République Algérienne Démocratique et Populaire
Ministère de l'Enseignement Supérieur et de la Recherche Scientifique

UNIVERSITÉ MOHAMED KHIDER, BISKRA
FACULTÉ des SCIENCES EXACTES et des SCIENCES de la NATURE et de la VIE
DÉPARTEMENT DE MATHÉMATIQUES

Thèse présentée en vue de l'obtention du Diplôme de

Doctorat en Mathématiques

Option: **Probabilités et Statistiques**

Par

Nacira Agram

Titre

Contrôle optimal stochastique à horizon infini

Devant le jury

Abdelhakim Necir	Pr.	UMK Biskra	**Président**
Brahim Mezerdi	Pr.	UMK Biskra	**Rapporteur**
Salah Eddine Rebiai	Pr.	ULK Batna	**Examinateur**
Djamel Meraghni	MC (A).	UMK Biskra	**Examinateur**
Boubakeur Labed	MC (A).	UMK Biskra	**Examinateur**

17 Décembre 2013

Dédicace

Pour mes chers Parents

Pour mes frères et soeurs

Un grand merci à tout qui m'a conduit à ce jour mémorable.

REMERCIEMENTS

En premier lieu, je tiens à remercier mon directeur de thèse, le Pr. Brahim Mezerdi pour la confiance qu'il m'a accordée en acceptant d'encadrer ce travail doctoral, par ses multiples conseils. Enfin, j'ai été extrêmement sensible à ses qualités humaines d'écoute et de compréhension tout au long de ce travail doctoral.
Mes remerciements s'adressent au Pr. Bernt Øksendal qui m'a permis de participer dans le groupe du Centre de Mathématiques Appliquées (CMA) de l'université d'Oslo (Pr. Frank Proske et Sven Haadem).
Grâce à lui j'ai pu valoriser mes travaux en me mettant en relation avec plusieurs spécialistes (Prs. Salah Eldin A. Mohammed, Giulia Di Nunno, Monique Jeanblanc, José luis Da Silva) et en m'assurant les meilleurs conditions pour mes déplacements et mes visites de recherche.
Je n'oublierai jamais l'aide, y compris financière, qui m'a été prodiguée par le Laboratoire de Mathématiques Appliquées (LMA) de Biskra.
Mes remerciements s'adressent au Pr. Abdelhakim Nécir, directeur du LMA de l'université de Biskra, qui a accepté d'être le président du jury de ma thèse; sans oublier qu'il a été mon enseignant durant mes études.
Je ne peux oublier Dr. Boubakeur Labed pour ses aides, ses disponibilités .Tout ce que je dis pour lui est insuffisant car il m'a enseigné durant tout le parcours de mon cursus. C'est un très grand plaisir pour moi qu'il fait partie de mon jury de thèse.
Je souhaite aussi adresser ma gratitude au Pr. Salah Eddine Rebiai de l'université de Batna, pour avoir accepté d'examiner mon travail et de faire partie du jury.
J'exprime ma gratitude au Dr. Djamel Meraghni pour avoir participé à ce jury. Ses multiples conseils m'ont été remarquables pendant chaque entretien.
Je n'oublierai jamais Mme. Samia Yakhlef pour ses aides et ses orientations ainsi que Mlle. Mabrouka Maamri.
Je ne peux oublier également, l'université Cadi Ayyad de Marrakech qui m'a donné l'occasion de connaître le Pr. Youssef Ouknine ainsi que tout son groupe.
En dernier, je tiens à remercier toute l'équipe du laboratoire de recherche de Biskra.

Contents

Dédicace **i**

Remerciements **ii**

Table of Contents **iii**

Introduction **1**

1 An introduction to jump processes **3**

1.1 Lévy process 3

1.1.1 Lévy-Khintchine formula 4

1.2 Stochastic integral with respect to Lévy process 5

1.2.1 Stochastic differential equations driven by Lévy processes 8

2 Stochastic control of jump diffusion processes **9**

2.1 Finite horizon 10

2.1.1 Sufficient conditions of optimality 11

2.1.2 Necessary conditions of optimality 14

2.1.3 Infinite horizon 17

2.2 Summary of the papers 17

2.2.1 Maximum principle for infinite horizon delay equations 17

2.2.2 Infinite horizon optimal control of FBSDEs with delay 18

3 Maximum principle for infinite horizon delay equations **19**

3.1 Introduction 19

3.2 Formulation of the problem 21

3.3 A sufficient maximum principle 22

3.3.1 Hamiltonian and time-advanced BSDEs for adjoint equations 23
3.3.2 A first sufficient maximum principle 24
3.3.3 A second sufficient maximum principle 28
3.4 A necessary maximum principle 39
3.5 Existence and uniqueness of the time-advanced BSDEs on infinite horizon . 43

4 Infinite horizon optimal control of FBSDEs with delay 53
4.1 Introduction 53
4.2 Setting of the problem 54
4.3 Sufficient maximum principle for partial information 59
4.4 Necessary conditions of optimality for partial information 63
4.5 Application to optimal consumption with respect to recursive utility . . . 68
4.5.1 A general optimal recursive utility problem 68
4.5.2 A solvable special case 71

Bibliography 75

Introduction

The problem of optimal control for delayed systems has had a lot of attention recently. Stochastic differential equations with delay are natural models for systems where the dynamics depends not only on the current value of the state but also on previous values. This is the case in population growth models in biology, for example. But it can also apply to other situations where there is some memory in the dynamics, for example economics and finance.

In general, stochastic delay differential equations are infinite dimensional, in the sense that the state of a solution needs infinitely many variables to be fully specified.

There has been also an increased interest in finance models which also include jumps. Such models can be represented in terms of stochastic differential equations driven by Lévy processes.

In the finite horizon case, maximum principles for delayed systems with jumps have been obtained by Øksendal & al [33], and for optimal control of forward-backward systems with jumps by Øksendal and Sulem [31]. In the infinite horizon case there is no terminal time condition for the BSDE, and it is not a priori clear how to compensate for this. It would be natural to guess that the terminal condition in the infinite horizon case would be a zero limit condition. But this is not correct. In fact, Halkin [21] provides a counterexample. Thus some care is needed in the infinite horizon case. We show that the missing terminal value condition should be replaced by a certain "transversality condition" in the infinite horizon case.

This is a thesis for the degree of Doctorate (PhD) in Mathematics: Probabilities and Statistics. It consists of two articles. It is organized as follows:

- In the first chapter, we give a short introduction to the jump processes.
- The second is devoted to study a stochastic control of jump diffusion processes in finite and infinite horizon.
- In the third and fourth chapters, we give the two papers obtained during this PhD.

Chapter 1

An introduction to jump processes

To model the sudden crashes in finance, it is natural to allow jumps in the model because this makes it more realistic. This models can be represented by Lévy processes which are used througout this work. This term (Lévy process) honours the work of the French mathematician Paul Lévy.

1.1 Lévy process

Definition 1.1.1 (Lévy process) *A process* $X = (X(t))_{t\geq 0}$ *defined on a probability space* $(\Omega, \mathcal{F}, \mathbb{P})$ *is said to be a Lévy process if it possesses the following properties:*

(i) The paths of X are $\mathbb{P}$-almost surely right continuous with left limits.

(ii) $\mathbb{P}(X(0) = 0) = 1$.

(iii) Stationary increments, i.e., for $0 \leqslant s \leqslant t$, $X(t) - X(s)$ has the same distribution as $X(t-s)$.

(iv) Independent increments, i.e., for $0 \leqslant s \leqslant t$, $X(t) - X(s)$ is independent of $X(u), u \leqslant s$.

Example 1.1.1 *The known examples are the standard Brownian motion and the Poisson process.*

Definition 1.1.2 (Brownian motion) *A stochastic process* $B = (B(t))_{t\geq 0}$ *on* $\mathbb{R}^d$ *is a Brownian motion if it is a Lévy process and if*

(i) *For all $t > 0$, has a Gaussian distribution with mean 0 and covariance matrix tId.*

(ii) *There is $\Omega_0 \in \mathcal{F}$ with $\mathbb{P}(\Omega_0) = 1$ such that, for every $\omega \in \Omega_0$, $B(t, \omega)$ is continuous in t.*

Definition 1.1.3 (Poisson process) *A stochastic process $N = (N(t))_{t \geq 0}$ on $\mathbb{R}$ such that*

$$\mathbb{P}\left[N(t) = n\right] = \frac{(\lambda t)^n}{n!} e^{-\lambda t}; \; n = 0, 1, ..$$

is a Poisson process with parameter $\lambda > 0$ if it is a Lévy process and for $t > 0$, $N(t)$ has a Poissson distribution with mean λt.

Remark 1.1.1

1. *Note that the properties of stationarity and independent increments imply that a Lévy process is a Markov process.*

2. *Thanks to almost sure right continuity of paths, one may show in addition that Lévy processes are also strong Markov processes.*

Any random variable can be characterized by its characteristic function. In the case of a Lévy process X, this characterization for all time t gives the Lévy-Khintchine formula and it is also called Lévy-Khintchine representation.

1.1.1 Lévy-Khintchine formula

The Lévy-Khintchine formula was obtained around 1930's by De Finetti and Kolmogorov in special cases where its simplest proof was given by Khintchine. It was later extended to more general cases by Lévy.

Theoreme 1.1.1 ***(The Lévy-Khintchine formula)***

1. *Let X be a Lévy process. Then its Fourier transform is*

$$E\left[e^{iuX(t)}\right] = e^{t\psi(u)}, \quad u \in \mathbb{R}(i = \sqrt{-1}), \tag{1.1}$$

with the characteristic exponent

$$\psi(u) := i\alpha u - \frac{1}{2}{\sigma_0}^2 u^2 + \int_{|z|<1} (e^{iuz} - 1 - iuz)\nu(dz) + \int_{|z|\geq 1} (e^{iuz} - 1)\nu(dz), \quad (1.2)$$

where the parameters $\alpha \in \mathbb{R}$ *and* ${\sigma_0}^2 \geqslant 0$ *are constants and* $\nu = \nu(dz)$, $z \in \mathbb{R}_0$, *is a* σ*-finite measure on* $\mathcal{B}(R_0)$ *satisfying*

$$\int_{\mathbb{R}_0} \min(1, z^2)\nu(dz) < \infty. \quad (1.3)$$

It follows that ν *is the Lévy measure of* X*.*

2. *Conversely, given the constants* $\alpha \in \mathbb{R}$, ${\sigma_0}^2 \geqslant 0$ *and the* σ*-finite measure* ν *on* $\mathcal{B}(\mathbb{R}_0)$ *such that* (1.3) *holds, the process* X *such that* (1.1)-(1.2) *is a Lévy process.*

Laplace transform

The Laplace transform can be obtained from the Fourier transform, as follows

$$E\left[e^{uX(t)}\right] = e^{t\psi(-iu)}, \quad u \in \mathbb{R}, \quad (1.4)$$

with

$$\psi(-iu) := \alpha u + \frac{1}{2}{\sigma_0}^2 u^2 + \int_{|z|<1} (e^{uz} - 1 - uz)\nu(dz) + \int_{|z|\geq 1} (e^{uz} - 1)\nu(dz). \quad (1.5)$$

1.2 Stochastic integral with respect to Lévy process

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a given probability space with the σ-algebra $(\mathcal{F}_t)_{t\geq 0}$ generated by the underline driven processes; Brownian motion $B(t)$ and an independent compensated Poisson random measure $\tilde{N}$, such that

$$\tilde{N}(dt, dz) := N(dt, dz) - \nu(dz)dt. \quad (1.6)$$

For any t, let $\tilde{N}(ds, dz)$, $z \in \mathbb{R}_0$, $s \leqslant t$, augmented for all the sets of P-zero probability.

For any $\mathcal{F}_t$-adapted stochastic process $\theta = \theta(t,z)$, $t \geqslant 0$, $z \in \mathbb{R}_0$ such that

$$E\left[\int_0^T \int_{\mathbb{R}_0} \theta^2(t,z)\nu(dz)dt\right] < \infty, \text{ for some } T > 0, \tag{1.7}$$

we can see that the process

$$M_n(t) := \int_0^t \int_{|z|\geq \frac{1}{n}} \theta(s,z)\tilde{N}(ds,dz), \quad 0 \leqslant t \leqslant T \tag{1.8}$$

is a martingale in $L^2(P)$ and its limit

$$M(t) := \lim_{n\to\infty} M_n(t) := \int_0^T \int_{\mathbb{R}_0} \theta(s,z)\tilde{N}(ds,dz), \quad 0 \leqslant t \leqslant T \tag{1.9}$$

in $L^2(P)$ is also a martingale. Moreover, we have the Itô isometry

$$E\left[\left(\int_0^T \int_{\mathbb{R}_0} \theta(s,z)\tilde{N}\left(ds,dz\right)\right)^2\right] = E\left[\left(\int_0^T \int_{\mathbb{R}_0} \theta^2(t,z)\nu(dz)dt\right)\right]. \tag{1.10}$$

Such processes can be expressed as the sum of two independent parts, a continuous part and a part expressible as a compensated sum of independent jumps. That is the Itô-Lévy decomposition.

Theoreme 1.2.1 (Itô-Lévy decomposion) *The Itô-Lévy decomposion for a Lévy process X is given by*

$$X(t) = b_0 t + \sigma_0 B(t) + \int_{|z|<1} z\tilde{N}(dt,dz) + \int_{|z|\geqslant 1} zN(dt,dz), \tag{1.11}$$

where $b_0, \sigma_0 \in \mathbb{R}$, $\tilde{N}(dt,dz)$ is the compensated Poisson random measure of $X(.)$ and $B(t)$ is an independent Brownian motion with the jump measure $N(dt,dz)$.

We assume that

$$E\left[X^2(t)\right] < \infty, \quad t \geqslant 0, \tag{1.12}$$

then

$$\int_{|z|\geq 1} |z|^2 \nu(dz) < \infty.$$

We can represent (1.11) as

$$X(t) = b_0^{'} t + \sigma_0 B(t) + \int_0^t \int_{\mathbb{R}_0} z \tilde{N}(ds, dz), \tag{1.13}$$

where $b_0^{'} = b_0 + \int_{|z|\geq 1} z\nu(dz)$. If $\sigma_0 = 0$, then a Lévy process is called a pure jump Lévy process.

Let us consider that the process $X(t)$ admits the stochastic integral representation as follows

$$X(t) = x + \int_0^t b(s)ds + \int_0^t \sigma(s)dB(s) + \int_0^t \int_{\mathbb{R}_0} \theta(s,z)\tilde{N}(ds, dz), \tag{1.14}$$

where $b(t)$, $\sigma(t)$, and $\theta(t,\cdot)$ are predictable processes such that, for all $t > 0, z \in \mathbb{R}_0$,

$$\int_0^t \left[|b(s)| + \sigma^2(s) + \int_{\mathbb{R}_0} \theta^2(s,z)\nu(dz) \right] ds < \infty \quad P - a.s. \tag{1.15}$$

Under this assumption, the stochastic integrals are well-defined and local martingales. If we strengthened the condition to

$$E\left[\int_0^t \left[|b(s)| + \sigma^2(s) + \int_{\mathbb{R}_0} \theta^2(s,z)\nu(dz) \right] ds \right] < \infty, \tag{1.16}$$

for all $t > 0$, then the corresponding stochastic integrals are martingales.

We call such a process an Itô–Lévy process. In analogy with the Brownian motion case, we use the short-hand differential notation

$$\begin{cases} dX(t) = b(t)dt + \sigma(t)dB(t) + \int_{\mathbb{R}_0} \theta(t,z)\tilde{N}(dt, dz), \\ X(0) = x \in \mathbb{R}. \end{cases} \tag{1.17}$$

The conditions satisfied by the coefficients to obtain existence and uniqueness of the solution of a SDEs with jumps, are given in the following theorem.

1.2.1 Stochastic differential equations driven by Lévy processes

By the Itô-Lévy decomposition, we can introduce the SDE for Lévy process.
For simplicity, we only consider the one dimentional case. The extension to several dimensions is straightforward.

