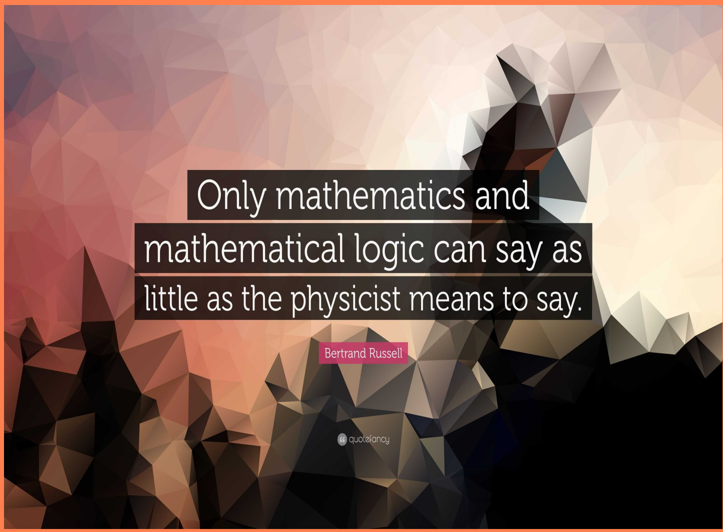

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Mathematical Logic

Courses and Exercises

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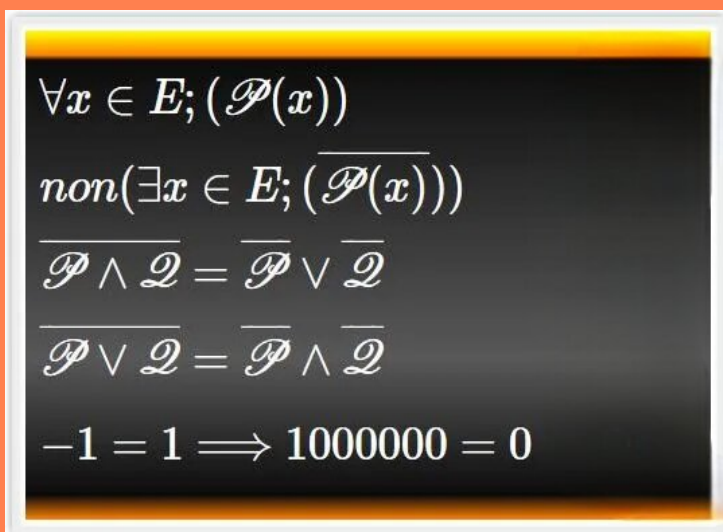


Contents

Introduction

This course material was created for second-year mathematics students. Follow the standard curriculum to help you build a solid understanding of essential logic concepts.

The main goal of this course is to introduce basic topics in logic, which are important for further studies in mathematics and related fields. By mastering these concepts, complex problems can be solved via a clear and logical approach.

We begin with propositional logic, where you will learn to break down and create logical statements, use logical operators, and work with concepts such as logical equivalence, implication, and truth tables.

Next, we move on to Predicate Logic, which expands on the ideas from Propositional Logic. Here, you will explore quantifiers, predicates, and the relationships between them, as well as how to structure arguments and translate everyday language into logical form.

The course will then cover types of reasoning, where you will understand the differences between deductive and inductive reasoning. This section also introduces various reasoning techniques commonly used in mathematical proofs and problem solving.

We will also discuss paradoxes that will challenge your understanding of logical consistency. You will explore famous paradoxes, such as the liar paradox and Russell's paradox, to think more critically about the limits of logical systems.

Finally, we introduce set theory, the fundamental language of mathematics. You will learn the basic operations of sets, such as union, intersection, and complementation, and move on to more complex topics such as Cartesian products, relations, and functions.

We hope that this material will be a valuable resource that will help you master key concepts in logic and apply them effectively in your mathematical studies.

Chapter 1 Fundamental terms

1.1 Terminology in Mathematics and Logic

1.1.1 Axiom

An axiom is a basic statement or proposition that is accepted as true without proof.

Example 1.1

Euclidean axioms Euclidean axioms are the fundamental assumption on which Euclidean geometry is based and were established by the ancient Greek mathematician Euclid. The most famous of these include the following:

- ✿ **Axiom 1**: A straight line can be drawn from any point to any other point.
- ✿ **Axiom 2**: A finite straight line can be extended indefinitely in a straight line.
- ✿ **Axiom 3**: A circle can be drawn with any center and radius.
- ✿ **Axiom 4**: All right angles are equal to each other.
- ✿ **Axiom 5 (Parallel Postulate)**: Given a line and a point not on it, there is one line parallel to the given line that passes through the point.

Peano's Axioms Peano's axioms are a set of axioms for natural numbers proposed by the Italian mathematician Giuseppe Peano. These axioms define the arithmetic properties of natural numbers:

- ✿ **Axiom 1**: 0 is a natural number.
- ✿ **Axiom 2**: Every natural number has a successor, which is also a natural number.
- ✿ **Axiom 3**: 0 is not the successor of any natural number.
- ✿ **Axiom 4**: Different natural numbers have different successors.
- ✿ **Axiom 5 (Induction Axiom)**: If a property holds for 0 and if, whenever it holds for a natural number, it also holds for its successor, then it holds for all natural numbers.

1.1.2 Proposition

A proposition is a statement that is true or false. In mathematics, a proposition is often a statement proven to be true on the basis of axioms and previously established theorems.

1.1.3 Assertion

An assertion is a declarative statement that is claimed to be true. It is similar to a proposition in logic and mathematics, although it may not always be accompanied by proof.

Example 1.2

- Every prime number is odd; False proposition.
- Every squared number is a positive number; true proposition.
- Close the door; It is not a proposition.

1.1.4 Definition

A definition in mathematics is a precise statement that explains the meaning of a term or concept. Definitions are used to establish the objects of study and the properties that they must have within a given mathematical framework. Definitions are foundational and must be unambiguous.

1.1.5 Theorem

A theorem is a statement (proposition) proven to be true through a logical sequence of reasoning on the basis of axioms, definitions, and previously established theorems. Theorems are the core results of mathematics.

1.1.6 Corollary

A corollary is a statement that follows readily from a previously proven theorem. It is often considered a direct consequence or an additional result that can be easily derived from a theorem.

1.1.7 Lemma

A lemma is a subsidiary or intermediate theorem that is used to prove a larger or more significant theorem. Lemmas are often stepping stones in the proof of theorems.

1.1.8 Conjecture

Conjecture is a statement or proposition that is suspected to be true on the basis of observations but for which no proof has yet been found. Conjectures are often the starting points for new areas of research in mathematics.

1.1.9 Fermat's Conjecture

Fermat's Conjecture, also known as Fermat's Last Theorem, is a famous problem in number theory. It states that there are no three positive integers a, b, and c that satisfy the equation $a^n + b^n = c^n$ for any integer value of $n > 2$. This conjecture was first proposed by Pierre de Fermat in 1637 and remained unproven until it was solved by Andrew Wiles in 1994.

1.1.10 Bertrand's Conjecture

Bertrand's Conjecture, proposed by Joseph Bertrand in 1845, states that for every integer $n > 1$, there is at least one prime number p such that $n < p < 2n$. This conjecture was later proven true by Chebyshev in 1850, and it is now known as Bertrand's Postulate.

Chapter 2 Propositional logic

Logic is the study of reasoning and argumentation. It involves analyzing the structure of statements and arguments to determine their validity or truthfulness. Logic provides the rules and techniques to distinguish between valid and invalid reasoning, ensuring that conclusions drawn from premises are logically sound. It is a fundamental branch of philosophy and mathematics and is widely used in computer science, linguistics, and other fields.

2.1 Proposition

Definition 2.1

A proposition is a statement to which it is possible to assign a true or false value.

Example 2.1

Consider the statement

- "6 is less than 4 " is a false proposition.
- "Algeria is an African country." is a true proposition.

Example 2.2

Consider the statement

- Mohammed has 3 children.

This is also a proposition. The statement is either true or false. You might not know if it is true or false, but it still has one of those truth values.

2.2 Logical basic operators and truth tables

2.2.1 Truth table

Definition 2.2

A truth table is a chart that lists all possible truth values of a statement such as pp, which can be either true or false. Shows how the truth value of a logical expression involving one or more statements changes depending on the truth values of those statements.

- "T or 1" stands for "True".
- "F or 0" stands for "False".

Let us examine the different types of truth tables as follows:

1. One proposition "P":

Proposition	P
True	T or 1
false	F or 0

2. Two propositions "P, Q":

P	Q
1	1
1	0
0	1
0	0

3. Three propositions "P, Q, R":

P	Q	R
1	1	1
1	1	0
1	0	1
1	0	0
0	1	1
0	1	0
0	0	1
0	0	0

2.2.2 Negation of a proposition

The negation of a proposition P is true if P is false and vice versa. We denote it by $\overline{P}$, which is "No P"

Example 2.3

If P represents "It is raining", then $\overline{P}$ represents "It is not raining."

Truth table

P	$\overline{P}$
1	0
0	1

2.2.3 Logic connectives

Logic connectives are symbols or words used to connect propositions or statements and form more complex expressions. These operators define the relationship between the truth values of the connected propositions, determining the truth value of the resulting compound statement. We define 4 as usual connectives.

1. Conjunction

The connective logic "and" is denoted by "$\wedge$"; therefore, the conjunction of two propositions "$P \wedge Q$" is true if both P and Q are true.

Truth table

p	q	$p \wedge q$
1	1	1
1	0	0
0	1	0
0	0	0

Example 2.4

$P \wedge Q$ (It is raining and it is cold) is true only if both "It is raining" and "It is cold" are true.

2. Disjunction

The logic connective "or" is denoted by "$\vee$"; then the disjunction of two propositions "$P \vee Q$" is true if at least one of P or Q is true.

Truth table

p	q	$p \vee q$
1	1	1
1	0	1
0	1	1
0	0	0

Example 2.5

$P \vee Q$ (It is raining or it is cold) is true if either "It is raining" or "It is cold" (or both) are true.

