



PEOPLE'S DEMOCRATIC REPUBLIC OF ALGERIA  
MINISTRY OF HIGHER EDUCATION AND SCIENTIFIC  
RESEARCH

University Mohamed Khider of Biskra  
Faculty of Exact Sciences  
Department of Mathematics

## Course of Mathematics 2

Intended for students in the first year

Sciences & Technologies  
Renewable energies

Presented by :Dahbia HAFAYED

College year 2025-2026

## *Semester2*

Teaching Unit: UEF 2.1.1

Subject: Mathematics 2

Total Hours: 67h30 (Lectures: 3h00, Tutorials: 1h30)

Credits: 6

Coefficient: 3

Objectives

This first mathematics course is primarily dedicated to standardizing the skill level of incoming university students. New concepts are introduced progressively to guide students towards more advanced mathematics. The topics covered in this course are fundamental and widely used in the fields of Science and Technology. Prerequisites A solid understanding of basic mathematical concepts from high school is recommended, including sets, functions, and equations.

Course Content:

Chapter 1: Matrices and Determinants (3 Weeks)

- 1.1 Matrices (definition and operations)
- 1.2 Matrix associated with a linear application
- 1.3 Linear application associated with a matrix
- 1.4 Change of basis and transition matrix

Chapter 2: Systems of Linear Equations (2 Weeks)

- 2.1 General concepts
- 2.2 Study of the set of solutions
- 2.3 Methods for solving a linear system:

- Cramer's method
- Inverse matrix method
- Gaussian elimination method

Chapter 3: Integrals (4 Weeks)

3.1 Indefinite integral and its properties

3.2 Integration of rational functions

3.3 Integration of exponential and trigonometric functions

3.4 Integration of polynomials

3.5 Definite integrals

Chapter 4: Differential Equations (4 Weeks)

4.1 Ordinary differential equations (ODEs)

4.2 First-order differential equations

4.3 Second-order differential equations

4.4 Second-order ordinary differential equations with constant coefficients

Chapter 5: Functions of Several Variables (2 Weeks)

5.1 Limit, continuity, and partial derivatives of a function

5.2 Differentiability

5.3 Double and triple integrals

Evaluation Method

Continuous Assessment: 40 Final Exam: 60

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# Introduction

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At the beginning of this booklet, we present the topics included in the second-semester program for first-year L.M.D. students in Science and Technology, specializing in Renewable Energies. The content is designed both to consolidate the knowledge acquired during the final year of secondary school and to introduce new chapters not previously studied. It provides a solid foundation for more advanced courses and serves as a bridge between elementary and higher-level subjects.

The material has been carefully adapted to meet the needs of Renewable Energy students, focusing on the mathematical concepts most relevant to their studies. Unnecessary complexity has been avoided in order to reduce difficulties while remaining faithful to the prescribed syllabus.

To address the challenges students often face in understanding this subject, numerous solved examples and practical applications are provided in the simplest possible form, while ensuring accuracy in the terminology used.

We sincerely hope that this booklet will be of benefit to our students and will also be appreciated by colleagues and specialists in the field.

## *Matrices and Determinates*

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### 1.1 Introduction of matrices

#### 1.1.1 Definition of a matrix

##### **Definition 1.1** (*Matrix*)

[9] Let  $m$  and  $n$  be two non-zero natural numbers. We call an  $m \times n$  matrix a matrix whose elements belong to the field  $\mathbb{K}$ , where  $\mathbb{K}$  can be either the set of real numbers ( $\mathbb{K} = \mathbb{R}$ ) or the set of complex numbers ( $\mathbb{K} = \mathbb{C}$ ). The matrix is arranged in a table with  $m$  rows and  $n$  columns, enclosed in brackets or braces, as follows:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}, \text{ or } A = \left[ \begin{array}{cccc} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{array} \right] = (a_{ij})_{m \times n}.$$

- We denote the matrix by the letters  $A, B, C, \dots$  or  $A = a_{ij}$ , where  $1 \leq i \leq m$  and  $1 \leq j \leq n$ .
- We call the element  $a_{ij}$  the coefficient or matrix element, and it is located in the  $i$  row and  $j$  column.
- If the matrix has  $m$  rows and  $n$  columns, we say that it has a dimension or rank of  $m \times n$ . We refer to this as the size or order  $m \times n$ .
- We denote the set of matrices of size  $m \times n$  with coefficients from the field  $\mathbb{K}$  by  $M_{m,n}(\mathbb{K})$ .

**Example 1.1**

$$A = \begin{pmatrix} 2 & 5 & -8 \\ 0 & -3 & -4 \\ 6 & 8 & 9 \end{pmatrix}.$$

**1.1.2 Special types of matrices**

There are several forms of matrices:

**Square matrix****Definition 1.2 (*Square matrix*)**

A matrix in which the number of rows is equal to the number of columns is called a square matrix. In that case, we denote the set of square matrices of size  $n \times n$  with elements from the field  $\mathbb{K}$  as  $M_n(\mathbb{K})$ .

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}$$

**Example 1.2**

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, B = \begin{pmatrix} 2 & 5 & -8 \\ 0 & -3 & -4 \\ 6 & 8 & 9 \end{pmatrix}.$$

**Diagonal matrix****Definition 1.3 (*Diagonal matrix*)**

A square matrix  $A = (a_{ij})_{n \times n}$  is called a diagonal matrix if each of its non-diagonal

elements is zero. That is,  $a_{ij} = 0$  if  $i \neq j$  and at least one element  $a_{ii} \neq 0$ .

$$A = \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix}.$$

### Example 1.3

$$A = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}, B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 9 \end{pmatrix}.$$

## Identity matrix

### Definition 1.4 (*Identity Matrix*)

A diagonal matrix whose diagonal elements are equal to 1 is called an identity matrix and is denoted by  $I_n$ . That is

$$a_{ij} = \begin{cases} 0 & \text{if } i \neq j, \\ 1 & \text{if } i = j. \end{cases}$$

Then

$$A = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}.$$

### Example 1.4

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

**Upper triangular matrix****Definition 1.5 (*Upper triangular matrix*)**

A square matrix is said to be an upper triangular matrix if  $a_{ij} = 0$  for all  $i \succ j$ .

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ 0 & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{pmatrix}.$$

**Example 1.5**

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{pmatrix}, B = \begin{pmatrix} 2 & 5 & -8 \\ 0 & -3 & -4 \\ 0 & 0 & 9 \end{pmatrix}.$$

**Lower triangular matrix****Definition 1.6 (*Lower Triangular matrix*)**

A square matrix is said to be an lower triangular matrix if  $a_{ij} = 0$  for all  $i \prec j$ .

$$A = \begin{pmatrix} a_{11} & 0 & \dots & 0 \\ a_{21} & a_{22} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}.$$

**Example 1.6**

$$A = \begin{pmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, B = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 6 & 8 & 9 \end{pmatrix}.$$

**Symmetric matrix****Definition 1.7 (*Symmetric matrix*)**

A square matrix  $A = (a_{ij})_{n \times n}$  is said to be symmetric if  $a_{ij} = a_{ji}$  for all  $i$  and  $j$ .

**Example 1.7**

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}, B = \begin{pmatrix} 2 & 5 & -8 \\ 5 & -3 & -4 \\ -8 & -4 & 9 \end{pmatrix}.$$

**Skew-symmetric matrix****Definition 1.8 (*Skew-symmetric matrix*)**

A square matrix  $A = (a_{ij})_{n \times n}$  is said to be skew-symmetric if  $a_{ij} = -a_{ji}$  for all  $i$  and  $j$ .

**Example 1.8**

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ -a_{12} & a_{22} & a_{23} \\ -a_{13} & -a_{23} & a_{33} \end{pmatrix}, B = \begin{pmatrix} 2 & 5 & -8 \\ -5 & -3 & -4 \\ 8 & 4 & 9 \end{pmatrix}.$$

**Zero matrix****Definition 1.9 (*Zero matrix*)**

A matrix whose all elements are zero is called a zero matrix, and the order zero matrix is denoted by  $0_{n \times m}$ .

$$A = \begin{pmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 \end{pmatrix} = 0.$$

**Example 1.9**

$$0_{3 \times 3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

**Row vector****Definition 1.10 (*Row vector*)**

A matrix consisting of a single row is called a row vector or row matrix.

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \end{pmatrix}.$$

**Example 1.10**

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \end{pmatrix}, B = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}.$$

**Column vector****Definition 1.11 (*Column vector*)**

A matrix consisting of a single column is called a column vector or column matrix.

$$A = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix}.$$

**Example 1.11**

$$A = \begin{pmatrix} a_{11} \\ a_{21} \\ a_{31} \end{pmatrix}, B = \begin{pmatrix} 1 \\ 4 \\ 7 \end{pmatrix}.$$

### Equality of two matrices

#### Definition 1.12 (*Equality of two matrices*)

Two matrices  $A = (a_{ij})_{m \times n}$  and  $B = (b_{ij})_{m \times n}$  are said to be equal if:

1. They are of the same order (if they have the same number of rows  $m$  and the same number of columns  $n$ ).
2. Their corresponding elements are equal ( $a_{ij} = b_{ij}$  for all  $1 \leq i \leq m$  and  $1 \leq j \leq n$ ).

#### Example 1.12

$$A = \begin{pmatrix} 1 & -1 \\ 2 & 0 \\ 3 & 5 \\ 4 & 0 \end{pmatrix} = B = \begin{pmatrix} 1 & -1 \\ 2 & 0 \\ 3 & 5 \\ 4 & 0 \end{pmatrix}.$$

### 1.1.3 Operation on matrices

#### Scalar multiple of a matrix

#### Definition 1.13 (*Scalar multiple of a matrix*)

Let  $k$  be a scalar ( $k \in \mathbb{R}, k \in \mathbb{C}$ ), then the scalar product of matrix  $A = (a_{ij})_{m \times n}$  is denoted by  $kA$  and given by  $kA = (ka_{ij})_{m \times n}$  (for all  $1 \leq i \leq m$  and  $1 \leq j \leq n$ )

$$kA = \begin{pmatrix} ka_{11} & ka_{12} & \dots & ka_{1n} \\ ka_{21} & ka_{22} & \dots & ka_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ ka_{m1} & ka_{m2} & \dots & ka_{mn} \end{pmatrix}.$$

#### Example 1.13

$$\text{Let } A = \begin{pmatrix} 1 & -2 & 3 \\ 4 & 5 & -4 \end{pmatrix}. \text{ Then } 3A = 3 \begin{pmatrix} 1 & -2 & 3 \\ 4 & 5 & -4 \end{pmatrix} = \begin{pmatrix} 9 & -6 & 9 \\ 12 & 15 & -12 \end{pmatrix}.$$

**Remark 1.1**

The matrix  $(-1) \times A$ , represented by  $-A$ , enables us to define the difference between two matrices  $A$  and  $B$ , both having the same number of rows and columns, as follows:

$$A - B = A + (-1) \times B.$$

**Example 1.14**

$$\text{Let } A = \begin{pmatrix} 2 & -1 \\ 0 & -4 \end{pmatrix}, B = \begin{pmatrix} 0 & 1 \\ -5 & -3 \end{pmatrix}.$$

Calculate  $-B$

$$-B = \begin{pmatrix} 0 & -1 \\ 5 & 3 \end{pmatrix}.$$

Then

$$A - B = \begin{pmatrix} 2 & -1 \\ 0 & -4 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 5 & 3 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ 5 & -1 \end{pmatrix}.$$

**Proposition 1.2**

Let  $A$ ,  $B$ , and  $C$  be three matrices, and  $\lambda$  and  $\mu$  are scalars.

1.  $A + B = B + A$ .
2.  $(A + B) + C = A + (B + C)$ .
3.  $\lambda(A + B) = \lambda A + \lambda B$ .
4.  $(\lambda + \mu)A = \lambda A + \mu A$ .
5.  $\lambda(\mu A) = (\lambda\mu)A$ .
6.  $IA = A$ .

**Example 1.15**

$$A = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, B = \begin{pmatrix} 1 & -3 \\ 1 & 5 \end{pmatrix},$$

Let  $2X + 3A = B$ . Find the matrix  $X$ ?

By remarking  $X = \frac{1}{2}(B - 3A)$ , then

$$B - 3A = \begin{pmatrix} 1 & -3 \\ 1 & 5 \end{pmatrix} - 3 \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 1-3 & -3+3 \\ 1+3 & 5-3 \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 4 & 2 \end{pmatrix}.$$

Hence

$$X = \frac{1}{2} \begin{pmatrix} -2 & 0 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 2 & 1 \end{pmatrix}.$$

**Addition of two matrices****Definition 1.14 (Addition of two matrices)**

Let  $A = (a_{ij})_{m \times n}$  and  $B = (b_{ij})_{m \times n}$  be two matrices with the same order, then the sum of the two matrices is given by

$$C = A + B = (c_{ij})_{m \times n} = (a_{ij})_{m \times n} + (b_{ij})_{m \times n} = (a_{ij} + b_{ij})_{m \times n}.$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \quad B = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{pmatrix}$$

Another written

$$\begin{pmatrix} c_{11} & c_{12} & \dots & c_{1n} \\ c_{21} & c_{22} & \dots & c_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ c_{m1} & c_{m2} & \dots & c_{mn} \end{pmatrix} = \begin{pmatrix} a_{11} + b_{11} & a_{12} + b_{12} & \dots & a_{1n} + b_{1n} \\ a_{21} + b_{21} & a_{22} + b_{22} & \dots & a_{2n} + b_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} + b_{m1} & a_{m2} + b_{m2} & \dots & a_{mn} + b_{mn} \end{pmatrix}$$

**Example 1.16**

$$A = \begin{pmatrix} 1 & -2 & 3 \\ 4 & 5 & -4 \end{pmatrix}, B = \begin{pmatrix} 3 & 0 & 2 \\ -1 & 1 & 8 \end{pmatrix}, A + B = \begin{pmatrix} 4 & -2 & 5 \\ 3 & 6 & 4 \end{pmatrix}.$$

**Multiplication****Definition 1.15** (*The product of a row matrix and a column matrix*)

Let  $V = (v_j)$  be a  $1 \times n$  row matrix (or a horizontal vector), and let  $W = (v_i)$  be a  $n \times 1$  column matrix (or a vertical vector). The product of the row vector  $V$  by the column vector  $W$  is the matrix  $1 \times 1$ , that is, it contains one coefficient of the form:

$$VW = (v_1, v_2, \dots, v_n) \begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{pmatrix} = v_1w_1 + v_2w_2 + \dots + v_nw_n.$$

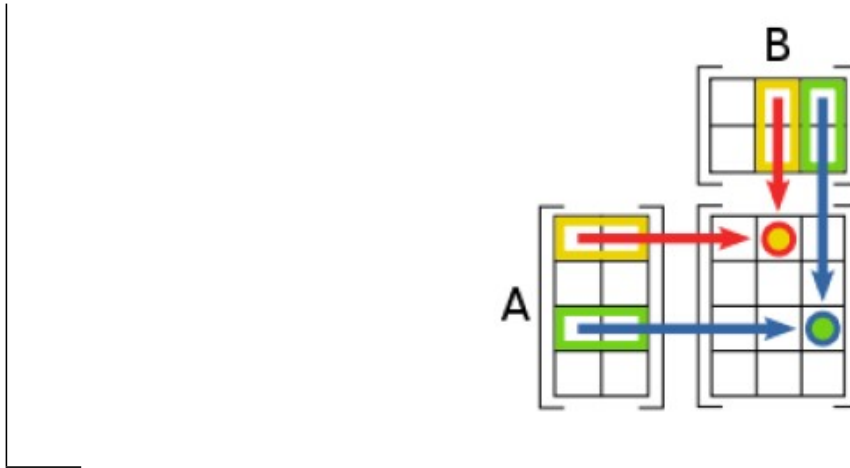
**Example 1.17**

$$V = \begin{pmatrix} -1 & 2 & -3 \end{pmatrix}, \text{ is } 1 \times 3, W = \begin{pmatrix} 5 \\ -1 \\ 0 \end{pmatrix}, \text{ is } 3 \times 1.$$

$$VW = \begin{pmatrix} -1 & 2 & -3 \end{pmatrix} \begin{pmatrix} 5 \\ -1 \\ 0 \end{pmatrix} = -7.$$

**Definition 1.16** (*Multiplication of two matrices*)

Let  $A = (a_{ij})$  be an  $m \times r$  matrix, and  $B = (b_{ij})$  be an  $r \times n$  matrix. The product of matrices  $A$  and  $B$  is a matrix  $C$  of dimensions  $n \times m$  with elements  $C = (c_{ij})$ , where  $c_{ij} = \sum_{k=1}^{k=r} a_{ik}b_{kj}$  (for all  $1 \leq i \leq m$  and  $1 \leq j \leq n$ ). We obtain the product of two matrices by multiplying each row of the first matrix by the columns of the second matrix.

**Example 1.18**

$$A = \begin{pmatrix} 1 & 2 & -3 \\ 0 & -5 & -1 \end{pmatrix}, B = \begin{pmatrix} 2 & 3 \\ -5 & 0 \\ 6 & -2 \end{pmatrix}.$$

Calculate  $AB$ ,  $BA$  is  $AB = BA$ ?

$$\begin{aligned} AB &= \begin{pmatrix} 1 & 2 & -3 \\ 0 & -5 & -1 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ -5 & 0 \\ 6 & -2 \end{pmatrix} \\ &= \begin{pmatrix} 1 \times 2 + 2 \times (-5) + (-3) \times 6 & 1 \times 3 + 2 \times 0 + (-3) \times (-2) \\ 0 \times 2 + (-5) \times (-5) + (-1) \times 6 & 0 \times 3 + (-5) \times 0 + (-1) \times (-2) \end{pmatrix} = \begin{pmatrix} -16 & 9 \\ 19 & 2 \end{pmatrix}. \\ BA &= \begin{pmatrix} 2 & 3 \\ -5 & 0 \\ 6 & -2 \end{pmatrix} \begin{pmatrix} 1 & 2 & -3 \\ 0 & -5 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 2 \times 1 + 3 \times 0 & 2 \times 2 + 3 \times (-5) & 2 \times (-3) + 3 \times (-1) \\ (-5) \times 1 + 0 \times 0 & (-5) \times 2 + 0 \times (-5) & (-5) \times (-3) + 0 \times (-1) \\ 6 \times 1 + (-2) \times 0 & 6 \times 2 + (-2) \times (-5) & 6 \times (-3) + (-2) \times (-1) \end{pmatrix} \\ &= \begin{pmatrix} 2 & -11 & -9 \\ -5 & -10 & 15 \\ 6 & 22 & -16 \end{pmatrix}. \end{aligned}$$

Then  $AB \neq BA$ .

**Remark 1.3**

Let  $A$ ,  $B$ , and  $C$  be three matrices.

1. To have the product  $AB$  defined, if the number of columns in matrix  $A$  must be equal to the number of rows in matrix  $B$ .
2.  $AB$  is not necessarily equal to  $BA$ ; matrix multiplication is not commutative in general.
3. Matrix multiplication is associative:  $A(BC) = (AB)C$ .
4. Matrix multiplication is distributive:  $A(B + C) = AB + AC$ .
5. If  $A$  is an  $m \times n$  matrix, then  $AI_n = I_m A = A$ .

**Definition 1.17 (integral power of matrices)**

Let  $A$  be a square matrix of order  $n$ , and let  $m$  be a positive integer. Then we define

$$A^m = A \times A \times A \dots A, (m \text{ times multiplication}).$$

**Transpose****Definition 1.18 (Transpose)**

If  $A = (a_{ij})_{m \times n}$  is an  $n \times m$  matrix, then the transposed matrix  $A$  is the matrix  $A^T = (a_{ji})_{n \times m}$  is an  $m \times n$ , where its rows become the columns of  $A$  and its columns become the rows of  $A$ , is represented by  $TA$  or  $A^T$ . That is

**Example 1.19**

$$A = \begin{pmatrix} 2 & 1 & -3 \\ 6 & 3 & -9 \end{pmatrix}, A^T = \begin{pmatrix} 2 & 6 \\ 1 & 3 \\ -3 & -9 \end{pmatrix}.$$

**Proposition 1.4 (Properties of the transpose)**

Let  $A$  and  $B$  be two matrices with the same number of rows and the same number

of columns.

1.  $(A + B)^T = A^T + B^T$ .
2.  $(A^T)^T = A$ .
3.  $(kA)^T = kA^T$ , for scalar  $k$ .
4.  $(AB)^T = B^T A^T$ .

### Example 1.20

By using the two matrices  $A$  and  $B$ , verify the transpose properties.

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 5 & -4 & 3 \\ 1 & -2 & -3 \end{pmatrix}, \quad B = \begin{pmatrix} -2 & 6 & -2 \\ -1 & 0 & 1 \\ -2 & 1 & 0 \end{pmatrix}.$$

### Definition 1.19

A square matrix  $A$  is said to be symmetric if  $A^T = A$ .

### Example 1.21

$$A = \begin{pmatrix} 1 & -1 & 1 \\ -1 & -4 & -2 \\ 1 & -2 & -3 \end{pmatrix},$$

$A$  is symmetric by the definition of a symmetric matrix. Then

$$A^T = \begin{pmatrix} 1 & -1 & 1 \\ -1 & -4 & -2 \\ 1 & -2 & -3 \end{pmatrix}.$$

That is  $A^T = A$ .

### Definition 1.20

A square matrix  $A$  is said to be skew-symmetric if  $A = -A^T$ .

**Example 1.22**

$$A = \begin{pmatrix} 1 & 3 & -1 \\ -3 & -5 & 8 \\ 1 & 8 & 9 \end{pmatrix}.$$

**Proposition 1.5 (*Properties of the transpose*)**

Let  $A$  and  $B$  be two symmetric matrices

1.  $AA^T$  and  $A^T A$  are both symmetric.
2.  $A + A^T$  is a symmetric matrix.
3.  $A - A^T$  is a skew-symmetric matrix.
4. If  $A$  is a symmetric matrix and  $m$  is any positive integer, then  $A^m$  is also symmetric.
5. If  $A$  is a skew-symmetric matrix, then odd integral powers of  $A$  are skew-symmetric, while positive even integral powers of  $A$  are symmetric.
6. If  $A$  and  $B$  are symmetric matrices, then
  - $(AB + BA)$  is symmetric.
  - $(AB - BA)$  is skew-symmetric.

**Trace****Definition 1.21 (*Trace*)**

The trace of a square matrix  $A$  is the sum of its main diagonal elements. It is denoted as  $Tr(A)$  and is defined by:

$$Tr(A) = \sum_{i=1}^{i=n} a_{ii}, \text{ where } a_{ii} \text{ are the diagonal elements of } A.$$

**Proposition 1.6 (*Important Properties*)**

1. *Linearity:*

$$\text{Tr}(A + B) = \text{Tr}(A) + \text{Tr}(B).$$

$$\text{Tr}(\lambda A) = \lambda \text{Tr}(A), \forall \lambda \in \mathbb{R}.$$

2. *Invariant under Transposition:*

$$\text{Tr}(A) = \text{Tr}(A^T).$$

3. *Product of Matrices:*

$$\text{Tr}(AB) = \text{Tr}(BA).$$

However, in general,

$$\text{Tr}(AB) \neq \text{Tr}(A)\text{Tr}(B).$$

## Inverse of a matrix

### Definition 1.22 (*Inverse of a matrix*)

Let  $A$  be a square matrix of size  $n \times n$ . We say that the matrix is invertible or has an inverse if there exists a square matrix  $B$  of the same size  $n \times n$  such that  $AB = BA = I$ , where  $I$  is the identity matrix. Then we call  $B$  inverse or inverted  $A$  inverse, and we symbolize it by the symbol  $A^{-1}$ .

### Remark 1.7

Inverse of a matrix if it exists is unique.

### Example 1.23

Demonstrated that matrix  $E$  accepts an inverse, which is  $F$ .

$$E = \begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix}, F = \begin{pmatrix} 4 & -3 \\ -1 & 1 \end{pmatrix}.$$

$$EF = \begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} 4 & -3 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 4-3 & -3+3 \\ 4-4 & -3+4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2.$$

Thus, the inverse of matrix  $E$  is  $F = E^{-1}$ .

**Remark 1.8**

1. Let  $A, B \in M_n(\mathbb{R})$ , if  $A$  and  $B$  are reversible, then  $AB$  is reversible, and we have  $(AB)^{-1} = B^{-1}A^{-1}$ .
2. Let  $A \in M_n(\mathbb{R})$ , if  $A$  is reversible, then  $A^T$  is reversible, and we have  $(A^T)^{-1} = (A^{-1})^T$ .

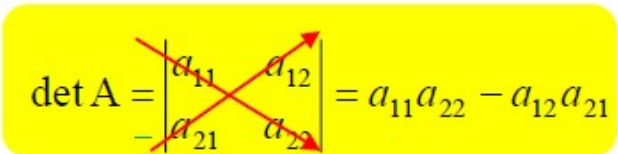
**1.1.4 Determinant of a square matrix****Definition 1.23 (Cofactor)**

[11] The cofactor of  $a_{ij}$  is denoted by  $A_{ij}$  and is equal to  $(-1)^{i+j} |M_{ij}|$ .

**Definition 1.24**

Let  $A = (a_{ij})_{n \times n}$  be a square matrix of order  $n$ , then the number  $\det A$  is called the determinant of the matrix  $A$ .

**Determinant of a  $2 \times 2$  matrix****Definition 1.25 (Determinant of a  $2 \times 2$  matrix)**



$$\det A = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}$$

**Example 1.24**

$$\det A = \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 2 \times 3 - 1 \times 1 = 5.$$

**Determinant of a  $3 \times 3$  matrix****Definition 1.26 (Determinant of a  $3 \times 3$  matrix)**

For the second column

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = -a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{22} \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{32} \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

For the first row

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$\det A = a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31} - a_{11}a_{23}a_{32} - a_{33}a_{21}a_{12}$$

### Example 1.25

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 2 & 0 & -4 \\ 2 & 1 & 0 \end{pmatrix}, B = \begin{pmatrix} 2 & 3 & 2 \\ -2 & -1 & 0 \\ -4 & 3 & 5 \end{pmatrix}.$$

**Determinant of a  $n \times n$  matrix.**

### Definition 1.27 (*Determinant of a $n \times n$ matrix*)

Let  $A = (a_{ij})$  be a square matrix  $n \times n$ , then  $\det A$  : given by the following statement:

For the column  $j$ :

$$\det A = \sum_{i=1}^n (-1)^{i+j} a_{ij} M_{ij}.$$

or, for the row  $i$ :

$$\det A = \sum_{j=1}^n (-1)^{i+j} a_{ij} M_{ij}.$$

where  $M_{ij}$  is the resulting determinant by omitting row  $i$  and column  $j$  of matrix  $A$ .

### Remark 1.9

1. The determinant value of the matrix is  $n \times n$ . It remains unchanged when applying any row or column operations.

2. The operation of row or column reduction results in the calculation of the determinant  $n \times n$  to calculate  $n$  of the determinants  $(n - 1) \times (n - 1)$  using reduced row echelon form. This leads us to the computation of  $n!$  from determinants  $1 \times 1$ .
3. The determinant of  $1 \times 1$  is of the form  $\det(a) = a$ .