Theoreme 1.2.2 (Existence and uniqueness) *Consider the following Lévy SDE in* $\mathbb{R}$:

$$\begin{cases} dX(t) = b(t, X(t))dt + \sigma(t, X(t))dB(t) + \int_{\mathbb{R}_0} \theta(t, X(t^-), z)\tilde{N}(dt, dz) \\ X(0) = x \in \mathbb{R}, \end{cases} \tag{1.18}$$

where

$$b : [0, T] \times \mathbb{R} \to \mathbb{R},$$
$$\sigma : [0, T] \times \mathbb{R} \to \mathbb{R},$$
$$\theta : [0, T] \times \mathbb{R} \times \mathbb{R}_0 \to \mathbb{R}.$$

We assume that the coefficients satisfy the following assumptions

1. *(At most linear growth) There exists a constant* $C_1 < \infty$ *such that*

$$\|\sigma(t, x)\|^2 + |b(t, x)|^2 + \int_{\mathbb{R}_0} |\theta(t, x, z)|^2 \nu d(z) \leq C_1(1 + |x|^2), \; x \in \mathbb{R}. \tag{1.19}$$

2. *(Lipschitz continuity) There exists a constant* $C_2 < \infty$ *such that*

$$\|\sigma(t, x) - \sigma(t, y)\|^2 + |b(t, x) - b(t, y)|^2 + \int_{\mathbb{R}_0} |\theta(t, x, z) - \theta(t, y, z)|^2 \nu d(z) \leq C_2(1 + |x - y|^2),$$

for all $x, y \in \mathbb{R}$.
Then there exists a unique càdlàg adapted solution $X(t)$ *such that* (1.12) *is satisfied.*

In the next chapters, we will study the use of those processes in the theory of optimal control.

Chapter 2

Stochastic control of jump diffusion processes

The fundamental problem of stochastic control is to solve the problem of minimizing a cost or maximizing a gain of the form

$$J(u) = E\left[\int_0^T f(t, X(t), u(t))dt + g(X(T))\right], \tag{2.1}$$

where $X(t)$ is a solution of a SDE (1.18) associated to an admissible control u and the terminal time is possibly infinite ($T \leq \infty$).

The two famous approach in solving control problems are the Bellman Dynamic Programming and Pontryagin's Maximum principles. The first method consists to find a solution of a stochastic partial differential equation (SPDE) which is not linear, verified by the value function. It is called Hamilton-Jacobi-Bellman (HJB) equation. We refer to [11] and [17] for more details about this method.

The second method which will be the center of our interest in this work which consists to find necessary conditions of optimality satisfied by an optimal control u^*.

The dynamic programming method works fine both in finite and infinite horizon, but if the system is not Markovian, then dynamic programming cannot be used, and one is left with the maximum principle approach. However, the classical formulation of this method is based on a finite horizon time T, because it is at this time T the terminal value of the BSDE is given. If the horizon is infinite, it is not clear how to compensate for this missing terminal condition?

2.1 Finite horizon

In the deterministic case, the maximum principle was introduced by Pontryagin & al [41] in the 1950's. Since then, a lot of works have been done for systems driven by Brownian motion such as Bismut [12], Kushner [22], Bensoussan [10] and Haussman [21].
Peng [38] derived a general stochastic maximum principle where the control domain is not necessarly convex and the diffusion coefficient can contain the control variable.
Mezerdi [28] generalized the principle of Kushner to the case of a SDE with non smooth drift.
It was also extended to systems with jumps by Tang & al [24], and later by Framstad & al [18].
We are interested in the study of the stochastic maximum principle with jumps for partial information where the consumer has less information than what's represented by the filtration generated by the underline driven processes. Let $u(t)$ be our control process required to have values in $\mathcal{U} \subset \mathbb{R}$, it is adapted to a given filtration $(\mathcal{E}_t)_{t\geq 0}$ such that $\mathcal{E}_t \subseteq \mathcal{F}_t$. For example, we can take $\mathcal{E}_t$ the delay information $\mathcal{F}_{(t-\gamma)^+}$ for $t \geq 0$, where $\gamma \geq 0$ is the given constant delay. However, there is an other case where the consumer has more information than what can be obtained by observing the driving processes such that $\mathcal{F}_t \subseteq \mathcal{E}_t$. This information is called inside information and the consumer is called an insider.
We study the stochastic maximum principle for jump diffusions which maximizes (2.1) with $T < \infty$

$$J(u) = E\left[\int_0^T f(t, X(t), u(t))dt + g(X(T))\right]. \tag{2.2}$$

The dynamic of the system satisfies the following controlled stochastic differential equation

$$\begin{cases} dX(t) = b(t, X(t), u(t))dt + \sigma(t, X(t))dB(t) + \displaystyle\int_{\mathbb{R}_0} \theta(t, X(t^-), u(t), z)\tilde{N}(dt, dz), \\ X(0) = x \in \mathbb{R}, \end{cases} \tag{2.3}$$

where

$$b : [0,T] \times \mathbb{R} \times \mathcal{U} \to \mathbb{R},$$
$$\sigma : [0,T] \times \mathbb{R} \times \mathcal{U} \to \mathbb{R},$$
$$\theta : [0,T] \times \mathbb{R} \times \mathcal{U} \times \mathbb{R}_0 \to \mathbb{R}.$$

We define the Hamiltonian $H : [0,T] \times \mathbb{R} \times \mathcal{U} \times \mathbb{R} \times \mathbb{R} \times \Re \times \Omega \to \mathbb{R};$ by

$$H(t,x,u,p,q,r(\cdot)) = f(t,x,u) + b(t,x,u)p + \sigma(t,x,u)q + \int_{\mathbb{R}_0} \theta(t,x,u,z)r(z)\nu(dz) \quad (2.4)$$

where $\Re$ is the set of the nice functions $r : \mathbb{R}_0 \to \mathbb{R}$ such that the corresponding integral in (2.4) converges. We assume that H is differentiable with respect to x and for all $u \in \mathcal{A}_{\mathcal{E}}$, then we can introduce the adjoint equation which is a BSDE as follows:

$$\begin{aligned} dp(t) = &-\frac{\partial H}{\partial x}(t, X(t), u(t), p(t), q(t), r(t,\cdot))dt + q(t)dB(t) \\ &+ \int_{\mathbb{R}_0} r(t,z)\tilde{N}(dt,dz); 0 \leqslant t \leqslant T, \end{aligned} \quad (2.5)$$

$$P(T) = g^{'}(X(T)). \quad (2.6)$$

To this problem, necessary and sufficient conditions of optimality were established by Baghery & Øksendal [5] in n dimention. For simplicity, we only consider the one dimentional case.

2.1.1 Sufficient conditions of optimality

We show that under some assumptions such as the concavity of H and g, maximizing the Hamiltonian leads to an optimal control.

Theoreme 2.1.1 (Sufficient maximum principle) *Let $\hat{u} \in \mathcal{A}_{\mathcal{E}}$ with corresponding state process $\hat{X}(t) = X^{(\hat{u})}(t)$ and suppose there exists a solution $(\hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot))$ of the corresponding adjoint equation* (2.5) – (2.6) ;

$$\begin{cases} d\hat{p}(t) = -\frac{\partial H}{\partial x}(t, \hat{X}(t), \hat{u}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot))dt + \hat{q}(t)dB(t) \\ \qquad\qquad + \int_{\mathbb{R}_0} \hat{r}(t,z)\tilde{N}(dt,dz); 0 \leqslant t \leqslant T, \\ \hat{p}(T) = g^{'}(\hat{X}(T)). \end{cases}$$

Suppose that the following growth conditions, hold

$$E\left[\int_0^T \left\{\hat{q}^2(t) + \int_{\mathbb{R}_0} \hat{r}^2(t,z)\nu(dz)\right\} \left(\hat{X}(t) - X(t)\right)^2 (t)dt\right] < \infty, \tag{2.7}$$

$$E\left[\int_0^T \left\{\sigma^2(t,\hat{X}(t),\hat{u}(t)) + \int_{\mathbb{R}_0} \theta^2(t,\hat{X}(t),\hat{u}(t),z)\nu(dz)\hat{p}^2(t)\right\} dt\right] < \infty, \tag{2.8}$$

and

$$E\left[\int_0^T \left|\frac{\partial H}{\partial u}(t,\hat{X}(t),\hat{u}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot))\right|^2\right] < \infty. \tag{2.9}$$

Moreover, assume that $H(t,x,u,\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot))$ *and* g *are concave with respect to* x, u *and* x *respectively.*

(Partial information maximum condition)

$$\begin{aligned} &E\left[\frac{\partial H}{\partial u}(t,\hat{X}(t),\hat{u}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot))|\mathcal{E}_t\right] \\ &= \max_{v\in\mathcal{U}} E\left[\frac{\partial H}{\partial u}(t,\hat{X}(t),v,\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot))|\mathcal{E}_t\right]. \end{aligned} \tag{2.10}$$

Then $\hat{u}$ *is an optimal control.*

Proof. Let $u \in \mathcal{A}_{\mathcal{E}}$. We can write $J(u) - J(\hat{u}) = I_1 + I_2$, where

$$I_1 = E\left[\int_0^T \left\{f(t,X(t),u(t)) - f(t,\hat{X}(t),\hat{u}(t))\right\} dt\right], \tag{2.11}$$

and

$$I_2 = E[g(X(T)) - g(\hat{X}(T))] \tag{2.12}$$

By the definition of H (2.4), we get

$$I_1 = I_{1,1} - I_{1,2} - I_{1,3} - I_{1,4},$$

with

$$I_{1,1} = E\left[\int_0^T \{H(t,X(t),u(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot)) - H(t,\hat{X}(t),\hat{u}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot))\}dt\right] \tag{2.13}$$

$$I_{1,2} = E\left[\int_0^T \{b(t,X(t),u(t)) - b(t,\hat{X}(t),\hat{u}(t))\}\hat{p}(t)dt\right] \tag{2.14}$$

$$I_{1,3} = E\left[\int_0^T \{\sigma(t,X(t),u(t)) - \sigma(t,\hat{X}(t),\hat{u}(t))\}\hat{q}(t)dt\right] \tag{2.15}$$

$$I_{1,4} = E\left[\int_0^T \int_{\mathbb{R}_0} \{\theta(t,X(t),u(t),z) - \theta(t,\hat{X}(t),\hat{u}(t),z)\}\hat{r}(t,z)\nu(dz)dt\right]. \tag{2.16}$$

Since H is concave, we have

$$\begin{aligned} &H(t,X(t),u(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot)) - H(t,\hat{X}(t),\hat{u}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot)) \\ &\le \frac{\partial H}{\partial x}(t,\hat{X}(t),\hat{u}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot))(X(t)-\hat{X}(t)) \\ &+ \frac{\partial H}{\partial u}(t,\hat{X}(t),\hat{u}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot))(u(t)-\hat{u}(t)). \end{aligned} \tag{2.17}$$

Using that H is maximal for $u = \hat{u}(t)$, and (2.9) , we get

$$\begin{aligned} 0 &\ge E\left[\frac{\partial H}{\partial u}(t,\hat{X}(t),u,\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot))|\mathcal{E}_t\right]_{u=\hat{u}(t)} (u(t)-\hat{u}(t)) \\ &= E\left[\frac{\partial H}{\partial u}(t,\hat{X}(t),\hat{u}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot))(u(t)-\hat{u}(t))|\mathcal{E}_t\right]. \end{aligned} \tag{2.18}$$

Combining (2.5), (2.7), (2.13), (2.17)and (2.18), we obtain

$$\begin{aligned} I_{1,1} &\le E\left[\int_0^T \frac{\partial H}{\partial x}(t,\hat{X}(t),\hat{u}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot))(X(t)-\hat{X}(t))ds\right] \\ &= -E\left[\int_0^T (X(t)-\hat{X}(t))d\hat{p}(t)\right] = -J_1. \end{aligned}$$

Using (2.6) and the fact that g is concave together with the Itô formula, gives

$$\begin{aligned}
I_2 &= E[g(X(T)) - g(\hat{X}(T))] \leq E[g^{'}(\hat{X}(T))(X(T) - \hat{X}(T))] \\
&= E[(X(T) - \hat{X}(T))\hat{p}(T)] \\
&= E\Bigg[\int_0^T (X(t) - \hat{X}(t))(-\frac{\partial H}{\partial x}(t, \hat{X}(t), \hat{u}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot)))dt \\
&+ \int_0^T \{b(t, X(t), u(t)) - b(t, \hat{X}(t), \hat{u}(t))\}\hat{p}(t)dt \\
&+ \int_0^T \{\sigma(t, X(t), u(t)) - \sigma(t, \hat{X}(t), \hat{u}(t))\}\hat{q}(t)dt \\
&+ \int_0^T \int_{\mathbb{R}_0} \{\theta(t, X(t), u(t), z) - \theta(t, \hat{X}(t), \hat{u}(t), z)\}\hat{r}(t,z)\nu(dz)dt\Bigg] \\
&= J_1 + I_{1,2} + I_{1,3} + I_{1,4}.
\end{aligned}$$

We conclude that

$$\begin{aligned}
J(u) - J(\hat{u}) &= I_1 + I_2 = I_{1,1} - I_{1,2} - I_{1,3} - I_{1,4} + I_2 \\
&\leq -J_1 - I_{1,2} - I_{1,3} - I_{1,4} + J_1 + I_{1,2} + I_{1,3} + I_{1,4} = 0.
\end{aligned}$$

Then $\hat{u}$ is an optimal control. ■

2.1.2 Necessary conditions of optimality

A drawback of the sufficient maximum principle is the concavity condition which does not always hold in applications. We prove now a result going in the other direction. We assume the folowing:

(A1) For all t_0, h such that $0 \leq t_0 < t_0 + h \leq T$, for all bounded $\mathcal{E}_{t_0}$-measurable random variable α, the control $\beta(t)$ defined by

$$\beta(t) := \alpha \mathbf{1}_{[t_0, t_0+h]}(t), \tag{2.19}$$

belongs to $\mathcal{A}_{\mathcal{E}}$.

(A2) For all $u, \beta \in \mathcal{A}_{\mathcal{E}}$ with β bounded, there exists $\delta > 0$ such that $\hat{u} + s\beta \in \mathcal{A}_{\mathcal{E}}$ for all $s \in (-\delta, \delta)$.

We define the derivative process $\xi(t) = \xi^{(u,\beta)}(t)$ by

$$\xi(t) = \frac{d}{ds} X^{(u+s\beta)}(t)|_{s=0}. \tag{2.20}$$

Note that

$$\xi(0) = 0.$$

Theoreme 2.1.2 (Necessary maximum principle) *Suppose that $\hat{u} \in \mathcal{A}_{\mathcal{E}}$ is a local maximum for $J(u)$, meaning that for all bounded $\beta \in \mathcal{A}_{\mathcal{E}}$, there exists $\delta > 0$ such that $\hat{u} + s\beta \in \mathcal{A}_{\mathcal{E}}$ for all $s \in (-\delta, \delta)$ and that*

$$\frac{d}{ds} J(\hat{u} + s\beta)|_{s=0} = 0. \tag{2.21}$$

Suppose there exists a solution $(\hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot))$ of (2.5) – (2.6).
Moreover, suppose that the following growth conditions, hold

$$E\left[\int_0^T \left\{ \hat{q}^2(t) + \int_{\mathbb{R}_0} \hat{r}^2(t,z)\nu(dz) \right\} \hat{\xi}^2(t)dt \right] < \infty, \tag{2.22}$$

$$\begin{aligned} &E\left[\int_0^T \left\{ \left(\frac{\partial \sigma}{\partial x}\right)^2 (t, \hat{X}(t), \hat{u}(t)) + \int_{\mathbb{R}_0} \left(\frac{\partial \theta}{\partial x}\right)^2 (t, \hat{X}(t), \hat{u}(t), z)\nu(dz) \right.\right. \\ &\left.\left. \left(\frac{\partial \sigma}{\partial u}\right)^2 (t, \hat{X}(t), \hat{u}(t)) + \int_{\mathbb{R}_0} \left(\frac{\partial \theta}{\partial u}\right)^2 (t, \hat{X}(t), \hat{u}(t), z)\nu(dz) \right\} \hat{p}^2(t)dt \right] < \infty. \end{aligned} \tag{2.23}$$

Then $\hat{u}$ is a stationary point for $E[H|\mathcal{E}_t]$ in the sense that for a.a.$t \in [0,T]$ we have

$$E\left[\frac{\partial H}{\partial u}(t, \hat{X}(t), \hat{u}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot))|\mathcal{E}_t\right] = 0. \tag{2.24}$$