3. Implication

An implication is a logical statement of the form "If P, then Q" and is denoted as $P \Rightarrow Q$ and called the conditional operator, or implicative operator. In this statement, P is called the antecedent, and Q is the consequent. The implication is considered false only when P is true and Q is false; in all other cases, it is true.

Truth table:

p	q	$p \Rightarrow q$
1	1	1
1	0	0
0	1	1
0	0	1

Example 2.6

- "If it is raining, then the ground is wet" $P \Rightarrow Q$ is true.
- Let P be the statement "A number xx is divisible by 4" and let Q be the statement "The number

xx is even."
Implication: "If a number xx is divisible by 4, then xx is even" $P \Rightarrow Q$. This is true because every number divisible by 4 is also even.

4. Equivalent Statement (Biconditional Statement)

An equivalent statement is a logical statement of the form "P if and only if Q" and is denoted as $"P \Leftrightarrow Q"$. This statement is true when both P and Q have the same truth value (both true or both false) and false otherwise.

Truth table

p	q	$p \Leftrightarrow q$
1	1	1
1	0	0
0	1	0
0	0	1

Example 2.7

Let P be the statement "A triangle is equilateral" and let Q be the statement "All sides of the triangle are equal."
Equivalence: "A triangle is equilateral if and only if all sides of the triangle are equal" $p \Leftrightarrow q$.
This is true because a triangle is equilateral if and only if all its sides are equal.

☞ Remark

1. The statement $P \Rightarrow Q$ (implication) is logically equivalent to the statement $\overline{P} \vee Q$.

$$\underbrace{(P \Rightarrow Q)}_{(1)} \Leftrightarrow \underbrace{(\overline{P} \vee Q)}_{(2)}$$

P	Q	(1)	$\overline{P}$	(2)	$(1) \Leftrightarrow (2)$
1	1	1	0	1	1
1	0	0	0	0	1
0	1	1	1	1	1
0	0	1	1	1	1

2. The reciprocal (or converse) of an implication $P \Rightarrow Q$ is the statement $Q \Rightarrow P$.
3. The contrapositive of an implication $P \Rightarrow Q$ is the statement $\overline{Q} \Rightarrow \overline{P}$.

$$\underbrace{(P \Rightarrow Q)}_{(1)} \Leftrightarrow \underbrace{(\overline{Q} \Rightarrow \overline{P})}_{(2)}$$

P	Q	$\overline{P}$	$\overline{Q}$	(1)	(2)	$(1) \Leftrightarrow (2)$
1	1	0	0	1	1	1
1	0	0	1	0	0	1
0	1	1	0	1	1	1
0	0	1	1	1	1	1

4. We have:

$$\underbrace{(P \Leftrightarrow Q)}_{0} \Leftrightarrow \underbrace{\underbrace{(P \Rightarrow Q)}_{1} \wedge \underbrace{(Q \Rightarrow P)}_{2}}_{3}$$

P	Q	(0)	(1)	(2)	(3)	$(0) \Leftrightarrow (3)$
1	1	1	1	1	1	1
1	0	0	0	1	0	1
0	1	0	1	0	0	1
0	0	1	1	1	1	1

2.3 Basic logical rules

1. $P \wedge \overline{P}$ is false.
2. $\overline{\overline{P}} \Leftrightarrow P$.
3. $\overline{(P \wedge Q)} \Leftrightarrow \overline{P} \vee \overline{Q}$. (This is the negation of conjunction).
4. $\overline{(P \vee Q)} \Leftrightarrow \overline{P} \wedge \overline{Q}$. (This is the negation of disjunction).
5. $P \wedge P \Leftrightarrow P$, $P \vee P \Leftrightarrow P$.
6. $P \vee \overline{P}$ is true.
7. $P \vee Q \Leftrightarrow Q \vee P$; $P \wedge Q \Leftrightarrow Q \wedge P$. (Commutative property).
8. $[P \wedge (Q \wedge R)] \Leftrightarrow [(P \wedge Q) \wedge R]$; $[P \vee (Q \vee R)] \Leftrightarrow [(P \vee Q) \vee R]$. (Associative Property).
9. $[P \wedge (Q \vee R)] \Leftrightarrow [(P \wedge Q) \vee (P \wedge R)]$; $[P \vee (Q \wedge R)] \Leftrightarrow [(P \vee Q) \wedge (P \vee R)]$. (Distributive property).
10. $\overline{(P \Rightarrow Q)} \Leftrightarrow (P \wedge \overline{Q})$. (Negation of implications).
11. $[(P \Rightarrow Q) \wedge (P \Rightarrow Q)] \Rightarrow (P \Rightarrow R)$. (Transitive property of $\Rightarrow$).
12. $[P \wedge (P \Rightarrow Q)] \Rightarrow Q$ is true.
13. $[P \Rightarrow (Q \Rightarrow R)] \Leftrightarrow [(P \wedge Q) \Rightarrow R]$.
14. $[(P \vee Q) \Rightarrow R] \Leftrightarrow (P \Rightarrow R) \wedge (Q \Rightarrow R)$.

2.4 Propositional Formulas

Definition 2.3

A propositional formula is an atomic proposition; it is composed of other propositions using connectors. ($\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$ $\dots$). The propositional calculus language is constructed as follows:

1. Propositions ($P, Q, R\dots$).
2. connecteurs ($\wedge$, $\vee$, $\Leftrightarrow$, $\dots$)

Example 2.8

- **Expression:** "$2+2=4 \vee 3+3=5$" Here, 2+2=4, 3+3=5 are atomic propositions. The formula is constructed using the logical connectors "or $\vee$.

☞ **Remark**

In the logic literature, we use several synonyms for propositional symbols; thus, propositional variables, atomic propositions, atomic formulas, or even atoms are all synonymous with propositions.

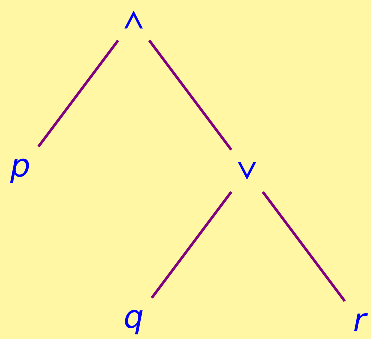
Definition 2.4

Let F_{pc} be the smallest set containing all the well-formed formulas or expressions in the propositional calculus where:

1. Every proposition is a formula.
2. If F is a formula, then $\overline{F}$ is also a formula.
3. If F, G are formulas, then $F \wedge G$, $F \vee G$ and $F \Rightarrow G$ are also formulas.

☞ **Remark**

The semantics of propositional calculus (also known as propositional logic) involve the interpretation and meaning of propositional formulas. Let us construct a parse tree for a simple propositional formula: $P \wedge (Q \vee R)$



Definition 2.5 Valuation

Let $P = (p, q, r\dots)$. A valuation (often denoted as υ) is an application $\upsilon : P \Rightarrow 0, 1$ that maps each propositional variable in a set P to a truth value—either true (1) or false (0).

Formula value:

- $\upsilon(\overline{F}) = 1$ iff $\upsilon(F) = 0$.
- $\upsilon(F \wedge G) = 1$ iff $\upsilon(F) = 1$ and $\upsilon(G) = 1$.

✿ $\upsilon(F \vee G) = 1$ iff $\upsilon(F) = 1$ or $\upsilon(G) = 1$.

✿ $\upsilon(F \Rightarrow G) = 0$ iff $\upsilon(F) = 1$ and $\upsilon(G) = 0$.

We can rewrite these values as follows:

✿ $\upsilon(\overline{F}) = 1 - \upsilon(F)$.

✿ $\upsilon(F \wedge G) = min(\upsilon(F), \upsilon(G))$.

✿ $\upsilon(F \vee G) = max(\upsilon(F), \upsilon(G))$.

✿ $\upsilon(F \Rightarrow G) == max(\upsilon(\overline{F}), \upsilon(G))$.

2.5 Formula Models

Definition 2.6

The valuation of a set of propositional variables is denoted as $Val(Prop)$.

Example 2.9

If $Prof = p, q, r, \dots$, then $Val(Prop)$ is presented in the following table:

p	q	r
1	1	1
1	1	0
1	0	1
1	0	0
0	1	1
0	1	0
0	0	1
0	0	0

Definition 2.7

A model formula of F is a valuation υ such as $\upsilon(F) = 1$. We denote the result as $mod(F)$

Example 2.10

Let us consider a proposition $F = (p \vee q) \wedge (p \vee \overline{r})$ and attempt to find its valuation.

Truth table of F:

p	q	r	$p \vee q$	$p \vee \bar{r}$	$F = (p \vee q) \wedge (p \vee \bar{r})$
0	0	0	0	1	0
0	0	1	0	0	0
0	1	0	1	1	1
0	1	1	1	0	0
1	0	0	1	1	1
1	0	1	1	1	1
1	1	0	1	1	1
1	1	1	1	1	1

Therefore, the F model is as follows:

$$mod(F) = \begin{array}{|c|c|c|} \hline p & q & r \\ \hline 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \\ \hline \end{array}$$

Definition 2.8 Satisfiability

A formula is satisfiable if there exists at least one model in which the formula is true; in other words, $\exists \upsilon : \upsilon(F) = 1$, i.e., $(mod(F) \neq \emptyset)$.

Definition 2.9

A formula F is not satisfiable if it does not accept a model, that is, $\forall \upsilon : \upsilon(F) = 0$.

Definition 2.10

A formula F is a tautology or valid if: For all possible valuations, F is evaluated as true, i.e. if $mod(F) = Val(F)$, then we denote $\models F$.

Example 2.11

$\models (P \vee \bar{P})$

Definition 2.11

A formula G is a consequence of a set of formulas F if: For every model, F is a model of G. $(mod(F) \in mod(G))$.

Example 2.12

$$(F \vee G) \models F$$
$$F \models F \vee G$$

2.6 Common normal forms

In logic, normal forms refer to standardized ways of representing logical formulas. These forms are useful for simplifying logical expressions and for various applications in logic.

Definition 2.12

- A literal is an atomic formula or the negation of an atomic formula (p or $\overline{p}$).
- A disjunctive clause is a disjunction of literals $\bigvee_{i=1}^{n} l_i = (l_1 \vee l_2 \vee \ldots \vee l_n)$ or the l_i are literals.
- A conjunctive clause is a conjunction of literals $\bigwedge_{i=1}^{n} l_i = (l_1 \wedge l_2 \wedge \ldots \wedge l_n)$ or l_i are literals.