### Properties of the determinant

#### Proposition 1.10 (*Properties of the determinant*)

1. The determinant of a matrix  $A$  and its transpose  $A^T$  are equal to  $\det(A) = \det(A^T)$ .
2. If  $A$  has a row (column) of zeros, then  $\det(A) = 0$ .
3. If  $A$  has two identical rows (or columns), then  $\det(A) = 0$ .
4. If  $A$  is a triangular matrix, then  $\det(A)$  is the product of the diagonal elements.
5.  $\det(I_n) = 1$ .
6.  $\det(AB) = \det(BA) = \det(A)\det(B)$ .
7.  $\det(A + B) \neq \det(A) + \det(B)$ .
8. For all  $\lambda \in \mathbb{R}$ ,  $\det(\lambda A) \neq \lambda \det(A)$ .
9. For all  $\lambda \in \mathbb{R}$ ,  $\det(\lambda A) = \det((\lambda I_n)A) = \det(\lambda I_n)\det(A) = \lambda^n \det(A)$ .
10. If  $A$  is invertible,  $\det(A^{-1}) = \frac{1}{\det(A)}$ .

### 1.1.5 Methods for calculating the inverse of a matrix

#### Direct method

##### Definition 1.28

Let  $A \in M_n(\mathbb{K})$ , we say that the matrix is invertible or has an inverse if there exists a square matrix  $A^{-1} \in M_n(\mathbb{K})$  such that  $AA^{-1} = A^{-1}A = I_n$ , where  $I_n$  is the identity matrix.

##### Example 1.26

Find the inverse of matrix  $A$ .

$$A = \begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix}.$$

$$AA^{-1} = \begin{pmatrix} 1 & 3 \\ 1 & 4 \end{pmatrix} \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} = \begin{pmatrix} x_{11} + 3x_{21} & x_{12} + 3x_{22} \\ x_{11} + 4x_{21} & x_{12} + 4x_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2.$$

Then

$$AA^{-1} = \begin{pmatrix} x_{11} + 3x_{21} & x_{12} + 3x_{22} \\ x_{11} + 4x_{21} & x_{12} + 4x_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \begin{cases} x_{11} + 3x_{21} = 1 \\ x_{11} + 4x_{21} = 0 \\ x_{12} + 3x_{22} = 0 \\ x_{12} + 4x_{22} = 1 \end{cases}$$

$$\Rightarrow x_{11} = 4, x_{21} = -1, x_{12} = -3, x_{22} = 1, A^{-1} = \begin{pmatrix} 4 & -3 \\ -1 & 1 \end{pmatrix}.$$

#### Cofactor method

##### Definition 1.29

Let  $A \in M_n(\mathbb{K})$  where  $\det(A) \neq 0$ , then we can calculate  $A^{-1}$  the inverse of the matrix  $A$ , as follows:

$$A^{-1} = \frac{1}{\det(A)} (\text{Cof } A)^T.$$

where  $1 \leq i, j \leq n$ ,  $\text{Cof } A = (c_{ij})$  and  $c_{ij} = (-1)^{i+j} M_{ij}$  and  $M_{ij}$  are the determinants obtained by deleting the row  $i$  and the column  $j$  of the matrix  $A$ .

**Proposition 1.11**

To calculate the inverse of a square matrix  $A$ , follow these steps:

1. Calculate  $\det(A)$ .
2. If  $\det(A) = 0$ , we conclude that  $A$  does not have an inverse, and we stop.  
However, if  $\det(A) \neq 0$ , proceed to step 3.
3. Calculate the resulting determinants,  $M_{ij}$ , by deleting the row  $i$ , and the column  $j$ .
4. Add alternating signs to the cofactor matrix  $\text{Cof } A$ .
5. Calculate the transpose of the cofactor matrix  $(\text{Cof } A)^T$ .
6. Divide all coefficients in  $(\text{Cof } A)^T$  by  $\det(A)$ .

**Example 1.27**

If possible, calculate the inverse of the matrix

$$A = \begin{pmatrix} 2 & 6 & 3 \\ -1 & -1 & 0 \\ -3 & -1 & 0 \end{pmatrix}.$$

**Solution 1.1**

1. Calculate  $\det(A)$

$$\det(A) = \begin{vmatrix} 2 & 6 & 3 \\ -1 & -1 & 0 \\ -3 & -1 & 0 \end{vmatrix} = 3(1 - 3) = -6.$$

2. Since  $\det(A) \neq 0$ , the matrix is invertible and

$$A^{-1} = \frac{(\text{Cof } A)^T}{\det(A)}.$$

3. Calculate  $\text{Cof } A = (c_{ij})_{1 \leq i, j \leq 3}$  with  $c_{ij} = (-1)^{i+j} M_{ij}$ :

$$\text{Cof } A = \begin{pmatrix} +M_{11} & -M_{12} & +M_{13} \\ -M_{21} & +M_{22} & -M_{23} \\ +M_{31} & -M_{32} & +M_{33} \end{pmatrix}$$

$$M_{11} = \begin{vmatrix} -1 & 0 \\ -1 & 0 \end{vmatrix} = 0 \quad M_{12} = \begin{vmatrix} -1 & 0 \\ -3 & 0 \end{vmatrix} = 0 \quad M_{13} = \begin{vmatrix} -1 & -1 \\ -3 & -1 \end{vmatrix} = -2$$

and

$$M_{21} = \begin{vmatrix} 6 & 3 \\ -1 & 0 \end{vmatrix} = 3 \quad M_{22} = \begin{vmatrix} 2 & 3 \\ -3 & 0 \end{vmatrix} = 9 \quad M_{23} = \begin{vmatrix} 2 & 6 \\ -3 & -1 \end{vmatrix} = 16$$

and

$$M_{31} = \begin{vmatrix} 6 & 3 \\ -1 & 0 \end{vmatrix} = 3 \quad M_{32} = \begin{vmatrix} 2 & 3 \\ -1 & 0 \end{vmatrix} = 3 \quad M_{33} = \begin{vmatrix} 2 & 6 \\ -1 & -1 \end{vmatrix} = 4$$

$$\text{Cof } A = \begin{pmatrix} 0 & 0 & -2 \\ -3 & 9 & -16 \\ 3 & -3 & 4 \end{pmatrix}$$

4. Calculate  $(\text{Cof } A)^T$ :

$$(\text{Cof } A)^T = \begin{pmatrix} 0 & -3 & 3 \\ 0 & 9 & -16 \\ -2 & -16 & 4 \end{pmatrix}$$

5. Divide all coefficients of  $(\text{Cof } A)^T$  by  $\det(A)$ :

$$A^{-1} = \frac{1}{\det(A)} (\text{Cof } A)^T = \begin{pmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{3}{2} & \frac{1}{2} \\ \frac{1}{3} & \frac{8}{3} & \frac{2}{3} \end{pmatrix}.$$

## 1.2 Representation of matrix by linear map

### Definition 1.30

A set of vectors  $(v_1, v_2, \dots, v_n)$  in  $V$  is a basis of  $V$  if

1.  $V = \text{span}(v_1, v_2, \dots, v_n)$ .
2. The vectors  $v_1, v_2, \dots, v_n$  are linearly independent.

### Example 1.28

Let  $e_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ . Show that  $\{e_1, e_2, e_3\}$  is a basis of  $\mathbb{R}^3$  "It is called the standard basis".

1. Clearly,  $\text{span}\{e_1, e_2, e_3\} = \mathbb{R}^3$ .
2.  $\{e_1, e_2, e_3\}$  are independent, because

$$\det \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = 1 \neq 0.$$

### Definition 1.31

Let  $E$  and  $F$  be two vector spaces of  $\mathbb{K}$ , let the application  $f : E \rightarrow F$  is called linear map if

$$f(u + v) = f(u) + f(v), \quad \forall u, v \in E.$$

$$f(\alpha u) = \alpha f(u), \quad \alpha \in \mathbb{K}, \quad \forall u \in E.$$

### Definition 1.32

Shows that given two vector spaces  $E$  and  $F$  and a basis  $(v_j)_{1 \leq j \leq n}$  of  $E$ , every linear map

$$f : E \rightarrow F,$$



**Example 1.29**

Let  $f$  be a linear map, defined by

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}^3$$

$$(x, y) \rightarrow f(x, y) = \begin{pmatrix} x - y \\ 2x + y \\ x + 3y \end{pmatrix}.$$

Determine  $M(f/B, C)$  the matrix associated with  $f$ , according to the standard bases  $\mathbb{R}^2$  and  $\mathbb{R}^3$ .

Let  $B = \{v_1, v_2\}$  the standard basis  $\mathbb{R}^2$

$$v_1 = (1, 0), v_2 = (0, 1)$$

Let  $C = \{w_1, w_2, w_3\}$  the standard basis  $\mathbb{R}^3$

$$w_1 = (1, 0, 0), w_2 = (0, 1, 0), w_3 = (0, 0, 1).$$

The dimension of the matrix  $M(f/B, C)$  is  $3 \times 2$ , i.e.,

$$M(f/B, C) \in M_{3 \times 2}(\mathbb{R})$$

$$\begin{cases} f(v_1) = f(1, 0) = (1, 2, 1) = 1(1, 0, 0) + 2(0, 1, 0) + 1(0, 0, 1) = w_1 + 2w_2 + w_3 \\ f(v_2) = f(0, 1) = (-1, 1, 3) = -1(1, 0, 0) + 1(0, 1, 0) + 3(0, 0, 1) = -w_1 + w_2 + 3w_3 \end{cases},$$

Then

$$M(f/B, C) = \begin{pmatrix} 1 & -1 \\ 2 & 1 \\ 1 & 3 \end{pmatrix}.$$

**Example 1.30**

Let  $f$  be a linear map, defined by

$$f : \mathbb{R}^2 \rightarrow \mathbb{R}^3$$

$$(x, y) \rightarrow f(x, y) = \begin{pmatrix} x - y \\ x + y \\ y - x \end{pmatrix}.$$

Determine  $M(f/B, C)$  the matrix associated with  $f$ .

Let  $B = \{v_1, v_2\}$  the basis of  $\mathbb{R}^2$

$$v_1 = (1, 1), v_2 = (-1, 1)$$

Let  $C = \{w_1, w_2, w_3\}$  the basis of  $\mathbb{R}^3$

$$w_1 = (1, 1, 1), w_2 = (0, 1, 0), w_3 = (-1, -1, 1).$$

Let  $B = \{v_1, v_2\}$  the basis of  $\mathbb{R}^2$

$$v_1 = (1, 1), v_2 = (-1, 1)$$

Let  $C = \{w_1, w_2, w_3\}$  the basis of  $\mathbb{R}^3$

$$w_1 = (1, 1, 1), w_2 = (0, 1, 0), w_3 = (-1, -1, 1).$$

The dimension of the matrix  $M(f/B, C)$  is  $3 \times 2$ , i.e.

$$M(f/B, C) \in M_{3 \times 2}(\mathbb{R}) = \begin{pmatrix} x - y \\ x + y \\ y - x \end{pmatrix}$$

Compute  $f(v_1)$

$$f(v_1) = f(1, 1) = (0, 2, 0) = a(1, 1, 1) + b(0, 1, 0) + c(-1, -1, 1),$$

Then

$$\begin{cases} 0 = a - c \\ 2 = a + b - c \\ 0 = a + c \end{cases} \Rightarrow \begin{cases} a = c = 0 \\ b = 2 \end{cases} \Rightarrow f(v_1) = f(1, 1) = (0, 2, 0).$$

Compute  $f(v_2)$

$$f(v_2) = f(-1, 1) = (-2, 0, 2) = a(1, 1, 1) + b(0, 1, 0) + c(-1, -1, 1).$$

Then

$$\begin{cases} -2 = a - c \\ 0 = a + b - c \\ 2 = a + c \end{cases} \Rightarrow \begin{cases} a = 0 \\ c = 2 - a = 2 \\ b = c - a = 2 \end{cases} \Rightarrow f(v_2) = f(-1, 1) = (0, 2, 2).$$

Then

$$M(f/B, C) = \begin{pmatrix} 0 & 0 \\ 2 & 2 \\ 0 & 2 \end{pmatrix}.$$

### Example 1.31

Let  $f$  be a linear map, defined by

$$f : \mathbb{R}^3 \rightarrow \mathbb{R}^2$$

$$(x, y, z) \rightarrow f(x, y, z) = \begin{pmatrix} x + y + z \\ x - 2y + 3z \end{pmatrix}.$$

Determine  $M(f/B, C)$  the matrix associated with  $f$ , according to the standard bases  $\mathbb{R}^3$  and  $\mathbb{R}^2$ .

Let  $B = \{v_1, v_2, v_3\}$  the standard basis  $\mathbb{R}^3$

$$v_1 = (1, 0, 0), v_2 = (0, 1, 0), v_3 = (0, 0, 1).$$

Let  $C = \{w_1, w_2\}$  the standard basis  $\mathbb{R}^2$

$$w_1 = (1, 0), w_2 = (0, 1).$$

The dimension of the matrix  $M(f/B, C)$  is  $2 \times 3$ , i.e.,

$$M(f/B, C) \in M_{2 \times 3}(\mathbb{R})$$

$$\begin{cases} f(v_1) = f(1, 0, 0) = (1, 1) = 1(1, 0) + 1(0, 1) = w_1 + w_2 \\ f(v_2) = f(0, 1, 0) = (1, -2) = 1(1, 0) - 2(0, 1) = w_1 - 2w_2 \\ f(v_3) = f(0, 0, 1) = (1, 3) = 1(1, 0) + 3(0, 1) = w_1 + 3w_2 \end{cases},$$

Then

$$M(f/B, C) = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -2 & 3 \end{pmatrix}.$$

**Definition 1.34**

Let  $E, F$ , and  $G$ , be three vectors spaces with respective bases  $B = (u_1, \dots, u_p)$  for  $E$ ,  $C = (v_1, \dots, v_n)$  for  $F$ , and  $D = (w_1, \dots, w_m)$  for  $G$ . Let  $f : E \rightarrow F$  and  $g : F \rightarrow G$  be linear maps.

As explained earlier,  $f : E \rightarrow F$  is determined by the images of the basis vectors  $u_k$ , and  $g : F \rightarrow G$  is determined by the images of the basis vectors  $v_j$ .

We would like to understand how  $gof : E \rightarrow G$  is determined by the images of the basis vector  $u_j$ . Then matrix associated of  $gof$ , defined bellow

$$M(gof/B, D) = M(g/C, D) M(f/B, C).$$

**Remark 1.13**

Note that we are considering linear maps  $f : E \rightarrow F$  and  $g : F \rightarrow G$ , instead of  $g : E \rightarrow F$  and  $f : F \rightarrow G$ , which yields the composition  $gof : E \rightarrow G$  instead of  $fog : E \rightarrow G$ .

**Remark 1.14**

1. The matrix associated for the zero application is the zero matrix.
2. The matrix associated for the neutral application is the unit matrix.
3. The matrix associated for application  $(f + g)$  is the sum the matrix associated for application  $f$ , and the matrix associated for application  $g$

$$M(f + g) = M(f) + M(g).$$

4. The matrix associated for application  $\lambda f$  is the matrix resulting from multiplying the matrix associated for application  $f$  and scalar  $\lambda$ .

$$M(\lambda f) = \lambda M(f).$$

5. The matrix associated for application  $gof$  is the matrix resulting from multiplying the matrix associated for application  $f$ , and the matrix associated for application  $g$

$$M(gof) = M(g) M(f).$$

6. If the matrix  $A$  is invertible and it is associated for application  $f$ , then  $A^{-1}$  is the matrix associated for application  $f^{-1}$ .

$$(M(f))^{-1} = M(f^{-1}).$$

## 1.3 Representation of linear map by matrix

### Definition 1.35

Let  $E$  and  $F$  be finite-dimensional vector spaces, Let  $B = (v_1, v_2, \dots, v_n)$  is a basis for  $E$  and that  $C = (w_1, w_2, \dots, w_m)$  is a basis for  $F$ . denote by  $[j]_E$  and  $[i]_F$  their  $n \times 1$  and  $m \times 1$  coordinate vectors with respect to the two bases of  $E$  and  $F$  respectively. Let  $f : E \rightarrow F$  be a linear map. An  $m \times n$  matrix  $[f]_{BC}$  such that, for any  $j \in E$ ,

$$[f(j)]_C = [f]_{BC}[j]_B.$$

is called a matrix of the linear map  $f$  with respect to the bases  $B$  and  $C$ .

Although it should already be clear, we highlight that the transformed vectors  $f(j)$  belong to  $F$  and their coordinate vectors  $[f(j)]_C$  are  $m \times 1$  vectors. Moreover,  $[f]_{BC}[j]_B$  is the matrix product of  $[f]_{BC}$  and  $[j]_B$ .

### Example 1.32

Consider this matrix

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \end{pmatrix}.$$

$\mathbb{R}^2$  vector space with standard basis  $\{e_1, e_2\}$ .  $\mathbb{R}^3$  vector space with standard basis  $\{e_1, e_2, e_3\}$ . There is a single linear map defined by

$$f : \mathbb{R}^3 \rightarrow \mathbb{R}^2.$$

Then

$$\begin{cases} f(e_1) = a_{11}e_1 + a_{21}e_2 = (2, 0) \\ f(e_2) = a_{12}e_1 + a_{22}e_2 = (1, 1) \\ f(e_3) = a_{13}e_1 + a_{23}e_2 = (-1, 1) \end{cases},$$

$$\begin{aligned} f(x, y, z) &= f(xe_1 + ye_2 + ze_3) \\ &= xf(e_1) + yf(e_2) + zf(e_3) \\ &= x(2, 0) + y(1, 1) + z(-1, 1) \\ &= (2x + y - z, y + z). \end{aligned}$$

Then, linear map is defined by

$$f(x, y, z) = (2x + y - z, y + z).$$

### Example 1.33

Consider this matrix

$$A = \begin{pmatrix} 2 & -1 & 3 \\ 0 & 1 & 2 \end{pmatrix}.$$

$\mathbb{R}^3$  vector space with standard basis  $\{e_1, e_2, e_3\}$ .  $\mathbb{R}^2$  vector space with standard basis  $\{e_1, e_2\}$ . There is a single linear map defined by

$$f : \mathbb{R}^3 \rightarrow \mathbb{R}^2.$$

Then

$$\begin{aligned} f(x, y, z) &= \left( A \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right)^T = \left( \begin{pmatrix} 2 & -1 & 3 \\ 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \right)^T \\ &= \begin{pmatrix} 2x - y + 3z \\ y + 2z \end{pmatrix}^T = (2x - y + 3z, y + 2z). \end{aligned}$$

Then, linear map is defined by

$$f(x, y, z) = (2x - y + 3z, y + 2z).$$

## 1.4 Change of basis, passage matrix.

### 1.4.1 Passage matrix

#### Definition 1.36

[9] Let  $E$  be a finite-dimensional vector space of dimension  $n$ . Let  $B_1(e_1, e_2, \dots, e_n)$  and  $B_2(e'_1, e'_2, \dots, e'_n)$  be two bases of  $E$ . The passage of matrix of  $B_1$  (old basis) to  $B_2$  (new basis), denoted by

$$P_{B_1}^{B_2} = a_{ij} \in M_n(E)$$

is defined by

$$e'_j = \sum_{i=1}^n a_{ij} e_i, \forall 1 \leq j \leq n.$$

#### Remark 1.15

The  $j$ -th column of the matrix  $P_{B_1}^{B_2}$  is formed by the coordinates of  $e'_j$  in the old basis  $B_1$ .

#### Example 1.34

Let  $E = \mathbb{R}^2$ , Consider the two bases of  $\mathbb{R}^2$

$$B_1 = (v_1, v_2) \text{ (standard basis)} : v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

$$B_2 = (w_1, w_2) : w_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}, w_2 = \begin{pmatrix} \frac{-3}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}.$$

Suppose that  $\alpha$  and  $\beta$  the components  $w_1$  in the basis  $B_1$ , i.e.,

$$\begin{aligned} w_1 = \alpha v_1 + \beta v_2 &\Rightarrow \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \begin{cases} \alpha = \frac{1}{\sqrt{2}} \\ \beta = \frac{1}{\sqrt{2}} \end{cases} \\ &\Rightarrow w_1 = \frac{1}{\sqrt{2}} v_1 + \frac{1}{\sqrt{2}} v_2. \end{aligned}$$

By same method, suppose that  $\alpha$  and  $\beta$  the components  $w_2$  in the basis  $B_1$ , i.e.,

$$w_2 = \alpha v_1 + \beta v_2 \Rightarrow \begin{pmatrix} \frac{-3}{\sqrt{2}} \\ \frac{3}{\sqrt{2}} \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix} \Rightarrow \begin{cases} \alpha = \frac{-3}{\sqrt{2}} \\ \beta = \frac{3}{\sqrt{2}} \end{cases}$$

$$\Rightarrow w_2 = \frac{-3}{\sqrt{2}}v_1 + \frac{3}{\sqrt{2}}v_2.$$

Then the passage of matrix of the basis  $B_1$  to the basis  $B_2$ , is

$$P_{B_1}^{B_2} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-3}{\sqrt{2}} \\ 1 & \frac{3}{\sqrt{2}} \end{pmatrix}.$$

### Theorem 1.16

Let  $E$  be a finite-dimensional vector space of dimension  $n$ . Let  $B_1(e_1, e_2, \dots, e_n)$  and

$B_2(e'_1, e'_2, \dots, e'_n)$  be two bases of  $E$ . Let  $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$  the vector of the components

of  $X$  in the basis  $B_1$  ( $X \in E_{B_1}$ ). Let  $X' = \begin{pmatrix} x'_1 \\ x'_2 \\ \vdots \\ x'_n \end{pmatrix}$  the vector of the components of

$X'$  in the base  $B_2$  ( $X' \in E_{B_2}$ ). Then

$$X = P_{B_1}^{B_2} X'.$$

**Proof:**

$$X = \sum_{i=1}^n x_i e_i = \sum_{j=1}^n x'_j e'_j = \sum_{j=1}^n x'_j \left( \sum_{i=1}^n a_{ij} e_i \right) = \sum_{i=1}^n \left( \sum_{j=1}^n a_{ij} x'_j \right) e_i. \text{ Conclude}$$

$$x_i = \sum_{j=1}^n a_{ij} x'_j \Leftrightarrow X = P_{B_1}^{B_2} X' \text{ or } X' = P_{B_2}^{B_1} X. \quad \blacksquare$$

### Properties of the passage of matrix

1. The passage of matrix is invertible, and

$$\left(P_{B_1}^{B_2}\right)^{-1} = P_{B_2}^{B_1}.$$

2.  $P_B^B = I_n$ .

3. If  $B_E, B'_E, B''_E$  are be three bases of  $E$ , then

$$P_{B_1}^{B_3} = P_{B_1}^{B_2} \times P_{B_2}^{B_3}.$$

### 1.4.2 Change of basis

The change of basis is a technique that allows us to express vector coordinates with respect to a "new basis" that is different from the "old basis" originally employed to compute coordinates.

#### The change-of-basis matrix

##### Theorem 1.17

Let  $E$  and  $F$  be two vector spaces. Let  $f : E \rightarrow F$  be linear map. If  $B_E$  and  $B'_E$  be two bases for  $E$ , and if  $B_F$  and  $B'_F$  be two bases for  $F$ . Then

$$M(f/B'_E, B'_F) = \left(P_{B'_F}^{B_F}\right)^{-1} M(f/B_E, B_F) \left(P_{B_E}^{B'_E}\right) = \left(P_{B'_F}^{B_F}\right) M(f/B_E, B_F) \left(P_{B'_E}^{B_E}\right)^{-1}.$$

We put

$$\begin{aligned} Q = P_{B'_F}^{B_F}, P = P_{B_E}^{B'_E} &\Rightarrow M(f/B'_E, B'_F) = (Q)^{-1} M(f/B_E, B_F) (P) \\ &\Rightarrow M(f/B'_E, B'_F) = (Q) M(f/B_E, B_F) (P)^{-1}. \end{aligned}$$

##### Remark 1.18

Case of an endomorphism

$$E = F, B_E = B_F = B, B'_E = B'_F = B'.$$

Then

$$Q = P \Rightarrow M(f) = (P) M(f) (P)^{-1}.$$

$P^{-1}$  is inverse of  $P$ .

### Example 1.35

Let  $E = \mathbb{R}^2$  and  $F = \mathbb{R}^3$  be two vector spaces. Let  $f$  be linear map, defined by

$$f : E \rightarrow F$$

$$(x, y) \rightarrow f(x, y) = \begin{pmatrix} x - y \\ x + y \\ y - x \end{pmatrix}.$$

- In  $\mathbb{R}^2$ , consider these bases

$$B_E = B_1 = \{u_1, u_2\}, u_1 = (1, 0), u_2 = (0, 1).$$

$$B'_E = B'_1 = \{u'_1, u'_2\}, u'_1 = (1, 1), u'_2 = (-1, 1).$$

- In  $\mathbb{R}^3$ , consider these bases

$$B_F = B_2 = \{v_1, v_2, v_3\}, v_1 = (1, 0, 0), v_2 = (0, 1, 0), v_3 = (0, 0, 1).$$

$$B'_F = B'_2 = \{v'_1, v'_2, v'_3\}, v'_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, v'_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, v'_3 = \begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix}.$$

Calculate  $M(f/B'_1, B'_2)$ .

### Solution 1.2

Let

$$M(f/B_1, B_2) = \begin{pmatrix} 1 & -1 \\ 1 & 1 \\ -1 & 1 \end{pmatrix}, \text{ and } P_{B_1}^{B'_1} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

Let

$$M(f/B'_1, B'_2) = \left(P_{B'_2}^{B_2}\right)^{-1} M(f/B_1, B_2) \left(P_{B_1}^{B'_1}\right) = \left(P_{B'_2}^{B_2}\right) M(f/B_1, B_2) \left(P_{B_1}^{B'_1}\right),$$

where

$$\left(P_{B'_2}^{B_2}\right) = a_{ij} \in M_3(\mathbb{R})$$

Such that each vector column of  $P_{B'_2}^{B_2}$  has coordinates expressed in basis  $B_2$ . Let

$$\begin{cases} v'_1 = v_1 + v_2 + v_3, \\ v'_2 = v_2, \\ v'_3 = -v_1 - v_2 + v_3, \end{cases} \Rightarrow \begin{cases} v_1 = \frac{1}{2}v'_1 - v'_2 - \frac{1}{2}v'_3, \\ v_2 = v'_2, \\ v_3 = \frac{1}{2}v'_1 + \frac{1}{2}v'_3, \end{cases}$$

Then

$$P_{B'_2}^{B_2} = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ -1 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \in M_3(\mathbb{R}).$$

Then

$$M(f/B'_1, B'_2) = \begin{pmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ -1 & 1 & 0 \\ -\frac{1}{2} & 0 & \frac{1}{2} \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 2 & 2 \\ 0 & 2 \end{pmatrix}.$$

## *Systems of Linear Equations*

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### 2.1 General

#### 2.1.1 Classical presentation

We now study how to find the solution of a system of  $m$  linear equations in  $n$  unknowns.

Consider the system of linear equations (in short SLE) in unknowns  $x_1, x_2, x_3, \dots, x_n$  as

$$(S) \begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m \end{cases},$$

is called a system of linear equations with  $n$  unknowns  $x_1, x_2, x_3, \dots, x_n$ .

If the constants  $b_1, b_2, b_3, \dots, b_n$  are all zero, then the system is said to be Homogeneous.

