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RESEARCH

University Mohamed Khider of Biskra
Faculty of Exact Sciences
Department of Mathematics

Course of Mathematics 1

Intended for students in the first year

Sciences & Technologies
Renewable energies

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Semester: 1

Teaching Unit: UEF 2.1.1

Subject: Mathematics 1

Total Hours: 67h30 (Lectures: 3h00, Tutorials: 1h30)

Credits: 6

Coefficient: 3

Objectives

This first mathematics course is primarily dedicated to standardizing the skill level of incoming university students. New concepts are introduced progressively to guide students towards more advanced mathematics. The topics covered in this course are fundamental and widely used in the fields of Science and Technology.

Prerequisites

A solid understanding of basic mathematical concepts from high school is recommended, including sets, functions, and equations.

Course Content

Chapter 1: Logic and Mathematical Reasoning (1 week)

1.1 Direct Reasoning

1.2 Reasoning by Contraposition

1.3 Proof by Contradiction

1.4 Reasoning by Counterexample

1.5 Proof by Induction

Chapter 2: Sets, Relations, and Applications (2 weeks)

2.1 Set Theory

2.2 Order Relations, Equivalence Relations

2.3 Injective, Surjective, and Bijective Applications.

Chapter 3: True-Value Variables and Their Functions (3 weeks)

3.1 Limit and Continuity of a Function

3.2 Derivative and Differentiability of a Function

Chapter 4: Application to Elementary Functions (3 weeks)

4.1 Power Function

4.2 Logarithmic Function

4.3 Exponential Function

4.4 Hyperbolic Function

4.5 Trigonometric Function

4.6 Inverse Function

Chapter 5: Limited Development (Taylor Series) (2 weeks)

5.1 Taylor's Formula

5.2 Limited Development

5.3 Applications

Chapter 6: Vector Space and Linear Maps (4 weeks)

6.1 Laws of Internal Composition

6.2 Vector Space, Basis, Dimension (definitions and elementary properties)

6.3 Linear Mapping, Kernel, Image, Rank

Evaluation Method

Continuous Assessment: 40%

Final Exam: 60%

Contents

Introduction	viii
1 <i>Mathematical Reasoning and Logic</i>	12
1.1 Numerical Sets	12
1.2 Propositional Logic	13
1.2.1 Proposition	13
1.2.2 Negation	13
1.3 Logical Connectives	14
1.3.1 Conjunction	14
1.3.2 Disjunction	14
1.3.3 Implication	15
1.3.4 Equivalence	16
1.3.5 Contrapositive	17
1.4 Quantifiers	18
1.4.1 Universal Quantifier	18
1.4.2 Existential Quantifier	18
1.4.3 Unique Existential Quantifier	19
1.5 Methods of Proof	19
1.5.1 Direct Methods	19
1.5.2 Contrapositive Proof	20
1.5.3 Proof by Contradiction (Absurd)	20
1.5.4 Proof by Counter-Example	22
1.5.5 Proof by Mathematical Induction	22

2	<i>Sets, Relations and Applications</i>	24
2.1	Set Theory	24
2.1.1	Application of Set Notation in conjunction with Quantifiers	25
2.1.2	Relationships Between Elements and Sets	25
2.1.3	Set Operations	27
2.2	Relations	32
2.2.1	Binary Relation	32
2.2.2	Equivalence Relation	33
2.2.3	Equivalences and Partitions	33
2.2.4	Order Relation	35
2.3	Applications	37
2.3.1	Image and Inverse Image of Subsets	39
2.3.2	Type of Application	40
2.3.3	Characteristics of an Applications	43
3	<i>Real-Valued Functions of a Real Variable: Properties of Functions</i>	47
3.1	Characterization of Functions	47
3.1.1	Even and Odd Function	47
3.1.2	Increasing and decreasing, monotonic function	47
3.2	Limit of Functions	48
3.2.1	Limit of a Function at a Point	48
3.2.2	Limit of a Function at Infinity	50
3.2.3	Properties of a Limit	51
3.2.4	Indeterminate Forms	53
3.2.5	Ways to Solve Limits	54
3.3	Continuous Functions	57
3.3.1	Continuity of a Function at a Point	57
3.3.2	Continuity of a Function in an Interval	58
3.3.3	Continuous Extension at a Point	59
3.3.4	Intermediate Value Theorem	60
3.3.5	Continuity of Composite Functions	60

3.3.6	Continuity of the Algebraic Combinations of Functions	60
3.4	Differentiability of Functions	62
3.4.1	Differentiability of a Function at a Point	62
3.4.2	Differentiability of a Function in an Interval	64
3.4.3	Algebra of Derivatives	65
3.4.4	Derivatives of Composite Functions	65
3.4.5	Derivative of Inverse Function	66
3.4.6	Table of Derivatives	66
3.4.7	Indeterminate Forms and Hospital's Rule	67
3.4.8	Rolle's Theorem	68
3.4.9	Mean Value Theorem	68
3.4.10	Higher Derivatives	69
4	<i>Application to Elementary Functions</i>	70
4.1	Exponential Function	70
4.1.1	Properties the Exponential Function	70
4.1.2	Study the Exponential Function	71
4.2	Logarithmic Function	72
4.2.1	Properties the Logarithmic Function	72
4.2.2	Study the Logarithmic Function	72
4.3	Trigonometric Functions	73
4.3.1	Sine Function	73
4.3.2	Arcsine Function	74
4.3.3	Cosine Function	75
4.3.4	Arccosine Function	76
4.3.5	Relation Between Sine and Cosine Functions	77
4.3.6	Tangent and Cotangent Functions	78
4.3.7	Tangent Function	78
4.3.8	Arctangent Function	78
4.3.9	Cotangent Function	79
4.3.10	Arccotangent Function	80

4.4	Hyperbolic Trigonometry and its Inverses	81
4.4.1	Hyperbolic Cosine and its Inverse	81
4.4.2	Hyperbolic Sine and its Inverse	82
4.4.3	Hyperbolic Tangent and Inverse	83
4.4.4	Hyperbolic Cotangent and its Inverse	83
4.4.5	Hyperbolic Trigonometry	84
5	<i>Limited Development</i>	85
5.1	Taylor Formula	85
5.1.1	Taylor Formula with the Rest of Lagrange	85
5.1.2	Taylor Formula with the Rest of Young	86
5.1.3	Taylor Formula with the Rest of Integral	87
5.1.4	Taylor Formula and Boundary Values of a Function	88
5.2	Limited Development	89
5.2.1	Publication Properties Limited	90
5.2.2	Limited Development at ∞	92
5.2.3	Application of Limited Development at 0	92
5.2.4	Operations on Limited Development	93
5.3	Application of Limited Developments	97
5.3.1	Application Example	97
5.3.2	Calculated Limits	97
6	<i>Vector Space and Linear Maps</i>	100
6.1	Vector Space	100
6.1.1	Subspace	101
6.1.2	Linear Combinations	102
6.1.3	Linear Independence and Linear Dependence	103
6.1.4	Generating Sets	104
6.1.5	Bases of a Vector Space	105
6.1.6	Dimension of a Vector Space	106
6.2	Linear Maps	107

6.2.1	Kernel of a Linear Map	108
6.2.2	Image of a Linear Map	109
6.2.3	Rank of a Linear Map	110

Introduction

Mathematics is a body of structured knowledge derived from logical reasoning applied to objects such as mathematical sets, numbers, and shapes, as well as the relationships and arithmetic operations that exist between these objects. The most important branches of mathematics include algebra, analysis, geometry, and mathematical logic. Whether in studies, work, or even in sports and games, mathematics helps bring organization and reduce randomness. It has been said about mathematics:

"Mathematics is the art of playing with numbers."

"Mathematics is food for the mind."

This document provides fundamental definitions, formulas, and connections for first-year Science and Technology students in Mathematics 1 based on the Ministry of Higher Education and Scientific Research curriculum. This module has six primary topics: The first chapter introduces basic logic principles, essential tools, and mathematical reasoning processes. The second chapter covers set theory and operations involving multiple sets. The text covers equivalence, order, and injective, surjective, and bijective functions. The third chapter covers real functions of a real variable, including limits, continuity, and differentiation. It also offers essential tables of well-known limits and derivatives. In the fourth chapter, several types of Taylor series, operations on finite expansions, and applying them to evaluate limits are covered. The fifth chapter covers common elementary functions in mathematics, including exponential, logarithmic, hyperbolic, and inverse functions. Additionally, it investigates constructing new functions by modifying old ones. In the sixth chapter, algebraic structures, such as groups and rings, are thoroughly explained, along with internal and exterior operations. It also covers vector spaces and linear transformations.

Mathematical Reasoning and Logic

1.1 Numerical Sets

In mathematics, we often examine sets comprised of real numbers. Several frequently encountered numerical sets are delineated as follows:

The set of Natural numbers, denoted as $\mathbb{N}$, is defined as follows:

$$\mathbb{N} = \{1, 2, 3, \dots\}.$$

$\mathbb{Z}$: the set of integers:

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}.$$

$\mathbb{D}$: the collection of decimal numbers (rationals with a denominator of 10^n):

Let $\mathbb{Q}$ be the set of rational numbers, defined as follows:

$$\mathbb{Q} = \left\{ \frac{p}{q} \mid p \in \mathbb{Z}, q \in \mathbb{Z}^* \right\}.$$

$\mathbb{R} \setminus \mathbb{Q}$: the set of Irrational numbers (numbers not in $\mathbb{Q}$).

$\mathbb{R}$: the set of Real numbers (numbers representable by finite or infinite decimal expansions).

$\mathbb{C}$: the set of Complex numbers:

$$\mathbb{C} = \{a + bi, \ a, b \in \mathbb{R}\}.$$

A complex number integrates a real number with a real multiple of i , where $i = \sqrt{-1}$.

The inclusion connections are:

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{D} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}.$$

The set of even numbers has elements such as $0, \pm 2, \pm 4, \pm 6$, etc., which are integers represented by the expression $2n$, where $n \in \mathbb{Z}$.

The set of odd numbers comprises integers that are not even. Consequently, odd numbers encompass $\pm 1, \pm 3, \pm 5, \dots$, which can be represented as $2n + 1$ for an integer n .

1.2 Propositional Logic

1.2.1 Proposition

Definition 1.1

[1] A proposition is a declarative statement that possesses a truth value of either true or false.

Corollary 1.1

Variables serve to denote propositions. The predominant variables employed are A , B , and r .

Example 1.1

1. A : "Every prime number is even" is a false proposition.
2. A : " $\sqrt{2}$ is a rational number" is a false proposition.
3. A : "2 is less than 4" is a true proposition

1.2.2 Negation

Definition 1.2

The negation of a proposition A is referred to as "not A " and is represented by $\bar{A}$.

Example 1.2

Provide the negation of the subsequent statements:

1. If A : "2 is even", then $\bar{A}$: "2 is not even".
 2. If A : " $2 + 2 = 5$ ", then $\bar{A}$: " $2 + 2 \neq 5$ ".
-

1.3 Logical Connectives

1.3.1 Conjunction

Definition 1.3

[1] When considering two propositions, A and B , their conjunction represents a logical statement that holds true exclusively under the condition that both A and B are true. The conjunction is represented by $\mathbf{A} \wedge \mathbf{B}$, which is interpreted as " A and B ".

We present the truth table for $A \wedge B$ as follows:

A	B	$A \wedge B$
1	1	1
1	0	0
0	1	0
0	0	0

Example 1.3

1. The statement "A triangle has three sides and a square has four sides" is a conjunction.
2. Let $A : "2 \leq 3"$ and $B : "22 \leq 32"$, the proposition $A \wedge B$ is true.

1.3.2 Disjunction

Definition 1.4

[2] A compound statement of the form " $\mathbf{A} \text{ or } \mathbf{B}$ " is known as a disjunction and it is denoted by $\mathbf{A} \vee \mathbf{B}$. The disjunction $A \vee B$ is false only when both A and B are false.

We give the truth table for $A \vee B$ as follows:

A	B	$A \vee B$
1	1	1
1	0	1
0	1	1
0	0	0

Example 1.4

1. An integer is a number that may be expressed as a natural number with either a positive or negative sign, denoting its position relative to 0 on a number line.
2. Consider the proposition $\bar{A} \vee \bar{B}$: " $2 > 3$ " $\vee$ " $2^2 > 3^2$ ". This assertion is incorrect.

1.3.3 Implication**Definition 1.5**

[1] A statement structured as "If ... then ..." is recognized as a conditional or an implication.

* A conditional statement comprises two elements:

- Antecedent (hypothesis or premise): the statement A ("If A ")
- Consequent (conclusion): the statement B ("then B ")

* A conditional statement can be expressed in several equivalent ways:

- If A , then B .
- A implies B .
- B if A .
- A only if B .
- A is sufficient for B .
- B is necessary for A .

* The implication is denoted symbolically as: $A \Rightarrow B$.

* The implication $A \Rightarrow B$ is false exclusively when A is true and B is false; in all other instances, it is true.

The truth table for $A \Rightarrow B$ is shown as follows:

A	B	$A \Rightarrow B$
1	1	1
1	0	0
0	1	1
0	0	1

Example 1.5

1. "If $1 \leq 3$ then $1 + 1 \leq 3 + 1$ " is a true implication.
2. "If $2 + 3 = 5$ then $3 \times 2 + 3 \times 3 = 20$ " is a false implication because the consequent is false while the antecedent is true.

Definition 1.6

[3] [converse of an implication] The converse of the implication The premise $A \Rightarrow B$ is equivalent to $B \Rightarrow A$.

1.3.4 Equivalence

Definition 1.7

Two mathematical assertions are equal if they possess identical truth values. A statement of the type "**A if and only if B**" is referred to as an equivalency or biconditional statement. It is typically abbreviated as "**A iff B**" and is expressed symbolically as $A \equiv B$ or $A \Leftrightarrow B$.

An equivalence is logically identical to the compound sentence "**A implies B, and B implies A.**" comprising two conditional expressions.

For the biconditional assertion to hold true, the truth values of A and B must be congruent (either both true or both false).

We present the truth table for $\mathbf{A} \Leftrightarrow \mathbf{B}$ as follows:

A	B	$A \Leftrightarrow B$
1	1	1
1	0	0
0	1	0
0	0	1

Example 1.6

1. The statement " $(1 = 1) \Leftrightarrow (0 = 0)$ " is valid; the statement " $(1 = 0) \Leftrightarrow (2 = 0)$ " is also valid; however, the statement " $(1 = 0) \Leftrightarrow (0 = 0)$ " is invalid.
2. For all real numbers x and y , the equivalence $y = x$ holds if and only if $\frac{y}{x} = 1$, which is valid under the condition that $x \neq 0$.
3. The equivalence statement $x = y \Leftrightarrow x^2 = y^2$ is not universally valid for all real numbers x and y . For instance, while $2^2 = (-2)^2$ holds true, it follows that $2 \neq -2$, thereby invalidating the biconditional relationship.

1.3.5 Contrapositive

Definition 1.8

[1] The contrapositive of the implication $A \Rightarrow B$ is the assertion $\neg B \Rightarrow \neg A$. It can be demonstrated that these two propositions are logically equivalent:

$$(A \Rightarrow B) \Longleftrightarrow (\neg B \Rightarrow \neg A).$$

Note: The statement $B \Rightarrow A$ is the converse of $A \Rightarrow B$, not the contrapositive, and they are generally not logically equivalent.

1.4 Quantifiers

1.4.1 Universal Quantifier

Definition 1.9

[2] A universal quantifier expresses the meaning "for all," "for any," "for each," or "for every" and is denoted by the symbol $\forall$.

Formally, to say that a predicate $D(x)$ is true for all values of the variable x in a domain E , we write:

$$\forall x \in E, D(x),$$

which reads as "for all x belonging to E , $D(x)$ is true."

Example 1.7

All natural numbers expressed as $2n + 1$ are classified as odd, which can be formally stated as:

$$\forall n, 2n + 1 \text{ is odd.}$$

1.4.2 Existential Quantifier

Definition 1.10

An existential quantifier expresses the meaning "there exists," "there is at least one," or "for some." It is denoted by the symbol $\exists$.

Formally, to say that a predicate $D(x)$ is true for some value(s) of the variable x in a domain E , we write:

$$\exists x \in E, D(x),$$

which reads as "there exists some x belonging to E such that $D(x)$ is true."

Example 1.8

There exists a natural number n such that $n \times n = n + n$, which may be expressed as:

$$\exists n, n \times n = n + n.$$

1.4.3 Unique Existential Quantifier

Definition 1.11

An existential quantifier denotes the concept of "there exists a unique," "there is precisely one," or "there exists solely one." It is represented by the symbol $\exists!$.

To formally express that there is precisely one value of the variable x within the domain E for which the predicate $D(x)$ holds true, we denote it as: which states, "there exists a unique x in E for which $D(x)$ is true."

Example: Set $D(x) : x + 2 = 5$.

1. $\forall x, D(x) : \text{"for all real numbers } x, x + 2 = 5\text{"}$, which is false.
2. $\exists x, D(x) : \text{"there is a real number } x \text{ such that } x + 2 = 5\text{"}$, which is true.
3. $\exists! x, D(x) : \text{"there is a unique real number } x \text{ such that } x + 2 = 5\text{"}$, which is true.

1.5 Methods of Proof

1.5.1 Direct Methods

Definition 1.12

Suppose that A is true, and we demonstrate that B is also true. This method of proving the implication $A \Rightarrow B$ is called a direct proof.

Example 1.9

Demonstrate the assertion: if n is even, then n^2 is even. Let the integer n be even. By definition, there exists an integer k such that $n = 2k$. Consequently,

$$n^2 = (2k)^2 = 4k^2 = 2(2k^2).$$

By defining $k' = 2k^2$, we obtain $n^2 = 2k'$, demonstrating that n^2 is even.

This demonstration explicitly demonstrates that the assumption of n being even logically results in n^2 also being even, exemplifying the framework of a direct proof.

1.5.2 Contrapositive Proof

Definition 1.13 (*Contrapositive*)

This proving approach relies on the logical equivalence of an implication and its contrapositive. The implication

$$A \Rightarrow B$$

is logically equivalent to its contrapositive

$$\bar{B} \Rightarrow \bar{A}.$$

Consequently, to demonstrate $A \Rightarrow B$, it is sufficient to establish the contrapositive $\bar{B} \Rightarrow \bar{A}$.

Example 1.10

We demonstrate the contrapositive statement: if n is odd, then n^2 is odd.

Suppose n is not even. Then n is odd, so there exists an integer $s \in \mathbb{Z}$ such that

$$n = 2s + 1.$$

Compute n^2 :

$$\begin{aligned} n^2 &= (2s + 1)^2 = 4s^2 + 4s + 1 \\ &= 2(2s^2 + 2s) + 1. \end{aligned}$$

Let $s' = 2s^2 + 2s$, which is an integer since $s \in \mathbb{Z}$. Thus,

$$n^2 = 2s' + 1,$$

which constitutes an odd integer. If n is odd, then n^2 is also odd. Given that the contrapositive is valid, the initial assertion holds: if n^2 is even, then n is even.

1.5.3 Proof by Contradiction (Absurd)

Definition 1.14

To demonstrate the veracity of a thesis A , we assume the contrary, that B is false, and subsequently illustrate that this assumption results in a contradiction or a falsehood.

Consequently, the first proposition A must be valid.

Example 1.11

Demonstrate that $\sqrt{2}$ is an irrational number. Let p : "The square root of 2 is irrational." To demonstrate p , assume the contrary, that p is false. Assume that $\sqrt{2}$ is a rational number. If $\sqrt{2}$ is rational, it can be represented as a fraction of two numbers a and b (with no common factors other than 1, and $b \neq 0$) :

$$\sqrt{2} = \frac{a}{b}.$$

Squaring both sides gives:

$$2 = \frac{a^2}{b^2} \Rightarrow a^2 = 2b^2.$$

This implies that a^2 is even because it equals $2b^2$, which is clearly even.

Example 1.12

Assuming p is false implies that $\bar{p}$ is true, which means $\sqrt{2}$ is rational. There exist integers a and b that share no common factors other than 1.

$$\exists a \in \mathbb{Z}, \exists b \in \mathbb{Z}^* : \sqrt{2} = \frac{a}{b} \Rightarrow a^2 = 2b^2$$

Then a^2 is an even number because it is equal to $2b^2$ which is clearly even.

Since a^2 is even, a must also be even (because the square of an odd number is odd).