Definition 2.13 Conjunctive Normal Form (CNF)

A formula is in conjunctive normal form (CNF) if it is a conjunction (AND) of disjunctions (ORs) of literals. Formally, it can be expressed as:

$$\bigwedge_{i=1}^{n} C_i$$

where each C_i is a disjunction (OR) of literals, i.e., $C_i = \bigvee_{j=1}^{m_i} L_{ij}$, and L_{ij} represents a literal.

Definition 2.14 Disjunctive Normal Form (DNF)

A formula is in disjunctive normal form (DNF) if it is a disjunction (OR) of conjunctions (AND) of literals. Formally, it can be expressed as:

$$\bigvee_{i=1}^{n} C_i$$

where each C_i is a conjunction (AND) of literals, that is, $C_i = \bigwedge_{j=1}^{m_i} L_{ij}$, and L_{ij} represents a literal.

Example 2.13

- $(\overline{p} \vee q \vee r) \wedge (\overline{q} \vee p) \wedge s$ is a CNF.
- $(\overline{p} \wedge q \wedge r) \vee (\overline{q} \wedge p) \vee s$ is a DNF.

☞ **Remark**

To transform a formula into a conjunctive normal form (CNF) or a disjunctive normal form (DNF), these steps are generally followed:

- Apply, if possible, the substitution of implications and equivalences with conjunctions and disjunctions.

- Replace implications $p \to q$ with $\neg p \vee q$.
- Replace the biconditionals $p \longleftrightarrow q$ with $(p \wedge q) \vee (\neg p \wedge \neg q)$.

- Move negations inward using De Morgan's laws
  - $\overline{(p \wedge q)} \equiv (\overline{p} \vee \overline{q})$
  - $\overline{(p \vee q)} \equiv (\overline{p} \wedge \overline{q})$
- Apply distributivity of conjunction and disjunction.
  - Use distribution to ensure that the formula is a conjunction of disjunctions. This step involves applying the distributive law:

$$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$$

- Simplification of the obtained formula:
  - Clauses containing two opposite literals are valid and can be removed from the CNF. [EX: $p \wedge q \wedge r \wedge \overline{q}$]
  - Repetitions of a literal within the same clause can be removed. [$\overline{p} \wedge q \wedge \overline{r} \wedge \overline{p}$]
- If in a clausal formula, a clause C_i is included in a clause C_j, then C_j can be removed from the set of clauses representing the CNF. ($C_i \models C_j$, i.e., C_j is a logical consequence).
  When C_i is included in C_j, all the literals in C_i are also present in C_j. In such a case, C_j is redundant because C_i alone is sufficient to satisfy the formula. (Ex: $C_i = p \vee q \vee r$ and $C_j = p \vee \overline{s} \vee t \vee q \vee r$)

Example 2.14

Let us have $F = p \wedge \overline{[q \Rightarrow (r \vee s)]} \wedge (p \vee q)$; find FNC.

$$\begin{aligned} F &= (\overline{p} \vee \overline{[\overline{q} \vee (r \vee s)]}) \wedge (p \vee q) \\ &= (\overline{p} \vee [q \wedge (\overline{r} \wedge \overline{s})]) \wedge (p \vee q) \\ &= (\overline{p} \vee [q \wedge \overline{r} \wedge \overline{s}]) \wedge (p \vee q) \\ &= (\overline{p} \vee q) \wedge (\overline{p} \vee \overline{r}) \wedge (\overline{p} \vee \overline{s}) \wedge (p \vee q) \end{aligned}$$

2.7 Remplacement theorem

We can see with the help of a truth table that $p \Rightarrow p$ is a tautology; if we substitute the formula $(r \vee s) \Rightarrow t$ into the variable p, we obtain

$$(r \vee s) \Rightarrow t \Rightarrow (r \vee s) \Rightarrow t$$

which is also a tautology. This means that if we replace all occurrences of a propositional variable in a tautology with any propositional formula, the resulting propositional formula is also a tautology.

Theorem 2.1

Let F be a propositional formula containing the variables $p_1, \ldots, p_n$, and let $F_i, i = 1.., n$ be any propositional formula. If F is a tautology, then the formula G obtained by replacing the occurrences *of the variables p_i, $i = 1.., n$ with F_i in F is also a tautology.*

Proof. If F is a tautology, by definition, F is true for all possible truth values of its variables $p_i, i = 1..., n$. In other words, no matter what truth values the variables $p_i, i = 1..., n$ take, FF is always true.

Now, consider substituting each variable p_i in F with a propositional formula F_i. Let us denote the resulting formula by G. The key idea is that since F is true regardless of the truth values of $p_i, i = 1..., n$, it will remain true even when each p_i is replaced by the corresponding F_i.

Since G is just F with these substitutions and F is true for all truth assignments to $p_i, i = 1..., n$, the formula G will also be true for all truth assignments to the components of the F_i formulas.

Therefore, G is also a tautology. ■

☞ **Remark** about the symbol $\models$

Note that the symbol is used in different situations; there is always a formula associated with this symbol.

- ✽ $p \models F$, where p is a model of F.
- ✽ $G \models F$, all the models of $G$are models of F; F is a logical consequence of G.
- ✽ $\models F$, where F is a tautology.

Chapter 3 Predicate Logic

Definition 3.1

Let E be a set. We say that a propositional form of a variable, defined in E, contains a variable x, which becomes a proposition for all given values of x in E.

Example 3.1

Let $E = \mathbb{N}$ and $P(x)$: x is the prime number.

- for $x = 2$, $P(2)$ is a prime number **True**.
- for $x = 4$, $P(4)$ is a prime number **False**.

Every propositional form has a variable called a predicate. For predicate logic, we need to introduce two symbols: the universal quantifier and the existential quantifier.

3.1 Quantifiers:

Let $P(x)$ be a propositional form defined in E and let $E_p = \{x \in E,\ P(x)\ True\}$.

- **Universal Quantifier** ($\forall$): Indicates that a statement applies to all the elements in a domain. For example, $\forall x\, P(x)$ means "For all $x \in E$, $P(x)$ is true."

$$\forall x \in E : P(x) \quad \text{is true} \Leftrightarrow E_p = E$$

- **Existential quantifier** ($\exists$): Indicates that there is at least one element in the domain for which the statement is true. For example, $\exists x\, P(x)$ means "There exists an $x \in E$ such that $P(x)$ is true."

$$\exists x \in E : P(x) \quad \text{is true} \Leftrightarrow E_p \neq \emptyset$$

These quantifiers extend propositional logic to predicate logic, allowing for more expressive statements about properties and relationships involving variables.

Example 3.2

To express the proposition P such as "for all real number x, there exists an integer n such that n is greater than x."

$$P : \ \forall x \in \mathbb{R}, \exists n \in \mathbb{N} : n \geq x$$

☞ Remark

- The order in which quantifiers appear significantly impacts the meaning of a statement. Changing the order can change the truth value of the statement.

Example 3.3

❀ $\forall x \in \mathbb{R}^*, \exists y \in \mathbb{R}^* : x.y = 1$. (True, for any $x \in \mathbb{R}^*$, you can choose $y = \frac{1}{x}$, which is also a nonzero real number. Thus, $x.y = x.\frac{1}{x} = 1.$).

❀ $\exists y \in \mathbb{R}^*, \forall x \in \mathbb{R}^* : x.y = 1$; (False, No single nonzero real number y can satisfy $x.y = 1$ for every $x \in \mathbb{R}^*$. The value of y depends on x and cannot be a fixed number for all x).

✿ Negating a universal quantifier turns it into an existential quantifier, and vice versa. This is a key principle in logical operations.

Example 3.4

The negation of "$P = \forall x \in \mathbb{R}, \exists n \in \mathbb{N}, n \geq x$" is $\overline{p} = \exists x \in \mathbb{R}, \forall n \in \mathbb{N}, n < x$.

Theorem 3.1

✿ $(\forall x \in E : P(x)) \wedge (\forall x \in E : Q(x)) \Leftrightarrow (\forall x \in E : P(x) \wedge Q(x))$

✿ $(\exists x \in E : P(x)) \vee (\exists x \in E : Q(x)) \Leftrightarrow (\exists x \in E : P(x) \vee Q(x))$

✿ $(\forall x \in E : P(x)) \vee (\forall x \in E : Q(x)) \Rightarrow (\forall x \in E : P(x) \vee Q(x))$

✿ $(\exists x \in E : P(x) \wedge Q(x)) \Rightarrow (\exists x \in E : P(x)) \wedge (\exists x \in E : Q(x))$

Proof. ✿ **($\Rightarrow$) Direction:**

❀ Assume that $\forall x \in E : P(x)$ and $\forall x \in E : Q(x)$ are true.

❀ This means that for every $x \in E$, $P(x)$ is true and $Q(x)$ is true.

❀ Therefore, for every $x \in E$, $P(x) \wedge Q(x)$ is true.

❀ Hence, $\forall x \in E : P(x) \wedge Q(x)$ is true.

($\Leftarrow$) Direction:

❀ Assume that $\forall x \in E : P(x) \wedge Q(x)$ is true.

❀ This means that for every $x \in E$, both $P(x)$ and $Q(x)$ are true.

❀ Therefore, $\forall x \in E : P(x)$ is true, and $\forall x \in E : Q(x)$ is true.

❀ Hence, $(\forall x \in E : P(x)) \wedge (\forall x \in E : Q(x))$ is true.

Since both directions hold, we have proven that:

$$(\forall x \in E : P(x)) \wedge (\forall x \in E : Q(x)) \Leftrightarrow (\forall x \in E : P(x) \wedge Q(x))$$

✿ **($\Rightarrow$) Direction:**

❀ Assume that $\exists x \in E : P(x)$ or $\exists x \in E : Q(x)$ is true.

❀ If $\exists x \in E : P(x)$ is true, then there exist some $x_0 \in E$ such that $P(x_0)$ is true.

❀ If $\exists x \in E : Q(x)$ is true, then there exist some $x_1 \in E$ such that $Q(x_1)$ is true.