#### 2.1.2 Write the system in matrix form

##### Definition 2.1

Let  $(a_{ij}) \in M_{m,n}(\mathbb{K})$  be the matrix of  $(S)$ ,  $B = (b_i)_{1 \leq i \leq m}$ ,  $X = (x_j)_{1 \leq j \leq n}$ . Then the system  $(S)$  can be put in matrix form as

$$AX = B,$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}, X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

### 2.1.3 Write the system with a linear application

#### Definition 2.2

[11] Let  $E$  and  $F$  be two vector spaces, with respective bases  $B = (u_1, \dots, u_m)$  for  $E$  and  $C = (v_1, \dots, v_n)$  for  $F$ . Let  $f : E \rightarrow F$  be linear map, with  $M(f/B, C) = A$ .

We note

$$X = \sum_{j=1}^n x_j u_j \in E, X = (x_i)_{1 \leq i \leq m}, B = \sum_{i=1}^m b_i v_i \in F.$$

The system  $(S)$  is written

$$f(X) = B.$$

#### Example 2.1

Consider the following system:

$$(S) \begin{cases} 2x + y - z + 2t = 1 \\ 3x + y + 2z - t = 0 \\ \sqrt{2}x - 2z + 5t = 8 \end{cases},$$

this system of three equations to four unknowns  $x, y, z, t$ .

$$A = \begin{pmatrix} 2 & 1 & -1 & 2 \\ 3 & 1 & 2 & -1 \\ \sqrt{2} & 0 & -2 & 5 \end{pmatrix}, X = \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix}, Y = \begin{pmatrix} 1 \\ 0 \\ 8 \end{pmatrix}.$$

Then the matrix form is

$$AX = B \Rightarrow \begin{pmatrix} 2 & 1 & -1 & 2 \\ 3 & 1 & 2 & -1 \\ \sqrt{2} & 0 & -2 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ t \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 8 \end{pmatrix}.$$

That is, changing the unknowns leads to a change in the coefficient matrix.

## 2.2 Study of all solutions

We call the solution of the system of existing linear equations  $(s_1, s_2, \dots, s_p)$  substituted  $x_1$  by  $s_1$  and  $x_2$  by  $s_2$  ...etc. In the system of linear equations, the equals are satisfied.

### Example 2.2

Consider the following system

$$(S) \begin{cases} x_1 - 3x_2 = 1 \\ -x_1 + 4x_2 = 9 \end{cases},$$

admits the solution  $(31, 10)$ ,

$$x_1 = 31, x_2 = 10.$$

### Remark 2.1

Searching for a pair  $(x, y)$  that satisfies many equations of the same order is searching the points of intersection of many lines.

### Example 2.3

The point  $(x, y)$  belongs to the intersection of two lines  $D_1$  and  $D_2$ .

$$a_{11}x + a_{12}y = b_1, \text{ the equation for line } D_1,$$

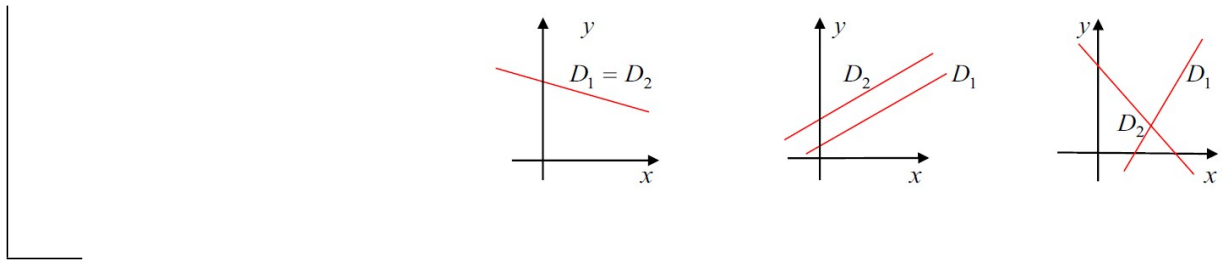
$$a_{21}x + a_{22}y = b_2, \text{ the equation for line } D_2,$$

Then this system

$$(S) \begin{cases} a_{11}x + a_{12}y = b_1 \\ a_{21}x + a_{22}y = b_2 \end{cases},$$

admits the solution  $(x, y)$ . There are three possibilities:

1. The lines  $D_1$  and  $D_2$  intersect at one point. Then the system accepts a unique solution.
2. The lines  $D_1$  and  $D_2$  are parallel. Then the system does not accept solutions.
3. The lines  $D_1$  and  $D_2$  are applicable. Then the system accepts an infinity of solutions.

**Remark 2.2**

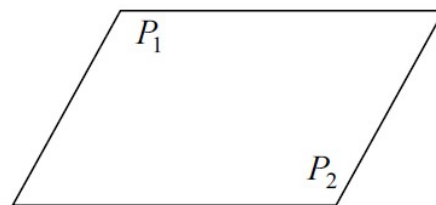
A linear equation with three unknowns  $(x, y, z)$  is the equation of a plane in space.

**Example 2.4**

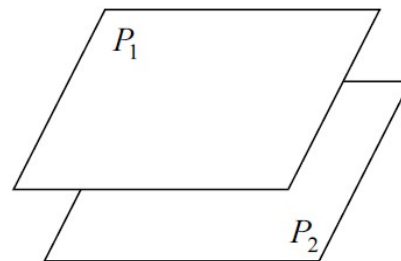
Let it be three systems with two equations and three unknowns in space

$$(1) \begin{cases} x + y + z = 1 \\ -x - y - z = -1 \end{cases}, (2) \begin{cases} x + y + z = 1 \\ -x - y - z = 1 \end{cases}, (3) \begin{cases} x + y + z = 1 \\ x - y + z = -1 \end{cases}.$$

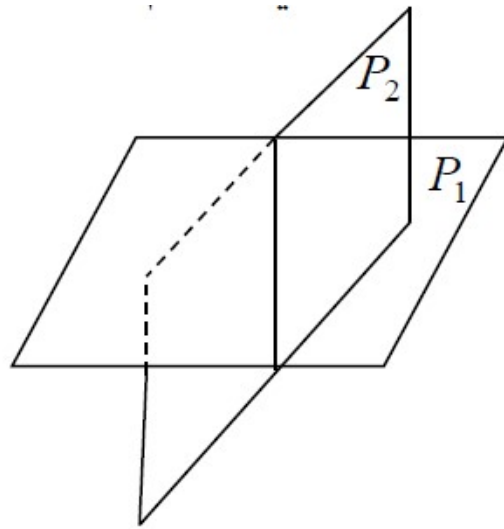
1. The equations of the first system represent the same plane. So the set of solutions is that plane.



2. The equations of the second system represent parallel planes. So the solution set is null.



3. The equations of the third system represent two planes intersecting in a line.



The system of three equations with three unknowns can accept a unique solution (in general, three planes intersect at a point in space).

### **Proposition 2.3**

Let  $S$  be a system of  $n$  linear equations with  $m$  unknowns on the form matrix

$$AX = B.$$

Then the system  $S$  does not accept solutions, or accept a unique solution, or accept infinity of solutions.

## **2.3 Methods of solving system of linear equations**

Solve a system of linear equations by using three methods

### **2.3.1 Solving by Cramer method**

Consider a system ( $S$ ) of  $n$  linear equations with  $n$  unknowns on the form matrix

$$AX = B.$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}, X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

Or

$$(S) \begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \dots + a_{nn}x_n = b_n \end{cases}$$

### Definition 2.3 (*Steps of Cramer method*)

Calculate determinants of matrix  $A$

$$\det A = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}$$

If  $\det A = 0$ , then the system  $(S)$  does not accept solutions or accept the infinity of solutions.

If  $\det A \neq 0$ , then the system  $(S)$  accepts a unique solution, as given by Crammer method

$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \text{ where } x_j = \frac{\det(A_j)}{\det(A)}, \text{ for all } 1 \leq j \leq n.$$

Another write

$$X_1 = \frac{\det A_1}{\det A}, X_2 = \frac{\det A_2}{\det A}, \dots, X_n = \frac{\det A_n}{\det A}.$$

Let  $\det(A_j)$   $j = 1, 2, \dots, n$  is the determinant obtained from  $A$  by replacing its

$j$  – th column with the column of the right-hand side. In other words:

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, A_1 = \begin{pmatrix} b_{11} & a_{12} & \cdots & a_{1n} \\ b_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix}, A_2 = \begin{pmatrix} a_{11} & b_{12} & \cdots & a_{1n} \\ a_{21} & b_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & b_{n2} & \cdots & a_{nn} \end{pmatrix}, \dots$$

### Example 2.5

Solving the system by Crammer method

$$(S) \begin{cases} 2x + y - z = 1 \\ x - y + z = 0 \\ x + y + 3z = 2 \end{cases}.$$

### Solution 2.1

The first step is to calculate  $\det(A)$

$$\det(A) = \begin{vmatrix} 2 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & 3 \end{vmatrix} = -12 \neq 0$$

Let  $\det A \neq 0$ , then the system (S) accept a unique solution, is given by Crammer method

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \text{ where } x_j = \frac{\det(A_j)}{\det(A)}, \text{ for all } 1 \leq j \leq 3.$$

The second step is to calculate  $\det(A_j)$ , for all  $1 \leq j \leq 3$

$$\det(A_1) = \begin{vmatrix} 1 & 1 & -1 \\ 0 & -1 & 1 \\ 2 & 1 & 3 \end{vmatrix} = -4, \det(A_2) = \begin{vmatrix} 2 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & 2 & 3 \end{vmatrix} = -8,$$

$$\det(A_3) = \begin{vmatrix} 2 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 1 & 2 \end{vmatrix} = -4.$$

The third step is to calculate

$$x_j = \frac{\det(A_j)}{\det(A)}, \text{ for all } 1 \leq j \leq 3,$$

$$x_1 = \frac{\det(A_1)}{\det(A)} = \frac{1}{3}, x_2 = \frac{\det(A_2)}{\det(A)} = \frac{2}{3}, x_3 = \frac{\det(A_3)}{\det(A)} = \frac{1}{3}.$$

Then the solution of the system (S) is

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \Rightarrow X = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}.$$

#### **Theorem 2.4**

A Crammer system has one and only one solution.

### **2.3.2 Solving by the inverse matrix method**

Consider a system (S) of  $n$  linear equations with  $n$  unknowns on the form matrix

$$AX = B.$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{pmatrix}, X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}.$$

$$(S) \begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + a_{n3}x_3 + \dots + a_{nn}x_n = b_n \end{cases}.$$

**Definition 2.4** (*Steps of inverse matrix method*)

Calculate the determinants of matrix  $A$

$$\det A = \begin{vmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{vmatrix}.$$

If  $\det A = 0$ , then the system  $(S)$  does not accept solutions or accept infinity of solutions.

If  $\det A \neq 0$ , then the system  $(S)$  accepts a unique solution, is given by

$$A^{-1}(AX) = A^{-1}B \implies X = A^{-1}B.$$

**Example 2.6**

Let's solve the system  $(S)$  by inverse matrix method

$$(S) \begin{cases} 2x + y - z = 1 \\ x - y + z = 0 \\ x + y + 3z = 2 \end{cases},$$

**Solution 2.2**

The first step is to calculate  $\det(A)$

$$\det(A) = \det \begin{pmatrix} 2 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & 3 \end{pmatrix} = -12 \neq 0$$

The second step is to calculate  $A^{-1}$ ,

$$A^{-1} = \frac{1}{\det(A)} (\text{Cof } A)^T,$$

Calculate  $\text{Cof } A$

$$\text{Cof } A = \begin{pmatrix} -1 & 1 & - & 1 & 1 & 1 & -1 \\ & 1 & 3 & & 1 & 3 & 1 & 1 \\ & 1 & -1 & & 2 & -1 & & 2 & 1 \\ - & 1 & 3 & & 1 & 3 & & 1 & 1 \\ & 1 & -1 & & 2 & -1 & & 2 & 1 \\ & -1 & 1 & & 1 & 1 & & 1 & -1 \end{pmatrix} = \begin{pmatrix} -4 & -2 & 2 \\ -4 & 7 & -1 \\ 0 & -3 & -3 \end{pmatrix}.$$

Then

$$(\text{Cof } A)^T = \begin{pmatrix} -4 & -4 & 0 \\ -2 & 7 & -3 \\ 2 & -1 & -3 \end{pmatrix}.$$

$$A^{-1} = -\frac{1}{12} \begin{pmatrix} -4 & -4 & 0 \\ -2 & 7 & -3 \\ 2 & -1 & -3 \end{pmatrix} = \begin{pmatrix} 4 & 4 & 0 \\ 2 & -7 & 3 \\ -2 & 1 & 3 \end{pmatrix}.$$

The third step is to calculate  $A^{-1}B$ ,

$$A^{-1}B = \frac{1}{12} \begin{pmatrix} 4 & 4 & 0 \\ 2 & -7 & 3 \\ -2 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{1}{3} \\ \frac{2}{3} \\ \frac{1}{3} \end{pmatrix}.$$

### 2.3.3 Solving by the Gauss-Seidel method

Consider a system ( $S$ ) of  $m$  linear equations with  $n$  unknowns on the form matrix

$$AX = B.$$

where

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}, X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}.$$

Or

$$(S) \begin{cases} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n = b_m \end{cases}.$$

**Definition 2.5 (Gauss-Seidel method)**

[9] Gauss-Seidel method is used to solve any type of linear equation in general form.

The Gaussian elimination method refers to a strategy used to obtain the row-echelon form of a matrix. The goal is to write matrix  $A$  with the number 1 as the entry down the main diagonal and all zeros below.

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \xrightarrow{\text{After Gaussian elimination}} A = \begin{bmatrix} 1 & b_{12} & b_{13} \\ 0 & 1 & b_{23} \\ 0 & 0 & 1 \end{bmatrix}.$$

**Definition 2.6 (Steps of Gauss-Seidel method)**

The first equation should have a leading coefficient of 1. Interchange rows or multiply by a constant, if necessary.

Use row operations to obtain zeros down the first column below the first entry of 1.

Use row operations to obtain a 1 in row 2, column 2.

Use row operations to obtain zeros down column 2, below the entry of 1.

Use row operations to obtain a 1 in row 3, column 3.

Continue this process for all rows until there is a 1 in every entry down the main diagonal and there are only zeros below.

If any rows contain all zeros, place them at the bottom.

**Example 2.7**

Solve the system by using the Gauss-Seidel method

$$(S) \begin{cases} 2x + 3y = 6 & (a) \\ x - y = \frac{1}{2} & (b) \end{cases}.$$

**Solution 2.3**

First, we write this as an augmented matrix

$$\left[ \begin{array}{cc|c} 2 & 3 & 6 \\ 1 & -1 & \frac{1}{2} \end{array} \right].$$

We want a 1 in row 1, column 1. This can be accomplished by interchanging rows 1 and 2

$$R_1 \leftrightarrow R_2 \rightarrow \left[ \begin{array}{cc|c} 1 & -1 & \frac{1}{2} \\ 2 & 3 & 6 \end{array} \right].$$

We now have a 1 as the first entry in row 1, column 1. Now let's obtain a 0 in row 2, column 1. This can be accomplished by multiplying row 1 by  $-2$  and then adding the result to row 2.

$$-2R_1 + R_2 = R_2 \rightarrow \left[ \begin{array}{cc|c} 1 & -1 & \frac{1}{2} \\ 0 & 5 & 5 \end{array} \right]$$

We only have one more step, to multiply row 2 by  $\frac{1}{5}$ :

$$\frac{1}{5}R_2 = R_2 \rightarrow \left[ \begin{array}{cc|c} 1 & -1 & \frac{1}{2} \\ 0 & 1 & 1 \end{array} \right]$$

The second row of the matrix represents  $y = 1$ . Back-substitute  $y = 1$  into the first equation

$$x - y = \frac{1}{2} \Rightarrow x - 1 = \frac{1}{2} \Rightarrow x = \frac{3}{2}.$$

The solution is the point  $\left(\frac{3}{2}, 1\right)$ .

**Example 2.8**

Consider solving the system  $(S)$  by Gauss-Seidel method

$$(S) \begin{cases} 2x + y - z = 1 & (a) \\ x - y + z = 0 & (b) \\ x + y + 3z = 2 & (c) \end{cases}.$$

Let's arrange for the coefficient of  $x$  in the first row to be equal to 1. To do this, we

swap the two rows (a) and (b) :

$$(S) \begin{cases} x - y + z = 0 & (b) \\ 2x + y - z = 1 & (a) \\ x + y + 3z = 2 & (c) \end{cases} .$$

We keep the first equation (b) as it is and add  $-2(b)$  to the second equation and  $-(b)$  to the third equation:

$$(S) \begin{cases} x - y + z = 0 & (b) \\ 3y - 3z = 1 & (a') \\ 2y + 2z = 2 & (c') \end{cases} .$$

Let's divide equation (a') by 3 to make the coefficient of y equal to 1

$$(S) \begin{cases} x - y + z = 0 & (b) \\ y - z = \frac{1}{3} & (a'') \\ 2y + 2z = 2 & (c') \end{cases} .$$

Let's add  $-2(a'')$  to equation (c'):

$$(S) \begin{cases} x - y + z = 0 & (b) \\ y - z = \frac{1}{3} & (a'') \\ 4z = \frac{4}{3} & (c'') \end{cases} .$$

From equation (c''), we get  $z = \frac{1}{3}$ , then from equation (a'') we deduce:

$$y = z + \frac{1}{3} = \frac{2}{3}.$$

Finally, from equation (b), we deduce:

$$x = y - z = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}.$$

## *Integrals*

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### 3.1 Indefinite integral and properties

#### 3.1.1 An antiderivative of a function

##### **Definition 3.1**

[1] Let  $f$  be a function defined on an interval  $I$  (where  $I$  is an open interval). An antiderivative of  $f$  on  $I$  is any function  $F$  defined and differentiable on  $I$  that satisfies the following property:

$$\forall x \in I, F'(x) = f(x).$$

##### **Example 3.1**

1. Let  $I = \mathbb{R}$  and  $f : I \rightarrow \mathbb{R}$  be the function defined by  $f(x) = x^4$ , then the function  $F : I \rightarrow \mathbb{R}$  defined by  $F(x) = \frac{1}{5}x^5$  is an antiderivative of  $f$  on  $I = \mathbb{R}$  since:

$$\forall x \in I, F'(x) = f(x) = x^4.$$

2. Let  $I = ]0, +\infty[$  and  $g : I \rightarrow \mathbb{R}$  be a function defined by  $g(x) = \sqrt{x}$ , then the function  $G : I \rightarrow \mathbb{R}$  defined by  $G(x) = \frac{2}{3}x\sqrt{x}$  is an antiderivative of  $g$  on  $I = ]0, +\infty[$  since:

$$\forall x \in I, G'(x) = g(x) = \sqrt{x}.$$

##### **Proposition 3.1**

Let  $f$  be a function defined and admit an antiderivative on  $I$  then:

1.  $f$  has infinitely many antiderivatives on  $I$ .
2. If  $F_1, F_2$  are two antiderivatives of  $f$  on  $I$ , then there exists a constant  $c \in \mathbb{R}$

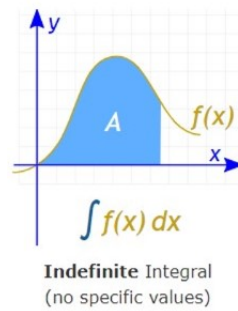
such that:

$$\forall x \in I, F_1(x) = F_2(x) + c.$$

### 3.1.2 Indefinite integral

#### Definition 3.2

[5] Let  $f$  be a function that is defined and admits an antiderivative on  $I$  ( $I$  is an open interval). The set of all antiderivatives of  $f$  on  $I$  is called the indefinite integral of  $f$  on  $I$ , then denoted by



#### Example 3.2

The indefinite integral of the function  $\frac{1}{x}$  on  $] -\infty, 0[$  is defined by:

$$\int \frac{dx}{x} = \ln(-x) + c_1, \forall c_1 \in \mathbb{R}.$$

The indefinite integral of the function  $\frac{1}{x}$  on  $]0, +\infty[$  is defined by:

$$\int \frac{dx}{x} = \ln(x) + c_2, \forall c_2 \in \mathbb{R}.$$

The indefinite integral of the function  $\frac{1}{x}$  on  $\mathbb{R}^*$  is defined by:

$$\int \frac{dx}{x} = \ln|x| + c, \forall c \in \mathbb{R}.$$

### 3.1.3 Existence of the indefinite integral

#### Theorem 3.2

Let  $f$  be a function defined on  $I$  ( $I$  is an open interval), then we have the following implication:

$$f \text{ is continuous on } I \Rightarrow f \text{ admits an antiderivative on } I.$$

Set of all the functions antiderivative defined on  $I$ , note by

$$\int f(x)dx = F(x) + c, \forall c \in \mathbb{R}, \forall x \in I, F'(x) = f(x).$$

#### Example 3.3

Let  $\cos(x)$  is an antiderivative of  $\sin(x)$  on  $\mathbb{R}$ , so we can define the indefinite integral of the function  $\sin(x)$  with the following

$$\int \sin(x)dx = -\cos(x) + c, \forall c \in \mathbb{R}.$$

### 3.1.4 Properties of the indefinite integral

#### Proposition 3.3

Let  $f$  and  $g$  be two continuous functions on  $I$ , then we have the following properties:

1.  $\int [f(x) \pm g(x)] dx = \int f(x)dx \pm \int g(x) dx.$
2.  $\int \lambda f(x)dx = \lambda \int f(x)dx, \forall \lambda \in \mathbb{R}.$
3.  $\left( \int f(x)dx \right)' = f(x).$
4.  $\int f'(x)dx = f(x) + c, \forall c \in \mathbb{R}.$

### 3.1.5 Indefinite integrals of some usual functions

| $\int f(x)dx$                                                                                | Domain of definition                                                 |
|----------------------------------------------------------------------------------------------|----------------------------------------------------------------------|
| $\int 0dx = c$                                                                               | $\mathbb{R}$                                                         |
| $\int adx = ax + c, (a \text{ cost})$                                                        | $\mathbb{R}$                                                         |
| $\int x^n dx = \frac{1}{n+1}x^{n+1} + c, (n \in \mathbb{Z} - \{-1\})$                        | $\mathbb{R}^*$                                                       |
| $\int \frac{1}{x} dx = \ln  x  + c$                                                          | $\mathbb{R}^*$                                                       |
| $\int x^\alpha dx = \frac{1}{\alpha+1}x^{\alpha+1} + c, (\alpha \in \mathbb{Z} - \{-1\})$    | $\mathbb{R}_+^*$                                                     |
| $\int e^x dx = e^x + c$                                                                      | $\mathbb{R}$                                                         |
| $\int a^x dx = \frac{1}{\ln(a)}a^x + c, (a \in \mathbb{R}_+^* - \{1\})$                      | $\mathbb{R}$                                                         |
| $\int \sin(x) dx = -\cos(x) + c$                                                             | $\mathbb{R}$                                                         |
| $\int \frac{1}{\cos^2(x)} dx = \tan(x) + c$                                                  | $\mathbb{R} - \left\{\frac{\pi}{2} + k\pi, k \in \mathbb{Z}\right\}$ |
| $\int \frac{1}{\sqrt{1-x^2}} dx = \arcsin(x) + c$                                            | $] -1, 1[$                                                           |
| $\int \frac{-1}{\sqrt{1-x^2}} dx = \arccos(x) + c$                                           | $] -1, 1[$                                                           |
| $\int \frac{1}{1+x^2} dx = \arctan(x) + c$                                                   | $\mathbb{R}$                                                         |
| $\int sh(x) dx = ch(x) + c$                                                                  | $\mathbb{R}$                                                         |
| $\int ch(x) dx = sh(x) + c$                                                                  | $\mathbb{R}$                                                         |
| $\int \frac{1}{ch^2(x)} dx = th(x) + c$                                                      | $\mathbb{R}$                                                         |
| $\int \frac{1}{\sqrt{1+x^2}} dx = \arg sh(x) + c = \ln(x + \sqrt{1+x^2}) + c$                | $\mathbb{R}$                                                         |
| $\int \frac{1}{\sqrt{x^2-1}} dx = \arg ch(x) + c = \ln(x + \sqrt{x^2-1}) + c$                | $]1, +\infty[$                                                       |
| $\int \frac{1}{1-x^2} dx = \arg th(x) + c = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right) + c$ | $] -1, +1[$                                                          |
| $\int f^4(x) f^n(x) dx = \frac{1}{1+n} f^{n+1}(x) + c, n \in \mathbb{Z}$                     | $D_{f'}$                                                             |
| $\int \frac{f^4(x)}{f^n(x)} dx = \frac{-1}{(n-1)f^{n-1}(x)} + c, (n \in \mathbb{Z} - \{1\})$ | /                                                                    |
| $\int \frac{f^4(x)}{f(x)} dx = \ln  f(x)  + c$                                               | /                                                                    |

## 3.2 Simple integration methods

### 3.2.1 Direct integration method

This method is based on the properties of indefinite integrals and formulas for transforming functions using primitive tables and two rules

$$\int \frac{f'(x)}{f(x)} dx = \ln |f(x)| + c.$$

And

$$\int f^{(n)}(x) dx = \frac{1}{n} f^{(n-1)}(x) + c.$$

#### Example 3.4

Compute

$$\int \left[ \cos^2(x) + \frac{1}{1+x^2} + \sqrt{x} \right] dx.$$

From indefinite integral property we have:

$$\int \cos^2(x) dx + \int \frac{1}{1+x^2} dx + \int \sqrt{x} dx.$$

Using the following trigonometric formula

$$\cos^2 x + \sin^2 x = 1, \cos(2x) = \cos^2 x - \sin^2 x.$$

Then

$$\cos(2x) = 2\cos^2(x) - 1 \Leftrightarrow \cos^2(x) = \frac{\cos(2x) + 1}{2} = \frac{1}{2}\cos(2x) + \frac{1}{2}. \quad (3.1)$$

On obtain

$$\int \cos^2(x) dx = \int \left[ \frac{1}{2}\cos(2x) + \frac{1}{2} \right] dx = \frac{1}{4}\sin(2x) + \frac{1}{2}x + c_1.$$

Using the table of antiderivatives, we obtain:

$$\int \frac{1}{1+x^2} dx = \arctan(x) + c_2.$$

Using the table of antiderivatives, we also get:

$$\int \sqrt{x} dx = \frac{2}{3}x^{\frac{3}{2}} + c_3 = \frac{2}{3}\sqrt{x^3} + c_3.$$

Then equation (3.1), write by

$$\begin{aligned} & \int \cos^2(x) dx + \int \frac{1}{1+x^2} dx + \int \sqrt{x} dx \\ &= \frac{1}{4} \sin(2x) + \frac{1}{2} x + c_1 + \arctan(x) + c_2 + \frac{2}{3} \sqrt{x^3} + c_3 \\ &= \frac{1}{4} \sin(2x) + \frac{1}{2} x + \arctan(x) + \frac{2}{3} \sqrt{x^3} + c, c \in \mathbb{R}. \end{aligned}$$

### 3.2.2 Integration By Substitution (Change of Variables)

The first formula

#### Proposition 3.4

Let  $f : J \rightarrow \mathbb{R}$  and  $g : I \rightarrow J$  be two functions such that:

- $I, J$  are two open intervals of  $\mathbb{R}$ .
- $f$  is a continuous function on  $J$ .
- $g$  is a differentiable function on  $I$ .