Therefore, we can write:

$$a = 2c \quad \text{for some integer } c.$$

Substituting back into the equation $a^2 = 2b^2$:

$$(2c)^2 = 2b^2 \Rightarrow 4c^2 = 2b^2 \Rightarrow 2c^2 = b^2.$$

This shows that b^2 is even, and hence b is even. Now, both a and b are even, which means they have a common factor of 2. This contradicts our initial assumption that a and b have no common factors other than 1. Since assuming $\sqrt{2}$ is rational. leads to a contradiction, our original proposition p must be true. Therefore, $\sqrt{2}$ is irrational.

1.5.4 Proof by Counter-Example

Definition 1.15

This proof strategy illustrates the falsity of a general statement by presenting a specific example in which the property does not hold. To demonstrate that the assertion $\forall x, P(x)$ is untrue, it suffices to identify a singular x for which $P(x)$ is not valid.

Example 1.13

Assertion: Demonstrate that the proposition "all triangles are obtuse" is incorrect. We present the following counterexample: an equilateral triangle, characterized by all angles measuring 60° . An obtuse triangle contains an angle over 90° , whereas all angles in an equilateral triangle are inferior than 90° ; thus, this triangle is not obtuse. Consequently, the assertion is inaccurate. Indeed, there exists an endless number of counterexamples; but, only one suffices to refute the assertion.

1.5.5 Proof by Mathematical Induction

Definition 1.16

To establish a proposition of the form $\forall n \in \mathbb{N}, Q(n)$, where n is a natural number, it is sufficient to demonstrate two steps:

- 1. **Base Case:** Demonstrate that $Q(n_0)$ holds for a specific initial value n_0 . The typical basic case is $n_0 = 0$. or $n_0 = 1$.*
- 2. **Inductive Step:** Demonstrate that if $Q(n)$ holds for an arbitrary n , then $Q(n + 1)$ also holds. Formally demonstrate that*

$$Q(n) \implies Q(n + 1).$$

Example 1.14

Demonstrate the subsequent formula for all natural integers n :

$$1 + 3 + 5 + 7 + \dots + (2n - 1) = n^2.$$

Let $P(n)$ be the statement: $1 + 3 + 5 + 7 + \dots + (2n - 1) = n^2$.

We will prove $\forall n \in \mathbb{N}, P(n)$ by induction:

Step 1: Base Case

For $n = 1$:

$$1 = 1^2,$$

which is true. So, $P(1)$ holds.

Step 2: Inductive Hypothesis

Assume that the statement $P(n)$ is true for some arbitrary $n \geq 1$, i.e.,

$$1 + 3 + 5 + 7 + \dots + (2n - 1) = n^2.$$

Step 3: Inductive Step

We need to prove that $P(n + 1)$ is true, that is

$$1 + 3 + 5 + 7 + \dots + (2n - 1) + [2(n + 1) - 1] = (n + 1)^2$$

Using the inductive hypothesis, the left-hand side becomes:

$$n^2 + [2(n + 1) - 1] = n^2 + (2n + 2 - 1) = n^2 + (2n + 1).$$

Simplify the right-hand side:

$$n^2 + 2n + 1 = (n + 1)^2.$$

Thus,

$$1 + 3 + 5 + 7 + \dots + (2n - 1) + (2n + 1) = (n + 1)^2,$$

which proves $P(n + 1)$.

Conclusion: By the principle of mathematical induction, the formula

$$1 + 3 + 5 + 7 + \dots + (2n - 1) = n^2$$

holds for all natural numbers $n \geq 1$.

Sets, Relations and Applications

2.1 Set Theory

Definition 2.1 (*Set*)

[1] A set K is a collection of items referred to as the elements of the set.

Definition 2.2 (*Element*)

If x is an element of the set K , we write $x \in K$. If x is not an element of K , we write $x \notin K$.

- . A set may alternatively be referred to as a **empty** set (or **null set**), which includes no elements. It is represented as ϕ (phi) or by the empty set notation $\{\}$.
- . The universe E is the most extensive set under examination, encompassing all other sets of interest.

Example 2.1

1. $\{\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}\}$ is a set containing five sets.
2. $\{x : x \in \{2, 3, 5\} \text{ and } x \leq 1\}$ is an empty set.

Definition 2.3

[3] [Cardinality] If a set K contains exactly n elements, where n is a non-negative integer, then K is classified as a finite set, and n is referred to as the cardinality of K .
Let $|A| = n$.

Example 2.2

1. A set of positive integers constitutes an infinite set.

$$2. K = \{1, 2, 7, 0\}, |A| = 4.$$

2.1.1 Application of Set Notation in conjunction with Quantifiers

At times, we explicitly limit the domain of a quantified statement through the application of set notation.

- We utilize $\forall x \in K, P(x)$ to signify that the property $P(x)$ is true for **all** elements x within the set K .
- We utilize $\exists x \in K, P(x)$ to signify that the property $P(x)$ is true for at least one element x within the set K .

2.1.2 Relationships Between Elements and Sets

Definition 2.4

[8] [Equality of Sets] Two sets K and S are equal if every element of K is an element of the set S , and note that every element within S is also an element of K . is represented as: $K = S$. Formally, this indicates:

$$K = S \Leftrightarrow \forall x, x \in K \Leftrightarrow x \in S.$$

Example 2.3

1. $\{1, 2, 3\} = \{2, 1, 3\}.$
2. $\{1, 2, 3, 4, 5\} = \{x \in \mathbb{N} : x \leq 5\}.$
3. $\{x \in \mathbb{R} : x^2 + 1 = 0\} = \phi.$

Definition 2.5 (Subset)

- A set K is a **subset** of S if and only if every element of K is contained within
-

S. We denote $K \subseteq S$ to signify that K is a subset of S , indicating that K is contained within S . Formally,

$$K \subseteq S \Leftrightarrow \forall x : x \in K \Rightarrow x \in S.$$

- A set K is considered a proper subset of a set S , denoted $K \subset S$, if it satisfies the conditions $K \subseteq S$ and $K \neq S$.
- To assert that K is **not** a subset of S , we utilize the negation of the statement $\forall x : x \in K \Rightarrow x \in S$, which can be expressed as follows:

$$\overline{\forall x : x \in K \Rightarrow x \in S} \Leftrightarrow \exists x : (x \in K) \wedge (x \notin S).$$

Therefore,

$$K \not\subseteq S \Leftrightarrow \exists x : (x \in K) \wedge (x \notin S).$$

Example 2.4

1. $\{12, 43, 66\} \subseteq \{12, 43, 66\}$.
2. $\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$.

Definition 2.6 (*Power Set*)

The power set of a set K comprises all possible subsets of K . The power set is represented as $\mathcal{P}(K)$. Formally, we define the power set of K as follows:

$$\mathcal{P}(K) = \{M \mid M \subseteq K\}.$$

Example 2.5

Determine the power set for the following sets:

$$\phi, \{1\}, \{1, 2\}.$$

If K is a set with $|K| = n$, then the cardinality of the power set of K , denoted $|\mathcal{P}(K)|$, is 2^n .

1. $\mathcal{P}(\phi) = \{\phi\}$ and $|\mathcal{P}(\phi)| = 1$.
2. $\mathcal{P}(\{1\}) = \{\phi, \{1\}\}$ and $|\mathcal{P}(\{1\})| = 2$.
3. $\mathcal{P}(\{1, 2\}) = \{\phi, \{1\}, \{2\}, \{1, 2\}\}$ and $|\mathcal{P}(\{1, 2\})| = 4$.

Proposition 2.1

If K is a set with $|K| = n$, then it follows that $|\mathcal{P}(K)| = 2^n$.

2.1.3 Set Operations

Definition 2.7 (*Set Intersection*)

The intersection of the sets K and S comprises all members that belong to both K and S . We denote the intersection of K and S as $K \cap S$. In a formal manner:

$$K \cap S = \{x, (x \in K) \wedge (x \in S)\}.$$

Example 2.6

$$\{1, 2, 3, 4\} \cap \{-3, 4, 5\} = \{4\}.$$

Definition 2.8 (*Set Disjoint*)

Two sets K and S are disjoint if $K \cap S = \phi$.

Example 2.7

$$\{2, 4, 6\} \cap \{8, 10, 12\} = \phi, \text{ so they are disjoint.}$$

Definition 2.9 (*Set Union*)

The union of two sets K and S , represented as $K \cup S$, comprises precisely all members that belong to either K or S (or to both). In a formal manner:

$$K \cup S = \{x, (x \in K) \vee (x \in S)\}.$$

Example 2.8

$$\text{Let } K = \{0, 1, 2, 3, 6\}, S = \{0, 1, 2, 4, 6, 9\}, K \cup S = \{0, 1, 2, 3, 4, 6, 9\}.$$

Lemma 2.1 (*Cardinality of Intersection and Union*)

For any two sets K and S , it holds that

$$|K \cup S| = |K| + |S| - |K \cap S|.$$

Lemma 2.2

For any sets K and S ,

1. $(K \cap S) \subseteq K$ and $(K \cap S) \subseteq S$
2. $K \subseteq (K \cup S)$ and $S \subseteq (K \cup S)$

Proposition 2.2

For any subsets K, S , and C of the universe E , the following statements hold true.

1. Commutative laws: $K \cap S = S \cap K, K \cup S = S \cup K$.
2. Associative laws: $(K \cap S) \cap C = S \cap (K \cap C), (K \cup S) \cup C = S \cup (K \cup C)$.
3. Distributive laws: $K \cap (S \cup C) = (K \cap S) \cup (K \cap C), K \cup (S \cap C) = (K \cup S) \cap (K \cup C)$.
4. Identity laws: $K \cup \phi = K, K \cap E = K$.
5. Domination laws: $K \cap \phi = \phi, K \cup E = E$.
6. Idempotent laws: $K \cup K = K, K \cap K = K$.

Definition 2.10 (*Set Difference*)

The difference between sets K and S is defined as the set of elements that belong to K but not to S . Formally:

$$K - S = \{x, (x \in K) \wedge (x \notin S)\}.$$

Example 2.9

$$\{1, 2, 3\} - \{3, 4, 5\} = \{1, 2\}.$$

Proposition 2.3

Let K and S subsets of the universe M

- $K - S \subset K$
- $K - K = \phi, K - \phi = K, \phi - K = \phi.$

Definition 2.11 (Set Complement)

Let K be a subset of the universal set M . The complement of set K relative to E , written as C_M^K , CK , or $\bar{K}$, comprises precisely those items that are absent from K . In a formal manner:

$$\bar{K} = C_M^K = M - K = \{x/x \in M \wedge x \notin K\}.$$

Example 2.10

Let the universe be denoted as $\mathbb{R}$.

1. $\overline{\{0\}} = \{x, x \neq 0\} = \mathbb{R}^*.$
2. $\overline{\mathbb{R}^-} = \{x, x \succ 0\} = (\mathbb{R}^+)^*.$

Proposition 2.4

Let K and S subsets of the universe E .

- $\bar{M} = C_M^M = \phi.$
- $\bar{\phi} = C_M^\phi = E.$
- $C_M^{C_M^K} = \bar{\bar{K}} = K.$
- $K \cup \bar{K} = M.$
- $K \cap \bar{K} = \phi.$
- $K \subset S \Rightarrow \bar{S} \subset \bar{K}.$
- $\overline{K \cap S} = \bar{K} \cup \bar{S}.$

$$\circ \overline{K \cup S} = \bar{K} \cap \bar{S}.$$

Definition 2.12 (Set Symmetric Difference)

The symmetric difference of set K and set S , represented as $K \Delta S$, comprises the elements that are present in either K or S , but not in both. In a formal manner:

$$\begin{aligned} K \Delta S &= \{x, (x \in K \wedge x \notin S) \vee (x \in S \wedge x \notin K)\} \\ &= (K - S) \cup (S - K). \end{aligned}$$

Proposition 2.5

Let K and S subsets of the universe E .

- $K \Delta S = S \Delta K$
 - $K \Delta \phi = K$
 - $K \Delta M = M - K$
 - $K \Delta K = \phi$
 - $K \Delta S = (K \cup S) - (K \cap S)$
 - $K \Delta S = \bar{K} \Delta \bar{S}$
-

Example 2.11

If $K = \{1, 2, 3, 4, 5, 10\}$ and $S = \{0, 1, 2, 3, 4, 5, 7, 8, 9\}$, then $K - S = \{10\}$ and $S - K = \{0, 7, 8, 9\}$. Hence $K \Delta S = \{0, 7, 8, 9, 10\}$.

Definition 2.13 (Set Cartesian Product)

The Cartesian product of the sets K and S , represented as $K \times S$, comprises all ordered pairs (x_1, x_2) such that $x_1 \in K$ and $x_2 \in S$:

$$K \times S = \{(x_1, x_2) \mid x_1 \in K, x_2 \in S\}$$

The equality in $K \times S$ is defined as: $(x_1, y_1) = (x_2, y_2) \Leftrightarrow x_1 = x_2 \wedge y_1 = y_2$. The Cartesian product of n sets $K_1, K_2, \dots, K_n$, called $K_1 \times K_2 \times \dots \times K_n$, is defined as

the set of ordered tuples: $(x_1, x_2, \dots, x_n)$ where $x_1 \in K_1, x_2 \in K_2, \dots, x_n \in K_n$. That is:

$$K_1 \times K_2 \times \dots \times K_n = \{(x_1, x_2, \dots, x_n) / x_1 \in K_1, x_2 \in K_2, \dots, x_n \in K_n\}.$$

$$K_1 \times K_2 \times \dots \times K_n = A \Rightarrow K_1 \times K_2 \times \dots \times K_n = K^n.$$

Example 2.12

Let $K = \{-2, 3\}$ and $S = \{0, -4, 2\}$

1. $K \times S = \{(-2, 0), (-2, -4), (-2, 2), (3, 0), (3, -4), (3, 2)\}$
2. $S \times K = \{(0, -2), (0, 3), (-4, -2), (-4, 3), (2, -2), (2, 3)\}$
3. $K \times S \neq S \times K$

Lemma 2.3 (*Cardinality of Cartesian Product*)

In general, if K_i 's are finite sets, we have:

$$|K_1 \times K_2 \times \dots \times K_n| = |K_1| \times |K_2| \times \dots \times |K_n|.$$

Proposition 2.6

If K and S are any sets, then

- $K \times S \neq S \times K$.
- $K \times \phi = \phi \times K = \phi$.

Example 2.13

The sets $K = \{1, 2, x\}$, $S = \{3, 4, y\}$, are given. Determine x and y , knowing that $\{1, 3\} \times \{2, 4\} \subseteq K \times S$

We form the sets $K \times S$ and $C = \{1, 3\} \times \{2, 4\}$.

$$K \times S = \{(1, 3), (1, 4), (1, y), (2, 3), (2, 4), (2, y), (x, 3), (x, 4), (x, y)\},$$

$$C = \{(1, 2), (1, 4), (3, 2), (3, 4)\}.$$

Because $\{1, 3\} \times \{2, 4\} \subseteq K \times S$, we obtain

$$(1, 2) \in C \Rightarrow (1, 2) \in K \times S \Rightarrow (1, 2) = (1, y) \Rightarrow y = 2,$$

$$(3, 4) \in C \Rightarrow (3, 4) \in K \times S \Rightarrow (3, 4) = (x, 4) \Rightarrow x = 3,$$

for $x = 3$ and $y = 2$, we have $(3, 4) \in K \times S$. Therefore: $x = 3$ and $y = 2$.

2.2 Relations

2.2.1 Binary Relation

Definition 2.14

A binary relation on a nonempty set K is a predicate R defined for ordered pairs (x, y) where $x, y \in K$. A binary relation on K is defined as a subset $R \subseteq K \times K$, comprising all pairs (x, y) for which the relation is true.

The graph of the relation R on K is defined as the subset $G \subseteq K \times K$ comprising precisely those pairs (x, y) for which $R(x, y)$ holds true. Conversely, for any graph $G \subseteq K \times K$, a relation R on K is defined by the criterion:

$$R(x, y) \text{ is true if and only if } (x, y) \in G.$$

Corollary 2.1

Let R be a binary relation defined on the set K . We define the relation xRy such that $(x, y) \in G$.

Example 2.14

$$1. \forall x, y \in \mathbb{R}, xRy \Leftrightarrow x \geq y.$$

$$2. \forall x, y \in \mathbb{R}, xRy \Leftrightarrow |x| = |y|.$$

Properties of a Relation

Let R be a binary relation defined on a set K .

- **Reflexive:** R is reflexive if and only if

$$\forall x \in K, xRx.$$

- **Symmetric:** R is symmetric if and only if

$$\forall x, y \in K, xRy \Rightarrow yRx.$$

- **Antisymmetric:** R is antisymmetric if and only if

$$\forall x, y \in K, (xRy) \wedge (yRx) \Rightarrow x = y.$$

- **Transitive:** R is transitive if and only if

$$\forall x, y, z \in K, (xRy) \wedge (yRz) \Rightarrow xRz.$$

2.2.2 Equivalence Relation

Definition 2.15

An equivalence relation is defined as a relation that satisfies the properties of reflexivity, symmetry, and transitivity.

Example 2.15

The equality relation $=$ on any set E is an equivalence relation.

- *Reflexive: $x = x$ for all x*
- *Symmetric: if $x = y$ then $y = x$*
- *Transitive: if $x = y$ and $y = z$ then $x = z$*

2.2.3 Equivalences and Partitions

Definition 2.16 (*Equivalences Partitions*)

*Given a partition $K_1, K_2, K_3, \dots$ of a set K , two elements x and y of K are said to be **equivalent**, with respect to that partition, if they belong to the same subset specified*

by the partition.

Definition 2.17 (Equivalence Class)

Let R be an equivalence relation on a set K . For any $x \in K$, the equivalence class of x is defined as the set

$$x = [x] = \{y \in K \mid xRy\}.$$

The set $[x]$ comprises all elements of K that are associated with x through the relation R .

Proposition 2.4

If R is an equivalence relation on K , then each $x \in A$ is a member of precisely one equivalence class.

Theorem 2.5

If R is an equivalence relation on a set K , then the set of equivalence classes forms a partition of K . Conversely, if P is a partition of K , then the relation defined by

$$xRy \Leftrightarrow \exists S \in P : x, y \in S$$

is an equivalence relation, with its equivalence classes corresponding to the elements of the partition.

Example 2.16

$\forall x, y \in \mathbb{R}, xRy \Leftrightarrow x^2 - x = y^2 - y$, R is the equivalence relation because:

1. R is reflexive

$$\begin{aligned} \forall x \in \mathbb{R} : xRx &\Leftrightarrow x^2 - x = x^2 - x \\ &\Leftrightarrow R \text{ is reflexive.} \end{aligned}$$

2. R is symmetric

$$\begin{aligned} \forall x, y \in \mathbb{R}, xRy &\Leftrightarrow x^2 - x = y^2 - y \\ &\Leftrightarrow y^2 - y = x^2 - x \\ &\Leftrightarrow R \text{ is symmetric.} \end{aligned}$$

3. R is transitive

$$\begin{aligned}\forall x, y, z \in \mathbb{R}, xRy \wedge yRz &\Leftrightarrow (x^2 - x = y^2 - y) \wedge (y^2 - y = z^2 - z) \\ &\Leftrightarrow x^2 - x = z^2 - z \\ &\Leftrightarrow R \text{ is transitive}\end{aligned}$$

-The equivalence class $0, 1, 2$.

$$\begin{aligned}\dot{0} &= \{y \in A, 0Ry\}, \\ 0Ry &\Leftrightarrow y^2 - y = 0 \\ &\Rightarrow \dot{0} = \{0, 1\}. \\ \dot{1} &= \{y \in A, 1Ry\}, \\ 1Ry &\Leftrightarrow y^2 - y = 0 \\ &\Rightarrow \dot{1} = \{0, 1\}. \\ \dot{2} &= \{y \in A, 2Ry\}, \\ 2Ry &\Leftrightarrow y^2 - y = 2 \\ &\Rightarrow \dot{2} = \{-1, 2\}.\end{aligned}$$

2.2.4 Order Relation

Definition 2.18

A binary relation R on a set K is classified as an order relation if it satisfies the properties of reflexivity, anti-symmetry, and transitivity.

Example 2.17

We define on $\mathbb{R}$ the relation T by:

$$xTy \Leftrightarrow x^3 - y^3 \leq 0.$$

Show that T is an order relation. To prove T is an order relation (partial order), we must show it is reflexive, antisymmetric, and transitive.

(a) Reflexivity: We must show xTx for all $x \in \mathbb{R}$.

$$xTx \Leftrightarrow x^3 - x^3 \leq 0$$

$$\Leftrightarrow 0 \leq 0 \quad \checkmark$$

This is true, so T is reflexive.