Proof. Put $\hat{X}(t) = X^{(\hat{u})}(t)$. Using (2.20) , we have

$$\begin{aligned} 0 &= \frac{d}{ds} J(\hat{u} + s\beta)|_{s=0} \\ &= E\left[\int_0^T \left\{ \frac{\partial f}{\partial x}(t, \hat{X}(t), \hat{u}(t)) \frac{d}{ds}(X^{\hat{u}+s\beta}(t))|_{s=0} \right.\right. \\ &\left.\left. + \frac{\partial f}{\partial u}(t, \hat{X}(t), \hat{u}(t))\beta(t) + g'(\hat{X}(T)) \frac{d}{ds}(X^{\hat{u}+s\beta}(T))|_{s=0} \right\} dt \right] \\ &= E\left[\int_0^T \left\{ \frac{\partial f}{\partial x}(t, \hat{X}(t), \hat{u}(t))\hat{\xi}(t) + \frac{\partial f}{\partial u}(t, \hat{X}(t), \hat{u}(t))\beta(t) + g'(\hat{X}(T))\hat{\xi}(T) \right\} dt \right]. \end{aligned} \tag{2.25}$$

By (2.6),(2.22),(2.23), and apply Itô's formula to $g(\hat{X}(t))\hat{\xi}(t)$, we have

$$\begin{aligned}
&E\left[g'(\hat{X}(T))\hat{\xi}(T)\right] = E\left[\hat{p}(T)\hat{\xi}(T)\right] \\
&= E\left[\int_0^T \left\{\hat{p}(t)\left(\frac{\partial b}{\partial x}(t,\hat{X}(t),\hat{u}(t))\hat{\xi}(t) + \frac{\partial b}{\partial u}(t,\hat{X}(t),\hat{u}(t))\beta(t)\right)\right.\right. \\
&+\hat{\xi}(t)(-\frac{\partial H}{\partial x}(t,\hat{X}(t),\hat{u}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot))) + \hat{q}(t)\left(\frac{\partial \sigma}{\partial x}(t,\hat{X}(t),\hat{u}(t))\hat{\xi}(t)\right. \\
&+\left.\frac{\partial \sigma}{\partial u}(t,\hat{X}(t),\hat{u}(t))\beta(t)\right) + \int_{\mathbb{R}_0}\hat{r}(t,z)\left(\frac{\partial \theta}{\partial x}(t,\hat{X}(t),\hat{u}(t))\hat{\xi}(t)\right. \\
&\left.\left.\left.+\frac{\partial \theta}{\partial u}(t,\hat{X}(t),\hat{u}(t),z)\beta(t)\right)\nu(dz)\right\}dt\right].
\end{aligned} \tag{2.26}$$

We have that

$$\frac{\partial H}{\partial x}(t,x,u,p,q,r) = \frac{\partial f}{\partial x}(t,x,u) + \frac{\partial b}{\partial x}(t,x,u)p + \frac{\partial \sigma}{\partial x}(t,x,u)q$$

$$+\int_{\mathbb{R}_0}\frac{\partial \theta}{\partial x}(t,x,u,z)r(t,z)\nu(dz),$$

and similarly for $\frac{\partial H}{\partial u}(t,x,p,q,r(\cdot))$.
Combining with (2.23) and (2.25), gives

$$\begin{aligned}
0 &= E\left[\int_0^T \left\{\frac{\partial f}{\partial u}(t,\hat{X}(t),\hat{u}(t)) + \hat{p}(t)\frac{\partial f}{\partial u}(t,\hat{X}(t),\hat{u}(t))\right.\right. \\
&\left.\left.+\hat{q}(t)\frac{\partial f}{\partial u}(t,\hat{X}(t),\hat{u}(t))\int_{\mathbb{R}_0}\hat{r}(t,z)\frac{\partial \theta}{\partial u}(t,\hat{X}(t),\hat{u}(t),z)\nu(dz)\right\}\beta(t)dt\right] \\
&= E\left[\int_0^T \frac{\partial H}{\partial u}(t,\hat{X}(t),\hat{u}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot))\beta(t)dt\right].
\end{aligned} \tag{2.27}$$

We apply the above to

$$\beta(s) := \alpha\mathbf{1}_{[t_0,t_0+h]}(s); \qquad s \in [0,T],$$

where $t_0 + h \leqslant T$ and α is bounded, $\mathcal{E}_{t_0}$-measurable. Then (2.27) leads to

$$E\left[\int_{t_0}^{t_0+h}\frac{\partial H}{\partial u}(s,\hat{X}(s),\hat{u}(s),\hat{p}(s),\hat{q}(s),\hat{r}(s,\cdot))\alpha ds\right] = 0.$$

Differentiating with respect to h at $h = 0$ gives

$$E\left[\frac{\partial H}{\partial u}(t, \hat{X}(t), \hat{u}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot))\alpha\right] = 0.$$

Since this holds for all bounded $\mathcal{E}_{t_0}$-measurable α. Using (2.9) ,we conclude that

$$E\left[\frac{\partial H}{\partial u}(t, \hat{X}(t), \hat{u}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot))|\mathcal{E}_{t_0}\right] = 0.$$

The result follows. ■

2.1.3 Infinite horizon

Conversly to the finite horizon case, there are few papers dealing with the infinite horizon stochastic optimal control problems. Haadem & al [19], introduced a maximum principle for infinite horizon jump diffusion processes for partial information. They proved necessary and sufficient maximum principles for this problem. The results obtained are applied to several problems which appear in finance.
However, Maslowski and Veverka [26] establish a sufficient stochastic maximum principle for infinite horizon discounted control problem. As an application, they study the controlled stochastic logistic equation of population dynamics.

In our papers, we show that in the infinite horizon case the missing terminal value should be replaced by an appropriate transversality condition at infinity, and we obtain corresponding infinite horizon sufficient and necessary versions of the maximum principle. We also extend the results to systems with delay. The results are illustrated by applications, e.g. to infinite horizon optimal consumption with respect to recursive utility.

2.2 Summary of the papers

This thesis contains two papers dealing with the theory of stochastic optimal control problems in infinite horizon, organized as follows:

2.2.1 Maximum principle for infinite horizon delay equations

- First we formulate the problem.

- Then we establish first and second sufficient maximum principles with an application to the optimal consumption rate from an economic quantity described by a stochastic delay equation.
- Next we formulate a necessary maximum principle and we prove an existence and uniqueness of the advanced backward stochastic differential equations on infinite horizon with jumps.

2.2.2 Infinite horizon optimal control of FBSDEs with delay

- We proceed to study infinite horizon control of forward-backward SDEs where we obtain sufficient and necessary maximum principles for this problem.
- Finally, we apply the result to a recursive utility optimisation problem.

Chapter 3

Maximum principle for infinite horizon delay equations

We prove a maximum principle of optimal control of stochastic delay equations on infinite horizon. We establish first and second sufficient stochastic maximum principles as well as necessary conditions for that problem. We illustrate our results with an application to the optimal consumption rate from an economic quantity.

3.1 Introduction

To solve the stochastic control problems, there are two approaches: the dynamic programming method (HJB equation) and the maximum principle.
In this paper, our system is governed by the stochastic differential delay equation (SDDE):

$$\begin{cases} dX(t) = b\left(t, X(t), Y(t), A(t), u(t)\right) dt \\ +\sigma\left(t, X(t), Y(t), A(t), u(t)\right) dB(t) \\ + \int_{\mathbb{R}_0} \theta\left(t, X(t), Y(t), A(t), u(t), z\right) \tilde{N}(dt, dz), & t \in [0, \infty)\,, \\ X(t) = X_0(t), & t \in [-\delta, 0], \\ Y(t) = X(t-\delta), & t \in [0, \infty), \\ A(t) = \int_{t-\delta}^{t} e^{-\lambda(t-r)} X(r) dr, & t \in [0, \infty), \end{cases} \tag{3.1}$$

with a corresponding performance functional;

$$J(u) = E\left[\int_0^\infty f\left(t, X(t), Y(t), A(t), u(t)\right)\ dt\right], \tag{1.2}$$

where $u(t)$ is the control process.

The SDDE is not Markovian so we cannot use the dynamic programming method. However, we will prove stochastic maximum principles for this problem. A sufficient maximum principle in infinite horizon with the trivial transversality conditions were treated by Haadem & al [19]. The natural transversality condition in the infinite case would be a zero limit condition, meaning in the economic sense that one more unit of good at the limit gives no additional value. But this property is not necessarily verified. In fact Halkin [20] provides a counterexample for a natural extension of the finite horizon transversality conditions. Thus some care is needed in the infinite horizon case. For the case of the natural transversality condition the discounted control problem was studied by Maslowski and Veverka [26].

In real life, delay occurs everywhere in our society. For example this is the case in biology where the population growth depends not only on the current population size but also on the size some time ago. The same situation may occur in many economic growth models. The stochastic maximum principle with delay has been studied by many authors. For example, Elsanosi & al [16] proved a verification theorem of variational inequality, Øksendal and Sulem [31] established the sufficient maximum principle for a certain class of stochastic control systems with delay in the state variable. In Haadem & al [19] an inifinite horizon system is studied, but without delay. In Chen and Wu [13], a finite horizon version of a stochastic maximum principle for a system with delay in both the state variable and the control variable is derived. In Øksendal & al [33] a maximum principle for systems with delay is studied in the finite horizon case. However, to our knowledge, no one has studied the infinite horizon case for delay equations.

For backward differential equations see Situ [43] and Li and Peng [23]. For the infinite horizon backward SDE see Peng and Shi [40], Pardoux [36], Yin [45], Barles & al [8] and Royer [42].

For more details about jump diffusion markets see Øksendal and Sulem [35] and for background and details about stochastic fractional delay equations see Mohammed and Scheutzow [30].

In this work, we establish two sufficient maximum principles and one necessary for the stochastic delay systems on infinite horizon with jumps.

Our paper is organized as follows. In the second section, we formulate the problem. The third section is devoted to the first and second sufficient maximum principles with an application to the optimal consumption rate from an economic quantity described by a stochastic delay equation. In the fourth section, we formulate a necessary maximum principle and we prove an existence and uniqueness of the advanced BSDEs on infinite horizon with jumps in the last section.

3.2 Formulation of the problem

Let $(\Omega, \mathcal{F}, P)$ be a probability space with filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions, on which an $\mathbb{R}$-valued standard Brownian motion $B(\cdot)$ and an independent compensated Poisson random measure $\tilde{N}(dt,dz) = N(dt,dz) - \nu(dz)dt$ are defined.

We consider the following stochastic control system with delay

$$\begin{cases} dX(t) & = b\left(t, X(t), Y(t), A(t), u(t)\right) dt \\ & +\sigma\left(t, X(t), Y(t), A(t), u(t)\right) dB(t) \\ & +\int_{\mathbb{R}_0} \theta\left(t, X(t), Y(t), A(t), u(t), z\right) \tilde{N}(dt,dz);\ t \in [0,\infty), \\ X(t) & = X_0(t),\ t \in [-\delta, 0], \\ Y(t) & = X(t-\delta),\ t \in [0,\infty), \\ A(t) & = \int_{t-\delta}^{t} e^{-\lambda(t-r)} X(r) dr,\ t \in [0,\infty), \end{cases} \tag{2.1}$$

where $X_0(t)$ is a given continuous (deterministic) function, and

$$\begin{aligned}
&\delta > 0,\ \lambda > 0 \text{ are given constants,}\\
&b : [0,\infty) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{U} \times \Omega \to \mathbb{R},\\
&\sigma : [0,\infty) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{U} \times \Omega \to \mathbb{R},\\
&\theta : [0,\infty) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{U} \times \mathbb{R}_0 \times \Omega \to \mathbb{R}
\end{aligned}$$

are given continuous functions such that for all t, $b(t,x,y,a,u)$, $\sigma(t,x,y,a,u)$, and $\theta(t,x,y,a,u,z)$ are $\mathcal{F}_t$-measurable for all $x \in \mathbb{R}$, $y \in \mathbb{R}$, $a \in \mathbb{R}$, $u \in \mathcal{U}$, and $z \in \mathbb{R}_0 := \mathbb{R} \setminus \{0\}$. We assume that b, σ, θ are C^1 (i.e., continuously differentiable/ Fréchet differentiable) with respect to x, y, a, u, and z for all t and a.a.ω. Let $\mathcal{E}_t \subset \mathcal{F}_t$ be a given subfiltration, representing the information available to the controller at time t. Let $\mathcal{U}$ be a nonempty subset of $\mathbb{R}$. We let $\mathcal{A}_{\mathcal{E}}$ denote a given family of admissible $\mathcal{E}_t$-adapted control processes. An element of $\mathcal{A}_{\mathcal{E}}$ is called an admissible control. The corresponding performance functional is

$$J(u) = E\left[\int_0^\infty f\left(t, X(t), Y(t), A(t), u(t)\right)\ dt\right], u \in \mathcal{A}_{\mathcal{E}}, \tag{2.2}$$

where we assume that

$$E\int_0^\infty \left\{\left|f\left(t, X(t), Y(t), A(t), u(t)\right)\right| + \left|\frac{\partial f}{\partial x_i}\left(t, X(t), Y(t), A(t), u(t)\right)\right|^2\right\} dt < \infty. \tag{2.3}$$

We also assume that f is C^1 with respect to x, y, a, u for all t and a.a. ω. The value function Φ is defined as

$$\Phi(X_0) = \sup_{u \in \mathcal{A}_{\mathcal{E}}} J(u). \tag{2.4}$$

An admissible control u^* is called an optimal control for (2.1) if it attains the maximum of $J(u)$ over $\mathcal{A}_{\mathcal{E}}$. (2.1) is called the state equation, and the solution $X^*(t)$ corresponding to u^* is called an optimal trajectory.

3.3 A sufficient maximum principle

In this section our objective is to establish a sufficient maximum principle.

3.3.1 Hamiltonian and time-advanced BSDEs for adjoint equations

We now introduce the adjoint equations and the Hamiltonian function for our problem. The Hamiltonian is defined by

$$H(t,x,y,a,u,p,q,r(\cdot)) = f(t,x,y,a,u) + b(t,x,y,a,u)p + \sigma(t,x,y,a,u)q + \int_{\mathbb{R}_0} \theta(t,x,y,a,u,z)r(z)\nu(dz), \tag{3.1}$$

where

$$H : [0,\infty) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{U} \times \mathbb{R} \times \mathbb{R} \times \Re \times \Omega \to \mathbb{R}$$

and $\Re$ is the set of functions r: $\mathbb{R}_0 \to \mathbb{R}$ such that the integral term in (3.1) converges and $\mathcal{U}$ is the set of possible control values.

We suppose that b, σ, and θ are C^1 functions with respect to (x,y,a,u) and that

$$E\left[\int_0^\infty \left\{ \left|\frac{\partial b}{\partial x_i}(t,X(t),Y(t),A(t),u(t))\right|^2 + \left|\frac{\partial \sigma}{\partial x_i}(t,X(t),Y(t),A(t),u(t))\right|^2 + \int_{\mathbb{R}_0} \left|\frac{\partial \theta}{\partial x_i}(t,X(t),Y(t),A(t),u(t))\right|^2 \nu(dz) \right\} dt \right] < \infty \tag{3.2}$$

for $x_i = x$, y, a, and u.

The adjoint processes $(p(t),q(t),r(t,\cdot))$, $t \in [0,\infty)$, $z \in \mathbb{R}_0$, are assumed to satisfy the equation

$$dp(t) = E\left[\mu(t) \mid \mathcal{F}_t\right] dt + q(t)dB(t) + \int_{\mathbb{R}_0} r(t,z)\tilde{N}(dt,dz), t \in [0,\infty), \tag{3.3}$$

where

$$\mu(t) = -\frac{\partial H}{\partial x}(t, X(t), Y(t), A(t), u(t), p(t), q(t), r(t,\cdot))$$
$$-\frac{\partial H}{\partial y}(t+\delta, X(t+\delta), Y(t+\delta), A(t+\delta), u(t+\delta), p(t+\delta), q(t+\delta), r(t+\delta,\cdot))$$
$$-e^{\lambda t}\left(\int_t^{t+\delta} \frac{\partial H}{\partial a}(s, X(s), Y(s), A(s), u(s), p(s), q(s), r(s,\cdot))\, e^{-\lambda s} ds\right). \tag{3.4}$$

Remark 3.3.1 *Note that we do not require a priori that the solution of (3.3)-(3.4) is unique.*

The following result is an infinite horizon version of Theorem 3.1 in [33].

3.3.2 A first sufficient maximum principle

Theoreme 3.3.1 *Let $\hat{u} \in \mathcal{A}_{\mathcal{E}}$ with corresponding state processes $\hat{X}(t)$, $\hat{Y}(t)$ and $\hat{A}(t)$ and adjoint processes $\hat{p}(t)$, $\hat{q}(t)$ and $\hat{r}(t,\cdot)$ assumed to satisfy the advanced BSDE (ABSDE)* (3.3)- (3.4)*. Suppose that the following assertions hold:*
(i)

$$\overline{\lim_{T\to\infty}} E\left[\hat{p}(T)(X(T) - \hat{X}(T))\right] \geq 0 \tag{3.5}$$

for all $u \in \mathcal{A}_{\mathcal{E}}$ with corresponding solution $X(t)$.

