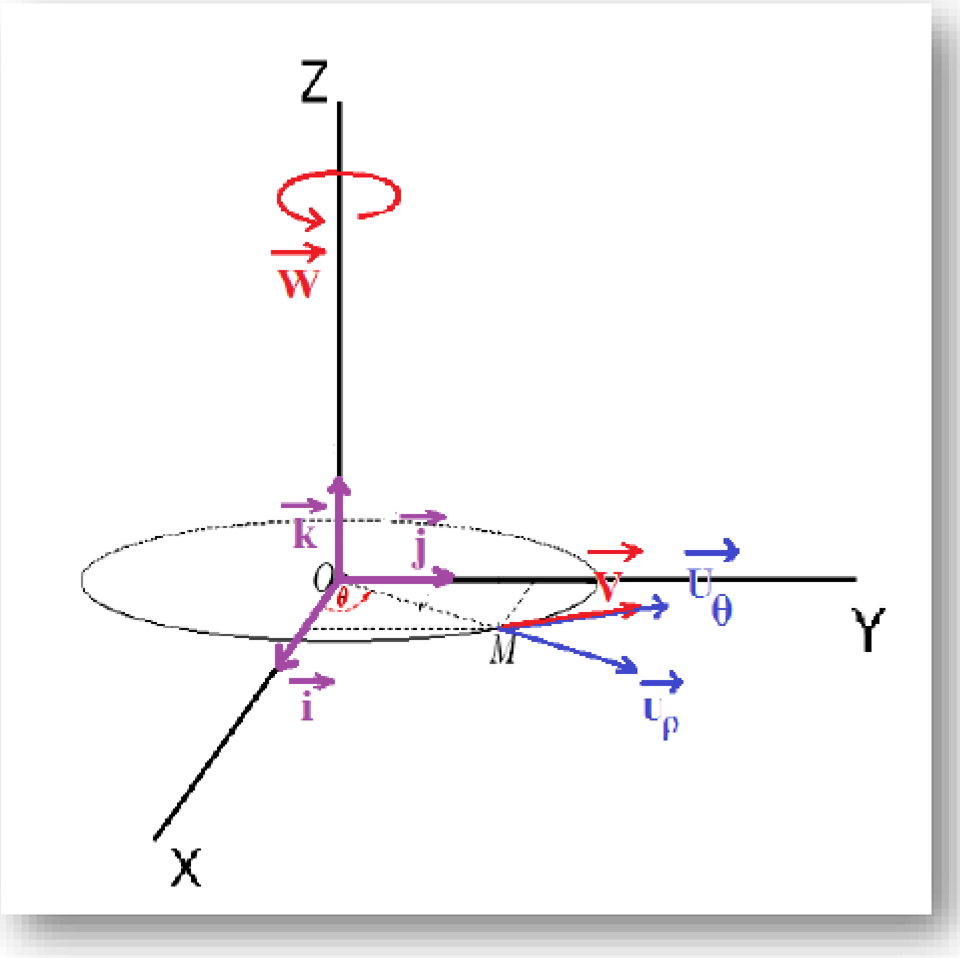


Fig.II.11. Angular rotation velocity vector.

II.7. Harmonic motion (rectilinear sinusoidal)

The motion of a solid is said to be rectilinear and sinusoidal if its time law is written in the form:

$x(t) = x_m \sin(wt + \varphi_0)$ (II.60) x: is also called the elongation of the solid at time t (m).

x_m : is the amplitude of the movement (m).

$\emptyset$= **(wt +** φ_0**)** is the phase at time t (rad).

φ_0 : initial phase, at t = 0 (rad).

w: is the pulsation of the movement (rad.s-1).

Rectilinear motion is periodic and sinusoidal with period T=w/2π **(s)** and a frequency f= 1/T= 2π/ w **(Hz).**

$$V = dx/dt = wx_0 \cos(wt + \varphi_0) \quad \text{(II.61)}$$

$$a = dv/dt = -w2x_0 \sin(wt + \varphi_0) \quad \text{(II.62)}$$

$$a = -w^2 x \quad \text{(II.63)}$$

II.8. Relative Motion

Any motion of a material point is related to the frame of reference used to study that motion. The position, path, velocity, and acceleration of the moving point (the moving material point) vary depending on the frame chosen by the observer. The concept of relative motion involves studying the motion of a material point M (a moving object) with respect to two frames of reference, R and R'. R represents the static (absolute) frame of reference, while R' represents the dynamic (relative) frame of reference.

R: The static (absolute) frame of reference, **R(oxyz)** → its base is ($\vec{i}$, $\vec{j}$, $\vec{k}$).

R': The dynamic (relative) frame of reference, **R'(o'x'y'z')** → its base is ($\vec{i'}$,$\vec{j'}$,$\vec{k'}$).

M: The moving object (the moving material point).

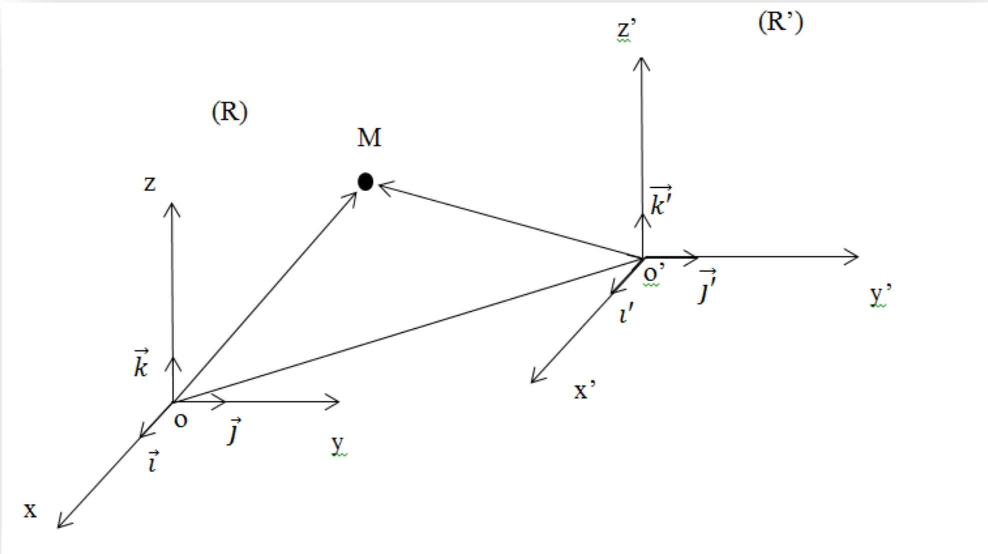


Fig.II.12. The static (absolute) frame of reference and the dynamic (relative) frame of reference.

The motion of a material point M with respect to an absolute frame of reference is called absolute motion.The motion of a material point M with respect to a relative frame of reference is called relative motion.The motion of a relative frame of reference (R') with respect to a frame of reference (R) is called acquired motion (or running motion).Position, velocity, and acceleration in absolute and relative frames of reference.The absolute (stationary) frame of reference (R): The absolute frame of reference is also called the reference frame. It is a stationary frame of reference in space, orthogonal, homogeneous (oxyz), and characterized by its unit vectors ($\vec{i}$, $\vec{j}$, $\vec{k}$). Its origin is the point O.The motion of a point M with respect to the absolute frame of reference (R) is called absolute motion.

Its velocity is called absolute velocity and is denoted by ($\overrightarrow{v_a}$). Its acceleration is called absolute acceleration and is denoted by ($\overrightarrow{a_a}$). The relative parameter (R'): The relative parameter R' is a moving object relative to the absolute parameter. It is orthonormal (o'x'yz') with an origin at o' and is characterized by unit vectors ((i', j', k')). The motion of point M relative to the relative parameter R' is called relative motion, its velocity is called relative velocity ($\overrightarrow{v_r}$) and its acceleration is called relative acceleration ($\overrightarrow{a_r}$). The relative parameter R' moves relative to the absolute parameter R with a velocity called the tractive velocity ($\overrightarrow{v_e}$) and an acceleration called the tractive acceleration ($\overrightarrow{a_e}$).The motion of the relative parameter R' relative to R can be translational, rotational, or both. We can summarize the absolute and relative magnitudes of the parameters R and R' in a table.

Table.II.1. summarize the absolute and relative magnitudes of the parameters R and R

The relative parameter R' (moving relative to R) ($\vec{i'}, \vec{j'}, \vec{k'}$)	The constant parameter R (absolute) ($\vec{i}, \vec{j}, \vec{k}$)	
$\overrightarrow{O'M} = x'\vec{i'} + y'\vec{j'} + z'\vec{k'}$	$\overrightarrow{OM} = x\vec{i} + y\vec{j} + z\vec{k}$	**Position vector**
$\overrightarrow{v_r} = \frac{d\overrightarrow{O'M}}{dt}$	$\overrightarrow{v_a} = \frac{d\overrightarrow{OM}}{dt}$	**Velocity vector**
$\overrightarrow{a_r} = \frac{d\overrightarrow{v_r}}{dt}$	$\overrightarrow{a_a} = \frac{d\overrightarrow{v_a}}{dt}$	**Acceleration vector**

The relationship between absolute and rational quantities (composition relationships):The relationship between the position vector in the rational coordinate system (R‘) and the position vector in the absolute coordinate system (R): The relationship between absolute and rational quantities (composition relationships):The relationship between the position vector in the rational coordinate system (R') and the position vector in the absolute coordinate system (R):

$$\overrightarrow{OM}(t) = \overrightarrow{OO'}(t) + \overrightarrow{O'M}(t). \tag{II.64}$$

The relationship between the velocity vector in the rational coordinate system (R') and the absolute coordinate system (R): The relationship between the velocity vector in the rational coordinate system (R') and the absolute coordinate system (R):

$$\vec{v_a}(t) = \frac{d\overrightarrow{OM}(t)}{dt}\Big/_R = \frac{d\overrightarrow{OO'}(t)}{dt}\Big/_R = \frac{d\overrightarrow{O'M}(t)}{dt}\Big/_R$$

$$\vec{v_a}(t) = \frac{d\overrightarrow{OO'}(t)}{dt}\Big/_R + \frac{d}{dt}\left(x'\vec{\imath'} + y'\vec{\jmath'} + z'\vec{k'}\right)$$

$$\vec{v_a}(t) = \frac{d}{dt}\overrightarrow{OO'}(t)\Big/_R + x'\frac{d\vec{\imath'}}{dt} + y'\frac{d\vec{\jmath'}}{dt} + z'\frac{d\vec{k'}}{dt} + \frac{dx'}{dt}\vec{\imath'} + \frac{dy'}{dt}\vec{\jmath'} + \frac{dz'}{dt}\vec{k'}$$

$$\vec{v_a}(t) = \vec{v_e}(t) + \vec{v_r}(t)$$

$$\vec{v_a}(t) = \vec{v_e}(t)_{R'/_R} + \vec{v_r}(t)\Big/_{R'} \tag{II.65}$$

($\vec{v_r}$): Relative velocity.

($\vec{v_e}$): Traction velocity (gained velocity), which is the velocity of the parameter R' with respect to R.

($\vec{v_a}$): Absolute velocity.

Note (1): The motion of the relative parameter (R') with respect to (R) may be translational, rotational, or both.

Note (2): The acceleration vector ($\vec{v_e}$) depends on the vector $\overrightarrow{OO'}$, the coordinates of the moving point, and the rotation of the relative parameter (R') with respect to the absolute parameter (R).

-The relationship between the acceleration vector in the relative parameter (R') and the absolute parameter (R)

$$\vec{a_a}(t) = \frac{d\vec{v_a}}{dt}\Big/_R = \frac{d^2\overrightarrow{OM}}{dt^2}\Big/_R = \frac{d\vec{v_e}}{dt}R'\Big/_R + \frac{d\vec{v_r}}{dt}\Big/_{R'}$$

$$\vec{a_a}(t) = \frac{d}{dt}\left[\frac{d}{dt}\overrightarrow{OO'}(t)\Big/_R + x'\frac{d\vec{\imath'}}{dt} + y'\frac{d\vec{\jmath'}}{dt} + z'\frac{d\vec{k'}}{dt} + \frac{dx'}{dt}\vec{\imath'} + \frac{dy'}{dt}\vec{\jmath'} + \frac{dz'}{dt}\vec{k'}\right]$$

$$\overrightarrow{a_a}(t) = \underbrace{\frac{d^2}{dt^2}\overrightarrow{OO'}/_R + x'\frac{d^2\overrightarrow{\imath'}}{dt^2} + y'\frac{d^2\overrightarrow{\jmath'}}{dt^2} + z'\frac{d^2\overrightarrow{k'}}{dt^2}}_{\overrightarrow{a_e}(t)} + \underbrace{\frac{d^2x'}{dt^2}\overrightarrow{\imath'} + \frac{d^2y'}{dt^2}\overrightarrow{\jmath'} + \frac{d^2z'}{dt^2}\overrightarrow{k'}}_{\overrightarrow{a_r}(t)}$$

$$+ \underbrace{2(\frac{dx'}{dt}.\frac{d\overrightarrow{\imath'}}{dt} + \frac{dy'}{dt}.\frac{d\overrightarrow{\jmath'}}{dt} + \frac{dz'}{dt}.\frac{d\overrightarrow{k'}}{dt})}_{\overrightarrow{a_c}(t)}$$

$$\overrightarrow{a_a}(t)/_R = \overrightarrow{a_e}(t)_{R'/_R} + \overrightarrow{a_r}(t)/_{R'} + \overrightarrow{a_c}(t)/_{R'}$$

$$\overrightarrow{a_a}(t) = \overrightarrow{a_e}(t) + \overrightarrow{a_r}(t) \qquad \text{(II.66)}$$

- $\overrightarrow{\boldsymbol{a_a}}$: Absolute acceleration vector
- $\overrightarrow{\boldsymbol{a_r}}$: Relative acceleration vector
- $\overrightarrow{\boldsymbol{a_e}}$: Translational (or acquired) acceleration vector
- $\overrightarrow{\boldsymbol{a_c}}$: Coriolis (additional) acceleration vector.

Exercise 1

A point object M moves according to the equations:

$$\begin{cases} x(t) = 5t \\ y(t) = 3t + 4 \end{cases}$$

1- Find the equation of the trajectory and its nature.

2- Find the position vectors $\overrightarrow{OM}$, the velocity $\overrightarrow{V}$ and the acceleration $\overrightarrow{a}$

Solution:

We are given the parametric equations of motion of the point object M:

$$\begin{cases} x(t) = 5t \\ y(t) = 3t + 4 \end{cases}$$

1. Equation and nature of the trajectory

Eliminate the parameter t.

From $x(t) = 5t$:

$$t = \frac{x}{5}$$

Substitute into $y(t)$:

$$y = 3\left(\frac{x}{5}\right) + 4$$

$$\boxed{y = \frac{3}{5}x + 4}$$

Nature of the trajectory

This is the equation of a straight line.

The trajectory is rectilinear motion.

2. Position vector $\overrightarrow{OM}(t)$

The position vector is:

$$\overrightarrow{OM}(t) = x(t)\vec{\imath} + y(t)\vec{\jmath}$$

$$\boxed{\overrightarrow{OM}(t) = 5t\,\vec{\imath} + (3t+4)\,\vec{\jmath}}$$

---

3. Velocity vector $\vec{V}(t)$

The velocity is the first derivative of the position vector:

$$\vec{V}(t) = \frac{d\overrightarrow{OM}}{dt}$$

$$\vec{V}(t) = 5\,\vec{\imath} + 3\,\vec{\jmath}$$

$$\boxed{\vec{V} = 5\,\vec{\imath} + 3\,\vec{\jmath}}$$

The velocity is constant.

4. Acceleration vector $\vec{a}(t)$

The acceleration is the derivative of velocity:

$$\vec{a}(t) = \frac{d\vec{V}}{dt}$$

$$\boxed{\vec{a} = \vec{0}}$$

Final Summary

- Trajectory equation:

$$y = \frac{3}{5}x + 4$$

- Nature of motion: Straight-line motion with constant velocity
- Position vector:

$$\overrightarrow{OM}(t) = 5t\,\vec{\imath} + (3t+4)\,\vec{\jmath}$$

- Velocity vector:

$$\vec{V} = 5\,\vec{\imath} + 3\,\vec{\jmath}$$

- Acceleration vector:

$$\vec{a} = \vec{0}$$

Exercise 02

According to the equations, the object M moves such that:

$$\begin{cases} \rho = 2ae^{\theta} \\ \theta = \omega t \end{cases}$$

where a and ω are positive constants.

1. Find the position, velocity, and acceleration vectors in polar coordinates, then express them in Cartesian coordinates.
2. Find the magnitude of the velocity vector $\| \vec{V} \|$.
3. Determine the tangential acceleration a_T and the normal acceleration a_N.
4. Find the radius of curvature R.
5. Given that $L(0) = 0$, calculate the arc length of the trajectory $L(\theta)$.

Solution:

1. Position, velocity, and acceleration vectors

1.1 Position vector

In polar coordinates

The position vector is:

$$\overrightarrow{OM} = \rho\,\vec{e}_{\rho}$$

$$\boxed{\overrightarrow{OM} = 2ae^{\theta}\,\vec{e}_{\rho}}$$

In Cartesian coordinates

Using:

$$x = \rho\cos\,\theta, y = \rho\sin\,\theta$$

$$\boxed{\begin{aligned} x &= 2ae^{\theta}\cos\,\theta \\ y &= 2ae^{\theta}\sin\,\theta \end{aligned}}$$

1.2 Velocity vector

In polar coordinates:

$$\vec{V} = \dot{\rho}\,\vec{e}_\rho + \rho\dot{\theta}\,\vec{e}_\theta$$

First derivatives:

$$\dot{\theta} = \omega, \dot{\rho} = \frac{d\rho}{d\theta}\dot{\theta}$$

$$\frac{d\rho}{d\theta} = 2ae^{\theta} \Rightarrow \dot{\rho} = 2a\omega e^{\theta}$$

So:

$$\boxed{\vec{V} = 2a\omega e^{\theta}(\vec{e}_\rho + \vec{e}_\theta)}$$

1.3 Acceleration vector

In polar coordinates:

$$\vec{a} = (\ddot{\rho} - \rho\dot{\theta}^2)\vec{e}_\rho + (\rho\ddot{\theta} + 2\dot{\rho}\dot{\theta})\vec{e}_\theta$$

Second derivatives:

$$\ddot{\theta} = 0$$

$$\ddot{\rho} = \frac{d}{dt}(2a\omega e^{\theta}) = 2a\omega^2 e^{\theta}$$

Substitute:

$$\ddot{\rho} - \rho\dot{\theta}^2 = 2a\omega^2 e^{\theta} - 2ae^{\theta}\omega^2 = 0$$

$$\rho\ddot{\theta} + 2\dot{\rho}\dot{\theta} = 4a\omega^2 e^{\theta}$$

$$\boxed{\vec{a} = 4a\omega^2 e^{\theta}\,\vec{e}_\theta}$$

2. Magnitude of velocity

$$\parallel \vec{V} \parallel = \sqrt{(2a\omega e^{\theta})^2 + (2a\omega e^{\theta})^2}$$

$$\boxed{\parallel \vec{V} \parallel = 2\sqrt{2}\,a\omega e^{\theta}}$$

3. Tangential and normal accelerations

Tangential acceleration a_T

$$a_T = \frac{d \parallel \vec{V} \parallel}{dt}$$

$$a_T = 2\sqrt{2}a\omega^2 e^\theta$$

$$\boxed{a_T = 2\sqrt{2}a\omega^2 e^\theta}$$

Normal acceleration a_N

$$a_N = \sqrt{\| \vec{a} \|^2 - a_T^2}$$

$$\| \vec{a} \| = 4a\omega^2 e^\theta$$

$$a_N = 2\sqrt{2}a\omega^2 e^\theta$$

$$\boxed{a_N = 2\sqrt{2}a\omega^2 e^\theta}$$

4. Radius of curvature R

$$R = \frac{\| \vec{V} \|^2}{a_N}$$

$$R = \frac{(2\sqrt{2}a\omega e^\theta)^2}{2\sqrt{2}a\omega^2 e^\theta}$$

$$\boxed{R = 2\sqrt{2}ae^\theta}$$

5. Arc length $L(\theta)$

For a polar curve:

$$L(\theta) = \int_0^\theta \sqrt{\rho^2 + \left(\frac{d\rho}{d\theta}\right)^2}\, d\theta$$

$$\rho = 2ae^\theta, \frac{d\rho}{d\theta} = 2ae^\theta$$

$$L(\theta) = \int_0^\theta 2\sqrt{2}ae^\theta\, d\theta$$

$$\boxed{L(\theta) = 2\sqrt{2}a(e^\theta - 1)}$$

Exercise 3

Frames: R(origin O) and R'(origin O').

Let $\overrightarrow{OM}(t)$be the position vector of point Mexpressed in $R$with Cartesian coordinates

$(x(t), y(t), z(t))$. Let the vector from O to O' be $\overrightarrow{OO'}(t) = (X_0(t), Y_0(t), Z_0(t))$. Assume R' is **translating only** with respect to R (no rotation of axes).