(b) Antisymmetry: We must show that if xTy and yTx , then $x = y$.

Assume:

$$xTy : x^3 - y^3 \leq 0 \quad \dots(1)$$

$$yTx : y^3 - x^3 \leq 0 \quad \dots(2)$$

From (1): $x^3 \leq y^3$

From (2): $y^3 \leq x^3$

Combining these: $x^3 \leq y^3$ and $y^3 \leq x^3$, which gives $x^3 = y^3$.

Since the function $f(t) = t^3$ is strictly increasing on $\mathbb{R}$ (injective), we have:

$$x^3 = y^3 \Rightarrow x = y$$

Therefore, T is antisymmetric.

(c) Transitivity: We must show that if xTy and yTz , then xTz .

Assume:

$$xTy : x^3 - y^3 \leq 0 \quad \Rightarrow x^3 \leq y^3 \quad \dots(1)$$

$$yTz : y^3 - z^3 \leq 0 \quad \Rightarrow y^3 \leq z^3 \quad \dots(2)$$

From (1) and (2), by transitivity of $\leq$ on real numbers: $x^3 \leq y^3 \leq z^3 \Rightarrow x^3 \leq z^3$

Therefore: $x^3 - z^3 \leq 0$

This means xTz , so T is transitive.

T is an order relation on $\mathbb{R}$

Note: This is actually a total order because for any $x, y \in \mathbb{R}$, either $x^3 \leq y^3$ or $y^3 \leq x^3$ (since $\mathbb{R}$ is totally ordered and $f(t) = t^3$ is strictly increasing). In fact, $xTy \Leftrightarrow x \leq y$.

Definition 2.19 (Total Order Relation)

A order relation R on a set A is called an **total order relation** if it satisfies one additional proposition

$$\forall x \in A, \forall y \in A : (xRy) \vee (yRx).$$

Definition 2.20 (Partial Order Relation)

If the relation is not total order relation, then is **partial order relation**.

Example 2.18

$\forall x, y \in \mathbb{R}, xRy \Leftrightarrow x \leq y$, is total order relation.

1. R is reflexive:

$$\forall x \in \mathbb{R}, x \leq x \Leftrightarrow xRx.$$

2. R is anti-symmetric:

$$\forall x, y \in \mathbb{R}, ((xRy) \wedge (yRx)) \Leftrightarrow ((x \leq y) \wedge (y \leq x)) \Rightarrow x = y.$$

3. R is transitive:

$$\begin{aligned} \forall x, y, z \in \mathbb{R}, ((xRy) \wedge (yRz)) &\Leftrightarrow ((x \leq y) \wedge (y \leq z)) \\ &\Leftrightarrow y \leq z \Rightarrow x \leq z \Leftrightarrow xRz. \end{aligned}$$

4. R is total order relation, because $\forall x, y \in \mathbb{R}, x \leq y$ or $y \leq x$.

2.3 Applications

Definition 2.21

[3] [Function] A function from a set M (the domain) to a set W (the codomain), represented as $g : E \rightarrow F$, is a rule that assigns to each element $x \in M$ precisely one unique member $y \in F$, denoted as xgy . We typically express this relationship between x and y as $y = g(x)$.

Definition 2.22

It is asserted that:

1. y is the **image** of x (under g).
2. x is the **pre-image** of y (under g).
3. g **maps** x onto y , and symbolize this statement by $x \xrightarrow{g} y$.
4. M is the **starting** set of g .
5. F is the **arrival set or codomain** of g .
6. G_g is the **graph** of the function g , given by:

$$G_g = \{(x, y) \in M \times F / y = g(x)\}.$$

Note that

$$\begin{aligned} g : M &\rightarrow F \\ x &\rightarrow y \end{aligned}$$

Definition 2.23 (Domain Definition)

The domain D_g of a function g from M to F comprises the elements $x \in M$ for which there exists a unique element $y \in F$ such that $y = g(x)$.

Definition 2.24

[3] [Application] An application g is a function of M in F whose domain definition D_g is equal to M .

Definition 2.25 (Application)

An application from a set M to another set F is a relation that assigns a unique element $y \in F$ to every element $x \in M$. In a formal manner, utilizing predicate logic:

$$g \text{ Application} \Leftrightarrow \begin{cases} 1) & \forall x \in M, \exists y \in F : y = g(x) \\ 2) & \forall x_1, x_2 \in M : x_1 = x_2 \Rightarrow g(x_1) = g(x_2). \end{cases}$$

We can write also,

$$g \text{ Application} \Leftrightarrow \forall x \in M, \exists! y \in F : y = g(x).$$

2.3.1 Image and Inverse Image of Subsets

Image of a Subset

Definition 2.26 (*Image of a Subset*)

Let $g : M \rightarrow F$ and consider the subset $K \subset M$. The **image** of the subset K under g , which we write $g(K)$, is the subset of F that consists of the images of the elements of K .

$$g(K) = \{g(x), x \in K\} \subset F.$$

$$y \in g(K) \Leftrightarrow \exists x \in K, y = g(x).$$

Inverse Image of a Subset

Definition 2.27 (*Inverse Image of a Subset*)

Let $g : M \rightarrow F$ and consider the subset $S \subset F$. The inverse image of the subset S under g , denoted $g^{-1}(S)$, is the subset of M comprising the pre-images of elements in S :

$$g^{-1}(S) = \{x \in M, g(x) \in S\} \subset M.$$

Example 2.19

Let $M = \{1, 2, 3, 4\}$ and $F = \{a, b, c\}$ and define an application $g : M \rightarrow F$ such that

$$g(1) = g(2) = a, \quad g(3) = g(4) = c.$$

Let $K \subset M, K = \{1, 2, 3\}$. Then $g(K) = \{a, c\}$. Also for example

$$g^{-1}(\{b\}) = \phi, g^{-1}(\{a, c\}) = K, g^{-1}(\{b, c\}) = \{3, 4\}.$$

Proposition 2.7

Consider an application $g : M \rightarrow F$, where K, S are subsets of M and C, D are subsets of F . The following properties are established:

1. $g(K \cap S) \subseteq g(K) \cap g(S)$.

2. $g(K \cup S) = g(K) \cup g(S)$.

3. $g^{-1}(K \cap S) = g^{-1}(K) \cap g^{-1}(S).$
4. $g^{-1}(K \cup S) = g^{-1}(K) \cup g^{-1}(S).$
5. $K \subseteq S \Rightarrow g(K) \subseteq g(S).$
6. $C \subseteq D \Rightarrow g^{-1}(C) \subseteq g^{-1}(D).$
7. $g^{-1}(\bar{C}) = \overline{g^{-1}(C)}.$

Example 2.20

Let $g(x) = 1 + x^2$. Find the following:

1. $g(\{-1, 1\}) = \{2\}.$
2. $g([-2, 2]) = g([-2, 0] \cup [0, 2]) = g([-2, 0]) \cup g([0, 2]) = [1, 5].$
3. $g([-2, 3]) = g([-2, 0] \cup [0, 3]) = g([-2, 0]) \cup g([0, 3]) = [1, 10].$
4. $g^{-1}(\{1, 5, 10\}) = \{-3, -2, 0, 2, 3\}.$
5. $g^{-1}([0, 1]) = \{0\}.$
6. $g^{-1}([2, 5]) = [-2, -1] \cup [1, 2].$

2.3.2 Type of Application

Injective Application

Definition 2.28

An injective function $g : M \rightarrow F$ is defined such that each element in the range corresponds to a unique element in the domain. In a formal context:

$$g \text{ is injective } \begin{cases} \forall x_1, x_2 \in M : x_1 \neq x_2 \Rightarrow g(x_1) \neq g(x_2), \\ \forall x_1, x_2 \in M : g(x_1) = g(x_2) \Rightarrow x_1 = x_2. \end{cases}$$

Example 2.21

Consider an application $g : \mathbb{R} \rightarrow \mathbb{R}$, $g(x) = 4x - 1$, we get

$$g(x_1) = g(x_2) \Rightarrow 4x_1 - 1 = 4x_2 - 1$$

$$\Rightarrow 4x_1 = 4x_2$$

$$\Rightarrow x_1 = x_2.$$

Therefore g is injective.

Remark 2.6

g is not injective $\Leftrightarrow \exists x_1, x_2 \in M : x_1 \neq x_2 \wedge g(x_1) = g(x_2)$.

Example 2.22

Consider an application $g : \mathbb{R} \rightarrow \mathbb{R}$, $g(x) = |x|$, by example

$$\exists x_1 = 2 \text{ and } x_2 = -2 : x_1 \neq x_2 \wedge g(x_1) = g(x_2) = 2.$$

Therefore g is not injective.

Surjective Application

Definition 2.29

An application $g : M \rightarrow W$ is surjective if and only if for every element $y \in W$, there exists an element $x \in M$ such that $y = g(x)$. This can be expressed as follows:

$$\forall y \in W, \exists x \in M : y = g(x).$$

Example 2.23

Reconsider the function $g : \mathbb{R} \rightarrow \mathbb{R}$ defined by $g(x) = 4x - 1$. We seek to determine if every element of $\mathbb{R}$ possesses a pre-image.

$$g(x) = y \Rightarrow 4x - 1 = y$$

$$\Rightarrow 4x = y + 1$$

$$\Rightarrow x = \frac{y + 1}{4} \in \mathbb{R}.$$

Consequently, g is surjective.

Remark 2.7

g is not surjective $\Leftrightarrow \exists y \in F, \forall x \in M : y \neq g(x)$.

Example 2.24

Consider an application $g : \mathbb{R}^* \rightarrow \mathbb{R}$, $g(x) = \frac{1}{x}$, by example

$$\exists y = 0, \forall x \in M \text{ where } \frac{1}{x} \neq 0.$$

Therefore g is not surjective.

Bijjective Application

Definition 2.30

An application $g : M \rightarrow W$ is **bijjective** if and only if it is both injective and surjective.

Definition 2.31

An application $g : M \rightarrow F$ is **bijjective** if and only if for every element $y \in W$, there is a unique element $x \in M$ with $y = g(x)$

$$\forall y \in W, \exists! x \in M : y = g(x).$$

Inverse Application

Definition 2.32

Let $g : M \rightarrow W$ be bijective application, then the **inverse application** of g is defined by $g^{-1}(y)$, which is the unique element $x \in M$ such that $g(x) = y$. Formally:

$$\begin{aligned} g^{-1} : F &\rightarrow M \\ y &\rightarrow g^{-1}(y) = x. \end{aligned}$$

Example 2.25

Consider the function $g : M \rightarrow W$ defined by $g(x) = 4x - 1$, which was examined in the preceding example. Since it is both injective and surjective, it follows that it is a bijection.

$$\begin{aligned} y = g(x) &\Leftrightarrow y = 4x - 1 \\ &\Leftrightarrow 4x = y + 1 \\ &\Leftrightarrow x = \frac{y + 1}{4} \\ &\Leftrightarrow g^{-1}(y) = \frac{y + 1}{4}. \end{aligned}$$

Proposition 2.8

Let $g : M \rightarrow W$ be a bijective function, then $g^{-1} : W \rightarrow M$ is also a bijective function.

2.3.3 Characteristics of an Applications**Equality-Extension-Restriction****Definition 2.33 (*Equality of Applications*)**

Two applications. Let $g = (K, S, R)$ and $h = (C, D, S)$. They are considered equal if and only if the following conditions are met: the domains are identical, i.e., $K = C$; the codomains are the same, i.e., $S = D$; and the graphics are equivalent, i.e., $G_g = G_h$. If $g, h : K \rightarrow S$, the equality $g = h$ is equivalent to $g(x) = h(x)$ for all $x \in K$. In other words:

$$g = h \Leftrightarrow \forall x \in K, g(x) = h(x).$$

Definition 2.34 (*Extension of an Application-Restriction of an Application*)

Let $g : X \rightarrow Y$ be a function, where K and S are sets with $X \subseteq K$ and $Y \subseteq S$. An extension of g to K is a function $h : K \rightarrow S$ such that $g(x) = h(x)$ for all $x \in X$. Alternatively, h is an extension of g to K if g is the restriction of h to X .

Operations on Applications**Definition 2.35**

Let $g : M \rightarrow W$ and $h : M \rightarrow W$ be two functions with a common domain M , and let α be a real number. Let $g + h$, $g - h$, $\alpha \times g$, $g \times h$, and $g \div h$ ($h(x) \neq 0$) represent the corresponding operations from M to F .

a) $(g + h)(x) = g(x) + h(x).$

b) $(g - h)(x) = g(x) - h(x).$

c) $(\alpha \times g)(x) = \alpha \times g(x).$

d) $(g \times h)(x) = g(x) \times h(x).$

e) $(g \div h)(x) = g(x) \div h(x)$, provided $h(x) \neq 0$.

Composition of Applications

Definition 2.36

If $g : M \rightarrow W$ and $h : W \rightarrow G$, then the composite application $h \circ g$ is defined as the application of g followed by h , such that

$$\begin{aligned} h \circ g : M &\rightarrow G \\ x &\rightarrow (h \circ g)(x) = h(g(x)). \end{aligned}$$

Example 2.26

If $g(x) = -4x + 9$ and $h(x) = 2x - 7$, find $(g \circ h)(x)$ and $(h \circ g)(x)$

$$\begin{aligned} (g \circ h)(x) &= g(h(x)) \\ &= -4h(x) + 9 \\ &= -4(2x - 7) + 9 \\ &= -8x + 37. \end{aligned}$$

Thus, $(g \circ h)(x) = -8x + 37$.

$$\begin{aligned} (h \circ g)(x) &= h(g(x)) \\ &= 2g(x) - 7 \\ &= 2(-4x + 9) - 7 \\ &= -8x + 18 - 7 \\ &= -8x + 11. \end{aligned}$$

Thus, $(h \circ g)(x) = -8x + 11$.

It is noted that $(g \circ h)(x)$ and $(h \circ g)(x)$ yield distinct results.

Proposition 2.9

Suppose g and h are application $g : M \rightarrow W$ and $h : W \rightarrow G$,

1. Composition is not Commutative: $h \circ g \neq g \circ h$.
2. If g and h are both injective applications, then $h \circ g$ is injective applications.
3. If g and h are both surjective applications, then $h \circ g$ is surjective applications.
4. If g and h are both bijective applications, then $h \circ g$ is bijective applications.

Proposition 2.10

Application properties. Let $g : M \rightarrow W$ we have:

1. $K \subset S \Rightarrow g(K) \subset g(S)$.
2. $g(K \cup S) = g(K) \cup g(S)$.
3. $g(K \cap S) \subset g(K) \cap g(S)$.

Definition 2.37 (*Inverse Composite Applications*)

If $g : M \rightarrow F$ and $h : W \rightarrow M$, then g and h are inverse applications of one another relative to composition iff

$$h \circ g = I_M \text{ and } g \circ h = I_W.$$

Example 2.27

Show that the application $h : \mathbb{R} \rightarrow \mathbb{R}$ defined by $h(x) = \frac{x-1}{2}$ is an inverse for the application $g : \mathbb{R} \rightarrow \mathbb{R}$ defined by $g(x) = 2x + 1$.

$$h(g(x)) = \frac{g(x) - 1}{2} = \frac{2x + 1 - 1}{2} = x.$$

$$g(h(x)) = 2h(x) + 1 = 2\left(\frac{x-1}{2}\right) + 1 = x.$$

Thus h is inverse the g .

Proposition 2.11

Let $g : M \rightarrow W$ and $h : W \rightarrow G$

1. If g has an inverse, then it is unique.

2. The composition $h \circ g$ of two invertible applications g and h is invertible. Moreover, the composition of the inverses in the reverse order

$$(h \circ g)^{-1} = g^{-1} \circ h^{-1}.$$

Real-Valued Functions of a Real Variable: Properties of Functions

3.1 Characterization of Functions

3.1.1 Even and Odd Function

Definition 3.1

[1] [Even Function] The function g is defined as even if:

$$\forall x \in D_g, \quad g(-x) = g(x).$$

Definition 3.2 (Odd Function)

The function g is defined as odd if:

$$\forall x \in D_g, \quad g(-x) = -g(x).$$

3.1.2 Increasing and decreasing, monotonic function

Definition 3.3

[1] [Increasing] The function is **Increasing** over an interval, if for each x_a and x_b in the interval,

$$x_a < x_b, \text{ and } g(x_a) \leq g(x_b).$$

A function is a **Strictly Increasing** function over an interval, if for each x_a and x_b in the interval,

$$x_a < x_b, \text{ and } g(x_a) < g(x_b).$$

Definition 3.4 (Decreasing)

This function is **Decreasing** over an interval, if for each x_a and x_b in the interval,

$$x_a < x_b, \text{ and } g(x_a) \geq g(x_b).$$

A function is a **Strictly Decreasing** function over an interval, if for each x_a and x_b in the interval,

$$x_a < x_b, \text{ and } g(x_a) > g(x_b).$$

Definition 3.5

A function is defined for I is called **Monotonic** if **Increasing or Decreasing** for I .

A function is defined for I is called **Strictly Monotonic** if **Strictly Increasing or Strictly Decreasing** for I .

Bounded and periodic function:

Definition 3.6 (**Bounded**)

A function g defined for I is called **bounded** if there exists a real number M such that

$$|g(x)| \leq M.$$

Definition 3.7 (**Periodic**)

A function g defined for D_g , and α is number real strictly positive, we said the function g is **periodic** function with period is α , if

$$\forall x \in D_g : \begin{cases} x + \alpha \in D_g, \\ g(x + \alpha) = g(x). \end{cases}$$

3.2 Limit of Functions

3.2.1 Limit of a Function at a Point

Definition 3.8

For a real valued function $g : D \rightarrow \mathbb{R}$, L is the limit as x approaches x_* if, for any small positive number ε , there exists a positive number δ such that if x is different from x_* and lies between $x_* - \delta$ and $x_* + \delta$, then $g(x)$ lies between $L - \varepsilon$ and $L + \varepsilon$.

Formally

$$\forall \varepsilon \succ 0, \exists \delta \succ 0, \forall x \in D : x \in]x_* - \delta, x_* + \delta[\Rightarrow g(x) \in]L - \varepsilon, L + \varepsilon[,$$

or we can write

$$\forall \varepsilon \succ 0, \exists \delta \succ 0, \forall x \in D : |x - x_*| \prec \delta \Rightarrow |g(x) - L| \prec \varepsilon.$$

If a function $g(x)$ has a limit value L as x approaches a finite value x_* , we write:

$$\lim_{x \rightarrow x_*} g(x) = L.$$

Example 3.1

Show that $\lim_{x \rightarrow 4} (2x - 1) = 7$. We have $g(x) = 2x - 1$, $x_* = 4$ and $L = 7$ and the question we must answer is "how close should x be to 4 if want to be sure that $g(x) = 2x - 1$ differs less than ε from $L = 7$?"

To figure this out we try to get an idea of how big $|g(x) - L|$ is:

$$|g(x) - L| = |(2x - 1) - 7| = 2|x - 4|.$$

So, if $2 \times |x - x_*| < \varepsilon$ then we have

$$|g(x) - L| < \varepsilon, \text{ i.e. } |x - x_*| < \frac{\varepsilon}{2} \Rightarrow |g(x) - L| < \varepsilon.$$

We can therefore choose $\delta = \frac{\varepsilon}{2}$. No matter what $\varepsilon > 0$ we are given our δ will also be positive,

$$\forall \varepsilon \succ 0, \exists \delta \succ 0, \forall x \in D : |x - 4| \prec \delta \Rightarrow |(2x - 1) - 7| \prec \varepsilon.$$

That shows that $\lim_{x \rightarrow 4} g(x) = 7$

Definition 3.9 (Left Limit and Right Limit)

a) We say that g has the left limit $L \in \mathbb{R}$ as x tends to x_* iff

$$\forall \varepsilon \succ 0, \exists \delta \succ 0, \forall x \in D : x \in]x_* - \delta, x_*[\Rightarrow g(x) \in]L - \varepsilon, L + \varepsilon[,$$

and in that case we write:

$$\lim_{x \rightarrow x_*^-} g(x) = L \text{ or } \lim_{x \rightarrow x_*^+} g(x) = L.$$

b) We say that g has the right limit $L \in \mathbb{R}$ as x tends to x_* iff

$$\forall \varepsilon > 0, \exists \delta > 0, \forall x \in D : x \in]x_*, x_* + \delta[\Rightarrow g(x) \in]L - \varepsilon, L + \varepsilon[,$$

and in that case we write:

$$\lim_{x \rightarrow x_*^-} g(x) = L \text{ or } \lim_{x \rightarrow x_*^+} g(x) = L.$$

Theorem 3.1 (*Existence of a Limit*)

Let g be a real valued function defined on a set $D \subset \mathbb{R}$, then $\lim_{x \rightarrow x_*} g(x)$ exists if and only if:

$$\lim_{x \rightarrow x_*^-} g(x) = \lim_{x \rightarrow x_*^+} g(x) = \lim_{x \rightarrow x_*} g(x).$$

Example 3.2

Let $g : [-1, 1] \rightarrow \mathbb{R}$ defined by:

$$g(x) \begin{cases} 0, & -1 \leq x \leq 0. \\ 1, & 0 < x \leq 1. \end{cases}$$

Then $\lim_{x \rightarrow 0} g(x)$ does not exist because $\lim_{x \rightarrow 0^-} g(x) = 0$ and $\lim_{x \rightarrow 0^+} g(x) = 1$.