(ii) The function

$$(x, y, a, u) \to H(t, x, y, a, u, \hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot))$$

is concave for each $t \in [0, \infty)$ a.s.
(iii)

$$E\left[\int_0^T \left\{\hat{q}^2(t)\left(\sigma(t) - \hat{\sigma}(t)\right)^2 + \int_{\mathbb{R}} \hat{r}^2(t,z)(\theta(t,z) - \hat{\theta}(t,z))^2 \nu(dz)\right\} dt\right] < \infty \tag{3.6}$$

for all $T < \infty$.

(iiii)

$$\max_{v\in\mathcal{U}} E\left[H\left(t,\hat{X}(t),\hat{X}(t-\delta),\hat{A}(t),v,\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot)\right)\Big|\mathcal{E}_t\right]$$
$$= E\left[H\left(t,\hat{X}(t),\hat{X}(t-\delta),\hat{A}(t),\hat{u}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot)\right)\Big|\mathcal{E}_t\right]$$

for all $t\in[0,\infty)$ *a.s.*

Then $\hat{u}$ *is an optimal control for the problem* (2.4).

Proof. Choose an arbitrary $u\in\mathcal{A}_{\mathcal{E}}$, and consider

$$J(u)-J(\hat{u})=I_1, \tag{3.7}$$

where

$$I_1 = E\left[\int_0^\infty \left\{f\big(t,X(t),Y(t),A(t),u(t)\big)-f\big(t,\hat{X}(t),\hat{Y}(t),\hat{A}(t),\hat{u}(t)\big)\right\}dt\right]. \tag{3.8}$$

By the definition (3.1) of H and the concavity, we have

$$\begin{aligned} I_1 \le E\Bigg[\int_0^\infty \Bigg\{ &\frac{\partial H}{\partial x}(t)(X(t)-\hat{X}(t))+\frac{\partial H}{\partial y}(t)(Y(t)-\hat{Y}(t))+\frac{\partial H}{\partial a}(t)(A(t)-\hat{A}(t)) \\ &+\frac{\partial H}{\partial u}(t)(u(t)-\hat{u}(t))-(b(t)-\hat{b}(t))\hat{p}(t)-(\sigma(t)-\hat{\sigma}(t))\hat{q}(t) \\ &-\int_{\mathbb{R}_0}(\theta(t,z)-\hat{\theta}(t,z))\hat{r}(t,z)\nu(dz)\Bigg\}dt\Bigg], \end{aligned} \tag{3.9}$$

where we have used the simplified notation

$$\frac{\partial H}{\partial x}(t)=\frac{\partial H}{\partial x}\left(t,\hat{X}(t),\hat{Y}(t),\hat{A}(t),\hat{u}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot)\right),$$

and similarly for b and σ.

Applying the Itô formula to $\hat{p}(t)(X(t)-\hat{X}(t))$ we get, by (3.5) and (3.6),

$$
\begin{aligned}
0 &\leq \overline{\lim_{T\to\infty}} E[\ \hat{p}(T)(X(T)-\hat{X}(T))] \\
&= \overline{\lim_{T\to\infty}} E\Bigg[\ \Bigg(\int_0^T (b(t)-\hat{b}(t))\hat{p}(t)dt + \int_0^T (X(t)-\hat{X}(t))E\left[\hat{\mu}(t)\mid \mathcal{F}_t\right]dt \\
&+ \int_0^T (\sigma(t)-\hat{\sigma}(t))\hat{q}(t)dt + \int_0^T\int_{\mathbb{R}_0} (\theta(t,z)-\hat{\theta}(t,z))\hat{r}(t,z)\nu(dz)dt\Bigg)\Bigg] \\
&= \overline{\lim_{T\to\infty}} E\Bigg[\ \Bigg(\int_0^T (b(t)-\hat{b}(t))\hat{p}(t)dt + \int_0^T (X(t)-\hat{X}(t))\hat{\mu}(t)dt \\
&+ \int_0^T (\sigma(t)-\hat{\sigma}(t))\hat{q}(t)dt + \int_0^T\int_{\mathbb{R}_0} (\theta(t,z)-\hat{\theta}(t,z))\hat{r}(t,z)\nu(dz)dt\Bigg)\Bigg].
\end{aligned}
\tag{3.10}
$$

Using the definition (3.4) of μ we see that

$$
\begin{aligned}
&\overline{\lim_{T\to\infty}} E\Bigg[\ \Bigg(\int_0^T (X(t)-\hat{X}(t))\hat{\mu}(t)dt\Bigg)\Bigg] \\
&= \overline{\lim_{T\to\infty}} E\Bigg[\ \Bigg(\int_\delta^{T+\delta} (X(t-\delta)-\hat{X}(t-\delta))\hat{\mu}(t-\delta)dt\Bigg)\Bigg] \\
&= \overline{\lim_{T\to\infty}} E\Bigg[\Bigg(-\int_\delta^{T+\delta} \frac{\partial H}{\partial x}(t-\delta)(X(t-\delta)-\hat{X}(t-\delta))dt \\
&- \int_\delta^{T+\delta} \frac{\partial H}{\partial y}(t)\left(Y(t)-\hat{Y}(t)\right)dt \\
&- \int_\delta^{T+\delta} \left(\int_{t-\delta}^{t} \frac{\partial H}{\partial a}(s)e^{-\lambda s}ds\right) e^{\lambda(t-\delta)}\left(X(t-\delta)-\hat{X}(t-\delta)\right)\Bigg)dt\Bigg].
\end{aligned}
\tag{3.11}
$$

Using Fubini and substituting $r = s - \delta$, we obtain

$$
\begin{aligned}
&\int_0^T \frac{\partial H}{\partial a}(s)(A(s) - \hat{A}(s))ds \\
&= \int_0^T \frac{\partial H}{\partial a}(s) \int_{s-\delta}^{s} e^{-\lambda(s-r)}(X(r) - \hat{X}(r))dr\ ds \\
&= \int_0^T \left(\int_r^{r+\delta} \frac{\partial H}{\partial a}(s) e^{-\lambda s} ds \right) e^{\lambda r}(X(r) - \hat{X}(r))\ dr \\
&= \int_0^{T+\delta} \left(\int_r^{r+\delta} \frac{\partial H}{\partial a}(s) e^{-\lambda s} ds \right) e^{\lambda(t-\delta)}(X(t-\delta) - \hat{X}(t-\delta))dt. \qquad (3.12)
\end{aligned}
$$

Combining (3.10), (3.11) and (3.12) we get

$$
\begin{aligned}
0 \leq \overline{\lim_{T\to\infty}} E\Big[\ \hat{p}(T)(X(T) - \hat{X}(T))\Big] = E\Bigg[\Bigg(&\int_0^\infty (b(t) - \hat{b}(t))\hat{p}(t)dt \\
&- \int_0^\infty \frac{\partial H}{\partial x}(t)\,(X(t) - \hat{X}(t))dt - \int_\delta^\infty \frac{\partial H}{\partial y}(t)\,(Y(t) - \hat{Y}(t))dt \\
&- \int_\delta^\infty \frac{\partial H}{\partial a}(t)\,(A(t) - \hat{A}(t))dt + \int_0^\infty (\sigma(t) - \hat{\sigma}(t))\hat{q}(t)dt \\
&+ \int_0^\infty \int_{\mathbb{R}_0} (\theta(t,z) - \hat{\theta}(t,z))\hat{r}(t,z)\nu(dz)dt \Bigg)\Bigg]. \qquad (3.13)
\end{aligned}
$$

Subtracting and adding $\int\limits_0^\infty \frac{\partial H}{\partial u}(t)(u(t)-\hat{u}(t))dt$ in (3.12) we conclude

$$
\begin{aligned}
0 \le \overline{\lim_{T\to\infty}} E\Big[\;\hat{p}(T)(X(T)-\hat{X}(T))\Big] = E\;\Bigg[\Bigg(&\int\limits_0^\infty (b(t)-\hat{b}(t))\hat{p}(t)dt \\
-\int\limits_0^\infty \frac{\partial H}{\partial x}(t)\,(X(t)-\hat{X}(t))dt &- \int\limits_\delta^\infty \frac{\partial H}{\partial y}(t)\,(Y(t)-\hat{Y}(t))dt \\
-\int\limits_\delta^\infty \frac{\partial H}{\partial a}(t)\,(A(t)-\hat{A}(t))dt &+ \int\limits_0^\infty (\sigma(t)-\hat{\sigma}(t))\hat{q}(t)dt \\
+\int\limits_0^\infty \int\limits_{\mathbb{R}_0} (\theta(t,z)-\hat{\theta}(t,z))\hat{r}(t,z)\nu(dz)dt& \\
-\int\limits_0^\infty \frac{\partial H}{\partial u}(t)(u(t)-\hat{u}(t))dt &+ \int\limits_0^\infty \frac{\partial H}{\partial u}(t)(u(t)-\hat{u}(t))dt\Bigg)\Bigg] \\
\le -I_1 + E\left[\int\limits_0^\infty E\left[\frac{\partial H}{\partial u}(t)(u(t)-\hat{u}(t)) \mid \mathcal{E}_t\right] dt\right]&.
\end{aligned}
$$

Hence

$$
I_1 \le E\left[\int\limits_0^\infty E\left[\frac{\partial H}{\partial u}(t) \mid \mathcal{E}_t\right](u(t)-\hat{u}(t))dt\right] \le 0.
$$

Since $u \in \mathcal{A}_{\mathcal{E}}$ was arbitrary, this proves Theorem 3.1. ■

3.3.3 A second sufficient maximum principle

We extend the result in Øksendal and Sulem [31] to infinite horizon with jump diffusions.

Consider again the system

$$\begin{cases} dX(t) = b\left(t, X(t), Y(t), A(t), u(t)\right) dt \\ +\sigma\left(t, X(t), Y(t), A(t), u(t)\right) dB(t) \\ + \int_{\mathbb{R}_0} \theta\left(t, X(t), Y(t), A(t), u(t), z\right) \tilde{N}(dt, dz), & t \in [0, \infty), \\ X(t) = X_0(t), & t \in [-\delta, 0], \\ Y(t) = X(t-\delta), & t \in [0, \infty), \\ A(t) = \int_{t-\delta}^{t} e^{-\lambda(t-r)} X(r) dr, & t \in [0, \infty). \end{cases}$$

We now give an Itô formula which is proved in [16] without jumps. Adding the jump parts is just an easy observation.

Lemma 3.3.1 (the Itô formula for delayed system) *Consider a function*

$$G(t) = F(t, X(t), A(t)), \tag{3.14}$$

where F is a function in $C^{1,2,1}(\mathbb{R}^3)$. Note that

$$A(t) = \int_{-\delta}^{0} e^{\lambda s} X(t+s) ds.$$

Then

$$\begin{aligned} dG(t) = {} & (LF)(t, X(t), Y(t), A(t), u(t))dt \\ & + \sigma(t, X(t), Y(t), A(t), u(t)) \frac{\partial F}{\partial x}(t, X(t), A(t)) dB(t) \\ & + \int_{\mathbb{R}_0} \Big\{ F(t, X(t^-) + \theta(t, X(t), Y(t), A(t), u(t), z), A(t^-)) \\ & - F(t, X(t^-), A(t^-)) \end{aligned} \tag{3.15}$$

$$
\begin{aligned}
&- \frac{\partial F}{\partial x}(t, X(t^-), A(t^-))\theta(t, X(t), Y(t), A(t), u(t), z)\Bigg\}\nu(dz)dt \\
&+ \int_{\mathbb{R}_0} \Bigg\{ F(t, X(t^-) + \theta(t, X(t), Y(t), A(t), u(t), z), A(t^-)) \\
&- F(t, X(t^-), A(t^-)) \Bigg\} \tilde{N}(dt, dz) \\
&+ [X(t) - \lambda A(t) - e^{-\lambda\delta} Y(t)] \frac{\partial F}{\partial a}(t, X(t), A(t))dt,
\end{aligned}
$$

where

$$
LF = LF(t, x, y, a, u) = \frac{\partial F}{\partial t} + b(t, x, y, a, u)\frac{\partial F}{\partial x} + \frac{1}{2}\sigma^2(t, x, y, a, u)\frac{\partial^2 F}{\partial x^2}.
$$

In particular, note that

$$
dA(t) = X(t) - \lambda A(t) - e^{-\lambda\delta} Y(t); \ t \geq 0. \tag{3.16}
$$

Now, define the Hamiltonian, $H^{'} : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathcal{U} \times \mathbb{R}^3 \times \mathbb{R}^3 \times \Re \to \mathbb{R}$, as

$$
\begin{aligned}
&H^{'}(t, x, y, a, u, p, q, r(\cdot)) \\
&= f(t, x, y, a, u) + b(t, x, y, a, u)p_1 + (x - \lambda a - e^{-\lambda\delta} y)p_3 \\
&+ \sigma(t, x, y, a, u)q_1 + \int_{\mathbb{R}_0} \theta(t, x, y, a, u, z)r(z)\nu(dz),
\end{aligned} \tag{3.17}
$$

where $p = (p_1, p_2, p_3)^T \in \mathbb{R}^3$ and $q = (q_1, q_2, q_3) \in \mathbb{R}^3$. For each $u \in \mathcal{A}_{\mathcal{E}}$ the associated adjoint equations are the following BSDEs in the unknown $\mathcal{F}_t$-adapted processes $(p(t), q(t), r(t, \cdot))$ given by

$$
\begin{aligned}
dp_1(t) &= -\frac{\partial H^{'}}{\partial x}(t, X(t), Y(t), A(t), u(t), p(t), q(t), r(t, \cdot))dt + q_1(t)dB(t) \\
&\quad + \int_{\mathbb{R}_0} r(t, z)\tilde{N}(dt, dz), \qquad (3.18)\\
dp_2(t) &= -\frac{\partial H^{'}}{\partial y}(t, X(t), Y(t), A(t), u(t), p(t), q(t), r(t, \cdot))dt, \qquad (3.19)\\
dp_3(t) &= -\frac{\partial H^{'}}{\partial a}(t, X(t), Y(t), A(t), u(t), p(t), q(t), r(t, \cdot))dt + q_3(t)dB(t), \qquad (3.20)
\end{aligned}
$$

Theoreme 3.3.2 (a second infinite horizon maximum principle for delay equations) *Suppose* $\hat{u} \in \mathcal{A}_{\mathcal{E}}$ *and let* $(\hat{X}(t), \hat{Y}(t), \hat{A}(t))$ *and* $(\hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot))$ *be corresponding solutions of* (3.18)-(3.20), *respectively. Suppose that*

$$(x, y, a, u) \mapsto H'(t, x, y, a, u, \hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot))$$

is concave for all $t \geq 0$ *a.s. and*

$$\begin{aligned} &E\left[H'(t, \hat{X}(t), \hat{Y}(t), \hat{A}(t), \hat{u}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot))|\mathcal{E}_t\right] \\ &= \max_{u\in\mathcal{U}} E\left[H'(t, \hat{X}(t), \hat{Y}(t), \hat{A}(t), u, \hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot))|\mathcal{E}_t\right]. \end{aligned} \tag{3.21}$$

Further, assume that

$$\overline{\lim_{T\to\infty}} E[\hat{p}_1(T)(X(T) - \hat{X}(T)) + \hat{p}_3(T)(A(T) - \hat{A}(T))] \geq 0. \tag{3.22}$$

In addition assume that

$$\hat{p}_2(t) = 0$$

for all t. *Then* $\hat{u}$ *is an optimal control for the control problem* (2.4).

