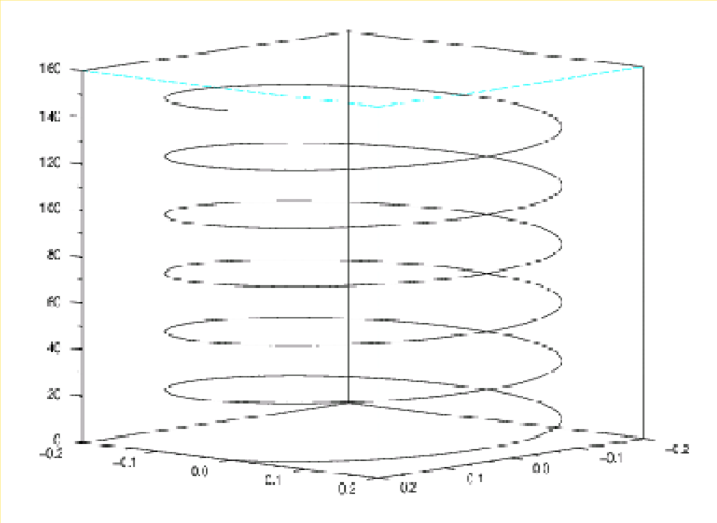

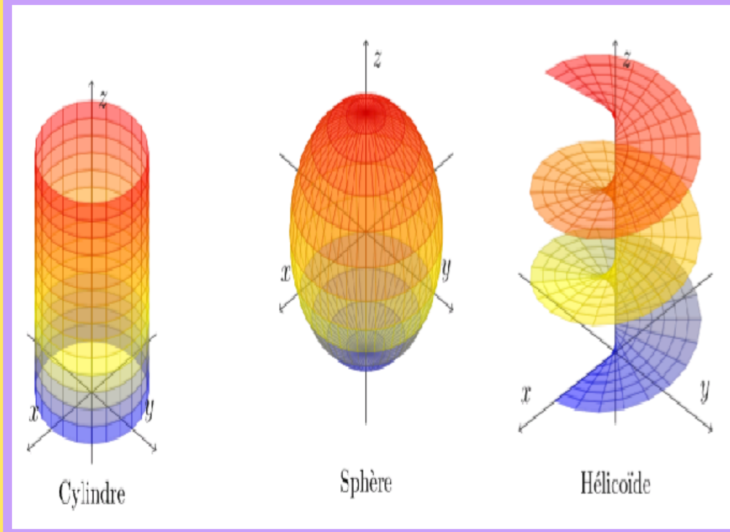

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Geometry

Courses and Exercises

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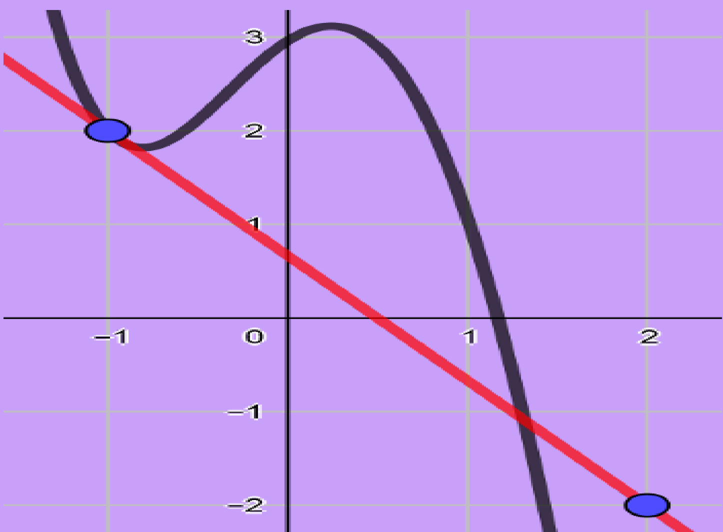

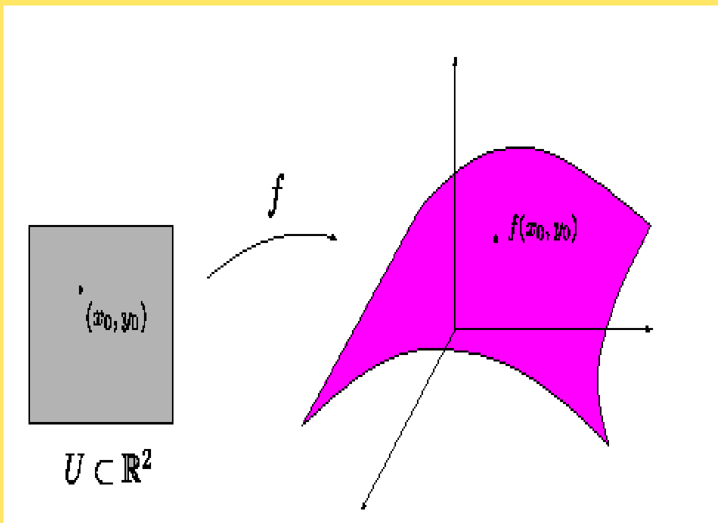

Contents

Introduction

Geometry is the branch of mathematics that studies shapes and figures in space, their relationships with each other, and their properties. Among the subbranches of geometry, analytical geometry uses algebraic and analytical calculations; it facilitates the study of the geometric properties of curves and surfaces and their geometric representation. There is also differential geometry which studies the local (in the neighborhood of a point) and essential properties of curves and surfaces, and there is also affine geometry which is a representation of the part of geometry that is learned in high school and which concerns the properties of lines, planes, distance, angle, etc. The essential difference from what we learned in high school is that geometry is fundamentally based on linear algebra.

This course in geometry is the result of twelve years of teaching in mathematics at Mohamed Khider University, Biskra. The first chapter is devoted to the study of parameterized curves. We begin by recalling some definitions of the characterization of a curve as well as the fundamental properties of the curve. We introduce the length of a curve, it is possible to parameterize a curve by its length. To explain what this parameterization by the curvilinear abscissa represents, the following equations are introduced through a moving frame, the Frenet frame, which is well adapted to the study of skew curves. We highlight the importance of the concepts, the curvature and torsion of a curve5. Second we are interested in the study of parameterized surfaces: such as regular surfaces and differential elements, and we introduce the following concepts (tangent plane and the normal line, to calculate the integral of a continuous function on a surface, the area of a surface, we use the first fundamental form of the surface which allows us to calculate the length of a curve drawn on the surface) In the third chapter, we introduce the curvilinear integral and its properties. In the last chapter, we are interested in the notions of affine geometry (affine spaces, barycenters, affine varieties, affine applications, affine frames, and affine transformations)

Chapter 1 Parametric Curves

Many mathematical models in physics and mechanics use the concept of a point mass and attempt to understand its movement over time along a path in space. We propose to study the behavior of curves from a geometric perspective, where we use a parametric description of curves. We look at the properties of curves that depend only on the path itself, not how it travels. These properties are called geometric properties.

1.1 Preliminaries

We use the standard notation for sets. therefore, we write $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ for the sets of natural numbers, integers and real numbers, respectively. This means that for $n \in \mathbb{N}$, we have the following notation:

$$\mathbb{R}^n = \{x = (x_1, x_2, \dots, x_n)/x_1 \in \mathbb{R}, x_2 \in \mathbb{R}, \dots, x_n \in \mathbb{R}\}$$

Additionally, the set $\mathbb{R}^n$ is considered with addition and multiplication by real numbers, for $x = (x_1, x_2, \dots, x_n$ $y = (y_1, y_2, \dots, y_n)$ and $k \in \mathbb{R}$, we define:

$$x + y = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n), \quad kx = (kx_1, kx_2, \dots, kx_n)$$

Some points will be specified:

$$\begin{aligned} e^1 &= (1, 0, 0 \dots, 0), \quad e^2 \\ &= (0, 1, 0 \dots, 0), \dots \quad , \quad e^n \\ &= (0, 0, 0 \dots.., 1) \end{aligned}$$

We look at the standard dot product:

$$\langle x, y \rangle = x_1 y_1 + x_2 y_2 \dots + x_n y_n = \sum_{i=1}^{n} x_i y_i$$

In addition, the norm and distance induced by this dot product are as follows:

$$\|x\| = \langle x, x \rangle^{\frac{1}{2}}, \quad dist(x, y) = \|x, y\|$$

Let us also remember the definition of the cross product in $\mathbb{R}^3$, for $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3)$

$$x \wedge y = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = (x_2\, y_3 - x_3\, y_2, x_3\, y_1 - x_1\, y_3, x_1\, y_2 - x_2\, y_1)$$

1.2 Definitions

Definition 1.1

A parameterized curve in $\mathbb{R}^n$ is a function

$$C: \quad I \longrightarrow \mathbb{R}^n$$
$$t \mapsto C(t) = (C_1(t), C_2(t), \ldots.., C_n(t))$$

where $I \subseteq \mathbb{R}$ is a nondegenerate interval and the components $C_i\,(i = 1 \ldots n)$ are continuous functions:

$$C_i : I \longrightarrow \mathbb{R}$$

The set $S = C(I) = \{C(t) : t \in I\}$ is called the geometric support of $C : I \longrightarrow \mathbb{R}^n$. We say that S is a geometric curve and that $C(t)$ is a parameterization of S. If $n = 2$: the curve S is planar. If $n = 3$: the curve S is a space curve.

☞ **Remark**

S is a geometric curve and C is a parameterization of S. The parameterized curve has more information than the geometric curve does.

1.2.1 Examples

Example 1.1

$$C : \left[0, \frac{\pi}{2}\right] \longrightarrow \mathbb{R}^2$$
$$t \longrightarrow C(t) = (\cos(t), \sin(t))$$

This is a plane curve. $C_1(t) = \cos(t)$, $C_2(t) = \sin(t)$, $C(0) = (1, 0)$, $C\left(\frac{\pi}{4}\right) = \left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$.

The support of C is an arc.

Example 1.2

The geometric support of the parameterized curve $C : \mathbb{R} \longrightarrow \mathbb{R}^3$ given by $C(t) = (k\cos(t), k\sin(t), t)$, where $k > 0$ is a helix.

Example 1.3

Line and segment in $\mathbb{R}^n$ Let p and q be two points in $\mathbb{R}^n$ such that $p \neq q$. There is a unique line that goes through p and q, which we denote as D_{pq}. A point $x \in \mathbb{R}^n$ is on this line if and only if there exists a $t \in \mathbb{R}$ such that $x - p = t(q - p)$.

Using t as a parameter, we can write the parameterized curve as follows:

$$C : \mathbb{R} \longrightarrow \mathbb{R}^n$$
$$t \longrightarrow C(t) = p + t(q - p)$$

The support of this curve is the line D_{pq}. This parameterization can be used to find a parameteri-

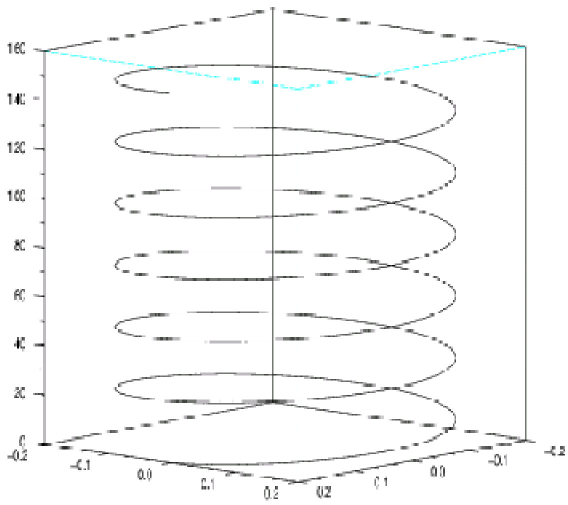

Figure 1.1. A helix

zation of the segment $[pq]$ as follows:

$$\gamma : [0,1] \longrightarrow \mathbb{R}^n$$
$$t \longrightarrow \gamma(t) = p + t(q-p)$$

We can also have another parameterization for the segment $[pq]$, for example:

$$\beta : \left[0, \frac{1}{5}\right] \longrightarrow \mathbb{R}^n$$
$$t \longrightarrow \beta(t) = p + 5t(q-p)$$

Example 1.4

Circle in $\mathbb{R}^2$ Consider the circle centered at $O(0,0)$ with radius ρ. We are looking for a parameterization $C : I \longrightarrow \mathbb{R}^2$ whose support should be the set

$$\{x = (x_1, x_2) \,|\, x_1^2 + x_2^2 = \rho^2\}$$

Let us use the angle formed by Ox and the horizontal axis as the parameter:

$$C : [0, 2\pi] \longrightarrow \mathbb{R}^2$$
$$t \longrightarrow C(t) = (\rho\cos t, \rho\sin t)$$

For a circle centered at $p = (p_1, p_2)$ with radius ρ, the parameterization will be be as follows: $C(t) = (p_1 + \rho\cos t, p_2 + \rho\sin t)$

From Example 3, we observe that two curves can have the same support but different parameterizations. This leads us to consider the following definition of a change in parameterization:

Definition 1.2

Let the curve $C : I \longrightarrow \mathbb{R}^n$. If $\varphi : \tilde{I} \longrightarrow I$ is a strictly monotonic continuous function (increasing or decreasing), then we obtain a new curve $\tilde{C}$:

$$\begin{aligned} \tilde{C} : \tilde{I} &\longrightarrow \mathbb{R}^n \\ s &\longrightarrow \tilde{C}(s) = C(\varphi(s)) \end{aligned}$$

equivalent to C (having the same support), and the transition from C to $\tilde{C}$ is called a change of parameterization.