3.2.2 Limit of a Function at Infinity

Definition 3.10

A function g has limit value L as x becomes large if one can demonstrate that for any positive number ε there exists a positive number A , such that all the values $g(x)$ of the function lies between $L - \varepsilon$ and $L + \varepsilon$ for $x > A$.

We write: $\lim_{x \rightarrow +\infty} g(x) = L$. One can similarly define the notion of a limit as x becomes large and negative: $\lim_{x \rightarrow -\infty} g(x) = L$.

Example 3.3

Let's compute

$$\lim_{x \rightarrow \infty} \frac{5x^2 + 1}{4x^2 + 3x - 2}.$$

We divide the numerator and denominator by x^2 , and you get

$$\lim_{x \rightarrow \infty} \frac{5x^2 + 1}{4x^2 + 3x - 2} = \lim_{x \rightarrow \infty} \frac{5 + \frac{1}{x^2}}{4 + \frac{3}{x} - \frac{2}{x^2}} = \frac{5}{4}.$$

Example 3.4

Compute

$$\lim_{x \rightarrow \infty} \frac{x}{x^5 - 2}.$$

We divide numerator and denominator by x^5 . This leads to

$$\lim_{x \rightarrow \infty} \frac{x}{x^5 - 2} = \lim_{x \rightarrow \infty} \frac{\frac{1}{x^4}}{1 - \frac{2}{x^5}} = \frac{0}{1} = 0.$$

3.2.3 Properties of a Limit

The following properties remain true if one replaces each limit by a one-sided limit, or a limit for $x \rightarrow \infty$.

Let g and h be two given functions whose limits for $x \rightarrow x_*$ we know,

$$\lim_{x \rightarrow x_*} g(x) = L_1, \lim_{x \rightarrow x_*} h(x) = L_2.$$

Then:

1.

$$\lim_{x \rightarrow x_*} (g + h)(x) = L_1 + L_2.$$

2.

$$\lim_{x \rightarrow x_*} (g \times h)(x) = L_1 \times L_2.$$

3.

$$\lim_{x \rightarrow x_*} (\lambda \times g)(x) = \lambda \times L_1.$$

4.

$$\lim_{x \rightarrow x_*} \frac{g(x)}{h(x)} = \frac{L_1}{L_2}, \text{ if } \lim_{x \rightarrow x_*} h(x) \neq 0.$$

Theorem 3.2

Suppose that

$$g(x) \leq h(x) \leq k(x).$$

(for all x) and that

$$\lim_{x \rightarrow x_*} g(x) = \lim_{x \rightarrow x_*} k(x).$$

Then

$$\lim_{x \rightarrow x_*} g(x) = \lim_{x \rightarrow x_*} h(x) = \lim_{x \rightarrow x_*} k(x).$$

Corollary 3.1

If $\lim_{x \rightarrow x_*} g(x) = 0$ and h is a bounded function. Then

$$\lim_{x \rightarrow x_*} (g \times h)(x) = 0.$$

Example 3.5

Compute

$$\lim_{x \rightarrow 0} x^2 \times \sin\left(\frac{1}{x}\right).$$

Since we have

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1.$$

Then

$$-x^2 \leq x^2 \times \sin\left(\frac{1}{x}\right) \leq x^2.$$

Since

$$\lim_{x \rightarrow 0} x^2 = \lim_{x \rightarrow 0} -x^2 = 0.$$

The corollary tells us that

$$\lim_{x \rightarrow 0} x^2 \times \sin\left(\frac{1}{x}\right) = 0.$$

3.2.4 Indeterminate Forms

Definition 3.11

A function g is said to have an indeterminate form at x_* (where x_* can be finite or infinite) if:

1. g is continuous on an interval including x_* , except possibly at x_* .
2. When we try to evaluate g at x_* we obtain one of the following forms:

$$\frac{0}{0}, \frac{\infty}{\infty}, 0 \times \infty, \infty - \infty.$$

Here 0 and ∞ represent **variable** quantities approaching the respective value, **NOT** constants with that value.

Example 3.6

By factoring

$$\begin{aligned} \lim_{x \rightarrow -1} \frac{x^2 - x - 2}{x^2 - 2x - 3} &= \frac{0}{0} \\ \lim_{x \rightarrow -1} \frac{x^2 - x - 2}{x^2 - 2x - 3} &= \lim_{x \rightarrow -1} \frac{(x-2)(x+1)}{(x-3)(x+1)} = \lim_{x \rightarrow -1} \frac{(x-2)}{(x-3)} \\ &= \frac{3}{4}. \end{aligned}$$

Example 3.7

By the conjugate

$$\begin{aligned} \lim_{x \rightarrow 4} \frac{\sqrt{x} - 2}{x - 4} &= \frac{0}{0} \\ \lim_{x \rightarrow 4} \frac{\sqrt{x} - 2}{x - 4} &= \lim_{x \rightarrow 4} \frac{(\sqrt{x} - 2)(\sqrt{x} + 2)}{(x - 4)(\sqrt{x} + 2)} = \lim_{x \rightarrow 4} \frac{(x - 4)}{(x - 4)(\sqrt{x} + 2)} \\ \lim_{x \rightarrow 4} \frac{1}{(\sqrt{x} + 2)} &= \frac{1}{4}. \end{aligned}$$

Example 3.8

By trigonometric identities

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{\sin(x)}{\sin(2x)} &= \frac{0}{0} \\ \lim_{x \rightarrow 0} \frac{\sin(x)}{\sin(2x)} &= \lim_{x \rightarrow 0} \frac{\sin(x)}{2 \sin(x) \cos(x)} = \lim_{x \rightarrow 0} \frac{1}{2 \cos(x)} \\ \lim_{x \rightarrow 0} \frac{1}{2 \cos(x)} &= \frac{1}{2}. \end{aligned}$$

3.2.5 Ways to Solve Limits

Euclidean Division

$\lim_{x \rightarrow 2} \frac{-3x^2 + 4x + 4}{x^2 - 5x + 6} = \frac{0}{0}$, by Euclidean division, we get

$$\begin{array}{r|l} \begin{array}{r} -3x^2 + 4x + 4 \\ -(-3x^2 + 6x + 0) \\ \hline -2x + 4 \\ -(-2x + 4) \\ \hline 0 \end{array} & \begin{array}{l} \frac{x-2}{-3x-2}, \end{array} \end{array} \quad \begin{array}{r|l} \begin{array}{r} x^2 - 5x + 6 \\ -(x^2 - 2x + 0) \\ \hline -3x + 6 \\ -(-3x + 6) \\ \hline 0 \end{array} & \begin{array}{l} \frac{x-2}{x-3} \end{array} \end{array}$$

Then $-3x^2 + 4x + 4 = (x - 2)(-3x - 2)$ and $x^2 - 5x + 6 = (x - 2)(x - 3)$, We get

$$\lim_{x \rightarrow 2} \frac{-3x^2 + 4x + 4}{x^2 - 5x + 6} = \lim_{x \rightarrow 2} \frac{(x - 2)(-3x - 2)}{(x - 2)(x - 3)} = \lim_{x \rightarrow 2} \frac{(-3x - 2)}{(x - 3)} = \frac{-8}{-1} = 8.$$

Analysis method $\lim_{x \rightarrow 2} \frac{-3x^2 + 4x + 4}{x^2 - 5x + 6} = \frac{0}{0}$. Then

$$\Delta_1 = 16 + 48 = 64$$

$$x_a = \frac{-b - \sqrt{\Delta}}{2a} = \frac{-4 - 8}{-6} = \frac{-12}{-6} = 2$$

$$x_a = \frac{-b + \sqrt{\Delta}}{2a} = \frac{-4 + 8}{-6} = \frac{4}{-6} = -\frac{2}{3}$$

$$-3x^2 + 4x + 4 = -3(x - 2)\left(x + \frac{2}{3}\right)$$

$$\Delta_1 = 25 - 24 = 1$$

$$x_a = \frac{-b - \sqrt{\Delta}}{2a} = \frac{5 - 1}{2} = \frac{4}{2} = 2$$

$$x_a = \frac{-b + \sqrt{\Delta}}{2a} = \frac{5 + 1}{2} = \frac{6}{2} = 3$$

$$x^2 - 5x + 6 = (x - 2)(x - 3)$$

Then

$$\begin{aligned}\lim_{x \rightarrow 2} \frac{-3x^2 + 4x + 4}{x^2 - 5x + 6} &= \lim_{x \rightarrow 2} \frac{-3(x-2)\left(x + \frac{2}{3}\right)}{(x-2)(x-3)} \\ &= \lim_{x \rightarrow 2} \frac{-3\left(x + \frac{2}{3}\right)}{(x-3)} \\ &= \frac{-3\left(2 + \frac{2}{3}\right)}{(2-3)} \\ &= \frac{-6-2}{-1} = 8.\end{aligned}$$

Multiplying by a Conjugate

For every x , $g(x) \leq h(x)$ then

If $\lim h(x) = -\infty$, then $\lim g(x) = -\infty$

If $\lim g(x) = +\infty$, then $\lim h(x) = +\infty$

$\lim_{x \rightarrow 5} \frac{\sqrt{2x-1}-3}{x-5} = \frac{0}{0}$, Multiplying by a conjugate, we get

$$\begin{aligned}\lim_{x \rightarrow 5} \frac{\sqrt{2x-1}-3}{x-5} &= \lim_{x \rightarrow 5} \frac{(\sqrt{2x-1}-3)(\sqrt{2x-1}+3)}{(x-5)(\sqrt{2x-1}+3)} \\ &= \lim_{x \rightarrow 5} \frac{2x-10}{(x-5)(\sqrt{2x-1}+3)} \\ &= \lim_{x \rightarrow 5} \frac{2(x-5)}{(x-5)(\sqrt{2x-1}+3)} \\ &= \lim_{x \rightarrow 5} \frac{2}{(\sqrt{2x-1}+3)} = \frac{2}{6} = \frac{1}{3}.\end{aligned}$$

Factoring

$\lim_{x \rightarrow +\infty} \frac{\sqrt{3x^2+3x-5}}{-2x+3} = \frac{+\infty}{-\infty}$, By extracting a common factor

$$\lim_{x \rightarrow +\infty} \frac{\sqrt{3x^2+3x-5}}{-2x+3} = \lim_{x \rightarrow +\infty} \frac{x\sqrt{3+\frac{3}{x}-\frac{5}{x^2}}}{x\left(-2+\frac{3}{x}\right)} = \lim_{x \rightarrow +\infty} \frac{\sqrt{3+\frac{3}{x}-\frac{5}{x^2}}}{\left(-2+\frac{3}{x}\right)} = -\frac{\sqrt{3}}{2}.$$

Derivative

Rule:

$$\lim_{x \rightarrow x_*} \frac{g(x) - g(x_*)}{x - x_*} = g'(x_*).$$

$$\lim_{x \rightarrow 0} \frac{\sin x}{x} = \frac{0}{0}, \text{ By definition of derivative}$$

$$g(x) = \sin x \text{ and } x_* = 0 \text{ then } g(0) = \sin(0) = 0,$$

$$g'(x) = \cos x \text{ then } g'(0) = \cos 0 = 1.$$

Hence

$$\begin{aligned} \lim_{x \rightarrow x_*} \frac{g(x) - g(x_*)}{x - x_*} &= g'(x_*) \\ \lim_{x \rightarrow 0} \frac{\sin x - 0}{x - 0} &= \cos 0 \\ \lim_{x \rightarrow 0} \frac{\sin x}{x} &= 1. \end{aligned}$$

Change the Variable

$$\lim_{x \rightarrow +\infty} x \sin\left(\frac{1}{x}\right) = +\infty \times 0, \text{ By changing the variable, we put}$$

$$t = \frac{1}{x} \Rightarrow x = \frac{1}{t}, \text{ and } x \rightarrow +\infty \Rightarrow t \rightarrow 0.$$

Then

$$\lim_{x \rightarrow +\infty} x \sin\left(\frac{1}{x}\right) = \lim_{t \rightarrow 0} \frac{1}{t} \sin(t) = 1.$$

Limiting Method

$$\lim_{x \rightarrow +\infty} \frac{x + \cos(5x^2 - 3)}{x + 1} = \frac{+\infty}{+\infty}, \text{ limiting method. Let}$$

$$-1 \leq \cos(5x^2 - 3) \leq 1.$$

Then

$$x - 1 \leq x + \cos(5x^2 - 3) \leq x + 1.$$

Hence

$$\frac{x - 1}{x + 1} \leq \frac{x + \cos(5x^2 - 3)}{x + 1} \leq \frac{x + 1}{x + 1} = 1.$$

$$\text{Since } \lim_{x \rightarrow +\infty} \frac{x - 1}{x + 1} = \lim_{x \rightarrow +\infty} 1 = 1, \text{ then } \lim_{x \rightarrow +\infty} \frac{x + \cos(5x^2 - 3)}{x + 1} = 1.$$

Comparison

$\lim_{x \rightarrow +\infty} \frac{x^3 + \exp(\sqrt{x})}{\ln x} = \frac{+\infty}{+\infty}$, By comparison, then

$$\frac{x^3 + \exp(\sqrt{x})}{\ln x} \geq \frac{x^3}{\ln x}.$$

Since

$$\lim_{x \rightarrow +\infty} \frac{x^3}{\ln x} = +\infty.$$

Then

$$\lim_{x \rightarrow +\infty} \frac{x^3 + \exp(\sqrt{x})}{\ln x} = +\infty.$$

Comparing the Growth Rate

Let α is number real strictly positive, then

$$\lim_{x \rightarrow 0^+} x^\alpha \ln x = 0, \lim_{x \rightarrow +\infty} \frac{\ln x}{x^\alpha} = 0, \lim_{x \rightarrow -\infty} x^\alpha \exp x = 0, \lim_{x \rightarrow +\infty} \frac{\exp x}{x^\alpha} = +\infty.$$

$$\lim_{x \rightarrow +\infty} \frac{3 \exp x}{4x^3 - 5x + 1} = \frac{+\infty}{+\infty - \infty}, \text{ By comparing the growth rate, then}$$

$$\begin{aligned} \lim_{x \rightarrow +\infty} \frac{3 \exp x}{4x^3 - 5x + 1} &= \lim_{x \rightarrow +\infty} \frac{3 \exp x}{x^3} \times \frac{x^3}{4x^3 - 5x + 1} \\ &= \lim_{x \rightarrow +\infty} \frac{3 \exp x}{x^3} \times \frac{x^3}{4x^3} \\ &= \frac{1}{4} \lim_{x \rightarrow +\infty} \frac{3 \exp x}{x^3} \\ &= +\infty. \end{aligned}$$

3.3 Continuous Functions

3.3.1 Continuity of a Function at a Point

Definition 3.12

[7]

◦ Let g be a real function on a subset of the real numbers. Then g is **continuous** at

x_* if

$$\begin{cases} 1) & x_* \in D_g, \\ 2) & \lim_{x \rightarrow 0} g(x) = g(x_*). \end{cases}$$

- In particular, if the left hand limit, right hand limit and the value of the function at $x = x_*$ exist and are equal to each other, i.e.,

$$\lim_{x \rightarrow x_*^-} g(x) = \lim_{x \rightarrow x_*^+} g(x) = g(x_*),$$

then g is said to be **continuous** at $x = x_*$.

- A function is **continuous** if it is continuous at every x_* in its domain D_g .
- If it is **not continuous** there, i.e. if either the limit does not exist or is not equal to $g(x_*)$ we will say that the function is **discontinuous** at x_* .

Example 3.9

Consider the function

$$g(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0, \\ 1, & x = 0. \end{cases}$$

The function is continuous at all $x = 0$, $\lim_{x \rightarrow 0} g(x) = \lim_{x \rightarrow 0} \frac{\sin x}{x} = g(0) = 1$.

3.3.2 Continuity of a Function in an Interval

Definition 3.13

1. g is said to be continuous in an open interval $]a, b[$ if it is continuous at every point x_* in this interval.
2. g is said to be continuous in the closed interval $[a, b]$ if
 - g is continuous in $]a, b[$.
 - g is right continuous at a point a , i.e. $\lim_{x \rightarrow a^+} g(x) = g(a)$.
 - g is left continuous at a point b , i.e. $\lim_{x \rightarrow b^-} g(x) = g(b)$.

Proposition 3.1

All polynomials, rational functions, trigonometric functions, the inverse trigonometric functions, the absolute value function, the exponential and logarithm functions are continuous everywhere within its domain.

3.3.3 Continuous Extension at a Point

We can redefine functions with removable discontinuities to obtain continuous functions.

Proposition 3.2

Let I be an interval, and $x_ \in I$. Let g be defined on $I - \{x_*\}$ such that*

$$\lim_{x \rightarrow x_*} g(x) = l \in \mathbb{R}.$$

Consider the function $\tilde{g}$:

$$\tilde{g}(x) = \begin{cases} g(x) & \text{for } x \in I - \{x_*\}, \\ l & \text{for } x = x_*, \end{cases}$$

then, the function $\tilde{g}$ is a continuous at x_ .*

Example 3.10

Find a continuous extension of the function $g(x) = \frac{\sin x}{x}$:

The domain of g is $D_g = \mathbb{R}^$, then g is discontinuous at $x = 0$ because $g(0)$ is not defined. Since $\lim_{x \rightarrow 0} g(x)$ exists, the discontinuity is removable.*

We know that $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$. For the function to be continuous at zero we need to define $g(0)$ we make $g(0) = \lim_{x \rightarrow 0} \tilde{g}(x) = \lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$, and redefine the function:

$$\tilde{g} : \tilde{g}(x) = \begin{cases} \frac{\sin x}{x} & \text{for } x \neq 0, \\ 1 & \text{for } x = 0. \end{cases}$$

We say $\tilde{g}$ is the continuous extension of g to $x = 0$.

3.3.4 Intermediate Value Theorem

Theorem 3.3

If a function g is continuous on a closed interval $[a, b]$, and if y_0 is some number between $g(a)$ and $g(b)$, then there is a number x_* in the interval $[a, b]$ such that $g(x_*) = y_0$.

Example 3.11

Use the **Intermediate Value Theorem** to prove $x^2 = 2$ has a root.

Consider the function $g : \mathbb{R} \rightarrow \mathbb{R}$ defined by: $g(x) = x^2$.

The function g is continuous on a closed interval $[1, 2]$.

One has $g(1) = 1$ and $g(2) = 4$. Since $g(1) \leq 2 \leq g(2)$, the intermediate value theorem with $a = 1, b = 2, y_0 = 2$ tells us that there is a number x_* between 1 and 2 such that $g(x_*) = 2$, i.e. for which $x_*^2 = 2$. So the theorem tells us that the square root of 2 exists.

3.3.5 Continuity of Composite Functions

Definition 3.14

[5] Let g and h be real valued functions such that $(g \circ h)$ is defined at x_* . If h is continuous at x_* and g is continuous at $h(x_*)$, then $(g \circ h)$ is continuous at x_* .

Example 3.12

Since both $g(x) = x^2 + 1$ and $h(x) = \cos x$ are continuous on $\mathbb{R}$.

Therefore, both $(g \circ h)(x) = \cos^2 x + 1$, and $(h \circ g)(x) = \cos(x^2 + 1)$ are continuous on $\mathbb{R}$.

3.3.6 Continuity of the Algebraic Combinations of Functions

Definition 3.15

If g and h are both continuous at x_* and λ is any constant, then each of the following functions is also continuous at x_* . The sum $g + h$, the difference $g - h$, the constant multiple λg , the product $g \times h$, the quotient $g \div h$, if $h(x_*) \neq 0$.