#### Definition 3.3 (*Practical method*)

To calculate the integral of type  $\int f(g(x))dx$ , we follow these steps:

1. We put  $t = g(x)$ .
2. Calculate the equation  $t = g(x)$ , we get  $x = h(t)$ .
3. Calculate  $dx = h'(t)dt$ .
4. We substitute in integration  $x$  and  $dx$  in terms of  $t$  and  $dt$ .
5. We calculate the indefinite integral

$$\int f(t)h'(t)dt.$$

6. Replacing in the result  $t$  by  $x$ , for indefinite integration.

**Example 3.5**

Calculate

$$I = \int \frac{x}{\sqrt{1+x^2}} dx.$$

We put

$$t = 1 + x^2 \Rightarrow dt = 2x dx.$$

Then

$$x = \sqrt{t-1} \Rightarrow dx = \frac{1}{2\sqrt{t-1}} dt.$$

Replacing in the integral  $I$  gives us:

$$I = \int \frac{1}{2\sqrt{t}} dt = \sqrt{t} + c, c \in \mathbb{R}.$$

Replacing  $t$  by  $x$ , we get

$$I = \sqrt{1+x^2} + c, c \in \mathbb{R}.$$

**The second formula****Proposition 3.5**

Let  $f : J \rightarrow \mathbb{R}$  and  $g : I \rightarrow J$  be two functions such that:

- $I, J$  are two open intervals of  $\mathbb{R}$ .
- $f$  is a continuous function on  $J$ .
- $g$  is a differentiable function on  $I$ .

If  $F(x)$  is an antiderivative of  $f(x)$  on  $J$ , then  $F(g(x))$  is an antiderivative of the function  $g'(x)f(g(x))$  on  $I$ .

**Definition 3.4**

Practical method: To calculate the integral of type  $\int g'(x)f(g(x))dx$ , we follow these steps:

1. We put  $t = g(x)$  and  $dt = g'(x)dx$ .
2. This gives the integral  $\int f(t)dt$ .

3. If  $F$  is an antiderivative of  $f$  then

$$\int f(t)dt = F(t) + c, c \in \mathbb{R}.$$

4. Replacing  $t$  by  $g(x)$  and  $dt$  by  $g'(x)dx$  gives the following result:

$$\int g'(x)f(g(x))dx = F(g(x)) + c, c \in \mathbb{R}.$$

### Example 3.6

Calculate

$$I = \int \frac{x}{\sqrt{1+x^2}}dx.$$

We put

$$t = 1 + x^2 \Rightarrow dt = 2xdx.$$

Replacing in the integral  $I$  gives us:

$$I = \int \frac{1}{2\sqrt{t}}dt = \sqrt{t} + c, c \in \mathbb{R} \Rightarrow I = \sqrt{1+x^2} + c, c \in \mathbb{R}.$$

### Example 3.7

Compute

$$I = \int \frac{1}{ch(x)}dx.$$

We have

$$ch(x) = \frac{e^x + e^{-x}}{2} \Leftrightarrow \frac{1}{ch(x)} = \frac{2}{e^x + e^{-x}} = \frac{2e^x}{e^{2x} + 1}.$$

Then

$$I = \int \frac{1}{ch(x)}dx = \int \frac{2e^x}{e^{2x} + 1}dx.$$

We put

$$t = e^x \Rightarrow dt = e^x dx.$$

Replacing in the integral  $I$  gives us:

$$I = 2 \int \frac{1}{t^2 + 1}dt = 2 \arctan(t) + c, c \in \mathbb{R} \Rightarrow I = 2 \arctan(e^x) + c, c \in \mathbb{R}.$$

### 3.2.3 Integration by parts method

#### Integration by parts several times

##### Definition 3.5 (*Integration by parts*)

Let  $f$  and  $g$  be two functions in class  $C^1$  on an open interval  $I$ , we have:

$$\int f(x) g'(x) dx = f(x) g(x) - \int f'(x) g(x) dx.$$

##### Example 3.8

Compute  $\int x e^{-2x} dx$  on  $\mathbb{R}$ . Let's put:

$$f(x) = x \longrightarrow f'(x) = 1.$$

$$g'(x) = e^{-2x} \longrightarrow g(x) = -\frac{1}{2} e^{-2x}.$$

According to the integration by parts formula, we obtain:

$$\int x e^{-2x} dx = -\frac{1}{2} x e^{-2x} + \frac{1}{2} \int e^{-2x} dx = -\frac{1}{2} x e^{-2x} - \frac{1}{4} e^{-2x} + c, c \in \mathbb{R}.$$

#### The choice of $f$ and $g'$ is not random

##### Proposition 3.6

To calculate

$$\int P_n(x) e^x dx, \text{ or } \int P_n(x) \cos x dx, \text{ or } \int P_n(x) \sin x dx$$

where  $P_n(x)$  is a polynomial of degree  $n$ .

We use integration by parts  $n$  times. The choice each time and in each case is as follows

$$\begin{array}{l} f(x) = P_n(x) \quad f(x) = P_n(x) \quad f(x) = P_n(x) \\ g'(x) = e^x \quad , \quad g'(x) = \cos x \quad , \quad g'(x) = \sin x \end{array}.$$

**Example 3.9**

To calculate  $I(x) = \int x^2 \cos x dx$ , using integration by parts, we put

$$\begin{array}{lcl} f(x) = & x^2 & \Rightarrow f'(x) = 2x \\ g'(x) = & \cos x & g(x) = \sin x \end{array}.$$

Then

$$I(x) = \int x^2 \cos x dx = x^2 \sin x - 2 \int x \sin x dx.$$

To calculate  $J(x) = \int x \sin x dx$ , we use a second time integration by parts, we put

$$\begin{array}{lcl} f(x) = & x & \Rightarrow f'(x) = 1 \\ g'(x) = & \sin x & g(x) = -\cos x \end{array}.$$

Then

$$J(x) = \int x \sin x dx = -x \cos x + \int \cos x dx = -x \cos x + \sin x + c.$$

Substituting into the statement  $I(x)$ , we find

$$\begin{aligned} I(x) &= x^2 \sin x - 2J(x) = x^2 \sin x - 2[-x \cos x + \sin x + c] \\ &= x^2 \sin x + 2x \cos x - 2 \sin x - 2c \\ &= x^2 \sin x + 2x \cos x - 2 \sin x - c_1. \end{aligned}$$

**Proposition 3.7**

To calculate

$$\begin{aligned} &\int P_n(x) \ln(x) dx, \text{ or } \int P_n(x) \arccos(x) dx, \text{ or } \int P_n(x) \arcsin(x) dx, \\ &\text{or } \int P_n(x) \arctan(x) dx, \text{ where } P_n(x) \text{ is a polynomial of degree } n. \end{aligned}$$

We use integration by parts only once, and the choice in each case should be as follows

$$\begin{array}{lcl} f(x) = & \ln(x) & f(x) = \arccos(x) \\ g'(x) = & e^x & g'(x) = \cos x \\ f(x) = & \arcsin(x) & f(x) = \arctan(x) \\ g'(x) = & \sin x & g'(x) = \sin x \end{array}.$$

**Example 3.10**

To calculate  $I(x) = \int \ln(x) dx$ , using integration by parts, we put

$$\begin{array}{rcl} f(x) = \ln(x) & \Rightarrow & f'(x) = \frac{1}{x} \\ g'(x) = 1 & & g(x) = x \end{array}$$

Then

$$I(x) = \int \ln(x) dx = x \ln(x) - \int \frac{x}{x} dx = x \ln(x) - \int 1 dx = x \ln(x) - x + c.$$

### 3.3 Integration of rational functions

**Definition 3.6 (*Integration of rational*)**

Let  $P(x)$  and  $Q(x)$  be two polynomials of degree  $n_1$  and  $n_2$  respectively. A rational function is any function of kind

$$f(x) = \frac{P(x)}{Q(x)}.$$

- If  $\deg(P) \geq \deg(Q)$  we say that  $f$  is an improper rational fraction.
- If  $\deg(P) < \deg(Q)$  we say that  $f$  is a proper rational fraction.

**Step I: Grade Control**

Let  $f(x) = \frac{P(x)}{Q(x)}$  be an improper rational fraction (i.e.  $P(x)$  and  $Q(x)$  are two polynomials of degree  $n_1$  and  $n_2$  respectively with  $\deg(P) \geq \deg(Q)$ ). The Euclidean division of  $P(x)$  by  $Q(x)$  yields:

$$f(x) = N(x) + \frac{R(x)}{Q(x)},$$

$$\text{where } \begin{cases} N(x) \text{ is a polynomial of degree } \deg(P) - \deg(Q). \\ R(x) \text{ is a polynomial of degree strictly less than } \deg(Q). \\ \frac{R(x)}{Q(x)} \text{ is proper rational fraction.} \end{cases}$$

We deduce that:

$$\int f(x)dx = \int \frac{P(x)}{Q(x)}dx = \int N(x)dx + \int \frac{R(x)}{Q(x)}dx.$$

Calculate

$$\int N(x)dx.$$

After, calculate

$$\int \frac{R(x)}{Q(x)}dx.$$

we pass to the second step.

### Step II: Analysis $Q(x)$ in $\mathbb{R}$

We write  $Q(x)$  in the form of first-order or second-order polynomial, and its Delta is negative ( $\Delta \prec 0$ ), i.e.

$$Q(x) = (x - \alpha_1)^{s_1} (x - \alpha_2)^{s_2} \cdots (x - \alpha_r)^{s_r} (x^2 + p_1x + q_1)^{m_1} (x^2 + p_2x + q_2)^{m_2} \cdots (x^2 + p_kx + q_k)^{m_k}.$$

For all  $1 \leq i \leq r$ , then  $\alpha_i$  is a real root of  $Q(x)$ , and  $s_i$  is the degree of its multiplication.

For all  $1 \leq j \leq k$ , then  $p_j^2 - 4q_j \prec 0$ .

### Step III: Decomposition $\frac{R(x)}{Q(x)}$

We write the fraction  $\frac{R(x)}{Q(x)}$  as the sum of fractions as follows:

$$\begin{aligned} \frac{R(x)}{Q(x)} &= \frac{A_{11}}{x - \alpha_1} + \frac{A_{12}}{(x - \alpha_1)^2} + \cdots + \frac{A_{1s_1}}{(x - \alpha_1)^{s_1}} + \frac{A_{21}}{x - \alpha_2} + \cdots + \frac{A_{2s_2}}{(x - \alpha_2)^{s_2}} \\ &+ \cdots + \frac{A_{r1}}{x - \alpha_r} + \cdots + \frac{A_{rs_r}}{(x - \alpha_r)^{s_r}} \\ &+ \frac{M_{11}x + N_{11}}{x^2 + p_1x + q_1} + \frac{M_{12}x + N_{12}}{(x^2 + p_1x + q_1)^2} + \cdots + \frac{M_{1m_1}x + N_{1m_1}}{(x^2 + p_1x + q_1)^{m_1}} \\ &+ \frac{M_{21}x + N_{21}}{x^2 + p_2x + q_2} + \frac{M_{22}x + N_{22}}{(x^2 + p_2x + q_2)^2} + \cdots + \frac{M_{2m_2}x + N_{2m_2}}{(x^2 + p_2x + q_2)^{m_2}} \\ &+ \cdots \\ &+ \frac{M_{k1}x + N_{k1}}{x^2 + p_kx + q_k} + \frac{M_{k2}x + N_{k2}}{(x^2 + p_kx + q_k)^2} + \cdots + \frac{M_{km_k}x + N_{km_k}}{(x^2 + p_kx + q_k)^{m_k}}. \end{aligned}$$

where  $A_{ij}$ ,  $M_{ij}$  and  $N_{ij}$  are constants, that we calculate by unifying the denominators and then matching or giving special values.

**Step *IIII*: Integration**

The integral  $I(x) = \int \frac{R(x)}{Q(x)} dx$ , depends on calculating the integration of simple rational fractions from both shapes:

$$I_1(x) = \int \frac{1}{(x - \alpha)^n} dx, I_2(x) = \int \frac{Mx + N}{(x^2 + px + q)^m} dx,$$

where  $n, m \in \mathbb{N}^*$ , and  $\Delta = p^2 - 4q < 0$ .

**3.3.1 Calculate the integral of the form  $I_1(x) = \int \frac{1}{(x - \alpha)^n} dx$** 

For  $n = 1$ , then

$$I_1(x) = \int \frac{1}{x - \alpha} dx = \ln |x - \alpha| + c.$$

For  $n > 1$ , then

$$I_1(x) = \int \frac{1}{(x - \alpha)^n} dx = \frac{1}{(1 - n)(x - \alpha)^{n-1}} + c.$$

**3.3.2 Calculate the integral of the form  $I_2(x) = \int \frac{Mx + N}{(x^2 + px + q)^m} dx$ , where  $m \in \mathbb{N}^*$ , and  $\Delta = p^2 - 4q < 0$ .**

If  $m = 1 \Rightarrow I = \int \frac{Mx + N}{x^2 + px + q} dx$ . We write the numerator in terms of the derivative of the denominator

$$Mx + N = \frac{M}{2}(2x + p) + N - \frac{Mp}{2}.$$

Then

$$I_2(x) = \frac{M}{2} \ln |x^2 + px + q| + \left(N - \frac{Mp}{2}\right) \frac{1}{\beta} \arctan\left(\frac{x + p/2}{\beta}\right) + C.$$

To calculate

$$I_3(x) = \int \frac{1}{x^2 + px + q} dx.$$

We try to write it as the famous integration

$$\int \frac{1}{t^2 + 1} dt.$$

We put

$$\beta^2 = \frac{-\Delta}{4} = \frac{4q - p^2}{4}.$$

Then

$$x^2 + px + q = \left(x + \frac{p}{2}\right)^2 + \beta^2 = \beta^2 \left[ \frac{\left(x + \frac{p}{2}\right)^2}{\beta^2} + 1 \right].$$

Then

$$I_3(x) = \int \frac{1}{\beta^2 \left[ \frac{\left(x + \frac{p}{2}\right)^2}{\beta^2} + 1 \right]} dx = \frac{1}{\beta^2} \int \frac{1}{\left(\frac{x + \frac{p}{2}}{\beta}\right)^2 + 1} dx.$$

Using a change of variable

$$t = \frac{x + \frac{p}{2}}{\beta} \Rightarrow x = t\beta - \frac{p}{2} \Rightarrow dx = \beta dt.$$

Then

$$\begin{aligned} I_3(x) &= \frac{1}{\beta} \int \frac{1}{[t^2 + 1]} dt = \frac{1}{\beta} \arctan(t) + c \\ &\Rightarrow I_3(x) = \frac{1}{\beta} \arctan\left(\frac{x + \frac{p}{2}}{\beta}\right) + c. \end{aligned}$$

Substituting into the statement  $I_2(x)$ , we get

$$I_2(x) = \frac{M}{2} \ln|x^2 + px + q| + \left(N - \frac{Mp}{2}\right) \frac{1}{\beta} \arctan\left(\frac{x + p/2}{\beta}\right) + C.$$

If  $m \geq 2 \Rightarrow I = \int \frac{Mx + N}{(x^2 + px + q)^m} dx$ . We write the numerator in terms of the derivative of the denominator

$$Mx + N = \frac{M}{2} (2x + p) + N - \frac{Mp}{2}.$$

Then

$$\begin{aligned} I &= \int \frac{Mx + N}{(x^2 + px + q)^m} dx = \int \frac{\frac{M}{2} (2x + p) + N - \frac{Mp}{2}}{(x^2 + px + q)^m} dx \\ &= \frac{M}{2} \int \frac{(2x + p)}{(x^2 + px + q)^m} dx + \left(N - \frac{Mp}{2}\right) \int \frac{1}{(x^2 + px + q)^m} dx \\ &= \frac{M}{2} I_1 + \left(N - \frac{Mp}{2}\right) I_2. \end{aligned}$$

Calculate  $I_1$ , we put

$$y = x^2 + px + q \Rightarrow dy = (2x + p) dx.$$

Then

$$\begin{aligned} I_1 &= \int \frac{(2x + p)}{(x^2 + px + q)^m} dx = \int \frac{1}{y^m} dy = \int y^{-m} dy = \frac{1}{1 - m} y^{1-m} \\ &= \frac{1}{(1 - m) y^{m-1}} = \frac{1}{(1 - m) (x^2 + px + q)^{m-1}}. \end{aligned}$$

So

$$I_2 = \int \frac{1}{(x^2 + px + q)^m} dx = \int \frac{1}{\left[\left(x + \frac{p}{2}\right)^2 \left(q - \frac{p^2}{4}\right)\right]^m} dy.$$

We put

$$t = x + \frac{p}{2} \Rightarrow dt = dx, s^2 = q - \frac{p^2}{4}.$$

Then

$$I_2 = \int \frac{1}{(t^2 + s^2)^m} dt = \frac{1}{(s^2)^m} \int \frac{1}{\left(\frac{t^2}{s^2} + 1\right)^m} dt = \frac{1}{s^{2m}} \int \frac{1}{\left(1 + \left(\frac{t}{s}\right)^2\right)^m} dt.$$

We put

$$\frac{t}{s} = \tan \gamma \Rightarrow t = s \tan \gamma \Rightarrow dt = s \left( \frac{1}{\cos^2 \gamma} \right) d\gamma.$$

Then

$$I_2 = \frac{1}{s^{2m}} \int \frac{s}{\cos^2 \gamma \left(1 + (\tan \gamma)^2\right)^m} d\gamma.$$

Let

$$\cos^2 \gamma + \sin^2 \gamma = 1 \Rightarrow 1 + \tan^2 \gamma = \frac{1}{\cos^2 \gamma}.$$

Then

$$I_2 = \frac{1}{s^{2m-1}} \int \frac{1}{\cos^2 \gamma \left(\frac{1}{\cos^2 \gamma}\right)^m} d\gamma = \frac{1}{s^{2m-1}} \int \frac{1}{\cos^2 \gamma \frac{1}{\cos^{2m} \gamma}} d\gamma = \frac{1}{s^{2m-1}} \int \cos^{2(m-1)} \gamma d\gamma.$$

Then

$$\begin{aligned} I &= \frac{M}{2} I_1 + \left(N - \frac{Mp}{2}\right) I_2 \\ &= \frac{M}{2} \left[ \frac{1}{(1-m)(x^2 + px + q)^{m-1}} \right] + \left(N - \frac{Mp}{2}\right) \left[ \frac{1}{s^{2m-1}} \int \cos^{2(m-1)} \gamma d\gamma \right]. \end{aligned}$$

Where

$$s = \frac{\sqrt{4q - p^2}}{2}, \gamma = \arctan\left(\frac{t}{s}\right), t = x + \frac{p}{2}.$$

**Example 3.11**

To calculate the integral of a form

$$I(x) = \int \frac{x+3}{x^2-3x+2} dx.$$

Let

$$x^2 - 3x + 2 = (x-2)(x-1).$$

Then

$$\begin{aligned} \frac{x+3}{x^2-3x+2} &= \frac{a}{(x-2)} + \frac{b}{(x-1)} \\ \frac{x+3}{x^2-3x+2} &= \frac{a(x-1) + b(x-2)}{(x-2)(x-1)} = \frac{(a+b)x - a - 2b}{(x-2)(x-1)}, \end{aligned}$$

Then

$$\begin{cases} a+b=1 \\ -a-2b=3 \end{cases} \Rightarrow \begin{cases} b=-4 \\ a=5 \end{cases},$$

Hence

$$\begin{aligned} I(x) &= \int \frac{x+3}{x^2-3x+2} dx = \int \frac{5}{(x-2)} - \frac{4}{(x-1)} dx \\ &= 5 \ln |x-2| - 4 \ln |x-1| + c. \end{aligned}$$

**Example 3.12**

Calculate these indefinite integrals of the rational functions.

1.  $\int \frac{1}{x^2(x-1)} dx.$
2.  $\int \frac{x+2}{x^2-3x-4} dx.$
3.  $\int \frac{-2x^3+7}{x^3+x^2-2} dx.$

Calculate these indefinite integrals of the rational functions.

**Solution 3.1**

1.  $\int \frac{1}{x^2(x-1)} dx$ . Let

$$\frac{1}{x^2(x-1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1}.$$

Then

$$\begin{aligned} \frac{1}{x^2(x-1)} &= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-1} = \frac{Ax(x-1)}{x^2(x-1)} + \frac{B(x-1)}{x^2(x-1)} + \frac{Cx^2}{x^2(x-1)} \\ &= \frac{Ax^2 - Ax + Bx - B + Cx^2}{x^2(x-1)} = \frac{Ax^2 + Cx^2 + Bx - Ax - B}{x^2(x-1)} \\ &= \frac{(A+C)x^2 + (B-A)x - B}{x^2(x-1)}. \end{aligned}$$

By matching, we find

$$\begin{cases} A+C=0 \\ B-A=0 \\ -B=1 \end{cases} \Rightarrow \begin{cases} C=-A=1 \\ A=B=-1 \\ B=-1 \end{cases}.$$

Then

$$\frac{1}{x^2(x-1)} = -\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x-1}.$$

Hence

$$\begin{aligned} \int \frac{1}{x^2(x-1)} dx &= \int -\frac{1}{x} - \frac{1}{x^2} + \frac{1}{x-1} dx = -\int \frac{1}{x} dx - \int \frac{1}{x^2} dx + \int \frac{1}{x-1} dx \\ &= -\ln|x| + \frac{1}{x} + \ln|x-1| + c. \end{aligned}$$

2.  $\int \frac{x+2}{x^2-3x-4} dx$ . Calculate  $\Delta$  of the polynomial  $x^2-3x-4$ :

$$\Delta = 25, x_1 = -1, x_2 = 4.$$

Then

$$\int \frac{x+2}{x^2-3x-4} dx = \int \frac{x+2}{(x+1)(x-4)} dx = \int \frac{A}{(x+1)} + \frac{B}{(x-4)} dx.$$

Let

$$\frac{A}{(x+1)} + \frac{B}{(x-4)} = \frac{A(x-4) + B(x+1)}{(x+1)(x-4)} = \frac{(A+B)x - 4A + B}{(x+1)(x-4)}.$$

By matching, we find

$$\begin{cases} A+B=1 \\ -4A+B=2 \end{cases} \Rightarrow \begin{cases} A = \frac{-1}{5} \\ B = \frac{6}{5} \end{cases}.$$

Then

$$\begin{aligned}\int \frac{x+2}{x^2-3x-4} dx &= \int \frac{\frac{-1}{5}}{(x+1)} + \frac{\frac{6}{5}}{(x-4)} dx \\ &= \frac{-1}{5} \int \frac{1}{(x+1)} dx + \frac{6}{5} \int \frac{1}{(x-4)} dx \\ &= \frac{-1}{5} \ln|x+1| + \frac{6}{5} \ln|x-4| + c.\end{aligned}$$

3.  $\int \frac{-2x^3+7}{x^3+x^2-2} dx$ . By division, we get

$$\begin{array}{r|l} -2x^3+7 & x^3+x^2-2 \\ - & \\ -2x^3-2x^2+4 & -2 \\ \hline 2x^2+3 & \end{array}$$

Then

$$\frac{-2x^3+7}{x^3+x^2-2} = -2 + \frac{2x^2+3}{x^3+x^2-2}.$$

Then  $x=1$  is solution of  $x^3+x^2-2$ , then

$$\begin{array}{r|l} x^3+x^2-2 & x-1 \\ - & \\ x^3-x^2 & x^2+2x+2 \\ \hline 2x^2-2 & \\ - & \\ 2x^2-2x & \\ \hline 2x-2 & \\ - & \\ 2x-2 & \\ \hline 0 & \end{array}$$

Then

$$\frac{2x^2+3}{x^3+x^2-2} = \frac{2x^2+3}{(x-1)(x^2+2x+2)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+2x+2}.$$

Then

$$\begin{aligned}\frac{2x^2+3}{x^3+x^2-2} &= \frac{Ax^2+2Ax+2A+Bx^2+Cx-Bx-C}{(x-1)(x^2+2x+2)} \\ &= \frac{(A+B)x^2+(2A-B+C)x+2A-C}{(x-1)(x^2+2x+2)}.\end{aligned}$$

By matching, we find

$$\begin{aligned} & \begin{cases} A + B = 2 \\ 2A - B + C = 0 \\ 2A - C = 3 \end{cases} \Rightarrow \begin{cases} B = 2 - A \\ 2A - B + C = 0 \\ C = 2A - 3 \end{cases} \\ & \Rightarrow \begin{cases} B = 1 \\ 2A + A + 2A - 3 - 2 = 0 \Rightarrow A = 1 \\ C = -1 \end{cases} . \end{aligned}$$

Then

$$\frac{2x^2 + 3}{x^3 + x^2 - 2} = \frac{1}{x - 1} + \frac{x - 1}{x^2 + 2x + 2}.$$

So,

$$\begin{aligned} \int \frac{-2x^3 + 7}{x^3 + x^2 - 2} dx &= - \int 2dx + \int \frac{2x^2 + 3}{x^3 + x^2 - 2} dx \\ &= - \int 2dx + \int \frac{1}{x - 1} + \frac{x - 1}{x^2 + 2x + 2} dx \\ &= - \int 2dx + \int \frac{1}{x - 1} dx + \int \frac{x - 1}{x^2 + 2x + 2} dx. \end{aligned}$$

Calculate

$$\begin{aligned} \int \frac{x - 1}{x^2 + 2x + 2} dx &= \int \frac{(x - 1) + 1 - 1}{x^2 + 2x + 2} dx = \int \frac{x + 1 - 1 - 1}{x^2 + 2x + 2} dx \\ &= \int \frac{x + 1 - 2}{x^2 + 2x + 2} dx = \int \frac{\frac{2}{2}(x + 1) - 2}{x^2 + 2x + 2} dx \\ &= \int \frac{\frac{1}{2}(2x + 2) - 2}{x^2 + 2x + 2} dx = \frac{1}{2} \int \frac{(2x + 2)}{x^2 + 2x + 2} - \frac{2}{x^2 + 2x + 2} dx. \end{aligned}$$

Then

$$\begin{aligned} \int \frac{-2x^3 + 7}{x^3 + x^2 - 2} dx &= - \int 2dx + \int \frac{1}{x - 1} dx + \int \frac{x - 1}{x^2 + 2x + 2} dx \\ &= - \int 2dx + \int \frac{1}{x - 1} dx + \frac{1}{2} \int \frac{(2x + 2)}{x^2 + 2x + 2} dx - \int \frac{2}{(x + 1)^2 + 1} dx \\ &= -2x + \ln|x| + \frac{1}{2} \ln|x^2 + 2x + 2| - 2 \arctan(x + 1) + c. \end{aligned}$$

## 3.4 Integration of exponential and trigonometric functions

### 3.4.1 Fractional rationals of trigonometric functions

The first form

$$I = \int f(\sin x, \cos x) dx.$$

Let  $f$  be a rational function,  $\sin x$  and  $\cos x$  are the trigonometric functions.