1. Express $\overrightarrow{O'M}(t) = \overrightarrow{OM'}(t)$ (coordinates of M in R') in terms of the given functions.
2. Define and compute:
   - $\vec{v}_e(t)$: the velocity of the moving frame (velocity of O' relative to O), i.e. $\frac{d}{dt}\overrightarrow{OO'}(t)$.
   - $\vec{v}(t)$: the velocity of M in R, i.e. $\frac{d}{dt}\overrightarrow{OM}(t)$.
   - $\vec{v}_a(t)$: the velocity of M relative to R' (i.e. derivative of $\overrightarrow{OM'}$ in time).
3. Show the relation linking these velocities.

<u>Solution:</u>

1. Vector decomposition (basic identity)

$$\overrightarrow{OM}(t) = \overrightarrow{OO'}(t) + \overrightarrow{O'M}(t).$$

So

$$\boxed{\overrightarrow{OM'}(t) = \overrightarrow{O'M}(t) = \overrightarrow{OM}(t) - \overrightarrow{OO'}(t).}$$

2. Velocities (time derivatives):

$$\vec{v}(t) = \frac{d}{dt}\overrightarrow{OM}(t), \vec{v}_e(t) = \frac{d}{dt}\overrightarrow{OO'}(t), \vec{v}_a(t) = \frac{d}{dt}\overrightarrow{OM'}(t).$$

Differentiate the identity $\overrightarrow{OM} = \overrightarrow{OO'} + \overrightarrow{OM'}$ to get:

$$\boxed{\vec{v}(t) = \vec{v}_e(t) + \vec{v}_a(t).}$$

Thus $\vec{v}_a(t) = \vec{v}(t) - \vec{v}_e(t)$.

(These hold because we assumed no rotation of axes. With rotation an extra $\vec{\omega} \times \overrightarrow{OM'}$ term appears.)

Chapter III : Dynamics

III.1. Introduction

Dynamics is one of the branches of classical mechanics that deals with the study of forces and torques and their effects on the motion of bodies. More precisely, dynamics focuses on the *causes of motion*.

The physicist Isaac Newton established the fundamental physical laws of dynamics, known as Newton's laws of motion, within specific reference frames called inertial frames.

III.2. Principle of Inertia (Newton's First Law of Motion)

Inertia is a physical term that describes a body's resistance to any change in its motion. Newton expressed this concept in his first law, known as the *Law of Inertia*.

The Principle of Inertia states that:

Every object continues in its state of rest or of uniform motion in a straight line unless compelled to change that state by an external unbalanced force.

This law was first clearly formulated by Sir Isaac Newton and is the foundation of classical mechanics.

This principle means that motion does not require a force to continue. A force is only required to:

- Start motion
- Stop motion
- Change speed
- Change direction

- Importance of the Principle of Inertia
  - Explains the need for seat belts in vehicles
  - Helps understand motion of planets and satellites
  - Forms the basis of Newton's Second and Third Laws
  - Essential in engineering, transport, and sports science.

III2.1. Types of Inertia

III.2.1.1 Inertia of Rest

The tendency of an object to remain at rest unless acted upon by a force.

Examples:

- A stationary football does not move until kicked.
- Dust particles fall off a carpet when it is shaken suddenly.

III.2.1.2 Inertia of Motion

The tendency of a moving object to continue moving with the same speed.

Examples:

- A person jumps out of a moving bus and falls forward.
- A cyclist continues to move for some distance even after stopping pedaling.

III.2.1.3. Inertia of Direction

The tendency of an object to continue moving in the same direction.

Examples:

- Passengers lean sideways when a car turns suddenly.
- Mud flies off the tires of a turning vehicle in a tangential direction.

III.3. Inertial Mass

Inertial mass is the measure of how strongly an object resists changes in its state of motion when a force is applied.

Inertial mass is the measure of an object's resistance to any change in its state of motion when a force is applied. It indicates how difficult it is to accelerate an object. According to Newton's second law of motion, inertial mass is defined as the ratio of the applied force to the acceleration produced in the object $(m = F/a)$. An object with a larger inertial mass requires a greater force to produce the same acceleration as an object with a smaller inertial mass. Thus, inertial mass provides a quantitative measure of the inertia of an object and remains constant regardless of the object's location or surroundings.

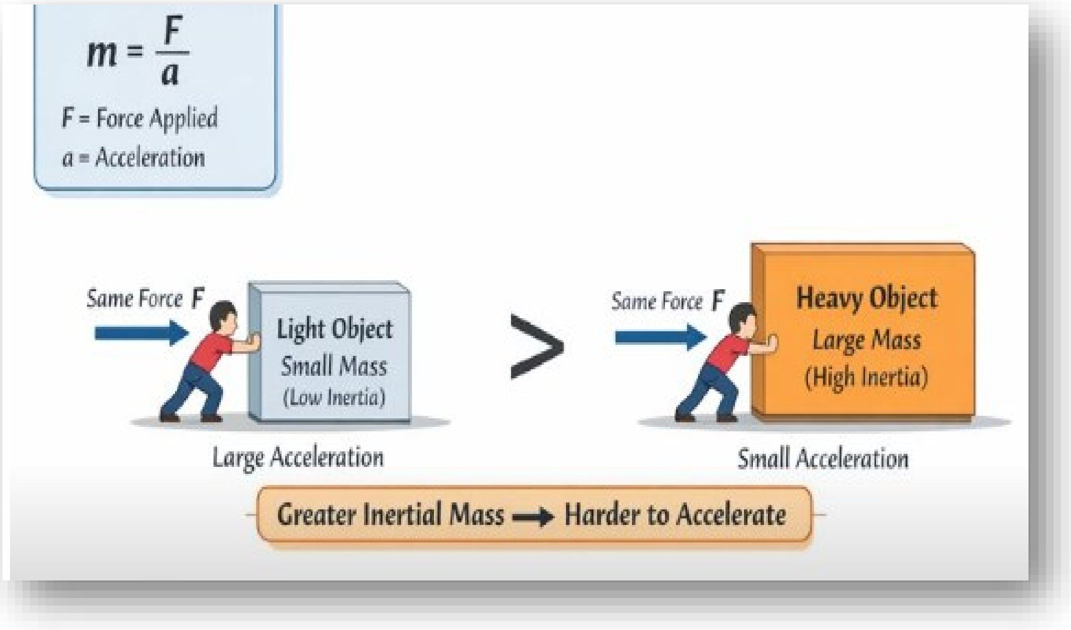


Fig.III.1. Inertial Mass.

III.4. Galilean Frame of reference

A Galilean frame of reference is a special kind of coordinate system used to observe and measure motion, where Newton's laws of motion are valid. In simpler words, it is a frame in which an object obeys the principle of inertia: it either stays at rest or moves in a straight line at constant speed unless a force acts on it.

A Galilean frame of reference (also called an inertial frame) is a frame of reference that is either at rest or moves with a constant velocity in a straight line relative to a fixed background.

- Constant velocity means there is no acceleration.
- If the frame accelerates or rotates, it is non-inertial (Newton's laws do not directly hold there).

III.4.1. Characteristics of a Galilean Frame

A Galilean frame of reference is an inertial frame in which Newton's laws of motion are valid. Its main characteristics are that it is either at rest or moves with constant velocity in a straight line, meaning it has no acceleration. In such a frame, objects obey the principle of inertia, and no fictitious forces are required to explain motion. All Galilean frames moving uniformly relative to one another are physically equivalent, so mechanical experiments give the same results in each frame. In these frames, velocity is relative while acceleration is absolute, and time is universal, passing at the same rate for all observers. These features make Galilean frames the fundamental basis for classical mechanics. So we can conclude that:

- No acceleration: The frame moves in a straight line with constant speed or is at rest.
- Newton's laws are valid: Objects behave according to the laws of motion.
- Equivalence of frames: All Galilean frames are physically equivalent for mechanics.
- Acceleration is absolute, velocity is relative: An object accelerating in a Galilean frame is real, but constant velocity is relative to the observer.

III.5. Momentum

Momentum is a fundamental concept in physics that describes the quantity of motion of an object. Momentum of an object is defined as the product of its mass and velocity:

$$\vec{P} = m\vec{v} \qquad \text{(III.1)}$$

Where:

- $\vec{P}$: momentum of the object
- m : mass of the object
- $\vec{v}$: velocity of the object

Momentum is a physical quantity that measures the motion of an object and is defined as the product of its mass and velocity. It is a vector quantity, which means it has both magnitude and direction, and it points in the direction of the object's motion. Momentum indicates how difficult it is to stop a moving object: the greater the mass or the higher the velocity, the greater the momentum. For example, a moving truck has much more momentum than a bicycle moving at the same speed because of its larger mass, and a fast-moving cricket ball has more momentum than a slowly rolling ball of the same size. Momentum plays an important role in understanding collisions, explosions, and other interactions between objects. In isolated systems, the total momentum is conserved, which is the basis of the law of conservation of momentum, making it a fundamental concept in mechanics. Momentum is a vector quantity, which means it has both magnitude and direction, and it is always in the direction of the object's motion.

III.5.1. Linear Momentum vector

The linear momentum vector $\overrightarrow{P}$ of a material point of mass m and velocity $\vec{v}$ is a vector quantity that has the same direction as velocity (see relation (III.1)). It is an important physical quantity because it combines two elements that characterize the motion of an object mass and velocity.

Total Linear Momentum of a System of Particles for a system consisting of several material points (particles), the total linear momentum is the vector sum of the linear momenta of all the individual particles.

If the system has n particles with masses $m_1, m_2, \ldots, m_n$ and velocities $\overrightarrow{v_1}, \overrightarrow{v_2}, \ldots, \overrightarrow{v_n}$, then the total momentum $\vec{P}$ of the system is:

$$\vec{P} = \sum_{i=1}^{n} m_i \overrightarrow{v_i} = m_1\overrightarrow{v_1} + m_2\overrightarrow{v_2} + \cdots + m_n\overrightarrow{v_n} \tag{III.2}$$

For a system consisting of several material points, the total linear momentum is:

$$\vec{P} = \sum m_i \overrightarrow{v_i} = \sum \overrightarrow{P_i} \tag{III.3}$$

Features

1. Vector Quantity
    - Total momentum has magnitude and direction, depending on the vector sum of individual momenta.
2. Depends on Mass and Velocity of All Particles
    - Each particle contributes to the total momentum according to its mass and velocity.
3. External Force and Momentum Change

- If no external force acts on the system, the total momentum remains constant (law of conservation of momentum):

$$\frac{dP}{dt} = F_{\text{external}} = 0 \Rightarrow P = \text{constant} \qquad \text{(III.4)}$$

III.5.2. The principle of conservation of momentum

The principle of conservation of momentum states that in an isolated system, where no external force acts, the total momentum of the system remains constant. Momentum is a vector quantity, so both its magnitude and direction are conserved. For a system of particles, the total momentum is the vector sum of the momenta of all individual particles, and this total does not change over time if the system is isolated. This principle can be observed in real-life situations such as collisions between frictionless billiard balls or when two ice skaters push off each other on a frictionless surface. Conservation of momentum is a fundamental law in classical mechanics, allowing us to analyze collisions, explosions, and other interactions without needing to know the internal forces, and it provides a basis for predicting the motion of objects after interactions.

In an inertial frame (R), if a system is isolated or quasi-isolated, then:

$$\sum \vec{F} = \vec{0} \tag{III.5}$$

Since

$$\sum \vec{F} = \frac{d\vec{P}}{dt} \tag{III.6}$$

it follows that

$$\frac{d\vec{P}}{dt} = \vec{0} \Rightarrow \vec{P} = constant = m\vec{v} = constant \tag{III.7}$$

Thus, the momentum of an isolated system is always conserved, it remains constant in both magnitude and direction:

$$\vec{P} = \overrightarrow{P'} \Rightarrow \sum \overrightarrow{P_i} = \sum \overrightarrow{P_i'} = \overrightarrow{P_1} + \overrightarrow{P_2} + \cdots = constant \tag{III.8}$$

<u>Second Law of Newton (Alternative Form):</u>

In an inertial frame, the rate of change of momentum is proportional to the resultant of the external forces acting on the object, and this change has the same direction as that resultant:

$$\sum \overrightarrow{F_{ext}} = \frac{d\vec{P}}{dt} \tag{III.9}$$

This is the second form of Newton's Second Law.

- **Case of Variable Mass:**

If the mass of an object is variable, the law is written as follows:

$$\overrightarrow{F_{ext}} = \frac{d\vec{P}}{dt} = \frac{d(m\vec{v})}{dt} = \vec{v}\frac{dm}{dt} + m\frac{d\vec{v}}{dt} \tag{III.10}$$

Therefore:

$$\vec{F} = \vec{v}\frac{dm}{dt} + m\vec{\gamma} \qquad \text{(III.11)}$$

Where:

$$\vec{\gamma} = \frac{d\vec{v}}{dt}$$

- **Case of Constant Mass:**

If the mass is constant, the fundamental principle of dynamics becomes:

$$\vec{F} = m\frac{d\vec{v}}{dt} \Rightarrow \vec{F} = m\vec{\gamma} \qquad \text{(III.12)}$$

Hence, any object of mass m and acceleration $\vec{\gamma}$ is subject to an external force $\vec{F}$, such that:

$$\vec{F} = \sum \overrightarrow{F_{ext}} = m\vec{\gamma} \qquad \text{(III.13)}$$

<u>**Special Cases:**</u>

- If the resultant force is constant, $\vec{F} = constant$, then the acceleration $\vec{\gamma} = \frac{\vec{F}}{m}$ is also constant, and the motion of the object is uniformly accelerated rectilinear motion.
- If the resultant force is zero, $\vec{F} = \vec{0}$, then the acceleration $\vec{\gamma} = \vec{0}$ and the motion of the object is inertial, i.e. $\vec{v} = constant$ (This corresponds to Newton's First Law, which is a special case of the Second Law.)

III.6. Notion of Force

A force is an action that can change the state of motion of an object or deform it. In other words, a force can cause an object at rest to start moving, a moving object to stop or change its velocity, or an object to change its shape. A force is a vector quantity that describes an influence capable of producing motion or changing the state of motion of a material point, or of deforming a physical object.

According to Newton's Second Law of Motion, force is defined as (see relation (III.12) This shows that force is directly proportional to the mass of the object and the acceleration produced.

If a material point is subject to several forces, the total effect on it is given by the *vector sum* of these forces. The single equivalent force is called the resultant force, expressed as (see relation (III.13)).

Characteristics of Force

1. Vector Quantity: Force has both magnitude and direction.
2. Causes Motion: It can start, stop, or change the motion of an object.
3. Can Cause Deformation: Force can compress, stretch, or twist an object.
4. Measured in Newtons (N): SI unit of force is 1 Newton = 1 kg·m/s^2.

III.7. Newton's Laws72

Newton's Laws of Motion are three essential principles that explain how objects behave in motion. The first law, also called the law of inertia, states that an object will remain stationary or continue moving in a straight line at a constant speed unless influenced by an external force, introducing the idea of inertia, which is the tendency of an object to resist changes in its motion. The second law defines how force affects motion, stating that the acceleration of an object is directly proportional to the total force applied and inversely proportional to its mass, represented mathematically as $F = ma$. The third law, or the action-reaction principle, states that for every action, there is an equal and opposite reaction, meaning forces always exist in pairs acting on separate objects. Collectively, these laws explain everyday events such as pushing a cart, a passenger moving forward in a stopping bus, or a rocket propelling forward as exhaust gases are ejected, and they form the basis of classical mechanics, allowing us to predict and analyze the movement of objects.

III.7.1. First Law (Law of Inertia)73

The first law states that: An object will remain at rest or continue moving in a straight line at constant speed unless acted upon by a net external force**.** This law introduces the concept of inertia, which is the tendency of an object to resist any change in its state of motion. Objects with greater mass have greater inertia and resist changes in motion more than lighter objects. In everyday life, this can be seen when a book stays on a table until pushed, when passengers lean forward as a bus stops suddenly, or when a ball continues rolling on a smooth surface until friction or another force slows it down. In simple terms, the first law explains that objects naturally resist changes in their motion.

$$\vec{F} = \sum \vec{F_i} = 0 \qquad \text{(III.14)}$$

The system is at rest (or in uniform motion if already moving).

III.7.2. Second Law (Fundamental Principle of Dynamics)73

The Second Law of Motion, also called the Fundamental Principle of Dynamics, states that the acceleration of an object is directly proportional to the net external force acting on it and inversely proportional to its mass. Mathematically, it is expressed as:

$$\vec{F} = m\vec{a}$$

where F is the net force applied, m is the mass of the object, and a is the acceleration produced. This law explains how the motion of an object changes when a force is applied, showing that heavier objects require more force to achieve the same acceleration as lighter ones. In daily life, it can be observed when pushing a full shopping cart is harder than an empty one, or when a car accelerates faster with more engine power. The second law provides a quantitative understanding of motion, forming the foundation for analyzing forces, predicting accelerations, and solving problems in mechanics.

The resultant of all forces acting on a body equals the product of its mass and acceleration.

$$\sum \vec{F_i} = m\vec{a_i} \tag{III.16}$$

III.7.3. Third Law (Action and Reaction)

The Third Law of Motion, also known as the Law of Action and Reaction, states that for every action, there is an equal and opposite reaction. This means that whenever an object exerts a force on another object, the second object simultaneously exerts a force of equal magnitude but in the opposite direction on the first object. These forces always occur in pairs and act on different objects, so they do not cancel each other. In everyday life, this can be seen when a person jumps off a boat, causing the boat to move backward, or when a rocket moves forward as it expels exhaust gases backward. The third law explains interactions between objects, the mechanics of propulsion, and is fundamental in understanding motion and forces in classical mechanics.

$$\overrightarrow{F_{12}} = -\overrightarrow{F_{21}} \tag{III.17}$$

For every action, there is an equal and opposite reaction (interaction between two objects).

Characteristics:

- The two forces have the same line of action.
- The two forces have the same nature of interaction (either attractive or repulsive).
- The forces are simultaneous, meaning they occur at the same time.
- Neither of the two forces can be considered the cause of the other.

III.8. Differential Equations of Motion

Differential equations of motion are mathematical equations that describe how the position, velocity, and acceleration of an object change over time under the influence of forces. They are derived from Newton's Second Law, which states that the net force on an object equals its mass times its acceleration ($F = ma = m\frac{d^2x}{dt^2}$). The equations are usually second-order differential equations because acceleration is the second derivative of displacement. The exact form depends on the forces acting on the object, such as gravity, friction, or spring forces. By solving these equations, we can determine the motion of the object as a function of time, such as a freely falling body under gravity or an oscillating mass-spring system. Differential equations of motion are fundamental in physics and engineering for predicting and analyzing the behavior of moving objects.

The equation that relates $\vec{v}(t)$and $\vec{F}(t)$is called the differential equation of motion:

$$\vec{\gamma} = \vec{a} = \frac{d\vec{v}}{dt} = \frac{\vec{F}}{m} \tag{III.18}$$

III.8.1 Differential Equation of Motion in Cartesian Coordinates

The differential equation of motion can be expressed in terms of the x, y, and z axes as separate component equations, for example:

$$\mathrm{m}\vec{\gamma} = m\vec{a} = \vec{F} \Rightarrow \begin{cases} m\ddot{x} = F_x & (ox) \\ m\ddot{y} = F_y & (oy) \\ m\ddot{z} = F_z & (oz) \end{cases} \tag{III.19}$$

$$\mathrm{m}(\overrightarrow{\gamma_x} + \overrightarrow{\gamma_y} + \overrightarrow{\gamma_z}) = m(\overrightarrow{F_x} + \overrightarrow{F_y} + \overrightarrow{F_z}) \tag{III.20}$$

III.8.2. Differential Equation of Motion in Cylindrical Coordinates

In cylindrical coordinates (r، θ، z) ,the motion is expressed as:

$$\mathrm{m}(\overrightarrow{a_\rho} + \overrightarrow{a_\theta} + \overrightarrow{a_z}) = m\left(\overrightarrow{F_\rho} + \overrightarrow{F_\varphi} + \overrightarrow{F_z}\right)$$

$$\Rightarrow \begin{cases} m(\ddot{\rho} - r\dot{\varphi}^2) = F_\rho & (\overrightarrow{U_\rho}) \\ m(\ddot{\rho} + 2\dot{\rho}\dot{\varphi}) = F_\varphi & (\overrightarrow{U_\varphi}) \\ m\ddot{z} = F_z & (\overrightarrow{U_z}) \end{cases} \tag{III.21}$$

III.8.3. Differential Equation of Motion in Spherical Coordinates

In spherical coordinates (r, θ, φ) ,the motion is expressed as:

$$\begin{cases} m\left(\ddot{r} - r\left(\dot{\theta}^2 + \dot{\varphi}^2 \sin\theta^2\right)\right) = F_r \\ m\left(r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\varphi}^2 \sin\theta\cos\theta\right) = F_\theta \\ m\left(r\ddot{\varphi}\sin\theta + 2\dot{r}\dot{\varphi}\sin\theta + 2r\dot{\theta}\dot{\varphi}\cos\theta\right) = F_\varphi \end{cases} \tag{III.22}$$

III.8.4. Differential Equations of Motion in Curvilinear (Tangential) Coordinates

In this case, the force can be decomposed into two components:

- one tangential, along the direction of velocity; the other normal (perpendicular) to it.

$$\mathrm{m}\vec{a} = m\vec{\gamma} = \vec{F} \Rightarrow m(\overrightarrow{a_T} + \overrightarrow{a_N}) = \overrightarrow{F_T} + \overrightarrow{F_N} \tag{III.23}$$

$$\Rightarrow \begin{cases} m|\overrightarrow{a_T}| = |\overrightarrow{F_T}| \Rightarrow m\frac{d\|\vec{v}\|}{dt} = |\overrightarrow{F_T}| \\ m|\overrightarrow{a_N}| = |\overrightarrow{F_N}| \Rightarrow m\left(\frac{v^2}{R}\right) = |\overrightarrow{F_N}| \end{cases} \tag{III.24}$$

$$\begin{cases} \overrightarrow{F_T}\text{:tangential force} \\ \overrightarrow{F_N}\text{:normal force} \end{cases}$$

Note:

- If $(\overrightarrow{a_T} \cdot \vec{v}) > 0$,the tangential force $\overrightarrow{F_T}$ has the same direction as velocity, and thus it is a driving force.
- If $(\overrightarrow{a_T} \cdot \vec{v}) < 0$,the tangential force $\overrightarrow{F_T}$ is opposite to the velocity, and it acts as a resistive force.