Example 3.13

Let $g : \mathbb{R} \rightarrow \mathbb{R}$ defined by:

$$g(x) = \begin{cases} x \sin\left(\frac{1}{x}\right), & x \neq 0, \\ 0, & x = 0. \end{cases}$$

Here is a continuous function on $\mathbb{R}$ because

- the inverse function of $x, x \rightarrow \frac{1}{x}$ is a continuous on $\mathbb{R}^*$.
- the sine function $x \rightarrow \sin x$ is a continuous on $\mathbb{R}$.
- the composite functions $x \rightarrow \sin\left(\frac{1}{x}\right)$ is a continuous on $\mathbb{R}^*$.
- the function $x \rightarrow x$ is a continuous on $\mathbb{R}$.
- the product function $x \rightarrow x \sin\left(\frac{1}{x}\right)$ is a continuous on $\mathbb{R}^*$.
- since $\lim_{x \rightarrow 0} x \sin\left(\frac{1}{x}\right) = g(0)$, then the function g is a continuous at $x_* = 0$.

3.4 Differentiability of Functions

3.4.1 Differentiability of a Function at a Point

Definition 3.16

[4] [Differentiability] Let g be a real valued function. Then g is said to be **differentiable** at $x_* \in D_g$ if

$$\lim_{x \rightarrow x_*} \frac{g(x) - g(x_*)}{x - x_*} = l \in \mathbb{R},$$

in that case the value l is called the derivative of g at x_* .

The derivative of g at x_* , if exists, is denoted by $g'(x_*)$.

Example 3.14

The function $g : x \rightarrow \sqrt{x}$ is differentiable at $x_* = 1$.

$$\begin{aligned} \lim_{x \rightarrow x_*} \frac{g(x) - g(x_*)}{x - x_*} &= \lim_{x \rightarrow 1} \frac{\sqrt{x} - 1}{x - 1} = \frac{0}{0} \\ \lim_{x \rightarrow 1} \frac{\sqrt{x} - 1}{x - 1} &= \lim_{x \rightarrow 1} \frac{(\sqrt{x} - 1)(\sqrt{x} + 1)}{(x - 1)(\sqrt{x} + 1)} \\ &= \lim_{x \rightarrow 1} \frac{(x - 1)}{(x - 1)(\sqrt{x} + 1)} \\ &= \lim_{x \rightarrow 1} \frac{1}{\sqrt{x} + 1} = \frac{1}{2}. \end{aligned}$$

Since g is differentiable at $x_* = 1$, then $g'(1) = \frac{1}{2}$.

Example 3.15

The function $g : x \rightarrow \frac{1}{x}$ is differentiable at $x_* = 2$.

$$\begin{aligned} \lim_{x \rightarrow x_*} \frac{g(x) - g(x_*)}{x - x_*} &= \lim_{x \rightarrow 2} \frac{\frac{1}{x} - \frac{1}{2}}{x - 2} = \frac{0}{0} \\ \lim_{x \rightarrow 2} \frac{\frac{1}{x} - \frac{1}{2}}{x - 2} &= \lim_{x \rightarrow 2} \frac{\frac{2-x}{2x}}{x - 2} \end{aligned}$$

Since g is differentiable at $x_* = 2$, then $g'(1) = -\frac{1}{4}$.

Definition 3.17

[4] [Left Differentiability and Right Differentiability]

- g is left differentiable at a point x_* , i.e.

$$\lim_{x \rightarrow x_*^-} \frac{g(x) - g(x_*)}{x - x_*} = g'_L(x_*).$$

- g is right differentiable at a point x_* , i.e.

$$\lim_{x \rightarrow x_*^+} \frac{g(x) - g(x_*)}{x - x_*} = g'_R(x_*).$$

Proposition 3.3

g is differentiable at a point x_* iff g is left differentiable and right differentiable at this point, i.e.

$$g'_L(x_*) = g'_R(x_*).$$

Definition 3.18

The function g is said to be differentiable at $x_* \in D_g$ iff

$$\lim_{h \rightarrow 0} \frac{g(x_* + h) - g(x_*)}{h} = g'(x_*).$$

Example 3.16

The derivative of $g(x) = x^2$ is $g'(x) = 2x$.

$$\lim_{h \rightarrow 0} \frac{g(x+h) - g(x)}{h} = \lim_{h \rightarrow 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \rightarrow 0} (2x+h) = 2x.$$

Theorem 3.4 (*Differentiability implies continuity*)

Suppose g is differentiable at $x_* \in D_g$. Then g is continuous at x_* .

Proof: Note that

$$\begin{aligned}
 g \text{ is differentiable at } x_* \in D_g &\Rightarrow \lim_{h \rightarrow 0} \frac{g(x_* + h) - g(x_*)}{h} = g'(x_*) \\
 &\Rightarrow \lim_{h \rightarrow 0} g(x_* + h) - g(x_*) = \lim_{h \rightarrow 0} h g'(x_*) \\
 &\Rightarrow \lim_{h \rightarrow 0} g(x_* + h) - g(x_*) = 0 \\
 &\Rightarrow \lim_{h \rightarrow 0} g(x_* + h) = g(x_*) \\
 &\Rightarrow g \text{ is continuous at } x_*.
 \end{aligned}$$

■

Remark 3.5

Every differentiable function is continuous, **but the converse is not true.**

Example 3.17

Consider the function

$$g(x) = |x| = \begin{cases} x, & x \geq 0, \\ -x, & x < 0. \end{cases}$$

This function is continuous at all x , but it is not differentiable at $x = 0$.

To see this try to compute the derivative at 0,

•

$$\lim_{x \rightarrow 0^-} \frac{|x| - |0|}{x - 0} = \lim_{x \rightarrow 0^-} \frac{-x}{x} = -1 \Rightarrow g'_L(0) = -1.$$

•

$$\lim_{x \rightarrow 0^+} \frac{|x| - |0|}{x - 0} = \lim_{x \rightarrow 0^+} \frac{x}{x} = 1 \Rightarrow g'_R(0) = 1.$$

• note that $g'_L(0) \neq g'_R(0)$.

3.4.2 Differentiability of a Function in an Interval

1. Differentiability in an open interval $]a, b[$:

A function g is said to be differentiable in the open interval $]a, b[$ if it is differentiable at every point $x_* \in]a, b[$. This means the derivative

$$g'(x_*) = \lim_{x \rightarrow x_*} \frac{g(x) - g(x_*)}{x - x_*}$$

exists for every x_* in $]a, b[$.

2. Differentiability in a closed interval $[a, b]$:

A function g is said to be differentiable on the closed interval $[a, b]$ if all of the following conditions hold:

- g is differentiable on the open interval $]a, b[$.
- g has a right-hand derivative at a , i.e., the limit

$$g'(a) = \lim_{x \rightarrow a^+} \frac{g(x) - g(a)}{x - a}$$

exists.

- g has a left-hand derivative at b , i.e., the limit

$$g'(b) = \lim_{x \rightarrow b^-} \frac{g(x) - g(b)}{x - b}$$

exists.

3.4.3 Algebra of Derivatives

If g, h are differentiable functions and λ is any constant, then

1. $(g + h)'(x) = g'(x) + h'(x)$.
2. $(g \times h)'(x) = g'(x) \times h(x) + g(x) \times h'(x)$.
3. $(\lambda \times g)'(x) = \lambda \times g'(x)$.
4. $\left(\frac{g}{h}\right)'(x) = \frac{g'(x) \times h(x) + g(x) \times h'(x)}{h^2(x)}$, if $h(x) \neq 0$.

3.4.4 Derivatives of Composite Functions

Definition 3.19

If g and h are differentiable, so is the composition $g \circ h$. The derivative of $g \circ h$ is given by:

$$(g \circ h)'(x) = g'(h(x)) \times h'(x).$$

Example 3.18

The function $g(x) = \sin 2x$ is the composition of two simpler functions, namely: $g(x) =$

$k(h(x))$ where $k(u) = \sin u$ and $h(x) = 2x$. Since k and h are differentiable then $k'(u) = \cos u$ and $h'(x) = 2$. Therefore the derivative of the composite functions rule implies that

$$g'(x) = (k(h(x)))' = k'(h(x)) \times h'(x) = (\cos 2x) \times 2 = 2 \cos 2x.$$

3.4.5 Derivative of Inverse Function

Definition 3.20

If g is a function with inverse function g^{-1} , then

$$(g^{-1})'(y) = \frac{1}{g'(x)}.$$

Example 3.19

The inverse of the function $g(x) = x^2$ with reduced domain $[0, +\infty[$ is $g^{-1}(y) = \sqrt{y}$.

Use the formula given above to find the derivative of g^{-1} . We have $g'(x) = 2x$, so that

$$(g^{-1})'(y) = \frac{1}{2x} = \frac{1}{2\sqrt{y}}.$$

3.4.6 Table of Derivatives

$g(x)$	$g'(x)$	$g(x)$	$g'(x)$
a^x	$a^x \ln a$	c	0
$\sin x$	$\cos x$	x	1
$\cos x$	$-\sin x$	$\sqrt{x}$	$\frac{1}{2\sqrt{x}}$
$\tan x$	$\frac{1}{\cos^2 x} = 1 + \tan^2 x$	$\sqrt[n]{x}$	$\frac{1}{n \times \sqrt[n]{x^{n-1}}}$
$\cot x$	$\frac{-1}{\sin^2 x} = -1 - \cot^2 x$	$\frac{1}{x}$	$-\frac{1}{x^2}$
$\arcsin x$	$\frac{1}{\sqrt{1-x^2}}$	$\frac{1}{x^n}$	$-\frac{n}{x^{n+1}}$
$\arccos x$	$\frac{-1}{\sqrt{1-x^2}}$	$\ln x$	$\frac{1}{x}$
$\arctan x$	$\frac{1}{1+x^2}$	$\exp x$	$\exp x$

3.4.7 Indeterminate Forms and Hospital's Rule

Hospital's Rule

Consider g and h are continuous functions on $[a, b]$ which are differentiable at every point in $]a, b[$, except possibly at $x_* \in [a, b]$. Assume that:

1.

$$h(x) \neq 0 \text{ and } h'(x) \neq 0 \text{ at every point in } [a, b].$$

2.

$$\lim_{x \rightarrow x_*} \frac{g(x)}{h(x)} = \frac{0}{0} \text{ or } \lim_{x \rightarrow x_*} \frac{g(x)}{h(x)} = \frac{\infty}{\infty}.$$

3.

$$\lim_{x \rightarrow x_*} \frac{g'(x)}{h'(x)} \text{ exists. Then } \lim_{x \rightarrow x_*} \frac{g(x)}{h(x)} \text{ exists and } \lim_{x \rightarrow x_*} \frac{g'(x)}{h'(x)} = \lim_{x \rightarrow x_*} \frac{g(x)}{h(x)}.$$

Remark 3.6

Note that the rule is also valid for one-sided limits and for limits at infinity or negative infinity.

In fact, for the special case in which $g(x_) = h(x_*) = 0$, g' and h' are continuous, and $h'(x_*) \neq 0$, it is easy to see why the rule is true.*

$$\begin{aligned} \lim_{x \rightarrow x_*} \frac{g'(x)}{h'(x)} &= \frac{g'(x_*)}{h'(x_*)} = \frac{\lim_{x \rightarrow x_*} \frac{g(x) - g(x_*)}{x - x_*}}{\lim_{x \rightarrow x_*} \frac{h(x) - h(x_*)}{x - x_*}} \\ &= \lim_{x \rightarrow x_*} \frac{g(x) - g(x_*)}{h(x) - h(x_*)} \\ &= \lim_{x \rightarrow x_*} \frac{g(x)}{h(x)}. \end{aligned}$$

Example 3.20

Find

$$\lim_{x \rightarrow 1} \frac{\ln x}{x - 1}.$$

$$\lim_{x \rightarrow 1} \frac{\ln x}{x - 1} = \frac{0}{0}.$$

Thus, we can apply Hospital's Rule:

$$\lim_{x \rightarrow x_*} \frac{g'(x)}{h'(x)} = \lim_{x \rightarrow 1} \frac{(\ln x)'}{(x-1)'} = \lim_{x \rightarrow 1} \frac{1}{x} = 1.$$

Then,

$$\lim_{x \rightarrow x_*} \frac{g(x)}{h(x)} = \lim_{x \rightarrow 1} \frac{\ln x}{x-1} = 1.$$

3.4.8 Rolle's Theorem

Definition 3.21

If a real-valued function g is continuous on a proper closed interval $[a, b]$, differentiable on the open interval $]a, b[$, and $g(a) = g(b)$, then there exists at least one c in the open interval $]a, b[$ such that $g'(c) = 0$.

Example 3.21

Verify Rolle's theorem for the function

$$g(x) = x^2 + 1, x \in [-1, 1].$$

The function $y = x^2 + 1$, as it is a polynomial function, is continuous in $[-1, 1]$ and differentiable in $] -1, 1[$. Also, $g(-1) = (-1)^2 + 1 = 1 + 1 = 2$ and $g(1) = (1)^2 + 1 = 1 + 1 = 2$. Thus, $g(-1) = g(1) = 2$. Hence, the function $g(x)$ satisfies all conditions of Rolle's theorem. Now, $g'(x) = 2x$. Rolle's theorem states that there is a point $c \in [-2, 2]$ such that

$$g'(c) = 0 \Rightarrow 2c = 0 \Rightarrow c = 0,$$

where $c = 0 \in] -1, 1[$.

3.4.9 Mean Value Theorem

Definition 3.22

Let g be a continuous function on the closed interval $[a, b]$, and differentiable on the open interval $]a, b[$, where $a < b$. Then there exists some c in $]a, b[$ such that

$$g'(c) = \frac{g(b) - g(a)}{b - a}.$$

The mean value theorem is a generalization of Rolle's theorem, which assumes $g(a) = g(b)$, so that the right-hand side above is zero.

3.4.10 Higher Derivatives

If the derivative $g'(x)$ of some function g exists for all x in the domain of g , then we have a new function, this function is called the **derivative function** of g , and it is denoted by g' . Now that we have agreed that the derivative of a function is a function, we can repeat the process and try to differentiate the derivative. The result, if it exists, is called the **second derivative** of g . It is denoted g'' . The derivative of the second derivative is called the **third derivative**, written g''' , and so on. The n -th derivative of g is denoted $g^{(n)}$. Thus

$$g^{(0)} = g, g^{(1)} = g', g^{(2)} = (g')', \dots, g^{(n)} = (g^{(n-1)})'.$$

Example 3.22

If $g(x) = \exp x$ then $g^{(1)}(x) = \exp x, g^{(2)}(x) = \exp x, \dots, g^{(n)} = \exp x$.

Application to Elementary Functions

We are going to deal mostly with elementary functions. These include:

Exponential functions e.g., e^x

Logarithmic functions e.g., $\ln x$, $\log x$

Trigonometric functions e.g., $\sin x$, $\cos x$, $\tan x$, etc.

Inverse trigonometric functions e.g., $\arcsin x$, $\arctan x$, etc.

We will also consider their sums, differences, products, quotients, and compositions.

Example 4.1

$$f(x) = \frac{\arcsin \sqrt{x^2 - 3}}{\ln(x^4 + 5x) - \tan \exp(\cos x)},$$

is an elementary function.

There are many non-elementary functions. For example,

$$f(x) = \int_0^x \frac{\sin t}{t} dt,$$

is not an elementary function.

4.1 Exponential Function

Definition 4.1

[2] The reciprocal function of $\ln :]0, +\infty[\rightarrow \mathbb{R}$ is called the exponential function, denoted by $\exp : \mathbb{R} \rightarrow]0, +\infty[$.

For $x \in \mathbb{R}$ we also note e^x for $\exp x$.

4.1.1 Properties the Exponential Function

The exponential function satisfies the following properties:

1. $\exp(\ln a) = a$ for all $a \succ 0$.
2. $\ln(\exp a) = a$ for all $a \in \mathbb{R}$.
3. $\forall a \in \mathbb{R}, \exp a \succ 0$.
4. $\forall a \in \mathbb{R}, \exp(-a) = \frac{1}{\exp a}$.
5. $\exp(a + b) = \exp(a) \times \exp(b)$.
6. $\exp(na) = (\exp a)^n$.
7. $a \succ b \Leftrightarrow \exp(a) \succ \exp(b)$.
8. $a \prec b \Leftrightarrow \exp(a) \prec \exp(b)$.
9. $\exp(1) = e \approx 2,71$.
10. $\exp(0) = 1$.
11. $\exp(a - b) = \frac{\exp a}{\exp b}$.
12. $a = b \iff \exp a = \exp b$.

4.1.2 Study the Exponential Function

Domain of definition $D_f =]-\infty, +\infty[$.

Limits

$$\lim_{x \rightarrow -\infty} \exp x = 0, \quad \lim_{x \rightarrow +\infty} \exp x = +\infty.$$

The exponential function is differential

$$(\exp x)' = \exp x, \text{ for all } x \in \mathbb{R}.$$

Sign of differentiability

$$\forall x \in \mathbb{R}, (\exp x)' \succ 0.$$

Remark 4.1

Exponential is a continuous, strictly increasing function of $\mathbb{R}$ for $]0, +\infty[$.

4.2 Logarithmic Function

Definition 4.2

Logarithmic function is denoted by $\ln :]0, +\infty[\rightarrow \mathbb{R}$ is reciprocal of exponential function $\exp : \mathbb{R} \rightarrow]0, +\infty[$.

4.2.1 Properties the Logarithmic Function

The Logarithmic function satisfies the following properties

1. $\ln(0) = -\infty, \ln(1) = 0, \ln(\exp(1)) = 1.$
2. $\forall a \in \mathbb{R}, \ln(\exp(a)) = a.$
3. $\forall a \in \mathbb{R}, \forall b \in \mathbb{R}_+^*, \ln(b) = a \Leftrightarrow b = \exp(a).$
4. $\forall a \in \mathbb{R}^+, \forall b \in \mathbb{R}^+, \exp(b) = a \Leftrightarrow b = \ln(a).$
5. $\ln(a) = \ln(b) \Leftrightarrow a = b.$
6. $\ln(a) \succ \ln(b) \Leftrightarrow a \succ b.$
7. $\forall a \succ 0, \exp(\ln(a)) = a.$
8. $\forall a \succ 1, \ln(a) \succ 0.$
9. $\forall a \in]0, 1[, \ln(a) \prec 0.$
10. $\ln(a \times b) = \ln(a) + \ln(b),$ for all $a, b \succ 0.$
11. $\ln\left(\frac{a}{b}\right) = \ln(a) - \ln(b),$ for all $a, b \succ 0.$
12. $\ln(a^n) = n \ln(a),$ (for all $n \in \mathbb{N}$), for all $a \succ 0.$

4.2.2 Study the Logarithmic Function

Domain of definition $D_f =]0, +\infty[$.

Limits

$$\lim_{x \rightarrow 0} \ln x = -\infty, \lim_{x \rightarrow +\infty} \ln x = +\infty.$$

The exponential function is differential

$$(\ln x)' = \frac{1}{x}, \text{ for all } x \in]0, +\infty[, \text{ and } \ln(1) = 0.$$

Sign of differentiability

$$\forall x \in]0, +\infty[, (\ln x)' = \frac{1}{x} \succ 0.$$

Remark 4.2

Logarithmic is a continuous, strictly increasing function of $]0, +\infty[$ for $\mathbb{R}$.

4.3 Trigonometric Functions

4.3.1 Sine Function

Definition 4.3

[3] Sine function is denoted by $\sin : \mathbb{R} \rightarrow [-1, 1]$.

Properties the Sine Function

The Sine function satisfies the following properties

1. $-1 \leq \sin x \leq 1, \forall x \in \mathbb{R}$.
2. $\sin(0) = 0, \sin\left(\frac{\pi}{2}\right) = 1, \sin(\pi) = 0, \sin\left(\frac{3\pi}{2}\right) = -1, \sin(2\pi) = 0$.
3. $\sin(x + \pi) = -\sin(x)$.

Study the Sine Function

1. Domain of definition $D_f = \mathbb{R}$.
2. Sine function is odd function because

$$\sin(-x) = -\sin(x).$$

3. Sine is periodic function with period is 2π

$$\sin(x + 2\pi) = \sin x.$$

4. Derivative of the sine function is

$$(\sin x)' = \cos x, \text{ for all } x \in \mathbb{R}.$$

4.3.2 Arcsine Function

Definition 4.4

Arcsine function is denoted by $\arcsin : [-1, 1] \rightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, is inverse of sine function.