Let

$$\begin{aligned} \sin x &= \sin\left(2\frac{x}{2}\right) = 2 \sin\left(\frac{x}{2}\right) \cos\left(\frac{x}{2}\right) = \frac{2 \sin\left(\frac{x}{2}\right) \cos\left(\frac{x}{2}\right)}{1} = \frac{2 \sin\left(\frac{x}{2}\right) \cos\left(\frac{x}{2}\right)}{\sin^2\left(\frac{x}{2}\right) + \cos^2\left(\frac{x}{2}\right)} \\ &= \frac{\frac{2 \sin\left(\frac{x}{2}\right) \cos\left(\frac{x}{2}\right)}{\cos^2\left(\frac{x}{2}\right)}}{\frac{\sin^2\left(\frac{x}{2}\right) + \cos^2\left(\frac{x}{2}\right)}{\cos^2\left(\frac{x}{2}\right)}} = \frac{2 \tan\left(\frac{x}{2}\right)}{1 + \tan^2\left(\frac{x}{2}\right)}. \end{aligned}$$

We put

$$t = \tan\left(\frac{x}{2}\right) \Rightarrow \frac{x}{2} = \arctan t \Rightarrow x = 2 \arctan(t) \Rightarrow dx = \frac{2}{1+t^2} dt.$$

Then

$$\sin t = \frac{2t}{1+t^2}.$$

The same method, we find

$$\cos t = \frac{1-t^2}{1+t^2}.$$

$$\tan t = \frac{2t}{1-t^2}.$$

Then

$$I = \int f(\sin x, \cos x) dx = \int f\left(\frac{2t}{1+t^2}, \frac{1-t^2}{1+t^2}\right) \frac{2}{1+t^2} dt.$$

**Example 3.13**

$$I = \int \frac{1}{2 + \cos x} dx.$$

We put

$$t = \tan\left(\frac{x}{2}\right) \Rightarrow \frac{x}{2} = \arctan t \Rightarrow x = 2 \arctan(t) \Rightarrow dx = \frac{2}{1+t^2} dt \Rightarrow \cos t = \frac{1-t^2}{1+t^2}.$$

Then

$$\begin{aligned} I &= \int \frac{1}{2 + \cos x} dx = \int \frac{1}{\left[2 + \left(\frac{1-t^2}{1+t^2}\right)\right]} \frac{2}{(1+t^2)} dt \\ &= \int \frac{2}{2(1+t^2) + (1-t^2)} dt = 2 \int \frac{1}{t^2 + 3} dt = \frac{2}{\sqrt{3}} \arctan\left(\frac{t}{\sqrt{3}}\right) + c. \end{aligned}$$

Then

$$I = \frac{2}{\sqrt{3}} \arctan\left(\frac{1}{\sqrt{3}} \tan\left(\frac{x}{2}\right)\right) + c.$$

**Example 3.14**

Calculate these indefinite integrals of the trigonometric rational.

1.  $\int \frac{1}{1 + \sin x} dx.$
2.  $\int \frac{\cos x}{2 - \cos^2 x} dx.$

Calculate these indefinite integrals of the trigonometric rational.

1.  $\int \frac{1}{1 + \sin x} dx.$  By change of the variable

$$t = \tan\left(\frac{x}{2}\right).$$

Then

$$\sin x = \frac{2t}{1+t^2}, dx = \frac{2}{1+t^2} dt.$$

Then

$$\begin{aligned} \int \frac{1}{1 + \sin x} dx &= \int \frac{1}{1 + \left[\frac{2t}{1+t^2}\right]} \frac{2}{(1+t^2)} dt = \int \frac{1}{\frac{1+t^2+2t}{(1+t^2)}} \frac{2}{(1+t^2)} dt \\ &= \int \frac{1}{\frac{t^2+2t+1}{(1+t^2)}} \frac{2}{(1+t^2)} dt = \int \frac{2}{t^2 + 2t + 1} dt = \int \frac{2}{(t+1)^2} dt \\ &= \frac{-2}{t+1} + c = \frac{-2}{\tan\left(\frac{x}{2}\right) + 1} + c. \end{aligned}$$

2.  $\int \frac{\cos x}{2 - \cos^2 x} dx$ . Let

$$\cos(\pi - x) = -\cos x.$$

We put

$$y = \sin x \Rightarrow dy = \cos x dx,$$

$$\sin^2 x + \cos^2 x = 1 \Rightarrow \cos^2 x = 1 - \sin^2 x.$$

Then

$$\begin{aligned} \int \frac{\cos x}{2 - \cos^2 x} dx &= \int \frac{1}{2 - [1 - \sin^2 x]} \cos x dx = \int \frac{1}{1 + \sin^2 x} \cos x dx \\ &= \int \frac{1}{1 + y^2} dy = \arctan y + c = \arctan(\sin x) + c. \end{aligned}$$

### The second form

$$I = \int \sin^p x \cos^q x dx, \quad p, q \in \mathbb{Z}.$$

To calculate the integral above, we use a change of variable according to the part of the numbers  $p$  and  $q$ . A distinction is made between

1. If  $p$  is odd, we put  $u = \cos x$ .
2. If  $q$  is odd, we put  $u = \sin x$ .
3. If  $p$  and  $q$  are odd, we put  $u = \cos 2x$ .
4. If  $p$  and  $q$  are even, we put  $u = \tan x$ .

Or we linear it integrating using the formulas

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \quad \cos^2 x = \frac{1 + \cos 2x}{2}.$$

**Example 3.15**

Calculate the integral

$$I = \int \sin^2 x \cos^4 x dx.$$

The integral is an even function with respect to the  $\sin x$  and  $\cos x$

$$\sin^2 x \cos^4 x = (1 - \cos^2 x) \cos^4 x = \cos^4 x - \cos^6 x = (\cos^2 x)^2 - (\cos^2 x)^3.$$

By linearizing the right line of this equality, we obtain

$$\begin{aligned} \sin^2 x \cos^4 x &= \left( \frac{1 + \cos 2x}{2} \right)^2 - \left( \frac{1 + \cos 2x}{2} \right)^3 \\ &= \frac{1}{8} [2(1 + \cos 2x)^2 - (1 + \cos 2x)^3] \\ &= \frac{1}{8} [2(1 + 2\cos 2x + \cos^2(2x)) - (1 + \cos^3 2x + 3\cos 2x(1 + \cos 2x))] \\ &= \frac{1}{8} [2 + 4\cos 2x + 2\cos^2(2x) - 1 - \cos^3 2x - 3\cos 2x - 3\cos^2 2x] \\ &= \frac{1}{8} [-\cos^3(2x) - \cos^2(2x) + \cos(2x) + 1]. \end{aligned}$$

Then

$$\begin{aligned} \int \sin^2 x \cos^4 x dx &= \frac{1}{8} \int [-\cos^3(2x) - \cos^2(2x) + \cos(2x) + 1] dx \\ &= \frac{1}{8} \left[ -\int \cos^3(2x) dx - \int \cos^2(2x) dx + \int \cos(2x) dx + \int 1 dx \right] \\ &= \frac{1}{8} \left[ -\int (1 - \sin^2(2x)) 2\cos(2x) dx - \int \frac{1 + \cos(4x)}{2} dx + \frac{1}{2} \sin(2x) + x \right]. \end{aligned}$$

Let

$$\cos^3(2x) = \cos^2(2x) \cos(2x) = (1 - \sin^2(2x)) \cos(2x).$$

Then

$$\begin{aligned} I &= \frac{1}{8} \left[ -\int (1 - \sin^2(2x)) \cos(2x) dx - \frac{x}{2} - \frac{\sin(4x)}{8} + \frac{1}{2} \sin(2x) + x \right] \\ &= \frac{1}{8} \left[ \int -\cos(2x) - \cos(2x) \sin^2(2x) dx + \frac{x}{2} - \frac{\sin(4x)}{8} + \frac{1}{2} \sin(2x) + x \right] \\ &= \frac{1}{8} \left[ -\frac{1}{2} \sin(2x) + \frac{1}{6} \sin^3(2x) + x - \frac{x}{2} - \frac{\sin(4x)}{8} + \frac{1}{2} \sin(2x) \right] \\ &= \frac{1}{8} \left[ \frac{x}{2} - \frac{\sin(4x)}{8} + \frac{1}{6} \sin^3(2x) \right]. \end{aligned}$$

---

**Integration the form**  $I = \int \sin(\alpha x) \cos(\beta x) dx$

The integral, which is a product of 2 trigonometric functions, could be transformed into a sum of trigonometric functions using the following formula:

$$\begin{aligned}\sin(\alpha x) \cos(\beta x) &= \frac{1}{2} [\sin((\alpha + \beta)x) + \sin((\alpha - \beta)x)], \\ \sin(\alpha x) \sin(\beta x) &= \frac{1}{2} [\cos((\alpha - \beta)x) - \cos((\alpha + \beta)x)], \\ \cos(\alpha x) \cos(\beta x) &= \frac{1}{2} [\cos((\alpha + \beta)x) + \cos((\alpha - \beta)x)].\end{aligned}$$

**Example 3.16**

$$I = \int \sin^2\left(\frac{x}{2}\right) \cos\left(\frac{3x}{2}\right) dx.$$

*Let*

$$\begin{aligned}\sin^2\left(\frac{x}{2}\right) &= \frac{1 - \cos x}{2} \\ \sin^2\left(\frac{x}{2}\right) \cos\left(\frac{3x}{2}\right) &= \left(\frac{1 - \cos x}{2}\right) \cos\left(\frac{3x}{2}\right) \\ &= \frac{1}{2} \left[ \cos\left(\frac{3x}{2}\right) - \cos(x) \cos\left(\frac{3x}{2}\right) \right].\end{aligned}$$

*Then*

$$I = \frac{1}{2} \int \left[ \cos\left(\frac{3x}{2}\right) - \cos\left(\frac{2x}{2}\right) \cos\left(\frac{3x}{2}\right) \right] dx = \frac{1}{2} \cos\left(\frac{3x}{2}\right) - \frac{1}{4} \left[ \cos\left(\frac{5x}{2}\right) - \cos\left(\frac{x}{2}\right) \right].$$

**Integration the form**  $I = \int e^{(\alpha x)} P(x) dx, \alpha \in \mathbb{R}, P(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_0$

Integrating  $n$  times per part, we get

$$\begin{aligned}I &= \int e^{(\alpha x)} P(x) dx = e^{(\alpha x)} \left[ \frac{P(x)}{\alpha} - \frac{P'(x)}{\alpha^2} + \frac{P''(x)}{\alpha^3} + \dots + (-1)^n \frac{P^{(n)}(x)}{\alpha^{(n+1)}} \right] + c \\ &= e^{(\alpha x)} Q(x) + c.\end{aligned}$$

**Example 3.17**

$$K = \int e^{2x} (x^2 + 3x + 5) dx.$$

So

$$K = Q(x) e^{2x} + c = (a_2 x^2 + a_1 x + a_0) e^{2x} + c.$$

Drifting we have:

$$\begin{aligned} \left[ (a_2 x^2 + a_1 x + a_0) e^{2x} \right]' &= (2a_2 x + a_1) e^{2x} + 2(a_2 x^2 + a_1 x + a_0) e^{2x} \\ &= [2a_2 x^2 + (2a_2 + 2a_1)x + (a_1 + 2a_0)] e^{2x} \\ &= (x^2 + 3x + 5) e^{2x}. \end{aligned}$$

Then

$$[2a_2 x^2 + (2a_2 + 2a_1)x + (a_1 + 2a_0)] e^{2x} = (x^2 + 3x + 5) e^{2x}.$$

Then

$$\begin{cases} 2a_2 = 1 \\ 2a_2 + 2a_1 = 3 \\ a_1 + 2a_0 = 5 \end{cases} \Rightarrow \begin{cases} a_2 = \frac{1}{2} \\ a_1 = \frac{3 - 2a_2}{2} = \frac{3 - 2 \cdot \frac{1}{2}}{2} = 1 \\ a_0 = \frac{5 - a_1}{2} = \frac{5 - 1}{2} = 2 \end{cases}.$$

Then

$$K = \left( \frac{1}{2} x^2 + x + 2 \right) e^{2x} + c.$$

**Example 3.18**

To calculate  $I(x) = \int e^{-2x} \cos(x) dx$ , we use integration by parts. We put

$$\begin{aligned} f'(x) &= \cos(x) & \Rightarrow & f(x) = \sin(x) \\ g(x) &= e^{-2x} & \Rightarrow & g'(x) = -2e^{-2x}. \end{aligned}$$

Then

$$I(x) = \int e^{-2x} \cos(x) dx = e^{-2x} \sin(x) dx + 2 \int e^{-2x} \sin(x) dx.$$

To calculate  $J(x) = \int e^{-2x} \sin(x) dx$  we use integration by parts. We put

$$\begin{aligned} f'(x) &= \sin(x) & \Rightarrow & f(x) = -\cos(x) \\ g(x) &= e^{-2x} & \Rightarrow & g'(x) = -2e^{-2x}. \end{aligned}$$

Then

$$J(x) = \int e^{-2x} \sin(x) dx = -e^{-2x} \cos(x) dx - 2 \int e^{-2x} \cos(x) dx = -e^{-2x} \cos(x) dx - 2I(x).$$

Substituting into the statement  $I(x)$ , we find

$$I(x) = e^{-2x} \sin(x) dx + 2 \left[ -e^{-2x} \cos(x) dx - 2I(x) \right]$$

$$I(x) = e^{-2x} \sin(x) dx - 2e^{-2x} \cos(x) dx - 4I(x)$$

$$5I(x) = e^{-2x} \sin(x) dx - 2e^{-2x} \cos(x) dx$$

$$I(x) = \frac{1}{5} \left( e^{-2x} \sin(x) dx - 2e^{-2x} \cos(x) dx \right) + c.$$

## 3.5 The integral of polynomials

### Definition 3.7

Let  $P_n(x)$  be a polynomials of degree  $n$ , the following form

$$P_n(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0, \text{ we have } \int x^\alpha dx = \frac{1}{1+\alpha} x^{\alpha+1} + c.$$

Then

$$\int P_n(x) dx = \frac{a_n}{n+1} x^{n+1} + \frac{a_{n-1}}{n} x^n + \dots + \frac{a_1}{2} x^2 + a_0 x + c.$$

### Example 3.19

$$\int (2x^2 + 1) dx = 2 \int x^2 dx + \int 1 dx = \frac{2}{3} x^3 + x + c.$$

## 3.6 Defined integral

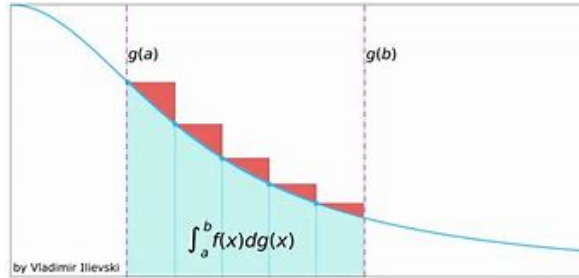
### 3.6.1 Riemann sum

#### Definition 3.8

Let  $f$  be a real-valued function defined on the interval  $[a, b]$ . The Riemann sum of  $f$  with respect to the tagged partition  $x_0, \dots, x_n$  together with  $t_0, \dots, t_{n-1}$  is

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \sum_{k=1}^{k=n} f(c_k) \Delta x_k,$$

where  $\Delta x_k = x_k - x_{k-1}$ , and  $c_k \in [x_{k-1}, x_k]$ .



### 3.6.2 Riemann integral

#### Definition 3.9

The function  $f$  is Riemann integral over the interval  $[a, b]$ , if

$$\lim_{n \rightarrow \infty} S_n = \lim_{n \rightarrow \infty} \sum_{k=1}^{k=n} f(c_k) \Delta x_k,$$

where  $\Delta x_k = x_k - x_{k-1}$ , and  $c_k \in [x_{k-1}, x_k]$ ,  $\int_a^b f(t) dt = \lim_{n \rightarrow \infty} S_n$  exists ( $\neq \pm\infty$ ).

#### Definition 3.10 (Defined integral)

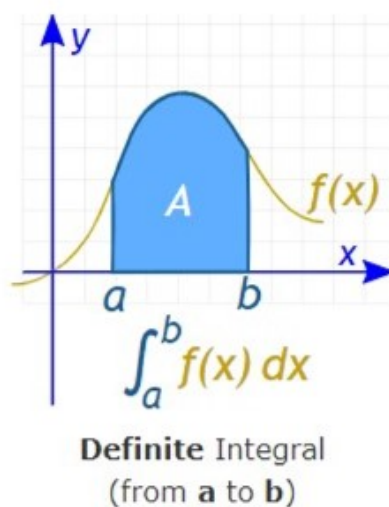
Let the function  $f(x)$  defined on interval  $[a, b]$ , let  $F$  be an antiderivative function of  $[a, b]$ . We call the defined integration for  $f$  with  $a$  and  $b$  is the value  $F(b) - F(a)$ , and note by

$$\int_a^b f(x) dx = [F(x)]_a^b = F(b) - F(a),$$

we call the numbers  $a$  and  $b$  the terms of the integration. The defined integral

$$\int_a^b f(x) dx,$$

geometrically represents the area between the curve of function  $f$ , the axis of intervals  $y = 0$ , and the lines with equations  $x = a$  and  $x = b$ .

**Example 3.20**

Calculated the following integral

$$\int_{\frac{1}{\sqrt{3}}}^1 \frac{1}{1+x^2} dx = \arctan(x) \Big|_{\frac{1}{\sqrt{3}}}^1 = \frac{\pi}{4} - \frac{\pi}{6} = \frac{\pi}{12}.$$

**3.6.3 Properties of the defined integral****Proposition 3.8**

Let  $f$  be a continuous function on the domain of  $[a, b]$ , Then

1.  $\int_a^a f(x) dx = 0.$
2.  $\int_a^b f(x) dx = - \int_b^a f(x) dx.$
3.  $\int_a^c f(x) dx + \int_c^b f(x) dx = \int_a^b f(x) dx, \text{ if } a \leq c \leq b.$

### 3.6.4 Properties of the integral of a positive function

#### Proposition 3.9

1. Let  $f$  be a continuous function on the domain of  $[a, b]$ , where  $a < b$ , such that  $\forall x \in [a, b], f(x) \geq 0$ , then  $\forall c \in [a, b]$  :

$$\int_a^c f(t) dt \geq 0.$$

2. Let  $f$  and  $g$  are a continuous functions on the domain of  $[a, b], \forall x \in [a, b], f(x) \leq g(x)$ , then

$$\int_a^b f(t) dt \leq \int_a^b g(t) dt.$$

3. Linearity of the integral: Let  $f$  and  $g$  be two integrable functions over  $[a, b]$  and  $\lambda, \mu \in \mathbb{R}$ , then

$$\int_a^b [\lambda f(t) + \mu g(t)] dt = \lambda \int_a^b f(t) dt + \mu \int_a^b g(t) dt.$$

4. Let  $f$  and  $g$  be two integrable functions over  $[a, b]$ , but in general

$$\int_a^b f(t) g(t) dt \neq \left( \int_a^b f(t) dt \right) \left( \int_a^b g(t) dt \right).$$

5. Let  $f$  be a integrable function over  $[a, b]$ , then

$$\left| \int_a^b f(t) dt \right| \leq \int_a^b |f(t)| dt.$$

### 3.6.5 Remarks on the computation of definite integrals

The techniques for computing definite integrals are the same as those used for indefinite integrals.

For example: **Change of variables**

$$\int_a^b f(x) dx = ?$$

We put

$$x = \Phi(t) \Rightarrow t = \Phi^{-1}(x) \Rightarrow dx = \Phi'(t) dt.$$

Then

$$\int_a^b f(x) dx = \int_{\Phi^{-1}(a)}^{\Phi^{-1}(b)} f(\Phi(t)) \Phi'(t) dt.$$

### Example 3.21

Calculate the following integral

$$\int_0^4 \frac{\sqrt{x}}{1+x} dx.$$

We put

$$t = \sqrt{x} \Rightarrow x = t^2 \Rightarrow dx = 2t dt.$$

Then

$$\begin{aligned} \int_0^4 \frac{\sqrt{x}}{1+x} dx &= \int_0^2 \frac{t}{1+t^2} 2t dt = 2 \int_0^2 \frac{t^2}{1+t^2} dt = 2 \int_0^2 \frac{t^2+1-1}{1+t^2} dt = 2 \int_0^2 \frac{1+t^2}{1+t^2} - \frac{1}{1+t^2} dt \\ &= 2 \int_0^2 1 dt - 2 \int_0^2 \frac{1}{1+t^2} dt = 2t \Big|_0^2 - 2 \arctan t \Big|_0^2 = 4 - 2 \arctan 2. \end{aligned}$$

### Example 3.22

We put  $\forall n \geq 0, I_n = \int_0^{\frac{\pi}{2}} \sin^n x dx$

1. Calculate  $I_1$  and  $I_2$ .

2. Proved that For all  $n \geq 2, I_n = \frac{n-1}{n} I_{n-2}$ .

### Solution 3.2

We put  $\forall n \geq 0, I_n = \int_0^{\frac{\pi}{2}} \sin^n x dx$ .

1. Calculate  $I_0$  and  $I_1$ .

$$I_0 = \int_0^{\frac{\pi}{2}} \sin^0 x dx = \int_0^{\frac{\pi}{2}} 1 dx = x \Big|_0^{\frac{\pi}{2}} = \frac{\pi}{2}.$$

And

$$I_1 = \int_0^{\frac{\pi}{2}} \sin^1 x dx = \int_0^{\frac{\pi}{2}} \sin x dx = -\cos x \Big|_0^{\frac{\pi}{2}} = -\cos\left(\frac{\pi}{2}\right) + \cos(0) = -0 + 1 = 1.$$

2. For all  $n \geq 2$ ,  $I_n = \frac{n-1}{n} I_{n-2}$ . By integration by parts. Let

$$\forall n \geq 0, I_n = \int_0^{\frac{\pi}{2}} \sin^n x dx.$$

We put

$$f(x) = \sin^{(n-1)} x \Rightarrow f'(x) = (n-1) \cos x \sin^{(n-2)} x.$$

$$g'(x) = \sin x \Rightarrow g(x) = -\cos x.$$

Then

$$\begin{aligned} I_n &= \int_0^{\frac{\pi}{2}} \sin^n x dx = -\cos x \sin^{(n-1)} x \Big|_0^{\frac{\pi}{2}} + \int_0^{\frac{\pi}{2}} \cos x (n-1) \cos x \sin^{(n-2)} x dx \\ &= 0 + (n-1) \int_0^{\frac{\pi}{2}} \cos^2 x \sin^{(n-2)} x dx. \end{aligned}$$

Let

$$\sin^2 x + \cos^2 x = 1 \Rightarrow \cos^2 x = 1 - \sin^2 x.$$

Then

$$\begin{aligned} I_n &= \int_0^{\frac{\pi}{2}} \sin^n x dx = (n-1) \int_0^{\frac{\pi}{2}} [1 - \sin^2 x] \sin^{(n-2)} x dx \\ &= (n-1) \int_0^{\frac{\pi}{2}} [1 - \sin^2 x] \sin^{(n-2)} x dx \\ &= (n-1) \left[ \int_0^{\frac{\pi}{2}} \sin^{(n-2)} x dx - \int_0^{\frac{\pi}{2}} \sin^n x dx \right] \\ &= (n-1) [I_{n-2} - I_n]. \end{aligned}$$

Then

$$\begin{aligned} I_n &= (n-1) I_{n-2} - (n-1) I_n \Rightarrow I_n + (n-1) I_n = (n-1) I_{n-2} \\ &\Rightarrow n I_n = (n-1) I_{n-2} \Rightarrow I_n = \frac{(n-1)}{n} I_{n-2}. \end{aligned}$$

## *Matrices and Determinates*

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### 4.1 Ordinary differential equations

#### Definition 4.1

[2] An ordinary differential equation (in short ODE) is called an equation establishing a relationship between the independent variable  $x$  and the unknown function:

$$y = f(x),$$

and its derivatives  $y', y'', y''', \dots, y^{(n)}$ , and we write

$$F\left(x, y, y', y'', y''', \dots, y^{(n)}\right) = 0, \text{ or } F\left(x, y, \frac{dy}{dx}, y'', y''', \dots, y^{(n)}\right) = 0.$$

#### Definition 4.2

[3] We call the order of an ordinary differential equation, the highest order of the derivative contained in that equation.

#### Example 4.1

The differential equation  $y' + 2y = 0$ , the first order.

The differential equation  $y'' - 2xy' + y = 0$ , the second order.

#### Definition 4.3

We apply the solution or integral of an ODE to any function  $y : I \subset \mathbb{R} \rightarrow \mathbb{R}$ . It is capable of being differentiated  $n$  times and satisfies

$$F\left(x, y, y', y'', y''', \dots, y^{(n)}\right) = 0.$$

## 4.2 First-order differential equations

### Definition 4.4

[4] A first-order ordinary differential equation is written in the form

$$F(x, y, y') = 0, \text{ or } y' = f(x, y).$$

### 4.2.1 Solving a first-order ODE

#### Differential equations with separated variables

### Definition 4.5

The ordinary differential equations with separated variables are in the following form:

$$M(x) dx + N(y) dy = 0.$$

By integrating this equation, we can find the general solution:

$$\int M(x) dx + \int N(y) dy = c.$$

### Example 4.2

$$x dx + y dy = 0 \Rightarrow \frac{x^2}{2} + \frac{y^2}{2} = c \Rightarrow x^2 + y^2 = 2c = k^2.$$

#### Differential equations are separable

### Definition 4.6

[7] The ordinary differential equations that are separable are defined by the following form:

$$M_1(x) N_1(y) dx + M_2(x) N_2(y) dy = 0.$$

If  $N_1(y) M_2(x) \neq 0$ , we can transform it into an equation with separated variables by dividing both sides of the equation by  $N_1(y) M_2(x)$ , and we find that

$$\frac{M_1(x) N_1(y)}{N_1(y) M_2(x)} dx + \frac{M_2(x) N_2(y)}{N_1(y) M_2(x)} dy = 0,$$

then

$$\frac{M_1(x)}{M_2(x)}dx + \frac{N_2(y)}{N_1(y)}dy = 0.$$

By integrating this equation, we can find the general solution:

$$\int \frac{M_1(x)}{M_2(x)}dx + \int \frac{N_2(y)}{N_1(y)}dy = c.$$

### Example 4.3

Let

$$(1+x)ydx + (1-y)xdy = 0.$$

Let's separate the variables by dividing both sides by  $xy$ . We find that

$$\begin{aligned} \frac{(1+x)y}{xy}dx + \frac{(1-y)x}{xy}dy &= 0 \Rightarrow \frac{(1+x)}{x}dx + \frac{(1-y)}{y}dy = 0 \\ &\Rightarrow \left(\frac{1}{x} + 1\right)dx + \left(\frac{1}{y} - 1\right)dy = 0. \end{aligned}$$

By integrating, we obtain

$$\int \left(\frac{1}{x} + 1\right)dx + \int \left(\frac{1}{y} - 1\right)dy = \int 0.$$

Then

$$\int \left(\frac{1}{x} + 1\right)dx + \int \left(\frac{1}{y} - 1\right)dy = \int 0dx \Rightarrow \ln|x| + x + \ln|y| - y = c.$$

Or

$$\ln|x| + \ln|y| + x - y = c \Rightarrow \ln|xy| + x - y = c.$$

This is the general solution.