III.9. Different types of Forces

III.9.1. Gravitational Forces (Law of Universal Gravitation)

Between any two material points, there exists a mutual force of attraction that is **d**irectly proportional to the product of their masses and inversely proportional to the square of the distance between them:

$$\vec{F} = G\frac{m_1 m_2}{r^2}\vec{U} \quad \text{(III.25)}$$

Where: G is the universal gravitational constant:

$G = 6.67 \times 10^{-11}\ Nm^2/kg^2$

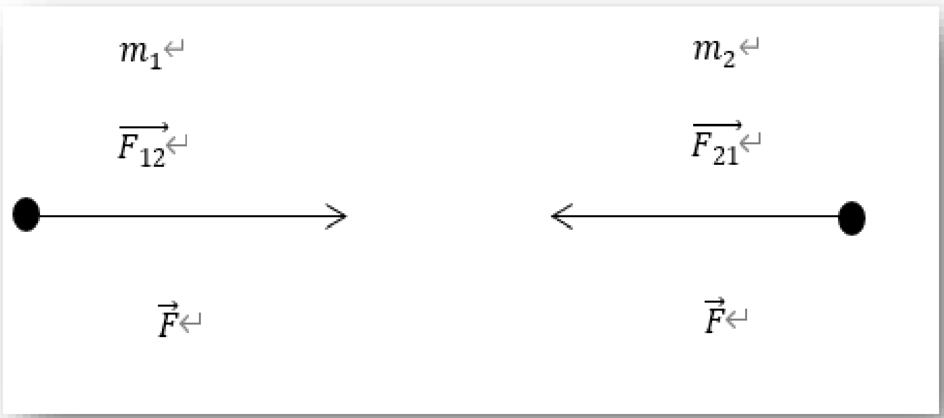


Fig.III.2. Gravitational Forces.

<u>Gravitational Field</u>

The Earth's gravitational force is called **weight** $(\vec{P})$.

Thus, we can determine the expression of $\vec{g}$)the gravitational field strength) at the surface of the Earth:

$$\vec{F} = \vec{P} = G\frac{M_T m}{R_T^2} = mg_0 \Rightarrow g_0 = \frac{GM_T}{R_T^2} \quad \text{(III.26)}$$

Where:

- M_T :Mass of the Earth $\rightarrow$ $M_T = 5.98 \times 10^{24}\ kg$

- R_T :Radius of the Earth $\rightarrow$ $R_T = 6.37 \times 10^6\ m$

  Therefore:

$$g_0 = 9.8\ N/kg$$

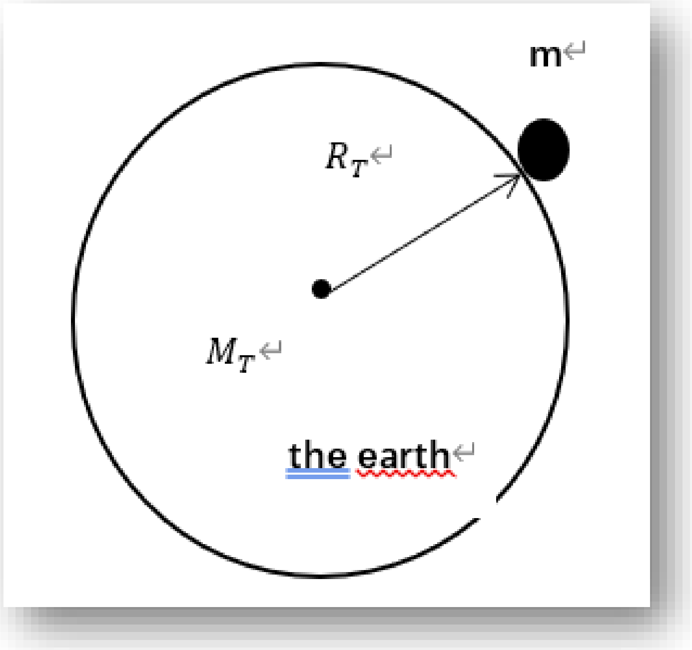


Fig.III.3. The Earth's gravitational force.

III.9.2. Electrostatic Force

It acts between two electric charges:

$$\vec{F} = \frac{qQ}{4\pi\varepsilon_0 r^2}\,\vec{u} \qquad \text{(III.27)}$$

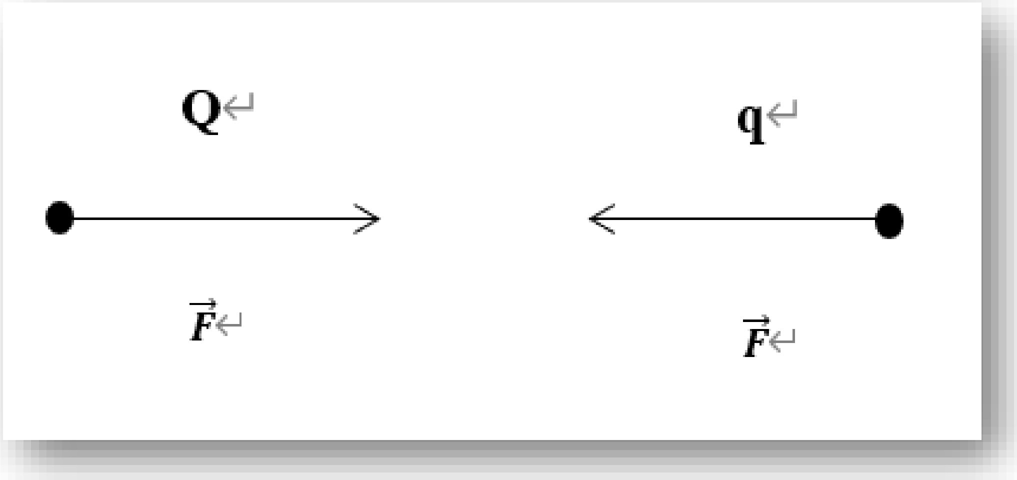


Fig.III.4. Electrostatic Force.

III.9.3. Electromagnetic Force

It is the force exerted on a charge $q$placed in an electric field $\vec{E}$and a magnetic field $\vec{B}$. It is called the **Lorentz Force**, and is expressed as:

$$\vec{F} = q(\vec{E} + \vec{v} \wedge \vec{B}) \qquad \text{(III.28)}$$

where $\vec{v}$is the velocity of the charge q.

III.9.4. Nuclear Force

This is the force acting inside the atomic nucleus, keeping it bound and stable.

III.9.5. Reaction Force on a Smooth Solid Support (Without Friction)

Let an object be placed on a table.

- $\vec{P}$:the weight of the object (due to gravity)
- $\vec{R}$:the reaction force of the table on the object

- (opposite to the weight)

Hence, the object remains at rest:

$$\vec{P} + \vec{R} = \vec{0} \qquad \text{(III.29)}$$

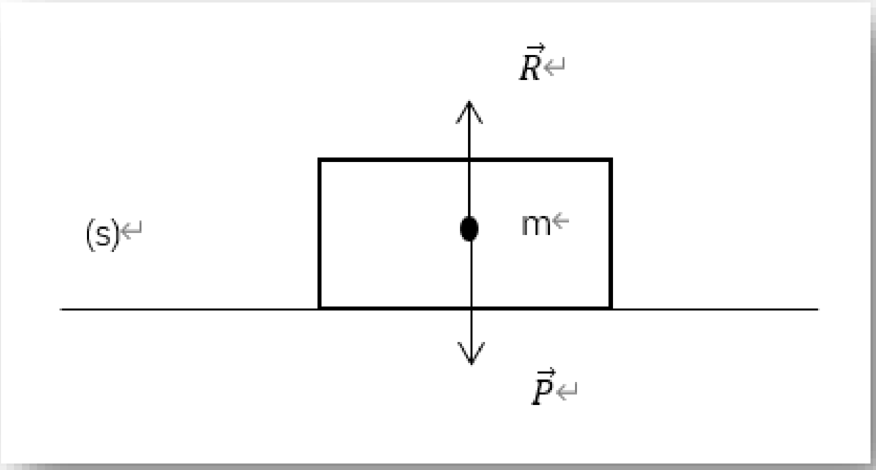


Fig.III.5. Reaction Force on a Smooth Solid Support (Without Friction).

III.9.6. Friction Force

It is a resistive force that appears as a tangential contact force, opposing motion.

It is called the frictional force.

III.9.6.1. Types of Friction

A. Static Friction

Static friction is the force that keeps a body at rest even when external forces tend to move it. In this case, the resultant of the static friction force balances the resultant of the external forces, and the object remains motionless.

<u>Example:</u>

Consider a body placed on a horizontal surface, subjected to four forces:

- The weight $\vec{P}$

- The normal reaction $\vec{N}$ of the surface
- The static friction $\overrightarrow{f_s}$ which prevents motion
- An applied force $\vec{F}$

For the object to remain at rest, according to Newton's Second Law: $\sum \vec{F} = \vec{0}$

$$\vec{P} + \vec{F} + \vec{N} + \vec{f_s} = \vec{0}$$

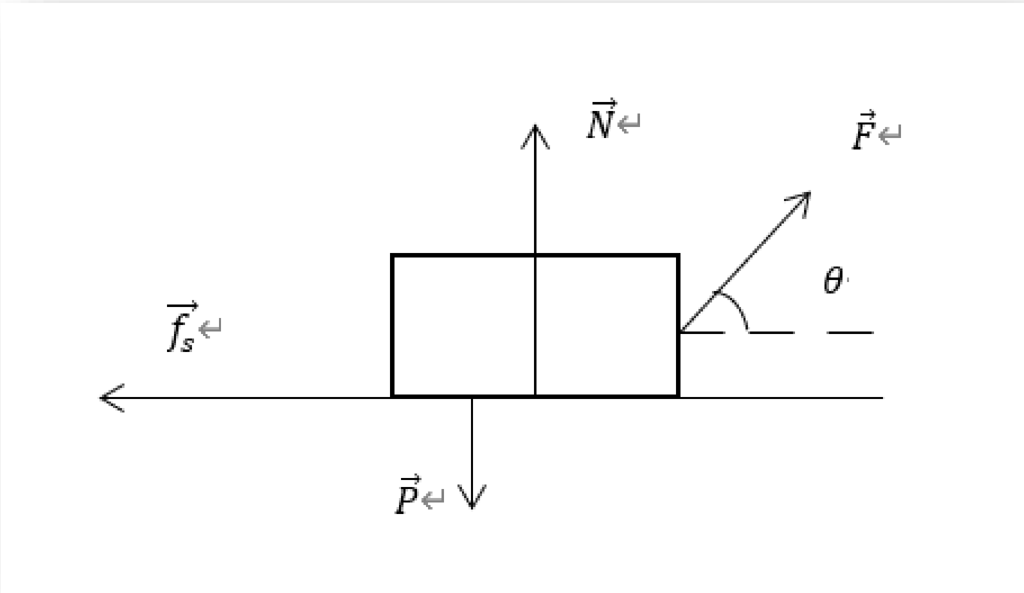


Fig.III.6. Static Friction.

Projecting on the vertical and horizontal axes gives:

$$\begin{cases} N - P + F\sin\theta = 0 \\ F\cos\theta - f_s = 0 \Rightarrow F\cos\theta = f_s \end{cases}$$

Static friction is proportional to the normal force N:

$$f_s = \mu_s N \quad \text{(III.30)}$$

Where:

μ_s is a positive constant called the coefficient of static friction.

B. Kinetic Friction

Once the object starts to move on a rough surface, the static friction becomes kinetic friction, and the relation becomes:

$$f_c = \mu_c N \tag{III.31}$$

where:

- f_c is the kinetic frictional force, always opposite to the direction of motion.
- μ_c is a positive constant called the coefficient of kinetic friction.

We observe that the kinetic frictional force is also proportional to the normal force N.

Note:

The coefficient of static friction is always greater than the coefficient of kinetic friction

$$\mu_s > \mu_c$$

For this reason, it is harder to set a stationary car in motion by pushing it, but once the car starts moving, a smaller force is needed to keep it moving.

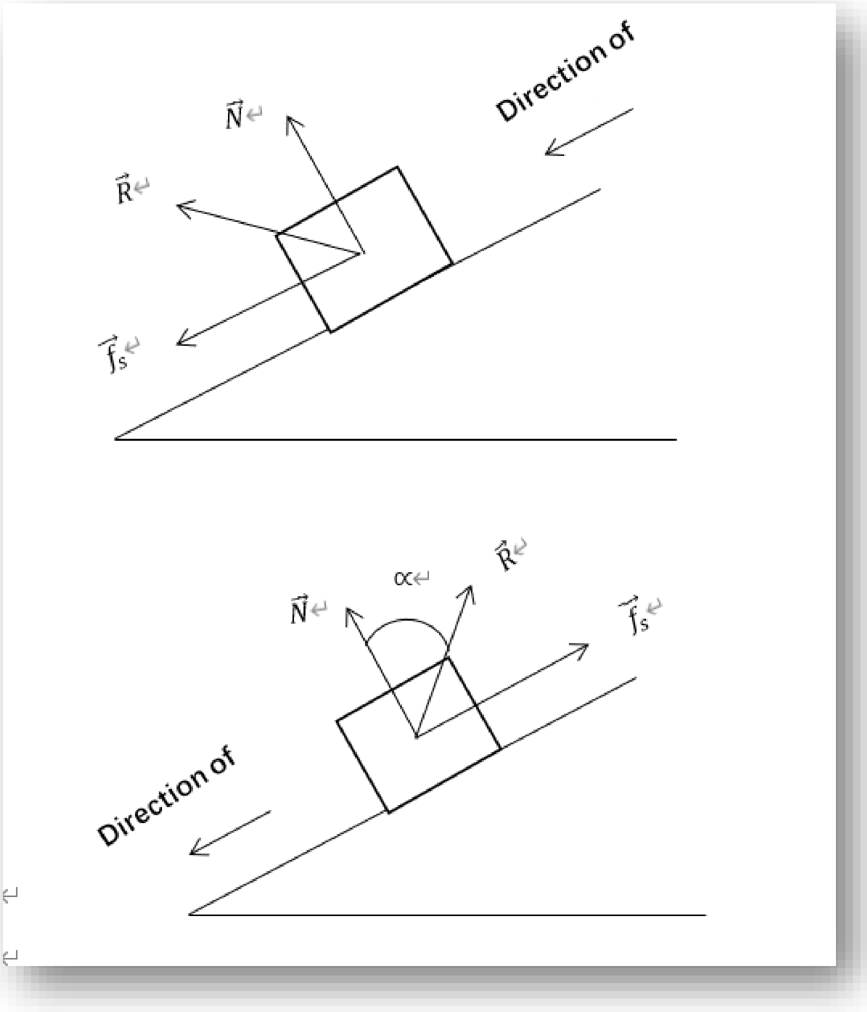


Fig.III.7. Kinetic Friction.

The total reaction force $(\vec{R})$ exerted by a surface on an object can be resolved into two perpendicular components:

- The first component, N: perpendicular to the contact surface, the normal reaction.
- The second component, f_s :parallel to the contact surface, represents the frictional force between the two objects.

From the figure, we have:

$$\tan\alpha = \frac{f_s}{N} = \mu_s \qquad \text{(III.32)}$$

$$\tan\alpha = \frac{f_c}{N} = \mu_c \qquad \text{(III.33)}$$

The angle α is called the angle of friction.

III.9.7. Viscous Friction

When a solid object moves through a fluid (a gas or a liquid) with a relatively low velocity, a frictional (drag) force appears, given by:

$$\vec{f} = -k\eta\vec{v} \qquad \text{(III.34)}$$

Where:

- $\vec{v}$:velocity of the body
- η: viscosity coefficient of the fluid $\left(\mathrm{kg}\backslash cdotpm^{-1}s^{-1}\right)$
- k :constant depending on the shape of the body

  For example, for a sphere, according to Stokes' law:

$$k = 6\pi R \qquad \text{(III.35)}$$

Hence:

$$\vec{f} = -6\pi R\eta\vec{v} \qquad \text{(III.36)}$$

III.9.8. Elastic Forces

Elastic forces appear in oscillatory motions, as they generate periodic movement:

$$F = -Kx \qquad \text{(III.37)}$$

Where k is the spring constant (elastic constant).

III.9.9. Tension Forces

Tension forces appear, for example, in the oscillation of a pendulum.

III.9.10. Application (The Inclined Plane)

A metal box of mass $m = 10\,kg$ is at rest on a wooden plank (initially horizontal). Let μ_s and μ_c be, respectively, the coefficients of static and kinetic friction between the box and the wooden plank.

Using a lever, we gradually raise one end of the plank, and we denote by α the angle that the plank makes with the horizontal.

We are asked to determine:

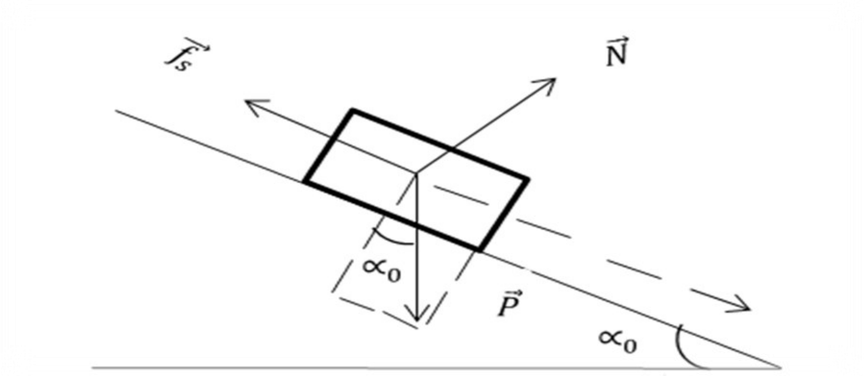
1. The maximum value α_0 of α for which we can raise the plank without the box slipping.
2. Starting from this critical equilibrium position, we raise the plank slightly so that the box begins to slide, find the acceleration of the box.
3. Then, we lower the plank gradually, i.e. we decrease the angle α; find the value α_1 for which the motion becomes uniform.

<u>**Solution**</u>

1. Finding the value of α_0:

At the limit of equilibrium (just before the box starts to slide), according to the fundamental law of dynamics:

$$\sum \vec{F} = \vec{0} \Rightarrow \vec{P} + \vec{N} + \vec{f_s} = \vec{0}$$



Projection on the x-axis (along the incline):

$$mg\sin\alpha_0 - f_s = 0 \ \Rightarrow\ f_s = mg\sin\alpha_0 \qquad (1)$$

Projection on the y-axis (perpendicular to the incline):

$$-mg\cos\alpha_0 + N = 0 \ \Rightarrow\ N = mg\cos\alpha_0 \qquad (2)$$

Since:

$$f_s = \mu_s N \qquad (3)$$

From (1), (2), and (3):

$$\mu_s = \frac{f_s}{N} = \frac{mg\sin\alpha_0}{mg\cos\alpha_0} = \tan\alpha_0$$

2. Calculating the acceleration a**:**

When the box begins to move, we apply Newton's second law:

$$\sum \vec{F} = m\vec{a} \ \Rightarrow\ \vec{P} + \vec{N} + \vec{f} = m\vec{a}$$

Projection on the x-axis:

$$P_x - f_c = ma \ \Rightarrow\ mg\sin\alpha - f_c = ma \qquad (1)'$$

Projection on the y-axis:

$$-P_y + N = 0 \ \Rightarrow\ -mg\cos\alpha + N = 0 \ \Rightarrow\ N = mg\cos\alpha \qquad (2)'$$

Since:

$$f_c = \mu_c N \qquad (3)'$$

From (1)', (2)', and (3)':

$$a = g\sin\alpha - \mu_c g\cos\alpha$$

$$\boxed{a = g(\sin\alpha - \mu_c\cos\alpha)} \qquad (*)$$

3. Finding the value of α_1:

For uniform motion, the acceleration is zero:

$$a = 0$$

Substituting into equation (*):

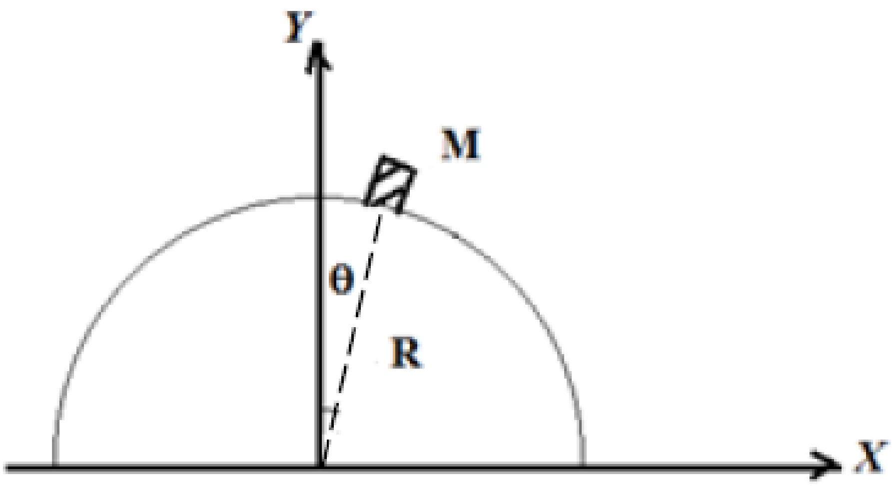
$$\mu_c = \frac{f_c}{N} = \frac{mg\sin\propto_1}{mg\cos\propto_1} = \tan\propto_1 \Leftarrow \begin{cases} mg\sin\propto_1 - f_c = 0 \Rightarrow f_c = mg\sin\propto_1 \\ -mg\cos\propto_1 + N = 0 \Rightarrow N = mg\cos\propto_1 \\ f_c = \mu_c N \Rightarrow \mu_c = \dfrac{f_c}{N} \end{cases}$$

$$\propto_1 = \arctan\mu_c$$

Exercise 01

An object of mass *m* slides without friction on the surface of an ice hemisphere of radius *R*. The object is released from the top of the hemisphere with an initial speed.