Example 1.5

Let $C : [a, b] \longrightarrow \mathbb{R}^n$. To obtain an equivalent curve $\tilde{C} : [0, 1] \longrightarrow \mathbb{R}^n$, it is sufficient to take:

$$\begin{aligned} \varphi : [0, 1] &\longrightarrow [a, b] \\ s &\longrightarrow \varphi(s) = a + s(b - a) \end{aligned}$$

1.3 Characteristic Equation

From the parameterization of a curve, we can sometimes obtain the equation that characterizes the geometric relationship of points in space. To do this, it is necessary to eliminate the parameters between the parametric equations. Let us cite some examples.

1.3.1 Examples

Example 1.6

Ellipse

Let the curve:

$$\begin{aligned} C : [0, 2\pi] &\longrightarrow \mathbb{R}^2 \\ t &\longrightarrow C(t) = (a \cos t, b \sin t) \end{aligned}$$

By eliminating t between the two components, we obtain the following equation:

$$\frac{x_1^2}{a^2} + \frac{x_2^2}{b^2} = 1$$

which is the equation of the ellipse with center $(0, 0)$ and radii a and b.

Example 1.7

Hyperbola Let the two curves:

$$\begin{aligned} \gamma : \mathbb{R} &\longrightarrow \mathbb{R}^2 \\ t &\longrightarrow \gamma(t) = (a \cosh t, b \sinh t) \end{aligned}$$

$$\beta : \mathbb{R} \longrightarrow \mathbb{R}^2$$
$$t \longrightarrow \beta(t) = (a\cosh t, -b\sinh t)$$

By eliminating t between the two components in both curves, we obtain the following equation:

$$\frac{x_1^2}{a^2} - \frac{x_2^2}{b^2} = 1$$

which is the equation of the hyperbola.

Example 1.8

Curve on a Sphere

Starting from spherical coordinates and creating a parameterization of a curve with its support on a sphere, it is sufficient to consider the following functions:

$$t \mapsto \Phi(t), t \mapsto \theta(t), t \in I$$

and define

$$C(t) = (\cos\theta(t)\cos\phi(t), \cos\theta(t)\sin\phi(t), \sin\theta(t))$$

1.4 Types of Curves

1.4.1 Simple Curve

A curve C is said to be simple if it does not intersect itself, in other words, if

$$\forall (t_1, t_2) \in I^2, t_1 \neq t_2 \Longrightarrow C(t_1) \neq C(t_2)$$

Example 1.9

The curve $C(t) = (\cos(t), \sin(t))$ with $C : [0, 2\pi] \longrightarrow \mathbb{R}^2$ is a simple curve.

1.4.2 Closed Curve

Let $a, b \in I$ be with $a < b$. Let $C : [a, b] \to \mathbb{R}^n$ be a parameterized curve such that $C(a) = C(b)$. In this case, we say that C is a closed curve.

☞ **Remark**

A simple curve is a curve that does not self-intersect; in other words, the path does not cross itself (except possibly if the curve is closed, in which case the only intersection point is the starting point, which is identical to the endpoint).

1.5 Regular Curves and Tangent Space

1.5.1 Rules for Calculating Derivatives (Recap)

- Let $f : I \longrightarrow \mathbb{R}^n$, $t \mapsto (f_1(t), f_2(t), \ldots, f_n(t))$, where each f_i is a real function. We say that f is differentiable if each f_i is differentiable. The derivative $\dot{f}$ is by definition the function $\dot{f} = (\dot{f}_1, \dot{f}_2, \ldots, \dot{f}_n)$.
- Let $f, g : I \longrightarrow \mathbb{R}^n$, $\lambda \in \mathbb{R}$, $A = (a_{ij})$ be an $m \times n$ matrix, and $\phi : I \longrightarrow J$ be a real function, then:
  - $\dfrac{d}{dt}(f(t) + g(t)) = \dot{f}(t) + \dot{g}(t)$
  - $\dfrac{d}{dt}(\lambda f(t)) = \lambda \dot{f}(t)$
  - $\dfrac{d}{ds}(f(\phi(s))) = \dot{\phi}(s)\dot{f}(\phi(s))$
  - $\dfrac{d}{dt}(Af(t)) = A\dot{f}(t)$
  - $\dfrac{d}{dt}\langle f(t), g(t)\rangle = \langle \dot{f}(t), g(t)\rangle + \langle f(t), \dot{g}(t)\rangle$
  - $\dfrac{d}{dt}(f(t) \wedge g(t)) = (\dot{f}(t) \wedge g(t)) + (f(t) \wedge \dot{g}(t)$

1.5.2 Regular Curves

Definition 1.3

Let $C : I \longrightarrow \mathbb{R}^n$ be a differentiable curve. The point $C(t_0)$ is said to be regular if $\dot{C}(t_0) \neq 0$.

The parameterized curve C is regular if all the points of $C(I)$ are regular.

Definition 1.4

A point $C(t_0)$ is said to be biregular if $\dot{C}(t_0)$ and $\ddot{C}(t_0)$ are linearly independent in $\mathbb{R}^n$.

The parameterized curve C is biregular if all the points of $C(I)$ are biregular.

Definition 1.5

A point $C(t_0)$ is said to be singular if $\dot{C}(t_0) = 0$.

Remark

If $C : I \to \mathbb{R}^n$ is regular and $\tilde{C} = C \circ \varphi$ is a reparameterization of C, then $\tilde{C}$ is regular. Indeed, we have:

$$\dot{\tilde{C}}(t) = \dot{C}(\varphi(t))\,\dot{\varphi}(t) \neq 0_{\mathbb{R}^n}$$

Example 1.10

(Cycloid) Let $C : \mathbb{R} \to \mathbb{R}^2$ be the curve defined by $C(t) = (t - \sin t, 1 - \cos t)$. To study the

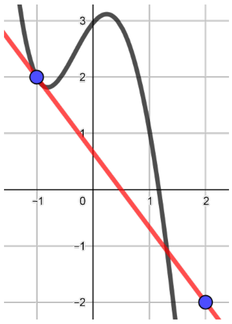

Figure 1.2. Tangent to a parameterized curve

regularity, we calculate $\dot{C}$, giving $\dot{C} = (1 - \cos t, \sin t)$, so:

$$\dot{C} = 0 \Longleftrightarrow \|\dot{C}\|^2 = 0 \Longleftrightarrow 2(1-\cos t) = 0 \Longleftrightarrow \cos t = 1 \Longleftrightarrow t = 2k\pi,\ k \in \mathbb{Z}$$

Thus, the singular points are $p_k = (2k\pi, 0)$.

1.5.3 Tangent Vectors

The tangent at a point $C(t_0)$ on a parameterized curve $C : I \longrightarrow \mathbb{R}^n$ is the limit of the lines that pass through $C(t)$ and $C(t_0)$ as t approaches t_0.

Definition 1.6

Let $C : I \to \mathbb{R}^n$ be a parameterized curve of class C^1 and regular. Then $\dot{C}(t_0)$ is the tangent vector to the curve C at the point $C(t_0)$.

1.5.4 Intersection and angle between two curves

Consider the two curves

$$\begin{aligned} \gamma : \quad & I \longrightarrow \mathbb{R}^n \\ t \quad & \mapsto \gamma(t) = (\gamma_1(t), \gamma_2(t), \ldots, \gamma_n(t)) \end{aligned}$$

$$\begin{aligned} \beta : \quad & J \longrightarrow \mathbb{R}^n \\ s \quad & \mapsto \beta(s) = (\beta_1(s), \beta_2(s), \ldots, \beta_n(s)) \end{aligned}$$

If there exists $t_0 \in I$ and $s_0 \in J$ such that $\gamma(t_0) = \beta(s_0)$, then the curves intersect, and the point $p = \gamma(t_0) = \beta(s_0)$ is the intersection point. The angle formed by the two tangent vectors at the intersection point, $\dot{\gamma}(t_0)$ and $\dot{\beta}(s_0)$, is defined as the angle between the two curves.

If θ is the angle formed, then:

$$\cos\theta = \frac{\langle \dot{\gamma}(t_0), \dot{\beta}(s_0) \rangle}{\|\dot{\gamma}(t_0)\| \|\dot{\beta}(s_0)\|}$$

Example 1.11

Consider the two curves

$$\begin{aligned} \gamma: \quad & \mathbb{R} \longrightarrow \mathbb{R}^2 \\ t \quad & \mapsto \gamma(t) = \left(t^2 - a^2, 2at\right) \end{aligned}$$

$$\begin{aligned} \beta: \quad & \mathbb{R} \longrightarrow \mathbb{R}^2 \\ s \quad & \mapsto \beta(s) = \left(b^2 - s^2, 2bs\right) \end{aligned}$$

Let us find the intersection points and calculate the angle at these points:

$$\gamma(t) = \beta(s) \Longleftrightarrow t^2 - a^2 = b^2 - s^2 \text{ and } 2at = 2bs$$

Substituting $s = \dfrac{at}{b}$, we obtain $t^2 = b^2$, which gives two solutions $t_1 = b$ and $t_2 = -b$. For $t_1 = b$, we have $s_1 = a$, and for $t_2 = -b$, we have $s_2 = -a$. Therefore, the two intersection points are $p_1 = \gamma(t_1) = (b^2 - a^2, 2ab)$ and $p_2 = \gamma(t_2) = (b^2 - a^2, -2ab)$.

To obtain the angle, we calculate the derivatives and apply the cosine formula:

$$\cos\theta_1 = \frac{\langle \dot{\gamma}(t_1), \dot{\beta}(s_1)\rangle}{\|\dot{\gamma}(t_1)\|\|\dot{\beta}(s_1)\|} = \frac{\langle (2b, 2a), (-2a, 2b)\rangle}{\sqrt{4b^2 + 4a^2}\sqrt{4b^2 + 4a^2}} = \frac{-4ab + 4ab}{4(a^2 + b^2)} = 0$$

Hence, $\theta_1 = \dfrac{\pi}{2}$, and similarly, $\theta_2 = \dfrac{\pi}{2}$.

1.6 Length of a Curve

To measure the length of a curve, a natural solution involves approximating this length by the length of a polygonal line with vertices on the curve. As the distance between two consecutive vertices is reduced, the length of the polygonal line becomes closer to the length of the curve.

Given points $a_0, a_1, \ldots, a_m$ in $\mathbb{R}^n$, the polygon $P = a_0 a_1 \ldots a_m$ formed by the segments $[a_k a_{k+1}]$ for $k = 0, \ldots, m-1$ has a length $L(P) = \sum_{k=0}^{m-1} \|a_{k+1} - a_k\|$.

Consider a regular curve $C : [a, b] \longrightarrow \mathbb{R}^n$ and $\sigma = \{t_0, t_1, \ldots, t_m\}$ a partition of $[a, b]$ with $t_0 = a < t_1 < \cdots < t_m = b$. We construct a polygon $P_\sigma = C(t_0)C(t_1)\ldots C(t_m)$, then $L(P_\sigma) = \sum_{k=0}^{m-1} \|C(t_{k+1}) - C(t_k)\|$.

If we denote $d_k = t_{k+1} - t_k$, then $L(P_\sigma) = \sum_{k=0}^{m-1} d_k \left\|\frac{1}{d_k}(C(t_{k+1}) - C(t_k))\right\|$.

As $\max_k d_k$ tends to 0, the polygon increasingly approximates the curve.

Definition 1.7

Let $C : [a, b] \longrightarrow \mathbb{R}^n$. We define the length of the curve on the interval $[a, b]$ as the quantity given by:

$$L_{[a,b]}(C) = \int_a^b \|\dot{C}(t)\|dt.$$

Example 1.12

The length of the helix parameterized by $C(t) = (k\cos(t), k\sin(t), t)$, where $k > 0$ and $C : [0, 2\pi] \longrightarrow \mathbb{R}^3$, is given by:

$$L_{[0,2\pi]}(C) = \int_0^{2\pi} \sqrt{k^2+1}dt = 2\pi\sqrt{k^2+1}.$$

This is because $\dot{C}(t) = (-k\sin t, k\cos t, 1) \Rightarrow \|\dot{C}(t)\| = \sqrt{k^2+1}$.

Proposition 1.1

The length of a curve does not depend on the chosen parameterization.

Proof. Let C be a curve, and let $C_1 : I \to \mathbb{R}^n$, $C_2 : J \to \mathbb{R}^n$ be two parameterizations of C. There exists a reparameterization $\varphi : J \to I$ such that $C_2 = C_1 \circ \varphi$, so:

$$L_J(C_2) = \int_J \|\dot{C}_2(t)\|dt = \int_J \|\dot{C}_1(\varphi(t))\|\dot{\varphi}(t)\,dt = \int_I \|\dot{C}_1(s)\|ds = L_I(C_1).$$

■

1.7 Curvilinear Abscissa

In differential geometry, the curvilinear abscissa is a type of algebraic variant of the arc length. We start with an origin from which we calculate lengths, assigning a sign to clearly determine the position on the curve: a certain distance before or after the initial point. The curvilinear abscissa is thus analogous to the abscissa on an oriented line.