### Differential equation homogeneous

#### Definition 4.7

[7] We say that a differential equation

$$y' = f(x, y),$$

is homogeneous if

$$\forall \lambda \in \mathbb{R}, f(\lambda x, \lambda y) = f(x, y).$$

We put

$$\lambda = \frac{1}{x}, \text{ where } x \neq 0.$$

Then

$$f(\lambda x, \lambda y) = f\left(\frac{1}{x}x, \frac{1}{x}y\right) = f\left(1, \frac{y}{x}\right).$$

### Method solution

#### Proposition 4.1 (*Method solution*)

We put

$$u = \frac{y}{x} \Rightarrow y = xu \Rightarrow y' = \frac{dy}{dx} = \frac{du}{dx}x + u.$$

Substituting in a differential equation yields

$$\begin{aligned} y' = f(1, u) = \frac{du}{dx}x + u &\Rightarrow y' = f(1, u) = \frac{du}{dx}x + u \Rightarrow f(1, u) - u = \frac{du}{dx}x \\ &\Rightarrow [f(1, u) - u] dx = x du \Rightarrow [f(1, u) - u] dx - x du = 0. \end{aligned}$$

This differential equation is separable.

$$[f(1, u) - u] dx = x du \Rightarrow \frac{du}{[f(1, u) - u]} = \frac{dx}{x} \Rightarrow \int \frac{du}{[f(1, u) - u]} = \int \frac{dx}{x} + c.$$

We find the function  $u(x)$ , and then substitute this function by  $y = u(x)x$ .

#### Example 4.4

Let

$$y' = e^{\frac{y}{x}} + \frac{y}{x}.$$

We put

$$u = \frac{y}{x} \Rightarrow y = xu \Rightarrow y' = \frac{dy}{dx} = \frac{du}{dx}x + u.$$

Substituting in a differential equation yields

$$u = \frac{y}{x} \Rightarrow y = xu \Rightarrow e^u + u = \frac{du}{dx}x + u \Rightarrow e^u = \frac{du}{dx}x.$$

*It is a separable equation. Therefore, by separating the variables, we find*

$$e^u = \frac{du}{dx}x \Rightarrow \frac{dx}{x} = \frac{du}{e^u} \Rightarrow \frac{dx}{x} = e^{-u}du \Rightarrow \frac{dx}{x} - e^{-u}du = 0.$$

*It is a separable equation. Solving by integration, we find*

$$\int \frac{dx}{x} - \int e^{-u}du = c \Rightarrow \ln|x| + e^{-u} = c.$$

*By substituting  $u = \frac{y}{x}$ , we find the general solution to the proposed equation to be*

$$\ln|x| + e^{-\frac{y}{x}} = c \Rightarrow e^{-\frac{y}{x}} = c - \ln|x| \Rightarrow -\frac{y}{x} = \ln(c - \ln|x|) \Rightarrow y = -x \ln(c - \ln|x|).$$

### Example 4.5

*Let*

$$y' = \frac{\sqrt{x^2 - y^2}}{x} + \frac{y}{x} = \frac{x\sqrt{1 - \left(\frac{y}{x}\right)^2}}{x} + \frac{y}{x} = \sqrt{1 - \left(\frac{y}{x}\right)^2} + \frac{y}{x}.$$

*We put*

$$u = \frac{y}{x} \Rightarrow y = xu \Rightarrow y' = \frac{dy}{dx} = \frac{du}{dx}x + u.$$

*Substituting in a differential equation yields*

$$u = \frac{y}{x} \Rightarrow y = xu \Rightarrow \sqrt{1 - u^2} + u = \frac{du}{dx}x + u \Rightarrow \sqrt{1 - u^2} = \frac{du}{dx}x.$$

*It is a separable equation. Therefore, by separating the variables, we find*

$$\sqrt{1 - (u)^2} = \frac{du}{dx}x \Rightarrow \frac{dx}{x} = \frac{du}{\sqrt{1 - u^2}} \Rightarrow \frac{dx}{x} = \frac{1}{\sqrt{1 - u^2}}du \Rightarrow \frac{dx}{x} - \frac{1}{\sqrt{1 - u^2}}du = 0.$$

*It is a separable equation. Solving by integration, we find*

$$\int \frac{dx}{x} - \int \frac{1}{\sqrt{1 - u^2}}du = c \Rightarrow \ln|x| - \arctan(u) = c.$$

*By substituting  $u = \frac{y}{x}$ , we find the general solution to the proposed equation to be*

$$\arctan\left(\frac{y}{x}\right) = \ln|x| + c.$$

**A first-order linear differential equation****Definition 4.8**

We say the first-order linear differential equation if it is written in the following general form

$$y' + P(x)y = Q(x),$$

where  $P(x)$  and  $Q(x)$  are continuous functions (or constants) over the interval  $I$  given.

If  $\forall x \in I, Q(x) = 0$ . Then we call the equation

$$y' + P(x)y = 0,$$

a homogeneous equation without a second term.

**Method solution**

The theory is based on the following principles:

**Proposition 4.2 (Method solution)**

For every first-order linear differential equation, its general solution can be written in the form

$$y_G = y_H + y_P,$$

where  $y_H$  is the general solution of the equation without the second term, and  $y_P$  is a particular solution of the equation with the second term.

**Proposition 4.3 (How to find  $y_H$ )**

If it suffices to solve the equation without the second term:

$$y' + P(x)y = 0 \Rightarrow \frac{dy}{dx} + P(x)y = 0 \Rightarrow \frac{dy}{y} + P(x)dx = 0.$$

By integration, we find:

$$\begin{aligned} \int \frac{dy}{y} + \int P(x)dx &= 0 \Rightarrow \ln|y| = -\int P(x)dx + c \\ &\Rightarrow y_H = ke^{-\int P(x)dx}, k : \text{real constant.} \end{aligned}$$

**Proposition 4.4 (*How to find  $y_P$* )**

We make the constant  $k$  appearing in the expression  $y_H$  a function of the variable  $x$ .

We put

$$k = k(x).$$

We suppose that

$$y_P = k(x) e^{-\int P(x) dx},$$

is a particular solution to the equation with the second term.

To determine  $k(x)$  differentiate the expression  $y_P$ , we find

$$y'_P = k'(x) e^{-\int P(x) dx} - k(x) P(x) e^{-\int P(x) dx} = [k'(x) - k(x) P(x)] e^{-\int P(x) dx},$$

and substitute it into the equation with the second term, we find:

$$\begin{aligned} y'_P + P(x) y_P &= Q(x) \\ [k'(x) - k(x) P(x)] e^{-\int P(x) dx} + P(x) k(x) e^{-\int P(x) dx} &= Q(x) \\ k'(x) e^{-\int P(x) dx} &= Q(x) \\ k'(x) &= Q(x) e^{\int P(x) dx}. \end{aligned}$$

So

$$k(x) = \int Q(x) e^{\int P(x) dx} dx.$$

Then

$$y_P = \left[ \int Q(x) e^{\int P(x) dx} dx \right] e^{-\int P(x) dx}.$$

Then the general solution can be written in the form

$$y_G = k e^{-\int P(x) dx} + \left[ \int Q(x) e^{\int P(x) dx} dx \right] e^{-\int P(x) dx}.$$

**Example 4.6**

Let

$$y' - \frac{y}{x} = x,$$

its general solution can be written in the form

$$y_G = y_H + y_P,$$

How to find  $y_H$ : If it suffices to solve the equation without the second term:

$$y' - \frac{1}{x}y = 0 \Rightarrow \frac{dy}{dx} - \frac{1}{x}y = 0.$$

By integrating, we find:

$$\begin{aligned} \int \frac{dy}{y} - \int \frac{1}{x} dx = 0 &\Rightarrow \ln |y| - \ln |x| = c \Rightarrow \ln |y| = \ln |x| + c \Rightarrow y_H = k e^{\int \ln |x| dx} \\ &\Rightarrow y_H = kx, k : \text{ real constant.} \end{aligned}$$

How to find  $y_P$ :

We put

$$y = k(x)x.$$

We suppose that

$$y_P = k(x)x,$$

is a particular solution to the equation with the second term.

To determine  $k(x)$  differentiate the expression  $y_P$ , we find

$$y'_P = k'(x)x + k(x).$$

After substituting it into the equation with the second term, we find:

$$k'(x)x + k(x) - \frac{xk(x)}{x} = x \Rightarrow k'(x) = 1 \Rightarrow k(x) = x \Rightarrow y_P = x^2.$$

Then

$$y_G = kx + x^2, k : \text{ real constant.}$$

**Example 4.7**

$$y' - \frac{y}{x} = \ln x. \quad (4.1)$$

The general solution of (H) :

$$y_G = y_P + y_H.$$

Since the particular solution  $y_P$  of (H) is not obvious, and  $y_H$  is the general solution of the equation without the right-hand side.  $(H') : y' - \frac{y}{x} = 0$  is a differential equation with separable variables. By remark,  $y = 0$  is the solution of  $(H')$ . If  $y \neq 0$ ,

$$y' - \frac{y}{x} = 0 \Leftrightarrow y' = \frac{y}{x} \Leftrightarrow \frac{dy}{dx} = \frac{y}{x} \Leftrightarrow \frac{1}{y} dy = \frac{1}{x} dx.$$

Then

$$\begin{aligned} \int \frac{1}{y} dy &= \int \frac{1}{x} dx \Rightarrow \ln |y| = \ln |x| + C, C \in \mathbb{R} \Rightarrow |y| = |x| e^C, C \in \mathbb{R} \\ \Rightarrow y &= \pm e^C x, C \in \mathbb{R} \Rightarrow y = kx, k \in \mathbb{R}^*. \end{aligned}$$

Because  $y = 0$  is solution of  $(H')$ , then

$$y_H = kx, k \in \mathbb{R}.$$

To find the particular solution  $y_P$ , we apply the method of variation of parameters.

This method involves replacing the constant  $K$  with a function  $K(x)$ . We put

$$y_P = K(x) x.$$

Then

$$y' = K'(x) x + K(x).$$

We replace  $y$  and  $y'$  in equation (H) to obtain  $K'(x)$  :

$$\begin{aligned} K'(x) x + K(x) - \frac{K(x) x}{x} &= \ln x \Rightarrow K'(x) x + K(x) - K(x) = \ln x \Rightarrow K'(x) x = \ln x \\ \Rightarrow K'(x) &= \frac{\ln x}{x} \Rightarrow K(x) = \int \frac{\ln x}{x} dx. \end{aligned}$$

To calculate it, we put

$$U = \ln x \Rightarrow dU = \frac{1}{x} dx.$$

Then

$$K(x) = \int \frac{\ln x}{x} dx = \int U dU = \frac{U^2}{2} + C = \frac{(\ln x)^2}{2} + C, C \in \mathbb{R}.$$

Then

$$y_P = K(x)x = \frac{(\ln x)^2}{2}x + Cx, C \in \mathbb{R}.$$

Therefore, the general solution of (H) is given by:

$$y_G = y_H + y_P = kx + \frac{(\ln x)^2}{2}x + Cx, C \in \mathbb{R}.$$

### Bernoulli equation

#### Definition 4.9 (*Bernoulli equation*)

We say the Bernoulli equation if it is written in the following general form:

$$y' + P(x)y = Q(x)y^n,$$

where  $P(x)$  and  $Q(x)$  are continuous functions (or constants) over the interval  $I$  given, and  $n \succ 1$ .

#### Proposition 4.5 (*Method solution*)

We convert it into a linear differential equation by dividing both sides by  $y^n$ ; it becomes in the form

$$y^{-n} \frac{dy}{dx} + P(x)y^{-n+1} = Q(x),$$

We put

$$z = y^{-n+1},$$

then

$$\frac{dz}{dx} = (-n+1)y^{-n} \frac{dy}{dx}.$$

By substituting and then multiplying both sides by  $(-n+1)$ , we find

$$y^{-n}(-n+1) \frac{dy}{dx} + (-n+1)P(x)y^{-n+1} = (-n+1)Q(x),$$

then

$$\frac{dz}{dx} + (-n+1)P(x)z = (-n+1)Q(x).$$

And it's a linear differential equation. We seek its general solution as mentioned in the previous paragraph, then substitute by

$$z = y^{-n+1}.$$

**Example 4.8**

Let

$$y' + xy = x^3y^3.$$

To solve it, we divide both sides by  $y^3$ , and we find

$$y^{-3}y' + xy^{-2} = x^3.$$

We put

$$z = y^{-2}.$$

Then

$$\frac{dz}{dx} = -2y^{-3}\frac{dy}{dx}.$$

By substituting and then multiplying both sides by  $-2$ , we find

$$-2y^{-3}y' - 2xz = -2x^3,$$

then

$$\frac{dz}{dx} - 2xz = -2x^3.$$

It is a quadratic equation whose general solution is in the form

$$z_G = z_H + z_P.$$

How to find  $z_H$ :

If it suffices to solve the equation without the second term:

$$\frac{dz}{dx} - 2xz = 0 \Rightarrow \frac{dz}{z} - 2xdx = 0.$$

By integrating, we find:

$$\begin{aligned} \int \frac{dz}{z} - \int 2xdx = 0 &\Rightarrow \ln|z| - x^2 = c \Rightarrow \ln|z| = x^2 + c \Rightarrow z_H = ke^{x^2} \\ &\Rightarrow z_H = ke^{x^2}, k : \text{ real constant.} \end{aligned}$$

How to find  $z_P$ :

We put

$$k = k(x).$$

We suppose that

$$z_P = k(x) e^{x^2},$$

is a particular solution to the equation with the second term.

To determine  $k(x)$  differentiate the expression  $y_P$ , we find

$$z'_P = k'(x) e^{x^2} + k(x) 2x e^{x^2}.$$

After substituting it into the equation with the second term, we find:

$$\frac{dz}{dx} - 2xz = -2x^3$$

$$k'(x) e^{x^2} + k(x) 2x e^{x^2} - 2x k(x) e^{x^2} = -2x^3. \Rightarrow k'(x) = -2x^3 e^{-x^2} \Rightarrow k(x) = \int -2x^3 e^{-x^2} dx.$$

To calculate it, we use integration by parts, where we put

$$f(x) = x^2 \Rightarrow f'(x) = 2x,$$

$$g'(x) = -2x e^{-x^2} \Rightarrow g(x) = e^{-x^2}.$$

Then

$$k(x) = \int -2x^3 e^{-x^2} dx = x^2 e^{-x^2} - \int 2x e^{-x^2} dx = x^2 e^{-x^2} + e^{-x^2}.$$

Then

$$z_P = x^2 + 1.$$

So

$$z_G = x^2 + 1 + k e^{x^2}, k : \text{ real constant.}$$

By substituting with  $z = y^{-2}$ . We find the general solution for the proposed equation in the form

$$y = \frac{1}{\sqrt{x^2 + 1 + k e^{x^2}}}.$$

## 4.3 Second-order differential equations with constant coefficients

It's an equation of the form

$$F(x, y, y', y'') = 0, \text{ or } y'' = f(x, y, y').$$

### 4.3.1 Second-order linear equation

#### Definition 4.10

[7] We call a second-order linear equation an equation of the form

$$ay'' + by' + cy = Q(x),$$

where  $a, b, c$  and  $f(x)$  are given functions of  $x$  or constants, and  $a_0 \neq 0$ . We will assume hereafter that  $a, b, c$  and  $f(x)$  are continuous,  $\forall x$ , and  $a = 1$ .

#### Homogeneous equation

#### Definition 4.11 (*Homogeneous equation*)

If  $Q(x) = 0$ , we call the equation

$$ay'' + by' + cy = 0,$$

is a homogeneous equation.

### 4.3.2 Method solution

The theory is based on the following principles:

#### Proposition 4.6

The general solution  $y_G$  to a second-order linear differential equation, can be written in the form:

$$y_G = y_H + y_P,$$

where  $y_H$  is the general solution of the equation without the second term ( $ay'' + by' + cy = 0$ ), and  $y_P$  is a particular solution of the equation with the second term.

**Proposition 4.7 (*How to find  $y_H$* )**

The general solution of

$$ay'' + by' + cy = 0.$$

The theory is based on the following principles: If  $y_1$  and  $y_2$  are particular solutions to the homogeneous linear differential equation

$$ay'' + by' + cy = 0.$$

and they are linearly independent constants:

$$\frac{y_1}{y_2} \neq cste.$$

Then, its general solution takes the form:

$$y_H = C_1 y_1 + C_2 y_2,$$

where  $C_1$  and  $C_2$  are real constants.

**Proposition 4.8 (*Research on two specific solutions  $y_1$  and  $y_2$* )**

From the shape

$$y = e^{kx},$$

where  $k$  is a real constant. To determine its value, it suffices to differentiate  $y$  twice and then substitute into the homogeneous equation, i.e.

$$y' = ke^{kx}, y'' = k^2 e^{kx}.$$

By substituting into the equation without the second term or the homogeneous term, we find

$$(ak^2 + bk + c) e^{kx} = 0.$$

where  $e^{kx} \neq 0$ , then  $ak^2 + bk + c = 0$ .

We name the equation

$$ak^2 + bk + c = 0.$$

An associated characteristic equation for the homogeneous differential equation. Its solutions depend on the discriminant  $\Delta = b^2 - 4ac$ . We identify three cases:

**If  $\Delta > 0$  :** then the characteristic equation admits two distinct real solutions:

$$k_1 = \frac{-b + \sqrt{\Delta}}{2a}, k_2 = \frac{-b - \sqrt{\Delta}}{2a}.$$

Then

$$y_1 = e^{k_1 x}, y_2 = e^{k_2 x}.$$

There are two distinct particular solutions for the homogeneous equation, that are linearly independent. Indeed

$$\frac{y_1}{y_2} = \frac{e^{k_1 x}}{e^{k_2 x}} = e^{(k_1 - k_2)x} \neq C \text{ ste, because } k_1 \neq k_2.$$

Therefore, the general solution to the homogeneous equation is

$$y_H = C_1 e^{k_1 x} + C_2 e^{k_2 x},$$

where  $C_1$  and  $C_2$  are real constants.

**If  $\Delta = 0$  :** then the characteristic equation admits a repeated real solution:

$$k = \frac{-b}{2a}.$$

Then

$$y_1 = e^{kx}, y_2 = xe^{kx}.$$

There are two distinct particular solutions for the homogeneous equation, that are linearly independent. Indeed

$$\frac{y_2}{y_1} = \frac{xe^{kx}}{e^{kx}} = x \neq C \text{ ste.}$$

Therefore, the general solution to the homogeneous equation is

$$y_H = C_1 e^{kx} + C_2 x e^{kx},$$

where  $C_1$  and  $C_2$  are real constants.

**If  $\Delta < 0$  :** Then the characteristic equation admits two complex conjugate solutions:

$$k_1 = \alpha + i\beta, k_2 = \alpha - i\beta,$$

where

$$\alpha = \frac{-b}{2a}, \beta = \frac{\sqrt{-\Delta}}{2a}.$$

Then

$$y_1 = e^{\alpha x} \cos(\beta x), y_2 = e^{\alpha x} \sin(\beta x).$$

There are two distinct particular solutions for the homogeneous equation, that are linearly independent. Indeed

$$\frac{y_2}{y_1} = \frac{e^{\alpha x} \sin(\beta x)}{e^{\alpha x} \cos(\beta x)} = \tan(\beta x) \neq \text{Cste},$$

because  $k_1 \neq k_2$ . Therefore, the general solution to the homogeneous equation is

$$y_H = e^{\alpha x} [C_1 \cos(\beta x) + C_2 \sin(\beta x)],$$

where  $C_1$  and  $C_2$  are real constants.

#### **Proposition 4.9 (How to find $y_P$ )**

A particular solution for

$$ay'' + by' + cy = Q(x).$$

There are two methods: a general method and a specific method.

#### **General method**

##### **Definition 4.12 (General method)**

We assign the constants appearing in the expression  $y_H$  continuing with respect to  $x$ , any:

We put

$$C_1 = C_1(x), C_2 = C_2(x).$$

And we assume that

$$y_P = C_1(x)y_1 + C_2(x)y_2.$$

It is a particular solution for

$$ay'' + by' + cy = Q(x).$$

To determine the constants  $C_1(x)$  and  $C_2(x)$ , we differentiate  $y_P$  twice with respect to  $x$ , then substitute into the equation with the second term, and finally we obtain:

$$\begin{cases} C_1'(x)y_1 + C_2'(x)y_2 = 0, \\ C_1'(x)y_1' + C_2'(x)y_2' = \frac{Q(x)}{a}. \end{cases}$$

And it is a linear equation in two variables that we solve using the substitution method.

#### Example 4.9

Let

$$y'' + 4y' + 3y = x.$$

The general solution can be written in the form

$$y_G = y_H + y_P.$$

Determining  $y_H$  : The general solution of

$$y'' + 4y' + 3y = 0.$$

The associated characteristic equation for it is

$$k^2 + 4k + 3 = 0.$$

Its discriminant is  $\Delta = 4 > 0$ , so it admits two roots

$$k_1 = -1, k_2 = -3.$$

Then

$$y_H = C_1e^{k_1x} + C_2e^{k_2x} \Rightarrow y_H = C_1e^{-x} + C_2e^{-3x},$$

where  $C_1$  and  $C_2$  are real constants.

Determining  $y_P$  : We put

$$C_1 = C_1(x), C_2 = C_2(x).$$

And we assume that

$$y_P = C_1(x) e^{-x} + C_2(x) e^{-3x}.$$

It is a particular solution for

$$y'' + 4y' + 3y = x.$$

To determine  $C_1(x)$  and  $C_2(x)$ , it suffices to solve the equation:

$$\begin{cases} C_1'(x) e^{-x} + C_2'(x) e^{-3x} = 0, \\ -C_1'(x) e^{-x} - 3C_2'(x) e^{-3x} = x. \end{cases}$$

By adding the two equations side by side, we find

$$-2C_2'(x) e^{-3x} = x \Rightarrow C_2'(x) = -\frac{x}{2} e^{3x},$$

$$C_1'(x) e^{-x} + C_2'(x) e^{-3x} = 0 \Rightarrow C_1'(x) = -C_2'(x) e^{-2x} \Rightarrow C_1'(x) = \frac{x}{2} e^{3x} e^{-2x} = \frac{x}{2} e^x.$$

We use integration by parts to calculate  $C_2(x)$ , where we put

$$f(x) = -\frac{x}{2} \Rightarrow f'(x) = -\frac{1}{2}$$

$$g'(x) = e^{3x} \Rightarrow g(x) = \frac{1}{3} e^{3x}.$$

$$-\int \frac{x}{2} e^{3x} dx = -\frac{x}{6} e^{3x} + \frac{1}{6} \int \frac{3}{3} e^{3x} dx = -\frac{x}{6} e^{3x} + \frac{1}{18} e^{3x} \Rightarrow C_2(x) = -\frac{x}{6} e^{3x} + \frac{1}{18} e^{3x}$$

We use integration by parts to calculate  $C_1(x)$ , where we put

$$f(x) = \frac{x}{2} \Rightarrow f'(x) = \frac{1}{2}$$

$$g'(x) = e^x \Rightarrow g(x) = e^x.$$

$$\int \frac{x}{2} e^x dx = \frac{x}{2} e^x - \frac{1}{2} \int e^x dx = \frac{x}{2} e^x - \frac{1}{2} e^x \Rightarrow C_1(x) = \frac{x}{2} e^x - \frac{1}{2} e^x.$$

And by substituting, we have

$$\begin{aligned} y_P &= C_1(x) e^{-x} + C_2(x) e^{-3x} \\ &= \left[ \frac{x}{2} e^x - \frac{1}{2} e^x \right] e^{-x} + \left[ -\frac{x}{6} e^{3x} + \frac{1}{18} e^{3x} \right] e^{-3x} \\ &= \frac{x}{2} - \frac{1}{2} - \frac{x}{6} + \frac{1}{18} \\ &= \frac{x}{3} - \frac{4}{9}. \end{aligned}$$

*So the general solution to the proposed equation is*

$$y_G = y_H + y_P = C_1(x) e^{-x} + C_2(x) e^{-3x} + \frac{x}{3} - \frac{4}{9},$$

*where  $C_1$  and  $C_2$  are real constants.*

### Specific method:

She relies of the second party  $Q(x)$ .

**If  $Q(x) = P_n(x) e^{\lambda x}$ , where  $P_n$  is a polynomial of degree  $n$ ,  $\lambda$  is a real constant**

**We distinguish three cases:**

#### Proposition 4.10

**First case:**  $\lambda$  is not a root of the characteristic equation, we look for a particular solution of the form

$$y_P = (A_0 + A_1x + \dots + A_nx^n) e^{\lambda x}.$$

**Second case:**  $\lambda$  is the simple root of the characteristic equation, we look for a particular solution of the form:

$$y_P = x(A_0 + A_1x + \dots + A_nx^n) e^{\lambda x}.$$

**Third case:**  $\lambda$  is the double root of the characteristic equation; we look for a particular solution of the form:

$$y_P = x^2(A_0 + A_1x + \dots + A_nx^n) e^{\lambda x}.$$

To determine the constants  $A_0, A_1, \dots, A_n$ , we differentiate the expression  $y_P$  twice and substitute it into the equation on the right-hand side, then equate.

#### Example 4.10

Let

$$y'' + 4y' + 3y = x.$$

The general solution can be written in the form

$$y_G = y_H + y_P.$$

$y_H$  has been computed in the previous example, then

$$y_H = C_1 e^{-x} + C_2 e^{-3x},$$

where  $C_1$  and  $C_2$  are real constants.

Determine  $y_P$  by using the special method.

We have the second term  $Q(X) = x$ , and it can be expressed in the form

$$Q(X) = P_n(x) e^{\lambda x},$$

with  $n = 1$  and  $\lambda = 0$ .

We note that  $\lambda = 0$  is not a root of the characteristic equation. Hence, we look for a particular solution of the form:

$$y_P = A_0 + A_1 x.$$

To determine  $A_0$  and  $A_1$ , we differentiate  $y_P$  twice to find

$$y_P' = A_1, y_P'' = 0.$$

Then we substitute the equation on the right-hand side

$$y'' + 4y' + 3y = x.$$

So we obtain

$$0 + 4A_1 + 3A_0 + 3A_1 x = x.$$

And by matching, we find

$$A_1 = \frac{1}{3}, A_0 = -\frac{4}{9}.$$

Then

$$y_P = -\frac{4}{9} + \frac{1}{3}x.$$

Therefore, the general solution to the proposed equation is

$$y_G = y_H + y_P = C_1 e^{-x} + C_2 e^{-3x} - \frac{4}{9} + \frac{1}{3}x,$$

where  $C_1$  and  $C_2$  are real constants.

If  $Q(x) = P(x)e^{\mu x} \cos wx + R(x)e^{\mu x} \sin wx$  where  $P$  and  $R$  are polynomials and  $\mu$  and  $w$  are real constants. We distinguish two cases:

**Proposition 4.11**

**First case:**  $(\mu + iw)$  Not a root of the characteristic equation; we look for a particular solution of the form:

$$y_P = R_1(x)e^{\mu x} \cos wx + R_2(x)e^{\mu x} \sin wx.$$

**Second case:**  $(\mu + iw)$  is a root of the characteristic equation; we look for a particular solution of the form:

$$y_P = x(R_1(x)e^{\mu x} \cos wx + R_2(x)e^{\mu x} \sin wx),$$

where  $R_1$  and  $R_2$  are polynomials of degree  $\max(\deg P, \deg P)$ . To determine them, we differentiate the expression  $y_P$  twice, then substitute it into the equation on the right-hand side, and finally match.