❀ In either case, $P(x_0) \vee Q(x_0)$ or $P(x_1) \vee Q(x_1)$ is true.

❀ Hence, $\exists x \in E : P(x) \vee Q(x)$ is true.

($\Leftarrow$) Direction:

❀ Assume that $\exists x \in E : P(x) \vee Q(x)$ is true.

❀ This means that there exist some $x_0 \in E$ such that $P(x_0)$ or $Q(x_0)$ is true.

❀ If $P(x_0)$ is true, then $\exists x \in E : P(x)$ is true.

❀ If $Q(x_0)$ is true, then $\exists x \in E : Q(x)$ is true.

❀ Hence, $(\exists x \in E : P(x)) \vee (\exists x \in E : Q(x))$ is true.

Since both directions hold, we have proven that:

$$(\exists x \in E : P(x)) \vee (\exists x \in E : Q(x)) \Longleftrightarrow (\exists x \in E : P(x) \vee Q(x))$$

✿ ❀ Assume that $\forall x \in E : P(x)$ or $\forall x \in E : Q(x)$ is true.

- ❖ **Case 1**: If $\forall x \in E : P(x)$ is true, then for every $x \in E$, $P(x)$ is true.
- ❖ Since $P(x)$ is true for all $x \in E$, $P(x) \vee Q(x)$ is also true for all $x \in E$.
- ❖ **Case 2**: If $\forall x \in E : Q(x)$ is true, then for every $x \in E$, $Q(x)$ is true.
- ❖ Since $Q(x)$ is true for all $x \in E$, $P(x) \vee Q(x)$ is also true for all $x \in E$.

❀ In either case, $\forall x \in E : P(x) \vee Q(x)$ is true. Thus, we have proven the following:

$$(\forall x \in E : P(x)) \vee (\forall x \in E : Q(x)) \Rightarrow (\forall x \in E : P(x) \vee Q(x))$$

❀ the reciprocal of Statement 3 is false. Let us have an example: $E = \mathbb{N}$, such that $P(x)$: x is even and $Q(x)$: x is odd; then,

$$\forall x \in \mathbb{N} : \quad (x \quad \text{is even} \vee$$

x $\quad$ is odd); $\quad True$. However, $[\forall x \in \mathbb{N} : \quad (x \quad \text{is even}; \quad False] \vee [\forall x \in \mathbb{N} : x \quad \text{is odd}; \quad False]$

✿ ❀ Assume that $\exists x \in E : P(x) \wedge Q(x)$ is true.

❀ This means that there exist some $x_0 \in E$ such that both $P(x_0)$ and $Q(x_0)$ are true.

❀ Since $P(x_0)$ is true, $\exists x \in E : P(x)$ is true.

❀ Since $Q(x_0)$ is true, $\exists x \in E : Q(x)$ is true.

❀ Hence, $(\exists x \in E : P(x)) \wedge (\exists x \in E : Q(x))$ is true.

❀ the reciprocal of Statement 4 is false. Let us have an example: $E = R$ and let $P(x) = x \geq 2$ and $Q(x) = x < 2$. Therefore, $\exists x \in \mathbb{R}, x \geq 2 \quad True \wedge \exists x \in \mathbb{R}, x < 2 \quad True$. However, $\exists x \in R, ((x \geq 2) \wedge (x < 2)) \;\; False$.

■

Chapter 4 Reasoning types

4.1 Mathematical Proof

There is no mathematical truth except within the framework of a given theory. For example, the assertion: "Every number has a strictly smaller number." an be analyzed differently depending on the context of the number set (real numbers, natural numbers, or complex numbers). Here is an explanation for each:

Real numbers: The assertion is true. For any real number x, there exists another real number y such that $y < x$. For example, if $x = 5$, you can always find a number such as 4.9 or even 4.999999.

Natural numbers: The assertion is false. The natural numbers typically start at 1 (or 0 in some definitions). For the smallest natural number (which is 1 in the usual definition), there is no strictly smaller natural number.

Complex numbers: Not Applicable: The concept of "strictly smaller" does not apply in the same way to complex numbers.

4.1.1 Proving $H \Rightarrow C$

In a given mathematical theory, solving a problem means demonstrating a proposition of the type ($H \Rightarrow C$). That is, proving C under the hypothesis H. To do this, one relies on the hypothesis H and the propositions of the relevant theory.

One establishes a sequence $Q_0, Q_1, \ldots, Q_n$ such that Q_0 is derived from the truth value of H, and the truth value of Q_i is deduced from the truth value of Q_{i-1}.

$$\begin{cases} H \Rightarrow Q_1 \\ (H \wedge Q_1) \Rightarrow Q_2 \\ \vdots \\ (H \wedge Q_1 \wedge \ldots \wedge Q_n) \Rightarrow C \end{cases}$$

Example 4.1

Let x be an integer ($x \in Z$). If x is an even integer, then x^2 is even.

Proof: Assume that x is an even integer. By definition, x can be written as:

$$x = 2k$$

where k is an integer. Square both sides of the equation:

$$x^2 = (2k)^2 = 4k^2$$

Factorize the right-hand side:

$$x^2 = 2(2k^2)$$

Since $2k^2$ is an integer (as k is an integer), let $m = 2k^2$. Thus, we can write:

$$x^2 = 2m$$

where m is an integer. Therefore, x^2 is of the form $2m$, which is the definition of an even number.

4.1.2 Proof of : $\forall x \in E: \; P(x)$

- To prove a statement of the form $\forall x \in E: \; P(x)$, where E is a set and $P(x)$ is a predicate (a statement involving x), a universal proof technique is typically needed. Let x be an arbitrary element of E. This means that x is any element you might choose from the set E, and we need to prove that $P(x)$. Suppose that $x \in E$. Since x is arbitrary, the proof shows that $P(x)$ holds for all x in E. Thus, you can conclude that $\forall x \in E: \; P(x)$ is true.
- To prove that $[\forall x \in E: \; P(x)]$ is false. It is sufficient to verify that $[\exists x \in E: \; \overline{P(x)}]$ is true. This means that find an arbitrary element $x_0 \in E$, which does not verify the property.

4.2 Direct Proof

- To directly prove an assertion of type $P \Rightarrow Q$, we suppose that we have P and that we prove that Q is true by using propositions of the theory concerned.
- To directly prove an assertion of type $P \wedge Q$, we prove P, and we prove Q.
- To directly prove an assertion of type $P \vee Q$, we can prove that one of the two assertions is true P or Q. Additionally, we can prove that one of them is false and that the other is true.

Example 4.2

- Suppose that x is an integer; prove that (x is odd and that $\vee$ x^2 is even) is true.
  We suppose that (x is odd) is false; then, x is even, and using the result of the previous example, we prove that x^2 is even.
- $(\forall x \in \mathbb{R}, x^2 \leq x)$ is false. We have a counterexample: $x = 2$ than $x^2 = 4 > x = 2$.

4.3 Indirect Proof

4.3.1 Reasoning by contrapositivity

Instead of proving the statement $P \Rightarrow Q$ directly, you prove its contrapositive: $\overline{Q} \Rightarrow \overline{P}$. Since a statement and its contrapositive are logically equivalent, proving one proves the other. Therefore, we suppose that $\overline{Q}$ is true and that we prove that $\overline{P}$.

Example 4.3

proving that if x^2 is odd, then x is odd. Proving by Contrapositive, i.e., we need to prove that x is even and that x^2 is even.

4.3.2 Reasoning by excellence

Proof by contradiction (also known as reasoning by absurdity) is based on the following statement:

$$[(\overline{P} \Rightarrow Q) \Rightarrow (\overline{P} \Rightarrow \overline{Q})] \Leftrightarrow P$$

This indicates that the entire expression on the left side $(\overline{P} \Rightarrow Q) \Rightarrow (\overline{P} \Rightarrow \overline{Q})$ is logically equivalent to the proposition P. Therefore, to prove that P is true, we need to confirm that the other proposition is true because a certain assertion is true. Therefore, we assume that $\overline{P}$ is true (P is false). This assumption leads you to simultaneously conclude that Q is true and that Q is false (which is a contradiction); then, the original assumption must be incorrect. Consequently, P must be true.

The statement sums up how proof by contradiction works. This shows that assuming that $\overline{P}$ (not P) leads to something impossible. (where Q and $\overline{Q}$ are both true), then P must be true.

Example 4.4

Prove that the set of prime numbers is infinite.

Let us use reasoning by contradiction: Suppose that the set $P = p_1, \ p_2 \dots p_n$ of prime numbers is finite and that we prove until we find a contradiction. Let us define a number

$$m = p_1.p_2...p_n + 1$$

By construction, m is not divisible by any of the primes $p_1, \ p_2 \dots p_n$. (i.e., m/p_i leaves a remainder of 1). Therefore, m must either be:

- A prime number itself (if it is greater than 1).
- Or, a composite number with prime factors not in our list.

In either case, m contradicts our original assumption that $p_1, \ p_2 \dots p_n$ are all prime numbers. Since assuming that the set of prime numbers is finite leads to a contradiction, our assumption must be false.

☞ Remark

- in this example, the Q assertion is the prime number $p_1, \ p_2 \dots p_n$.
- This example proves that the analysis of a problem is not always obvious

4.3.3 Proof by disjunction of Cases

We have:

$$[(Q \Rightarrow P) \wedge (\overline{Q} \Rightarrow P)] \Leftrightarrow P$$

To prove that a given assertion P is true, it is sufficient to find the assertion Q such that $(Q \Rightarrow P)$ and $(\overline{Q} \Rightarrow P)$ are true. In general, we mention that the assertion Q comes naturally from the problem analysis. In the proof, we write:

- 1^{st} case: Assume we have Q and verify P.
- 2^{nd} case: Assume that we have $\overline{Q}$ and verify P.

Example 4.5

Suppose $\lambda < 0$, proving:

$$\forall (x, y) \in \mathbb{R}^2 : max(\lambda x, \lambda y) = \lambda min(x, y)$$

Solution: Suppose that x and y are two real numbers.