Definition 1.8

Let $C : I \longrightarrow \mathbb{R}^n$ be a curve of class C^1 parameterized by $t_0 \in I$. The curvilinear abscissa from the parameter point t_0 is the function $S : I \longrightarrow \mathbb{R}^n$ given by:

$$s(t) = \int_{t_0}^{t} \|\dot{C}(v)\|dv \quad , \forall t \in I$$

Definition 1.9

A parameterization $C : I \longrightarrow \mathbb{R}^n$ of a geometric curve is said to be normal (or by a curvilinear abscissa) if for any $[t_1, t_2] \subset I$, the length of the geometric curve between the points $C(t_1)$ and

$C(t_2)$ is exactly $(t_2 - t_1)$:

$$L_{[t_1,t_2]}(C) = t_2 - t_1$$

Remark
The curvilinear abscissa $s(t)$ of the geometric curve $C(I)$ is the length of the curve between the points $C(t_0)$ and $C(t)$.

Theorem 1.1

Let $C : [a, b] \longrightarrow \mathbb{R}^n$ be a regular curve of class C^1 with length L. Then there exists a reparameterization $\varphi : [0, L] \longrightarrow [a, b]$ such that the curve $\gamma : [0, L] \longrightarrow \mathbb{R}^n$, defined by $\gamma(s) = C(\varphi(s))$, satisfies $\|\dot{\gamma}(t)\| = 1$.

Proof. Let $\psi : [a, b] \longrightarrow [0, L]$ be defined as follows:

$$\psi(t) = L_{[a,t]}[C] = \int_a^t \|\dot{C}(\tau)\| d\tau.$$

Since $\|\dot{C}(\tau)\|$ implies that ψ is monotonically increasing, it admits an inverse $\psi^{-1} : [0, L] \longrightarrow [a, b]$. Setting $\varphi = \psi^{-1}$, from the definition of ψ, we have $\dot{\psi}(t) = \|\dot{C}(t)\|$, hence $\dot{\varphi}(s) = \dfrac{1}{\|\dot{C}(\varphi(s))\|} > 0$.

Therefore, if we take $\gamma(s) = C(\varphi(s))$, then $\dot{\gamma}(s) = \dot{\varphi}(s)\dot{C}(\varphi(s))$, and thus $\|\dot{\gamma}(s)\| = 1$. ■

Example 1.13

Let $C :]0, 1[\longrightarrow \mathbb{R}^2$ be the curve defined by $t \longrightarrow C(t) = \left(t, \sqrt{1 - t^2}\right)$.
The curvilinear abscissa is as follows:

$$\begin{aligned}
\psi(t) &= \int_{t_0}^t \|\dot{C}(\tau)\| d\tau \\
&= \int_{t_0}^t \|(1, \frac{-\tau}{\sqrt{1-\tau^2}})\| d\tau \\
&= \int_{t_0}^t \sqrt{1 + \left(\frac{-\tau}{\sqrt{1-\tau^2}}\right)^2} d\tau
\end{aligned}$$
$$\psi(t) = \arcsin(t)$$

The function $\arcsin :]0, 1[\longrightarrow \left]0, \frac{\pi}{2}\right[$ is a bijection. The reparameterization $\tilde{C} = C \circ \psi^{-1} : \left]0, \frac{\pi}{2}\right[\rightarrow \mathbb{R}^2$ is given by:

$$\tilde{C}(s) = C(\sin s) = (\sin s, \cos(s))$$

Definition 1.10

A curve $C : [0, L] \longrightarrow \mathbb{R}^n$ satisfies $\|\dot{C}\| = 1$ and is said parameterized by the arc length or curvilinear abscissa.

Definition 1.11

Unit tangent Vector Let $C : I \longrightarrow \mathbb{R}^n$ be a parameterized curve. The unit tangent vector is defined as follows:

$$T_C(t) = \frac{1}{\|\dot{C}\|}, \quad t \in I$$

Definition 1.12

Let $C : I \longrightarrow \mathbb{R}^n$ be a regular curve. The curvature of C at point $C(t)$ is defined by $k(t) = \frac{\|\dot{T}_C(t)\|}{\|\dot{C}\|}$, and $R = \frac{1}{k(t)}$ is the radius of curvature.

1.8 Frenet Frame

Let $C : I \longrightarrow \mathbb{R}^3$ be a regular curve satisfying $k(t) \neq 0$. We define the following vectors:

$$N_C(t) = \frac{\dot{T}_C(t)}{\|\dot{T}_C(t)\|}$$

$$B_C(t) = T_C(t) \wedge N_C(t)$$

The set $\{T_C(t), N_C(t), B_C(t)\}$ forms an orthonormal frame in $\mathbb{R}^3$, called the Frenet frame.

1.8.1 Frenet Formulas

The Frenet frame associated with a regular curve C of nonzero curvature satisfies the following relations:

- $\frac{\dot{T}_C(t)}{\|\dot{C}(t)\|} = k(t)N_C(t)$
- $\frac{\dot{N}_C(t)}{\|\dot{C}(t)\|} = -k(t)T_C(t) + \tau(t)B_C(t)$
- $\frac{\dot{B}_C(t)}{\|\dot{C}(t)\|} = -\tau(t)N_C(t)$

where $\tau : I \longrightarrow \mathbb{R}$, associated with the curve and defined by $\tau(t) = \frac{\langle \dot{N}_C(t), B_C(t)\rangle}{\|\dot{C}(t)\|}$, is called the curve torsion. To compute the Frenet frame, curvature, and torsion, it is often useful to proceed in a different manner than in theory.

1.9 Frenet Frame

Let $C : I \longrightarrow \mathbb{R}^3$ be a regular curve satisfying $k(t) \neq 0$. We define the vectors as follows:

$$\begin{aligned}
B_C(t) &= \frac{\dot{C}(t) \wedge \ddot{C}(t)}{\left\|\dot{C}(t) \wedge \ddot{C}(t)\right\|} \\
N_C(t) &= B_C(t) \wedge T_C(t) \\
k(t) &= \frac{\left\|\dot{C}(t) \wedge \ddot{C}(t)\right\|}{\left\|\dot{C}(t)\right\|^2} \\
\tau(t) &= \frac{\left\langle \dot{C}(t) \wedge \ddot{C}(t), \dddot{C}(t)\right\rangle}{\left\|\dot{C}(t) \wedge \ddot{C}(t)\right\|^2}
\end{aligned}$$

If the curve C is parameterized by curvilinear abscissa and $k(t) \neq 0$, then the formulas simplify to:

$$\begin{aligned}
T_C(t) &= \dot{C}(t), \\
N_C(t) &= \frac{\ddot{C}(t)}{\left\|\ddot{C}(t)\right\|} \\
k(t) &= \left\|\ddot{C}(t)\right\|, \\
B_C(t) &= \frac{\dot{C}(t) \wedge \ddot{C}(t)}{\left\|\ddot{C}(t)\right\|}, \\
\tau(t) &= \frac{\left\langle \dot{C}(t) \wedge \ddot{C}(t), \dddot{C}(t)\right\rangle}{\left\|\ddot{C}(t)\right\|^2}
\end{aligned}$$

1.9.1 Curve in a Plane

Let $C : I \longrightarrow \mathbb{R}^3$ be a regular curve satisfying $k(t) \neq 0$, for all $t \in I$.

Theorem 1.2

The curve is located in a plane if and only if $\tau(t) = 0$.

Proof. ✿ $\tau(t) = 0 \Longrightarrow$ The curve is located in a plane.

We have $\dfrac{\dot{B}_C(t)}{\|\dot{C}(t)\|} = -\tau(t)N_C(t)$, so $\dot{B}_C(t) = 0$, hence $B_C(t) = \text{constant}$. Let $B_C(t) = (b_1, b_2, b_3)$.

Let (π) be the plane that contains a point $C(t_0)$ of the curve, and whose normal vector is $B_C(t) = (b_1, b_2, b_3)$. We need to show that $C(t) \in (\pi)$ for all $t \in I$.

We must show that: $\langle C(t)-C(t_0), B_C(t)\rangle = 0$, for all $t \in I$.
Since $B_C(t) = \text{constant}$ and $T_C(t) \perp B_C(t)$, we have $\dot{B}_C(t) = 0$ and $\langle T_C(t), B_C(t)\rangle = 0$, hence:

$$\begin{aligned}
\frac{d}{dt}\langle C(t)-C(t_0), B_C(t)\rangle &= \langle \dot{C}(t), B_C(t)\rangle + \underbrace{\langle C(t)-C(t_0), \dot{B}_C(t)\rangle}_{=0} \\
&= \langle \|\dot{C}(t)\| T_C(t), B_C(t)\rangle \\
&= \|\dot{C}(t)\| \underbrace{\langle T_C(t), B_C(t)\rangle}_{=0} \\
&= 0
\end{aligned}$$

Thus, $\langle C(t)-C(t_0), B_C(t)\rangle = \text{constant}$, and for t_0, we have $\langle C(t_0)-C(t_0), B_C(t)\rangle = 0$, so $\langle C(t)-C(t_0), B_C(t)\rangle = 0$ for all $t \in I$. Therefore, C is contained in the plane (π) that contains $C(t_0)$ and is orthogonal to $B_C(t)$.

✿ The curve is located in a plane $\Longrightarrow \tau(t) = 0$.
Let (π) be the plane with the equation $ax+by+dz+e=0$. If $C(t) = (C_1(t), C_2(t), C_3(t))$ is contained in (π), then $aC_1(t)+bC_2(t)+dC_3(t)+e=0$. If we differentiate this equation twice, we obtain:

$$\begin{aligned}
a\dot{C}_1(t)+b\dot{C}_2(t)+d\dot{C}_3(t) &= 0 \\
a\ddot{C}_1(t)+b\ddot{C}_2(t)+d\ddot{C}_3(t) &= 0
\end{aligned}$$

Thus, $(a,b,d) \perp \dot{C}(t)$ and $(a,b,d) \perp \ddot{C}(t)$, so the vector $B_C(t)$, which is also orthogonal to $\dot{C}(t)$ and $\ddot{C}(t)$, is parallel to (a,b,d) for all $t \in I$. Since $B_C(t)$ is unitary, it is constant, hence $\dot{B}_C(t) = 0$ and ultimately $\tau(t) = \dfrac{\langle \dot{B}_C(t), N_C(t)\rangle}{\|\dot{C}(t)\|} = 0$.

■

Curvature Circle Let $C : I \to \mathbb{R}^3$ be a regular curve. The curvature $k(t)$ and the radius of curvature $R = \dfrac{1}{k(t)}$ have been defined. Now, we define the center of the curvature circle as follows: the center ω_0 of the curvature circle is located on the line directed by the normal vector $N_C(t)$ and is calculated by the formula: $\omega_0 = C(t_0) + RN_C(t_0)$.

Chapter 2 Parameterized surfaces

Classical surfaces in Euclidean geometry (spheres, cones, cylinders, etc.) naturally appear as graphs of functions of two variables. Here, we address the representation of these geometric objects in parametric form, as sets described by a family of curves in space.

Definition 2.1

Let F be the function defined by

$$\begin{aligned} F : D \subset \mathbb{R}^2 &\longrightarrow \mathbb{R}^n \\ (u,v) &\longmapsto (f_1(u,v), f_2(u,v), \ldots, f_n(u,v)), \end{aligned}$$

where D is a domain in $\mathbb{R}^2$ (an open connected set in $\mathbb{R}^2$), and the functions f_i are continuous for all $i = 1, \ldots, n$.
The set $S = f(D)$ is called a parametric surface.
We say that the surface S is differentiable if the functions f_i are differentiable.