**Example 4.11**

Let

$$y'' - 2y' + y = 3e^{2x} \cos x.$$

The general solution to the from

$$y_G = y_H + y_P.$$

Determinant  $y_H$  : The general solution for

$$y'' - 2y' + y = 0.$$

The corresponding characteristic equation is:

$$k^2 - 2k + 1 = 0.$$

Its discriminant is  $\Delta = 0$ , so it accepts a double root

$$k = 1.$$

Then

$$y_H = C_1 e^x + C_2 x e^x,$$

where  $C_1$  and  $C_2$  are real constants.

Determining  $y_P$  : We have the second term of the equation.

$$Q(x) = 3e^{2x} \cos x.$$

It can be expressed in the general form

$$Q(x) = P(x) e^{\mu x} \cos wx + R(x) e^{\mu x} \sin wx.$$

with

$$w = 1, \mu = 2, P(x) = 3, R(x) = 0.$$

We remark that

$$(\mu + iw = 2 + i).$$

It is not a root of the characteristic equation, and since

$$\max(\deg P, \deg R) = 0.$$

We search for a particular solution of the form

$$y_P = R_1 e^{2x} \cos x + R_2 e^{2x} \sin x,$$

where  $R_1$  and  $R_2$  are constants, to determine them, we differentiate an expression

$$\begin{aligned} y_P' &= (R_1 e^{2x} \cos x)' + (R_2 e^{2x} \sin x)' \\ &= 2R_1 e^{2x} \cos x - R_1 e^{2x} \sin x + 2R_2 e^{2x} \sin x + R_2 e^{2x} \cos x \\ &= (2R_1 + R_2) e^{2x} \cos x + (2R_2 - R_1) e^{2x} \sin x. \end{aligned}$$

$$\begin{aligned} y_P'' &= (2R_1 + R_2) (e^{2x} \cos x)' + (2R_2 - R_1) (e^{2x} \sin x)' \\ &= (2R_1 + R_2) (2e^{2x} \cos x - e^{2x} \sin x) \\ &\quad + (2R_2 - R_1) (2e^{2x} \sin x + e^{2x} \cos x) \\ &= 2(2R_1 + R_2) e^{2x} \cos x - (2R_1 + R_2) e^{2x} \sin x \\ &\quad + 2(2R_2 - R_1) e^{2x} \sin x + (2R_2 - R_1) e^{2x} \cos x \\ &= (3R_1 + 4R_2) e^{2x} \cos x + (3R_2 - 4R_1) e^{2x} \sin x. \end{aligned}$$

And if we substitute it into the equation for the second term, we find

$$\begin{aligned} &(3R_1 + 4R_2) e^{2x} \cos x + (3R_2 - 4R_1) e^{2x} \sin x \\ &- 2 \left( (2R_1 + R_2) e^{2x} \cos x + (2R_2 - R_1) e^{2x} \sin x \right) + R_1 e^{2x} \cos x + R_2 e^{2x} \sin x \\ &= 3e^{2x} \cos x \Rightarrow 2R_2 e^{2x} \cos x - 2R_1 e^{2x} \sin x = 3e^{2x} \cos x. \end{aligned}$$

Then, by comparison, we find

$$\begin{cases} 2R_2 = 3, \\ -2R_1 = 0 \end{cases} \Rightarrow \begin{cases} R_2 = \frac{3}{2}, \\ R_1 = 0. \end{cases}$$

Then

$$y_P = \frac{3}{2}e^{2x} \sin x.$$

And thus, the general solution to the proposed equation is

$$y_G = \frac{3}{2}e^{2x} \sin x + C_1 e^x + C_2 x e^x,$$

where  $C_1$  and  $C_2$  are real constants.

If  $Q(x) = f_1(x) + f_2(x) + \dots + f_n(x)$ . If  $y_1$  is the solution of this equation:  $ay'' + by' + cy = f_1(x)$ .

If  $y_2$  is the solution of this equation:  $ay'' + by' + cy = f_2(x)$ .

.....

If  $y_n$  is the solution of this equation:  $ay'' + by' + cy = f_n(x)$ .

Then the equation

$$ay'' + by' + cy = Q(x),$$

accept a particular solution in the form

$$y_P = y_1 + y_2 + \dots + y_n.$$

#### Example 4.12

Let

$$y'' + y = e^{2x} + 5xe^{-x} - 3 \sin x.$$

The general solution is the from

$$y_G = y_H + y_P.$$

Determinant  $y_H$  : The general solution for

$$y'' + y = 0.$$

The corresponding characteristic equation is:

$$k^2 + 1 = 0.$$

Its discriminant is  $\Delta = -4$ , so it admits two complex conjugate solutions, where  $\alpha = 0$ .

Then

$$\beta = \frac{\sqrt{-\Delta}}{2} = 1.$$

Then

$$y_H = C_1 \cos x + C_2 \sin x.$$

where  $C_1$  and  $C_2$  are real constants.

Determining  $y_P$  :

$$y_P = y_1 + y_2 + y_3.$$

where  $y_1, y_2$  and  $y_3$  are the particular solutions of the following equations:

$$y_1'' + y_1 = e^{2x}, y_2'' + y_2 = 5xe^{-x}, y_3'' + y_3 = -3 \sin x,$$

with respectively

Determining  $y_1$  :

We have the second term of the equation.

$$Q(x) = P_n(x) e^{\lambda x} = e^{2x}, \text{ with } n = 0 \text{ and } \lambda = 2.$$

We remark  $\lambda = 2$  is not a solution to the equation, then we search for a particular solution of the form

$$y_1 = A_0 e^{2x}.$$

To determine  $A_0$ , we differentiate  $y_1$  twice, and we find

$$y_1' = 2A_0 e^{2x}, y_1'' = 4A_0 e^{2x}.$$

And substitute it into the equation for the second term, and by comparison, we find

$$4A_0 e^{2x} + A_0 e^{2x} = e^{2x} \Rightarrow 5A_0 e^{2x} = e^{2x} \Rightarrow 5A_0 = 1 \Rightarrow A_0 = \frac{1}{5}.$$

Then

$$y_1 = \frac{1}{5} e^{2x}.$$

Determining  $y_2$  :

We have the second term of the equation.

$$Q(x) = P_n(x) e^{\lambda x} = 5xe^{-x}, \text{ with } n = 1 \text{ and } \lambda = -1.$$

We remark that  $\lambda = -1$  is not a solution to the equation, then we search for a particular solution of the form

$$y_2 = (A_0 + xA_1) e^{-x}.$$

To determine  $A_0$  and  $A_1$ , we differentiate  $y_2$  twice, and we find

$$y_2' = -(A_0 + xA_1) e^{-x} + A_1 e^{-x} = (A_1 - A_0 - xA_1) e^{-x}.$$

$$y_2'' = -(A_1 - A_0 - xA_1) e^{-x} - A_1 e^{-x} = (A_0 - 2A_1 + xA_1) e^{-x}.$$

And if we substitute it into the equation for the second term, we find

$$(A_0 - 2A_1 + xA_1) e^{-x} + (A_0 + xA_1) e^{-x} = 5xe^{-x}$$

$$(A_0 - 2A_1 + xA_1 + A_0 + xA_1) e^{-x} = 5xe^{-x}$$

$$(2A_0 - 2A_1 + 2xA_1) e^{-x} = 5xe^{-x}$$

$$2A_0 - 2A_1 + 2xA_1 = 5x.$$

By comparison, we find

$$\begin{cases} 2A_0 - 2A_1 = 0, \\ 2A_1 = 5, \end{cases} \Rightarrow \begin{cases} A_0 = A_1 = \frac{5}{2}, \\ A_1 = \frac{5}{2}. \end{cases}$$

Then

$$y_2 = \left(\frac{5}{2} + x\frac{5}{2}\right) e^{-x} = \frac{5}{2}(1+x) e^{-x}.$$

Determining  $y_3$  :

We have the second term of the equation.

$$Q(x) = -3 \sin x.$$

It can be expressed in the general form

$$Q(x) = P(x) e^{\mu x} \cos wx + R(x) e^{\mu x} \sin wx.$$

with

$$w = 1, \mu = 0, P(x) = 0, R(x) = -3.$$

We remark that

$$(\mu + iw = i).$$

It is not a root of the characteristic equation, and since

$$\max(\deg P, \deg R) = 0.$$

We search for a particular solution of the form

$$y_3 = x(R_1 \cos x + R_2 \sin x),$$

where  $R_1$  and  $R_2$  are constants, to determine them, we differentiate  $y_3$  twice

$$\begin{aligned} y_3' &= (R_1 \cos x + R_2 \sin x) + x(-R_1 \sin x + R_2 \cos x) \\ &= R_1 \cos x + xR_2 \cos x + R_2 \sin x - xR_1 \sin x \\ &= (R_1 + xR_2) \cos x + (R_2 - xR_1) \sin x, \end{aligned}$$

$$\begin{aligned} y_3'' &= ((R_1 + xR_2) \cos x + (R_2 - xR_1) \sin x)' \\ &= R_2 \cos x - (R_1 + xR_2) \sin x - R_1 \sin x + (R_2 - xR_1) \cos x \\ &= (2R_2 - xR_1) \cos x - (2R_1 + xR_2) \sin x. \end{aligned}$$

And if we substitute it into the equation for the second term, we find

$$\begin{aligned} (2R_2 - xR_1) \cos x - (2R_1 + xR_2) \sin x + x(R_1 \cos x + R_2 \sin x) &= -3 \sin x \\ (2R_2 - xR_1 + xR_1) \cos x - (2R_1 + xR_2 - xR_2) \sin x &= -3 \sin x \\ 2R_2 \cos x - 2R_1 \sin x &= -3 \sin x. \end{aligned}$$

Then, by comparison, we find

$$\begin{cases} 2R_2 = 0, \\ -2R_1 = -3 \end{cases} \Rightarrow \begin{cases} R_2 = 0, \\ R_1 = \frac{3}{2}. \end{cases}$$

Then

$$y_3 = \frac{3}{2}x \cos x.$$

Then the equation admits a particular solution of the form

$$y_P = \frac{1}{5}e^{2x} + \frac{5}{2}(1+x)e^{-x} + \frac{3}{2}x \cos x.$$

And thus, the general solution to the proposed equation is

$$y_G = C_1 \cos x + C_2 \sin x + \frac{1}{5}e^{2x} + \frac{5}{2}(1+x)e^{-x} + \frac{3}{2}x \cos x.$$

where  $C_1$  and  $C_2$  are real constants.

## *Functions of Several Variables*

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In the applied field, single-variable functions are rare, while multivariable functions are common.

**For example:**

The area of a rectangle with length  $x$  and width  $y$  is a function of two variables,  $x$  and  $y$ .

The volume of a rectangular prism with dimensions  $x$ ,  $y$ , and  $z$  is a function of three variables.

Temperature and density at every point in a three-dimensional room are functions of three variables.

### 5.1 Multivariable Function

#### Definition 5.1

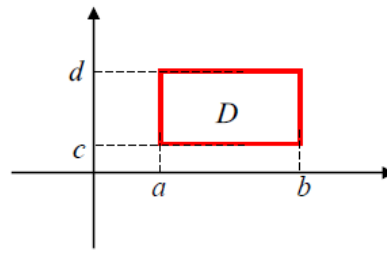
[3] [Multivariable Function] A real-valued **multivariable function** or a function of  $n$  variables, is any real function  $f$  defined from  $\mathbb{R}^n$  to  $\mathbb{R}$ , which assigns to each element in  $\mathbb{R}^n$  at most one real value.

$$\begin{aligned} f : \mathbb{R}^n &\rightarrow \mathbb{R}, \\ (x_1, x_2, \dots, x_n) &\rightarrow f(x_1, x_2, \dots, x_n). \end{aligned}$$

#### Example 5.1

$$\begin{aligned} f : \mathbb{R}^2 &\rightarrow \mathbb{R}, \\ (x, y) &\rightarrow 2(x + y), \end{aligned}$$

It is a real-valued function of two real variables, representing the perimeter of a rectangle with length  $x$  and width  $y$ .



### 5.1.1 Domain Definition of the Function

#### Definition 5.2

[1] We call  $D_f$  the domain definition of the function  $f$ , it is the set of points  $M = (x_1, x_2, \dots, x_n)$  in  $\mathbb{R}^n$  for which the multivariable function  $f$  has a real value. In other words:

$$D_f = \{M = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n / f(M) \in \mathbb{R}\} \subset \mathbb{R}^n.$$

#### Example 5.2

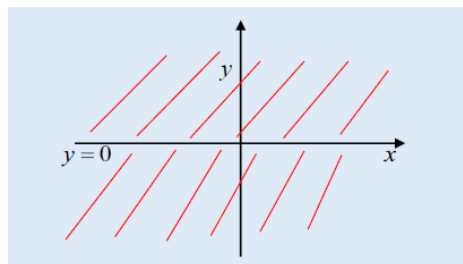
Let  $f$  be a real-valued function of two variables, where:

$$f(x, y) = \frac{x}{y},$$

Then

$$D_f = \{(x, y) \in \mathbb{R}^2 / y \neq 0\} = \mathbb{R} \times \mathbb{R}^*.$$

It can be represented graphically as follows:



It represents all the points of the plane except the  $x$  (i.e., where  $y = 0$ ) the shaded-out part in red.

**Example 5.3**

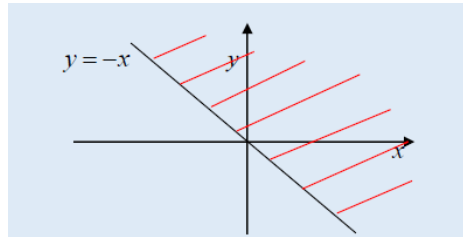
Let  $f$  be a real-valued function of two variables, where:

$$f(x, y) = \sqrt{x + y},$$

then

$$D_f = \{(x, y) \in \mathbb{R}^2 / x + y \geq 0\} = \{(x, y) \in \mathbb{R}^2 / y \geq -x\}.$$

It can be represented graphically as follows:



It represents all the points of the plane lying above the line with the equation  $y = -x$ , the shaded-out part in red.

**Example 5.4**

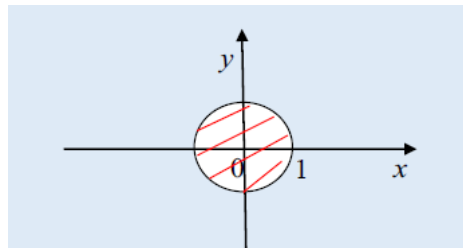
Let  $f$  be a real-valued function of two variables, where:

$$f(x, y) = \sqrt{1 - x^2 - y^2},$$

then

$$D_f = \{(x, y) \in \mathbb{R}^2 / 1 - x^2 - y^2 \geq 0\} = \{(x, y) \in \mathbb{R}^2 / x^2 + y^2 \leq 1\}.$$

It can be represented graphically as follows:



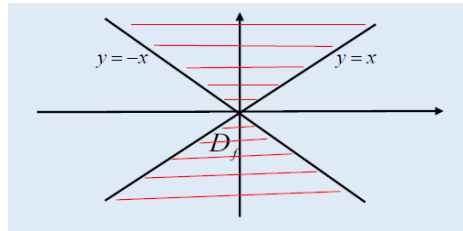
It represents all the points of the closed disk centered at the origin  $(0, 0)$  with a radius of 1, excluding the cross-hatched (red) portion.

**Example 5.5**

Find, and then represent graphically, the domains of the two functions  $f$  and  $g$ , denoted respectively by  $D_f$  and  $D_g$ , which are defined as follows:

$$f(x, y) = \ln \left( \frac{x+y}{y-x} \right), g(x, y) = \sqrt{f(x, y)}.$$

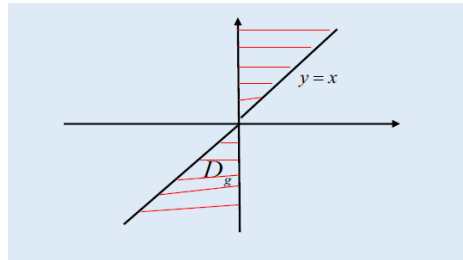
$$\begin{aligned} D_f &= \left\{ (x, y) \in \mathbb{R}^2 / \frac{x+y}{y-x} > 0 \right\} \\ &= \left\{ (x, y) \in \mathbb{R}^2 / (x+y > 0 \wedge y-x < 0) \vee (x+y < 0 \wedge y-x > 0) \right\} \\ &= \left\{ (x, y) \in \mathbb{R}^2 / (y > -x \wedge y < x) \vee (y < -x \wedge y > x) \right\}. \end{aligned}$$



$$D_g = \left\{ (x, y) \in \mathbb{R}^2 / f(x, y) \geq 0 \right\} = \left\{ (x, y) \in \mathbb{R}^2 / \ln \left( \frac{x+y}{y-x} \right) \geq 0 \right\} = \left\{ (x, y) \in \mathbb{R}^2 / \frac{x+y}{y-x} \geq 1 \right\}.$$

If  $y - x > 0 \Rightarrow y > x$ , then  $x + y \geq y - x \Rightarrow x \geq 0$ .

If  $y - x < 0 \Rightarrow y < x$ , then  $x + y \leq y - x \Rightarrow x \leq 0$ .



### 5.1.2 Geometric Representation of a Real-Valued Multivariable Function

**Definition 5.3**

[3] Let  $f$  be a real-valued function of  $n$  real variables. We call the set of points that belong to

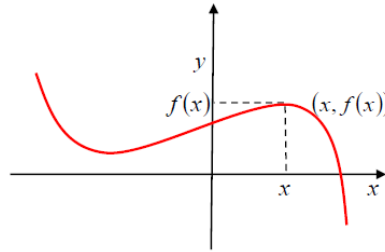
$$(x_1, x_2, \dots, x_n, f(x_1, x_2, \dots, x_n)),$$

where  $(x_1, x_2, \dots, x_n) \in D_f$ . The graph of the function  $f$ , denoted by  $G_f$ , is defined as:

$$G_f = \{(x_1, x_2, \dots, x_n, f(x_1, x_2, \dots, x_n)) / (x_1, x_2, \dots, x_n) \in D_f\} \subset \mathbb{R}^{n+1}.$$

**Definition 5.4 (For  $n = 1$ )**

We represent the graph of a real-valued function of a single real variable by a curve in the plane  $\mathbb{R}^2$ . In this case, we plot the points with coordinates  $(x, f(x))$  :



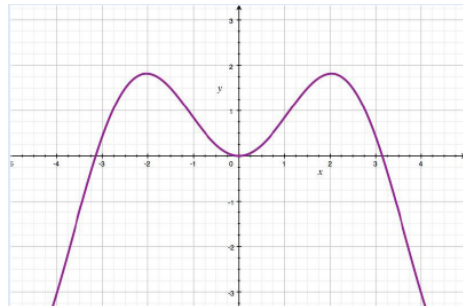
That is, the graph consists of all points whose first coordinate is the variable  $x$ , and whose second coordinate is the value of the function  $f(x)$ .

**Example 5.6**

The graph of the real-valued function of one variable

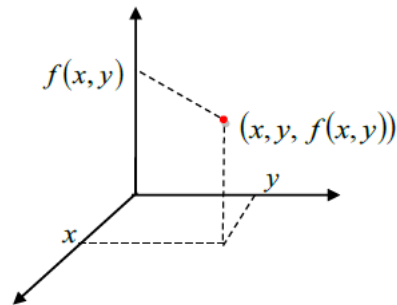
$$x \rightarrow x \sin x,$$

is represented in the plane  $\mathbb{R}^2$  as follows:

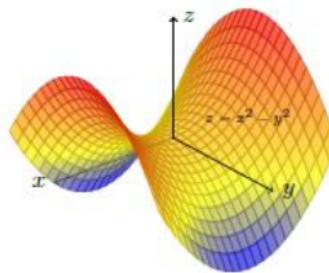


**Definition 5.5** (*For  $n = 2$* )

That is, for a real-valued function of two real variables, the graph is represented as a surface in three-dimensional space  $\mathbb{R}^3$ . We plot the points with coordinates  $(x, y, f(x, y))$ :

**Example 5.7**

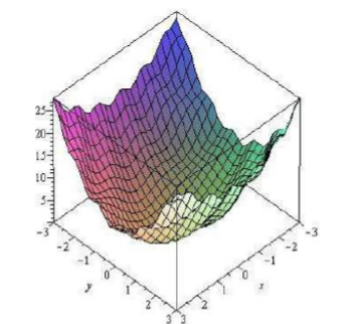
The graph of the real-valued function  $f$  of two variables  $f(x, y) = x^2 - y^2$  is represented in the three-dimensional space  $\mathbb{R}^3$  as follows:

**Example 5.8**

The graph of the real function of two variables  $f$  defined by

$$f(x, y) = x^2 + 2y^2 + \sin(2xy),$$

is represented in the space  $\mathbb{R}^3$  as follows:



**Definition 5.6 (For  $n \geq 3$ )**

That is, for real-valued functions of three or more variables, there is no clear way to represent their graphs geometrically. It becomes very difficult to obtain a visual or graphical representation of such functions in space.

However, there is another method to represent real-valued functions of two or three variables.

**Definition 5.7**

Let  $f$  be a real-valued function of two variables. For a real number  $k$ , the set of all points in  $D_f$  whose image under  $f$  equals  $k$  is called a level curve (also known as an isoline) of the function  $f$  corresponding to the level  $k$ . This set is denoted by  $L_f$  or  $C_f$ , and it is defined as:

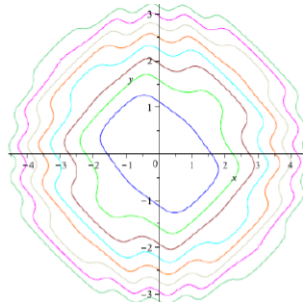
$$L_f = \{(x, y) \in D_f / f(x, y) = k\} \subset \mathbb{R}^2.$$

**Example 5.9**

The level curves  $k = \{0, 2.5, \dots, 17.5, 20\}$  of the real-valued function  $f$  of two variables, defined as

$$f(x, y) = x^2 + 2y^2 + \sin(2xy),$$

We represent it in the space  $\mathbb{R}^2$  as follows:

**Definition 5.8**

Let  $f$  be a real function of three variables, and let  $a$  be a real number. We call the set of points in the domain  $D_f$  of  $f$  whose images under  $f$  are equal to  $a$  the level surface of  $f$  corresponding to  $a$ , and we denote it by  $S_a$ , as:

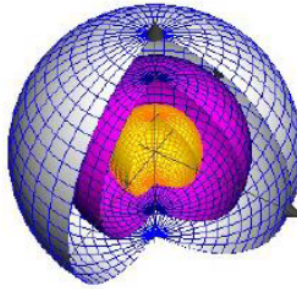
$$S_a = \{(x, y, z) \in D_f / f(x, y, z) = a\} \subset \mathbb{R}^3.$$

**Example 5.10**

The level surfaces  $a \in \{1, 2, 3\}$  of the real-valued function of three variables defined by

$$f(x, y, z) = x^2 + y^2 + z^2.$$

in  $\mathbb{R}^3$  are represented in space (with a longitudinal section so that the interior surfaces become visible) as follows:



## 5.2 Limit of a Real-Valued Function of Two Variables

**Definition 5.9**

Let  $f$  be a real-valued function of two variables defined on a subset of  $\mathbb{R}^2$ , and let  $(a, b)$  be a point in  $\mathbb{R}^2$ . We say that the function  $f$  has the limit  $l$  at  $(x, y)$  if  $f(x, y)$  can be made arbitrarily close to  $l$  whenever  $(x, y)$  is sufficiently close to  $(a, b)$  (but not necessarily equal to  $(a, b)$ ). This is written as:

$$\lim_{(x,y) \rightarrow (a,b)} f(x, y) = l.$$

So

$$\left( \lim_{(x,y) \rightarrow (a,b)} f(x, y) = l \right) \\ \Leftrightarrow \left( \forall \varepsilon > 0, \exists \delta > 0, \forall (x, y) \in D_f : (x - a)^2 + (y - b)^2 < \delta^2 \Rightarrow |f(x, y) - l| < \varepsilon \right)$$

**Remark 5.1**

We say that the pair  $(x, y)$  approaches  $(a, b)$ , this means that  $x$  approaches  $a$ , and  $y$  approaches  $b$ .

### 5.2.1 Operations on Limits

#### Theorem 5.2

Let  $f$  and  $g$  be two real-valued functions defined on a neighborhood of  $(a, b) \in \mathbb{R}^2$ , and suppose that

$$\begin{aligned}\lim_{(x,y) \rightarrow (a,b)} [f(x, y) + g(x, y)] &= \lim_{(x,y) \rightarrow (a,b)} f(x, y) + \lim_{(x,y) \rightarrow (a,b)} g(x, y). \\ \lim_{(x,y) \rightarrow (a,b)} [f(x, y) \times g(x, y)] &= \lim_{(x,y) \rightarrow (a,b)} f(x, y) \times \lim_{(x,y) \rightarrow (a,b)} g(x, y). \\ \lim_{(x,y) \rightarrow (a,b)} \alpha f(x, y) &= \alpha \times \lim_{(x,y) \rightarrow (a,b)} f(x, y), \alpha \in \mathbb{R}. \\ \lim_{(x,y) \rightarrow (a,b)} \left[ \frac{f(x, y)}{g(x, y)} \right] &= \frac{\lim_{(x,y) \rightarrow (a,b)} f(x, y)}{\lim_{(x,y) \rightarrow (a,b)} g(x, y)}, \lim_{(x,y) \rightarrow (a,b)} g(x, y) \neq 0.\end{aligned}$$

## 5.3 Continuity of a Real-Valued Function of Two Variables

#### Definition 5.10

[1] Let  $f$  be a real-valued function of two variables defined on a domain  $D_f$ , and let  $(a, b) \in D_f$ .

- We say that the function  $f$  is continuous at the point  $(a, b)$  if:

$$\lim_{(x,y) \rightarrow (a,b)} f(x, y) = f(a, b).$$

- We say that the function  $f$  is continuous on its domain  $D_f$  if it is continuous at every point  $(a, b) \in D_f$ .

#### Example 5.11

Let the function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be defined by  $f(x, y) = xy$ , is continuous at the point  $(0, 0)$

$$\lim_{(x,y) \rightarrow (0,0)} f(x, y) = 0 = f(0, 0).$$

**Remark 5.3**

- We say that a function  $f$  is discontinuous or not continuous at a point  $(a, b)$  if:

$$\lim_{(x,y) \rightarrow (a,b)} f(x, y) \text{ does not exist or } \lim_{(x,y) \rightarrow (a,b)} f(x, y) \neq f(a, b).$$

- The graph of a discontinuous function necessarily contains a hole or a break.
- From the properties of limits and the definition of continuity, we conclude that: The sum, difference, product, and quotient (when the denominator is nonzero) of continuous functions are also continuous on their domains.
- Every polynomial is a continuous function on the set of real numbers.
- Every rational function (i.e., a ratio of polynomials) is continuous on its domain.
- If  $f$  is a real-valued function of two variables, continuous on its domain  $D_f$ , and  $g$  is a real-valued function of one variable, defined and continuous on the range (image) of  $f$ , then the composite function  $h = g \circ f$ , defined by:

$$h(x, y) = g(f(x, y)),$$

is continuous on  $D_f$ .