- 1^{st} case: Let $x \leq y$ then $\lambda x \geq \lambda y$ and $max(\lambda x, \lambda y) = \lambda x = \lambda min(x, y)$.
- 2^{nd} case: Let $x > y$ then $\lambda x < \lambda y$ and $max(\lambda x, \lambda y) = \lambda y = \lambda min(x, y)$

This is the result.

☞ **Remark**

✿ In the above example, the assertion $Q = x \leq y$ comes naturally in the evaluation of $min(x, y)$.

✿ The method presented can be generalized easily to the study of more than two cases.

4.3.4 Proof by induction

This type of demonstration applies to propositions of the form $X = \forall n \in \mathbb{N}, P(n)$. It is often more effective to use a proof by induction rather than a classical proof. The property of induction is presented here as a consequence of the construction of $\mathbb{N}$.

They rely on the following arithmetic theorem.

Every subset of $\mathbb{N}$ that satisfies the following conditions:

✿ $0 \in X$.

✿ $\forall x \in \mathbb{N}; (x+1) \in X$

is identical to $\mathbb{N}$ and can be expressed in the following form.

Induction property Let P be a statement (or assertion) about the integer variable $n \in \mathbb{N}$. We have the following proposition:

$$P(0) \wedge (\forall k \in \mathbb{N} : P(k) \Rightarrow P(k+1)) \Rightarrow (\forall n \in \mathbb{N}, P(n))$$

Proof. Let $X = n \in \mathbb{N}/P(n)$ and suppose that we have $P(0) \wedge (\forall k \in \mathbb{N} : P(k) \Rightarrow P(k+1))$; then,

✿ $0 \in X$

✿ $\forall k \in \mathbb{N} : k \in X \Rightarrow k+1 \in X$

on the basis of the arithmetic theorem $X = \mathbb{N}$. Therefore, $\forall n \in \mathbb{N}, P(n)$. ■

The principle of mathematical induction is often taken as an axiom in the construction of natural numbers, but we can understand its validity through a logical argument.

1. **Initialization:** Let us assume that $P(0)$ (or $P(1)$) is true.
2. **Inductive Hypothesis:** Assume that$P(k)$ is true for some arbitrary $k \in \mathbb{N}$.
3. **Inductive Step:** Using the induction hypothesis, you must show that $P(k+1)$ is true. This step connects the truth of $P(k)$ to the truth of $P(k+1)$.

☞ **Remark**

Let us highlight two commonly encountered variants, which are consequences of the property of recurrence:

✿ Incomplete Recurrence Property:

$$P(n_0) \wedge (\forall k \geq n_0 : P(k) \Rightarrow P(k+1)) \Rightarrow (\forall n \geq n_0 P(n))$$

✿ finite Recurrence Property:

$$P(n_0) \wedge \ (\forall k \in \{n_0,\ \ldots,\ m-1\} : P(k) \Rightarrow P(k+1)) \Rightarrow (\forall n \in \{n_0,\ \ldots,\ m\} P(n))$$

Chapter 5 Paradox

Definition 5.1

A paradox begins with true statements and uses correct logic, but ends with contradictory or impossible conclusions. These contradictions arise from the way the statements are linked together. In real life, paradoxes such as these do not actually occur and cannot be fully understood or resolved. They often involve statements that refer to themselves, such as "This sentence is false." If the sentence is true, then it is false, but if it is false, then it has to be true.

Origin of the word "paradox" The word "paradox" comes from the Greek word "paradoxon," which means something surprising or contrary to what we expect. It comes from:

"Para-" means "beside" or "beyond" "Doxon" means "opinion" or "belief".

Thus, "paradoxon" means "beyond belief," showing that it is something that goes against what we normally think.

5.1 Paradox of the business card

Imagine a business card with two sides:

- Side 1: "The statement on the other side is false."
- Side 2: "The statement on the other side is true."

Reasoning:

- If the statement on Side 1 is true, then Side 2's statement ("The statement on the other side is true") is false. However, if Side 2's statement is false, Side 1's statement is also false, which creates a contradiction.
- If the statement on Side 1 is false, then Side 2's statement is true. However, if Side 2's statement is true, then Side 1's statement should be true, again creating a contradiction.

Therefore, both statements contradict each other regardless of which side you start with.

5.2 Paradox of the Mouse and the Rabbit

Imagine a situation where:

- The Mouse says: "The rabbit is lying."
- The rabbit says, "The mouse is telling the truth."

Reasoning:

- If the Mouse is telling the truth (i.e., the Rabbit is lying), then the Rabbit's statement ("The mouse is telling the truth") is false. This means that the mouse is not telling the truth, which creates a contradiction.
- If the mouse is lying (i.e., the Rabbit is not lying), then the Rabbit's statement ("The mouse is telling the truth") is true. However, if the Rabbit's statement is true, then the mouse must be telling the truth, again creating a contradiction.

5.3 Russell's Paradox

Russell's paradox arises in set theory, which is a branch of mathematical logic that addresses collections of objects, called sets. The paradox was discovered by the philosopher and logician Bertrand Russell in 1901. It highlights a fundamental problem with naive set theory, which allows for the creation of any set defined by a property.

Paradox Statement: Consider the set of all sets that do not contain themselves as a member. Let us call this set A.

Definition of A: A is the set of all sets that do not contain themselves as a member.

Reasoning:

- If A contains itself as a member, then it should not contain itself. This creates a contradiction because A cannot both contain and not contain itself simultaneously.
- If A does not contain itself as a member, then it should contain itself. This also creates a contradiction because A cannot both contain and contain itself simultaneously.

Logical Formulation:

Define A as follows:

$$A = \{X | X \text{ is a set and } x \notin X\}$$

Ask whether RR is a member of itself:

- If $A \in A$: By the definition of A, A should not be a member of itself. This is a contradiction.
- If $A \notin A$: By the definition of A, A should be a member of itself. This is also a contradiction.

5.4 Barber Paradox

Situation: In a village, there is a barber who shaves exactly those people who do not shave themselves. Does the barber shave himself?

Reasoning:

- Assume the barber does shave himself: According to the rule, the barber only shaves those who do not shave themselves. If the barber shaves himself, then he must be someone who shaves himself, which means that, according to the rule, he should not shave himself. This creates a contradiction because if he shaves himself, he should not be shaving himself.
- Assume that the barber does not shave himself: According to the rule, the barber shaves those who do not shave themselves. If the barber does not shave himself, then he should be shaved by the barber because he fits the condition of "not shaving himself." This creates another contradiction, because if he does not shave himself, he must be shaved by himself, which contradicts the assumption.

5.5 Liar Paradox

The liar paradox is a classic example of a self-referential paradox. It arises when someone makes a statement that refers to itself in a way that creates a contradiction. For example, a person says "I am a liar." **Reasoning**

- If the statement "I am a liar" is true:

If someone is telling the truth about being a liar, then they must always lie. However, if they always lie, then the statement "I am a liar" must be false, because they would be lying when they said it. This creates a contradiction because if the statement is true, it must be false.

- If the statement "I am a liar" is false:
  If the person lies about being a liar, then they must be telling the truth. However, if they are telling the truth, then the statement "I am a liar" should be true, which again creates a contradiction.

5.6 Grelling–Nelson Paradox

The Grelling–Nelson paradox is a self-referential paradox that arises in the context of linguistics, specifically dealing with the classification of adjectives. It was formulated by the German logician Kurt Grelling and philosopher Leonard Nelson in 1908. The paradox revolves around two types of adjectives:

- **Autological Adjectives:** These are adjectives that describe themselves. For example, the word "short" is short, so it is autological. Similarly, "English" is an English word, so it is autological.
- **Heterological Adjectives:** These are adjectives that do not describe themselves. For example, "long" is not a long word, so it is heterological. Likewise, "French" is not a French word, so it is heterological.

The paradox arises when one tries to classify the word "heterological" itself.

Reasoning:

- Assume "heterological" is heterological: If "heterological" is heterological, then it means that it does not describe itself. However, if "heterological" does not describe itself, then it must be true that "heterological" is heterological. This leads to a contradiction because the word is supposed to describe something it does not describe.
- Assume that "heterological" is not heterological (i.e., it is autological): If "heterological" is autological, then it describes itself. However, if it describes itself, it should be heterological, because it describes words that do not describe themselves. This, again, leads to a contradiction.

Chapter 6 Set theory

6.1 Set Theory

6.1.1 Definition

A set is a collection of objects that can be enumerated or defined by a property. A set is often denoted by an uppercase letter (e.g., A, B), and its elements are denoted by lowercase letters (e.g., x, y, z).

- Some sets have special notations, such as $\mathbb{C}$ for complex numbers, $\mathbb{N}$ for natural numbers, and $\mathbb{R}$ for real numbers.
- When we enumerate the elements of a set, we say that the set is defined by extension. For example, $E = \{a, b, c\}$.
- When we define a set by a property, we say that the set is defined by comprehension. For example, $E = \{x \in A \mid P(x)\}$.
- The set that contains no elements is called the empty set, denoted by $\emptyset$.
- A set that contains only one element is called a singleton.

Example 6.1

- $A = \{1, 3, 5, 7, 9\}$ (defined by extension).
- $B = \{2^0, 2^1, 2^2, \ldots\} = \{2^i \mid i \in \mathbb{N}\}$ (also defined by extension).

When the number of elements in a set becomes too large or when there is an infinite number of elements, we can define it by comprehension:

$$[2, 15] = \{x \in \mathbb{R} \mid 2 \leq x \leq 15\} \text{ (defined by comprehension).}$$

$$2\mathbb{N} = \{n \in \mathbb{N} | \exists k \in \mathbb{N} : n = 2k\}$$

6.1.2 Membership, Inclusion

Definition 6.1 Membership (Appartenance)

Let E be a nonempty set.

- Membership refers to the relationship between an element and a set. If $x \in E$ means that "x is an element of E". Equivalent expressions are "x is a member of E", "x belongs to E", and "x is in E".
-

Example 6.2

If $A = \{1, 2, 3\}$, then $1 \in A$ (1 is a member of the set A), but $4 \notin A$ (4 is not a member of set A).

Definition 6.2 Inclusion

Inclusion refers to the relationship between two sets. A set F is said to be a subset of set E if every element of F is also an element of E. This is denoted as $F \subseteq E \Leftrightarrow \forall x \in F, x \in E$.