Properties the Arcsine Function

The Arcsine function satisfies the following properties

$$1. \forall x \in [-1, 1], \forall y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\sin y = x \Leftrightarrow y = \arcsin x.$$

$$2. \forall x \in [-1, 1]$$

$$\sin [\arcsin x] = x.$$

$$3. \forall y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\arcsin [\sin y] = y.$$

$$4.$$

$$\cos (\arcsin x) = \sqrt{1 - x^2}.$$

Because $\cos^2 x + \sin^2 x = 1$, we put $x = \arcsin x$, we get

$$\cos^2 (\arcsin x) + \sin^2 (\arcsin x) = 1$$

$$(\cos (\arcsin x))^2 + (\sin (\arcsin x))^2 = 1$$

$$(\cos (\arcsin x))^2 = 1 - x^2$$

$$\cos (\arcsin x) = +\sqrt{1 - x^2},$$

because $\arcsin x \in [-1, 1]$, and $\cos ([-1, 1]) = [0, \pi]$.

Study the Arcsine Function

1. Domain of definition $D_f = [-1, 1]$.
2. Arcsine function is odd function because

$$\arcsin(-x) = -\arcsin(x).$$

3. Derivative of the arcsine function is

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}, \text{ for all } x \in [-1, 1].$$

Derivative of inverse function is $(f^{-1})'(y) = \frac{1}{f'(x)}$,

Let $f^{-1} = \arcsin$, $f = \sin$, we put $y = \sin x$, $x = \arcsin y$, then

$$(\arcsin)'(y) = \frac{1}{\sin'(x)} = \frac{1}{\cos(x)} = \frac{1}{\cos(\arcsin y)} = \frac{1}{\sqrt{1-x^2}}, \text{ because}$$

$$\cos(\arcsin y) = \sqrt{1-x^2}.$$

4.3.3 Cosine Function

Definition 4.5

Cosine function is denoted by $\cos : \mathbb{R} \rightarrow [-1, 1]$.

Properties the Cosine Function

The Cosine function satisfies the following properties

1. $-1 \leq \cos x \leq 1, \forall x \in \mathbb{R}$.
2. $\cos(0) = 1, \cos\left(\frac{\pi}{2}\right) = 0, \cos(\pi) = -1, \cos\left(\frac{3\pi}{2}\right) = 0, \cos(2\pi) = 1$.
3. $\cos(x + \pi) = -\cos(x)$.

Study the Cosine Function

1. Domain of definition $D_f = \mathbb{R}$.

2. Cosine function is even function because

$$\cos(-x) = \cos(x).$$

3. Cosine is periodic function with period is 2π

$$\cos(x + 2\pi) = \cos x.$$

4. Derivative of the cosine function is

$$(\cos x)' = -\sin x, \text{ for all } x \in \mathbb{R}.$$

4.3.4 Arccosine Function

Definition 4.6

Arccosine function is denoted by $\arccos : [-1, 1] \rightarrow [0, \pi]$, is inverse of cosine function.

Properties the Arccosine Function

The arccosine function satisfies the following properties

1. $\forall x \in [-1, 1], \forall y \in [0, \pi]$

$$\cos y = x \Leftrightarrow y = \arccos x.$$

2. $\forall x \in [-1, 1]$

$$\cos(\arccos x) = x.$$

3. $\forall y \in [0, \pi]$

$$\arccos[\cos y] = y.$$

4.

$$\sin(\arccos x) = \sqrt{1 - x^2}.$$

Because $\cos^2 x + \sin^2 x = 1$, we put $x = \arccos x$, we get

$$\begin{aligned}\cos^2 (\arccos x) + \sin^2 (\arccos x) &= 1 \\ (\cos (\arccos x))^2 + (\sin (\arccos x))^2 &= 1 \\ (\sin (\arccos x))^2 &= 1 - x^2 \\ \sin (\arccos x) &= +\sqrt{1 - x^2}, \\ \text{because } \arccos x &\in [-1, 1], \text{ and } \sin ([-1, 1]) = [0, \pi].\end{aligned}$$

Study the Cosine Function

1. Domain of definition $D_f = [-1, 1]$.
2. Cosine function is even function because

$$\arccos (-x) = \arccos (x).$$

3. Derivative of the arccosine function is

$$(\arccos x)' = \frac{-1}{\sqrt{1 - x^2}}, \text{ for all } x \in [-1, 1].$$

$$\text{Derivative of inverse function is } (f^{-1})'(y) = \frac{1}{f'(x)},$$

Let $f^{-1} = \arccos$, $f = \cos$, we put $y = \cos x$, $x = \arccos y$, then

$$(\arccos)'(y) = \frac{1}{\cos'(x)} = -\frac{1}{\sin(x)} = -\frac{1}{\sin(\arccos y)} = -\frac{1}{\sqrt{1 - x^2}}, \text{ because}$$

$$\sin(\arccos y) = \sqrt{1 - x^2}.$$

4.3.5 Relation Between Sine and Cosine Functions

The fundamental relationship in trigonometry is:

$$\cos^2 x + \sin^2 x = 1, \forall x \in \mathbb{R}.$$

Addition formula $\forall a, b \in \mathbb{R}$, we have

$$\sin(a + b) = \sin a \cos b + \sin b \cos a.$$

$$\sin(a - b) = \sin a \cos b - \sin b \cos a.$$

$$\cos(a + b) = \cos a \cos b - \sin a \sin b.$$

$$\cos(a - b) = \cos a \cos b + \sin a \sin b.$$

Duplication formula $\forall x \in \mathbb{R}$, we have

$$\sin 2x = 2 \sin x \cos x.$$

$$\cos 2x = \cos^2 x - \sin^2 x = -1 - 2 \sin^2 x.$$

4.3.6 Tangent and Cotangent Functions

4.3.7 Tangent Function

Definition 4.7

We call *tangent* the function $\tan$ defined by

$$\tan x = \frac{\sin x}{\cos x}, \forall x \in \mathbb{R} - \left\{ \frac{\pi}{2} + k\pi, k \in \mathbb{Z} \right\}.$$

Study the Tangent Function

1. Domain of definition $D_f = \mathbb{R} - \left\{ \frac{\pi}{2} + k\pi, k \in \mathbb{Z} \right\}$.
2. Tangent function is odd function because

$$\tan(-x) = \frac{\sin(-x)}{\cos(-x)} = -\frac{\sin x}{\cos x} = -\tan x.$$

3. Derivative of the tangent function is

$$(\tan x)' = \frac{1}{\cos^2 x} = 1 + \tan^2 x, \text{ for all } x \in \mathbb{R} - \left\{ \frac{\pi}{2} + k\pi, k \in \mathbb{Z} \right\}.$$

4. Limits

$$\lim_{x \rightarrow \frac{\pi}{2}} \tan x = \lim_{x \rightarrow \frac{\pi}{2}} \frac{\sin x}{\cos x} = +\infty, \quad \lim_{x \rightarrow -\frac{\pi}{2}} \tan x = \lim_{x \rightarrow -\frac{\pi}{2}} \frac{\sin x}{\cos x} = -\infty.$$

4.3.8 Arctangent Function

Definition 4.8

Arctangent function is denoted by $\arctan : \mathbb{R} \rightarrow \left[-\frac{\pi}{2}, \frac{\pi}{2} \right]$, is inverse of tangent function.

Properties the Arctangent Function

The arctangent function satisfies the following properties

$$1. \forall x \in \mathbb{R}, \forall y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\tan y = x \Leftrightarrow y = \arctan x.$$

$$2. \forall x \in \mathbb{R}$$

$$\tan(\arctan x) = x.$$

$$3. \forall y \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$$

$$\arctan[\tan y] = y.$$

Study the Arctangent Function

1. Domain of definition $D_f = \mathbb{R}$.

2. Arctangent function is odd function because

$$\arctan(-x) = -\arctan x.$$

3. Derivative of the arctangent function is

$$(\arctan x)' = \frac{1}{1+x^2}, \text{ for all } x \in \mathbb{R}.$$

$$\text{Derivative of inverse function is } (f^{-1})'(y) = \frac{1}{f'(x)},$$

Let $f^{-1} = \arctan$, $f = \tan$, we put $y = \tan x$, $x = \arctan y$, then

$$(\arctan)'(y) = \frac{1}{\tan'(x)} = \frac{1}{1+\tan^2(x)} = \frac{1}{1+\tan^2(\arctan y)} = \frac{1}{1+y^2}.$$

4.3.9 Cotangent Function

Definition 4.9

We call cotangent the function $\cot$ defined by

$$\cot x = \frac{\cos x}{\sin x}, \forall x \in \mathbb{R} - \{k\pi, k \in \mathbb{Z}\}.$$

Study the Cotangent Function

1. Domain of definition $D_f = \mathbb{R} - \{k\pi, k \in \mathbb{Z}\}$.

2. Cotangent function is odd function because

$$\cot(-x) = \frac{\cos(-x)}{\sin(-x)} = -\frac{\cos x}{\sin x} = -\cot x.$$

3. Derivative of the cotangent function is

$$(\cot x)' = -\frac{1}{\sin^2 x} = -(1 + \cot^2 x), \text{ for all } x \in \mathbb{R} - \{k\pi, k \in \mathbb{Z}\}.$$

4. Limits

$$\lim_{x \nearrow \pi} \cot x = \lim_{x \nearrow \pi} \frac{\cos x}{\sin x} = \frac{-1}{0^-} = -\infty, \lim_{x \rightarrow 0} \cot x = \lim_{x \rightarrow 0} \frac{\cos x}{\sin x} = \frac{1}{0^+} = +\infty.$$

Remark 4.3

For all $x \in \mathbb{R} - \left(\left\{ \frac{\pi}{2} + k\pi, k \in \mathbb{Z} \right\} \cup \{k\pi, k \in \mathbb{Z}\} \right)$, we have the reality

$$\cot x = \frac{1}{\tan x}.$$

4.3.10 Arccotangent Function

Definition 4.10

Arccotangent function is denoted by $\operatorname{arccot} : \mathbb{R} \rightarrow]0, \pi[$, is inverse of tangent function.

Properties the Arccotangent Function

The arccotangent function satisfies the following properties

1. $\forall x \in \mathbb{R}, \forall y \in]0, \pi[$

$$\cot y = x \Leftrightarrow y = \operatorname{arccot} x.$$

2. $\forall x \in \mathbb{R}$

$$\cot(\operatorname{arccot} x) = x.$$

3. $\forall y \in]0, \pi[$

$$\operatorname{arccot}[\cot y] = y.$$

Study the Arctangent Function

1. Domain of definition $D_f = \mathbb{R}$.
2. Arccotangent function is odd function because

$$\operatorname{arccot}(-x) = \frac{\sin(-x)}{\cos(-x)} = -\frac{\sin x}{\cos x} = -\operatorname{arccot} x.$$

3. Derivative of the arccotangent function is

$$(\operatorname{arccot} x)' = -\frac{1}{1+x^2}, \text{ for all } x \in \mathbb{R}.$$

Derivative of inverse function is $(f^{-1})'(y) = \frac{1}{f'(x)}$,

Let $f^{-1} = \operatorname{arccot}$, $f = \cot$, we put $y = \cot x$, $x = \operatorname{arccot} y$, then

$$(\operatorname{arccot})'(y) = \frac{1}{\cot'(x)} = -\frac{1}{1+\cot^2(x)} = -\frac{1}{1+\cot^2(\operatorname{arccot} y)} = -\frac{1}{1+y^2}.$$

4.4 Hyperbolic Trigonometry and its Inverses

4.4.1 Hyperbolic Cosine and its Inverse

For $x \in \mathbb{R}$, the hyperbolic cosine is:

$$\cosh x = \frac{\exp(x) + \exp(-x)}{2}.$$

The function $\cosh : [0, +\infty[\rightarrow [1, +\infty[$.

Derivative of the hyperbolic cosine function is

$$\cosh' x = \frac{\exp(x) - \exp(-x)}{2} = \sinh x.$$

The inverse function $\cosh$ is $\arg \cosh : [1, +\infty[\rightarrow [0, +\infty[$, and this function is strictly creasing and continuous, by the forms

$$\arg \cosh x = \ln(x + \sqrt{x^2 - 1}), \quad \forall x \geq 1.$$

$\arg \cosh$ is derivable and derivative is

$$\arg \cosh' x = \frac{1}{\sqrt{x^2 - 1}}, \quad \forall x \succ 1.$$

Because

$$\begin{aligned}\arg \cosh' x &= \left(\ln \left(x + \sqrt{x^2 - 1} \right) \right)' \\ &= \frac{1 + \frac{2x}{2\sqrt{x^2 - 1}}}{x + \sqrt{x^2 - 1}} = \frac{1 + \frac{x}{\sqrt{x^2 - 1}}}{x + \sqrt{x^2 - 1}} \\ &= \frac{\frac{x + \sqrt{x^2 - 1}}{\sqrt{x^2 - 1}}}{x + \sqrt{x^2 - 1}} = \frac{1}{\sqrt{x^2 - 1}}, \forall x \succ 1.\end{aligned}$$

4.4.2 Hyperbolic Sine and its Inverse

For $x \in \mathbb{R}$, the hyperbolic sine is:

$$\sinh x = \frac{\exp(x) - \exp(-x)}{2}.$$

The function $\sinh : \mathbb{R} \rightarrow \mathbb{R}$.

Derivative of the hyperbolic sine function is

$$\sinh' x = \frac{\exp(x) + \exp(-x)}{2} = \cosh x.$$

The inverse function $\sinh$ is $\arg \sinh : \mathbb{R} \rightarrow \mathbb{R}$, and this function is strictly increasing and continuous, by the forms

$$\arg \sinh x = \ln \left(x + \sqrt{x^2 + 1} \right), \forall x \in \mathbb{R}.$$

$\arg \sinh$ is derivable and derivative is

$$\arg \sinh' x = \frac{1}{\sqrt{x^2 + 1}}, \forall x \in \mathbb{R}.$$

Because

$$\begin{aligned}\arg \sinh' x &= \left(\ln \left(x + \sqrt{x^2 + 1} \right) \right)' \\ &= \frac{1 + \frac{2x}{2\sqrt{x^2 + 1}}}{x + \sqrt{x^2 + 1}} = \frac{1 + \frac{x}{\sqrt{x^2 + 1}}}{x + \sqrt{x^2 + 1}} \\ &= \frac{\frac{x + \sqrt{x^2 + 1}}{\sqrt{x^2 + 1}}}{x + \sqrt{x^2 + 1}} = \frac{1}{\sqrt{x^2 + 1}}.\end{aligned}$$

Limits

$$\lim_{x \rightarrow -\infty} \sinh x = -\infty, \lim_{x \rightarrow +\infty} \sinh x = +\infty.$$

4.4.3 Hyperbolic Tangent and Inverse

For $x \in \mathbb{R}$, the hyperbolic cosine is:

$$\tanh x = \frac{\sinh x}{\cosh x}.$$

The function $\tanh : \mathbb{R} \rightarrow]-1, 1[$.

Derivative of the hyperbolic tangent function is

$$\tanh' x = 1 - \tanh^2 x = \frac{1}{\cosh^2 x}.$$

Because

$$\begin{aligned} \tanh' x &= \left(\frac{\sinh x}{\cosh x} \right)' = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} \\ &= \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = 1 - \tanh^2 x = \frac{1}{\cosh^2 x}. \end{aligned}$$

The inverse function $\tanh$ is $\arg \tanh :]-1, 1[\rightarrow \mathbb{R}$, and this function is strictly creasing and continuous, by the forms

$$\arg \tanh x = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right), \forall x \in]-1, 1[.$$

$\arg \tanh$ is derivable and derivative is

$$\arg \tanh' x = \frac{1}{1-x^2}, \forall x \in]-1, 1[.$$

Because

$$\begin{aligned} \arg \tanh' x &= \left(\frac{1}{2} \ln \left(\frac{1+x}{1-x} \right) \right)' \\ &= \left(\frac{1}{2} [\ln(1+x) - \ln(1-x)] \right)' \\ &= \frac{1}{2} \left[\frac{1}{1+x} + \frac{1}{1-x} \right] \\ &= \frac{1}{2} \left[\frac{2}{1-x^2} \right] = \frac{1}{1-x^2}. \end{aligned}$$

4.4.4 Hyperbolic Cotangent and its Inverse

For $x \in \mathbb{R}$, the hyperbolic cotangent is:

$$\coth x = \frac{\cosh x}{\sinh x}.$$

The function $\coth : \mathbb{R}^* \rightarrow \mathbb{R} - [-1, 1]$.

Derivative of the hyperbolic cotangent function is

$$\coth' x = - \left(1 + \coth^2 x \right) = - \frac{1}{\sinh^2 x}.$$

Because

$$\begin{aligned} \coth' x &= \left(\frac{\cosh x}{\sinh x} \right)' = \frac{\sinh^2 x - \cosh^2 x}{\sinh^2 x} \\ &= \frac{\sinh^2 x}{\sinh^2 x} - \frac{\cosh^2 x}{\sinh^2 x} = 1 - \coth^2 x = - \frac{1}{\sinh^2 x}. \end{aligned}$$

The inverse function $\coth$ is $\arg \coth : \mathbb{R} - [-1, 1] \rightarrow \mathbb{R}^*$, and this function is strictly creasing and continuous, by the forms

$$\arg \coth x = \frac{1}{2} \ln \left(\frac{1+x}{1-x} \right), \forall |x| > 1.$$

$\arg \coth$ is derivable and derivative is

$$\arg \coth' x = \frac{1}{1-x^2}, \forall |x| > 1.$$

Because

$$\begin{aligned} \arg \coth' x &= \left(\frac{1}{2} \ln \left(\frac{1+x}{x-1} \right) \right)' \\ &= \left(\frac{1}{2} [\ln(1+x) - \ln(x-1)] \right)' \\ &= \frac{1}{2} \left[\frac{1}{1+x} - \frac{1}{x-1} \right] \\ &= \frac{1}{2} \left[\frac{-2}{x^2-1} \right] = \frac{1}{1-x^2}. \end{aligned}$$

4.4.5 Hyperbolic Trigonometry

$$\sinh' x = \cosh x, \cosh' x = \sinh x.$$

$$\cosh^2 x - \sinh^2 x = 1.$$

$$\cosh(a+b) = \cosh a \cosh b + \sinh a \sinh b.$$

$$\cosh(2a) = \cosh^2 a + \sinh^2 a = 2 \cosh^2 a - 1 = 2 \sinh^2 a + 1.$$

$$\sinh(a+b) = \sinh a \cosh b + \cosh a \sinh b.$$

$$\sinh(2a) = 2 \sinh a \cosh a.$$

$$\tanh(a+b) = \frac{\tanh a + \tanh b}{1 + \tanh a \tanh b}.$$

Limited Development

5.1 Taylor Formula

5.1.1 Taylor Formula with the Rest of Lagrange

Definition 5.1

If f is a numerical function of the class C^n that includes I the number x_0 such that it accepts $f^{(n)}$ differentiation on I , then for every number x of I there is a number c , completely confined between x and x_0 , such that

$$\begin{aligned} f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 \\ + \dots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}. \end{aligned} \quad (5.1)$$

Definition 5.2

[2] The relationship (5.1) is called the propagation or Taylor formula with the rest Lagrange of the order n of the function f between x and x_0 , we written

$$f(x) = P_n(x) + R_n(x),$$

where

$$P_n(x) = \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k, R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} (x - x_0)^{n+1}.$$

Proposition 5.1 (*Special Case*)

In the special case $x_0 = 0$, the relationship (5.1) is called the Maclaurin's formula with the rest Lagrange of the order n of the function f , we written

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \dots + \frac{f^{(n)}(0)}{n!}x^n + \frac{f^{(n+1)}(c)}{(n+1)!}x^{n+1}.$$

Remark 5.1

1. The Maclaurin's formula is often used to calculate approximate values.
2. If f is a polynomial function. From the degree n , then the rest

$$\frac{f^{(n+1)}(c)}{(n+1)!} (x - x_0)^{n+1} = 0,$$

we written

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!} (x - x_0) + \frac{f''(x_0)}{2!} (x - x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n.$$

Example 5.1

Consider the function $f(x) = \exp x$. Calculate Taylor formula of this function near the point $x_0 = 1$, four degree

$$\begin{aligned} \exp x &= 1 + \frac{\exp(1)}{1!} (x - 1) + \frac{\exp(1)}{2!} (x - 1)^2 + \frac{\exp(1)}{3!} (x - 1)^3 \\ &+ \frac{\exp(1)}{4!} (x - 1)^4 + \frac{\exp(c)}{5!} (x - 1)^5. \end{aligned}$$