Proof. To simplify notation we put

$$\zeta(t) = (X(t), Y(t), A(t))$$

and

$$\hat{\zeta}(t) = (\hat{X}(t), \hat{Y}(t), \hat{A}(t)).$$

Let

$$I := J(\hat{u}) - J(u) = E\left[\int_0^\infty (f(t, \hat{\zeta}(t), \hat{u}(t)) - f(t, \zeta(t), u(t)))dt\right].$$

Then we have that

$$
\begin{aligned}
I &= \overline{\lim_{T\to\infty}} E\left[\int_0^T (H'(t,\hat{\zeta}(t),\hat{u}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot)) - H'(t,\zeta(t),u(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot))dt\right] \\
&- E\left[\int_0^T (b(t,\hat{\zeta}(t),\hat{u}(t)) - b(t,\zeta(t),u(t))\hat{p}_1(t)dt\right] \\
&- E\left[\int_0^T \{(\hat{X}(t) - \lambda\hat{A}(t) - e^{-\lambda\delta}\hat{Y}(t)) - (X(t) - \lambda A(t) - e^{-\lambda\delta}Y(t))\}\hat{p}_3(t)dt\right] \\
&- E\left[\int_0^T \{\sigma(t,\hat{\zeta}(t),\hat{u}(t)) - \sigma(t,\zeta(t),u(t))\}\hat{q}_1(t)dt\right] \\
&- E\left[\int_0^T \int_{\mathbb{R}_0} (\theta(t,\hat{\zeta}(t),\hat{u}(t),z) - \theta(t,\zeta,u(t),z))\hat{r}(t,z)\nu(dz)dt\right] \\
&=: I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}
\tag{3.23}
$$

Since $(\zeta,u) \to H'(t,\zeta,u,\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot))$ is concave, we have by (3.21) that

$$
\begin{aligned}
&H'(t,\zeta,u,\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot)) - H'(t,\hat{\zeta},\hat{u},\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot)) \\
&\leq \nabla_\zeta H'(t,\hat{\zeta},\hat{u},\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot))\cdot(\zeta-\hat{\zeta}) + \frac{\partial H'}{\partial u}(t,\hat{\zeta},\hat{u},\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot))\cdot(u-\hat{u}) \\
&\leq \nabla_\zeta H'(t,\hat{\zeta},\hat{u},\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot))\cdot(\zeta-\hat{\zeta}),
\end{aligned}
$$

where $\nabla_\zeta H' = (\frac{\partial H'}{\partial x}, \frac{\partial H'}{\partial y}, \frac{\partial H'}{\partial a})$. From this we get that

$$
\begin{aligned}
I_1 &\geq \overline{\lim_{T\to\infty}} E\left[\int_0^T -\nabla_\zeta H'(t,\hat{\zeta}(t),\hat{u}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot))\cdot(\zeta(t)-\hat{\zeta}(t))dt\right] \\
&= \overline{\lim_{T\to\infty}} E\left[\int_0^T (\zeta(t)-\hat{\zeta}(t))d\hat{p}(t)\right] \\
&= E\left[\int_0^\infty (X(t)-\hat{X}(t))d\hat{p}_1(t) + \int_0^\infty (A(t)-\hat{A}(t))d\hat{p}_3(t)\right].
\end{aligned}
\tag{3.24}
$$

From (3.18), (3.19), and (3.20) we get that

$$
\begin{aligned}
0 &\geq -\overline{\lim_{T\to\infty}} E[\hat{p}_1(T)(X(T)-\hat{X}(T)) + \hat{p}_3(T)(A(T)-\hat{A}(T))] \\
&= -\overline{\lim_{T\to\infty}} E\Bigg[\int_0^T (X(t)-\hat{X}(t))d\hat{p}_1(t) + \int_0^T \hat{p}_1(t)d(X(t)-\hat{X}(t)) \\
&+ \int_0^T \Big[\sigma(t,\zeta(t),u(t)) - \sigma(t,\hat{\zeta}(t),\hat{u}(t))\Big]\hat{q}_1(t)dt \\
&+ \int_0^T \int_{\mathbb{R}_0} (\theta(t,\zeta(t),u(t),z) - \theta(t,\hat{\zeta},\hat{u}(t),z))\hat{r}(t,z)\nu(dz)dt \\
&+ \int_0^T (A(t)-\hat{A}(t))d\hat{p}_3(t) + \int_0^T \hat{p}_3(t)d(A(t)-\hat{A}(t))\Bigg].
\end{aligned}
\tag{3.25}
$$

Combining this with (3.23) and (3.24) and using (3.16), we have that

$$
-I = I_1 + I_2 + I_3 + I_4 + I_5 \leq 0.
$$

Hence $J(\hat{u}) - J(u) = I \geq 0$, and $\hat{u}$ is an optimal control for our problem. ■

Example 3.3.1 (a nondelay infinite horizon example) *Let us first consider a nondelay example. Assume we are given the performance functional*

$$
J(u) = E\left[\int_0^\infty e^{-\rho t}\frac{1}{\gamma}u^\gamma(t)dt\right] \tag{3.26}
$$

and the state equation

$$
\begin{cases}
dX(t) &= [X(t)\mu - u(t)]\,dt \\
 &\quad +\sigma(t,X(t),u(t))dB(t), t \geq 0, \\
X(0) &= X_0 > 0,
\end{cases}
\tag{3.27}
$$

where $X_0 > 0$, $\gamma \in (0,1)$, $\rho > 0$, and $\mu \in \mathbb{R}$ are given constants. We assume that

$$
\mu\gamma < \rho. \tag{3.28}
$$

Here $u(t) \geq 0$ is our control. It can be interpreted as the consumption rate from a cash flow $X(t)$. The performance $J(u)$ is the total expected discounted utility of the consumption.

For u to be admissible we require that $E\big[X(t)\big] \geq 0$ for all $t \geq 0$.

In this case the Hamiltonian (3.17) takes the form

$$H'(t,x,u,p,q) = e^{-\rho t}\frac{1}{\gamma}u^{\gamma} + [x\mu - u]p + \sigma(t,x,u)q, \tag{3.29}$$

so that we get the partial derivative

$$\frac{\partial H'}{\partial u}(t,u,x,p,q) = e^{-\rho t}u^{\gamma-1} - p + \frac{\partial \sigma}{\partial u}q.$$

Therefore, if $\dfrac{\partial H'}{\partial u} = 0$ we get

$$p(t) = e^{-\rho t}u^{\gamma-1}(t) + \frac{\partial \sigma}{\partial u}(t,X(t),u(t))q(t). \tag{3.30}$$

We now see that the adjoint equation is given by

$$dp(t) = -\left[\mu p(t) + \frac{\partial \sigma}{\partial x}(t,X(t),u(t))q(t)\right]dt + q(t)dB(t).$$

Now assume that

$$\sigma(t,x,u) = \sigma_0(t)x \tag{3.31}$$

for some bounded adapted process $\sigma_0(t)$. Let us try to choose $q = 0$. Then

$$dp(t) = -\mu p(t)dt,$$

which gives

$$p(t) = p(0)e^{-\mu t}$$

for some constant $p(0)$. Hence, by (3.30)

$$\hat{u}(t) = p^{\frac{1}{\gamma-1}}(0)e^{\frac{(\mu-\rho)t}{1-\gamma}} \tag{3.32}$$

for all $t > 0$. Inserting $\hat{u}(t)$ into the dynamics of $\hat{X}(t)$, we get that

$$d\hat{X}(t) = \left[\mu\hat{X}(t) - p^{\frac{1}{\gamma-1}}(0)e^{\frac{1}{\gamma-1}(\rho t - \mu t)}\right] dt + \sigma_0(t)\hat{X}(t)dB(t).$$

So

$$\hat{X}(t) = \left[\hat{X}(0)\Gamma(t) - p^{\frac{1}{\gamma-1}}(0)\int_0^t \frac{\Gamma(t)}{\Gamma(s)} \exp\left(\frac{(\mu-\rho)s}{1-\gamma}\right) ds\right], \tag{3.33}$$

where

$$\Gamma(t) = \exp\left(\int_0^t \sigma_0(s)dB(s) + \mu t - \frac{1}{2}\int_0^t \sigma_0^2(s)ds\right). \tag{3.34}$$

Hence

$$E\left[\hat{X}(t)\right] = e^{\mu t}\left[\hat{X}(0) - p^{\frac{1}{\gamma-1}}(0)\int_0^t \exp\left(\frac{(\mu\gamma-\rho)s}{1-\gamma}\right) ds\right].$$

Therefore, to ensure that $E\left[\hat{X}(t)\right]$ is nonnegative, we get the optimal $\hat{p}(0)$ as

$$\hat{p}(0) = \left[\frac{\hat{X}(0)}{\int_0^\infty \exp\left(\frac{(\mu\gamma-\rho)s}{1-\gamma}\right) ds}\right]^{\gamma-1}. \tag{3.35}$$

We now see that $\overline{\lim}_{T\to\infty} E[\hat{p}(T)\hat{X}(T)] = 0$, so that we have

$$\overline{\lim_{T\to\infty}} E[\hat{p}(T)(X(T) - \hat{X}(T))] \geq 0.$$

This tells us that $\hat{u}$ with $p(0) = \hat{p}(0)$ given by (3.35), the control $\hat{u}$ given by (3.32) is indeed an optimal control.

Example 3.3.2 (an infinite horizon example with delay) *Now let us consider a case where we have delay. This is an infinite horizon version of Example 1 in Øksendal and Sulem [31]. Let*

$$J(u) = E\left[\int_0^\infty e^{-\rho t}\frac{1}{\gamma}u(t)^\gamma dt\right], \tag{3.36}$$

and define

$$\begin{cases} dX(t) & = dX^{(u)}(t) = [X(t)\mu + Y(t)\beta + \alpha A(t) - u(t)]dt \\ & +\sigma(t, X(t), Y(t), A(t), u(t))dB(t), t \geq 0, \\ X(t) & = X_0(t) > 0, t \in [-\delta, 0]. \end{cases} \tag{3.37}$$

We want to find a consumption rate $u^*(t)$ *such that*

$$J(u^*) = \sup\left\{J(u); E\big[X^{(u)}(t)\big] \geq 0 \textit{ for all } t \geq 0\right\}. \tag{3.38}$$

Here $\gamma \in (0,1)$, $\rho, \delta \geq 0$, *and* $\beta \in \mathbb{R}$ *are given constants.*

In this case the Hamiltonian (3.17) takes the form

$$\begin{aligned} H^{'}(t,x,u,y,a,p,q) = e^{-\rho t}\frac{1}{\gamma}u^{\gamma} + [x\mu + \beta y + \alpha a - u]p_1 \\ + [x - \lambda a - e^{-\lambda\delta}y]p_3 + \sigma(t,x,y,a,u)q_1, \end{aligned} \tag{3.39}$$

so that we get the partial derivative

$$\frac{\partial H^{'}}{\partial u}(t,x,u,y,a,p,q) = e^{-\rho t}u^{\gamma-1} - p_1 + \frac{\partial \sigma}{\partial u}q_1.$$

This, together with the maximality condition, gives that

$$p_1(t) = e^{-\rho t}u(t)^{\gamma-1} + \frac{\partial \sigma}{\partial u}q_1.$$

We now see that the adjoint equations are given by

$$\begin{aligned} dp_1(t) &= -\left[\mu p_1(t) + p_3(t) + \frac{\partial \sigma}{\partial x}q_1(t)\right]dt + q_1(t)dB(t), \\ dp_2(t) &= -\left[\beta p_1(t) - e^{-\lambda\delta}p_3(t) + \frac{\partial \sigma}{\partial y}q_1(t)\right]dt, \\ dp_3(t) &= -\left[\alpha p_1(t) - \lambda p_3(t) + \frac{\partial \sigma}{\partial a}q_1(t)\right]dt + q_3(t)dB(t). \end{aligned}$$

Since the coefficents in front of p_1 and p_3 are deterministic we can choose $q_1 = q_3 = 0$. Since we want $p_2(t) = 0$, we then get

$$p_1(t) = \frac{e^{-\lambda\delta}}{\beta}p_3(t),$$

which gives us that

$$dp_1(t) = -\left[\mu p_1(t) + \beta e^{\lambda\delta} p_1(t)\right] dt,$$
$$dp_3(t) = -\left[\frac{\alpha}{\beta} e^{-\lambda\delta} p_3(t) - \lambda p_3(t)\right] dt,$$

and

$$u(t) = e^{\frac{\rho t}{\gamma-1}} p_1^{\frac{1}{\gamma-1}}(t). \tag{3.40}$$

Hence, to ensure that

$$p_1(t) = \frac{e^{-\lambda\delta}}{\beta} p_3(t) \tag{3.41}$$

we need that

$$\alpha = \beta e^{\lambda\delta}(\mu + \lambda + \beta e^{\lambda\delta}). \tag{3.42}$$

So

$$p_1(t) = p_1(0) e^{-(\mu+\beta e^{\lambda\delta})t} \tag{3.43}$$

for some constant $p_1(0)$. Hence by (3.40) we get

$$u(t) = u_{p_1(0)} = p_1(0)^{\frac{1}{\gamma-1}} \exp\left(\frac{(\mu + \beta e^{\lambda\delta} - \rho)t}{1-\gamma}\right) \tag{3.44}$$

for all $t > 0$ and some $p_1(0)$. Now assume that

$$\alpha = 0, \text{ i.e. } \lambda + \beta e^{\lambda\delta} = -\mu \tag{3.45}$$

and that

$$\sigma(t, X(t), Y(t), A(t), u(t)) = \kappa A(t) \ (\kappa \text{ constant}). \tag{3.46}$$

Then (3.37) gets the form

$$\begin{cases} dX(t) = [\mu X(t) + \beta Y(t) - u(t)]dt + \kappa A(t) dB(t), \ t \geq 0, \\ X(t) = X_0(t), \ t \in [-\delta, 0], \end{cases} \tag{3.47}$$

and

$$p_1(t) = p_1(0) e^{\lambda t}. \tag{3.48}$$

Let θ be the unique solution of the equation

$$\mu + \theta + |\beta| e^{\theta\delta} = 0. \tag{3.49}$$

Then by Corollary 4.1 in Mohammed and Scheutzow [30] the top a.s. Lyapunov exponent λ_1 of the solution $X^{(0)}(t)$ of the stochastic delay equation (3.47) corresponding to $u = 0$ satisfies the inequality

$$\lambda_1 \leq -\theta + \frac{\kappa^2}{2|\beta|} e^{|\theta|\delta}. \tag{3.50}$$

Therefore we see that

$$\begin{aligned} &\lim_{T\to\infty} \hat{p}_1(T)\hat{X}(T) \leq \lim_{T\to\infty} \hat{p}_1(T)\hat{X}^{(0)}(T) \\ &\leq \text{ const. } \lim_{T\to\infty} \exp\left(- \left(-\lambda + \theta - \frac{\kappa^2}{2|\beta|} e^{|\theta|\delta} \right) T \right) = 0 \end{aligned}$$

if

$$\lambda + \frac{\kappa^2}{2|\beta|} e^{|\theta|\delta} < \theta. \tag{3.51}$$

By (3.41) condition (3.51) also implies that

$$\lim_{T\to\infty} \hat{p}_3(T)\hat{A}(T) = 0. \tag{3.52}$$

We conclude that (3.22) holds. It remains to determine the optimal value of $\hat{p}_1(0)$. To maximize the expected utility of the consumption (3.36), we choose $\hat{p}_1(0)$ as big as possible under the constraint that $E\big[\hat{X}(t)\big] \geq 0$ for all $t \geq 0$. Hence we put

$$\hat{p}_1(0) = \sup\left\{ p_1(0); E\big[X^{(\hat{u})}(t)\big] \geq 0 \text{ for all } t \geq 0 \right\}, \tag{3.53}$$

where

$$\begin{cases} dX^{(\hat{u})}(t) &= \left[X^{(\hat{u})}(t)\mu + Y^{(\hat{u})}(t)\beta - p_1^{\frac{1}{\gamma-1}}(0) \exp\left(\frac{-(\lambda+\rho)t}{1-\gamma} \right) \right] dt \\ & \quad +\kappa A^{(\hat{u})}(t) dB(t), \;\; t \geq 0, \\ X^{(\hat{u})}(t) &= X_0(t) > 0, \;\; t \in [-\delta, 0]. \end{cases} \tag{3.54}$$

In this case, however, in lack of a solution formula for $E\big[X^{(u)}(t)\big]$, we are not able to find an explicit expression for $\hat{p}_1(0)$, as we could in Example 3.3.1. We conclude that our candidate for the optimal control is given by

$$\hat{u}(t) = \hat{p}_1^{\frac{1}{\gamma-1}}(0) \exp\left(\frac{-(\lambda+\rho)t}{1-\gamma} \right).$$

3.4 A necessary maximum principle

In addition to the assumptions in sections 3.2 and 3.3.1, we now assume the following:
(A_1) For all $u \in \mathcal{A}_{\mathcal{E}}$ and all $\beta \in \mathcal{A}_{\mathcal{E}}$ bounded, there exists $\epsilon>0$ such that

$$u + s\beta \in \mathcal{A}_{\mathcal{E}} \qquad \text{for all } s \in (-\epsilon, \epsilon).$$