- If F is a subset of E but F is not equal to E, we say that F is a proper subset of E and write $F \subset E$.
- $E = F \Leftrightarrow F \subset E \wedge E \subset F$
- $\mathbb{P}(E) = \{F | F \subset E\}$
- $x \in A \cup B \Leftrightarrow (x \in A) \wedge (x \in B)$
- $x \in A \cap B \Leftrightarrow (x \in A) \vee (x \in B)$
- $x \in \overline{A} \Leftrightarrow x \notin A$
- $\overline{A \cap B} = \overline{A} \cup \overline{B}$
- $\overline{A \cup B} = \overline{A} \cap \overline{B}$
- $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
- $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

☞ **Remark**
A negation denotes the complement, union by $\vee$, and intersection by $\wedge$. We can prove these rules easily via the truth table:

A	B	$\overline{A}$	$\overline{B}$	$A \cap B$	$\overline{A \cap B}$	$\overline{A} \cup \overline{B}$
1	1	0	0	1	0	0
1	0	0	1	0	1	1
0	1	1	0	0	1	1
0	0	1	1	0	1	1

Example 6.3

If $F = \{1, 2\}$ and $E = \{1, 2, 3\}$, then $F \subseteq E$ (F is a subset of E). However, $E \nsubseteq A$ because not all the elements of E are in F.

6.2 Cartesian product

Definition 6.3

The Cartesian product of two sets $E \neq \emptyset$ and $F \neq \emptyset$, denoted as $E \times F$, is the set of all ordered pairs (x, y), where $x \in E$ and $y \in F$. Formally, it is defined as:

$$E \times F = \{(x, y) | \, x \in E and y \in F\}$$

For example, if $A = 1, 2$ and $B = x, y$, then the Cartesian product $A \times B$ is:

$$A \times B = \{(1, x), (1, y), (2, x), (2, y)\}$$

Proposition 6.1

Let $A,\ B,\ C$ be three sets

✽ $A \times (B \cup C) = (A \times B) \cup (A \times C)$

✽ $A \times (B \cap C) = (A \times B) \cap (A \times C)$

☞ **Remark**

Let $A,\ E_i$ be nonempty sets.

✽ $\prod_{i=1}^{n} E_i = \{(x_1,\ x_2,\ \ldots,\ x_n) | x_i \in E_i \forall i = \overline{1,n}\}$

✽ $A^n = A \times \ldots . \times A$

Definition 6.4

Given two sets E and F, a graph of a function (or functional graph) is a subset $X \subset E \times F$ that satisfies the following condition:

$$\forall (a,b) \in X and (a',b') \in X,\ a = a' \Rightarrow b = b'$$

6.3 Binary Relation

Definition 6.5

Let E and F be two sets. A **binary relation** R from E to F is a subset of the Cartesian product $E \times F$. That is,

$$R \subseteq E \times F$$

This means that a binary relation R consists of ordered pairs (a,b), where a is an element of E and b is an element of F.

☞ **Remark**

If $(a,b) \in R$, we say that a **is related to** b **by** R, and we often write this as:

$$a\,R\,b$$

Example 6.4

1. **Equality Relation**: Let $E = F = \mathbb{R}$ (the set of real numbers). The relation $R = \{(x,y) \mid x = y\}$ defines the equality relation on $\mathbb{R}$.
2. **Less Than Relation**: Let $E = F = \mathbb{R}$. The relation $R = \{(x,y) \mid x < y\}$ defines the "less than" relation on $\mathbb{R}$.
3. **Divisibility Relation**: Let $E = F = \mathbb{Z}$ (the set of integers). The relation $R = \{(a,b) \mid a \text{ divides } b\}$ defines the divisibility relation on $\mathbb{Z}$.

☞ **Remark** Properties of Binary Relations

Binary relations can have various properties depending on the specific sets E and F and the nature of the relation R. Some important properties include the following:

✽ **Reflexivity**: R is reflexive if $\forall a \in E,\ (a,a) \in R$.

✿ **Symmetry**: R is symmetric if $\forall(a,b) \in R,\ (b,a) \in R$.

✿ **Antisymmetry**: R is antisymmetric if $\forall(a,b) \in R$ and $(b,a) \in R$, then $a = b$.

✿ **Transitivity**: R is transitive if $\forall(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$.

6.4 Application

Definition 6.6

An **application** is a special type of binary relation between two sets, where every element in the first set (called the **domain**) is associated with exactly one element in the second set (called the **codomain**).
Formally, an application f from a set E to a set F is a relation such that:

$$f : E \to F$$

This means that for each $a \in E$, there exists a unique $b \in F$ such that $f(a) = b$.

Example 6.5

Let $E = \{1, 2, 3\}$ and $F = \{a, b, c\}$. A function $f : E \to F$ can be defined as:

$$f(1) = a, \quad f(2) = b, \quad f(3) = c$$

6.4.1 Image of a Function

The **image** of an application f, also known as the **range**, is the set of all values that f takes as its output. In other words, it is the set of all the elements in the codomain F that are mapped to by at least one element of the domain E.

Formally, if $f : E \to F$, the image of f, denoted by $f(E)$ or $\mathrm{Im}(f)$, is:

$$\mathrm{Im}(f) = \{f(a) \mid a \in E\}$$

Example 6.6

Using the previous example where $f(1) = a, f(2) = b$, and $f(3) = c$, the image of f is as follows:

$$\mathrm{Im}(f) = \{a, b, c\}$$

6.4.2 Direct Image of a Subset

Definition 6.7

The **direct image** (or **forward image**) of a subset $A \subseteq E$ under an application $f : E \to F$ is the set of all the elements in F that are images of the elements of A.
Formally, for $A \subseteq E$, the direct image $f(A)$ is:

$$f(A) = \{f(a) \mid a \in A\}$$

Example 6.7

Let $A = \{1, 2\}$. The direct image of A under the function f is as follows:

$$f(A) = \{f(1), f(2)\} = \{a, b\}$$

6.4.3 Inverse Image of a Subset

Definition 6.8

The **inverse image** (or **preimage**) of a subset $B \subseteq F$ under a function $f : E \to F$ is the set of all the elements in E that map to the elements of B under f.
Formally, for $B \subseteq F$, the inverse image $f^{-1}(B)$ is:

$$f^{-1}(B) = \{a \in E \mid f(a) \in B\}$$

Example 6.8

Let $B = \{b, c\}$. The inverse image of B under the function f is as follows:

$$f^{-1}(B) = \{a \in E \mid f(a) \in \{b, c\}\}$$

If $f(2) = b$ and $f(3) = c$, then:

$$f^{-1}(B) = \{2, 3\}$$

Proposition 6.2

Let $f : E \to F$ be an application

- ✽ $\forall A,\ B \subset F\ \ f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$
- ✽ $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$
- ✽ $f(A \cap B) \subset f(A) \cap F(B),\quad f(A \cup B) = f(A) \cup f(B)$
- ✽ $f(f^{-1}(B)) \subset B$
- ✽ $A \subset f^{-1}(f(A))$

Chapter 7 Exercises

Exercise 7.1

A flan contains 9% sugar. What is the percentage of sugar in 6 flans?

- 54%
- 9%
- 50%

Exercise 7.2

A snail tries to reach the top of a 12-meter pole. It climbs 3 meters each day but slips down 2 meters each night. How long will it take to reach the top of the pole?

- 12
- 11
- 10
- 9

Exercise 7.3

Using Newton's Binomial, compute:

$$\sum_{k=0}^{100}(-1)^k 2^{-k} C_{100}^k$$

- 0
- 2^{-100}
- 2^{100}
- 100

Exercise 7.4

Let

$$A = \{(x, y) \in \mathbb{R}^2 : 2x - y = 1\}$$
$$B = \{(t+1, 2t+1) : t \in \mathbb{R}\}$$

What can we say about A and B?

- $A \quad B$
- $B \quad A$
- $A \neq B$
- $A = B$

Exercise 7.5

Rewrite the following formulas using only the connectors $(-, \wedge)$:

- $p \Rightarrow (q \vee r)$;
- $\overline{p} \vee q \vee \overline{r}$;
- $\overline{p} \Rightarrow (\overline{q} \Leftrightarrow r)$.

Exercise 7.6

Given that p represents the proposition "X esteems Y" and that q represents the proposition "Y esteems X", formalize the following sentences in propositional calculus:

- X does not estimate Y
- X esteems Y and Y esteems X
- X estimates Y but Y does not estimate X
- X esteems Y or Y esteems X
- If X estimate Y, then Y estimate X

Exercise 7.7

Are the following assertions true or false? Provide their negations.

- $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, x + y > 0$
- $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x + y > 0$
- $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, y^2 > x$
- $\forall \epsilon \in \mathbb{R}^{*+}, \exists \alpha \in \mathbb{R}^{*+}, |x| < \alpha \Rightarrow |x^2| < \epsilon$

Exercise 7.8

We introduce the connector $|$ (Sheffer's stroke), whose truth table is given by

P	Q	$P \mid Q$
1	1	0
1	0	1
0	1	1
0	0	1

- Express the connector $|$ in terms of $\wedge$
- Is the connector $|$ commutative $(P \mid Q = Q \mid P)$? Is it associative $((P \mid Q) \mid R = P \mid (Q \mid R))$?
- Provide truth tables for the formulas $P \mid P$, $(P \mid Q) \mid (P \mid Q)$, $(P \mid P) \mid (Q \mid Q)$, $P \mid (Q \mid Q)$
- Express the following formulas $P \mid P$, $(P \mid Q) \mid (P \mid Q)$, $(P \mid P) \mid (Q \mid Q)$, $P \mid (Q \mid Q)$ in terms of the connectors $\wedge, \vee, \Rightarrow, \ldots$

Exercise 7.9

Consider the following two proposition formulas:

$$(p \vee (q \wedge r)) \wedge (q \vee r) \tag{7.1}$$

$$((q \Rightarrow p) \vee \neg(p \Leftrightarrow q)) \wedge (r \wedge \neg p) \tag{7.2}$$

- Construct the truth table for the two formulas
- Given that p and q are true and that r is false, find the truth values of the two propositional formulas (1) and (2)

Exercise 7.10

Complete the sentences exclusively with the words BECAUSE, IF, THEN, AS, and THEREFORE:

- The number a is less than 5 .... a is less than 3;
- I am Algerian .... I am not European;
- $y^2 = 9$ .... $y = 3$;
- ..... two lines are perpendicular to the same line, ... they are parallel;
- ...... I is the midpoint of $[AB]$, we have $AI = IB$.