Example 2.1

The cylinder, sphere, and helicoid in $\mathbb{R}^3$ are parameterized by the functions

$$\begin{aligned} \phi(u_1,u_2) &= (r\cos u_1, r\sin u_1, u_2), \quad u_1 \in [0, 2\pi[, \quad u_2 \in I \subset \mathbb{R} \\ \varphi(u_1,u_2) &= (r\cos u_1 \sin u_2, r\sin u_1 \sin u_2, r\cos u_2), \quad u_1 \in [0,2\pi[, \quad u_2 \in [0,\pi[\\ \psi(u_1,u_2) &= (u_1\cos u_2, u_1 \sin u_2, u_2), \quad u_2 \in [0,2\pi], \quad u_1 \in \mathbb{R} \end{aligned}$$

2.1 Revolution Surface

Consider a curve $C : I \longrightarrow \mathbb{R}^3$ given by $C(t) = (0, C_2(t), C_3(t))$. By rotating the plane $\{(x,y,z) \mid x = 0\}$ around the z-axis, the curve sweeps out a surface. Each point on the curve C generates a circle around the point $(0,0,C_3(t))$ with radius $C_2(t)$ in a plane parallel to the plane $\{(x,y,z) \mid z = 0\}$. The

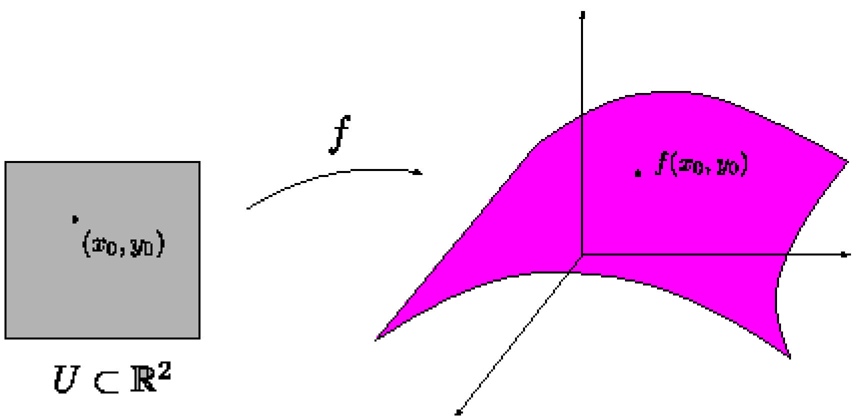

Figure 2.1. Parameterized surface

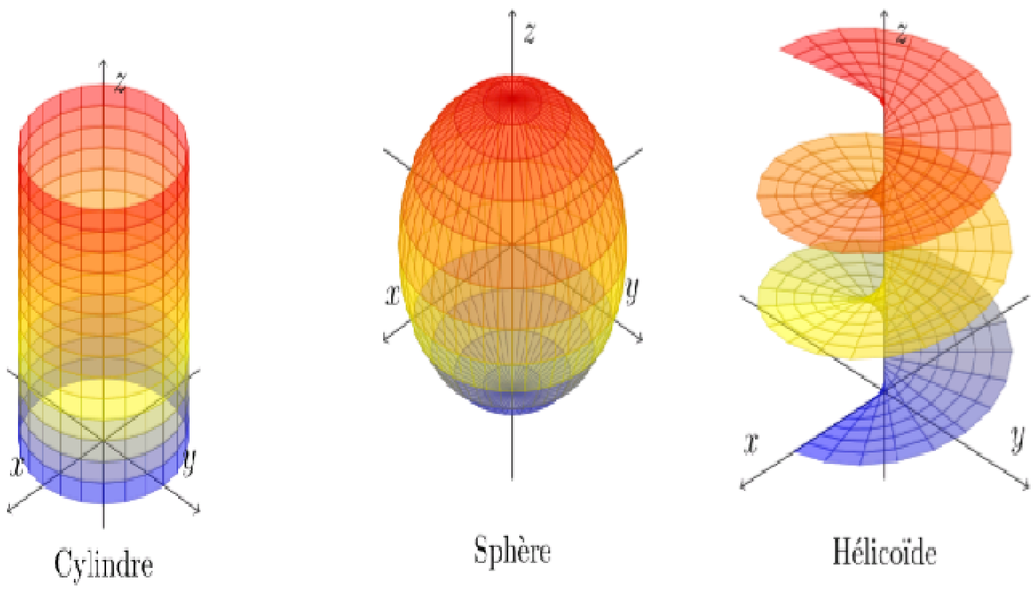

Figure 2.2. Examples of surfaces

union of these circles forms a surface S parameterized by:

$$\begin{array}{rcl} f : D = [0, 2\pi] \times I & \longrightarrow & \mathbb{R}^3 \\ (u, v) & \longmapsto & (C_2(v)\cos u, C_2(v)\sin u, C_3(v)) \end{array}$$

2.2 Curves on a Surface

Let S be the surface parameterized by $f : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ and let $C : I \longrightarrow D \subset \mathbb{R}^2$ be a curve. Curve γ defined by $\gamma : I \longrightarrow \mathbb{R}^3$ $\gamma(t) = f(C(t))$ is a curve on the surface S.

2.3 Regular Surfaces and Tangent Vectors

Definition 2.2

A surface S parametrized by $f : D \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ is said to be regular at the point $(u_0, v_0) \in D$ if the tangent vectors to the surface at the point $f(u_0, v_0)$, denoted by

$$f_u(u_0, v_0) = \frac{\partial f(u_0, v_0)}{\partial u}, \quad f_v(u_0, v_0) = \frac{\partial f(u_0, v_0)}{\partial v},$$

are linearly independent (and therefore nonzero); that is, if their cross product is nonzero:

$$f_u(u_0, v_0) \wedge f_v(u_0, v_0) \neq 0.$$

A point $f(u_0, v_0)$ is said to be singular if the two tangent vectors at this point are linearly dependent, i.e., if

$$f_u(u_0, v_0) \wedge f_v(u_0, v_0) = 0.$$

Example 2.2

The surface parameterized by $f(u,v)=(u^2,v^2,uv)\in\mathbb{R}^3$, with $u,v\in\mathbb{R}$, is singular at $(0,0)$, because

$$\left\{\begin{array}{l}\partial_u f(u,v)=(2u,0,v)\\ \text{and}\\ \partial_v f(u,v)=(0,2v,u)\end{array}\right. \Longrightarrow (\partial_u f\wedge\partial_v f)(u,v)=(-2v^2,-2u^2,4uv).$$

Thus, the vector $(\partial_u f\wedge\partial_v f)(u,v)$ vanishes at $(0,0)$. The surface S is regular in $\mathbb{R}^2-\{(0,0)\}$.

Example 2.3

The graph of a function $h:\mathbb{R}^2\to\mathbb{R}$ gives a surface S parameterized by

$$f(u,v)=(u,v,h(u,v))$$

with $(u,v)\in D_h\subset\mathbb{R}^2$. If h is of class C^1, then the surface S is regular everywhere:

$$\left\{\begin{array}{l}\partial_u f(u,v)=(1,0,\partial_u h(u,v))\\ \text{and}\\ \partial_v f(u,v)=(0,1,\partial_v h(u,v))\end{array}\right. \Rightarrow (\partial_u f\wedge\partial_v f)(u,v)=(-\partial_u h(u,v),-\partial_v h(u,v),1)\neq\vec{0}.$$

Definition 2.3

Let S be a surface parameterized by $f:u\times v\longrightarrow\mathbb{R}^3$, and let $m_0=f(u_0,v_0)$ be a regular point of S. We define:

- **Tangent plane** to S at the point m_0 as the plane generated by the vectors $f_u(u_0,v_0)$ and $f_v(u_0,v_0)$ and passing through m_0:

$$\begin{aligned} T_{m_0}S &= m_0+\operatorname{span}\left(f_u(u_0,v_0),f_v(u_0,v_0)\right)\\ T_{m_0}S &= \left\{f(u_0,v_0)+\lambda f_u(u_0,v_0)+\mu f_v(u_0,v_0)\mid\lambda,\mu\in\mathbb{R}\right\}.\end{aligned}$$

- **Normal (unit) vector** at m_0 as the vector

$$N_S(u_0,v_0)=\frac{f_u(u_0,v_0)\wedge f_v(u_0,v_0)}{\|f_u(u_0,v_0)\wedge f_v(u_0,v_0)\|}.$$

The tangent plane to S at (u_0,v_0) contains the tangent line to all regular curves on S passing through $f(u_0,v_0)$. Indeed, if $r(t)=f(u(t),v(t))$ is such a curve, and $(u_0,v_0)=(u(t_0),v(t_0))$, then

$$r'(t_0)=\frac{\partial f(u_0,v_0)}{\partial u}u'(t_0)+\frac{\partial f(u_0,v_0)}{\partial v}v'(t_0)\in\operatorname{span}\left(\partial_u f(u_0,v_0),\partial_v f(u_0,v_0)\right).$$

Remark
By definition, the three vectors $(\partial_u f(u_0, v_0), \partial_v f(u_0, v_0), N_S(u_0, v_0))$ form a direct basis for the space above the point $f(u_0, v_0)$ on the surface (i.e., a movable coordinate system), but this basis is neither orthogonal nor normal.

Definition 2.4

The tangent space to a surface parameterized by $f : U \to \mathbb{R}^3$ at the point $m_0 = f(u_0, v_0)$ is the affine space denoted $T_{m_0}S$ (with $S = f(U \times V)$) passing through m_0 and generated by the vectors

$$f_u(u_0, v_0) \text{ and } f_v(u_0, v_0).$$

In practice, the tangent space $T_{m_0}S$ also refers to the vector space that defines the affine space mentioned above, namely the vector space generated by the vectors $\dfrac{\partial f(u_0, v_0)}{\partial u}$ and $\dfrac{\partial f(u_0, v_0)}{\partial v}$.

2.4 Metric Tensors

Definition 2.5

Let S be the surface defined by $f : D \subset \mathbb{R}^2 \to \mathbb{R}^3$. The matrix

$$G(u, v) = \begin{pmatrix} \langle f_u, f_u \rangle & \langle f_u, f_v \rangle \\ \langle f_v, f_u \rangle & \langle f_v, f_v \rangle \end{pmatrix}$$

is called the metric tensor matrix (matrix of the first fundamental form).

Proposition 2.1

Let S be the surface defined by $f : D \subset \mathbb{R}^2 \to \mathbb{R}^3$, and let $C : I \to D$ and $d : J \to D$ be two curves. We denote the two curves on the surface S by $\gamma = f \circ C$ and $\delta = f \circ d$.

- *The length of the curve γ on the surface S is given by:*

$$L(\gamma) = \int_I \sqrt{\langle G(C(t))\dot{C}(t), \dot{C}(t) \rangle}\, dt$$

- *The angle α between the two curves γ and δ intersecting at $f(u)$ where $u = C(t_0) = d(s_0)$ is computed as follows:*

$$\cos\alpha = \frac{\langle G(u)\dot{d}(s_0), \dot{C}(t_0) \rangle}{\sqrt{\langle G(u)\dot{C}(t_0), \dot{C}(t_0) \rangle}\sqrt{\langle G(u)\dot{d}(s_0), \dot{d}(s_0) \rangle}}$$

Definition 2.6

The area of the surface $S = f(I \times J)$ parameterized by $f : I \times J \to \mathbb{R}^3$ is as follows:

$$\text{Area}(S) = \int_I \int_J \sqrt{\det G(u, v)}\, du\, dv$$

Proposition 2.2

Let S be a regular surface defined by $f : I \times J \to \mathbb{R}^3$. Then:

$$\|(f_u \wedge f_v)(u,v)\| = \sqrt{\det G(u,v)}$$

Proof. We have:

$$\begin{aligned}
\|(f_u \wedge f_v)(u,v)\|^2 &= \langle f_u \wedge f_v, f_u \wedge f_v \rangle \\
&= \langle f_u, f_u \rangle \langle f_v, f_v \rangle - \langle f_u, f_v \rangle \langle f_v, f_u \rangle \\
&= \det G(u,v)
\end{aligned}$$

■

Example 2.4

Let $g : [0, 2\pi[\times [0, 2\pi[\to \mathbb{R}^3$ be parameterized by:

$$g(u,v) = \begin{pmatrix} (k + r\cos u)\cos v \\ (k + r\cos u)\sin v \\ r\sin u \end{pmatrix}$$

The area of this surface is given by:

$$\begin{aligned}
\text{Area}(S) &= \int_0^{2\pi} \int_0^{2\pi} \|g_u(u,v) \wedge g_v(u,v)\| \, du\, dv \\
&= \int_0^{2\pi} \int_0^{2\pi} |r(k + r\cos u)| \, du\, dv \\
&= \int_0^{2\pi} (2\pi r k) \, dv \\
&= 4\pi^2 r k.
\end{aligned}$$

Theorem 2.1

Let (D, f, S) be a differentiable surface and $G(u,v)$ its matrix of the first fundamental form. Then:

- *The angle between two curves C and d intersecting at a point $U \in D$ and the angle between their images $\gamma = f \circ C$ and $\delta = f \circ d$ at the point $p = f(U)$ are equal (**f preserves angles**) if and only if there exists $\lambda : D \to \mathbb{R}$ such that:*

$$G(u,v) = \lambda(u,v) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

- *The length of a curve on D and the length of its image on S are equal (**f preserves lengths**) if and only if $\forall (u,v) \in D$:*

$$G(u,v) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

✿ *The areas of any* $\Omega \subset D$ *and the area of its image* $f(\Omega)$ *are equal (**f preserves areas**) if and only if* $\det G(u, v) = 1$.

Chapter 3 Curvilinear integral (Line Integral)

3.1 Line Integral of a Vector Field

Definition 3.1

A vector field in the plane or space consists of assigning, at each point (x, y) (or (x, y, z) in space), a vector $g(x, y)$ (or $g(x, y, z)$).

Definition 3.2

Let $t \mapsto C(t) = (x(t), y(t))$, for $t \in [a, b]$, be a parameterized curve where the vector $\dot{C}$ is continuous. Let $(x, y) \mapsto g(x, y)$ be a continuous vector field. The line integral of g along C, denoted $\int_C g(x, y)\,dx\,dy$, is defined by

$$\int_C g(x, y)\,dx\,dy = \int_a^b g(C(t)) \cdot \dot{C}(t)\,dt.$$

Example 3.1

Let $g : (x_1, x_2) \in \mathbb{R}^2 \to (x_1x_2, x_1+x_2)$ and let C be the line segment joining the points $A = (0, 1)$ and $B = (1, 1)$, oriented from A to B. The parameterization of C is

$$\varphi : t \in [0, 1] \to (t, 1) \in \mathbb{R}^2.$$

For $t \in [0, 1]$, we have:

$$\dot{\varphi}(t) = (1, 0) \quad \text{and} \quad g(\varphi(t)) = (t, 1 + t).$$

Thus,

$$g(\varphi(t)) \cdot \dot{\varphi}(t) = t$$

and, therefore,

$$\int_C g(x_1, x_2)\,dx_1\,dx_2 = \int_0^1 t\,dt = \frac{1}{2}.$$

☞ **Remark**

In some contexts, the line integral $\int_C g(X) \cdot dX$ is referred to as the circulation of the vector field g along the curve C.