1. Determine the forces acting on the object, then calculate the normal reaction force **N** at point **M** as a function of *m*, *g*, and the angle θ, where *g* represents the acceleration due to gravity.
2. Find the angle at which the object leaves the surface of the hemisphere.



Solution:

1. Forces acting on the object

At a general point M on the hemisphere, two forces act on the object:

1. Weight:

$$\vec{p} = m\vec{g}$$

directed vertically downward.

2. Normal reaction force N:

   Exerted by the surface, directed radially outward, perpendicular to the surface.

2. Expression of the normal reaction force $N(\theta)$

Radial direction (toward the center)

The radial component of the weight is:

$$mg\cos\,\theta$$

Applying Newton's second law in the radial direction:

$$mg\cos\,\theta - N = m\frac{v^2}{R}$$

So,

$$\boxed{N = mg\cos\,\theta - m\frac{v^2}{R}}$$

3. Speed as a function of θ

Since there is no friction, mechanical energy is conserved.

Height loss:

$$h = R(1 - \cos\,\theta)$$

Energy conservation:

$$\frac{1}{2}mv^2 = mgR(1 - \cos\,\theta$$

Thus,

$$\boxed{v^2 = 2gR(1 - \cos\theta)}$$

4. Normal force as a function of θ

Substitute v^2 into the expression for N:

$$N = mg\cos\theta - m\frac{2gR(1 - \cos\theta)}{R}$$

$$N = mg\cos\theta - 2mg(1 - \cos\theta)$$

$$\boxed{N = mg(3\cos\theta - 2)}$$

5. Angle at which the object leaves the surface

The object leaves the hemisphere when the normal force becomes zero:

$$N = 0$$

$$3\cos\theta - 2 = 0$$

$$\cos\theta = \frac{2}{3}$$

$$\boxed{\theta = \cos^{-1}\left(\frac{2}{3}\right)}$$

Numerical value:

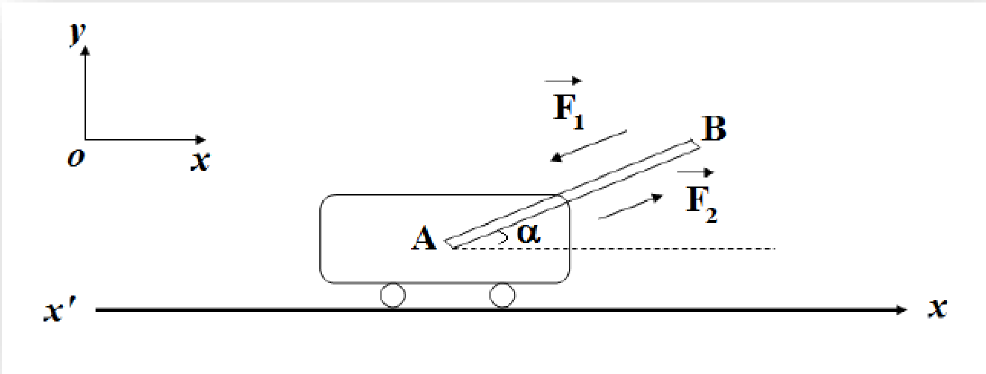
$$\theta \approx 48.2°$$

Exercise 2:

A man moves a cart of mass m on a rough horizontal road with a coefficient of friction μ.

He pushes the cart forward with a force $\vec{F}_1$, causing it to accelerate at $\vec{\gamma}$. Then he pushes the cart backward with a force $\vec{F}_2$, causing it to accelerate at the same rate.

- prove that one of the two forces is greater than the other.



Solution:

1. Forces acting on the cart

The weight of the cart is:

$$p = mg$$

The normal force depends on whether the man pushes downward or upward:

- Pushing forward: suppose he leans on the cart, adding a vertical component $F_1 \sin\ \theta$downward.

  Then the normal force is:

$$N_1 = mg + F_1 \sin\ \theta$$

- Pushing backward: he may pull up on the handle, reducing the normal force:

$$N_2 = mg - F_2 \sin\ \theta$$

Friction is:

$$f = \mu N$$

So:

1. Forward push:

$$F_1 \cos\ \theta - \mu N_1 = m\gamma$$

2. Backward push:

$$F_2\cos\,\theta - (-\mu N_2) = m\gamma$$

or more clearly, friction opposes motion, so backward push:

$$F_2\cos\,\theta - \mu N_2 = m\gamma$$

2. Solve for the forces

Forward push:

$$F_1\cos\,\theta - \mu(mg + F_1\sin\,\theta) = m\gamma$$

$$F_1\cos\,\theta - \mu mg - \mu F_1\sin\,\theta = m\gamma$$

$$F_1(\cos\,\theta - \mu\sin\,\theta) = m\gamma + \mu mg$$

$$\boxed{F_1 = \frac{m\gamma + \mu mg}{\cos\,\theta - \mu\sin\,\theta}}$$

Backward push:

$$F_2\cos\,\theta - \mu(mg - F_2\sin\,\theta) = m\gamma$$

$$F_2\cos\,\theta - \mu mg + \mu F_2\sin\,\theta = m\gamma$$

$$F_2(\cos\,\theta + \mu\sin\,\theta) = m\gamma + \mu mg$$

$$\boxed{F_2 = \frac{m\gamma + \mu mg}{\cos\,\theta + \mu\sin\,\theta}}$$

3. Compare F_1 and F_2

- Denominator for F_1: $\cos\,\theta - \mu\sin\,\theta$
- Denominator for F_2: $\cos\,\theta + \mu\sin\,\theta$

Since $\mu > 0$ and $\sin\,\theta > 0$:

$$\cos\,\theta - \mu\sin\,\theta < \cos\,\theta + \mu\sin\,\theta$$

Dividing the same numerator by smaller denominator → larger force

$$\Rightarrow F_1 > F_2$$

Chapter IV: Rotational Motion

IV.1. Introduction

The curved motion of a mass depends on both the force applied to it and the radius of curvature of its path. The effect of the force is maximal when it is applied perpendicular to the radius of curvature. To facilitate the study of this motion, we introduce a new dynamic quantity called angular momentum, which takes into account both the force and the radius of curvature. This concept allows us to derive the equation of motion, especially when a material point moves around a given axis.

IV.2. Angular momentum

Angular momentum ($\vec{L}$) is a measure of the rotational motion of an object around a point or axis. Mathematically, for a particle:

$$\vec{L} = \vec{r} \wedge \vec{p} \tag{IV.1}$$

Where:

- $\vec{r}$: position vector relative to the reference point
- $\vec{p}$: linear momentum ($\vec{p} = m\vec{v}$)
- $\wedge$: cross product (gives a vector perpendicular to both $\vec{r}$and $\vec{p}$)

So, the angular momentum depends on:

- How far the object is from the axis or point (r)
- Its mass (m)
- Its velocity (v)
- The angle between $\vec{r}$ and $\vec{v}$

Angular momentum is a fundamental concept in physics that describes the rotational motion of an object around a specific point or axis. For a single object, such as a ball or a planet, angular momentum is defined as the cross product of the position vector (from the reference point to the object) and the linear momentum of the object. This means it depends not only on the mass and velocity of the object, but also on how far it is from the point of rotation and the direction of its motion relative to that point. The magnitude of the angular momentum is given by:

$$L = r\,p\sin\,\theta = r\,m\,v\sin\,\theta \qquad \text{(VI.2)}$$

where:

θ is the angle between the position vector $\vec{r}$ and the velocity vector $\vec{v}$.

- If the object moves directly away or toward the point ($\theta = 0°$ or $180°$), $L = 0$
- If it moves perpendicular to the radius ($\theta = 90°$), L is maximum

This shows that if the object moves directly toward or away from the point, the angular momentum is zero, while it is maximum when the motion is perpendicular to the radius. The direction of the angular momentum vector is perpendicular to the plane formed by the position and velocity vectors, following the right-hand rule. One of the most important properties of angular momentum is its conservation: if no external torque acts on the object, its angular momentum remains constant. This principle explains many natural phenomena, such as why a spinning ball speeds up when it moves closer to the rotation axis.

IV.2.1. Theory of Angular Momentum

The theory of angular momentum explains how objects rotate around a point or axis and how their rotational motion is related to their mass, velocity, and position relative to the axis. For a single object, angular momentum is defined as the cross product of the position vector ($\vec{r}$) and the linear momentum ($\vec{p} = m\vec{v}$) of the object see relationship (IV.1). This shows that angular momentum depends not only on the mass and velocity of the object, but also on the distance from the axis and the direction of motion relative to that axis. The magnitude of angular momentum is calculating by the relationship (VI.2). The direction of angular momentum is perpendicular to the plane formed by $\vec{r}$ and $\vec{v}$, determined by the right-hand rule. A central principle in this theory is the conservation of angular momentum: if no external torque acts on a system, its angular momentum remains constant. This principle explains many real-world phenomena, such as why a spinning ice skater accelerates when pulling in her arms or why planets maintain their orbits around the Sun. In essence, the theory of angular momentum provides a fundamental understanding of rotational motion and the forces that govern it.

Let M be a material point of mass m, moving in a Galilean frame R with origin O. The angular momentum of this material point M with respect to O is defined as the following vector quantity:

$$\vec{L}_0(M) = \overrightarrow{OM} \wedge m\vec{v} \qquad \text{(IV.3)}$$

where $\vec{p} = m\vec{v}$is the linear momentum vector of the material point, and $\vec{v}$is its velocity relative to R. The angular momentum is measured in $[\text{kgm}^2/s]$.

The angular momentum is the cross product of the position vector and the linear momentum vector, so it is a vector perpendicular to both the position vector and the velocity vector. Therefore, it is perpendicular to the trajectory of the material point M (see Fig.IV.1).

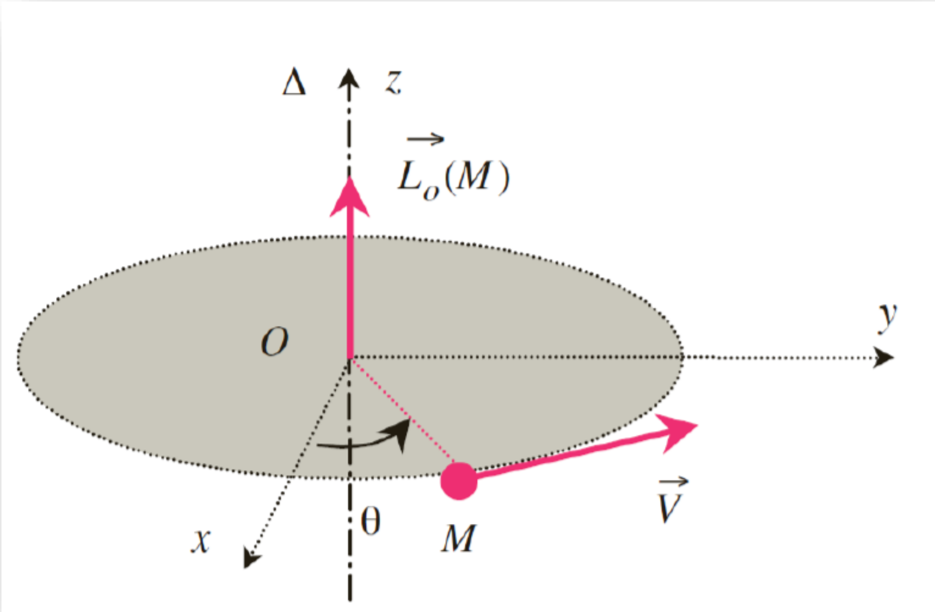


Fig.1: Angular momentum.

IV.2.2. Properties of Angular Momentum

- If $\vec{L}_0$ = constant⇒ for all t, $\overrightarrow{OM} \perp \sum\vec{F}_{ext}$

  (The angular momentum is conserved when the net external torque is zero.)

- If $\vec{L}_0$is constant ⇒ the motion is planar (the trajectory lies in a plane).

- If $\vec{L}_0 = \vec{0}$⇒ the motion is straight-line

  (This happens when $\overrightarrow{OM} \parallel \vec{v}$, so the position vector is collinear with the velocity vector).

IV.3. Moment of Force

The moment of force, also called torque, is a measure of the tendency of a force to rotate an object about a point or axis. It depends not only on the magnitude of the applied force but also on how far from the point of rotation the force is applied and the direction of the force relative to the lever arm. Mathematically, the moment of force ($\vec{M}$) is expressed as the cross product of the position vector ($\vec{r}$) from the axis to the point of application and the force vector ($\vec{F}$) applied.

$\vec{r}$ is the distance from the axis to where the force acts, and the magnitude of the moment is given by:

$$M = r\,F \sin\,\theta \tag{IV.4}$$

where θ is the angle between $\vec{r}$ and $\vec{F}$. The direction of the moment vector is perpendicular to the plane formed by $\vec{r}$ and $\vec{F}$, following the right-hand rule.

The concept of the moment of force is fundamental in mechanics, as it explains why forces applied at different points or angles cause different rotational effects. For example, pushing a door near its hinge hardly opens it, but applying the same force at the edge opposite the hinge produces a strong rotation. In essence, the moment of force links the magnitude, point of application, and direction of a force to the rotational effect it produces.

Let M be a material point of mass m, moving in a Galilean frame R with origin O, and let $\vec{F}$ be a force acting on M. The torque of the force $\vec{F}$ with respect to point O is defined as the following vector quantity:

$$\vec{M}_O(\vec{F}) = \overrightarrow{OM} \wedge m\vec{a} \tag{IV.5}$$

In the International System of Units (SI), the torque of a force is measured in Newton-meters (N·m).

Note:

Any force whose line of action passes through the point O has zero torque with respect to O.

IV.2.2. Relation Between Angular Momentum and Torque

Angular momentum ($\vec{L}$) and torque ($\vec{M}$) are closely related concepts in rotational motion. For a single object, angular momentum is defined as the cross product of the position vector and the object's linear momentum. Torque, on the other hand, is the cross product of the position vector and the force applied on the object. The connection between them comes from Newton's second law applied to rotation. The rate of change of angular momentum of the object is equal to the net torque acting on it. In a Galilean frame R, the theory of angular momentum can be applied as follows:

$$\frac{d\vec{L}_O}{dt} = \frac{d\left(\overrightarrow{OM} \wedge m\vec{v}\right)}{dt} = \frac{d\overrightarrow{OM}}{dt} \wedge m\vec{v} + \overrightarrow{OM} \wedge m\frac{d\vec{v}}{dt}$$
$$= \vec{v} \wedge m\vec{v} + \overrightarrow{OM} \wedge m\vec{a} = \overrightarrow{OM} \wedge m\vec{a}$$

where $\vec{a}$ is the acceleration of point M.

By applying the fundamental principle of dynamics, we have:

$$\frac{d\vec{L}_0}{dt} = \overrightarrow{OM} \wedge m\vec{a} = \overrightarrow{OM} \wedge \vec{F} \qquad \text{(IV.6)}$$

Thus, we obtain the angular momentum theorem:

$$\frac{d\vec{L}_0}{dt} = \sum \vec{M}_0(\vec{F}) \qquad \text{(IV.7)}$$

This means that the time derivative of the angular momentum of a material point M with respect to a fixed point O in a Galilean frame R equals the sum of the torques of the external forces acting on M relative to the same point O.

This theorem is also valid in a non-Galilean frame, provided that the torques of the inertial forces.

The relationship (IV.7) means that:

- If no torque acts on the object ($\vec{M} = 0$), its angular momentum is conserved.

The object continues to rotate with the same angular momentum.

- If a torque is applied, it changes the object's angular momentum, either by altering the magnitude (speed of rotation) or the direction of the angular momentum vector.

Example: Consider a ball tied to a string and spinning in a circle.

- Pulling the string closer to the center reduces the radius (r), but because no external torque acts on the ball, its angular momentum $\vec{L} = mrv$ is conserved.
- If a force is applied perpendicular to the radius (like pushing the ball tangentially), it creates a torque, changing the angular momentum and increasing the speed of rotation.

In essence, torque is the cause, and angular momentum is the effect: torque changes angular momentum over time, while angular momentum describes the state of rotational motion of the object. This fundamental relationship is central to understanding rotations in physics, from spinning tops to planets orbiting stars.

IV.3.1. Properties of Moment of Force (Torque)

The moment of force, or torque, describes the tendency of a force applied to an object to cause it to rotate around a specific point or axis. Unlike a straight-line force, which only moves an object linearly, torque produces rotational motion. For a single object, the torque depends on three factors: the magnitude of the applied force, the distance from the point of rotation to where the force is applied (the lever arm), and the angle between the force and the line connecting the point of rotation to the object. Mathematically, torque is expressed as $\vec{M} = \vec{r} \times \vec{F}$, or in magnitude $M = rF\sin\,\theta$, where $\theta$is the angle between the position vector $\vec{r}$and the force $\vec{F}$. This formula shows that torque is maximum when the force is applied perpendicular to the lever arm ($\theta = 90°$) and zero when the force is applied along the line from the rotation point to the object ($\theta = 0°$ or $180°$).

Torque is a vector quantity, meaning it has both magnitude and direction, with the direction perpendicular to the plane formed by the object's position and the applied force, determined by the right-hand rule. Torques from multiple forces acting on the same object can be added vectorially to determine the total rotational effect. The axis or point of rotation is critical: applying the same force at the same point can produce different torques if measured from different rotation points. Torque is directly related to the rotational acceleration of the object: the larger the torque, the faster the object rotates, following the principle $\vec{M} = I\vec{\alpha}$, where $I$is the object's moment of inertia and $\alpha$is angular acceleration. Practical examples include pushing a wheel at different points, where applying force farther from the center produces a stronger rotation, or turning a wrench, where applying force perpendicular to the handle maximizes the torque on a single bolt.

In short, for a single object, the properties of torque—its dependence on force, distance, angle, vector nature, axis of rotation, and rotational effect—explain how forces cause objects to rotate and how their rotational motion can be controlled efficiently.

IV.4. Moment of Inertia

The moment of inertia of an object is a measure of its resistance to changes in rotational motion about a specific axis. While mass measures how hard it is to change an object's linear motion, the moment of inertia measures how hard it is to change its rotational motion. For a single object treated as a particle, the moment of inertia is calculated as:

$$I = mr^2 \tag{IV.8}$$

Here, m is the mass of the object and r is the perpendicular distance from the axis of rotation. This formula shows that the farther the object is from the axis, the larger its moment of inertia, and therefore the more torque is needed to change its rotational speed.

The moment of inertia is not just about mass, but about how the mass is distributed relative to the axis. For example, two objects with the same mass can have very different moments of inertia if one is close to the axis and the other is farther away. This explains why a spinning wheel with heavy rims is harder to start or stop than one with mass concentrated near the center.

For a system of particles or a rigid object, the total moment of inertia is:

$$I = \sum_i m_i r_i^2 \qquad \text{(IV.9)}$$

Note:

- The larger the moment of inertia, the harder it is to change the rotational speed.
- Moment of inertia depends on both mass distribution and distance from the axis.