Exercise 7.11

In $\mathbb{R}^2$, we define the sets:

$$F_1 = \{(x, y) \in \mathbb{R}^2 \mid y \leq 0\}$$
$$F_2 = \{(x, y) \in \mathbb{R}^2 \mid xy \geq 1, \quad x \geq 0\}$$

Let M_1M_2 denote the usual distance between two points M_1 and M_2 in $\mathbb{R}^2$. Evaluate the following propositions:

- $\forall \varepsilon > 0, \exists M_1 \in F_1, \exists M_2 \in F_2$ such that $M_1M_2 < \varepsilon$
- $\exists M_1 \in F_1, \exists M_2 \in F_2$ such that $\forall \varepsilon > 0$ we have $M_1M_2 < \varepsilon$
- $\exists \varepsilon > 0, \forall M_1 \in F_1, \forall M_2 \in F_2$ we have $M_1M_2 < \varepsilon$
- $\forall M_1 \in F_1, \forall M_2 \in F_2, \exists \varepsilon > 0$ such that $M_1M_2 < \varepsilon$

When they are false, provide their negations.

Exercise 7.12

- Give a sufficient but not necessary condition for an integer to be strictly greater than 10;
- Give a necessary but not sufficient condition for an integer to be divisible by 6;
- Give a necessary and sufficient condition (CNS) for an integer n to be divisible by 3.

Exercise 7.13

where the error is as follows:

✿

$$
\begin{aligned}
-20=-20 \implies 16-36=25-45 &\implies 4^2-4\times 9=5^2-5\times 9 && (7.3)\\
&\implies 4^2-2\times 4\times \frac{9}{2}=5^2-2\times 5\times \frac{9}{2} && (7.4)\\
&\implies 4^2-2\times 4\times \frac{9}{2}+\frac{81}{4}=5^2-2\times 5\times \frac{9}{2}+\frac{81}{4} && (7.5)\\
&\implies (4-\frac{9}{2})^2=(5-\frac{9}{2})^2 && (7.6)\\
&\implies 4-\frac{9}{2}=5-\frac{9}{2} && (7.7)\\
&\implies 4=5 \quad !!! && (7.8)
\end{aligned}
$$

✿ We have $i^2=-1$; thus,

$$
\begin{aligned}
-1\times -1=1 &\implies i^2\times i^2=1 && (7.9)\\
&\implies (i^2)^2=1 && (7.10)\\
&\implies i^2=1 \quad !!!! && (7.11)
\end{aligned}
$$

Exercise 7.14

Three people, (A), (B), and (C), each have a different profession: pharmacist, dentist, or surgeon. Given that the following implications are true, their professions should be determined:

✿ A is a surgeon $\Rightarrow$ B is a dentist

✿ A is a dentist $\Rightarrow$ B is a pharmacist

✿ B is not a surgeon $\Rightarrow$ C is a dentist

Exercise 7.15

We define the connector denoted by $\oplus$ as follows: $p \oplus q$ is true if and only if exactly one of the propositions p and q is true.

✿ Express the truth table for $\oplus$

✿ Show that $p\oplus q=(p\wedge \overline{q})\vee(\overline{p}\wedge q)$

✿ Show that $p\oplus q=\overline{p\Leftrightarrow q}$

✿ Determine if the connector $\oplus$ is commutative ($p\oplus q\equiv q\oplus p$) and if it is associative ($[(p\oplus q)\oplus r]\equiv[p\oplus(q\oplus r)]$)?

Exercise 7.16

Let f be a function from $\mathbb{R}$ to $\mathbb{R}$. The following statements are expressed via quantifiers, and their negation is then provided.

- f is the zero function.
- The equation $f(x) = 0$ has a solution.
- The equation $f(x) = 0$ has exactly one solution.
- f is the identity function on $\mathbb{R}$.
- The graph of f intersects line $y = x$.
- f is increasing on $\mathbb{R}$.

Exercise 7.17

Let the following propositions:
$P : \forall x \in \mathbb{R}, (f(x) = 0 \wedge g(x) = 0)$
Q: $(\forall x \in \mathbb{R}, f(x) = 0) \wedge (\forall x \in \mathbb{R}, g(x) = 0)$
$R : \forall x \in \mathbb{R}, (f(x) = 0 \vee g(x) = 0)$
$S : (\forall x \in \mathbb{R}, f(x) = 0) \vee (\forall x \in \mathbb{R}, g(x) = 0)$
$T : (\forall x \in \mathbb{R}, f(x).g(x) = 0) \Rightarrow [(\forall x \in \mathbb{R}, f(x) = 0) \wedge (\forall x \in \mathbb{R}, g(x) = 0)]$

- Are P and Q equivalent?
- Are R and S equivalent?
- Translate the proposition T into everyday language.
- Show with an example that the proposition T is false.

Exercise 7.18

Let a and b be two nonzero natural numbers. Show that

$$(\exists k \in \mathbb{N} \mid b = ka \wedge \exists l \in \mathbb{N} \mid a = lb) \Rightarrow a = b$$

Exercise 7.19

Let $n \geq 1$ be a natural number. Consider $n + 1$ real numbers $x_0, x_1, \ldots, x_n$ satisfying $0 \leq x_0 \leq x_1 \leq \ldots \leq x_n \leq 1$.
We want to prove by contradiction the property: There are two of these reals that are at a distance of less than $\frac{1}{n}$.

- Write using quantifiers and the values $x_i - x_{i-1}$ a logical formula equivalent to the property.
- Write the negation of this logical formula.
- Write a proof by contradiction of the property (you can show that $x_n - x_0 > 1$).

Exercise 7.20

Let the proposition
If the integer $n^2 - 1$ is not divisible by 8, then the integer n is even.

- Write the contrapositive of the previous proposition.
- Show that any odd integer n can be written in the form $n = 4k + r$ with $k \in \mathbb{N}$ and $r \in \{1, 3\}$.
- Prove the contrapositive.

✿ Have we proven the property of the statement?

Exercise 7.21

Prove by induction that:

✿ $\forall n \in \mathbb{N} - \{0,1\}, 1 + \frac{1}{2} + \ldots + \frac{1}{n^2} > \frac{3n}{2n+1}$

✿ If x is a real number such that $(x + \frac{1}{x}) \in \mathbb{Z}$, then for all $n \geq 1$, $(x^n + \frac{1}{x^n}) \in \mathbb{Z}$

Exercise 7.22

For $n \in \mathbb{N}$, define two properties:

$$P_n : 3 \text{ divides } (4^n - 1) \text{ and } Q_n : 3 \text{ divides } (4^n + 1)$$

✿ Prove that for all $n \in \mathbb{N}$, $P_n \Rightarrow P_{n+1}$ and $Q_n \Rightarrow Q_{n+1}$.

✿ Show that P_n is true for all $n \in \mathbb{N}$.

✿ What do you think about the assertion: $\exists n_0 \in \mathbb{N}, \forall n \in \mathbb{N}, n \geq n_0 \Rightarrow Q_n$?

Exercise 7.23

Prove by induction that:

✿ $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \forall n \in \mathbb{N}^*$

✿ $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \forall n \in \mathbb{N}^*$

Exercise 7.24

Let f be a function from $\mathbb{R}$ to $\mathbb{R}$. The following statements are expressed via quantifiers, and their negation is then provided.

✿ f is the zero function.

✿ The equation $f(x) = 0$ has a solution.

✿ The equation $f(x) = 0$ has exactly one solution.

✿ f is the identity function on $\mathbb{R}$.

✿ The graph of f intersects line $y = x$.

✿ f is increasing on $\mathbb{R}$.

✿ f is bounded.

✿ f never vanishes.

✿ f is periodic.

✿ f is not the zero function.

✿ f takes on all values in $\mathbb{N}$.

Exercise 7.25

Show that

$\forall \varepsilon > 0, \exists n_0 \in \mathbb{N}$ such that $n \geq n_0 \Rightarrow 2 - \varepsilon < \dfrac{2n+1}{n+2} < 2 + \varepsilon$

Exercise 7.26

Let $P = \{n \in \mathbb{N} \mid n \text{ be even }\}$ $PR = \{n \in \mathbb{N} \mid n \text{ is prime }\}$ $Div(x, y) \equiv x$ divides y

- Express the following sentences via quantifiers:
  - There exists an integer that is less than or equal to all others.
  - There does not exist an integer greater than or equal to all others, but for every integer, there exists one that is strictly greater.
  - Every even integer is equal to the sum of two prime integers.
  - The set of prime integers is unbounded.
- Explain in words the meaning of each of the following formulas and state whether they are verified:
  - $\forall (x, y) \in \mathbb{N}^2, (x, y) \in P^2 \Rightarrow (x + y) \in P$
  - $\forall (x, y) \in \mathbb{N}^2, \exists z \in \mathbb{N} : (Div(x, z) \wedge Div(z, y))$
- For each of the following predicates, provide an equivalent formula that uses only the constant symbols 0 and 1, the functions $+$ and $\times$, and the equality relation $=$:
  - $x \in P$
  - $Div(x, y)$
  - $x \in PR$ (You may use $Div(x, y)$)

Exercise 7.27

Is A a logical consequence of B?

- $B = (p \wedge q) \vee \left(\overline{p} \wedge r\right)$ and $A = (q \vee r)$
- $B = (p \Rightarrow q) \vee \left(p \Rightarrow \overline{r}\right)$ and $A = \overline{p}$

Exercise 7.28

We introduce the connector $\triangle$ whose truth table is given by:

p	q	$p \triangle q$
1	1	0
1	0	0
0	1	0
0	0	1

- Give a conjunctive normal form and a disjunctive normal form of the formula $p \triangle q$.
- Determine if the connector $\triangle$ is commutative ($p \triangle q \equiv q \triangle p$) and if it is associative ($[(p \triangle q) \triangle r] \equiv [p \triangle (q \triangle r)]$).