Theorem 3.1

The circulation of a vector field along a curve does not depend on the chosen parameterization of the curve.

Proof. Replace the parameter t with $\tau = \varphi(t)$, and let $C_1(\tau) = C(t)$. The vector $\dot{C}(t)$ changes to

$v_1(\tau) = \dfrac{v(t)}{\dot{\varphi}(t)}$. Then:

$$\begin{aligned}\int_C g(X)\cdot dX &= \int_J g(C_1(\tau))\cdot v_1(\tau)\,d\tau \\ &= \int_I g(C_1(\varphi(t)))\cdot v_1(\varphi(t))\cdot \dot{\varphi}(t)\,dt \\ &= \int_I g(r(t))\cdot v(t)\,dt \\ &= \int_C g(X)\cdot dX.\end{aligned}$$

■

Corollary 3.1

Let C be a curve parameterized by its arc length, and let $g(s)$ be the unit tangent vector to C. Then the circulation of a vector field g along C can be written as

$$\text{circulation}(g, C) = \int g\cdot u(C(s))\,ds.$$

Thus,

$$\text{circulation}(g, C) = \int_I g\cdot u\,ds.$$

3.1.1 Case of Vector Fields Derived from a Potential

Theorem 3.2

Let f be a function, and $t \mapsto C(t)$, $t \in I$, be a parameterized curve. For any t_0 and $t_1 \in I$,

$$\text{circulation}(\nabla f, C[t_0, t_1]) = f(C(t_1)) - f(C(t_0)).$$

In other words, the circulation of a vector field derived from a potential depends only on the initial and final states, not on the chosen path. When a material point moves in a potential, the work done by the force equals the change in potential energy between the final and initial states.

3.2 Differential Forms of Degree 1

Consider a parameterized curve defined by the following equations:

$$\begin{cases} x = C_1(t) \\ y = C_2(t) \end{cases} \quad t \in [a, b]$$

Let $F = p(x, y)\,dx + q(x, y)\,dy$ be a differential form defined in a domain containing C. Replacing x with $C_1(t)$ and y with $C_2(t)$, we have:

$$\begin{cases} dx = dC_1(t) = \dot{C}_1(t)\,dt \\ dy = dC_2(t) = \dot{C}_2(t)\,dt \end{cases}$$

Let:

$$\begin{cases} p_1(t) = p(C_1(t), C_2(t)) \\ q_2(t) = q(C_1(t), C_2(t)) \end{cases}$$

The form g becomes:

$$g = p_1(t) \cdot \dot{C}_1(t) + q_2(t) \cdot \dot{C}_2(t).$$

Consider the integral:

$$I = \int_a^b \left[p_1(t) \cdot \dot{C}_1(t) + q_1(t) \cdot \dot{C}_2(t) \right] dt.$$

Proposition 3.1

The integral I does not depend on the chosen parameter, but only on the curve C and the form g.

Definition 3.3

The integral I is called the line integral of the form $g = p\,dx + q\,dy$ along the curve C, and we denote it by $\int_C g$ or $\int_C (p\,dx + q\,dy)$.

Thus:

$$\int_C (p\,dx + q\,dy) = \int_a^b \left[p(C_1(t), C_2(t))\, \dot{C}_1(t) + q(C_1(t), C_2(t))\, \dot{C}_2(t) \right] dt$$

☞ **Remark**

Consider the case of a curve C in space $\mathbb{R}^3$ parameterized by:

$$\begin{cases} x = C_1(t) \\ y = C_2(t) \\ z = C_3(t) \end{cases} \quad t \in [a, b]$$

and a differential form $g = p\,dx + q\,dy + r\,dz$. The line integral of g along C is denoted by:

$$I = \int_C (p\,dx + q\,dy + r\,dz)$$

and it is equal to:

$$I = \int_C \left[p(C_1(t), C_2(t), C_3(t))\, \dot{C}_1(t) + q(C_1(t), C_2(t), C_3(t))\, \dot{C}_2(t) + r(C_1(t), C_2(t), C_3(t))\, \dot{C}_3$$

which does not depend on the choice of parameter.

Proposition 3.2

Let $\vec{V}$ be a vector field with components $p(x, y)$ and $q(x, y)$, and let $M(t)$ be the point $(C_1(t), C_2(t))$ on the curve, with $\dot{\vec{M}}(t)$ a tangent vector to the curve C, with components

$\left(\dot{C}_1(t), \dot{C}_2(t)\right)$. *Then:*

$$\int_C (p\,dx + q\,dy) = \int_a^b \vec{V}(C_1(t), C_2(t)) \cdot \dot{\vec{M}}(t)\,dt$$

Proof. We have:

$$\vec{V}(C_1(t), C_2(t)) \cdot \dot{\vec{M}} = p(C_1(t), C_2(t))\,\dot{C}_1(t) + q(C_1(t), C_2(t))\,\dot{C}_2(t)$$

Thus:

$$\begin{aligned}\int_a^b \vec{V}(C_1(t), C_2(t)) \cdot \dot{\vec{M}}(t)\,dt &= \int_a^b \left[p(C_1(t), C_2(t))\,\dot{C}_1(t) + q(C_1(t), C_2(t))\,\dot{C}_2(t)\right] dt \\ &= \int_C (p\,dx + q\,dy).\end{aligned}$$

■

3.3 Properties of the Line Integral

The following properties hold:

✿

$$\int_C (g_1 + g_2) = \int_C g_1 + \int_C g_2 \quad \text{and} \quad \int_C \lambda g = \lambda \int_C g$$

✿ Let g be a scalar field defined and continuous along a curve C. Let $\vec{c}_1 : [a, b]$ and $\vec{c}_2 : [c, d]$ be two equivalent parameterizations of C. Then:

$$\int_C g\,dX = \int_a^b g(\vec{c}_1(t)) \cdot \|\dot{\vec{c}}_1(t)\|\,dt = \int_c^d g(\vec{c}_2(u)) \cdot \|\dot{\vec{c}}_2(u)\|\,du.$$

3.4 Area of a Planar Region

Let D be a region in the plane bounded by a closed curve C, without double points. Assume that the curve C is oriented in the positive (counterclockwise) direction.

Theorem 3.3

The area of the region D is equal to the line integral:

$$S = \int_C x\,dy = -\int_C y\,dx = \frac{1}{2}\int_C (x\,dy - y\,dx).$$

Example 3.2

Let us recalculate the area of the ellipse bounded by the curve given by:

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \text{with} \quad a,b>0$$

We use the parametric representation:

$$x=a\cos t, \quad y=b\sin t, \quad t\in[0,2\pi], \quad a,b>0$$

The area of the ellipse is as follows:

$$\begin{aligned}
S &= \frac{1}{2}\int_C (x\,dy-y\,dx)\\
&= \frac{1}{2}\int_0^{2\pi}(a\cos t\,d(b\sin t)-b\sin t\,d(a\cos t))\\
&= \frac{1}{2}\int_0^{2\pi}\left(ab\cos^2 t\,dt+ab\sin^2 t\,dt\right)\\
&= \frac{1}{2}\int_0^{2\pi} ab\left(\cos^2 t+\sin^2 t\right)dt\\
&= \frac{1}{2}\int_0^{2\pi} ab\,dt\\
&= \pi ab
\end{aligned}$$

Chapter 4 Affine geometry

4.1 Affine Spaces

Definition 4.1

Let E be a vector space over $\mathbb{K}$ and Ω be a nonempty subset of E. We say that Ω is an affine space over E if and only if there exists a function $\varphi : \Omega \times \Omega \to E$ such that:

- $\forall (A, B, C) \in \Omega^3, \varphi(A, C) = \varphi(A, B) + \varphi(B, C)$
- $\forall x \in E, \forall A \in \Omega$, there exists a unique $B \in \Omega$ such that $\varphi(A, B) = x$

The elements of Ω are called points, and E is called the direction of Ω.

Remark

- We write $\overrightarrow{MN}$ instead of $\varphi(M, N)$.
- The first condition in the definition is called **Chasles' Relation**.
- The second condition in the definition is called the **Bijectivity Condition**.

Example 4.1

Let E be a vector space over $\mathbb{K}$. The following function is used:

$$\begin{aligned} \varphi : E \times E &\longrightarrow E \\ (u, v) &\longmapsto v - u \end{aligned}$$

defines an affine space on E over itself. Indeed, for all $(A, B, C) \in E^3$, $\varphi(A, B) + \varphi(B, C) = (B - A) + (C - B) = C - A = \varphi(A, C)$.
Let $u, v \in E$. We seek $w \in E$ such that $v = \varphi(u, w)$. We have $\varphi(u, w) = w - u$, so $v = w - u$, hence $w = v + u$. Therefore, the second condition is satisfied.
In conclusion, any vector space is an affine space over itself, and we have $\overrightarrow{MN} = N - M$.

4.1.1 Centroid

Let Ω be an affine space over a vector space E, and let $\{A_i\}_{i=1,\ldots,n} \subset \Omega$, $\{\alpha_i\}_{i=1,\ldots,n} \subset \mathbb{K}$.

Definition 4.2

The function

$$\begin{aligned} f : \Omega &\longrightarrow E \\ M &\longmapsto f(M) = \sum_{j=1}^{n} \alpha_j \overrightarrow{MA_j} \end{aligned}$$

is called the Leibniz vector function associated with the system $\{A_i, \alpha_i\}_{i=1,\ldots,n}$.

Theorem 4.1

Let f be the Leibniz vector function associated with the system $\{A_i, \alpha_i\}_{i=1,\dots,n}$. Then:

- *If $\sum_{j=1}^{n} \alpha_j = 0$, then f is constant.*
- *If $\sum_{j=1}^{n} \alpha_j \neq 0$, then f is bijective.*

Proof. f is constant if and only if $\forall M, N \in \Omega, f(M) - f(N) = 0$.

We have:

$$\begin{aligned} f(M)-f(N) &= \sum_{j=1}^{n} \alpha_j \overrightarrow{MA_j} - \sum_{j=1}^{n} \alpha_j \overrightarrow{NA_j} \\ &= \sum_{j=1}^{n} \alpha_j (\overrightarrow{MA_j} - \overrightarrow{NA_j}) \\ &= \sum_{j=1}^{n} \alpha_j (\overrightarrow{MA_j} + \overrightarrow{A_jN}) \\ &= \left(\sum_{j=1}^{n} \alpha_j\right) \overrightarrow{MN} \\ &= 0 \qquad \left(\text{since } \sum_{j=1}^{n} \alpha_j = 0\right) \end{aligned}$$

f is bijective if and only if it is injective and surjective.

- Injectivity:

$$\begin{aligned} f(M)=f(N) &\Longrightarrow f(M)-f(N)=0 \\ &\Longrightarrow \sum_{j=1}^{n} \alpha_j \overrightarrow{MA_j} - \sum_{j=1}^{n} \alpha_j \overrightarrow{NA_j} = 0 \\ &\Longrightarrow \sum_{j=1}^{n} \alpha_j (\overrightarrow{MA_j} + \overrightarrow{A_jN}) = 0 \\ &\Longrightarrow \left(\sum_{j=1}^{n} \alpha_j\right) \overrightarrow{MN} = 0 \\ &\Longrightarrow \overrightarrow{MN} = 0 \qquad \left(\text{since } \sum_{j=1}^{n} \alpha_j \neq 0\right) \\ &\Longrightarrow M = N \end{aligned}$$

- Surjectivity:

  Suppose for contradiction that f is not surjective. Then there exists $y \in E$, such that for all $M \in \Omega$, $y \neq f(M)$, meaning for any $\{\alpha_j\} \subset \mathbb{K}$ with $\sum_{j=1}^{n} \alpha_j \neq 0$ and any $\{A_j\} \subset \Omega$, $y \neq \sum_{j=1}^{n} \alpha_j \overrightarrow{MA_j}$. Take

$\alpha_1 = 1$ and $\alpha_j = 0$ for $j = 2, \dots, n$, then $f(M) = \overrightarrow{MA_1}$. By assumption, $y \neq \overrightarrow{MA_1}$ for all $M \in \Omega$, which contradicts the condition of the affine space. Therefore, f is surjective.

■

Definition 4.3

Let f be the Leibniz vector function associated with the system $\{A_i, \alpha_i\}_{i=1,\dots,n}$ such that $\sum_{j=1}^{n} \alpha_j \neq 0$. The centroid of the system $\{A_i, \alpha_i\}_{i=1,\dots,n}$ is the point $G \in \Omega$ such that $f(G) = 0$, that is, $\sum_{j=1}^{n} \alpha_j \overrightarrow{GA_j} = 0$.