(A_2) For all t_0, h such that $0 \le t_0 < t_0 + h \le T$ and all bounded $\mathcal{E}_{t_0}$-measurable random variables α, the control process $\beta(t)$ defined by

$$\beta(t) = \alpha 1_{[t_0, t_0+h]}(t) \tag{4.1}$$

belongs to $\mathcal{A}_{\mathcal{E}}$.
(A_3) The derivative process

$$\xi(t) := \frac{d}{ds} X^{u+s\beta}(t) \mid_{s=0} \tag{4.2}$$

exists and belongs to $L^2(m \times P)$, where m denotes the Lebesgue measure on $\mathbb{R}$.
It follows from (2.1) that

$$d\xi(t) = \left\{ \frac{\partial b}{\partial x}(t)\xi(t) + \frac{\partial b}{\partial y}(t)\xi(t-\delta) + \frac{\partial b}{\partial a}(t) \int_{t-\delta}^{t} e^{-\lambda(t-r)}\xi(r)dr + \frac{\partial b}{\partial u}(t)\beta(t) \right\} dt \tag{4.3}$$

$$+ \left\{ \frac{\partial \sigma}{\partial x}(t)\xi(t) + \frac{\partial \sigma}{\partial y}(t)\xi(t-\delta) + \frac{\partial \sigma}{\partial a}(t) \int_{t-\delta}^{t} e^{-\lambda(t-r)}\xi(r)dr + \frac{\partial \sigma}{\partial u}(t)\beta(t) \right\} dB(t)$$

$$+ \int_{\mathbb{R}_0} \left\{ \frac{\partial \theta}{\partial x}(t,z)\xi(t) + \frac{\partial \theta}{\partial y}(t,z)\xi(t-\delta) + \frac{\partial \theta}{\partial a}(t,z) \int_{t-\delta}^{t} e^{-\lambda(t-r)}\xi(r)dr + \frac{\partial \theta}{\partial u}(t,z)\beta(t) \right\} \tilde{N}(dt,dz),$$

where, for simplicity of notation, we define

$$\frac{\partial}{\partial x} b(t) := \frac{\partial}{\partial x} b(t, X(t), X(t-\delta), A(t), u(t)),$$

and use that

$$\frac{d}{ds}Y^{u+s\beta}(t)\mid_{s=0}=\frac{d}{ds}X^{u+s\beta}(t-\delta)\mid_{s=0}=\xi(t-\delta), \tag{4.4}$$

and

$$\begin{aligned}\frac{d}{ds}A^{u+s\beta}(t)\mid_{s=0} &= \frac{d}{ds}\left(\int_{t-\delta}^{t} e^{-\lambda(t-r)}X^{u+s\beta}(r)dr\right)\mid_{s=0} \\ &= \left(\int_{t-\delta}^{t} e^{-\lambda(t-r)}\frac{d}{ds}X^{u+s\beta}(r)dr\right)\mid_{s=0} \\ &= \int_{t-\delta}^{t} e^{-\lambda(t-r)}\xi(r)dr.\end{aligned} \tag{4.5}$$

Note that

$$\xi(t)=0 \text{ for } t\in[-\delta,0]. \tag{4.6}$$

Theoreme 3.4.1 (Necessary maximum principle) *Suppose that $\hat{u}\in\mathcal{A}_{\mathcal{E}}$ with corresponding solutions $\hat{X}(t)$ of (2.1)-(2.2) and $\hat{p}(t)$, $\hat{q}(t)$, and $\hat{r}(t,\cdot)$ of (3.2)-(3.3), and corresponding derivative process $\hat{\xi}(t)$ given by (4.2). Assume that for all $u\in\mathcal{A}_{\mathcal{E}}$ with corresponding $(X(t),p(t),q(t),r(t,\cdot))$ the following hold:*

$$\begin{aligned}&E\Bigg[\int_0^T \hat{p}^2(t)\Bigg\{\Big(\frac{\partial\sigma}{\partial x}\Big)^2(t)\hat{\xi}^2(t)+\Big(\frac{\partial\sigma}{\partial y}\Big)^2(t)\hat{\xi}^2(t-\delta)+\Big(\frac{\partial\sigma}{\partial a}\Big)^2(t)\Big(\int_{t-\delta}^{t} e^{-\lambda(t-r)}\hat{\xi}(r)dr\Big)^2 \\ &+\Big(\frac{\partial\sigma}{\partial u}\Big)^2(t)+\int_{\mathbb{R}_0}\Big\{\Big(\frac{\partial\theta}{\partial x}\Big)^2(t,z)\hat{\xi}^2(t)+\Big(\frac{\partial\theta}{\partial y}\Big)^2(t,z)\hat{\xi}^2(t-\delta) \\ &+\Big(\frac{\partial\theta}{\partial a}\Big)^2(t,z)\Big(\int_{t-\delta}^{t} e^{-\lambda(t-r)}\hat{\xi}(r)dr\Big)^2+\Big(\frac{\partial\theta}{\partial u}\Big)^2(t,z)\Big\}\nu(dz)\Bigg\}dt \\ &+\int_0^T \hat{\xi}^2(t)\Big\{\hat{q}^2(t)+\int_{\mathbb{R}_0}\hat{r}^2(t,z)\ \nu(dz)\Big\}dt\Bigg]<\infty, \text{ for all } T<\infty\end{aligned} \tag{4.7}$$

and

$$\lim_{T\to\infty} E\left[\hat{p}(T)\hat{\xi}(T)\right] = 0. \tag{4.8}$$

Then the following assertions are equivalent:

(i) *For all bounded* $\beta \in \mathcal{A}_{\mathcal{E}}$,

$$\frac{d}{ds} J(\hat{u} + s\beta) \mid_{s=0} = 0.$$

(ii) *For all* $t \in [0, \infty)$,

$$E\left[\frac{\partial H}{\partial u}\Big(t,\ \hat{X}(t), \hat{Y}(t), \hat{A}(t), u, \hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot)\Big)\Big|\mathcal{E}_t\right]_{u=\hat{u}(t)} = 0 \ a.s.$$

Proof. Suppose that assertion (i) holds. Then

$$\begin{aligned}
0 &= \frac{d}{ds} J(\hat{u} + s\beta) \mid_{s=0} \\
&= \frac{d}{ds} E\left[\int_0^\infty f(t, X^{\hat{u}+s\beta}(t), Y^{\hat{u}+s\beta}(t), A^{\hat{u}+s\beta}(t), \hat{u}(t) + s\beta(t) dt\right] \mid_{s=0} \\
&= E\left[\int_0^\infty \left\{\frac{\partial f}{\partial x}(t)\xi(t) + \frac{\partial f}{\partial y}(t)\xi(t-\delta) + \frac{\partial f}{\partial a}(t)\int_{t-\delta}^t e^{-\lambda(t-r)}\xi(r)dr + \frac{\partial f}{\partial u}(t)\beta(t)\right\} dt\right].
\end{aligned}$$

We know by the definition of H that

$$\frac{\partial f}{\partial x}(t) = \frac{\partial H}{\partial x}(t) - \frac{\partial b}{\partial x}(t)p(t) - \frac{\partial \sigma}{\partial x}(t)q(t) - \int_{\mathbb{R}_0} \frac{\partial \theta}{\partial x}(t,z)r(t,z)\nu(dz) \tag{4.9}$$

and the same for $\frac{\partial f}{\partial y}(t)$, $\frac{\partial f}{\partial a}(t)$, and $\frac{\partial f}{\partial u}(t)$.

Applying the Itô formula to $\hat{p}(t)\hat{\xi}(t)$, we obtain by (4.8) and (4.9)

$$\begin{aligned}
0 &= \lim_{T\to\infty} E\left[\hat{p}(T)\hat{\xi}(T)\right] \\
&= E\Bigg[\int_0^\infty \hat{p}(t) \ \left\{\frac{\partial b}{\partial x}(t)\xi(t) + \frac{\partial b}{\partial y}(t)\xi(t-\delta) + \frac{\partial b}{\partial a}(t)\int_{t-\delta}^t e^{-\lambda(t-r)}\xi(r)dr + \frac{\partial b}{\partial u}(t)\beta(t)\right\} dt
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^\infty \xi(t)E\left(\mu(t)\mid\mathcal{F}_t\right)dt\\
&+\int_0^\infty q(t)\left\{\frac{\partial\sigma}{\partial x}(t)\xi(t)+\frac{\partial\sigma}{\partial y}(t)\xi(t-\delta)+\frac{\partial\sigma}{\partial a}(t)\int_{t-\delta}^{t}e^{-\lambda(t-r)}\xi(r)dr+\frac{\partial\sigma}{\partial u}(t)\beta(t)\right\}dt\\
&+\int_0^\infty\int_{\mathbb{R}_0} r(t,z)\Big\{\frac{\partial\theta}{\partial x}(t,z)\xi(t)+\frac{\partial\theta}{\partial y}(t,z)\xi(t-\delta)+\frac{\partial\theta}{\partial a}(t,z)\int_{t-\delta}^{t}e^{-\lambda(t-r)}\xi(r)dr\\
&+\frac{\partial\theta}{\partial u}(t,z)\beta(t)\Big\}\nu(dz)dt\Big]\\
&=-\frac{d}{ds}J(\hat{u}+s\beta)\mid_{s=0}+E\left(\int_0^\infty\frac{\partial H}{\partial u}(t)\beta(t)dt\right).
\end{aligned}
$$

Therefore

$$
E\left(\int_0^\infty\frac{\partial H}{\partial u}(t)\beta(t)dt\right)=\frac{d}{ds}J(\hat{u}+s\beta)\Big|_{s=0}. \tag{4.10}
$$

Now apply this to

$$
\beta(t)=\alpha 1_{[t_0,t_0+h]}(t)\ ,
$$

where α is bounded and $\mathcal{E}_{t_0}$-measurable, $0\le t_0<t_o+h\le T$. Then if (i) holds we get

$$
E\left(\int_{t_0}^{t_0+h}\frac{\partial H}{\partial u}(t)dt\ \alpha\right)=0.
$$

Differentiating with respect to h at 0, we have

$$
E\left(\frac{\partial H}{\partial u}(t_0)\ \alpha\right)=0.
$$

This holds for all $\mathcal{E}_{t_0}$-measurable α and hence we obtain that

$$
E\left(\frac{\partial H}{\partial u}(t_0)\mid\mathcal{E}_{t_0}\right)=0.
$$

This proves that assertion (i) implies (ii).

To complete the proof, we need to prove the converse implication; which is obtained since every bounded $\beta \in \mathcal{A}_{\mathcal{E}}$ can be approximated by linear combinations of controls β of the form (4.1). ■

3.5 Existence and uniqueness of the time-advanced BSDEs on infinite horizon

The main result in this section refers to the existence and uniqueness for (3.3) – (3.4) where the coefficients satisfy a Lipschitz condition.

Given a positive constant δ, denote by $D([0,\delta],\mathbb{R})$ the space of all càdlàg paths from $[0,\delta]$ into $\mathbb{R}$. For a path $X(\cdot) : \mathbb{R}_+ \to \mathbb{R}$, X_t will denote the function defined by $X_t(s) = X(t+s)$ for $s \in [0,\delta]$. Let $\mathcal{H} = L^2(\nu)$ be the set of all functions $r : \mathbb{R}_0 \to \mathbb{R}$ such that $\int_{\mathbb{R}_0} r^2(z)\nu(dz) < \infty$. Consider the L^2 space $V_1 := L^2([0,\delta] \to \mathbb{R}; ds)$ and $V_2 := L^2([0,\delta] \to \mathcal{H}; ds)$. Let

$$F : \mathbb{R}_+ \times \mathbb{R} \times \mathbb{R} \times V_1 \times \mathbb{R} \times \mathbb{R} \times V_1 \times \mathcal{H} \times \mathcal{H} \times V_2 \times \Omega \to \mathbb{R}$$

be a function satisfying the following Lipschitz condition: There exists a positive constant C such that

$$\begin{aligned}
&|F(t,p_1,p_2,p,q_1,q_2,q,r_1,r_2,r,\omega) - F(t,\bar{p}_1,\bar{p_2},\bar{p},\bar{q}_1,\bar{q}_2,\bar{q},\bar{r}_1,\bar{r}_2,\bar{r},\omega)| \\
&\le C(|p_1-\bar{p}_1| + |p_2-\bar{p}_2| + |p-\bar{p}|_{V_1} + |q_1-\bar{q}_1| + |q_2-\bar{q}_2| + |q-\bar{q}|_{V_1} \\
&+ |r_1-\bar{r}_1|_{\mathcal{H}} + |r_2-\bar{r}_2|_{\mathcal{H}} + |r-\bar{r}|_{V_2}) \text{ a.s.} \qquad (5.1)
\end{aligned}$$

Assume that $(t,\omega) \to F(t,p_1,p_2,p,q_1,q_2,q,r_1,r_2,r,\omega)$ is predictable for all $p_1,p_2,p,q_1,q_2,q,r_1,r_2,r$. Further we assume the following:

$$E\int_0^\infty e^{\lambda t}|F(t,0,0,0,0,0,0,0,0,0)|^2 dt < \infty \qquad (5.2)$$

for all $\lambda \in \mathbb{R}$. We now consider the following BSDE in the unknown $\mathcal{F}_t$-adapted, $\mathbb{R}\times\mathbb{R}\times\mathcal{H}$-

valued process $(p(t), q(t), r(t) = r(t, \cdot))$:

$$dp(t) = E[F(t, p(t), p(t+\delta), p_t, q(t), q(t+\delta), q_t, r(t), r(t+\delta), r_t)|\mathcal{F}_t]dt \\ + q(t)dB(t) + \int_{\mathbb{R}_0} r(t,z)\tilde{N}(dt, dz), \tag{5.3}$$

where

$$E\left[\int_0^\infty e^{\lambda t}|p(t)|^2 dt\right] < \infty \tag{5.4}$$

for all $\lambda \in \mathbb{R}$.

Theoreme 3.5.1 (Existence and uniqueness) *Assume the conditions (5.1)-(5.2) are fulfilled. Then the BSDE (5.3) - (5.4) admits a unique solution $(p(t), q(t), r(t, \cdot))$ such that*

$$E\left[\int_0^\infty e^{\lambda t}\{|p(t)|^2 + |q(t)|^2 + \int_{\mathbb{R}_0} |r(t,z)|^2 \nu(dz)\}dt\right] < \infty$$

for all $\lambda > 0$.

Proof.