- It plays the same role in rotational motion that mass plays in linear motion:

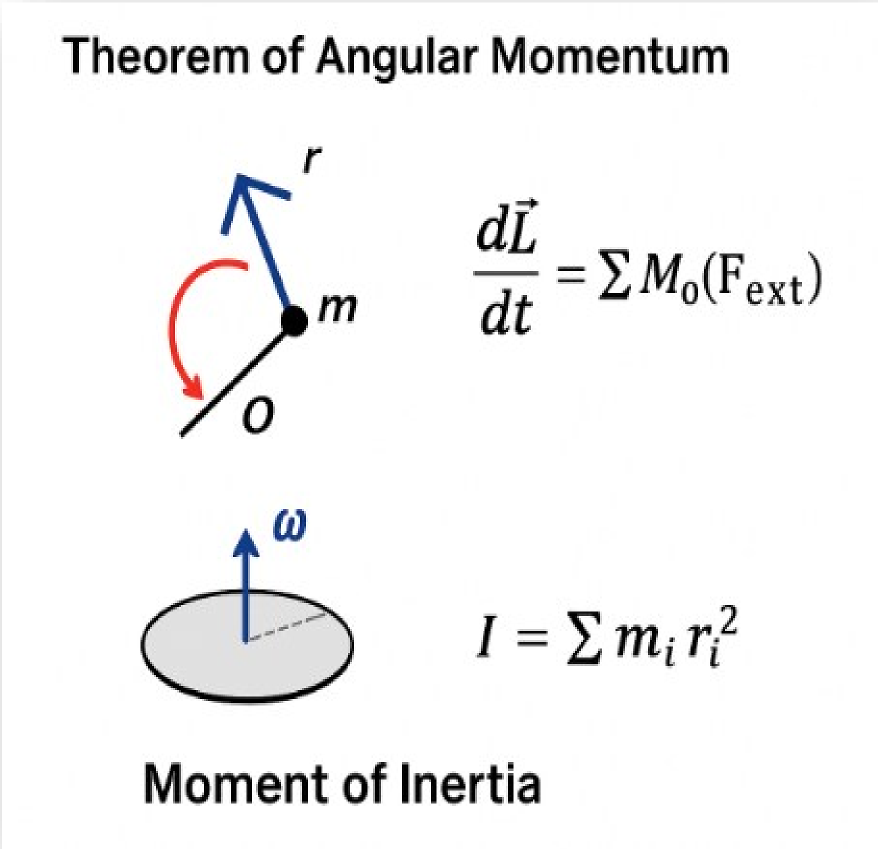


Fig.IV.2. The moment of inertia.

The moment of inertia plays a central role in rotational dynamics, appearing in Newton's rotational law:

$$\vec{M} = I\vec{\alpha} \qquad \text{(IV.10)}$$

where $\vec{M}$ is the torque applied to the object and $\vec{\alpha}$ is the angular acceleration it produces. This equation shows that the larger the moment of inertia, the more torque is needed to achieve the same angular acceleration.

IV.5. Application

IV.5.1. Simple Pendulum

A simple pendulum consists of a small object M (considered a material point) with mass m, suspended from the end of a string of length l(assumed massless and inextensible), with the other end fixed at a point O. The size of the object is negligible compared to the string length, and friction is neglected.

1. Determine the equilibrium position of the string.
2. Displace the pendulum from its equilibrium and release it from rest (initial velocity = 0). Find the differential equation of motion using the fundamental principle of dynamics (or torque/momentum theorem).

Solution

- The system is the material point $M$of mass m.
- The reference frame is Galilean, with point Ofixed.
- All forces acting on $M$are:
    - Weight: $\vec{P} = m\vec{g}$
    - Tension: $\vec{T}$

1- Equilibrium Position

Applying the fundamental principle of dynamics:

$$\vec{P} + \vec{T} = m\vec{a}$$

At equilibrium:

$$T = -P = -mg \implies \vec{P} + \vec{T} = \vec{0} \implies \sum \vec{F} = 0 \implies \vec{a} = 0$$

2- Find the differential equation of motion

Method 1: Using the fundamental principle of dynamics

When the pendulum moves (Fig. b), the path of the body is part of a circle. The best coordinates are polar coordinates (r = l):

The acceleration vector in polar coordinates is:

$$\vec{a} = (\ddot{r} - r\dot{\theta}^2)\overrightarrow{u_r} + (2\dot{r}\dot{\theta} + r\ddot{\theta})\overrightarrow{u_\theta}$$

Since $r = l$is constant, it becomes:

$$\vec{a} = (-l\dot{\theta}^2)\overrightarrow{u_r} + (l\ddot{\theta})\overrightarrow{u_\theta}$$

Then:

$$\vec{P} + \vec{T} = m\vec{a} = m(-l\dot{\theta}^2\overrightarrow{u_r} + l\ddot{\theta}\overrightarrow{u_\theta})$$

Projecting forces onto $\overrightarrow{u_r}$ and $\overrightarrow{u_\theta}$:

$$\vec{T} = -T\overrightarrow{u_r}$$

$$\vec{P} = mg\cos\,\theta\,\overrightarrow{u_r} - mg\sin\,\theta\overrightarrow{u_\theta}$$

We get two equations:

<u>**Radial:**</u>

$$mg\cos\,\theta - T = ml\dot{\theta}^2$$

<u>**Tangential:**</u>

$$-mg\sin\,\theta = ml\ddot{\theta}$$

Which leads to the nonlinear differential equation:

$$\ddot{\theta} + \frac{g}{l}\sin\,\theta = 0$$

For small angles (θ small), $\sin\,\theta \approx \theta$, giving the linearized equation:

$$\ddot{\theta} + \frac{g}{l}\theta = 0$$

Defining $\omega^2 = g/l$, we write:

$$\ddot{\theta} + \omega^2\theta = 0$$

The solution is:

$$\theta(t) = \theta_{\max}\sin(\omega t + \theta_0)$$

<u>Method 2:</u> Using Angular Momentum (Torque)

Torque of the tension relative to O:

$$\vec{M}_O(\vec{T}) = (\overrightarrow{OM}) \times \vec{T} = l\widehat{U}_r \times (-T\overrightarrow{u_r}) = 0$$

Torque of the weight relative to O:

$$\vec{M}_O(\vec{P}) = (\overrightarrow{OM}) \times \vec{P} = l\overrightarrow{u_r} \times (mg\cos\,\theta\,\overrightarrow{u_r} - mg\sin\,\theta\,\overrightarrow{u_\theta}) = -mgl\sin\,\theta\,\vec{k}$$

Angular momentum relative to O:

$$\vec{L}_O = (\overrightarrow{OM}) \times m\vec{v} = l\overrightarrow{u_r} \times (ml\dot{\theta}\overrightarrow{u_\theta}) = ml^2\dot{\theta}\,\vec{k}$$

Time derivative of angular momentum:

$$\frac{d\vec{L}_O}{dt} = ml^2\ddot{\theta}\,\vec{k}$$

Applying the angular momentum theorem:

$$\frac{d\vec{L}_O}{dt} = \sum\vec{M}_O(\vec{F}) \Rightarrow ml^2\ddot{\theta} = -mgl\sin\,\theta$$

$$\ddot{\theta} + \frac{g}{l}\sin\,\theta = 0 \Rightarrow \ddot{\theta} + \frac{g}{l}\theta = 0 \;\text{ (small angles)}$$

Tension as a function of angle:

$$T = m(g\cos\,\theta - l\dot{\theta}^2)$$

Exercice 01

An object of mass m= 6 Kg is given a position vector:

$$\vec{r} = (3t^2 - 6t)\vec{\imath} + (-4t^3)\vec{\jmath} + (3t + 2)\vec{k}$$

Find:

1. The force $\vec{F}_1$ acting on the body.
2. The torque (moment of force) $\vec{F}_2$ with respect to the origin.
3. The momentum $\vec{P}$ of the body and its angular momentum $\vec{L}_0$ with respect to the origin.
4. Verify that:

$$\vec{F} = \frac{d\vec{P}}{dt} \quad \text{and} \quad \overrightarrow{M_0}(\vec{F}) = \frac{d\vec{L}}{dt}$$

Solution

1. Force acting on the object:

$$\vec{F} = m\vec{\gamma} = m\ddot{\vec{r}} \Rightarrow \vec{F} = 36\hat{\imath} - 144t\hat{\jmath}$$

2. Torque of the force relative to the origin:

$$\overrightarrow{M}_0(\vec{F}) = \vec{r} \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3t^2 - 6t & -4t^3 & 3t + 2 \\ 36 & -144t & 0 \end{vmatrix}$$

$$\overrightarrow{M}_0(\vec{F}) = (432t^2 + 288t)\hat{\imath} + (108t + 72)\hat{\jmath} + (-288t^3 + 864t^2)\hat{k}$$

3. Linear momentum of the object:

$$\vec{P} = m\vec{v} = 36(t - 1)\hat{\imath} - 72t^2\hat{\jmath} + 18\hat{k}$$

4. Angular momentum of the object relative to the origin:

$$\vec{L}_0 = \vec{r} \times \vec{P} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3t^2 - 6t & -4t^3 & 3t + 2 \\ 36(t - 1) & -72t^2 & 18 \end{vmatrix}$$

$$\vec{L}_0 = (144t^3 + 144t^2)\hat{\imath} + (54t^2 + 72t - 72)\hat{\jmath} + (-72t^4 + 288t^3)\hat{k}$$

Also:

$$\vec{P} = 36(t - 1)\hat{\imath} - 72t^2\hat{\jmath} + 18\hat{k}$$

$$\frac{d\vec{P}}{dt} = 36\hat{\imath} - 144t\hat{\jmath} = \vec{F}$$

and similarly,

$$\vec{M}_0(\vec{F}) = \frac{d\vec{L}_0}{dt}$$

Exersice02

Exercise: Force, Torque, and Angular Momentum of an Object

An object of mass 4 kg moves in space. Its position vector as a function of time is:

$\vec{r}(t) = (2t^2 - t)\,\vec{\imath} + (t^3 - 2t)\,\vec{\jmath} + (3t + 1)\,\vec{k}$ meters

1. Find the velocity $\vec{v}(t)$and acceleration $\vec{a}(t)$of the object.
2. Determine the force acting on the object.
3. Compute the linear momentum $\vec{P}(t)$of the object.
4. Find the torque of the force relative to the origin: $\vec{M}_0 = \vec{r} \times \vec{F}$.
5. Compute the angular momentum of the object relative to the origin: $\vec{L}_0 = \vec{r} \times \vec{P}$.
6. Verify that $\vec{F} = \frac{d\vec{P}}{dt}$ and $\vec{M}_0 = \frac{d\vec{L}_0}{dt}$.

Solution

Position vector:

$$\vec{r}(t) = (2t^2 - t)\,\vec{\imath} + (t^3 - 2t)\,\vec{\jmath} + (3t + 1)\,\vec{k}$$

Velocity vector:

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = (4t - 1)\,\vec{\imath} + (3t^2 - 2)\,\vec{\jmath} + 3\,\vec{k}$$

Acceleration vector:

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = 4\vec{\imath} + 6t\,\vec{\jmath} + 0\,\vec{k}$$

2- Force Acting on the Object

$$\vec{F} = m\vec{a} = 4 \cdot (4\,\vec{\imath} + 6t\,\hat{\jmath} + 0\,\vec{k}) = 16\,\vec{\imath} + 24t\,\vec{\jmath} + 0\,\vec{k}\ \text{N}$$

3- Linear Momentum of the Object

$$\vec{P} = m\vec{v} = 4 \cdot (4t - 1)\vec{\imath} + 4 \cdot (3t^2 - 2)\vec{\jmath} + 4 \cdot 3\vec{k}$$

$$= (16t - 4)\vec{\imath} + (12t^2 - 8)\vec{\jmath} + 12\vec{k}$$

4- Torque of the Force Relative to the Origin

$$\vec{M}_0 = \vec{r} \times \vec{F}$$

$$\vec{M}_0 = [-24t(3t + 1)]\vec{\imath} + [48t + 16]\vec{\jmath} + [-16t^3 + 24t(2t^2 - t) + 32t]\vec{k}$$

5- Angular Momentum of the Object Relative to the Origin

$$\vec{L}_0 = \vec{r} \times \vec{P}$$

$$\vec{L}_0 = [12t^3 - 24t - (3t + 1)(12t^2 - 8)]\vec{\imath} + [-24t^2 + 12t + (3t + 1)(16t - 4)]\vec{\jmath} + [-(16t - 4)(t^3 - 2t) + (2t^2 - t)(12t^2 - 8)]\vec{k}$$

6- Verification of Newton's Laws

- Time derivative of linear momentum:

$$\frac{d\vec{P}}{dt} = 16\vec{\imath} + 24t\vec{\jmath} + 0\vec{k} = \vec{F}$$

- Time derivative of angular momentum:

$$\frac{d\vec{L}_0}{dt} = \vec{M}_0$$

This confirms that:

$$\vec{F} = \frac{d\vec{P}}{dt}, \vec{M}_0 = \frac{d\vec{L}_0}{dt}$$

Exercice 03

An object of mass m = 3 kg has the velocity vector:

$$\vec{v} = (4\text{t} - 5)\vec{\imath} + (-2\text{t} + 1)\vec{\jmath} + 5\,\vec{k}$$

Find:

1. The force $\vec{F}_1$ acting on the object.
2. The torque (moment of force) $\vec{F}_2$ with respect to the origin.
3. The momentum $\vec{P}$ of the body and its angular momentum $\vec{L}_0$ with respect to the origin.

Solution

1- Force acting on the object $\vec{F}_1$

First, compute the acceleration:

$$\vec{a}(t) = \frac{d\vec{v}}{dt}$$

$$\vec{a}(t) = 4\vec{\imath} - 2\,\vec{\jmath} + 0\,\vec{k}$$

Now apply Newton's second law:

$$\vec{F}_1 = m\vec{a}$$

$$\boxed{\vec{F}_1 = 12\,\vec{\imath} - 6\,\vec{\jmath}}$$

2- Momentum of the object $\vec{P}$

$$\vec{P} = m\vec{v}$$

$$\vec{P}(t) = 3(4t - 5)\,\vec{\imath} + 3(-2t + 1)\,\hat{\jmath} + 3(5)\,\hat{k}$$

$$\boxed{\vec{P}(t) = (12t - 15)\,\vec{\imath} + (-6t + 3)\,\vec{\jmath} + 15\vec{k}}$$

3- Position vector of the object

Since:

$$\vec{v} = \frac{d\vec{r}}{dt}$$

Integrate each component (with $\vec{r}(0) = 0$):

$$\vec{r}(t) = (2t^2 - 5t)\,\vec{\imath} + (-t^2 + t)\,\vec{\jmath} + 5t\,\vec{k}$$

4- Torque (moment of force) with respect to the origin $\vec{F}_2$

Torque is defined as:

$$\vec{F}_2 = \vec{M}_0 = \vec{r} \times \vec{F}_1$$

$$\vec{M}_0 = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 2t^2 - 5t & -t^2 + t & 5t \\ 12 & -6 & 0 \end{vmatrix}$$

Result:

$$\boxed{\vec{M}_0 = 30t\,\vec{\imath} + 60t\,\vec{\jmath} + (12t^2 - 30t)\,\vec{k}}$$

5- Angular momentum of the object with respect to the origin $\vec{L}_0$

$$\vec{L}_0 = \vec{r} \times \vec{P}$$

$$\vec{L}_0 = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ 2t^2 - 5t & -t^2 + t & 5t \\ 12t - 15 & -6t + 3 & 15 \end{vmatrix}$$

After calculation:

$$\boxed{\vec{L}_0 = (15t^2)\,\vec{\imath} + (30t^2)\,\vec{\jmath} + (4t^3 - 15t^2)\,\hat{k}}$$

6- Verification (important result)

- Force–momentum relation:

$$\frac{d\vec{P}}{dt} = 12\,\vec{\imath} - 6\,\vec{\jmath} = \vec{F}_1$$

- Torque–angular momentum relation:

$$\frac{d\vec{L}_0}{dt} = \vec{M}_0$$

Chapter V :
Work, Power, Energy

V.1. Introduction

To move an object from one place to another, a force must be applied, causing its speed to change from zero to a noticeable value. In this case, we say that the applied force has performed work during the body's displacement, and the body has gained kinetic energy from this work.

In this chapter, we define the work of a force and the theorem of kinetic energy. We also define a special type of forces called conservative forces, as well as potential energy.

In physics, mechanical work is the amount of energy required to move an object by applying a force over a certain distance.

Mechanical work depends on:

- The force magnitude $\vec{F}$
- The displacement of the point of application from A to B
- The angle (α) between the force vector $\vec{F}$ and the displacement vector $\overrightarrow{AB}$

A force is said to be constant if:

- Its magnitude (intensity) remains constant.
- Its direction remains constant.

V.2. Work and Power of a Force

Work and power of a force describe how forces cause motion and how fast energy is transferred. Work is done by a force when it causes an object to move in the direction of the force. It is equal to the product of the force and the displacement in the direction of the force, so if a force acts but there is no movement, no work is done. Power measures how quickly work is done or energy is transferred. It depends on both the amount of work done and the time taken to do it. A force that does the same work in less time has greater power. Together, work and power help explain not only whether a force can move an object, but also how efficiently and quickly it does so.

V.2.1. Work of a Constant Force

If a force $\vec{F}$ acts on an object and moves it from point A to another point B, we say that the force has done work.

We define the elementary work (corresponding to a small displacement) and denote it by δW for the force $\vec{F}$ when the point M undergoes an elementary displacement $d\vec{r}$, by the following relation:

$$\delta W = \vec{F} \cdot d\vec{r} \qquad \text{(V.1)}$$

The unit of work in the International System of Units is the Joule (J), which equals a Newton-meter: 1 Nm = 1 J.

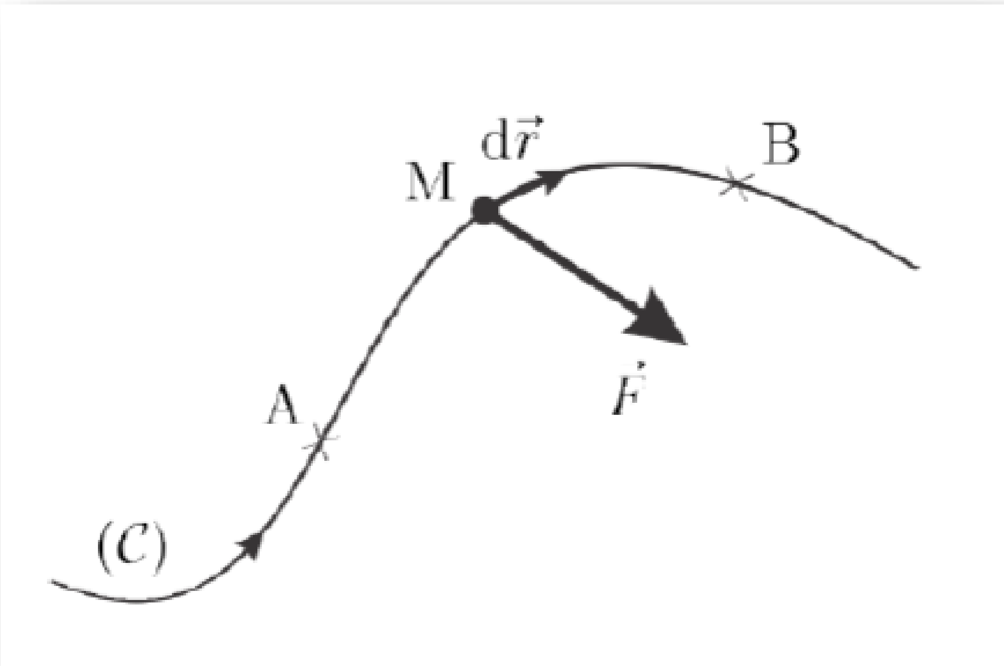


Fig.V1. shows the force $\vec{F}$ and the elementary displacement $d\vec{r}$.

If a material point moves along the path C from point A to point B, then the work of this force $\vec{F}$during this displacement is the sum of all the elementary displacements between the starting point A and the ending point B. This sum includes an infinite number of terms corresponding to the elementary displacements, which is equivalent to the integral between points A and B. We write the total work as:

$$W_{AB}(\vec{F}) = \int_A^B \delta W = \int_A^B \vec{F} \cdot d\vec{r} \qquad \text{(V.2)}$$

We notice that, in the general case, the work done by the force $\vec{F}$ for the displacement of point M from A to B depends on the path taken. Some forces do not depend on the path; these are called conservative forces.

Notes:

- Work is a scalar quantity, so it can be positive, negative, or zero.
- When the force $\vec{F}$is perpendicular to the path at every moment, $\vec{F} \cdot d\vec{r} = 0$, meaning the work of this force is zero.
- If the work of the force $\vec{F}$is positive, i.e., $W_{AB}(\vec{F}) > 0$, it is called driving work; in this case, the force increases the speed of point M.
- If the work of the force $\vec{F}$is negative, i.e., $W_{AB}(\vec{F}) < 0$, it is called resisting work; in this case, the force decreases the speed of point M.
- If F_x, F_y, F_z are the Cartesian components of the force vector $\vec{F}$, and $dx, dy, dz$are the Cartesian components of the elementary displacement $d\vec{r}$, then the work can be written as:

$$W_{AB}(\vec{F}) = \int_A^B \vec{F} \cdot d\vec{r} = \int_A^B (F_x dx + F_y dy + F_z dz) \qquad \text{(V.3)}$$

The work of a constant force $\vec{F}$, when a point of application moves from point A to point B, is given by the dot product:

$$W_{A\to B}(\vec{F}) = \vec{F}\cdot\overrightarrow{AB} = F\,AB\cos\,\alpha \tag{V.4}$$

Where:

- F is the magnitude of the force,
- AB is the magnitude of the displacement,
- $\alpha$is the angle between the force vector $\vec{F}$and the displacement vector $\overrightarrow{AB}$.