Properties

- For any $\lambda \in \mathbb{K} - \{0\}$, the centroid of the system $\{A_i, \lambda\alpha_i\}_{i=1,\dots,n}$ is the same as the centroid of the system $\{A_i, \alpha_i\}_{i=1,\dots,n}$.
- If E, F is a partition of the set $\{1, 2, \dots, n\}$ and $\beta = \sum_{j\in E} \alpha_j \neq 0$, $\gamma = \sum_{j\in F} \alpha_j \neq 0$. If G_1 is the centroid of the system $\{A_j, \alpha_j\}_{j\in E}$, G_2 is the centroid of the system $\{A_j, \alpha_j\}_{j\in F}$, and G is the centroid of the system $\{A_j, \alpha_j\}_{j=1,\dots,n}$, then G is the centroid of the system $\{(G_1, \beta), (G_2, \gamma)\}$.

4.1.2 Affine Varieties

Definition 4.4

Let Ω be an affine space over E, and let F be a nonempty subset of Ω. We say that F is an affine variety of Ω if and only if

$$\exists A \in F : \text{the set } H = \{\overrightarrow{AM} \mid M \in F\} \text{ is a vector subspace of } E.$$

H is called the direction of F, and the affine variety F is also referred to as an affine subspace.

Proposition 4.1

If F is an affine variety of an affine space Ω, then for all $B \in F$, the set $H' = \{\overrightarrow{BM} \mid M \in F\}$ is a vector subspace of E (the direction of F does not depend on the choice of the point A).

Proof. F is an affine variety, so $\exists A \in F$ such that the set $H = \{\overrightarrow{AM} \mid M \in F\}$ is a vector subspace of E. Let $H' = \{\overrightarrow{BM} \mid M \in F\}$. We will show that $H = H'$.

- $H' \subset H$?

$$\begin{aligned} x \in H' &\implies \exists M \in F : x = \overrightarrow{BM} \\ &\implies \overrightarrow{BM} = \overrightarrow{BA} + \overrightarrow{AM} \\ &\implies \overrightarrow{BM} = (\overrightarrow{AM} - \overrightarrow{AB}) \in H \qquad \text{(since } H \text{ is a vector subspace)} \end{aligned}$$

- $H \subset H'$?

  Let $\overrightarrow{AM}$ and $\overrightarrow{AB}$ be elements of H. Since H is a vector subspace, we can represent any vector $x \in H$

as follows: $\forall x \in H, \exists M \in F : x = \overrightarrow{AM} - \overrightarrow{AB}$, thus $x = \overrightarrow{BA} + \overrightarrow{AM} = \overrightarrow{BM} \in H'$.

■

Definition 4.5

- The dimension of an affine space is the dimension of the vector space over which it is defined.
- The dimension of an affine variety is the dimension of its direction.
- Any affine variety of dimension 1 (resp. 2) is called a line (resp. plane) affine.
- If Ω is an affine space of dimension n, then any affine variety of dimensions $(n-1)$ is called a hyperplane.
- Singletons in Ω are affine varieties of dimension 0.

Theorem 4.2

The nonempty intersection of any family of affine varieties is an affine variety, and its direction is the intersection of all the directions of that family.

Proof. Let $\{F_i\}_{i\in I}$ be a family of affine varieties, and let $\{H_i\}_{i\in I}$ be their directions. Let $A \in F = \bigcap_{i\in I} F_i$. $A \in F \Longrightarrow A \in F_i, \forall i \in I$, so we can write the H_i with respect to point A: $H_i = \{\overrightarrow{AM} \mid M \in F_i\}$, which are vector subspaces of E. We know that $W = \bigcap_{i\in I} H_i$ is a vector subspace of E. Let $H = \{\overrightarrow{AM} \mid M \in F\}$ and show that $W = H$.

- $H \subset W$?
  $x \in H \Longrightarrow \exists M \in F : x = \overrightarrow{AM}$ and $M \in F \Longleftrightarrow M \in F_i, \forall i \in I$, hence $x = \overrightarrow{AM} \in H_i, \forall i \in I$, so $x \in W$.
- $W \subset H$?
  $x \in W \Longrightarrow x \in H_i, \forall i \in I$. Since we are in an affine space, there exists a unique $M \in F_i$ such that $x = \overrightarrow{AM}, \forall i \in I$. Thus $M \in F$ and $x = \overrightarrow{AM}$, so $x \in H$.

Thus we have shown that $F = \bigcap_{i\in I} F_i$ is an affine variety with direction $H = \bigcap_{i\in I} H_i$. ■

Definition 4.6

- Let $\varnothing \neq A \subset \Omega$. The affine variety generated by A is the intersection of all affine varieties containing A, denoted by $Aff(A)$.
- We say that the family $\{A_i\}_{i=0,\dots,l}$ ($l+1$ points) of an affine space Ω is affinely independent if the affine variety generated by them is of dimension l.
- Let F and F' be two varieties with directions H and H', respectively. We say that F and F' are parallel if and only if $H = H'$ and we write $F \parallel F'$.

Proposition 4.2

- *If $F \parallel F'$, then $F = F'$ or $F \cap F' = \varnothing$.*
- *For any point $A \in \Omega$, there exists a unique affine variety F_A containing A and parallel to F.*

Proof. Suppose that $F \parallel F'$, then that $H = H'$. We want to show that $F = F'$ or $F \cap F' = \varnothing$. Assume that $F \cap F' \neq \varnothing$ and that $F = F'$.
Let $A \in F \cap F'$. We can write the directions H and H' in terms of the point A, since the directions

do not depend on the choice of the point. $H = \{\overrightarrow{AM} \mid M \in F\}$ and $H' = \{\overrightarrow{AN} \mid N \in F'\}$. We have $H = H' \Longrightarrow F = F'$ because $M \in F \Leftrightarrow \overrightarrow{AN} \in H = H' \Leftrightarrow M \in F'$.

✿ Let H be the direction of F. Define the set $F_A = \{M \in \Omega \mid \overrightarrow{AM} \in H\}$. Let $A \in F_A$. The direction of F_A is then given by $H' = \{\overrightarrow{AM} \mid M \in F_A\}$. To show that F_A is an affine variety parallel to F, it is sufficient to show that $H = H'$.
$x \in H' \Leftrightarrow x = \overrightarrow{AM}, M \in F_A \Leftrightarrow x = \overrightarrow{AM}, \overrightarrow{AM} \in H \Leftrightarrow x \in H$.

■

4.2 Affine Applications

4.2.1 Definitions and Properties

Definition 4.7

Let Ω and Λ be two affine spaces over E and F, respectively. We say that a function $f : \Omega \to \Lambda$ is affine if and only if there exists a unique linear function φ from E to F such that: $\forall (M, N) \in \Omega^2$: $\overrightarrow{f(M)f(N)} = \varphi(\overrightarrow{MN})$.

Example 4.2

For the affine space $\mathbb{R}$ over itself, the function $f : \mathbb{R} \to \mathbb{R}$ given by $f(x) = 2x - 1$ is affine. Indeed, for all $(x, y) \in \mathbb{R}^2$, we have $\overrightarrow{f(x)f(y)} = f(y) - f(x) = 2(y - x) = 2\overrightarrow{xy}$, so the associated linear function $\varphi : \mathbb{R} \to \mathbb{R}$ is $\varphi(p) = 2p$.

Proposition 4.3

✿ *The composition of two affine functions is an affine function.*

✿ *Let $f : \Omega \to \Lambda$ and φ its associated linear function, then*

- ❀ *φ is injective if and only if f is injective.*
- ❀ *φ is surjective if and only if f is surjective.*

✿ *The image of an affine variety under an affine function is an affine variety.*

Proof. ✿ Let Ω, Λ, and Δ be three affine spaces over E, F, and G, respectively, and let $\Omega \xrightarrow{f} \Lambda \xrightarrow{g} \Delta$ be two affine functions with associated linear functions φ and ψ from E to F and F to G, respectively. Then
$\overrightarrow{(g \circ f)(M)(g \circ f)(N)} = \overrightarrow{g(f(M))g(f(N))} = \psi(\overrightarrow{f(M)f(N)}) = \psi(\varphi(\overrightarrow{MN}))$.
Therefore, $g \circ f$ is affine with an associated linear function $\psi \circ \varphi$.

✿ ❀ Is φ injective $\Longrightarrow$ is f injective?
We have $f(M) = f(N) \Longrightarrow \overrightarrow{f(M)f(N)} = 0 \Longrightarrow \varphi(\overrightarrow{MN}) = 0$. Since φ is injective, $\overrightarrow{MN} = 0_E$, thus $M = N$, so f is injective.
Is f injective $\Longrightarrow$ is φ injective?
We have $\varphi(\overrightarrow{MN}) = \varphi(\overrightarrow{M'N'}) \Longrightarrow \varphi(\overrightarrow{MN} - \overrightarrow{M'N'}) = 0_E$. Since $(\overrightarrow{MN} - \overrightarrow{M'N'}) \in E$ and Ω is an affine space over E, $(S, T) \in \Omega$ exists such that $\overrightarrow{MN} - \overrightarrow{M'N'} = \overrightarrow{ST}$.
Thus, $\varphi(\overrightarrow{ST}) = 0_E \Longrightarrow \overrightarrow{f(S)f(T)} = 0_E \Longrightarrow S = T$ (since f is injective) $\Longrightarrow \overrightarrow{ST} = 0_E \Longrightarrow \overrightarrow{MN} = \overrightarrow{M'N'}$.

❀ Is f surjective $\Longrightarrow$ is φ surjective?

Let $S \in \Lambda$ and $y \in F$. Then, there exists $S' \in \Lambda$ such that $y = \overrightarrow{SS'}$ and since f is surjective: $(M, N) \in \Omega^2$ such that $S = f(M)$ and $S' = f(N)$. Thus, $y = \overrightarrow{f(M)f(N)} = \varphi(\overrightarrow{MN})$. Therefore, for all $y \in F$, there exists $x \in E$ such that $y = \varphi(x)$, so φ is surjective.

Is φ surjective $\Longrightarrow$ is f surjective?

Let $S \in \Omega$ and $N \in \Omega$ such that $f(N) = S'$. Let $y = \overrightarrow{SS'} \in F$. Then there exists $x \in E$ such that $y = \overrightarrow{SS'} = \varphi(x)$. We have $N \in \Omega$ and $x \in E \Longrightarrow$ there exists $M \in \Omega$ such that $x = \overrightarrow{MN}$. Thus, $\varphi(x) = \varphi(\overrightarrow{MN}) = \overrightarrow{f(M)f(N)} = \overrightarrow{SS'}$. Since $f(N) = S'$, it follows that $f(M) = S$, hence f is surjective.

✿ Let $f : \Omega \to \Lambda$ be an affine function with an associated linear function $\varphi : E \to F$ and G be an affine variety in Ω with direction $H = \{\overrightarrow{AM} \mid M \in G\}$.

Is $f(G)$ an affine variety in Λ?

Since H is a direction in Ω, for any $A \in G$, we have $f(A) = f(M)$ and $\overrightarrow{f(A)f(N)} = \varphi(\overrightarrow{AM})$ with $A \in G$ and $M \in G$. We have:

$\forall (M, N) \in \Omega^2$ $\overrightarrow{f(A)f(N)} = \overrightarrow{f(A)f(N)} = \varphi(\overrightarrow{AM})$. Thus, $\overrightarrow{f(A)f(N)} \in F$ is a direction in Λ and $f(G)$ is an affine variety in Λ.

■

4.3 Affine Subspaces

4.3.1 Definitions and Properties

Definition 4.8

Let F be a subspace of an affine space Ω over a field E. We say that F is an affine subspace of Ω if there exists a subspace H of E such that F is the affine variety with direction H and point $A \in \Omega$.

Example 4.3

In $\mathbb{R}^3$, let F be the affine subspace defined by the plane $2x + 3y - z = 4$ with point $A = (0, 0, 4)$. Then F is an affine subspace of $\mathbb{R}^3$ with the direction $H = \{\overrightarrow{AM} \mid M \in F\}$, where H is the direction of the plane.

Proposition 4.4

✿ *The intersection of two affine subspaces is an affine subspace.*

✿ *The union of two affine subspaces is an affine subspace if and only if one of the affine subspaces is contained in the other.*

✿ *Every affine subspace is contained in an affine subspace of higher dimension.*

Proof. ✿ Let F_1 and F_2 be two affine subspaces of Ω with directions H_1 and H_2, respectively. Their intersection $F_1 \cap F_2$ is also an affine subspace with direction $H = H_1 \cap H_2$.

Proof: For $A \in F_1 \cap F_2$, the direction H of $F_1 \cap F_2$ is the intersection of the directions H_1 and H_2. Thus, H is a subspace of E. Therefore, $F_1 \cap F_2$ is an affine subspace with direction H.

✿ Let F_1 and F_2 be two affine subspaces of Ω. Suppose that F_1 and F_2 are not contained in each other. Then $F_1 \cup F_2$ is not an affine subspace, as the directions of F_1 and F_2 do not coincide.

Conversely, if $F_1 \subseteq F_2$ or $F_2 \subseteq F_1$, then $F_1 \cup F_2 = F_2$ or $F_1 \cup F_2 = F_1$, respectively, so $F_1 \cup F_2$ is an affine subspace.