- All the previous results can be generalized to real-valued functions of several variables.

**5.4 Partial Derivatives****Definition 5.11**

Let  $f$  be a real-valued function of  $n$  real variables, and let

$$(a_1, a_2, \dots, a_n) \in D_f.$$

For each  $i = 1, 2, \dots, n$ , the partial derivative of  $f$  with respect to the variable  $x_i$  at the point  $(a_1, a_2, \dots, a_n)$ , is defined by the limit:

$$\lim_{h \rightarrow 0} \frac{f(a_1, \dots, a_{i-1}, a_i + h, a_{i+1}, \dots, a_n) - f(a_1, \dots, a_n)}{h}.$$

If this limit exists. We then denote it by:

$$\frac{\partial f}{\partial x_i}(a_1, a_2, a_3, \dots, a_n) \text{ or } f'_{x_i}(a_1, a_2, a_3, \dots, a_n).$$

### 5.4.1 Partial Derivatives of Two-Variable Functions

#### Definition 5.12

Let  $f$  be a real-valued function of two variables, and let  $(a, b)$  be a pair of  $D_f$ .

- The partial derivative of  $f$  with respect to  $x$  at  $(a, b)$  is defined as the limit:

$$f'_x(a, b) = \frac{\partial f}{\partial x}(a, b) = \lim_{h \rightarrow 0} \frac{f(a + h, b) - f(a, b)}{h},$$

If the limit exists.

- The partial derivative of  $f$  with respect to  $y$  at  $(a, b)$  is defined as the limit:

$$f'_y(a, b) = \frac{\partial f}{\partial y}(a, b) = \lim_{h \rightarrow 0} \frac{f(a, b + h) - f(a, b)}{h},$$

If the limit exists.

### 5.4.2 How to Compute a Partial Derivative in Two Variables

#### Definition 5.13

Let  $f$  be a real-valued function of two variables.

- To compute the partial derivative of  $f$  with respect to  $x$ , denoted by  $f'_x(x, y)$ , we consider  $y$  as a constant and differentiate  $f$  normally with respect to  $x$ .
- To compute the partial derivative of  $f$  with respect to  $y$ , denoted by  $f'_y(x, y)$ , we consider  $x$  as a constant and differentiate  $f$  normally with respect to  $y$ .

**Example 5.12**

Let the function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  defined as  $f(x, y) = 2x^3y^2$ , then

$$\frac{\partial f}{\partial x}(x, y) = 6x^2y^2, \frac{\partial f}{\partial y}(x, y) = 4x^3y.$$

**Example 5.13**

Let the function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  defined as  $f(x, y) = \ln(x^2 + y^2 + 1)$ , then

$$\frac{\partial f}{\partial x}(x, y) = \frac{2x}{x^2 + y^2 + 1}, \frac{\partial f}{\partial y}(x, y) = \frac{2y}{x^2 + y^2 + 1}.$$

**Proposition 5.4**

Let  $f$  and  $g$  be two real-valued functions of  $n$  real variables, and let  $\alpha$  be a real number. Then, the following properties hold:

$$\begin{aligned} \frac{\partial (f + g)}{\partial x_i}(a) &= \frac{\partial f}{\partial x_i}(a) + \frac{\partial g}{\partial x_i}(a), \\ \frac{\partial (f \times g)}{\partial x_i}(a) &= \frac{\partial f}{\partial x_i}(a) g(a) + \frac{\partial g}{\partial x_i}(a) f(a), \\ \frac{\partial (\alpha f)}{\partial x_i}(a) &= \alpha \frac{\partial f}{\partial x_i}(a), \\ \frac{\partial \left(\frac{f}{g}\right)}{\partial x_i}(a) &= \frac{\frac{\partial f}{\partial x_i}(a) g(a) - \frac{\partial g}{\partial x_i}(a) f(a)}{(g(a))^2}. \end{aligned}$$

**Example 5.14**

Let  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$  be a function defined by

$$f(x, y, z) = x \cos(y) + z \sin(y).$$

Then

$$\frac{\partial f}{\partial x}(x, y, z) = \cos(y), \frac{\partial f}{\partial y}(x, y, z) = -x \sin(y) + z \cos(y), \frac{\partial f}{\partial z}(x, y, z) = \sin(y).$$

## 5.5 Differentiation

### Definition 5.14

- Let  $f$  be a real-valued function of two real variables  $x$  and  $y$ . The differential of  $f$ , denoted by  $df$ , is defined as

$$df = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy.$$

- Let  $f$  be a real-valued function of  $n$  real variables. If  $f$  has continuous partial derivatives, then  $f$  is said to be differentiable, and we call  $df$  the differential of  $f$ , is given by:

$$df = \sum_{i=1}^{i=n} \frac{\partial f}{\partial x_i} dx_i.$$

### Definition 5.15

Let  $f$  be a real-valued function of  $n$  real variables, and suppose that  $f$  has continuous partial derivatives. Let  $a \in D_f$ . We say that  $f$  is differentiable at the point  $a$  is defined as the linear map, denoted by  $df(a)$ , where:

$$df(a) = \sum_{i=1}^{i=n} \frac{\partial f(a)}{\partial x_i} dx_i.$$

as the linear map (linear application) that best approximates the change in  $f$  near the point  $a$ .

### Proposition 5.5

Let  $f$  and  $g$  be differentiable functions at a point  $a \in \mathbb{R}^n$ , and let  $\alpha, \beta \in \mathbb{R}$  be real numbers. Then the following properties hold:

$$d(f + g)(a) = df(a) + dg(a).$$

$$d(\alpha f)(a) = \alpha df(a).$$

$$d(f \times g)(a) = f(a) dg(a) + g(a) df(a).$$

**Remark 5.6**

Let  $U$  be a subset of  $\mathbb{R}^n$ , and let  $f : U \rightarrow \mathbb{R}$  be a function.

- We say that  $f$  is differentiable on  $U$  if it is differentiable at every point of  $U$ .
- Every function that is differentiable at a point of  $U$  is also continuous at that point.

**Example 5.15**

Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be a function defined by

$$f(x, y) = -18y^2 - 17xy + 6x^2.$$

Then

$$\frac{\partial f}{\partial x}(x, y) = -17y + 12x, \quad \frac{\partial f}{\partial y}(x, y) = -36y - 17x.$$

So

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = (-17y + 12x)dx + (-36y - 17x)dy.$$

## 5.6 Double Integral

**Definition 5.16**

Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be a function, and let  $D$  be a closed and bounded domain of  $\mathbb{R}^2$ .

We call the double integral of  $f$  over  $D$  denoted by:

$$\iint_D f(x, y) dx dy.$$

**Theorem 5.7**

The double integral has the same linearity properties as the single integral.

Let  $f$  and  $g$  be two continuous and bounded functions of two variables defined on a closed and bounded domain

$D$  of  $\mathbb{R}^2$ . For any real numbers  $\alpha$  and  $\beta$  of  $\mathbb{R}$ , we have:

$$\iint_D (\alpha f + \beta g)(x, y) dx dy = \iint_D \alpha f(x, y) dx dy + \iint_D \beta g(x, y) dx dy.$$

$$\forall (x, y) \in D, f \geq 0 \Rightarrow \iint_D f(x, y) dx dy \geq 0.$$

$$\iint_D f(x, y) dx dy = \iint_{D_1} f(x, y) dx dy + \iint_{D_2} f(x, y) dx dy, D = D_1 \cup D_2.$$

where  $D_1$  and  $D_2$  are disjoint.

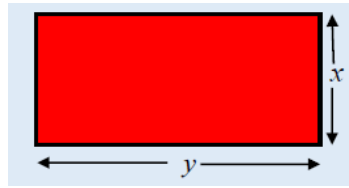
### 5.6.1 How to compute a double integral

#### Definition 5.17

A double integral is computed according to the shape of the domain of integration  $D$ :

**If  $D$  is a rectangle, that is:**

$$D = [a, b] \times [c, d].$$



Then

$$\iint_D f(x, y) dx dy = \int_a^b \left[ \int_c^d f(x, y) dy \right] dx = \int_c^d \left[ \int_a^b f(x, y) dx \right] dy.$$

#### Example 5.16

For all  $D = [0, 2] \times [0, 1]$ , and  $f(x, y) = x^2 + y$ . Then

$$\begin{aligned} \iint_D f(x, y) dx dy &= \int_0^2 \left[ \int_0^1 (x^2 + y) dy \right] dx = \int_0^2 \left( x^2 y + \frac{y^2}{2} \right) \Big|_0^1 dx \\ &= \int_0^2 \left( x^2 + \frac{1}{2} \right) dx = \left( \frac{1}{3} x^3 + \frac{1}{2} x \right) \Big|_0^2 = \frac{8}{3} + 1 = \frac{11}{6}. \end{aligned}$$

And

$$\begin{aligned} \iint_D f(x, y) dx dy &= \int_0^1 \left[ \int_0^2 (x^2 + y) dx \right] dy = \int_0^1 \left[ \left( \frac{1}{3} x^3 + yx \right) \Big|_0^2 \right] dy \\ &= \int_0^1 \left( \frac{8}{3} + 2y \right) dy = \left( \frac{8}{3} y + y^2 \right) \Big|_0^1 = \frac{8}{3} + 1 = \frac{11}{6}. \end{aligned}$$

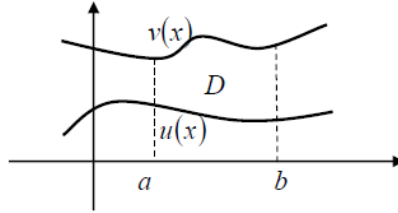
**Definition 5.18** (If  $D$  is a simple domain with respect to the  $ox$  axis (vertical simple

That is:

$$D = [a, b] \times [u(x), v(x)],$$

where  $u$  and  $v$  are continuous functions on  $[a, b]$  and satisfy

$$\forall x \in [a, b], u(x) \prec v(x).$$



Then

$$\iint_D f(x, y) dx dy = \int_a^b \left[ \int_{u(x)}^{v(x)} f(x, y) dy \right] dx.$$

**Example 5.17**

Compute the double integral

$$\iint_D x^2 dx dy,$$

where

$$D = \{(x, y) \in \mathbb{R}^2, x \in [0, 1], y \in [0, \sqrt{x}]\}.$$

Then

$$\begin{aligned} \iint_D x^2 dx dy &= \int_0^1 x^2 \left[ \int_0^{\sqrt{x}} 1 dy \right] dx = \int_0^1 x^2 [y]_0^{\sqrt{x}} dx = \int_0^1 x^2 \sqrt{x} dx = \int_0^1 x^{\frac{5}{2}} dx \\ &= \frac{2}{7} x^{\frac{7}{2}} \Big|_0^1 = \frac{2}{7}. \end{aligned}$$

**Example 5.18**

Compute the double integral

$$\iint_D \sqrt{y(x-2y-1)} dx dy,$$

1. Find the domain of the function  $f$ , and then represent it graphically.
2. Calculate the following double integral

$$\iint_D (2x^2 - x) \exp(xy) dx dy.$$

where

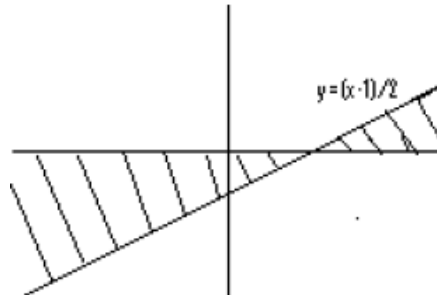
$$\Delta = \{(x, y) \in \mathbb{R}^2, 1 \leq x \leq 2\}.$$

Consider the double integral

$$\iint_D \sqrt{y(x-2y-1)} dx dy.$$

1. The domain of the function  $f$ :

$$D = \left\{ (x, y) \in \mathbb{R}^2, \left( y \geq 0 \wedge y \leq \frac{x-1}{2} \right) \vee \left( y \leq 0 \wedge y \geq \frac{x-1}{2} \right) \right\}.$$



2. Calculate the following double integral, where

$$\begin{aligned} \iint_D (2x^2 - x) \exp(xy) dx dy &= \int_1^2 (2x - 1) \left[ \int_0^{\frac{x-1}{2}} x \exp(xy) dy \right] dx \\ &= \int_1^2 (2x - 1) \left[ \exp\left(\frac{x^2 - x}{2}\right) - 1 \right] dx \\ &= 2 \int_1^2 \left(\frac{2x - 1}{2}\right) \exp\left(\frac{x^2 - x}{2}\right) dx - \int_1^2 (2x - 1) dx \\ &= 2 \exp\left(\frac{x^2 - x}{2}\right) \Big|_1^2 - x^2 - x \Big|_1^2 \\ &= 2 \exp(1) - 4. \end{aligned}$$

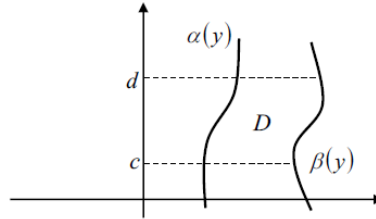
**Definition 5.19** (If  $D$  is simple domain with respect to the  $oy$  axis (vertical simple region))

That is

$$D = [\alpha(y), \beta(y)] \times [c, d],$$

where  $\alpha$  and  $\beta$  are continuous functions on  $[c, d]$  and satisfy

$$\forall y \in [c, d], \alpha(y) \prec \beta(y).$$



Then

$$\iint_D f(x, y) dx dy = \int_c^d \left[ \int_{\alpha(y)}^{\beta(y)} f(x, y) dx \right] dy.$$

In each of the three cases above, we convert the double integral into two nested single (iterated) integrals. The integration is performed from the inside out: at each step we integrate with respect to one variable while treating the other variable as constant.

### Example 5.19

Calculate the value of the following double integral

$$\iint_D 2x \exp(-y) dx dy.$$

where

$$D = \{(x, y) \in \mathbb{R}^2 / x \geq 0, y \leq 1, x - y \leq 0\}.$$

Let  $0 \leq x \leq y \leq 1$ , then

$$\begin{aligned} \iint_D 2x \exp(-y) dx dy &= \int_0^1 \left( \int_x^1 2x \exp(-y) dy \right) dx = \int_0^1 2x \left( \int_x^1 \exp(-y) dy \right) dx \\ &= \int_0^1 2x - \exp(-y) \Big|_x^1 dx = \int_0^1 2x [\exp(-x) - \exp(-1)] dx \\ &= 2 \int_0^1 x \exp(-x) dx - \exp(-1). \end{aligned}$$

Integrate by parts (or use a standard antiderivative):

$$g(x) = x, g'(x) = 1,$$

$$H'(x) = \exp(-x), H(x) = -\exp(-x),$$

then

$$\int_0^1 x \exp(-x) dx = -x \exp(-x) \Big|_0^1 - \exp(-x) \Big|_0^1 = -2 \exp(-1) + 1.$$

Then

$$\iint_D 2x \exp(-y) dx dy = -5 \exp(-1) + 2.$$

### Remark 5.8

If the domain  $D$  can be represented in two simple forms "simple with respect to the  $y$  axis ( $oy$ ) or simple with respect to the  $x$  axis ( $ox$ )" then the value of the double integral does not change.

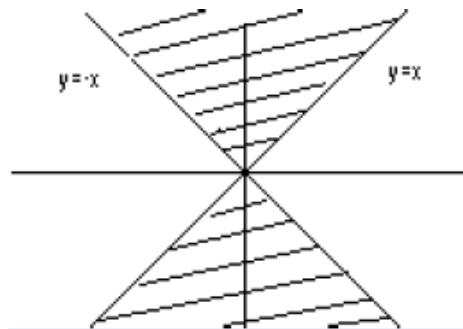
## 5.6.2 Change of Variables

### Definition 5.20

Let  $f$  be a function of two variables  $x$  and  $y$ .

Every point  $(x, y)$  in the plane  $\mathbb{R}^2$  can be represented by polar coordinates  $(r, \theta)$  as follows:

$$\begin{cases} x = r \cos \theta, \\ y = r \sin \theta. \end{cases}$$



If the function  $f$  is continuous on a closed and bounded domain  $D$  of  $\mathbb{R}^2$ , then

$$\iint_D f(x, y) dx dy = \iint_{\Delta} f(r \cos \theta, r \sin \theta) r dr d\theta.$$

### Example 5.20

Let the function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be defined by

$$f(x, y) = \sqrt{3 - y^2 - x^2}.$$

1. Find the domain of  $f$ .
2. Using a change of variables, calculate the area of the region  $D_f$ .

Let the function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be defined by

$$f(x, y) = \sqrt{3 - y^2 - x^2}.$$

1. The domain definition of  $f$  is

$$D_f = \{(x, y) \in \mathbb{R}^2, 3 - y^2 - x^2 \geq 0\}$$

$$3 - y^2 - x^2 = 0 \Rightarrow (x - 0)^2 + (y - 0)^2 = (\sqrt{3})^2,$$

is the circle centered at the origin  $(0, 0)$  with radius  $\sqrt{3}$ . We use polar coordinates

$$\begin{cases} x = r \cos \theta = \sqrt{3} \cos \theta, \\ y = r \sin \theta = \sqrt{3} \sin \theta. \end{cases}$$

Hence,  $D_f$  is transformed into

$$D_f = \{(r, \theta) \in \mathbb{R}^2 / 0 \leq \theta \leq 2\pi, 0 \leq r \leq \sqrt{3}\}$$

$$3 - y^2 - x^2 = 0 \Rightarrow (x - 0)^2 + (y - 0)^2 = (\sqrt{3})^2,$$

2. Compute the following

$$\int_0^{2\pi} \int_0^{\sqrt{3}} r dr d\theta = \left( \int_0^{2\pi} d\theta \right) \left( \int_0^{\sqrt{3}} r dr \right) = 2\pi \frac{r^2}{2} \Big|_0^{\sqrt{3}} = 3\pi.$$

## 5.7 Triple Integral

### Definition 5.21

Let  $\mathbb{R}^3 \rightarrow \mathbb{R}$  be a function, and let  $\Delta$  be a closed and bounded domain of  $\mathbb{R}^3$ .

### 5.7.1 How to Compute a Triple Integral

#### Definition 5.22

*If  $\Delta$  is represented in the form*

$$(x, y, z) \in \Delta \Leftrightarrow \begin{cases} x \in [a, b], \\ y \in [u(x), v(x)], \\ z \in [\alpha(x, y), \beta(x, y)], \end{cases}$$

*where  $u, v, \alpha, \beta$  are continuous functions such that*

$$\begin{aligned} \forall x \in [a, b], u(x) < v(x), \\ \forall (x, y) \in [a, b] \times [u(x), v(x)], \alpha(x, y) < \beta(x, y), \end{aligned}$$

*We denote by*

$$\iiint_{\Delta} f(x, y, z) dx dy dz = \int_a^b \left[ \int_{u(x)}^{v(x)} \left[ \int_{\alpha(x, y)}^{\beta(x, y)} f(x, y, z) dz \right] dy \right] dx.$$

*For the triple integral of the function  $f$  over  $\Delta$ .*

#### Definition 5.23

*If  $\Delta$  is represented in the form:*

$$(x, y, z) \in \Delta \Leftrightarrow \begin{cases} y \in [a, b], \\ x \in [u(y), v(y)], \\ z \in [\alpha(x, y), \beta(x, y)], \end{cases}$$

*where  $u, v, \alpha, \beta$  are continuous functions such that*

$$\begin{aligned} \forall y \in [a, b], u(y) < v(y) \\ \forall (y, x) \in [a, b] \times [u(y), v(y)], \alpha(x, y) < \beta(x, y), \end{aligned}$$

*We denote by*

$$\iiint_{\Delta} f(x, y, z) dx dy dz = \int_a^b \left[ \int_{u(y)}^{v(y)} \left[ \int_{\alpha(x, y)}^{\beta(x, y)} f(x, y, z) dz \right] dx \right] dy.$$

**Definition 5.24**

**If  $\Delta$  is represented in the form:**

$$(x, y, z) \in \Delta \Leftrightarrow \begin{cases} z \in [a, b], \\ x \in [u(z), v(z)], \\ y \in [\alpha(x, z), \beta(x, z)], \end{cases}$$

where  $u, v, \alpha, \beta$  are continuous functions such that

$$\begin{aligned} \forall z \in [a, b], u(z) < v(z) \\ \forall (z, x) \in [a, b] \times [u(z), v(z)], \alpha(x, z) < \beta(x, z), \end{aligned}$$

We denote by

$$\iiint_{\Delta} f(x, y, z) dx dy dz = \int_a^b \left[ \int_{u(z)}^{v(z)} \left[ \int_{\alpha(x, z)}^{\beta(x, z)} f(x, y, z) dy \right] dx \right] dz.$$

**Definition 5.25**

**If  $\Delta$  is represented in the form:**

$$(x, y, z) \in \Delta \Leftrightarrow \begin{cases} z \in [a, b], \\ y \in [u(z), v(z)], \\ x \in [\alpha(y, z), \beta(y, z)], \end{cases}$$

where  $u, v, \alpha, \beta$  are continuous functions such that

$$\forall z \in [a, b], u(z) < v(z), \forall (z, y) \in [a, b] \times [u(z), v(z)], \alpha(y, z) < \beta(y, z),$$

We denote by

$$\iiint_{\Delta} f(x, y, z) dx dy dz = \int_a^b \left[ \int_{u(z)}^{v(z)} \left[ \int_{\alpha(y, z)}^{\beta(y, z)} f(x, y, z) dx \right] dy \right] dz.$$

**Definition 5.26**

**If  $\Delta$  is represented in the form:**

$$(x, y, z) \in \Delta \Leftrightarrow \begin{cases} y \in [a, b], \\ z \in [u(y), v(y)], \\ x \in [\alpha(y, z), \beta(y, z)], \end{cases}$$

where  $u, v, \alpha, \beta$  are continuous functions such that

$$\forall y \in [a, b], u(y) \prec v(y), \forall (y, z) \in [a, b] \times [u(y), v(y)], \alpha(y, z) \prec \beta(y, z),$$

We denote by

$$\iiint_{\Delta} f(x, y, z) dx dy dz = \int_a^b \left[ \int_{u(y)}^{v(y)} \left[ \int_{\alpha(y, z)}^{\beta(y, z)} f(x, y, z) dx \right] dz \right] dy.$$

**Definition 5.27**

**If  $\Delta$  is represented in the form:**

$$(x, y, z) \in \Delta \Leftrightarrow \begin{cases} x \in [a, b], \\ z \in [u(x), v(x)], \\ y \in [\alpha(x, z), \beta(x, z)], \end{cases}$$

where  $u, v, \alpha, \beta$  are continuous functions such that

$$\forall x \in [a, b], u(x) \prec v(x), \\ \forall (x, z) \in [a, b] \times [u(x), v(x)], \alpha(x, z) \prec \beta(x, z),$$

We denote by

$$\iiint_{\Delta} f(x, y, z) dx dy dz = \int_a^b \left[ \int_{u(x)}^{v(x)} \left[ \int_{\alpha(x, z)}^{\beta(x, z)} f(x, y, z) dy \right] dz \right] dx.$$

**Example 5.21**

Let the function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be defined by

$$f(x, y) = \sqrt{y^2 - x^2}.$$

1. Find the domain of  $f$  and sketch its graph.
2. Compute the following triple integral  $\Delta$  to complete

$$\iiint_{\Delta} \left( (f(x, y))^2 + x^2 \right) dx dy dz.$$

where

$$\Delta = \left\{ (x, y, z) \in \mathbb{R}^3 \Leftrightarrow x \in [0, 1], y \in [0, x], z \in [0, \sqrt{y}] \right\}.$$

Let the function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be defined by

$$f(x, y) = \sqrt{y^2 - x^2}.$$

1. The domain definition of  $f$  is

$$\begin{aligned} D_f &= \left\{ (x, y) \in \mathbb{R}^2, y^2 - x^2 \geq 0 \right\} = \left\{ (x, y) \in \mathbb{R}^2, (y - x)(y + x) \geq 0 \right\} \\ &= \left\{ (x, y) \in \mathbb{R}^2, [(y \geq x) \wedge (y \geq -x)] \vee [(y \leq x) \wedge (y \leq -x)] \right\}. \end{aligned}$$

2. Compute the following triple integral  $\Delta$

$$\begin{aligned} &\int_0^1 \int_0^x \int_0^{\sqrt{y}} \left( \left( \sqrt{y^2 - x^2} \right)^2 + x^2 \right) dz dy dx \\ &= \int_0^1 \int_0^x \int_0^{\sqrt{y}} y^2 dz dy dx = \int_0^1 \int_0^x \left( y^{\frac{5}{2}} \right) dy dx \\ &= \int_0^1 \left( \frac{2}{7} y^{\frac{7}{2}} \right) \Big|_0^x dx = \left( \frac{2}{7} \times \frac{2}{9} x^{\frac{9}{2}} \right) \Big|_0^1 = \frac{4}{63}. \end{aligned}$$

**Example 5.22**

Let the function  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$  be defined by

$$f(x, y, z) = x.$$

Compute the following triple integral

$$\iiint_{\Delta} x dx dy dz.$$

where

$$\Delta = \{(x, y, z) \in \mathbb{R}^3 \Leftrightarrow x \in [0, 1], y \in [0, z], z \in [x, 2x]\}.$$

Let the function  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$  be defined by

$$f(x, y, z) = x.$$

Compute the following triple integral

$$\begin{aligned} \int_0^1 \int_0^z \int_x^{2x} f(x, y, z) dz dy dx &= \int_0^1 \int_x^{2x} \int_0^z x dy dz dx = \int_0^1 \int_x^{2x} x z dz dx \\ &= \int_0^1 \frac{1}{2} z^2 x \Big|_x^{2x} dx = \int_0^1 \frac{3}{2} x^3 dx = \left( \frac{3}{8} x^4 \right) \Big|_0^1 = \frac{3}{8}. \end{aligned}$$

**Example 5.23**

Let the function  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$  be defined by

$$f(x, y, z) = \frac{x}{\sqrt{y - x^2}}.$$

Compute the following triple integral

$$\iiint_{\Delta} \frac{x}{\sqrt{y - x^2}} dx dy dz.$$

where

$$\Delta = \{(x, y, z) \in \mathbb{R}^3 \Leftrightarrow x \in [0, \sqrt{y}], y \in [0, z^2], z \in [1, 2]\}.$$

Let the function  $f : \mathbb{R}^3 \rightarrow \mathbb{R}$  be defined by

$$f(x, y, z) = \frac{x}{\sqrt{y - x^2}}.$$

Compute the following triple integral  $\Delta$

$$\begin{aligned} & \int_0^{\sqrt{y}} \int_0^{z^2} \int_1^2 f(x, y, z) dz dy dx \\ &= \int_0^{\sqrt{y}} \int_0^{z^2} \int_1^2 \frac{x}{\sqrt{y - x^2}} dy dz dx = \int_1^2 \int_0^{z^2} \int_0^{\sqrt{y}} \left( -\sqrt{y - x^2} \right) dx dy dz \\ &= \int_1^2 \int_0^{z^2} \sqrt{y} dy dz = \int_1^2 \left. \frac{2}{3} y^{\frac{3}{2}} \right|_0^{z^2} dz = \int_1^2 \frac{2}{3} z^3 dz = \left. \left( \frac{1}{6} z^4 \right) \right|_1^2 = \frac{5}{2}. \end{aligned}$$

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