Existence:

Step 1:

First, assume F is independent of its second, third, and fourth parameters.
Set $q^0(t) := 0$, $r^0(t, \cdot) := 0$. For $n \geq 1$, define $(p^n(t), q^n(t), r^n(t, \cdot))$ to be the unique solution of the following BSDE:

$$dp^n(t) = E\left[F(t, q^{n-1}(t), q^{n-1}(t+\delta), q_t^{n-1}, r^{n-1}(t, \cdot), r^{n-1}(t+\delta, \cdot), r_t^{n-1}(\cdot)) \mid \mathcal{F}_t\right] dt \\ + q^n(t)dB(t) + \int_{\mathbb{R}_0} r^n(t,z)\tilde{N}(dt, dz), \tag{5.5}$$

for $t \in [0, \infty)$ such that

$$E\left[\int_0^\infty e^{\lambda t}|p^n(t)|^2 dt\right] < \infty.$$

The triples $(p^n(t), q^n(t), r^n(t, \cdot))$ exist by Theorem 3.1 in Haadem & al [19].
Our goal is to show that $(p^n(t), q^n(t), r^n(t, \cdot))$ forms a Cauchy sequence. By the Itô formula

we get that

$$
\begin{aligned}
0 = E\Bigg[& e^{\lambda t}|p^{n+1}(t) - p^n(t)|^2 + \int_t^\infty \lambda e^{\lambda s}|p^{n+1}(s) - p^n(s)|^2 ds \\
& + \int_t^\infty e^{\lambda s}|q^{n+1}(s) - q^n(s)|^2 ds \\
& + \int_t^\infty e^{\lambda s}\int_{\mathbb{R}_0} |\big(r^{n+1}(s,z) - r^n(s,z)\big)|^2 ds\nu(dz) \\
& + 2\int_t^\infty e^{\lambda s}\left(p^{n+1}(s) - p^n(s)\right) E\left[F^n(s) - F^{n-1}(s) \mid \mathcal{F}_s\right] ds\Bigg].
\end{aligned}
$$

Rearranging, using that for all $a, b \in \mathbb{R}$: $2ab \leq \frac{a^2}{\epsilon} + \epsilon b^2$, we have by the Lipschitz requirement (5.1)

$$
\begin{aligned}
& E\left[e^{\lambda t}|p^{n+1}(t) - p^n(t)|^2\right] \\
& + E\left[\int_t^\infty \lambda e^{\lambda s}|p^{n+1}(s) - p^n(s)|^2 ds\right] \\
& + E\left[\int_t^\infty e^{\lambda s}|q^{n+1}(s) - q^n(s)|^2 ds\right] \\
& + E\left[\int_t^\infty \int_{\mathbb{R}_0} e^{\lambda s}|\big(r^{n+1}(s,z) - r^n(s,z)\big)|^2 \nu(dz) ds\right] \\
& \leq C_\epsilon E\left[\int_t^\infty e^{\lambda s}|p^{n+1}(s) - p^n(s)|^2 ds\right] \\
& + \epsilon 6 E\left[\int_t^\infty e^{\lambda s}|q^n(s) - q^{n-1}(s)|^2 ds\right] \\
& + \epsilon 6 E\left[\int_t^\infty e^{\lambda s}|q^n(s+\delta) - q^{n-1}(s+\delta)|^2 ds\right] \\
& + \epsilon 6 E\left[\int_t^\infty e^{\lambda s}\int_s^{s+\delta} |q^n(u) - q^{n-1}(u)|^2 du ds\right] \\
& + \epsilon 6 E\left[\int_t^\infty e^{\lambda s}|r^n(s) - r^{n-1}(s)|^2_{\mathcal{H}} ds\right] \\
& + \epsilon 6 E\left[\int_t^\infty e^{\lambda s}|r^n(s+\delta) - r^{n-1}(s+\delta)|^2_{\mathcal{H}} ds\right] \\
& + \epsilon 6 E\left[\int_t^\infty e^{\lambda s}\int_s^{s+\delta} |r^n(u) - r^{n-1}(u)|^2_{\mathcal{H}} du ds\right],
\end{aligned}
$$

where $C_\epsilon = \frac{C^2}{\epsilon}$ and we used the abbreviation

$$F^n(t) := F(t, q^n(t), q^n(t+\delta), q_t^n, r^n(t,\cdot), r^n(t+\delta,\cdot), r_t^n(\cdot)).$$

Note that

$$\begin{aligned}
&E\left[\int_t^\infty e^{\lambda s}|q^n(s+\delta) - q^{n-1}(s+\delta)|^2 ds\right] \\
&\leq e^{-\lambda\delta} E\left[\int_t^\infty e^{\lambda s}|q^n(s) - q^{n-1}(s)|^2 ds\right].
\end{aligned}$$

Using Fubini

$$\begin{aligned}
&E\left[\int_t^\infty \int_s^{s+\delta} e^{\lambda s}|q^n(u) - q^{n-1}(u)|^2 du ds\right] \\
&\leq E\left[\int_t^\infty \int_{u-\delta}^{u} e^{\lambda s}|q^n(u) - q^{n-1}(u)|^2 ds du\right] \\
&\leq \frac{1}{\lambda}(1 - e^{-\lambda\delta}) E\left[\int_t^\infty e^{\lambda s}|q^n(s) - q^{n-1}(s)|^2 ds\right] \\
&\leq E\left[\int_t^\infty e^{\lambda s}|q^n(s) - q^{n-1}(s)|^2 ds\right].
\end{aligned}$$

Similarly for $r^n - r^{n-1}$. It now follows that

$$\begin{aligned}
&E[e^{\lambda t}|p^{n+1}(t) - p^n(t)|^2] \\
&+ E\left[\int_t^\infty e^{\lambda s}|q^{n+1}(s) - q^n(s)|^2 ds\right] \\
&+ E\left[\int_t^\infty \int_{\mathbb{R}_0} e^{\lambda s}|\big(r^{n+1}(s,z) - r^n(s,z)\big)|^2 \nu(dz) ds\right] \\
&\leq (C_\epsilon - \lambda) E\left[\int_t^\infty e^{\lambda s}|p^{n+1}(s) - p^n(s)|^2 ds\right] \\
&+ \epsilon 6(2 + e^{-\lambda\delta}) E\left[\int_t^\infty e^{\lambda s}|q^n(s) - q^{n-1}(s)|^2 ds\right] \\
&+ \epsilon 6(2 + e^{-\lambda\delta}) E\left[\int_t^\infty e^{\lambda s}|r^n(s) - r^{n-1}(s)|_{\mathcal{H}}^2 ds\right].
\end{aligned}$$

Choosing $\epsilon = \frac{1}{12(2+e^{-\lambda\delta})}$ we get

$$
\begin{aligned}
&E[e^{\lambda t}|p^{n+1}(t) - p^n(t)|^2] \\
&+ E[\int_t^\infty e^{\lambda s}|q^{n+1}(s) - q^n(s)|^2 ds] \\
&+ E[\int_t^\infty \int_{\mathbb{R}_0} e^{\lambda s}|\big(r^{n+1}(s,z) - r^n(s,z)\big)|^2 \nu(dz)ds] \\
&\leq (C_\epsilon - \lambda)E[\int_t^\infty e^{\lambda s}|p^{n+1}(s) - p^n(s)|^2 ds] + \frac{1}{2}E[\int_t^\infty e^{\lambda s}|q^n(s) - q^{n-1}(s)|^2 ds] \\
&+ \frac{1}{2}E[\int_t^\infty e^{\lambda s}|r^n(s) - r^{n-1}(s)|^2_{\mathcal{H}} ds]. \qquad (5.6)
\end{aligned}
$$

From this we deduce that

$$
\begin{aligned}
&-\frac{d}{dt}\left(e^{(C_\epsilon-\lambda)t}E\left[\int_t^\infty e^{\lambda s}|p^{n+1}(s) - p^n(s)|^2 ds\right]\right) \\
&+ e^{(C_\epsilon-\lambda)t}E\left[\int_t^\infty e^{\lambda s}|q^{n+1}(s) - q^n(s)|^2\right] ds \\
&+ e^{(C_\epsilon-\lambda)t}E\left[\int_t^\infty \int_{\mathbb{R}_0} e^{\lambda s}|\big(r^{n+1}(s,z) - r^n(s,z)\big)|^2 \nu(dz)ds\right] \\
&\leq \frac{1}{2}e^{(C_\epsilon-\lambda)t}E\left[\int_t^\infty e^{\lambda s}|q^n(s) - q^{n-1}(s)|^2 ds\right] \\
&+ \frac{1}{2}e^{(C_\epsilon-\lambda)t}E\left[\int_t^\infty e^{\lambda s}|r^n(s) - r^{n-1}(s)|^2_{\mathcal{H}} ds\right].
\end{aligned}
$$

Integrating the last inequality we get that

$$
\begin{aligned}
&E\left[\int_0^\infty e^{\lambda t}|p^{n+1}(t) - p^n(t)|^2 dt\right] \\
&+ \int_0^\infty e^{(C_\epsilon-\lambda)t}E\left[\int_t^\infty e^{\lambda s}|q^{n+1}(s) - q^n(s)|^2 ds\right] dt \\
&+ \int_0^\infty e^{(C_\epsilon-\lambda)t}E\left[\int_t^\infty \int_{\mathbb{R}_0} e^{\lambda s}|\big(r^{n+1}(s,z) - r^n(s,z)\big)|^2 \nu(dz)ds\right] dt \\
&\leq \frac{1}{2}\int_0^\infty e^{(C_\epsilon-\lambda)t}E\left[\int_t^\infty e^{\lambda s}|q^n(s) - q^{n-1}(s)|^2 ds\right] dt \\
&+ \frac{1}{2}\int_0^\infty e^{(C_\epsilon-\lambda)t}E\left[\int_t^\infty e^{\lambda s}|r^n(s) - r^{n-1}(s)|^2_{\mathcal{H}} ds\right] dt. \qquad (5.7)
\end{aligned}
$$

So that

$$\begin{aligned}
&\int_0^\infty e^{(C_\epsilon-\lambda)t}E\left[\int_t^\infty e^{\lambda s}|q^{n+1}(s)-q^n(s)|^2ds\right]dt\\
&+\int_0^\infty e^{(C_\epsilon-\lambda)t}E\left[\int_t^\infty\int_{\mathbb{R}_0} e^{\lambda s}|\big(r^{n+1}(s,z)-r^n(s,z)\big)|^2\nu(dz)ds\right]dt\\
&\leq \frac{1}{2}\int_0^\infty e^{(C_\epsilon-\lambda)t}E\left[\int_t^\infty e^{\lambda s}|q^n(s)-q^{n-1}(s)|^2ds\right]dt\\
&+\frac{1}{2}\int_0^\infty e^{(C_\epsilon-\lambda)t}E\left[\int_t^\infty e^{\lambda s}|r^n(s)-r^{n-1}(s)|^2_{\mathcal{H}}ds\right]dt.
\end{aligned}$$

This gives that

$$\begin{aligned}
&\int_0^\infty e^{(C_\epsilon-\lambda)t}E\left[\int_t^\infty e^{\lambda s}|q^{n+1}(s)-q^n(s)|^2ds\right]dt\\
&+\int_0^\infty e^{(C_\epsilon-\lambda)t}E\left[\int_t^\infty\int_{\mathbb{R}_0} e^{\lambda s}|\big(r^{n+1}(s,z)-r^n(s,z)\big)|^2\nu(dz)ds\right]dt\\
&\leq \frac{1}{2^n}C_3
\end{aligned}$$

if $\lambda > \frac{C}{\epsilon}$. It then follows from (5.7) that

$$E\left[\int_0^\infty e^{\lambda t}|p^{n+1}(t)-p^n(t)|^2dt\right]\leq\frac{1}{2^n}C_3.$$

From (5.6) and (5.7), we now get

$$\begin{aligned}
&E\left[\int_t^\infty\int_{\mathbb{R}_0} e^{\lambda s}|\big(r^{n+1}(s,z)-r^n(s,z)\big)|^2\nu(dz)ds\right]\\
&+E\left[\int_t^\infty e^{\lambda s}|q^{n+1}(s)-q^n(s)|^2ds\right]\leq\frac{1}{2^n}C_3nC_\epsilon.
\end{aligned}$$

From this we conclude that there exist progressively measurable processes $(p(t),q(t),r(t,\cdot))$, such that

$$\lim_{n\to\infty} E\left[e^{\lambda t}|p^n(t)-p(t)|^2 dt\right]=0,$$
$$\lim_{n\to\infty} E\left[\int_0^\infty e^{\lambda t}|p^n(t)-p(t)|^2 dt\right]=0,$$
$$\lim_{n\to\infty} E\left[\int_0^\infty e^{\lambda t}|q^n(t)-q(t)|^2 dt\right]=0,$$
$$\lim_{n\to\infty} E\left[\int_t^\infty \int_{\mathbb{R}_0} e^{\lambda s}|r^n(s,z)-r(s,z)|^2 \nu(dz)ds\right]=0.$$

Letting $n\to\infty$ in (5.5) we see that $(p(t),q(t),r(t,\cdot))$ satisfies

$$\begin{aligned} dp(t) &= E\big[F(t,q(t),q(t+\delta),q_t,r(t,\cdot),r(t+\delta,\cdot),r_t(\cdot)) \mid \mathcal{F}_t\big]dt \\ &\quad + q(t)dB(t) + \int_{\mathbb{R}_0} r(t,z)\tilde{N}(dt,dz), \end{aligned}$$

for all $t>0$.

Step 2:

General F.

Let $p^0(t)=0$. For $n\geq 1$ define $(p^n(t),q^n(t),r^n(t,\cdot))$ to be the unique solution to the following ABSDE:

$$\begin{aligned} dp^n(t) &= E\left[F(t,p^{n-1}(t),p^{n-1}(t+\delta),p_t^{n-1},q^n(t),q^n(t+\delta),q_t^n,r^n(t),r^n(t+\delta),r_t^n)|\mathcal{F}_t\right]dt \\ &\quad + q^n(t)dB(t) + \int_{\mathbb{R}_0} r^n(t,z)\tilde{N}(dz,dt) \end{aligned}$$

for $t\in[0,\infty)$. The existence of $(p^n(t),q^n(t),r^n(t,\cdot))$ was proved in Step 1.

By using the same arguments as above, we deduce that

$$\begin{aligned} &E\left[e^{\lambda t}|p^{n+1}(t)-p^n(t)|^2\right] \\ &+ E\left[\int_t^\infty e^{\lambda s}|q^{n+1}(s)-q^n(s)|^2 ds\right] \\ &+ E\left[\int_t^\infty e^{\lambda s}\int_{\mathbb{R}_0} |\big(r^{n+1}(s,z)-r^n(s,z)\big)|^2\nu(dz)ds\right] \\ &\leq (C_\epsilon-\lambda)E\left[\int_t^\infty e^{\lambda s}|p^{n+1}(s)-p^n(s)|^2 ds\right] + \frac{1}{2}E\left[\int_t^\infty e^{\lambda s}|p^n(s)-p^{n-1}(s)|^2 ds\right]. \end{aligned}$$

This implies that

$$-\frac{d}{dt}\left(e^{(C_\epsilon-\lambda)t}E\left[\int_t^\infty e^{\lambda s}|p^{n+1}(s)-p^n(s)|^2 ds\right]\right) \leq \frac{1}{2}e^{(C_\epsilon-\lambda)t}E\left[\int_t^\infty e^{\lambda s}|p^n(s)-p^{n-1}(s)|^2 ds\right].$$

Integrating from 0 to ∞, we get

$$E\left[\int_0^\infty e^{\lambda s}|p^{n+1}(s)-p^n(s)|^2 ds\right] \leq \frac{1}{2}\int_0^\infty e^{(C_\epsilon-\lambda)t}E\left[\int_t^\infty e^{\lambda s}|p^n(s)-p^{n-1}(s)|^2 ds\right] dt.$$

So if $\lambda \geq C_\epsilon$ then by iteration we see that

$$E\left[\int_0^\infty e^{\lambda s}|p^{n+1}(s)-p^n(s)|^2 ds\right] \leq \frac{K}{2^n(\lambda - C_\epsilon)^n}$$

for some constant K.