The unit of work in the International System of Units (SI) is the Joule (J), $1\text{ J} = 1\text{ Nm}$

In electricity, we also use the electron volt (eV), $1\text{ eV} = 1.6\times10^{-19}\text{ J}$

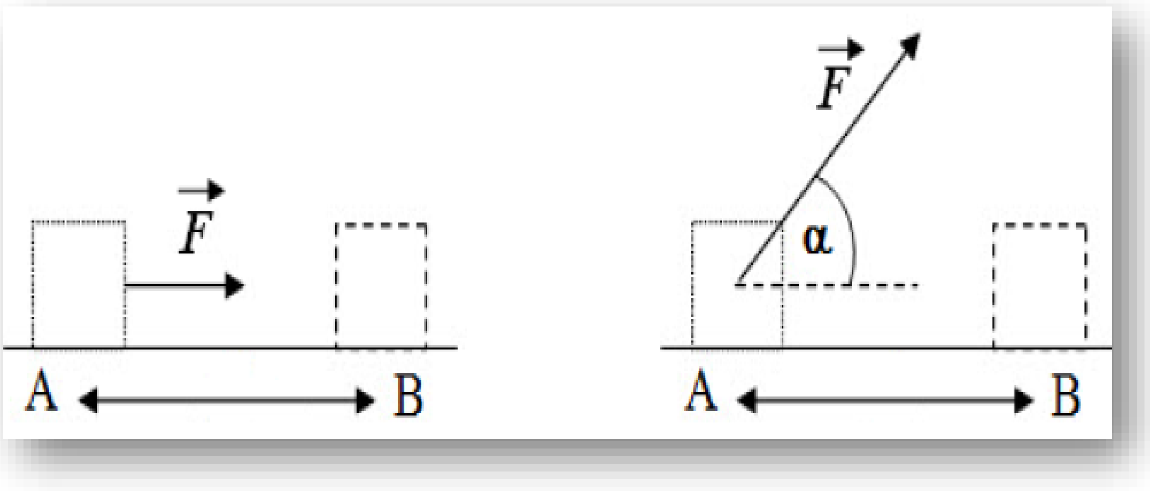


Fig.V.2. The work of a constant force $\vec{F}$.

Note

The work of a constant force does not depend on the path taken by the point of application.

$W_{A\to B}(\vec{F})$ it has the same value whether the path is straight or curved (see Fig.V.3).

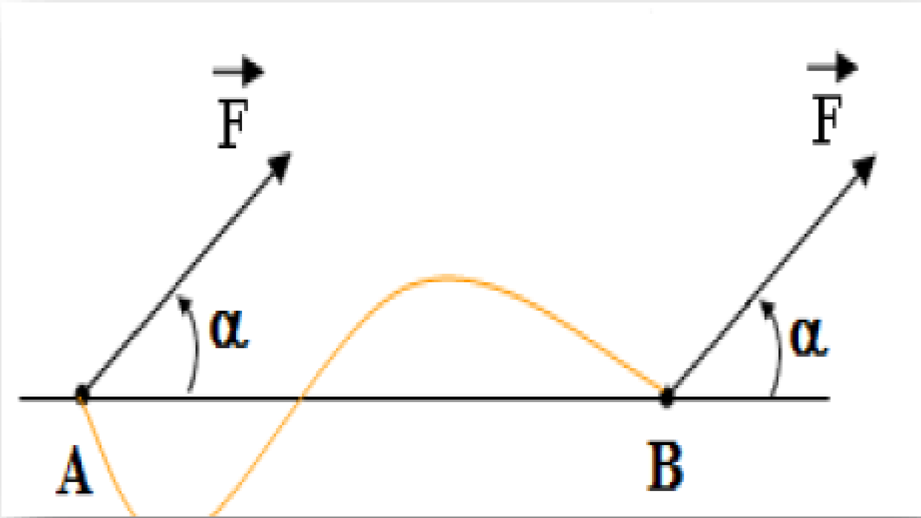


Fig.V.3. The work of a constant force does not depend on the path.

V.2.1.1. Work of the Weight Force

The weight force of an object, which represents the Earth's gravitational attraction on that object, can be considered a constant force within a limited region of space. We encounter this force every time we study a mechanical system on Earth, and therefore it is essential to know how to calculate the work of this force.

We want to find the expression of the work done by the weight force for the displacement of a material point m of mass M moving from point **A** to point **B** (see Fig.V.4). We consider that the displacement occurs along different possible paths.

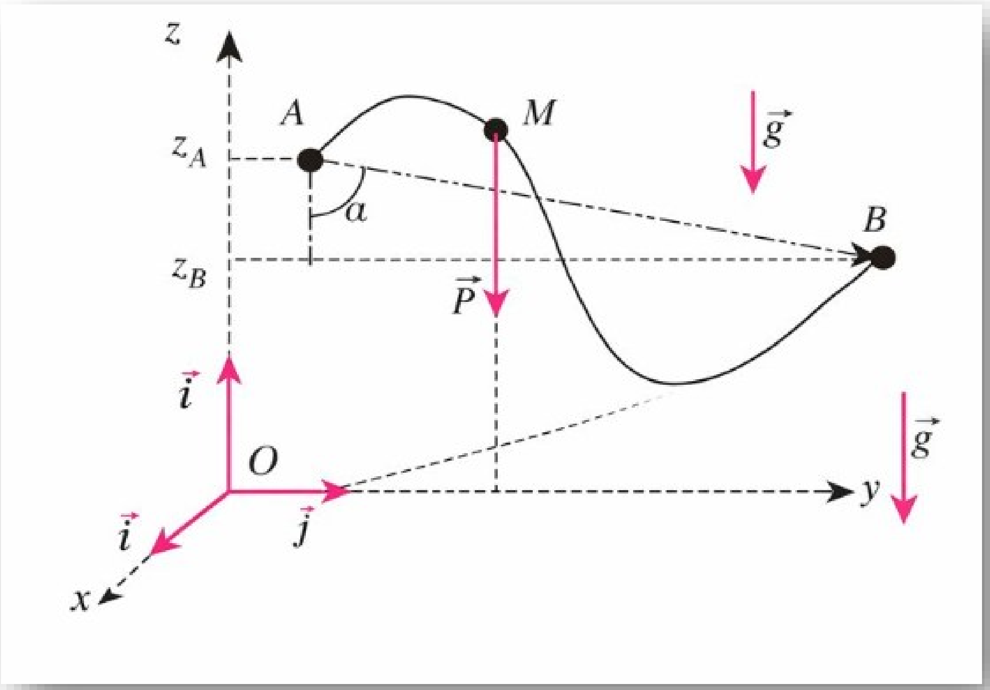


Fig.V.4. Represents the work of the weight force during the displacement from A to B.

Since the weight force is a constant force, and according to relation (V.1), we can write:

$$W_{AB}(\vec{P}) = \vec{P} \cdot \overrightarrow{AB} \qquad \text{(V.5)}$$

The dot product can be calculated in two ways:

► Using the cosine of the angle α:

$$(W_{AB}(\vec{P}) = \vec{P} \cdot \overrightarrow{AB} = mg \cdot AB \cdot \cos\, \alpha \qquad \text{(V.6)}$$

► Using the components of the two vectors $\vec{P}$ and $\overrightarrow{AB}$:

$$(W_{AB}(\vec{P}) = \vec{P} \cdot \overrightarrow{AB} = -mg(z_B - z_A) \quad \text{(V.7)}$$

This is identical to the previous expression because:

$$AB \cdot \cos\ \alpha = (z_A - z_B) \quad \text{(V.8)}$$

Thus, we can write the work of the weight force between the two points A and B as:

$$W_{AB}(\vec{P}) = mg(z_A - z_B) \quad \text{(V.9)}$$

From this, the work of the weight force depends only on the two points A and B; that is, it does not depend on the path taken by point M during its movement from A to B. It depends only on the difference in height between them. (This means the weight force is a conservative force.)

► When the body moves from bottom to top:

If $z_B > z_A$, then the work of the weight force is negative, meaning it is resistive work.

► When the body moves from top to bottom:

If $z_B < z_A$, then the work of the weight force is positive, meaning it is driving work.

V.2.2 Work of a Non-Constant Force

If the force $\vec{F}$ has a variable magnitude, then its work cannot be calculated using the expression $\vec{F} \cdot \overrightarrow{AB}$. Instead, the path must be divided into elementary displacements. We call the work of the force during an elementary displacement the elementary work dW.

V.2.2.1. Elementary Work

Let the object **M** move along an arbitrary path during a very smal time interval dt.

The displacement from position M to $M'$is represented by:

$$\overrightarrow{MM'} = d\vec{r} \tag{V.10}$$

The elementary work of the force $\vec{F}$ is defined as:

$$dW = \vec{F} \cdot d\vec{r} = F\,dr\cos\,\alpha \tag{V.11}$$

Where:

- $F\cos\alpha = F_T$ is the tangential component of the force.

Thus:

$$dW = F_T\,dr \tag{V.12}$$

Here, F_T is the tangential component of the force acting on the moving object.

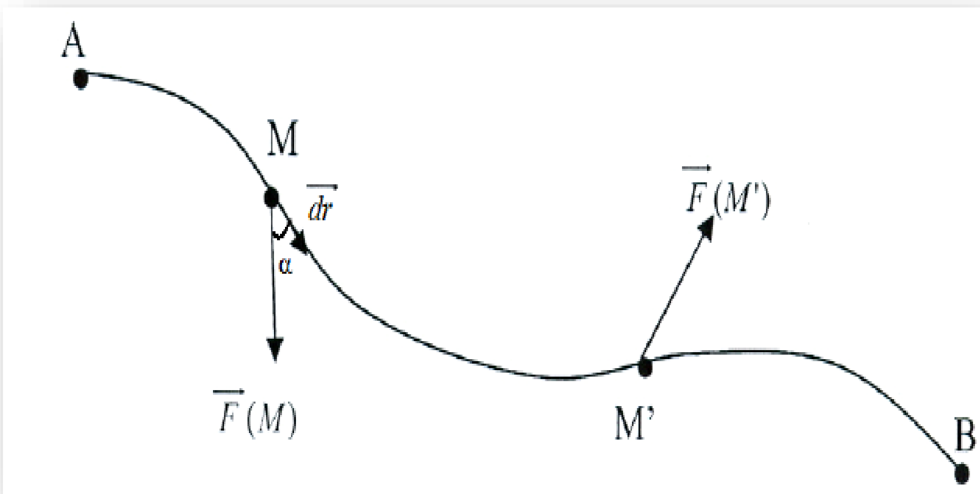


Fig.V.5. F_T is the tangential component of the force acting on the moving object.

This means that the tangential component of the acting force is the one that produces work.

As for the component perpendicular to the path, it produces no work, meaning its work is equal to zero.

$dW = 0 \Leftrightarrow \vec{F} \perp d\vec{r}, \theta = \frac{\pi}{2}$.

Any force perpendicular to the path of motion has zero work.

V.2.2.2. Driving force

- Positive Work

If $0 < \theta < \frac{\pi}{2}$ then $dW > 0$ and the force $\vec{F}$ is a driving force (the work is positive).

-Negative Work (Resistive force)

If $\frac{\pi}{2} < \theta < \pi$ then $dW < 0$ and the force $\vec{F}$ is a resistive force (the work is negative).

V.2.2.3. Work of the Spring Tension Force

A massless elastic spring with spring constant k, and natural length l_0(its length when it is empty, i.e., in the rest position), is fixed at one end, while a mass $m$is attached to the other end. The mass and the spring are placed on a horizontal surface (Fig.V.6).

In this figure:

- In **(a)** the spring is shown at its natural length, meaning in its original unstretched position.
- In **(b)** the spring is shown in its new extended length, as well as the associated elementary displacement $d\vec{l}$.

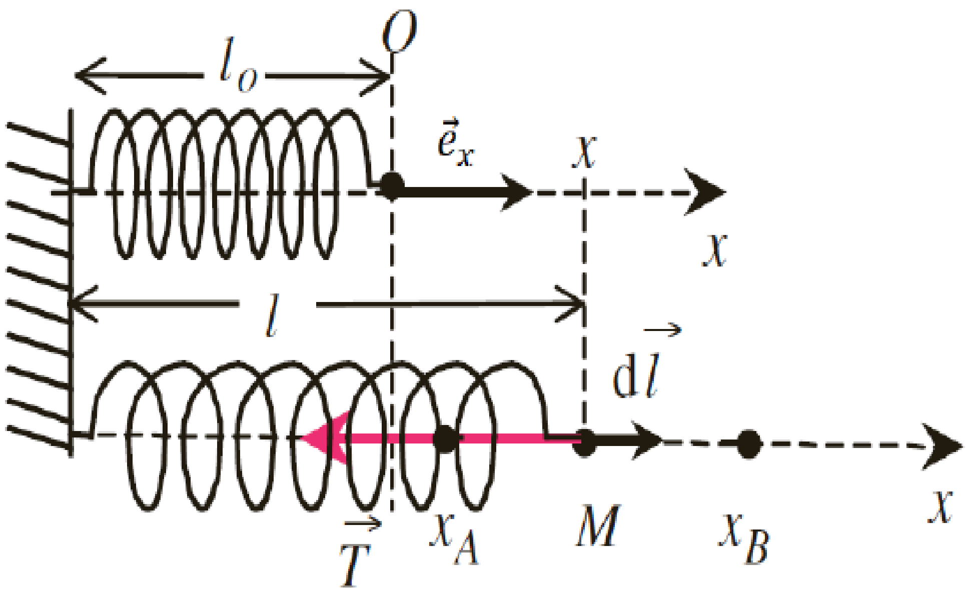


Fig.V.6. Work of the Spring Tension Force.

Using the notation shown in Fig.V.6, we write:

$$\vec{T} = -K(l - l_0)\,\vec{\imath} = -K\,\Delta l\,\vec{\imath} = -k\,x\,\vec{\imath} \qquad \text{(V.13)}$$

We consider an elementary displacement

$$d\vec{l} = dx\,\vec{\imath} \qquad \text{(V.14)}$$

of the end M of the spring. The expression of the elementary work then becomes:

$$\delta W = \vec{T}\cdot d\vec{l} = -k\,x\,\vec{\imath}\cdot dx\,\vec{\imath} = -k\,x\,dx \qquad \text{(V.14)}$$

The work of the spring tension force when point M (the end of the spring) moves from position x_A to position x_B is given by:

$$W_{AB}(\vec{T}) = \int_A^B -kx\,dx = -k\int_A^B x\,dx = -k[\frac{x^2}{2}]_{x_A}^{x_B} = \frac{1}{2}kx_A^2 - \frac{1}{2}kx_B^2 \quad \text{(V.15)}$$

We notice that the work of the spring tension force does not depend on the path taken, but depends only on the initial and final positions.

V.2.3. Power

The definition of the work of a force does not give any importance to the passage of time. Whether you perform a certain amount of work in one second or in one hour, the work itself is the same. However, we often need to know how fast the work is being done. For this reason, the concept of power is introduced.

Power is defined as the ratio of the work done to the time required to do it. It is a scalar quantity.

V.2.3.1. Average Power

If W is the work done during the time interval Δt, the average power of the force—denoted by P_m is expressed as:

$$P_m = \frac{W}{\Delta t} \quad \text{(V.16)}$$

In the International System (SI), power is measured in watts (W), which corresponds to 1 joule of work done in 1 second.

V.2.3.2. Instantaneous Power

Let δWbe the work done by the force $\vec{F}$during the elementary time interval dt.

The power of this force at instant t, called instantaneous power, is written as:

$$P(t) = \frac{\delta W}{dt} = \vec{F}\cdot\frac{d\vec{r}}{dt} = \vec{F}\cdot\vec{v} \quad \text{(V.17)}$$

This is the expression of the instantaneous power of the force $\vec{F}$.

V.3. Kinetic Energy

Kinetic Energy is the energy an object has because it is moving. If an object is at rest, its kinetic energy is zero, and as it starts moving, it gains kinetic energy. The amount of kinetic energy depends on two factors: the mass of the object and its speed (velocity). Heavier objects have more kinetic energy than lighter ones moving at the same speed, and objects moving faster have much more kinetic energy than slower ones because kinetic energy is proportional to the square of the speed.

This means that if the speed of an object doubles, its kinetic energy becomes four times greater. Kinetic energy is measured in joules (J). In everyday life, kinetic energy can be observed in moving cars, flowing water, flying bullets, or wind turning the blades of a windmill. When an object stops moving, its kinetic energy is not destroyed but is transformed into other forms of energy, such as heat, sound, or potential energy.

Let **M** be a material point of mass **m**, moving in an inertial frame **R**, under the action of several forces whose resultant is $\vec{F}$.

We calculate the elementary work of this force:

$$\delta W = \vec{F} \cdot d\vec{r} \qquad \text{(V.18)}$$

By applying the fundamental principle of dynamics (Newton's second law), we have:

$$\vec{F} = m\frac{d\vec{v}}{dt} \qquad \text{(V.19)}$$

By substituting into the previous equation, we obtain:

$$\delta W = m\,\frac{d\vec{v}}{dt} \cdot d\vec{r} = m\,\frac{d\vec{v}}{dt} \cdot \vec{v}\,dt = m\,\vec{v} \cdot d\vec{v}$$

We have:

$v^2 = \vec{v} \cdot \vec{v}$ Type equation here.

By differentiating this expression with respect to time, we get:

$$2v\,dv = 2\,\vec{v} \cdot d\vec{v} \;\Rightarrow\; v\,dv = \vec{v} \cdot d\vec{v}$$

Therefore, we can write:

$$\delta W = m\,\vec{v} \cdot d\vec{v} = m\,v\,dv = m\,d(\frac{1}{2}v^2) = d(\frac{1}{2}mv^2)$$

We define the kinetic energy of a material point $M$of mass $m$moving with velocity $\vec{v}$in a Galilean reference frame as:

$$E_c = \frac{1}{2}mv^2 \tag{V.20}$$

V.3.1. Kinetic Energy Theorem

The work of the resultant of the forces acting on a material point in a Galilean frame, during its displacement from point A to point B, is equal to the change in its kinetic energy between these two positions. We have:

$\delta W = d(\frac{1}{2}mv^2) = d(E_c)$

The work of the force $\vec{F}$when the material point Mmoves from position $A$to position $B$is given by:

$$W_{AB}(\vec{F}) = \int_A^B \delta W = \int_A^B d(E_c) = E_c(B) - E_c(A) = \Delta E_c \tag{V.21}$$

Thus, the kinetic energy theorem is written as:

$$W_{AB}(\vec{F}) = E_c(B) - E_c(A) = \Delta E_c \qquad \text{(V.22)}$$

V.3.2. Properties of Kinetic Energy

- Every moving object possesses kinetic energy.
- Kinetic energy is directly proportional to the mass m of the object.
- Kinetic energy is directly proportional to the square of the velocity v^2.
- Kinetic energy depends on the reference frame from which the motion is observed.

<u>Note</u>

The expression for kinetic energy given here is valid only for translational motion of the object.

V.4. Potential Energy

Potential energy is a form of energy related to the position of a system.

By changing the position, this energy may increase (the system stores energy) or decrease (the system releases energy outward).

Just as kinetic energy is defined through its change related to the work of all forces, potential energy is defined through the work of a specific type of forces: those whose work does not depend on the path taken. These forces are called conservative forces.

The work $W_{AB}(\vec{F})$of a conservative force $\overrightarrow{F}$does not depend on the path followed during the displacement, but depends only on the initial state (point A) and the final state (point B). This work can be expressed as a state function E_p(a function that depends only on the state of the system), called potential energy. By definition: for a conservative force $\vec{F}$, there exists a state function $E_p$such that:

$$W_{AB}(\vec{F}) = E_p(A) - E_p(B) = -\Delta E_p \tag{V.23}$$

The change in potential energy between points A and B equals the negative of the work of the conservative force applied between these two points.
In other words, the work of a conservative force equals the decrease of potential energy.
We notice from relation (V.23) that the change in potential energy $[E_p(A) - E_p(B)]$ between positions A and B exactly corresponds to the work from A to B (taking positions in the same order) of the conservative force associated with this energy. Forces $\overrightarrow{F}$ that derive from a potential energy are called conservative forces, because their work is transformed into the potential energy of the system.
Thus, we can write:

$$\vec{F} = -\nabla E_p \quad \Rightarrow \quad \text{rot}\,\vec{F} = 0 \tag{V.24}$$

The relation (V.24) is the test for a conservative force. In other words, it is sufficient to check the following equations to verify that the force $\overrightarrow{F}$ derives from a potential:

$$\frac{\partial F_x}{\partial y} = \frac{\partial F_y}{\partial x}, \frac{\partial F_x}{\partial z} = \frac{\partial F_z}{\partial x}, \frac{\partial F_z}{\partial y} = \frac{\partial F_y}{\partial z} \tag{V.25}$$

From relation (V.25), we get:

$$\vec{F} = -\nabla E_p = -\frac{\partial E_p}{\partial x}\vec{\imath} - \frac{\partial E_p}{\partial y}\vec{\jmath} - \frac{\partial E_p}{\partial z}\vec{k} \tag{V.26}$$

In Cartesian coordinates, we can write:

$$\vec{F} = F_x\vec{\imath} + F_y\vec{\jmath} + F_z\vec{k} \tag{V.27}$$

From this, we have:

$$F_x = -\frac{\partial E_p}{\partial x}, F_y = -\frac{\partial E_p}{\partial y}, F_z = -\frac{\partial E_p}{\partial z} \quad (V.28)$$

Note: If the potential energy depends on only one variable, for example $x$alone, we can write:

$$\vec{F} = -\frac{dE_p}{dx}\,\vec{\imath} \quad (V.29)$$

In this case, the force acts in the direction of decreasing potential energy.