- For any affine subspace F of Ω, consider an affine subspace G of Ω such that $\dim(G) > \dim(F)$. This is always possible by considering the affine space Ω as a whole.

■

4.3.2 Formula of an Affine Map

Let:

- Ω be an affine space over E,
- Λ be an affine space over F,
- $R = \{O, e_1, e_2, \ldots, e_n\}$ be an affine basis of Ω,
- $R' = \{O, e'_1, e'_2, \ldots, e'_m\}$ be an affine basis of Λ,
- $f : \Omega \longrightarrow \Lambda$ be an affine map,
- $\varphi : E \longrightarrow F$ be the associated linear map of f,
- A be the matrix of φ,
- $b = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$ be the components of $f(O)$ in R',
- $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ be the components of a point M in Ω in R,
- $Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}$ be the components of $f(M)$.

We have $X = \overrightarrow{OM}$, $\overrightarrow{f(O)f(M)} = Y - b$, and $\overrightarrow{f(O)f(M)} = \varphi(\overrightarrow{OM}) = AX$, hence $Y - b = AX$, so $Y = AX + b$. This last formula is called the formula of the affine map f.

4.4 Affine Transformations

4.4.1 Fixed Points

Definition 4.9

Let Ω be an affine space over E, and let $f : \Omega \to \Omega$ be an affine map.
A point p in Ω is called a fixed point if $f(p) = p$.

Proposition 4.5

The set of fixed points of an affine map f is an affine variety with direction $\ker(Id_E - \varphi)$, where φ is the linear map associated with f.

Proof. Let p and q be two fixed points of f. We have $f(p) = p$ and $f(q) = q$, thus $\overrightarrow{f(p)f(q)} = \varphi(\overrightarrow{pq}) \Leftrightarrow \overrightarrow{pq} = \varphi(\overrightarrow{pq}) \Leftrightarrow \overrightarrow{pq} - \varphi(\overrightarrow{pq}) = 0_E \Leftrightarrow (Id_E - \varphi)(\overrightarrow{pq}) = 0_E \Leftrightarrow \overrightarrow{pq} \in \ker(Id_E - \varphi)$.
Therefore, $H = \ker(Id_E - \varphi)$ is the direction of the set of fixed points, and it is a vector subspace of E, so the set of fixed points is an affine variety with direction $\ker(Id_E - \varphi)$. ∎

4.4.2 Affine Projection

Definition 4.10

Let G and F be two affine varieties in an affine space Ω, such that the direction of one is the complement of the other. The projection onto F parallel to G is the map $P : \Omega \to \Omega$, which is defined as follows:
If G' is the unique affine variety passing through the point p and parallel to G, then $p' = P(p) = F \cap G'$, noting that the intersection is a point because the directions of the varieties F and G' are direct sums in E.

Proposition 4.6

If $P : \Omega \to \Omega$ is a projection, then:

- ✿ $P \circ P = P$,
- ✿ *P is affine.*

Proof. ✿ Let P be the projection onto the affine variety F parallel to G. We have $p' = P(p) = F \cap G'$, where G' is the unique affine variety passing through the point p and parallel to G. $(P \circ P)(p) = P(P(p)) = P(p')$. G' is also the unique affine variety passing through the point p' and parallel to G, so it intersects F at p', thus $P(p') = p' = P(p)$.

✿ Let P be the projection onto the affine variety F parallel to G, with H and H' being the directions of G and F, respectively. To show that P is affine, we need to find a linear map $\varphi : E \to E$ such that $\forall (M, N) \in \Omega^2 : \overrightarrow{P(M)P(N)} = \varphi(\overrightarrow{MN})$.
Let $P(M) = M'$ and $P(N) = N'$. We have:

$$\begin{aligned} \overrightarrow{MN} &= \overrightarrow{MM'} + \overrightarrow{M'N'} + \overrightarrow{N'N} \\ &= \left(\overrightarrow{MM'} + \overrightarrow{N'N}\right) + \overrightarrow{M'N'} \end{aligned}$$

$$= \left(\overrightarrow{MM'} + \overrightarrow{N'N}\right) + \overrightarrow{P(M)P(N)}$$

Here, $\left(\overrightarrow{MM'} + \overrightarrow{N'N}\right) \in H$ and $\overrightarrow{P(M)P(N)} \in H'$, and $E = H \oplus H'$. If we set $\varphi(\overrightarrow{MN})$, φ is the projection onto H' parallel to H, so φ is linear, hence P is affine.

■

4.4.3 Affine Translations and Homothety

Definition 4.11

Let Ω be an affine space over E. We know that:
$\forall x \in E, \forall M \in \Omega$, there exists a unique M' such that $x = \overrightarrow{MM'}$. We define the map $T_x : \Omega \to \Omega$ by $T_x(M) = M'$, so $\overrightarrow{MT_x(M)} = x$. T_x is an affine map, its associated linear map is the identity, and it is called an **affine translation** with vector x.

Definition 4.12

Let Ω be an affine space over E, $O \in \Omega$, and $\lambda \in \mathbb{K} - \{1\}$. For $M \in \Omega$, there exists a unique $M' \in \Omega$ such that $\overrightarrow{OM'} = \lambda\overrightarrow{OM}$. We define the map $H_\lambda : \Omega \to \Omega$ by $H_\lambda(M) = M'$, so $\overrightarrow{OH_\lambda(M)} = \lambda\overrightarrow{OM}$. H_λ is an affine map, its associated linear map is the homothety of quotient λ, and it is called an **affine homothety** with center O and quotient λ.

☞ **Remark**

✽ If $x \neq 0$, then the translation T_x has no fixed points.

✽ Point O is the only fixed point of the homothety with center O and quotient λ.

Chapter 5 Exercises

Exercise 5.1

Let E be a vector space over $\mathbb{R}$, and let $n \in \mathbb{N}$ such that $n \geq 2$ and $\{V_1, V_2, \ldots, V_n\} \subset E$.
We define:

$$W_1 = V_1 + V_2, \quad W_2 = V_2 + V_3, \quad W_3 = V_3 + V_4, \ldots, W_{n-1} = V_{n-1} + V_n, \quad W_n = V_n + V_1$$

Assume that n is odd. Show that:

If $\{V_1, V_2, \ldots, V_n\}$ is linearly independent, then $\{W_1, W_2, \ldots, W_n\}$ is also linearly independent.

Exercise 5.2

$$\text{Let} u_1 = \begin{pmatrix} c \\ a \\ b \\ c \end{pmatrix}, u_2 = \begin{pmatrix} a \\ c \\ c \\ b \end{pmatrix}, u_3 = \begin{pmatrix} b \\ c \\ c \\ a \end{pmatrix}, u_4 = \begin{pmatrix} c \\ b \\ a \\ c \end{pmatrix}$$

- What relationship must a, b, c satisfy these vectors to form a linearly independent set?
- When these vectors form a linearly dependent set, give in each case a verified linear combination.

Exercise 5.3

Let $F = \left\{(x, y, z) \in \mathbb{R}^3 \mid x - 2y + z = 0\right\}$

- Show that F is a subspace of $\mathbb{R}^3$.
- Find a basis of F and deduce $\dim F$.

Exercise 5.4

Let the matrix $m(d, a, b, c) = \begin{bmatrix} d & -a & -b & -c \\ a & b & -c & b \\ b & c & d & -a \\ c & -b & a & d \end{bmatrix}$ and let the set

$$F = \{m(d, a, b, c) \in M_4(\mathbb{R}) \mid a, b, c, d \in \mathbb{R}\}$$

- Show that F is a subspace of $M_4(\mathbb{R})$.
- Find a basis of F and deduce $\dim F$.
- Show that multiplication is closed in F.
- Compute the product $m(d, a, b, c) \cdot m(d, -a, -b, -c)$ and deduce the inverse of $m(d, a, b, c)$.

Exercise 5.5

We present the following parameterized curves:

- $I=[\frac{\pi}{2},\pi]$, $c(t)=(\cos t,\sin t)$;
- $I=[-\pi,\frac{\pi}{2}]$, $c(t)=(\cos t,\sin t)$;
- $I=[0,2]$, $c(t)=(t^3,t^3)$;
- $I=[-1,1]$, $c(t)=(t^2,t^2)$;
- $I=[\frac{3\pi}{2},3\pi]$, $c(t)=(\cos(\frac{t}{3}),\sin(\frac{t}{3}))$.

For each case:

- Sketch the image of the curve and indicate the direction of traversal with an arrow.
- Find curves C_B defined on $I=[0,1]$ that give the same images.
- Find curves C_C defined on $I=[0,1]$ that give the same images, but this time with $C_C(0)=C_B(1)$.
- Find curves C_D defined on $I=[0,5]$ that give the same images.

Exercise 5.6

- Parameterize a square $abcd$ in $\mathbb{R}^n$ with side length s, such that if $c(t_1)$ and $c(t_2)$ are on the same segment, the length of the segment $[c(t_1)c(t_2)]$ is equal to $|t_1-t_2|$.
- Same question and condition for a rectangle $abcd$ in $\mathbb{R}^n$ with $\|b-a\|=\|d-c\|=r$ and $\|c-b\|=\|d-c\|=s$.
- Same question and condition for a triangle abc in $\mathbb{R}^n$ with $\|b-a\|=q$, $\|c-b\|=r$, and $\|a-c\|=s$.

Exercise 5.7

Let the curve $C\colon[0,2\pi]\to\mathbb{R}^2$ be defined by $C(t)=(e^t\cos t,e^t\sin t)$.

- Determine the points of intersection of the curve C with the circle D centered at the origin and with radius r.
- Calculate the angle between curve C and circle D.
- Calculate the length of the curve C.

Exercise 5.8

Let the curve $C\colon[0,1]\to\mathbb{R}^2$ be defined by

$$C(t)=b_0(t)\begin{pmatrix}\frac{-1}{3}\\ \frac{-2}{3}\end{pmatrix}+b_1(t)\begin{pmatrix}\frac{-1}{3}\\ 0\end{pmatrix}+b_2(t)\begin{pmatrix}0\\ \frac{5}{3}\end{pmatrix}+b_3(t)\begin{pmatrix}\frac{5}{3}\\ \frac{17}{3}\end{pmatrix},$$

where

$$b_0(t)=(1-t)^3,\quad b_1(t)=3t(1-t)^2,\quad b_2(t)=3t^2(1-t),\quad b_3(t)=t^3.$$

- Calculate $\dddot{C}$.
- Calculate the length of C over the interval $[0, \beta]$ where $\beta \in]0, 1]$ (hint: calculate $(2 + 6t + 5t^2)^2$).

Exercise 5.9

Let the curve

$$C : \mathbb{R} \to \mathbb{R}^3, \quad t \longmapsto (R\cos t, R\sin t, at),$$

where $a > 0$ and $R > 0$.

- What does this curve represent geometrically?
- Determine a parameterization by arc length.
- Calculate the Frenet frame.
- Calculate the curvature k_a of the curve C at every point. What do you notice?
- Calculate the torsion τ_a of the curve C at every point. What do you notice?
- Calculate the limits $\lim\limits_{a\to 0} k_a$ and $\lim\limits_{a\to 0} \tau_a$.

Exercise 5.10

Let a and b be two strictly positive real numbers. Consider the curve C: $[0, 2\pi] \to \mathbb{R}^3$ given by

$$C(t) = (a\cos t + b\cos 3t, a\sin t - b\sin 3t, 2\sqrt{ab}\sin 2t).$$

- Determine the regular points of the curve C.
- Show that the image of the curve C is included in a sphere, and determines that sphere.
- Determine the points of intersection of C with the equator of the sphere (solve the equation $z(t) = 0$).
- For which value(s) of b does the curve have a vertical tangent at the points of intersection with the equator?
- Calculate the Frenet frame, the curvature, and the torsion.

Exercise 5.11

Let the curve C: $\mathbb{R} \to \mathbb{R}^3$ be defined by $C(t) = (e^t, e^{-t}, \sqrt{2}t)$.

- Show that C is regular, and calculate the unit tangent vector $T_C(t)$.
- Calculate the length $L(C)$ of curve C between the points $C(0)$ and $C(t)$.
- Give the parameterization by the arc length of the curve C.
- Calculate the curvature k and the torsion τ of C, then the ratio $\dfrac{\tau}{k}$.
- Calculate the normal vector $N_C(t)$ and the binormal vector $B_C(t)$.

Exercise 5.12

Let C be the parameterized curve given by:

$$\begin{aligned} C \;:\; & \mathbb{R} \to \mathbb{R}^3 \\ & t \mapsto \left(\frac{1}{2}\sin(t), \left(\sin\left(\frac{t}{2}\right)\right)^2, \frac{\sqrt{3}}{2}t\right) \end{aligned}$$

- Show that the curve C is parameterized by the arc length and deduce that it is regular.
- Compute the Frenet frame $\{T_C(t), N_C(t), B_C(t)\}$.
- Calculate the curvature $k(t)$ and the torsion $\tau(t)$.