- Give formulas equivalent to $\overline{p}, (p \wedge q), (p \vee q), (p \Rightarrow q)$ using only the connector $\triangle$ and the variables p and q.
- Write $[p \wedge (q \wedge r)]$ in terms of p, q, r, and $\triangle$.

Exercise 7.29

Convert the following formulas into conjunctive and disjunctive normal forms:

- $((\overline{p} \vee q) \vee r) \Rightarrow r$;
- $p \Rightarrow ((\overline{q} \vee r) \Rightarrow s)$;
- $(p \Leftrightarrow q) \wedge (\overline{p \Rightarrow q} \vee \overline{q \Rightarrow p})$.

Exercise 7.30

Let the formula F be:

$$(p \vee q) \wedge (r \vee s),$$

and the system S be:

$$(x-7)(y-2) = 0 \wedge (x-5)(y-3) = 0,$$

- Express F in disjunctive normal form (DNF);
- Use the DNF form to solve the system S.

Exercise 7.31

Each of the following propositional formulas is used:

$$(\overline{p \vee q}) \vee (\overline{p \wedge q}),$$

$$(p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r)),$$

- Construct its truth table;
- Indicate whether it is satisfiable, a tautology, or a contradiction.

Exercise 7.32

Let the propositional formula F be:

$$(p \Rightarrow (q \Rightarrow r)) \Rightarrow (r \vee \overline{p}),$$

- Construct its truth table;
- Indicate whether it is satisfiable, valid (a tautology), or unsatisfiable (a contradiction);
- Does the formula F have a model? If so, find $mod(F)$;
- Provide the conjunctive normal form (CNF) and disjunctive normal form (DNF) of the formula F.

Exercise 7.33

A formula F is said to be contingent if it is satisfiable and not tautology.

- What can be said about $mod(F)$ when F is contingent?

- Let F and G be two propositional formulas. What do you think about the following statements:
  - If F is contingent, then $\overline{F}$ is also contingent;
  - If F and G are contingent, then $F \vee G$ and $F \wedge G$ are contingent;
  - If $F \vee G$ is unsatisfiable, then F and G are unsatisfiable;
  - If $F \vee G$ is a tautology, then F and G are tautologies;
  - If $F \vee G$ is a tautology, then either F is a tautology or G is a tautology.

Exercise 7.34

Let $P_0, P_1; P_2, ..., P_{2k+1}$ be propositions and let the propositional formula F be:

$$P_0 \wedge \overline{P_1} \wedge P_2 \wedge \overline{P_3} \wedge .... \wedge \overline{P_{2k+1}}$$

- Is F satisfiable?
- Do we have $F \models [(P_1 \Rightarrow P_2) \Leftrightarrow (P_4 \vee P_5)]$?
- Do we have $F \models [(\overline{P_0} \wedge P_1) \vee (\overline{P_2} \wedge P_3) \vee (\overline{P_4} \wedge P_5) \vee ... \vee (\overline{P_{2k}} \wedge P_{2k+1})]$?
- Do we have $F \models [(P_0 \vee P_1) \wedge (P_2 \vee P_3) \wedge ... \wedge (P_{2k} \vee P_{2k+1})]$?

Exercise 7.35

- Show that $(A \Rightarrow B) \Leftrightarrow [B \Leftrightarrow (A \vee B)]$
  - Deduces that if $\models (A \Rightarrow B)$, then $B \Leftrightarrow (A \vee B)$; in this case, we say that A is a redundant element in the propositional form $A \vee B$
- Let the formula be $F = (P \wedge \overline{Q} \wedge \overline{R}) \vee (\overline{P} \wedge Q) \vee (P \wedge R) \vee (Q \wedge R)$
  - Construct the truth table for F.
  - Deduces the redundant elements of F.
  - Provide a simplified form of F.

Exercise 7.36

We introduce the Sheffer stroke | (NAND) with the truth table given by:

p	q	p|q
1	1	0
1	0	1
0	1	1
0	0	1

- Provide a conjunctive normal form and a disjunctive normal form of the formula $p|q$.
- Is the Sheffer stroke commutative ($p|q \equiv q|p$)? and is it associative ($[(p|q)|r] \equiv [p|(q|r)]$)?
- Provide the truth tables for the formulas $(p|p)$, $((p|p)|p)$, and $[(p|p)|(p|p)]$
- Provide formulas equivalent to $\overline{p}$, $(p \wedge q)$, $(p \vee q)$, $(p \Rightarrow q)$ that use only the Sheffer stroke and the variables p and q.

✿ Let the function *Shef* be defined as follows:

Shef (p) = p

$$Shef(\overline{p}) = (Shef(p))|(Shef(p))$$

$$Shef(p \wedge q) = ((Shef(p))|(Shef(q))|((Shef(p))|(Shef(q))$$

$$Shef(p \vee q) = ((Shef(p))|(Shef(p))|((Shef(q))|(Shef(q))$$

$$Shef(p \Rightarrow q) = Shef(p)|(Shef(q))|(Shef(q))$$

Find the result of $Shef[p \vee (q \vee r)]$

Exercise 7.37

✿ Let p and q be two logical propositions. Show that

1. $\left[(p \Rightarrow q) \wedge \overline{q}\right] \models \overline{p}$
2. $[(p \Rightarrow q) \wedge q]\ \ p$

✿ Are the following arguments logically correct?

❀ If I am guilty, I must be punished; I must not be punished. Therefore, I am not guilty.

❀ If I am guilty, I must be punished; I must be punished. Therefore, I am guilty.

Exercise 7.38

In my painter's kit, I have paint tubes in the following colors: red, blue, yellow, orange, gray, and black. Let R be the proposition "I have a Red paint tube", B for Blue, J for Yellow, O for Orange, G for Gray, and N for Black.

Write the following propositional formulas using the propositions defined above:

✿ F_1: I do not have Yellow.

✿ F_2: If I have Red, then I have neither Black nor Gray.

✿ F_3: Of the three colors, blue, yellow, and red, I have at least two.

✿ F_4: Of the two colors Gray and Orange, I have exactly one.

Suppose that $[F_1 \wedge F_2 \wedge F_3 \wedge F_4]$ is satisfiable. Is it possible to deduce the contents of my kit? If so, provide the contents; if not, explain why.

Exercise 7.39

Three colleagues, Ahmed, Brahim, and Cherif, have lunch together every day. The following statements are true:

- If Ahmed orders a dessert, Brahim orders one as well.
- Every day, Brahim or Cherif, but not both, order a dessert.
- Ahmed or Cherif, or both, order a dessert every day.
- If Cherif orders a dessert, Ahmed does the same.

- Express the problem data as propositional formulas.
- What can we infer about who orders a dessert?
- Could we reach the same conclusion by removing one of the four statements?

Exercise 7.40

You are walking in a maze and suddenly find yourself in front of three possible paths: the left path is paved with gold, the path in front of you is paved with marble, and the right path is made of small stones. Each path is guarded by a guardian. You speak to the guardians, and here is what they say:

$F1$: Guardian of the gold path: "This path will lead you directly to the center. Moreover, if the stones lead you to the center, then the marble also leads to the center."
$F2$: Guardian of the marble path: "Neither the gold nor the stones will lead you to the center."
$F3$: Guardian of the stone path: "Follow the gold and you will reach the center, follow the marble and you will be lost."

- Formalize these 3 statements via the propositional variables defined below:
  G: the gold path leads to the center
  M: the marble path leads to the center
  S: the stone path leading to the center
- Given that all the guardians are liars, can you choose a path with certainty that it will lead you to the center of the maze?
- If so, which path would you choose?

Exercise 7.41

Brown, Jones, and Smith are accused of tax fraud. They swear the following:

BROWN: Jones is guilty and Smith is innocent. (I)
JONES: If Brown is guilty, then Smith is also guilty. (II)
SMITH: I am innocent, but at least one of the other two is guilty. (III)

Let B, J, and S represent the statements: B: Brown is innocent, J: Jones is innocent, and S: Smith is innocent.

- Express each suspect's testimony in logical symbolism.
- Calculate the truth values of the three obtained formulas.
- Are the testimonies of the three suspects compatible (simultaneously satisfiable)?

- Does the testimony of one suspect follow from the testimony of another? Which two testimonies are involved?
- Assuming that all are innocent, which one would have committed perjury?

Exercise 7.42

You are presented with three boxes, each carrying a label with an inscription. We denote by

$C1$: ”The gold is in box 1”

$C2$: ”The gold is in box 2”

$C3$: “The gold is in box 3”

- Using propositions $C1$, $C2$, and $C3$, formalize the following statements:
  - Label 1 on box 1: ”The gold is not here”
    Label 2 on box 2: ”The gold is not here”
    Label 3 on box 3: ”The gold is in box 2”
  - $F1$: ”Exactly one box contains the gold”
  - $F2$: “Exactly one label’s inscription is true; the others are false”
- If statements $F1$ and $F2$ are true, where is the gold?

Exercise 7.43

- - What does $p \models q$ mean? Recall the definition of $p \models q$.
  - Show that:
    - $(p \models (A \wedge B)) \Leftrightarrow [(p \models A) \wedge (p \models B)]$.
    - $[(A \wedge B) \models (p \Rightarrow q)] \Leftrightarrow [(A \wedge B \wedge p) \models q]$.
- Consider the following sentences:
  $F1$: For Ahmed to pass the logic exam, it is necessary and sufficient that, first, he attends the class, second, he stops talking to his neighbor, and finally, he listens to his teacher.
  $F2$: If Ahmed listens to the teacher, then he attends the class and stops talking to his neighbor.
  $F3$: It is necessary and sufficient for Ahmed to listen to the teacher to pass the logic exam.
  - Formalize these 3 sentences using the propositional variables defined below and only the connectors $-$, $\Rightarrow$, $\wedge$, and $\vee$:
    - R: Ahmed passes the logic exam
    - A: Ahmed attends the class
    - C: Ahmed stops talking to his neighbor
    - E: Ahmed listens to the teacher
  - shows that $[(F1 \wedge F2) \models F3]$.

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