V.4.1. Examples of Potential Energy

V.4.1.1. Gravitational Potential Energy

When we lift an object from the surface of the Earth to a certain height, we must do work against the Earth's gravity.

The object begins to store this work as potential energy (or positional energy).

This form of energy is associated with the Earth's gravitational field and is called gravitational potential energy.

It depends on the height above the Earth's surface. The object gains this energy as its height increases.

When we release the object, the stored potential energy gradually transforms into kinetic energy, causing the object’s speed to increase while its height decreases.

For an object moving upward, we have:

$$\vec{F} = \vec{P} = -mg\,\vec{k} \quad (V.30)$$

The potential energy is given by:

$$E_p(z) = -\int \vec{F} \cdot d\vec{r} + C = -\int -mg\, dz + C = mgz + C$$

By choosing the reference point $z = 0 \Rightarrow E_p = 0$, we get:

$$E_p(z) = mgz \qquad \text{(V.31)}$$

Note:

- In general, if the object rises above its original position, its gravitational potential energy increases.
- If the object falls below its original position, its gravitational potential energy decreases.

V.4.1.2. Elastic (Spring) Potential Energy

Consider a spring in its rest position, with one end fixed and the other free.

Assume the spring is massless, and a rigid object of mass M moves on a frictionless horizontal surface with velocity $\vec{V}$ parallel to the spring's axis and collides with the free end of the spring (see Fig.V.7).

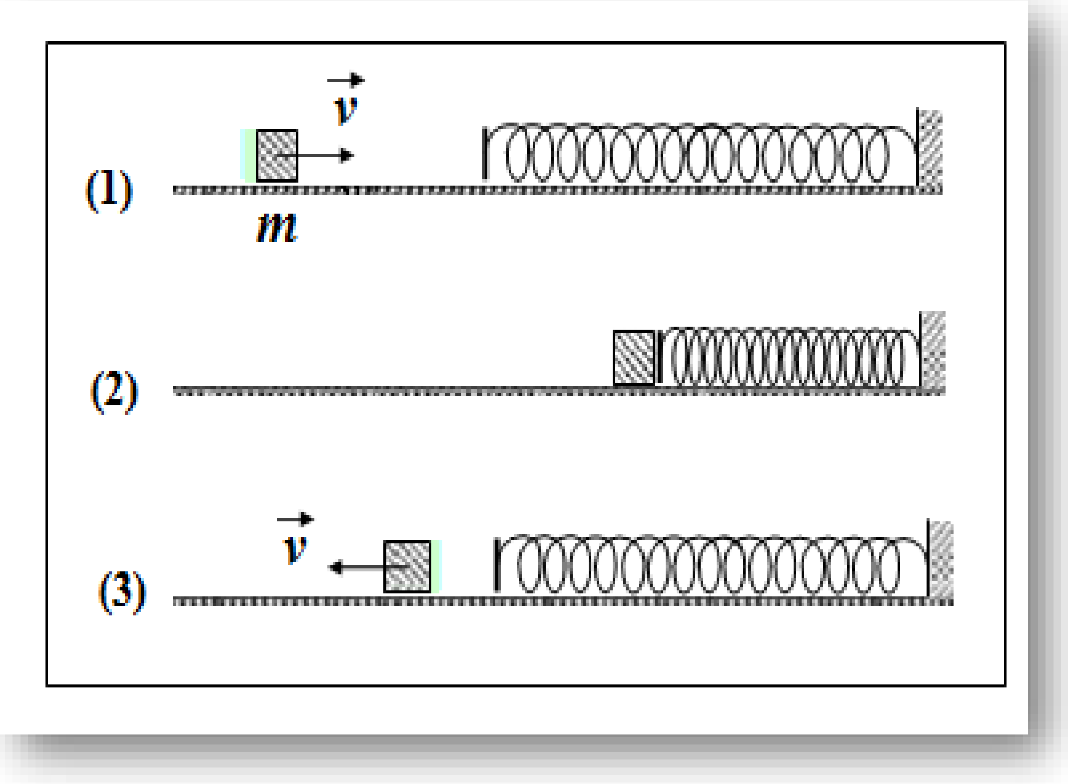


Fig.V.7. Elastic (Spring) Potential Energy.

- The object's speed decreases, and thus its kinetic energy decreases.
- The spring compresses due to its elasticity, exerting a restoring force that varies with the displacement.
- The spring begins to store energy, which will later appear again as the spring pushes the object.

This stored energy is called the elastic potential energy of the spring, denoted E_p.

For a spring force, we have:

$$\vec{F} = \vec{T} = -kx\,\vec{i} \qquad \text{(V.32)}$$

The potential energy is given by:

$$E_p = -\int \vec{F} \cdot d\vec{r} + C = -\int -kx\,dx + C = \frac{1}{2}kx^2 + C$$

Choosing the equilibrium position $x = 0 \Rightarrow E_p = 0$, we get:

$$C = 0 \quad \Rightarrow \quad E_p = \frac{1}{2}kx^2 \tag{V.34}$$

Notes

- The unit of elastic potential energy is the joule (J).
- The spring constant k is measured in newtons per meter (N/m).
- The extension or compression x of the spring is measured in meters (m).

V.4.1.3. Electric Potential Energy

The force between two point charges Q and q separated by a distance r is given by Coulomb's law:

$$\vec{F} = \frac{1}{4\pi\varepsilon_0}\frac{Qq}{r^2}\,\vec{u} = k\frac{Qq}{r^2}\vec{u} \tag{V.35}$$

Following the same principle as before, the electric potential energy is:

$$E_p = -\int \vec{F} \cdot d\vec{r} = -k\frac{Qq}{r} + C$$

By choosing the reference point at infinity ($r \to \infty$), we have:

$$E_p \to 0 \quad \Rightarrow \quad C = 0$$

Thus, the electric potential energy becomes:

$$E_p = -k\frac{Qq}{r} \tag{V.36}$$

V.4.1.4. Gravitational Potential Energy

Consider an object of mass $m$moving under the influence of the gravitational force $\vec{F}_g$applied by the Earth (or another massive object of mass M_T) see Fig.V.8, which is a conservative force:

$$\vec{F}_g(r) = -\frac{GM_T m}{r^2}\,\vec{u} \qquad \text{(V.37)}$$

Here, $\vec{u} = \frac{\vec{r}}{r}$is the unit vector along the radial direction.

We can also write:

$$\vec{F}_g(r) = -\frac{GM_T m}{r^3}\,\vec{r} \qquad \text{(V.38)}$$

From the definition of potential energy:

$$\vec{F}_g(r) = -\frac{dE_p}{dr}\,\vec{u}$$

$$\Rightarrow \quad \frac{dE_p}{dr} = \frac{GM_T m}{r^2} \qquad \text{(V.39)}$$

Integrating:

$$E_p(r) = \int \frac{GM_T m}{r^2}\,dr = -\frac{GM_T m}{r} + C$$

By choosing the reference point at infinity ($r \to \infty$):

$E_p \to 0 \quad \Rightarrow \quad C = 0$

Thus, the gravitational potential energy is:

$$E_p = -\frac{GM_T m}{r} \tag{V.40}$$

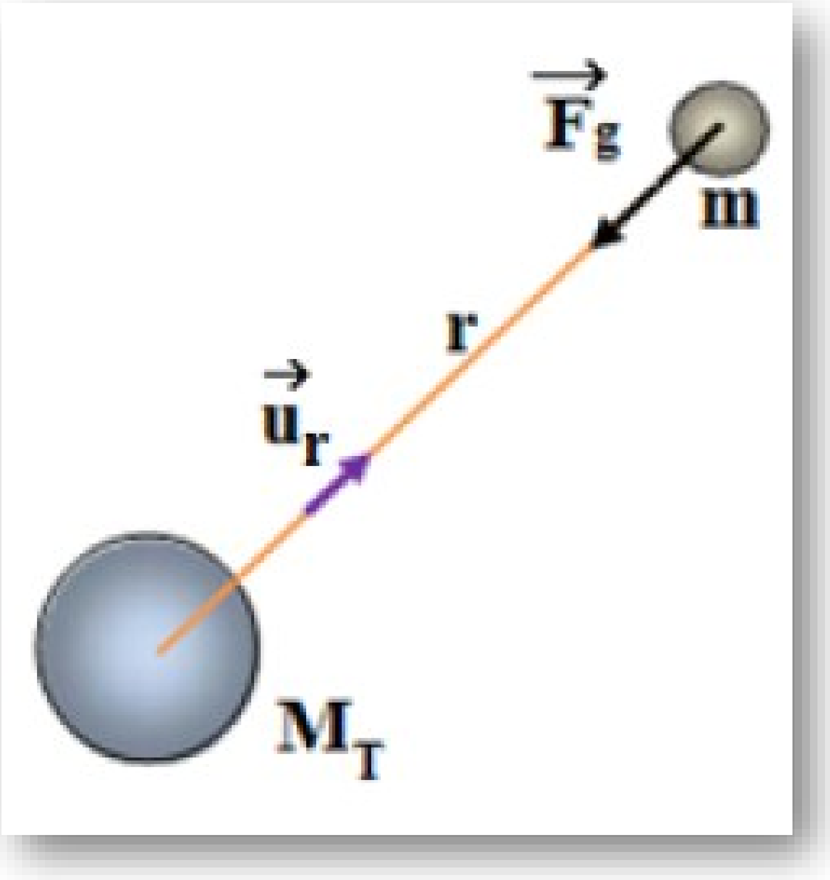


Fig.V.8. The gravitational force $\vec{F}_g$ applied by the Earth.

V.5. Conservative and Non-Conservative Forces

V.5.1. Conservative Forces

Conservative forces are those whose work does not depend on the path followed during the displacement, but depends only on the initial position (starting point) and the final position (ending point).

Examples we have encountered in this chapter include:

- Work of the weight force
- Work of the spring tension force
- Work of a constant force (fixed magnitude and direction).

V.5.1. Non-Conservative Forces

Non-conservative forces are forces whose work depends on the path followed.
In general, forces that depend on time or on the velocity vector are non-conservative.

Examples:

- Friction forces: The work of these forces is always resistive ($W < 0$), meaning they dissipate energy.

For example, for solid friction between two objects, this force constantly opposes motion, has a constant magnitude, and acts in the opposite direction of displacement.
The work of a solid friction force is calculated as:

$$\delta W = \vec{F} \cdot d\vec{r} = -F\, dr \quad \Rightarrow \quad W_{AB}(\vec{F}) = -\int_{A}^{B} F\, dr = -F\text{AB}$$

Here, AB is the actual distance traveled between A and B.
This distance clearly depends on the path, which illustrates why the force is non-conservative.

- Similarly, for fluid friction:

When a material object moves in a fluid, the fluid particles collide with the surface of the object, producing a resisting force. There is no simple law for this force; only empirical laws exist under specific conditions.

- The conditions depend on the motion of the object relative to the fluid, i.e., the motion of the fluid around the object.
- The resisting force opposes the velocity of the object and is independent of position.

- The force is proportional to the velocity for small speeds and proportional to the square of the velocity for high speeds.
- A particle can be subjected simultaneously to conservative and non-conservative forces.

V.6. principle of Energy Conservation

V.6.1. Mechanical Energy

The total mechanical energy E_M is the sum of:

$$E_M = E_C + E_P \tag{V.41}$$

Where:

- E_C: kinetic energy
- E_P: potential energy

Between points A and B:

$$\Delta E_M = E_M(B) - E_M(A) = \sum W_{A\to B}(\vec{F}) \tag{V.42}$$

V.6.1.1 Mechanical Energy Theorem

A. In the case of conservative forces

If an object is in free fall, i.e., subjected only to its weight and released from a certain height, its gravitational potential energy decreases while its kinetic energy increases. However, the sum of these two energies remains constant throughout the motion; it is a conserved quantity (mechanical energy is conserved over time). This is expressed as:

$$E_M = E_C + E_P = \text{constant} \Rightarrow \Delta E_M = 0 \tag{V.43}$$

This means that the change in kinetic energy equals the opposite of the change in potential energy:

$$\Delta E_C = -\Delta E_P \tag{V.44}$$

In other words, if the system is mechanically isolated, mechanical energy is conserved.

Example:

Consider a spring of constant k. If we stretch it downward by a distance $x$and then release it, it will oscillate under the restoring force of the spring.

- Its kinetic energy at a given instant:

$E_C(x) = \frac{1}{2}mv^2$

- Its potential energy stored in the spring:

$E_P(x) = \frac{1}{2}kx^2$

- Its total mechanical energy:

$E_T(x) = E_C(x) + E_P(x) = E_0$

To determine this energy, we can consider a special position at the beginning of motion, where $x = a$ and the velocity is zero:

$E(a) = \frac{1}{2}ka^2 = E_0$

The energy stored in the spring was given by the work we did to stretch the mass to position $x = a$. Afterward, the spring continues to oscillate between positions $+a$ and $-a$.

- If there is friction, it gradually dissipates energy, and the oscillations eventually stop.

Result:

In the Fig.V.9 any decrease in the spring's potential energy $E(x)$ is accompanied by an equal increase in kinetic energy, showing a continuous exchange between kinetic and potential energies: energy lost by one form is gained by the other.

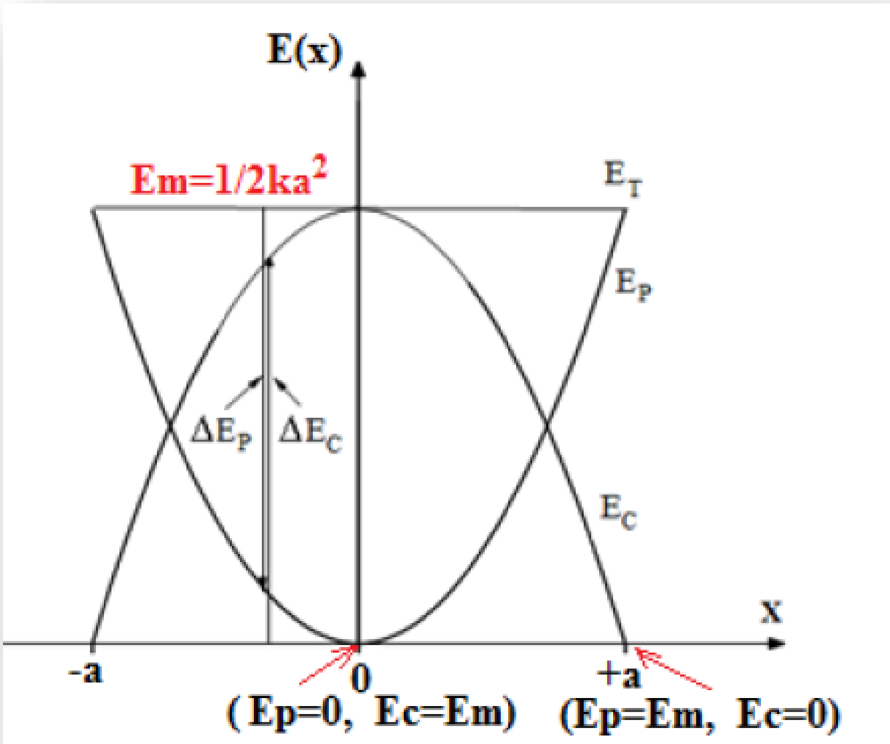


Fig.V.9. Continuous exchange between kinetic and potential energies.

B. In the case of non-conservative forces

The sum of the works of non-conservative external forces applied to a moving system between two points A and B equals the change in the system's total mechanical energy between these points:

$$E_M(B) - E_M(A) = \sum W_{A\to B}(\vec{F}_{NC}) \Rightarrow \Delta E_M = \sum W_{A\to B}(\vec{F}_{NC}) \quad \text{(V.44)}$$

From this, we observe that the total mechanical energy is not constant. Its change is not zero, but rather equals the work done by non-conservative forces, which represents energy loss (or dissipation) from the system.

V.7. Collisions between Particles

A collision is an event in which two or more objects come into contact and exert forces on each other for a short time. During a collision, the objects' velocities change due to these interaction forces. Collisions are analyzed using the laws of momentum and energy.

There are two main types of collisions. In an elastic collision, both momentum and kinetic energy are conserved, meaning the objects bounce off each other without any loss of kinetic energy. In an inelastic collision, momentum is conserved but kinetic energy is not conserved; some of it is transformed into other forms such as heat, sound, or deformation. A special case is a perfectly inelastic collision, where the objects stick together and move as one after the collision.

In a Galilean reference frame, consider two independent material particles with masses $m_1$and m_2. Each particle, when considered individually, is mechanically isolated or quasi-isolated, and therefore moves in a straight line with uniform motion (according to the principle of inertia). Their initial velocities are $\vec{v}_1$and $\vec{v}_2$, respectively.

If the two particles meet and collide (see Fig.V.10), then after the collision (which lasts for a very short time), the particles are again isolated or quasi-isolated, so they move in straight lines with new velocities $\vec{v}_1'$ and $\vec{v}_2'$.

V.7.1.1 Elastic Collision

In an elastic collision, both momentum and kinetic energy are conserved.

A. Conservation of Momentum

$$m_1v_1 + m_2v_2 = m_1v`_1 + m_2v`_2 \quad (V.45)$$

B. Conservation of Kinetic Energy

$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}m_1v`_1^2 + \frac{1}{2}m_2v`_2^2 \quad (V.46)$$

From these two equations, the final velocities $v_1$and $v_2$can be calculated.

Another important result for elastic collisions is the equation of relative motion:

$$v_1 - v_2 = -(v`_1 - v`_2) \quad (V.47)$$

This means the relative speed of approach equals the relative speed of separation.

V.7.1.2. Inelastic Collision

In an inelastic collision, momentum is conserved, but kinetic energy is not conserved.

A. Conservation of Momentum

$$m_1v_1 + m_2v_2 = m_1v`_1 + m_2v`_2$$

B. Kinetic Energy

$$\frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 \neq \frac{1}{2}m_1v`_1^2 + \frac{1}{2}m_2v`_2^2 \quad (V.48)$$

Some kinetic energy is converted into other forms like heat, sound, or deformation.

V.7.2. Properties of the collision

Before and after the collision, m_1 is mechanically isolated. During the instant of collision, m_1 is influenced by m_2(an external force acting on it), so it is not isolated. The same applies to m_2. If we consider the system composed of both masses together (m_1 and m_2), the system is isolated from the beginning to the end (before, during, and after the collision). The mutual forces between the particles are internal forces of the system. This setup allows us to use conservation laws (momentum and, depending on the type of collision, kinetic energy) to determine the post-collision velocities $\vec{v}_1'$and $\vec{v}_2'$.

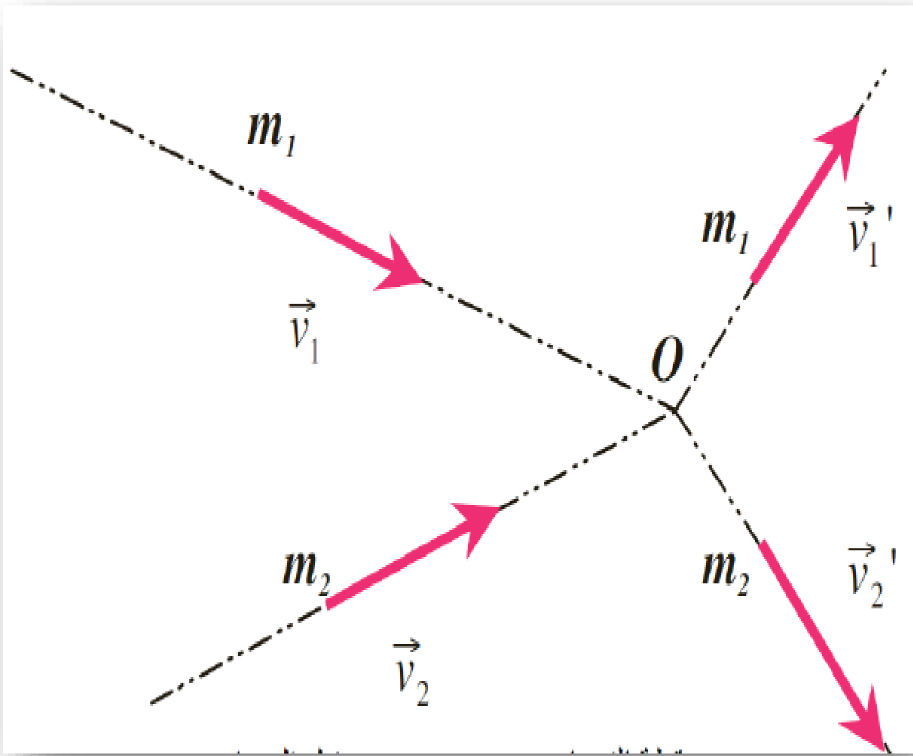


Fig.V.10. collision.