5.1.2 Taylor Formula with the Rest of Young**Definition 5.3**

[3] If $f : I \rightarrow R$ is a numerical function of the class C^n that includes I the number x_0 such that it accepts $f^{(n)}$ differentiation on I , then for every number x of I there is a function ε , completely confined between x and x_0 , such that

$$\begin{aligned} f(x) &= f(x_0) + \frac{f'(x_0)}{1!} (x - x_0) + \frac{f''(x_0)}{2!} (x - x_0)^2 \\ &+ \dots + \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n + (x - x_0)^n \varepsilon(x). \end{aligned} \tag{5.2}$$

Definition 5.4

The relationship (5.2) is called the propagation or Taylor formula with the rest Young

of the order n of the function f between x and x_0 , we written

$$f(x) = P_n(x) + R_n(x),$$

where

$$P_n(x) = \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k, R_n(x) = (x - x_0)^n \varepsilon(x).$$

Remark 5.2

1. We using Taylor formula with the rest Young formula is often used to calculate limits.
2. If f is a polynomial function. From the degree n , then the rest

$$\varepsilon(x)(x - x_0)^n = O((x - x_0)^n),$$

we written

$$f(x) = f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \dots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + O((x - x_0)^n).$$

Example 5.2

Consider the function $f(x) = \cos x$. Calculate Taylor formula of this function near the point $x_0 = \pi$, with four degree, there are exist function $\lim_{x \rightarrow \pi} \varepsilon(x) = 0$.

$$\cos x = -1 + \frac{1}{2!}(x - \pi)^2(x - \pi)^3 - \frac{1}{4!}(x - \pi)^4 + \varepsilon(x)(x - \pi)^4.$$

5.1.3 Taylor Formula with the Rest of Integral

Definition 5.5

If $f : I \rightarrow \mathbb{R}$ is a numerical function of the class C^{n+1} that includes I the number x_0 such that it accepts $f^{(n)}$ differentiation on I , completely confined between x and x_0 ,

such that

$$\begin{aligned} f(x) = & f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 \\ & + \dots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \frac{1}{n!} \int_{x_0}^x f^{(n+1)}(t) (x - t)^n dt. \end{aligned} \quad (5.3)$$

Definition 5.6

The relationship (5.3) is called the propagation or Taylor formula with the rest integrate of the order n of the function f in x_0 , we written

$$f(x) = P_n(x) + R_n(x),$$

where

$$P_n(x) = \sum_{k=0}^n \frac{f^{(k)}(x_0)}{k!} (x - x_0)^k, R_n(x) = \frac{1}{n!} \int_{x_0}^x f^{(n+1)}(t) (x - t)^n dt.$$

5.1.4 Taylor Formula and Boundary Values of a Function

Definition 5.7

If $f : I \rightarrow \mathbb{R}$ is a numerical function of the class C^n that includes I the number x_0 and $n \geq 1$, such that

$$f(x_0) = f^{(n)}(x_0) \neq 0, f'(x_0) = f''(x_0) = \dots = f^{(n-1)}(x_0) = 0.$$

1. If n is even, then

a) If $f^{(n)}(x_0) < 0$, then $f(x_0)$ is a local maximum value.

b) If $f^{(n)}(x_0) > 0$, then $f(x_0)$ is a local minimum value.

2. If n is odd, then $M(x_0, y_0)$ is an inflection point for the function curve.

Remark 5.3 (Special Case)

If f is a numerical function of the class C^2 that includes I the number x_0 , such that

$f'(x_0) = 0$ and $f''(x_0) \neq 0$, then

1. If $f''(x_0) < 0$, then $f(x_0)$ is a local maximum value.
2. If $f''(x_0) > 0$, then $f(x_0)$ is a local minimum value.

Example 5.3

1. Consider the function $f(x) = x + \frac{1}{x}$, of the class $C^2(\mathbb{R}^*)$ and

$$f'(x) = 1 - \frac{1}{x^2}, f''(x) = \frac{2}{x^3}, \forall x \in \mathbb{R}^*.$$

- Let $f'(-1) = 0$ and $f''(-1) = -2 < 0$. Then $f'(-1) = -2$ is a local maximum value.
- Let $f'(1) = 0$ and $f''(1) = 2 > 0$. Then $f'(1) = 2$ is a local minimum value.

2. Consider the function $f(x) = x^4 - x^3$, of the class $C^n(\mathbb{R})$ and $n \geq 1$ and

$$f'(0) = f''(0) = 0, \forall x \in \mathbb{R}. \text{ Also } f'''(0) = -6 \neq 0.$$

Hence $M(0, 0)$ is an inflection point for the function curve.

5.2 Limited Development

Let f be a numerical function defined on an open interval I , and $x_0 \in I$, or (defined on $\mathbb{R} - \{x_0\}$), and n a number natural. It is said that f has a limited Development of order n at x_0 , if there exist the real numbers $a_0, a_1, \dots, a_n$, and the function ε defined on I verifies $\lim_{x \rightarrow x_0} \varepsilon(x) = 0$, such that for every x in I , we have

$$\begin{aligned} f(x) &= a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + (x - x_0)^n \varepsilon(x) \quad (5.4) \\ &= \sum_{i=0}^{i=n} a_i(x - x_0)^i + R(x), R(x) = (x - x_0)^n \varepsilon(x), \text{ where } \lim_{x \rightarrow x_0} R(x) = 0. \end{aligned}$$

Remark 5.4

The relationship (5.4) can be formulated as:

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + O((x - x_0)^n),$$

$$\lim_{x \rightarrow x_0} \frac{O((x - x_0)^n)}{(x - x_0)^n} = 0.$$

Example 5.4

Let $\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2} = \frac{1}{2}$, we put $\varepsilon(x) = \frac{1}{2} - \frac{1 - \cos x}{x^2} = 0$, we have $\lim_{x \rightarrow x_0} \varepsilon(x) = 0$. Then

$$\cos x = 1 - \frac{1}{2}x^2 + x^2\varepsilon(x),$$

this is limited Development of order 2 of the function $\cos$ at $x_0 = 0$.

5.2.1 Publication Properties Limited**Definition 5.8**

[2] [Necessary condition] If f accepts a limited development of order $n \geq 1$ at $x_0 \Rightarrow$
 $\lim_{x \rightarrow x_0} f(x) = a_0$.

Example 5.5

Let $f(x) = \frac{1}{(1-x)^2}$, then $\lim_{x \rightarrow 1} \frac{1}{(1-x)^2} = +\infty$. Hence, f does not accept a limited development when $x_0 = 1$.

Theorem 5.5

If f accepts a limited development of order $n \geq 1$ at x_0 and f is continuous at the point x_0 , i.e.,

$$f(x) = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n + (x - x_0)^n \varepsilon(x),$$

$$\lim_{x \rightarrow x_0} f(x) = f(x_0) = a_0.$$

Then f is differentiable at the point x_0 and $f'(x_0) = a_1$, because

$$\begin{aligned} & \lim_{x \rightarrow x_0} \frac{f(x) - f(x_0)}{x - x_0} \\ &= \lim_{x \rightarrow x_0} \left[a_1 + a_2(x - x_0)^1 + \dots + a_n(x - x_0)^{n-1} + (x - x_0)^{n-1} \varepsilon(x) \right] \\ &= a_1. \end{aligned}$$

Definition 5.9

If f accepts a limited development of order n at x_0 , as form:

$$\begin{aligned} f(x) &= a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \dots + a_n(x - x_0)^n \\ &\quad + (x - x_0)^n \varepsilon(x), \\ \lim_{x \rightarrow x_0} \varepsilon(x) &= 0. \end{aligned}$$

This development is unique.

Definition 5.10

If f is a function of the class C^n at the point x_0 , Taylor formula with the rest of Young of the order n of the function f at the point x_0 is the limited development of the order n of the function f at x_0 , where

$$a_k = \frac{f^{(k)}(x_0)}{k!}, k = 0, \dots, n.$$

Definition 5.11

If f is a function that accepts a limited development of rank n at the point x_0 , then f accepts a final development of rank (for $p \leq n$) at the point x_0 .

Definition 5.12

If f is a function that accepts a limited development of rank n at the point 0, we have

- If f is an even function, then the polynomial $P_n(x)$ for the function f is an even polynomial.
- If f is an odd function, then the polynomial $P_n(x)$ for the function f is an odd polynomial.

5.2.2 Limited Development at ∞

Definition 5.13

Let f be a numerical function defined on an interval I as the form $]a, +\infty[$ or $]-\infty, a[$, and n a natural number. It is said that f has a limited Development of order n at $+\infty$ (or $-\infty$) if the function accepts

$$g : t \rightarrow f\left(\frac{1}{t}\right), \text{ where } t = \frac{1}{x}.$$

Limited development of order n at 0.

Example 5.6

Develop the function f defined by $f(x) = \frac{x}{x-1}$ of the order 3 at $+\infty$, develop the function:

$$g : t \rightarrow f\left(\frac{1}{t}\right).$$

i.e.

$$g(t) = f\left(\frac{1}{t}\right) = \frac{\frac{1}{t}}{\frac{1}{t}-1} = \frac{1}{t} \frac{1}{\frac{1-t}{t}} = \frac{1}{1-t},$$

the function g of order 3 at the point 0. Let

$$g(t) = f\left(\frac{1}{t}\right) = \frac{1}{1-t} = 1 + t + t^2 + t^3 + t^3\varepsilon(t).$$

Then the limited development of the function f of order 3 at $+\infty$ is

$$f(x) = 1 + \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \frac{1}{x^3}h(t),$$

where

$$\lim_{x \rightarrow +\infty} h(x) = \lim_{x \rightarrow +\infty} \varepsilon\left(\frac{1}{x}\right) = 0.$$

5.2.3 Application of Limited Development at 0

1. $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots + x^n + O(x^n).$
2. $\frac{1}{1+x} = 1 - x + x^2 - x^3 + \dots + (-1)^n x^n + O(x^n).$

-
3. $\ln(1-x) = -x + \frac{x^2}{2} - \frac{x^3}{3} - \dots - \frac{x^n}{n} + O(x^n).$
 4. $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} + \dots + (-1)^{n-1} \frac{x^n}{n} + O(x^n).$
 5. $\exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + O(x^n).$
 6. $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + (-1)^n \frac{x^{2n}}{(2n)!} + O(x^{2n+1}).$
 7. $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + (-1)^n \frac{x^{2n+1}}{(2n+1)!} + O(x^{2n+2}).$
 8. $\tan(x) = x + \frac{x^3}{3} + \frac{2x^5}{15} + \frac{17x^7}{315} + O(x^8).$
 9. $\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots + \frac{x^{2n}}{(2n)!} + O(x^{2n+1}).$
 10. $\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots + \frac{x^{2n+1}}{(2n+1)!} + O(x^{2n+2}).$

5.2.4 Operations on Limited Development

Let f and g be two numerical functions, each of which accepts a limited development of order n at the point x_0 :

$$f(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \dots + a_n(x-x_0)^n + O_1((x-x_0)^n)$$

$$f(x) = P(x) + O_1((x-x_0)^n).$$

$$g(x) = b_0 + b_1(x-x_0) + b_2(x-x_0)^2 + \dots + b_n(x-x_0)^n + O_2((x-x_0)^n)$$

$$g(x) = Q(x) + O_2((x-x_0)^n).$$

Sum

The function $\alpha f + \beta g$ (where $(\alpha, \beta) \in \mathbb{R}^2$), accepts limited development of the order n at x_0 , as the form

$$(\alpha f + \beta g)(x) = \alpha P(x) + \beta Q(x) + O((x-x_0)^n).$$

Example 5.7

Find the limited development of two functions $\cosh(x)$ and $\sinh(x)$ of order 4 at the point 0?

Limited development of two functions $\exp(x)$ and $\exp(-x)$ of order 4 at the point 0

$$P(x) = \exp(x) = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + O_1(x^4).$$

$$Q(x) = \exp(-x) = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + O_2(x^4).$$

Then the limited development of $\cosh(x)$ and $\sinh(x)$ are

$$\sinh(x) = \frac{\exp(x) - \exp(-x)}{2} = x + \frac{x^3}{3!} + O(x^4).$$

$$\cosh(x) = \frac{\exp(x) + \exp(-x)}{2} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + O(x^4).$$

Product

The function $f \times g$ accepts a limited development of order n at x_0 , as the form

$$(f \times g)(x) = P(x) \times Q(x) + O((x - x_0)^n).$$

Example 5.8

Find product two functions $\cos(x) \times \sqrt{1+x}$ of order 3 at the point 0?

Limited development of two functions $\cos x$ and $\sqrt{1+x}$ of order 2 at the point 0

$$P(x) = \cos(x) = 1 - \frac{x^2}{2} + O_1(x^2).$$

$$Q(x) = \sqrt{1+x} = 1 + \frac{x}{2} - \frac{x^2}{8} + O_2(x^2).$$

Then the limited development of $\cos(x) \times \sqrt{1+x}$ is

$$\begin{aligned} \cos(x) \times \sqrt{1+x} &= \left(1 - \frac{x^2}{2} + O_1(x^2)\right) \left(1 + \frac{x}{2} - \frac{x^2}{8} + O_2(x^2)\right) \\ &= 1 + \frac{x}{2} - \frac{x^2}{8} - \frac{x^2}{2} + O(x^2). \\ &= 1 + \frac{x}{2} - \frac{5x^2}{8} + O(x^2). \end{aligned}$$

Quotient

If $\lim_{x \rightarrow x_0} g(x) \neq 0$, then the function $\frac{f}{g}$ accepts a limited development of order n at x_0 , as the form

$$\left(\frac{f}{g}\right)(x) = \frac{P(x)}{Q(x)} + O((x - x_0)^n).$$

Example 5.9

Let $\tan x = \frac{\sin x}{\cos x}$. Find the limited development of the function f of order 5 at the point 0?

The limited development of two functions $\sin x$ and $\cos x$ are

$$\begin{aligned} Q(x) = \cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + O_1(x^5). \\ P(x) = \sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} + O_2(x^5). \end{aligned}$$

Let

$$\begin{aligned} \frac{1}{Q(x)} &= \frac{1}{\cos(x)} = \frac{1}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} + O_1(x^5)} \\ &= \frac{1}{1 - \left[\frac{x^2}{2!} - \frac{x^4}{4!} + O_1(x^5)\right]} \end{aligned}$$

By using limited development of the function $\frac{1}{1-x}$, of order 5 at the point 0,

$$\begin{aligned} \frac{1}{Q(x)} &= \frac{1}{\cos(x)} \\ &= 1 + \left[\frac{x^2}{2!} - \frac{x^4}{4!} + O_1(x^5)\right] + \left[\frac{x^2}{2!} - \frac{x^4}{4!} + O_1(x^5)\right]^2 + O_1(x^5) \\ &= 1 + \frac{x^2}{2!} - \frac{x^4}{4!} + \frac{6x^4}{4!} + O_1(x^5) \\ &= 1 + \frac{x^2}{2} + \frac{5x^4}{24} + O_1(x^5). \end{aligned}$$

Then

$$\begin{aligned}
 \tan x &= P(x) \times \frac{1}{Q(x)} \\
 &= \left(x - \frac{x^3}{6} + \frac{x^5}{120} + O_2(x^5) \right) \left(1 + \frac{x^2}{2} + \frac{5x^4}{24} + O_1(x^5) \right) \\
 &= x + \frac{x^3}{2} + \frac{5x^5}{24} - \frac{x^3}{6} - \frac{x^5}{12} + \frac{x^5}{120} + O(x^5) \\
 &= x + \frac{x^3}{2} - \frac{x^3}{6} + \frac{25x^5}{120} - \frac{10x^5}{120} + \frac{x^5}{120} + O(x^5) \\
 &= x + \frac{(3-1)x^3}{6} + \frac{(25-10+1)x^5}{120} + O(x^5) \\
 &= x + \frac{2x^3}{6} + \frac{16x^5}{120} + O(x^5) \\
 &= x + \frac{x^3}{3} + \frac{2x^5}{15} + O(x^5).
 \end{aligned}$$

Composition

Let f be a numerical function that accepts a limited development of order n at the point x_0 .

Let g be a numerical function that accepts a limited development of order n at the point x_1 .

The function $g \circ f$ accepts a limited development of order n at x_0 , as the form

$$(g \circ f)(x) = Q(P(x)) + O((x - x_0)^n).$$

Example 5.10

Let $f(x) = \ln(x+1)$, Find the limited development of the function f of order 3 at the point 0?

We put

$$f(x) = \ln(x+1), g(t) = \sin t.$$

The limited development of two functions f and g are

$$\begin{aligned}
 f(x) &= \ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} + O(x^3), \\
 g(t) &= \sin t = t - \frac{t^3}{6} + O(t^3).
 \end{aligned}$$

Then

$$\begin{aligned}
 g(f(x)) &= \sin(\ln(x+1)) \\
 &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{\left(x - \frac{x^2}{2} + \frac{x^3}{3} + O(x^3)\right)^3}{6} + O(x^3) \\
 &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^3}{6} + O(x^3) \\
 &= x - \frac{x^2}{2} + \frac{x^3}{6} + O(x^3).
 \end{aligned}$$

5.3 Application of Limited Developments

5.3.1 Application Example

Example 5.11

Let

$$f(x) = \frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots + x^n + O(x^{n+1})$$

Derivative:

$$f'(x) = \frac{-1}{(1-x)^2} = 1 + 2x + 3x^2 + \dots + nx^{n-1} + O(x^n).$$

5.3.2 Calculated Limits

Theorem 5.6 (*Limit*)

If the function f has a limited development, i.e.

$$\begin{aligned}
 f(x) &= a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + a_3(x - x_0)^3 + \dots + a_n(x - x_0)^n \\
 &\quad + O((x - x_0)^n).
 \end{aligned}$$

Then

$$\lim_{x \rightarrow x_0} f(x) = a_0.$$

Theorem 5.7 (Unique)

If the function f has a limited development of order n , then this limited development is unique.

Example 5.12

Calculate

$$\lim_{x \rightarrow 0} \frac{3x^2 \sin^2 x}{\ln(1+x) - \tan x + \frac{1}{2} \sin^2 x}.$$

Let

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + O(x^4).$$

$$\sin(x) = x - \frac{x^3}{3!} + O(x^4).$$

$$\tan(x) = x + \frac{x^3}{3} + O(x^4).$$

$$3x^2 \sin^2 x = 3x^2 (x + O(x))^2 = 3x^4 + O(x^4),$$

$$\begin{aligned} & \ln(1+x) - \tan x + \frac{1}{2} \sin^2 x \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} - \left(x + \frac{x^3}{3}\right) + \frac{1}{2} \left(x - \frac{x^3}{3!}\right)^2 + O(x^4) \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} - x - \frac{x^3}{3} + \frac{x^2}{2} - \frac{x^4}{3!} + O(x^4) \\ &= -\frac{x^2}{2} + \frac{x^2}{2} - \frac{x^4}{4} - \frac{x^4}{6} + O(x^4) \\ &= -\frac{5x^4}{12} + O(x^4). \end{aligned}$$

Then

$$\frac{3x^2 \sin^2 x}{\ln(1+x) - \tan x + \frac{1}{2} \sin^2 x} = \frac{3x^4}{-\frac{5x^4}{12}} + O(x^4).$$

Hence

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{3x^2 \sin^2 x}{\ln(1+x) - \tan x + \frac{1}{2} \sin^2 x} &= \lim_{x \rightarrow 0} \frac{3x^4}{-\frac{5x^4}{12}} + O(x^4) \\ &= \lim_{x \rightarrow 0} -12 \times \frac{3}{5} + O(x^4) = -\frac{36}{5}. \end{aligned}$$

Example 5.13

calculate

$$\lim_{x \rightarrow 0} \frac{x(1 + \cos x) - 2 \tan x}{2x - \sin x - \tan x} = \frac{0}{0}.$$

By the limited development

$$\sin x = x - \frac{x^3}{6} + O(x^4), \cos x = 1 - \frac{x^2}{2} + O(x^3).$$

$$\tan x = x + \frac{x^3}{3} + O(x^4).$$

Then

$$\begin{aligned} \lim_{x \rightarrow 0} \frac{x(1 + \cos x) - 2 \tan x}{2x - \sin x - \tan x} &= \lim_{x \rightarrow 0} \frac{x \left(1 + 1 - \frac{x^2}{2} + O(x^3)\right) - 2 \left(x + \frac{x^3}{3} + O(x^4)\right)}{2x - \left(x - \frac{x^3}{6} + O(x^4)\right) - \left(x + \frac{x^3}{3} + O(x^4)\right)} \\ &= \lim_{x \rightarrow 0} \frac{2x - \frac{x^3}{2} - 2x - \frac{2x^3}{3} + O(x^4)}{2x - x + \frac{x^3}{6} - x - \frac{x^3}{3} + O(x^4)} \\ &= \lim_{x \rightarrow 0} \frac{-\frac{x^3}{2} - \frac{2x^3}{3} + O(x^4)}{\frac{x^3}{6} - \frac{x^3}{3} + O(x^4)} \\ &= \lim_{x \rightarrow 0} \frac{-\frac{7}{6}x^3}{-\frac{1}{6}x^3} = 7. \end{aligned}$$

Vector Space and Linear Maps

6.1 Vector Space

Underlying every vector space (to be defined shortly) is a **scalar field** B .