Exercise 5.13

Let the following parameterized curve be given:

$$\begin{aligned} C \;:\; & \mathbb{R} \to \mathbb{R}^3 \\ & t \mapsto \left(\cos\left(\frac{t}{\sqrt{2}}\right), \sin\left(\frac{t}{\sqrt{2}}\right), \frac{t}{\sqrt{2}}\right) \end{aligned}$$

- Show that the curve C is parameterized by the arc length and deduce that it is regular.
- Compute the Frenet frame $\{T_C(t), N_C(t), B_C(t)\}$.
- Calculate the curvature $k(t)$ and the torsion $\tau(t)$.

Exercise 5.14

Let C be a circle in the (yz) plane with center $(0, 3, 3)$ and radius $r = 2$.

- Consider CY, a cylinder of height h with base circle C. Find D and $f : D \to \mathbb{R}^3$ such that $f(D) = CY$.
- Consider the torus T obtained by rotating the circle C around the oz axis. Find D and $f : D \to \mathbb{R}^3$ such that $f(D) = T$.

Exercise 5.15

Consider the parameterized surface defined by

$$f : \mathbb{R}^2 \to \mathbb{R}^3, \quad f(u, v) = ((u+v)^2, v, 2(u+v)).$$

- Calculate the vectors f_u and f_v. Are there any nonregular points?
- Find the Cartesian equation of the tangent plane at the point $f(1, 0)$.

Exercise 5.16

Let S be the surface defined by $f : \mathbb{R}^2 \to \mathbb{R}^3$

$$f(u, v) = (u\cos v, u\sin v, v).$$

Find the equation of the tangent plane to S at the point $f(u, v)$.

Exercise 5.17

Let S be the surface parameterized by

$$f : \mathbb{R}^2 \to \mathbb{R}^3, \quad f(u,v) = (u^2, v^2, uv).$$

Given the points $a = (2,2)$, $b = (2,4)$, $c = (1,2)$, and $d = (3,4)$:

- Parameterize the line segment $\gamma : [-1,1] \to \mathbb{R}^2$ such that $\gamma(-1) = a, \gamma(1) = b$.
- Parameterize the line segment $\beta : [-1,1] \to \mathbb{R}^2$ such that $\beta(-1) = c, \beta(1) = d$.
- Find the intersection of γ and β.
- Parameterize the images of these curves as seen on the surface S.
- Calculate the matrix G of the metric tensor.
- Using the matrix G, calculate the angle between these two curves on the surface.

Exercise 5.18

Consider the ellipsoid parameterized by

$$f : [0,2\pi] \times [0,\pi] \to \mathbb{R}^3, \quad f(u,v) = (\cos u \sin v, \sin u \sin v, \frac{\sqrt{5}}{5} \cos v).$$

- Study the regularity of this surface.
- Calculate the area of the surface.
- Calculate the principal curvatures.
- Calculate the Gaussian curvature.

Exercise 5.19

Consider two real numbers $0 < r < R$. We consider the torus parameterized by

$$f : [0,2\pi] \times [0,2\pi] \to \mathbb{R}^3, \quad f(u,v) = ((R + r\cos u)\cos v, (R + r\cos u)\sin v, r\sin u).$$

- Show that this surface is regular.
- Calculate the area of this surface.
- What do the curves $C_1 = \{f(0,v) \mid v \in [0,2\pi]\}$ and $C_2 = \{f(u,0) \mid u \in [0,2\pi]\}$ represent geometrically?
- Show that the tangent vectors f_u and f_v are orthogonal.
- Calculate the Gaussian curvature.

Exercise 5.20

Consider a surface $f : D \to \mathbb{R}^3$, where $D = \{(u,v) \in \mathbb{R}^2 \mid v > 0\}$, with the metric tensor matrix given by:

$$G(u,v) = \begin{pmatrix} \frac{1}{v^2} & 0 \\ 0 & \frac{1}{v^2} \end{pmatrix}.$$

Consider the curve $C : [a,b] \to D$ given by $C(t) = (0,t)$ with $0 < a < b$.

✿ Calculate the length $L(\gamma)$ of the curve $\gamma(t) = f(C(t))$.

✿ What is the value of $L(\gamma)$ when $a \to 0$?

Exercise 5.21

Let S be the parameterized surface given by:

$$\begin{aligned} f \ : \ & [0, 2\pi] \times \mathbb{R}_+^* \to \mathbb{R}^3 \\ & (u, v) \mapsto (e^{-v}\cos u, e^{-v}\sin u, v) \end{aligned}$$

✿ Show that the surface S is regular.

✿ Compute the first and second fundamental forms.

✿ Calculate the surface area.

Exercise 5.22

Let S be the following parameterized surface:

$$\begin{aligned} f \ : \ & [0, 1] \times [0, 1] \to \mathbb{R}^3 \\ & (u, v) \mapsto \left(\frac{1}{2}(u+v), \frac{1}{2}(u-v), uv\right) \end{aligned}$$

✿ Show that the surface S is regular.

✿ Compute the first fundamental form.

✿ Calculate the surface area.

Exercise 5.23

Let S be the following parameterized surface:

$$\begin{aligned} f \ : \ & [0, 1] \times [0, 2\pi] \to \mathbb{R}^3 \\ & (u, v) \mapsto ((1+u)\cos v, (1+u)\sin v, u) \end{aligned}$$

✿ Compute the first fundamental form.

✿ Show that the surface S is regular.

✿ Calculate the surface area.

Exercise 5.24

evaluate the given line integral. Follow the direction of C

✿ $\displaystyle\int_C (3x^2 - 2y)ds$, where C is the line segment from $(3, 6)$ to $(1, -1)$.

✿ $\displaystyle\int_C (2yx^2 - 4x)ds$, where C is the lower half of the circle centered at the origin of radius 3 with clockwise rotation.

✿ $\displaystyle\int_C (xy - 4z)ds$, where C is the line segment from $(1, 1, 0))$ to $(2, 3, -2))$

Exercise 5.25

evaluate the given line integral. Follow the direction of C

- $\displaystyle\int_C [2ydx + (1-x)dy]$, where C is the portion of $y = 1 - x^3$ from $x = -1$ to $x = 2$.
- $\displaystyle\int_C \sqrt{1+y}dy$, where C is the portion of $y = \exp 2x$ from $x = 0$ to $x = 2$.
- $\displaystyle\int_C [x^2dy - yzdz]$, where C is the line segment from $(4, -1, 2))$ to $(1, 7, -1))$

Exercise 5.26

evaluate the given line integral. Follow the direction of C

- $\displaystyle\int_C \overrightarrow{F} \cdot d\overrightarrow{r}$, where $\overrightarrow{F}(x, y) = y^2\overrightarrow{i} + (3x-6y)\overrightarrow{j}$ and where C is the line segment from $(3, 7)$ to $(0, 12)$.
- $\displaystyle\int_C \overrightarrow{F} \cdot d\overrightarrow{r}$, where $\overrightarrow{F}(x, y) = (x+y)\overrightarrow{i} + (1-x)\overrightarrow{j}$ and where C is the portion of $\dfrac{x^2}{4} + \dfrac{y^2}{9} = 1$ that is in the 4^{th} quadrant with the counterclockwise rotation.
- $\displaystyle\int_C \overrightarrow{F} \cdot d\overrightarrow{r}$, where $\overrightarrow{F}(x, y) = y^2\overrightarrow{i} + (x^2 - 4)\overrightarrow{j}$ and where C is the portion of $y = (x-1)^2$ $x = 0$ to $x = 3$.

Exercise 5.27

Let α, β and γ be positive constants. Let:

$$F = -\alpha y^2\overrightarrow{i} + \beta x^2\overrightarrow{j}$$

a particle of mass m moving on the xy-plane, under the action of F. Find the work done by F on the particle in moving it from the Cartesian origin O to the point $(1, 1)$, in each of the following cases.

- Directly from O to $(1, 0)$, then directly from $(1, 0)$ to $(1, 1)$.
- Directly from O to $(0, 1)$, then directly from $(0, 1)$ to $(1, 1)$.
- Move the particle with velocity $v = \gamma(\overrightarrow{i} + \overrightarrow{j})$.

Exercise 5.28

Let $\Omega = \left\{(x, y) \in \mathbb{R}^2 \mid x + y + 4 = 0\right\}$ and let the application be

$$\begin{array}{rcl} \varphi & : & \mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R} \\ ((x, y), (x', y')) & \longmapsto & \dfrac{(x'-x) + (y-y')}{2} \end{array}$$

- Determine φ restricted to $\Omega \times \Omega$.
- Show that φ restricted to $\Omega \times \Omega$ defines an affine space on Ω over $\mathbb{R}$.

Exercise 5.29

Let $\Omega = \mathbb{R}_+^*$ and let

$$\begin{array}{rccl} \varphi & : & \Omega \times \Omega \longrightarrow \mathbb{R} \\ (x,y) & \longmapsto & \ln\left(\frac{y}{x}\right) \end{array}$$

✿ Verify that φ defines an affine space on Ω over $\mathbb{R}$.

Exercise 5.30

Let Ω be an affine space over a vector space E on $\mathbb{k}$. Let $O \in F \subset \Omega$, and $(A,B) \in \Omega^2$ with $(\alpha,\beta) \in \mathbb{k}^2$.

✿ What do the points $(A_1, B_1) \in \Omega^2$ such that

$$\overrightarrow{OA_1} = (\alpha+\beta)\overrightarrow{OA}, \quad \overrightarrow{OB_1} = (\alpha+\beta)\overrightarrow{OB}$$

represent?

✿ Assume $\alpha+\beta \neq 0$ and let $M \in \Omega$ such that $\overrightarrow{OM} = \alpha\overrightarrow{OA} + \beta\overrightarrow{OB}$. Show that M is the barycenter of $[(A_1,\alpha),(B_1,\beta)]$.

✿ Assume $\alpha+\beta = 0$. Let $\lambda \in \mathbb{k}' - \{0,1\}$.

❀ What does the point $C \in \Omega$ such that $\overrightarrow{OC} = \frac{1}{\lambda}\overrightarrow{OA}$ represent?

❀ Show that the point $M \in \Omega$ such that $\overrightarrow{OM} = \alpha\overrightarrow{OA} + \beta\overrightarrow{OB}$ is the barycenter of $[(C_1,\alpha\lambda),(B_2,\beta)]$, where

$$\overrightarrow{OC_1} = (\alpha\lambda+\beta)\overrightarrow{OC}, \quad \overrightarrow{OB_2} = (\alpha\lambda+\beta)\overrightarrow{OB}.$$

✿ Assume that for any system of points (A_i,α_i) in F, the barycenter of this system belongs to F. Define $H = \left\{\overrightarrow{OM} \mid M \in F\right\}$. Show that H is a subspace of E.

Exercise 5.31

Let $F = \left\{(x,y) \in \mathbb{R}^2 \mid x+y+4=0\right\}$.

✿ Is F a subspace of $\mathbb{R}^2$?

✿ Show that F is an affine manifold of $\mathbb{R}^2$.

Exercise 5.32

Let

$$A \in M_{m,n}(\mathbb{R}), \quad b \in \mathbb{R}^m, \quad F = \left\{X \in \mathbb{R}^n \mid AX = b\right\}.$$

✿ Is F a subspace of $\mathbb{R}^n$?

✿ Show that F is an affine manifold of $\mathbb{R}^n$.

✿ Express the dimension of F as a function of $\operatorname{rank} A$.

Exercise 5.33

Let the application be

$$\begin{array}{rcl} f & : & \mathbb{R}^2 \longrightarrow \mathbb{R}^2 \\ \begin{pmatrix} x \\ y \end{pmatrix} & \longmapsto & \begin{pmatrix} 3x+5y \\ -2x-3y-2 \end{pmatrix}. \end{array}$$

- Show that f is affine.
- Find the fixed points of f.
- Find $f \circ f$.

Exercise 5.34

Let Ω be an affine space over E, and let $f : \Omega \longrightarrow \Omega$ be an affine map.
I) Let the application $g : \Omega \longrightarrow E$ be defined as

$$M \mapsto g(M) = \overrightarrow{Mf(M)}.$$

- Show that g is affine.
- Under what condition is g bijective?

Exercise 5.35

Let Ω be an Euclidean affine space over E, and let $f : \Omega \longrightarrow \Omega$ be an affine map that preserves distance. Let $A \in \Omega$, and define the application $\varphi : E \longrightarrow E$ as follows: for all $x \in E$, $\varphi(x) = \overrightarrow{f(A)f(M)}$, where M satisfies $x = \overrightarrow{AM}$.

- Show that $\varphi(0_E) = 0_E$.
- Show that $\langle \varphi(x), \varphi(y) \rangle = \langle x, y \rangle$.
- Show that $\|\varphi(x) - \varphi(y)\| = \|x - y\|$.

Exercise 5.36

Let Ω be an affine space over $E = F \oplus G$, and let $\varphi : E \longrightarrow E$ be such that:

$$\text{For all } x = u + v \text{ with } \begin{cases} u \in F \\ v \in G \end{cases} : \varphi(x) = u - v$$

- Show that φ is linear.
- Let $O \in \Omega$ and let $f : \Omega \longrightarrow \Omega$ be the map such that $\overrightarrow{Of(M)} = \varphi\left(\overrightarrow{OM}\right)$, $\forall M \in \Omega$.
  - Show that f is affine.
  - Study the fixed points of f.

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