Exercise 1:

I) A force $\vec{F}$ acts in the Oxy plane on an object whose position coordinates in the plane are given by:

$\vec{r}(x,y) = y\,\vec{\imath} + 2xy\,\vec{\jmath}$

The object moves along the following paths from the origin O to the point $P(1,2)$:

1. Path 1: Straight-line path OP.
2. Path 2: Broken path OAP, where $A(1,0)$ is a point on the Ox axis.
3. Path 3: Parabolic path given by the equation $y = 2x^2$.

Calculate the work done by the force $\vec{F}$ in each case, and comment on the results obtained.

II) An object of mass m is subjected to a force field:

$\vec{F} = (-4x + y)\,\vec{\imath} + (x - z)\,\vec{\jmath} - y\,\vec{k}$

Show that the force $\vec{F}$ is derived from a potential energy E_p (i.e., that the force is conservative).

Solution:

I) Work done by a force along different paths. The force field in the Oxy plane is:

$\vec{F}(x,y) = y\,\vec{\imath} + 2xy\,\vec{\jmath}$

The infinitesimal displacement vector is:

$d\vec{r} = dx\,\vec{\imath} + dy\,\vec{\jmath}$

The work done by a variable force is:

$W = \int \vec{F} \cdot d\vec{r}$

So,

$\vec{F} \cdot d\vec{r} = y\,dx + 2xy\,dy$

Path 1: Straight-line path OP from $O(0,0)$ to $P(1,2)$.

Parametrization:

$x = t, y = 2t, 0 \leq t \leq 1$

$dx = dt, dy = 2dt$

Substitute into the work expression

$\vec{F} \cdot d\vec{r} = y\,dx + 2xy\,dy$

$$= (2t)(dt) + 2(t)(2t)(2dt)$$

$$= 2t\,dt + 8t^2\,dt$$

Integrate:

$$W_1 = \int_0^1 (2t + 8\,t^2)\,dt$$

$$W_1 = \left[t^2 + \frac{8}{3}t^3\right]_0^1$$

$$\boxed{W_1 = \frac{11}{3}}$$

Path 2: Broken path OAP

- Segment OA: $O(0,0) \rightarrow A(1,0)$

Here:

$y = 0, dy = 0$

$\vec{F} \cdot d\vec{r} = y\,dx + 2xy\,dy = 0$

$W_{OA} = 0$

- Segment AP: $A(1,0) \rightarrow P(1,2)$

Here:

$x = 1, dx = 0$

$\vec{F} \cdot d\vec{r} = y\,dx + 2xy\,dy = 2(1)y\,dy = 2y\,dy$

$W_{AP} = \int_0^2 2y\,dy = [y^2]_0^2 = 4$

-Total work (Path 2)

$\boxed{W_2 = 4}$

Path 3: Parabolic path $y = 2x^2$ from $x = 0$to $x = 1$

$dy = 4x\,dx$

Substitute into the work expression:

$\vec{F} \cdot d\vec{r} = y\,dx + 2xy\,dy$

$= (2x^2)\,dx + 2x(2x^2)(4x\,dx)$

$= 2x^2\,dx + 16x^4\,dx$

Integrate:

$W_3 = \int_0^1 (2\,x^2 + 16x^4)\,dx$

$= \left[\frac{2}{3}x^3 + \frac{16}{5}x^5\right]_0^1$

$\boxed{W_3 = \frac{58}{15}}$

Comment on the results

$W_1 = \frac{11}{3}, W_2 = 4, W_3 = \frac{58}{15}$

The work done depends on the path taken between the same points. Therefore, the force is not conservative.

II) Showing that the force is conservative

Given:

$\vec{F} = (-4x + y)\,\vec{\imath} + (x - z)\,\vec{\jmath} - y\,\vec{k}$

A force is conservative if:

$\nabla \times \vec{F} = \vec{0}$

Compute the curl

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \partial_x & \partial_y & \partial_z \\ -4x + y & x - z & -y \end{vmatrix}$$

Components:

- $\partial_y(-y) - \partial_z(x - z) = (-1) - (-1) = 0$
- $\partial_z(-4x + y) - \partial_x(-y) = 0 - 0 = 0$
- $\partial_x(x - z) - \partial_y(-4x + y) = 1 - 1 = 0$

$\nabla \times \vec{F} = \vec{0}$

Conclusion

Since:

$\nabla \times \vec{F} = \vec{0}$

The force is conservative. It is derived from a potential energy E_p such that:

$\vec{F} = -\nabla E_p$

Exercise 2:

A force $\vec{F}$acts on a particle in the Oxy-plane, and the position vector of the particle is:

$$\vec{r}(x, y) = x^2\,\hat{\imath} + xy\,\hat{\jmath}$$

The particle moves from the origin $O(0,0)$to the point $P(1,2)$along the following paths:

1. Path 1: Straight-line path OP
2. Path 2: Broken path OAP, where $A(1,0)$lies on the x-axis
3. Path 3: Parabolic path given by $y = 2x^2$

Tasks:

- Calculate the work done by the force $\vec{F}$along each path.
- Comment on whether the force is path-dependent or path-independent.

II) Checking if a Force is Conservative

A particle of mass $m$is subjected to a force field:

$$\vec{F} = (2x - 3y)\hat{\imath} + (x + 4z)\hat{\jmath} + (2y - z)\hat{k}$$

Tasks:

- Show that this force is conservative.
- If it is conservative, find the potential energy function $E_p(x, y, z)$corresponding to this force.

Solution:

I) Work Done by a Force Along Different Paths

Given:

$\vec{r}(x, y) = x^2\hat{\imath} + xy\hat{\jmath}$,particle moves from $O(0,0)$ to $P(1,2)$

Force: Let's assume the force is:

$\vec{F}(x, y) = 3x\hat{\imath} + 2y\hat{\jmath}$(to make the problem solvable, similar to standard exercises)

Work formula:

$$W = \int_C \vec{F} \cdot d\vec{r}$$

Here, $d\vec{r} = dx\hat{\imath} + dy\hat{\jmath}$.

Path 1: Straight-Line Path OP

Parametrize the straight line:

The line from $O(0,0)$to $P(1,2)$can be written as:

$$y = 2x, 0 \leq x \leq 1$$

So: $dy = 2dx$

Differential displacement:

$$d\vec{r} = dx\hat{\imath} + dy\hat{\jmath} = dx\hat{\imath} + 2dx\hat{\jmath} = (1\hat{\imath} + 2\hat{\jmath})dx$$

Work integral:

$$W = \int_0^1 \vec{F} \cdot d\vec{r} = \int_0^1 (3x\,\hat{\imath} + 2y\hat{\jmath}) \cdot (1\hat{\imath} + 2\hat{\jmath})dx$$

Substitute $y = 2x$:

$$\vec{F} \cdot d\vec{r} = (3x)(1) + 2(2x)(2) = 3x + 8x = 11x$$

$$W = \int_0^1 11x\,dx = \frac{11}{2}x^2\ |_0^1 = 5.5\text{ J}$$

Work along straight line:

$$\boxed{W_1 = 5.5\text{ J}}$$

Path 2: Broken Path $O \to A(1,0) \to P(1,2)$

- Segment OA: along x-axis ($y = 0$)
  - $dy = 0,\ dx = dx$
  - $\vec{F} \cdot d\vec{r} = (3x\hat{\imath} + 2y\hat{\jmath}) \cdot (dx\hat{\imath} + 0) = 3xdx$
  - Integrate $x: 0 \to 1$:

$$W_{OA} = \int_0^1 3xdx = \frac{3}{2} = 1.5\text{ J}$$

- Segment AP: vertical ($x = 1,\ dx = 0,\ dy = dy$)
  - $\vec{F} \cdot d\vec{r} = (3x\hat{\imath} + 2y\hat{\jmath}) \cdot (0\hat{\imath} + dy\hat{\jmath}) = 2ydy$
  - Integrate $y: 0 \to 2$:

$$W_{AP} = \int_0^2 2ydy = y^2 \,|_0^2 = 4\text{ J}$$

Total work along broken path:

$$W_2 = W_{OA} + W_{AP} = 1.5 + 4 = 5.5\text{ J}$$

Path 3: Parabolic Path $y = 2x^2$

- $dy = 4xdx$

$$d\vec{r} = dx\hat{\imath} + dy\hat{\jmath} = dx\hat{\imath} + 4xdx\hat{\jmath} = (1\hat{\imath} + 4x\hat{\jmath})dx$$

- $\vec{F} \cdot d\vec{r} = (3x\hat{\imath} + 2y\hat{\jmath}) \cdot (1\hat{\imath} + 4x\hat{\jmath})$
- Substitute $y = 2x^2$:

$$\vec{F} \cdot d\vec{r} = 3x + 2(2x^2)(4x) = 3x + 16x^3 = 3x + 16x^3$$

$$W_3 = \int_0^1 (3x + 16\,x^3)dx = \frac{3}{2} + 4 = 5.5\ \mathrm{J}$$

Observation

- $W_1 = W_2 = W_3 = 5.5\ \mathrm{J}$

  Work is path-independent, so the force is conservative.

II) Check if a 3D Force is Conservative

Force:

$$\vec{F} = (2x - 3y)\hat{\imath} + (x + 4z)\hat{\jmath} + (2y - z)\hat{k}$$

Step 1: Check curl

A force is conservative if:

$$\nabla \times \vec{F} = 0$$

Compute components:

$$\vec{F} = (F_x, F_y, F_z) = (2x - 3y, x + 4z, 2y - z)$$

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ F_x & F_y & F_z \end{vmatrix}$$

component: $\partial_y F_z - \partial_z F_y = \partial_y(2y - z) - \partial_z(x + 4z) = 2 - 4 = -2 \neq 0$

Since curl $\neq 0$, the force is not conservative.

- Work is path-independent, so the force in part I is conservative.

  Part II: 3D force

- $\vec{F} = (2x - 3y, x + 4z, 2y - z)$

- $\nabla \times \vec{F} \neq 0$, so force is non-conservative.

Exercise 2:

A spring of spring constant $k = 400$ N/mis compressed by 0.20 m.

1. Calculate the elastic potential energy stored in the spring.
2. Is the spring force constant or variable? Explain briefly.

Solution:

Given:

- Spring constant: $k = 400$ N/m
- Compression: $x = 0.20$ m

1) Elastic Potential Energy

The elastic potential energy stored in a spring is given by Hooke's law formula:

$$E_p = \frac{1}{2}kx^2$$

Substitute the values:

$$E_p = \frac{1}{2}(400)(0.20)^2$$

$$E_p = 200 \times 0.04$$

$$\boxed{E_p = 8 \text{ J}}$$

So, the spring stores 8 Joules of elastic potential energy.

2) Is the spring force constant or variable?

According to Hooke's law:

$$F = kx$$

- The force depends linearly on the displacement x.
- As the spring is compressed or stretched, Fchanges proportionally to x.

Therefore, the spring force is variable, not constant.

Answer:

- Elastic potential energy: 8 J
- Spring force: variable (depends on the displacement)

Exercise 3 : (Spring Potential Energy)

A spring has a spring constant $k = 500$ N/mand is compressed by 0.15 m.

1. Calculate the elastic potential energy stored in the spring.
2. Is the spring force constant or variable? Explain briefly.

Solution:

Step 1: Use the elastic potential energy formula

$$E_c = \frac{1}{2}kx^2$$

$$E_c = \frac{1}{2}(500)(0.15)^2$$

$$E_c = 250 \cdot 0.0225$$

$$E_c = 5.625 \text{ J}$$

Step 2: Nature of the spring force

- The spring force $F = kx$is directly proportional to the displacement.
- Since $k$is constant, the force varies with x, i.e., it is variable in magnitude, but always acts linearly with compression/extension.

Exercise 3:

A 1 kg object is thrown vertically upward with a speed of 10 m/s.

1. Calculate the maximum height reached.
2. What is the kinetic energy at the highest point?

Solution:

Given

- Mass: $m = 1$ kg
- Initial speed: $v_i = 10$ m/s
- Gravity: $g = 9.8$ m/s^2

1) Maximum height using conservation of mechanical energy

Mechanical energy is conserved:

$E_c^{\text{initial}} + E_p^{\text{initial}} = E_c^{\text{max height}} + E_p^{\text{max height}}$

At launch ($y = 0$):

$E_c^{\text{initial}} = \frac{1}{2}mv_i^2 = \frac{1}{2}(1)(10^2) = 50$ J

$E_p^{\text{initial}} = mgh = 1 \cdot 9.8 \cdot 0 = 0$

At maximum height, the object stops ($v = 0$):

$E_c^{\text{max height}} = 0$

$E_p^{\text{max height}} = mgh_{\max}$

By conservation of energy:

$E_c^{\text{initial}} + E_p^{\text{initial}} = E_c^{\text{max height}} + E_p^{\text{max height}}$

$50 + 0 = 0 + (1)(9.8)h_{\max}$

$h_{\max} = \frac{50}{9.8} \approx 5.10 \text{ m}$

2) Kinetic energy $E_c$at the highest point

At the maximum height, the object stops:

$\boxed{E_c^{\text{max height}} = 0 \text{ J}}$

Exercise 1

A ball of mass **2 kg** moving with a velocity of **6 m/s** collides elastically with another ball of mass **3 kg** at rest on a frictionless surface.

1. Find the velocities of both balls after the collision.
2. Verify conservation of momentum and kinetic energy.

Solution:

- $m_1 = 2$ kg, initial velocity $v_1 = 6$ m/s
- $m_2 = 3$ kg, initial velocity $v_2 = 0$ m/s

The collision is elastic, so momentum and kinetic energy (E_c) are conserved.

Step 1: Conservation of Momentum

$$m_1 v_1 + m_2 v_2 = m_1 v_1' + m_2 v_2'$$

$$(2)(6) + (3)(0) = 2v_1' + 3v_2'$$

$$2v_1' + 3v_2' = 12(1)$$

Step 2: Conservation of Kinetic Energy (E_c)

$$\frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2 = \frac{1}{2}m_1 v_1'^2 + \frac{1}{2}m_2 v_2'^2$$

$$\frac{1}{2}(2)(6^2) = \frac{1}{2}(2)v_1'^2 + \frac{1}{2}(3)v_2'^2$$

$$v_1'^2 + \frac{3}{2}v_2'^2 = 36(2)$$

Step 3: Solve Equations

From (1):

$$v_1' = \frac{12 - 3v_2'}{2}$$

Substitute into (2):

$$\left(\frac{12 - 3v_2'}{2}\right)^2 + \frac{3}{2}v_2'^2 = 36$$

Solving this gives:

$$v_2' = 4.8 \text{ m/s}$$

Substitute into (1):

$$v_1' = \frac{12 - 3(4.8)}{2} = -1.2 \text{ m/s}$$

Exercise 2: (One-Dimensional Inelastic Collision)

A 1.5 kg cart moving at 8 m/s collides with a 2.5 kg cart at rest. After the collision, the two carts stick together.

1. Find the final velocity of the combined system.
2. Calculate the loss in kinetic energy.

Solution:

- $m_1 = 1.5 \text{ kg}, \ v_1 = 8 \text{ m/s}$
- $m_2 = 2.5 \text{ kg}, \ v_2 = 0 \text{ m/s}$

Let the common final velocity after collision be v'.

1. Final Velocity of the Combined System

Conservation of Momentum

$$m_1v_1 + m_2v_2 = (m_1 + m_2)v'$$

$$(1.5)(8) + (2.5)(0) = (1.5 + 2.5)v'$$

$$12 = 4v'$$

$$\boxed{v' = 3 \text{ m/s}}$$

2. Loss in Kinetic Energy

Initial Kinetic Energy ($E_{c,i}$)

$$E_{c,i} = \frac{1}{2} m_1 v_1^2$$

$$E_{c,i} = \frac{1}{2} (1.5)(8^2) = 48 \text{ J}$$

Final Kinetic Energy ($E_{c,f}$)

$$E_{c,f} = \frac{1}{2} (m_1 + m_2) v'^2$$

$$E_{c,f} = \frac{1}{2} (4)(3^2) = 18 \text{ J}$$

Loss in Kinetic Energy

$$\Delta E_c = E_{c,i} - E_{c,f}$$

$$\boxed{\Delta E_c = 48 - 18 = 30 \text{ J}}$$

Exercise 3: (Two-Dimensional Collision)

A **0.5 kg** ball moving at **10 m/s** along the x-axis collides with another **0.5 kg** ball at rest. After collision, the first ball moves at **6 m/s** at an angle of **30°** to the x-axis.

1. Find the velocity and direction of the second ball after collision.
2. State whether the collision is elastic or inelastic.

Solution:

- Mass of ball 1: $m_1 = 0.5$ kg
- Initial velocity of ball 1: $v_1 = 10$ m/s along x-axis
- Mass of ball 2: $m_2 = 0.5$ kg
- Initial velocity of ball 2: $v_2 = 0$ m/s
- After collision, ball 1 moves at $v_1' = 6$ m/sat 30° to x-axis
- Find: $v_2'$and its direction

- Determine if collision is elastic

Step 1: Conservation of Momentum

Momentum is a vector, so we conserve it separately along x and y axes.

X-axis

$$m_1 v_1 + m_2 v_2 = m_1 v_1' \cos\,\theta_1 + m_2 v_2' \cos\,\theta_2$$

$$(0.5)(10) + (0.5)(0) = (0.5)(6)\cos\,30^\circ + (0.5)v_2' \cos\,\theta_2$$

$$5 = 3 \cdot 0.866 + 0.5 v_2' \cos\,\theta_2$$

$$5 = 2.598 + 0.5 v_2' \cos\,\theta_2$$

$$0.5 v_2' \cos\,\theta_2 = 5 - 2.598 = 2.402$$

$$v_2' \cos\,\theta_2 = 4.804(1)$$

Y-axis

$$0 = m_1 v_1' \sin\,\theta_1 + m_2 v_2' \sin\,\theta_2$$

$$0 = 0.5 \cdot 6 \cdot \sin\,30^\circ + 0.5 \cdot v_2' \sin\,\theta_2$$

$$0 = 3 \cdot 0.5 + 0.5 v_2' \sin\,\theta_2$$

$$0 = 1.5 + 0.5 v_2' \sin\,\theta_2$$

$$v_2' \sin\,\theta_2 = -3(2)$$

The negative sign indicates ball 2 moves downward (opposite y-direction).

Step 2: Find Magnitude and Angle of v_2'

$$v_2' = \sqrt{(v_2' \cos\,\theta_2)^2 + (v_2' \sin\,\theta_2)^2}$$

$$v_2' = \sqrt{(4.804)^2 + (-3)^2}$$

$$v_2' = \sqrt{23.08 + 9} = \sqrt{32.08} \approx 5.66\ \text{m/s}$$

Direction: Angle below x-axis

$$\tan\,\theta_2 = \frac{v_2' \sin\,\theta_2}{v_2' \cos\,\theta_2} = \frac{-3}{4.804} \approx -0.6248$$

$$\theta_2 \approx -32^\circ(\text{below x-axis})$$

Step 3: Check if Collision is Elastic

Initial kinetic energy ($E_{c,i}$):

$$E_{c,i} = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}(0.5)(10^2) + 0 = 25\ \text{J}$$

Final kinetic energy ($E_{c,f}$):

$$E_{c,f} = \frac{1}{2}m_1v_1'^2 + \frac{1}{2}m_2v_2'^2$$

$$E_{c,f} = 0.25(36) + 0.25(32.08) = 9 + 8.02 \approx 17.02\ \text{J}$$

$E_{c,f} < E_{c,i} \Rightarrow$ collision is inelastic.

Exercise :

A 3 kg cart moving at 5 m/s collides with a 2 kg cart at rest on a frictionless track. After the collision, the two carts stick together and move as one.

1. Find the final velocity of the combined carts.
2. Calculate the kinetic energy lost during the collision.

Solution:

Step 1: Use Conservation of Momentum

$$m_1v_1 + m_2v_2 = (m_1 + m_2)v'$$

$$(3)(5) + (2)(0) = (3 + 2)v'$$

$$15 = 5v' \quad \Longrightarrow \quad v' = 3\ \text{m/s}$$

Step 2: Initial and Final Kinetic Energy

- Initial E_c:

$$E_{c,i} = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 = \frac{1}{2}(3)(5^2) + 0 = 37.5\ \text{J}$$

- Final E_c:

$$E_{c,f} = \frac{1}{2}(m_1 + m_2)v'^2 = \frac{1}{2}(5)(3^2) = 22.5\ \text{J}$$

- Kinetic energy lost:

$$\Delta E_c = 37.5 - 22.5 = 15\ \text{J}$$

Exercise 5:

A 2 kg ball is moving at 8 m/s along the x-axis. It collides with a 3 kg ball moving at 6 m/s along the y-axis. After the collision, the two balls stick together.

1. Find the magnitude and direction of the final velocity of the combined mass.
2. Calculate the kinetic energy lost during the collision.

Solution:

 Treat momentum as a vector:

- x-direction: $m_1 v_{1x} + m_2 v_{2x} = (m_1 + m_2) v'_x$
- y-direction: $m_1 v_{1y} + m_2 v_{2y} = (m_1 + m_2) v'_y$

 Use Pythagoras to find the magnitude of the final velocity:

$$v' = \sqrt{v_x'^2 + v_y'^2}$$

 Use trig to find the direction:

$$\theta = \tan^{-1}\left(\frac{v'_y}{v'_x}\right)$$

 Compute initial and final kinetic energies to find energy lost:

$$\Delta E_c = E_{c,i} - E_{c,f}$$

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