A field is a set of elements where the four basic operations $+$, $-$, $\times$, $\div$ are defined, with their usual properties (commutativity, associativity, distributivity).

Definition 6.1

[1] A **vector space** over a field B is a nonempty set D of objects, called vectors, on which are defined two operations:

1. An internal operation (vector addition)

$$\begin{aligned} + : D \times D &\rightarrow D \\ (x, y) &\rightarrow x + y \end{aligned}$$

2. An external scalar (scalars multiplication)

$$\begin{aligned} \cdot : B \times D &\rightarrow D \\ (\beta, x) &\rightarrow \beta x \end{aligned}$$

such that the following properties are satisfied:

1. $\forall \mathbf{u}, d \in D, u + d = d + u.$
2. $\forall \mathbf{u}, d, w \in D, (u + d) + w = u + (d + w).$
3. $\forall \mathbf{u} \in D, \exists 0_d \in D, u + 0 = u, (0_d \text{ is the zero vector}).$
4. $\forall \mathbf{u} \in D, \exists (-u) \in D, u + (-u) = 0_d.$
5. $\forall \mathbf{u}, d \in D, \forall \beta \in B, \beta(u + d) = \beta u + \beta d.$

$$6. \forall \mathbf{u} \in D, \forall \beta, \gamma \in B, (\beta + \gamma)u = \beta u + \gamma u.$$

$$7. \forall \mathbf{u} \in D, \forall \beta, \gamma \in B, (\beta\gamma)u = \beta(\gamma u).$$

$$8. \forall \mathbf{u} \in D, 1u = u.$$

Example 6.1

Prove that operation is commutative and accept a neutral element in $\mathbb{R}$.

$$\forall x, y \in \mathbb{R} : x * y = x(1 - y) + y(1 - x).$$

$*$ commutative

$$\forall x, y \in \mathbb{R} : x * y = x(1 - y) + y(1 - x)$$

$$\forall x, y \in \mathbb{R} : y * x = y(1 - x) + x(1 - y)$$

$$= x(1 - y) + y(1 - x)$$

$$= x * y$$

e is the neutral element of $\mathbb{R}$. Hence $*$ commutative, then find e on one side:

$$\forall x, e \in \mathbb{R} : x * e = e * x = x$$

$$x(1 - e) + e(1 - x) = x$$

$$x - xe + e - xe = x$$

$$-2xe + e = 0$$

$$e(1 - 2x) = 0 \implies e = 0 \vee x = \frac{1}{2}.$$

6.1.1 Subspace

Definition 6.2 (*Subspace*)

[3] A **subspace** of a vector space D is a nonempty subset F of D that has three properties:

$$1. 0_d \in F.$$

$$2. F \text{ is closed under vector addition. That is, } \forall \mathbf{u}, d \in F, u + d \in F.$$

3. F is closed under multiplication by scalars. That is, $\forall \mathbf{u} \in F, \forall \beta \in B, \beta \mathbf{u} \in F$.

Definition 6.3 (*Subspace*)

A **subspace** of a vector space D is a nonempty subset F of D if and only if:

$$\forall \mathbf{u}, d \in F, \forall \beta, \gamma \in B : (\beta \mathbf{u} + \gamma d) \in F.$$

Remark 6.1

Properties (1, 2, 3) guarantee that a subspace F of D is itself a vector space, under the vector space operations already defined in D .

Example 6.2

1. $\mathbb{R}^{n-1}$ is a subspace of $\mathbb{R}^n$.
2. $\{0_d\}$ is a subspace of D .
3. D is a subspace of D .

6.1.2 Linear Combinations

Definition 6.4

Let D be a vector space. We say that the vector u is a **linear combination** of the vectors $d_1, d_2, \dots, d_n$ of D if

$$\exists \beta_1, \beta_2, \dots, \beta_n \in B : \mathbf{u} = \beta_1 d_1 + \beta_2 d_2 + \dots + \beta_n d_n.$$

Example 6.3

Express $u = (2, 1)$ in $\mathbb{R}^2$ over $\mathbb{R}$ as a linear combination of the vectors $d_1 = (1, -1)$ and $d_2 = (-1, 2)$.

Let β_1, β_2 be scalars such that

$$u = \beta_1 d_1 + \beta_2 d_2$$

$$(2, 1) = \beta_1(1, -1) + \beta_2(-1, 2)$$

$$(2, 1) = (\beta_1 - \beta_2, -\beta_1 + 2\beta_2)$$

$$\beta_1 - \beta_2 = 2 \wedge -\beta_1 + 2\beta_2 = 1$$

$$\beta_1 = 2 + \beta_2 \wedge -2 - \beta_2 + 2\beta_2 = 1$$

$$\beta_1 = 5 \wedge \beta_2 = 3.$$

Then

$$u = 3d_1 + 5d_2.$$

6.1.3 Linear Independence and Linear Dependence

Definition 6.5 (*Linear Independence*)

Let the set $U = \{d_1, d_2, \dots, d_n\} \subset D$, a vector space. We say that U is **linearly independent** if all scalars $\beta_1, \beta_2, \dots, \beta_n$ are zero for which $\beta_1 d_1 + \beta_2 d_2 + \dots + \beta_n d_n = 0_D$. That is

$$\forall \beta_1, \beta_2, \dots, \beta_n \in B : \beta_1 d_1 + \beta_2 d_2 + \dots + \beta_n d_n = 0_D \implies \beta_1 = \beta_2 = \dots = \beta_n = 0_B.$$

Example 6.4

The set $U = \{(-1, 0), (2, 1)\}$ is **linearly independent**.

Let $\beta_1, \beta_2 \in \mathbb{R}$

$$\beta_1(-1, 0) + \beta_2(2, 1) = (0, 0)$$

$$-\beta_1 + 2\beta_2 = 0 \wedge \beta_2 = 0$$

$$\beta_1 = \beta_2 = 0.$$

Definition 6.6 (*Linear Dependence*)

The set $U = \{d_1, d_2, \dots, d_n\}$ is **linearly dependent** if there are scalars $\beta_1, \beta_2, \dots, \beta_n$

not all zero for which $\beta_1 d_1 + \beta_2 d_2 + \cdots + \beta_n d_n = 0_D$. That is:

$$\exists \beta_1, \beta_2, \dots, \beta_n \in B : \beta_i \neq 0, i \in \{1, 2, \dots, n\} \wedge \beta_1 d_1 + \beta_2 d_2 + \cdots + \beta_n d_n = 0_D.$$

Remark 6.2

- $\{u\}$ is **linearly independent** $\Leftrightarrow u \neq 0_D$
- $0_D \in U = \{d_1, d_2, \dots, d_n\} \implies U$ is **linearly dependent**.

Example 6.5

The set $U = \{(1, 0), (2, 0)\}$ is **linearly dependent**.

Let $\beta_1, \beta_2 \in \mathbb{R}$

$$\beta_1(1, 0) + \beta_2(2, 1) = (0, 0)$$

$$\beta_1 + 2\beta_2 = 0 \wedge 0 = 0$$

$$\beta_1 = -2\beta_2$$

$$\exists \beta_1 = 1 \wedge \beta_2 = -\frac{1}{2} \wedge \beta_1(1, 0) + \beta_2(2, 0) = (0, 0).$$

6.1.4 Generating Sets

Definition 6.7 (*Generating Sets*)

Given a vector space D , a finite set of vectors $U = \{d_1, d_2, \dots, d_n\} \subset D$ is called a system of **generators** if every vector $u \in D$ can be expressed as a linear combination of vectors of U :

$$\forall u \in D, \exists \beta_1, \beta_2, \dots, \beta_n \in B : u = \beta_1 d_1 + \beta_2 d_2 + \cdots + \beta_n d_n$$

and we write $D = [U]$.

Example 6.6

The set $U = \{(1, 1, 1), (1, 1, 0), (1, 0, 0)\}$ is a system of **generators** of $\mathbb{R}^3$:

Let $d = (x, y, z) \in \mathbb{R}^3$, we check the scalars $\beta, \gamma, \delta \in \mathbb{R}$ such that

$$\beta(1, 1, 1) + \gamma(1, 1, 0) + \delta(1, 0, 0) = (x, y, z)$$

Then

$$\beta + \gamma + \delta = x, \quad \beta + \gamma = y, \quad \beta = z$$

Hence

$$\delta = x - y, \quad \gamma = y - z, \quad \beta = z$$

Then

$$d = z(1, 1, 1) + (y - z)(1, 1, 0) + (x - y)(1, 0, 0).$$

6.1.5 Bases of a Vector Space

Definition 6.8 (*Bases*)

[1] Given a vector space D , a finite set of vectors $U = \{d_1, d_2, \dots, d_n\} \subset D$ is called a **basis** of D if it has two properties:

1. U is **linearly independent**.
2. U is a **generator** of D , that is $D = [U]$.

Example 6.7

Let $\mathbb{R}^n$ be a vector space over $\mathbb{R}$.

1. If $n = 1$, the basis of $\mathbb{R}$ is the set $U_1 = \{1\}$. $\mathbb{R}$ is generated by U_1 .

$\forall \beta \in \mathbb{R} : \beta \cdot 1 = 0 \implies \beta = 0$. Thus the set U_1 is linearly independent.

2. If $n = 2$, the basis of $\mathbb{R}^2$ is the set $U_2 = \{(1, 0), (0, 1)\}$. $\mathbb{R}^2$ is generated by U_2 .

$\forall \beta, \gamma \in \mathbb{R} : \beta \cdot (1, 0) + \gamma \cdot (0, 1) = (\beta, \gamma) = (0, 0) \implies \beta = \gamma = 0$. Thus the set U_2 is linearly independent.

3. If $n = 3$, the basis of $\mathbb{R}^3$ is the set $U_3 = \{(1, 0, 0), (0, 1, 0), (0, 0, 1)\}$. $\mathbb{R}^3$ is generated by U_3 .

$\forall \beta, \gamma, \delta \in \mathbb{R} : \beta(1, 0, 0) + \gamma(0, 1, 0) + \delta(0, 0, 1) = (\beta, \gamma, \delta) = (0, 0, 0) \implies \beta = \gamma = \delta = 0$. Thus the set U_3 is linearly independent.

Proposition 6.1

If U is a basis, every vector can be written as a linear combination of its elements in a unique way.

Example 6.8

Let the set $U = \{(1, 0), (1, 1)\}$, we can write any vector of $\mathbb{R}^2$ as a linear combination of $(1, 0)$ and $(1, 1)$ in a unique way.

6.1.6 Dimension of a Vector Space

Definition 6.9 (*Dimension*)

If a vector space D has a basis with finite number of elements, then every other basis of D has the same number of elements. This number is called the dimension of D . We write

$$\dim D = n.$$

Proposition 6.2

If we know that a vector space has dimension n ($\dim D = n$), then:

- Every basis consists of exactly n vectors (but not every set of n vectors is a basis!).
- Every system of generators has to contain at least n vectors.
- If a system of generators consists of n vectors, then it is a basis.
- If a set of n vectors is linearly independent, then it is a basis.

Proposition 6.3

Let D be a vector space of dimension n , and let F be a subspace of D . Then $\dim F \leq \dim D$. Furthermore

$$\text{if } \dim F = \dim D \text{ then } F = D.$$

Example 6.9

The set of three vectors $\{(1, -1, 1), (0, 2, -1), (3, 0, 1)\}$ is a basis of $\mathbb{R}^3$.

Let $3 = \dim \mathbb{R}^3 = 3$. Let $\beta_1, \beta_2, \beta_3 \in \mathbb{R}$

$$\beta_1(1, -1, 1) + \beta_2(0, 2, -1) + (3, 0, 1)\beta_3 = (0, 0, 0)$$

$$(\beta_1 + 3\beta_3, -\beta_1 + 2\beta_2, \beta_1 - \beta_2 + \beta_3) = (0, 0, 0)$$

Then

$$\beta_1 + 3\beta_3 = 0 \wedge -\beta_1 + 2\beta_2 = 0 \wedge \beta_1 - \beta_2 + \beta_3 = 0$$

$$\beta_3 = -\frac{\beta_1}{3} \wedge \beta_2 = \frac{\beta_1}{2} \wedge \beta_1 - \frac{\beta_1}{2} - \frac{\beta_1}{3} = 0$$

$$\beta_1 = \beta_2 = \beta_3 = 0$$

Example 6.10

The set of three vectors $\{(1, 2), (-1, 2), (3, 1)\}$ is not a basis of $\mathbb{R}^2$ because $3 > \dim \mathbb{R}^2 = 2$.

6.2 Linear Maps**Definition 6.10**

[1] A **linear map** g from a vector space D into a vector space H is a rule that assigns to each vector x in D a unique vector $g(x)$ in H , such that:

$$g : D \rightarrow H$$

$$x \rightarrow g(x)$$

$$1. \forall x_1, x_2 \in D : g(x_1 + x_2) = g(x_1) + g(x_2).$$

$$2. \forall x \in D, \forall \beta \in \mathbb{R} : g(\beta x) = \beta g(x).$$

Proposition 6.4

Here are some simple properties of linear maps $g : D \rightarrow H$.

$$1. g(0_D) = 0_H.$$

2. $g(-x) = -g(x)$.
3. If D_1 is a subspace of D , then $g(D_1)$ is a subspace of H .
4. If H_1 is a subspace of H , then $g^{-1}(H_1)$ is a subspace of D .
5. The composite map of two linear maps is a linear map.

Example 6.11

$g : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ defined as $g(x, y) = (3x - y, 2y)$ is **linear map**:

1.

$$\begin{aligned}
 g((x_1, y_1) + (x_2, y_2)) &= g(x_1 + x_2, y_1 + y_2) \\
 &= (3(x_1 + x_2) - (y_1 + y_2), 2(y_1 + y_2)) \\
 &= (3x_1 + 3x_2 - y_1 - y_2, 2y_1 + 2y_2) \\
 &= (3x_1 - y_1 + 3x_2 - y_2, 2y_1 + 2y_2) \\
 &= (3x_1 - y_1, 2y_1) + (3x_2 - y_2, 2y_2) \\
 &= g((x_1, y_1) + (x_2, y_2)) = g(x_1, y_1) + g(x_2, y_2).
 \end{aligned}$$

2.

$$g(\beta(x, y)) = g((\beta x, \beta y)) = (3(\beta x) - (\beta y), 2(\beta y)) = \beta(3x - y, 2y) = \beta g((x, y))$$

Example 6.12

$g : \mathbb{R} \rightarrow \mathbb{R}$ defined as $g(x) = 2x + 1$ is **not linear map**, because $g(0) = 1 \neq 0$.

6.2.1 Kernel of a Linear Map

Definition 6.11 (Kernel)

The **kernel** (or **null space**) of such a g , denoted by $\ker g$, is the set of all x in D such that

$$g(x) = 0_H$$

(the zero vector in H)

$$\ker g = \{x \in D, g(x) = 0_H\} = g^{-1}(\{0_H\}).$$

6.2.2 Image of a Linear Map

Definition 6.12 (*Image*)

[1] The **image** of g , denoted by $\text{Im } g$, is the set of all vectors in H of the form $g(x)$ for some x in D .

$$\text{Im } g = \{g(x), x \in D\} = g(D).$$

Remark 6.3

Let $g : D \rightarrow H$ be a linear map, then

1. The **kernel** of g is a subspace of D .
2. The **image** of g is a subspace of H .
3. g is **injective** $\iff \ker g = \{0_D\}$.
4. g is **surjective** $\iff \text{Im } g = g(D) = H$.

Example 6.13

Prove that map $g : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ defined as $g(x, y, z) = (x + y, z)$ is surjective and not injective?

1. g is injective $\iff \ker g = \{0_{\mathbb{R}^3}\}$

$$\begin{aligned} \ker g &= \{(x, y, z) \in \mathbb{R}^3, g(x, y, z) = (0, 0)\} \\ &= \{(x, y, z) \in \mathbb{R}^3, (x + y, z) = (0, 0)\} \\ &= \{(x, y, z) \in \mathbb{R}^3, x = -y \wedge z = 0\}. \end{aligned}$$

Remark $(-1, 1, 0) \in \ker g$, then $\ker g \neq \{0_{\mathbb{R}^3}\}$. Hence g is not injective.

2. g is surjective $\iff \text{Im } g = g(\mathbb{R}^3) = \mathbb{R}^2$

$$\begin{aligned}\text{Im } g &= \{g(x, y, z), (x, y, z) \in \mathbb{R}^3\} \\ &= \{(x + y, z), (x, y, z) \in \mathbb{R}^3\} \\ &= \{x(1, 0) + y(1, 0) + z(0, 1) \mid (x, y, z) \in \mathbb{R}^3\}\end{aligned}$$

Hence, $\text{Im } g$ is generated by two vectors $(1, 0), (1, 0)$ which are the canonical basis of $\mathbb{R}^2$. Then $\text{Im } g = \mathbb{R}^2$ and g is surjective.

Proposition 6.4

Let $g : D \rightarrow H$ be a linear map, with D finite-dimensional. Then:

$$\dim D = \dim \ker g + \dim \text{Im } g.$$

6.2.3 Rank of a Linear Map

Definition 6.13 (*Rank*)

[1] The rank of a linear map g is the dimension of its image, written $\text{rank } g$:

$$\text{rank } g = \dim \text{Im } g.$$

Example 6.14

Find $\text{Im } g$, $\ker g$ and $\text{rank } g$ of the map $g : \mathbb{R}^4 \rightarrow \mathbb{R}^3$ defined as $g(x, y, z, t) = (x - y, z + t, x - y + z)$.

1.

$$\begin{aligned}\text{Im } g &= \{g(x, y, z, t), (x, y, z, t) \in \mathbb{R}^4\} \\ &= \{(x - y, z + t, x - y + z), (x, y, z, t) \in \mathbb{R}^4\} \\ &= \{(x - y)(1, 0, 1) + t(0, 1, 0) + z(0, 1, 1) \mid (x, y, z, t) \in \mathbb{R}^4\}\end{aligned}$$

Hence, $\text{Im } g$ is generated by three vectors $\{(1, 0, 1), (0, 1, 0), (0, 1, 1)\}$ which are the canonical basis of $\mathbb{R}^3$. Then $\text{Im } g = \mathbb{R}^3$.

Let $\beta_1, \beta_2, \beta_3 \in \mathbb{R}$

$$\beta_1 (1, 0, 1) + \beta_2 (0, 1, 0) + (0, 1, 1) \beta_3 = (0, 0, 0)$$

$$(\beta_1, \beta_2 + \beta_3, \beta_1 + \beta_3) = (0, 0, 0)$$

$$\beta_1 = 0 \wedge \beta_2 + \beta_3 = 0 \wedge \beta_1 + \beta_3 = 0$$

$$\beta_1 = \beta_2 = \beta_3 = 0$$

Therefore, the set $\{(1, 0, 1), (0, 1, 0), (0, 1, 1)\}$ is linearly independent and it is the basis of $\text{Im } g$.

2.

$$\begin{aligned} \ker g &= \{(x, y, z, t) \in \mathbb{R}^4, g(x, y, z, t) = (0, 0, 0)\} \\ &= \{(x, y, z, t) \in \mathbb{R}^4, (x - y, z + t, x - y + z) = (0, 0, 0)\} \\ &= \{(x, y, z, t) \in \mathbb{R}^4, (x - y = 0 \wedge z + t = 0 \wedge x - y + z = 0)\} \\ &= \{(x, y, z, t) \in \mathbb{R}^4, (x = y \wedge z = -t \wedge z = 0)\} \\ &= \{(x, x, 0, 0), x \in \mathbb{R}\} = \{x(1, 1, 0, 0), x \in \mathbb{R}\}. \end{aligned}$$

Then $\ker g = \text{span} \{(1, 1, 0, 0)\}$.

3.

$$\text{rank } g = \dim \text{Im } g = 3.$$

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