Uniqueness:

In order to prove the uniqueness, we assume that there are two solution triples $(p^1(t), q^1(s), r^1(s,\cdot))$ and $(p^2(t), q^2(s), r^2(s,\cdot))$ to the ABSDE

$$\begin{aligned} dp(t) &= E\left[F(t, p(t), p(t+\delta), p_t, q(t), q(t+\delta), q_t, r(t), r(t+\delta), r_t) \mid \mathcal{F}_t\right] dt \\ &\quad + q(t)dB(t) + \int_{\mathbb{R}_0} r(t,z)\tilde{N}(dt,dz); t \in [0,\infty), \end{aligned}$$

where

$$E\left[\int_0^\infty e^{\lambda\, t}\, |p(t)|^2\, dt\right] < \infty$$

and

$$\lambda \geq \frac{3C^2}{\epsilon} + \frac{1}{2}.$$

By the Itô formula, we have

$$
\begin{aligned}
& E\left[e^{\lambda t}\left|p^1(t)-p^2(t)\right|^2\right]+E\left[\int_t^{\infty}\lambda\, e^{\lambda s}\left|p^1(s)-p^2(s)\right| ds\right] \\
& + E\left[\int_t^{\infty} e^{\lambda s}\left|q^1(s)-q^2(s)\right|^2 ds\right]+E\left[\int_t^{\infty} e^{\lambda s}\int_{\mathbb{R}_0}\left|r^1(s,z)-r^2(s,z)\right|^2 ds\nu(dz)\right] \\
& = 2E\Bigg[\int_t^{\infty} e^{\lambda s}\Bigg[\left|p^1(s)-p^2(s)\right| \\
& \times\Bigg(E\left[F(s,p^1(s),p^1(s+\delta),p^1_s,q^1(s),q^1(s+\delta),q^1_s,r^1(s),r^1(s+\delta),r^1_s)\mid\mathcal{F}_s\right] \\
& - E\left[F(s,p^2(s),p^2(s+\delta),p^2_s,q^2(s),q^2(s+\delta),q^2_s,r^2(s),r^2(s+\delta),r^2_s)\mid\mathcal{F}_s\right]\Bigg)\Bigg]ds\Bigg] \\
& \leq 2E\Bigg[\int_t^{\infty} e^{\lambda s}\Bigg[\left|p^1(s)-p^2(s)\right| \\
& \times C\Bigg(\left|p^1(s)-p^2(s)\right|+\left|p^1(s+\delta)-p^2(s+\delta)\right|+\int_s^{s+\delta}\left|p^1(u)-p^2(u)\right| du \\
& +\left|q^1(s)-q^2(s)\right|+\left|q^1(s+\delta)-q^2(s+\delta)\right|+\int_s^{s+\delta}\left|q^1(u)-q^2(u)\right| du \\
& +\left|r^1(s)-r^2(s)\right|^2_{\mathcal{H}}+\left|r^1(s+\delta)-r^2(s+\delta)\right|^2_{\mathcal{H}}+\int_s^{s+\delta}\left|r^1(u)-r^2(u)\right|^2_{\mathcal{H}} du\Bigg)\Bigg]ds\Bigg].
\end{aligned}
$$

By the above inequalities for (p,q,r) and the fact that $2ab \leq \frac{a^2}{\epsilon}+\epsilon b^2$, we have that

$$
\begin{aligned}
& E\left[e^{\lambda t}\left|p^1(t)-p^2(t)\right|^2\right]+E\left[\int_t^{\infty} e^{\lambda s}\left|q^1(s)-q^2(s)\right|^2 ds\right] \\
& + E\left[\int_t^{\infty} e^{\lambda s}\int_{\mathbb{R}_0}\left|r^1(s,z)-r^2(s,z)\right|^2 ds\nu(dz)\right] \\
& \leq \left(\frac{3C^2}{\epsilon}-\lambda\right) E\left[\int_t^{\infty} e^{\lambda s}\left|p^1(s)-p^2(s)\right|^2 ds\right]
\end{aligned}
$$

$$+ \left(2 + e^{-\lambda\,\delta}\right) \epsilon E \left[\int_t^\infty e^{\lambda s} \left|p^1(s) - p^2(s)\right|^2 ds \right]$$
$$+ \left(2 + e^{-\lambda\,\delta}\right) \epsilon\ E \left[\int_t^\infty e^{\lambda s} \left[\left|q^1(s) - q^2(s)\right|^2 + \left|r^1(s,z) - r^2(s,z)\right|_{\mathcal{H}}^2 \right] ds \right].$$

Taking ϵ such that $(2 + e^{-\lambda\delta})\ \epsilon = \frac{1}{2}$

$$E\left[e^{\lambda t}\left|p^1(t) - p^2(t)\right|^2\right] + E\left[\int_t^\infty e^{\lambda s}\left|q^1(s) - q^2(s)\right|^2 ds\right]$$
$$+ E\left[\int_t^\infty e^{\lambda\,s} \int_{\mathbb{R}_0} \left|r^1(s,z) - r^2(s,z)\right|^2 ds\nu(dz)\right]$$
$$\leq \left(\frac{3C^2}{\epsilon} - \lambda + \frac{1}{2}\right) E\left[\int_t^\infty e^{\lambda s}\left|p^1(s) - p^2(s)\right|^2 ds\right]$$
$$+ \frac{1}{2} E\left[\int_t^\infty \left|q^1(s) - q^2(s)\right|^2 ds\right]$$
$$+ \frac{1}{2} E\left[\int_t^\infty \left|r^1(s,z) - r^2(s,z)\right|_{\mathcal{H}}^2 ds\right].$$

We get

$$E\left[e^{\lambda t}\left|p^1(t) - p^2(t)\right|^2\right] + \frac{1}{2} E\left[e^{\lambda s}\left|q^1(s) - q^2(s)\right|^2 ds\right]$$
$$+ \frac{1}{2} E\left[\int_t^\infty e^{\lambda s} \int_{\mathbb{R}_0} \left|r^1(s,z) - r^2(s,z)\right|^2 ds\nu(dz)\right]$$
$$\leq \left(\frac{3C^2}{\epsilon} - \lambda + \frac{1}{2}\right) E\left[\int_t^\infty e^{\lambda s}\left|p^1(s) - p^2(s)\right|^2 ds\right].$$

Using the fact that $\lambda \geq \frac{3C^2}{\epsilon} + \frac{1}{2}$, we obtain for all $t \in [0, \infty)$,

$$E\left[e^{\lambda t}\left|p^1(t) - p^2(t)\right|^2\right] = 0,$$

which proves that $p^1(t)$ and $p^2(t)$ are indistinguishable. ■

Chapter 4

Infinite horizon optimal control of FBSDEs with delay

We consider a problem of optimal control of an infinite horizon system governed by forward-backward stochastic differential equations with delay. Sufficient and necessary maximum principles for optimal control under partial information in infinite horizon are derived. We illustrate our results by an application to a problem of optimal consumption with respect to recursive utility from a cash flow with delay.

4.1 Introduction

One of the problems posed recently and which has got a lot of attention is the optimal control of forward-backward stochastic differential equations (FBSDEs). This theory was first developed in the early 90s by [4], [25], [39] and others.

The paper [39] established the maximum principle of FBSDE in the convex setting and later it was studied by many authors such as [3], [27], [32], [34], [44]. For the existence of an optimal control of FBSDEs, see [6].

The optimal control problem of FBSDE has interesting applications especially in finance like in option pricing and recursive utility problems. The latter was introduced by [14] and for more details about the recursive utility maximization problems, we refer to [9], [39].

The recursive utility is a solution of the backward stochastic differential equation (BSDE) which is not necessarily linear. The BSDE was studied by [38], [37] etc.

All the papers above where dealing with finite horizon FBSDEs. Other related stochastic control publications dealing with finite horizon only are [7], [29] and [24].

Related papers dealing with infinite horizon control, but either without FB systems or without delay, are [2], [19], [26], [40] and [45].

We will study this problem by using a version of the maximum principle which is a combination of the infinite horizon maximum principle in [2] and the finite horizon maximum principle for FBSDEs in [32] and [27]. We extend an application in [27] to infinite horizon and in [33] for the FBSDE.

We emphasize that although the current paper has similarities with [2], the fact that we are considering forward-backward systems and not just forward systems creates a new situation. In particular, we now get additional transversality conditions involving the additional adjoint process λ. See Theorem 3.1 and Theorem 4.1.

In this paper we obtain a sufficient and a necessary maximum principle for infinite horizon control of FBSDEs with delay. As an illustration we solve explicitly an infinite horizon optimal consumption problem with recursive utility.

The partial results mentioned above indicate that it should be possible to prove a general existence and uniqueness theorem for controlled infinite horizon FBSDEs with delay. However, this is difficult problem and we leave this for future research.

4.2 Setting of the problem

Let $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, P)$ be a complete filtered probability space on which a one-dimensional standard Brownian motion $B(t)$ and an independent compensated Poisson random measure $\tilde{N}(dt, da) = N(dt, da) - \nu(da)dt$ are defined. We assume that $\mathbb{F}$ is the natural filtration, made right continuous generated by the processes B and N.

We study the following infinite horizon coupled forward-backward stochastic differential equations control system with delay:

FORWARD EQUATION in the unknown measurable process $X^u(t)$:

$$\begin{aligned} dX(t) &= dX^u(t) = b\left(t, \mathbf{X}^u(t), u(t)\right) dt + \sigma\left(t, \mathbf{X}^u(t), u(t)\right) dB(t) \\ &\quad + \int_{\mathbb{R}_0} \theta\left(t, \mathbf{X}^u(t), u(t), a\right) \tilde{N}(dt, da); t \in [0, \infty), \\ X(t) &= X_0(t); \qquad\qquad t \in [-\delta, 0], \end{aligned} \tag{4.1}$$

where

$$\mathbf{X}^u(t) = \left(X^u(t), X_1^u(t), X_2^u(t)\right),$$

with

$$X_1^u(t) = X^u(t-\delta),\ \ X_2^u(t) = \int_{t-\delta}^{t} e^{-\rho(t-r)} X^u(r) dr,$$

and X_0 is a given continuous (and deterministic) function on $[-\delta, 0]$.

BACKWARD EQUATION in the unknown measurable processes $Y^u(t), Z^u(t), K^u(t, \cdot)$:

$$\begin{aligned} dY^u(t) &= -g\left(t, \mathbf{X}^u(t), Y^u(t), Z^u(t), u(t)\right) dt + Z^u(t) dB(t) \\ &\quad + \int_{\mathbb{R}_0} K^u(t, a)\, \tilde{N}(dt, da); t \in [0, \infty). \end{aligned} \tag{4.2}$$

We interpret the infinite horizon BSDE (4.2) in the sense of Pardoux [38] i.e. for all $T < \infty$, the triple $(Y^u(t), Z^u(t), K^u(t, \cdot))$ solves the equation

$$\begin{aligned} Y^u(t) &= Y(T) + \int_t^T g\left(s, \mathbf{X}^u(s), Y^u(s), Z^u(s), u(s)\right) ds - \int_t^T Z^u(s) dB(s) \\ &\quad - \int_t^T \int_{\mathbb{R}_0} K^u(s, a)\, \tilde{N}(ds, da);\ 0 \le t \le T. \end{aligned} \tag{4.3}$$

We call the process $(Y^u(t), Z^u(t), K^u(t, \cdot))$ the solution of (4.3) if it also satisfies

$$E[\sup_{t \ge 0}\ e^{\kappa t} \left(Y^u\right)^2(t) + \int_0^\infty e^{\kappa t}((Z^u)^2(t) + \int_{\mathbb{R}_0} \left(K^u\right)^2(t, a)\, \nu(da))dt] < \infty \tag{4.4}$$

for some constant $\kappa > 0$. We refer the reader to Section 4 in [38] for assumptions of the coefficients that insure the existence and uniqueness of the solution of the FBSDE system. Note that (4.4) implies in particular that $\lim_{t\to\infty} Y^u(t) = 0$.
Throughout this paper, we introduce the following notations

$$
\begin{aligned}
&\delta > 0, \rho > 0 \text{ are given constants,}\\
&b : [0,\infty) \times \mathbb{R}^3 \times \mathcal{U} \times \Omega \to \mathbb{R},\\
&\sigma : [0,\infty) \times \mathbb{R}^3 \times \mathcal{U} \times \Omega \to \mathbb{R},\\
&g : [0,\infty) \times \mathbb{R}^5 \times \mathcal{U} \times \Omega \to \mathbb{R},\\
&\mathbb{R}_0 := \mathbb{R} - \{0\},\\
&\theta, K : [0,\infty) \times \mathbb{R}^3 \times \mathcal{U} \times \mathbb{R}_0 \times \Omega \to \mathbb{R},\\
&f : [0,\infty) \times \mathbb{R}^5 \times \mathcal{R} \times \mathcal{U} \times \Omega \to \mathbb{R},\\
&h : \mathbb{R} \to \mathbb{R},
\end{aligned}
$$

where the coefficients b, σ, θ and g are Fréchet differentiable (C^1) with respect to the variables $(\boldsymbol{x}, y, z, u)$. Here $\mathcal{R}$ is the set of all functions $k : \mathbb{R}_0 \to \mathbb{R}$. In the following, we will for simplicity suppress the dependence on $\omega \in \Omega$ in the notation.

Note that if g does not depend on $Z^u(s)$ then the Itô representation theorem for Lévy processes (see [15]), implies that equation (4.3) is equivalent to the equation

$$Y^u(t) = E[Y(T) + \int_t^T g\left(s, \mathbf{X}^u(s), Y^u(s), u(s)\right) ds \mid \mathcal{F}_t];\ t \le T, \text{ for all } T < \infty. \tag{4.5}$$

Let $\mathbb{E} = \{\mathcal{E}_t\}_{t\ge 0}$ with $\mathcal{E}_t \subseteq \mathcal{F}_t$ for all $t \ge 0$ be a given subfiltration, representing the information available to the controller at time t.

Let $\mathcal{U}$ be a non-empty convex subset of $\mathbb{R}$. We let $\mathcal{A} = \mathcal{A}_{\mathcal{E}}$ denote a given locally convex family of admissible $\mathbb{E}$-predictable control processes u with values in $\mathcal{U}$, such that the corresponding solution (X^u, Y^u, Z^u, K^u) of (4.1) – (4.5) exist and

$$E[\int_0^\infty |X^u(t)|^2\, dt] < \infty.$$

The corresponding performance functional is

$$J(u) = E[\int_0^\infty f\left(t, \mathbf{X}(t)\right)\ dt + h(Y(0))], \tag{4.6}$$

where $f\left(t, \mathbf{X}(t)\right)$ is a short-hand notation for $f\left(t, \mathbf{X}^u(t), Y^u(t), Z^u(t), K^u\left(t, \cdot\right), u(t)\right)$.

We assume that the functions f and h are Fréchet differentiable (C^1) with respect to the variables $(\boldsymbol{x}, y, z, k(\cdot), u)$ and $Y(0)$, respectively, and f satisfies

$$E[\int_0^\infty |f(t,\mathbf{X}(t))\ |dt] < \infty, \text{ for all } u \in \mathcal{A}. \tag{4.7}$$

The optimal control problem is to find an optimal control $u^* \in \mathcal{A}$ and the value function $\Phi : C([-\delta, 0]) \to \mathbb{R}$ such that

$$\Phi(X_0) = \sup_{u\in\mathcal{A}} J(u) = J(u^*). \tag{4.8}$$

We will study this problem by using a version of the maximum principle which is a combination of the infinite horizon maximum principle in [2] and the finite horizon maximum principle for FBSDEs in [34] and [32].

The Hamiltonian

$$H : [0,\infty) \times \mathbb{R}^5 \times L^2(\nu) \times \mathcal{U} \times \mathbb{R}^3 \times L^2(\nu) \to \mathbb{R}$$

is defined by

$$\begin{aligned} H(t,\boldsymbol{x},y,z,k(\cdot),u,\lambda,p,q,r(\cdot)) &= f(t,\boldsymbol{x},y,z,k,u) + g(t,\boldsymbol{x},y,z,u)\lambda \\ &+ b(t,\boldsymbol{x},u)p + \sigma(t,\boldsymbol{x},u)q + \int_{\mathbb{R}_0} \theta(t,\boldsymbol{x},u,a)r(a)\nu(da). \end{aligned} \tag{4.9}$$

We assume that the Hamiltonian H is Fréchet differentiable (C^1) in the variables $\boldsymbol{x}, y, z, k$ and u.

We also assume that for all t the Fréchet derivative of $H(t,\mathbf{X}^u(t),Y^u(t),Z^u(t),k,u(t),p(t),q(t),r(t,\cdot))$ with respect to k, denoted by $\nabla_k H(t,\cdot)$, as a random measure is absolutely continuous with respect to ν, with Radon-Nikodym derivative $\frac{d\nabla_k H}{d\nu}$ satisfying

$$E[\int_0^T \int_{\mathbb{R}} \left|\frac{d\nabla_k H}{d\nu}(t,a)\right|^2 \nu(da)dt] < \infty, \text{ for all } T < \infty.$$

See Appendix A in [34] for details.

We associate to the problem (4.8) the following pair of forward-backward SDEs in the adjoint processes $\lambda(t)$, ($p(t), q(t), r(t,\cdot)$):

ADJOINT FORWARD EQUATION:

$$\begin{cases} d\lambda(t) = \dfrac{\partial H}{\partial y}(t)\ dt + \dfrac{\partial H}{\partial z}(t)\ dB(t) + \displaystyle\int_{\mathbb{R}_0} \frac{d\nabla_k H}{d\nu}(t,a)\tilde{N}(dt,da) \\ \lambda(0) = h^{'}(Y(0)) \end{cases} \tag{4.10}$$

where we have used the short hand notation

$$\tfrac{\partial H}{\partial y}(t) = \tfrac{\partial}{\partial y} H(t, \mathbf{X}^u(t), y, Z^u(t), K^u(t,\cdot), u(t), \lambda(t), p(t), q(t), r(t,\cdot)) \mid_{y=Y(t)}$$

and similarly with $\frac{\partial H}{\partial z}(t)$, $\frac{\partial H}{\partial x}(t), ...$

ADJOINT BACKWARD EQUATION:

$$dp(t) = E[\mu(t) \mid \mathcal{F}_t]dt + q(t)dB(t) + \int_{\mathbb{R}_0} r(t,a)\tilde{N}(dt,da); t \in [0,\infty) \tag{4.11}$$

where

$$\mu(t) = -\tfrac{\partial H}{\partial x}(t) - \tfrac{\partial H}{\partial x_1}(t+\delta) - e^{\rho t}(\int_t^{t+\delta} \tfrac{\partial H}{\partial x_2}(s)e^{-\rho s}ds). \tag{4.12}$$

with terminal condition as in (4.4), i.e.

$$E[\sup_{t\geq 0}\ e^{\kappa t}p^2(t) + \int_0^\infty e^{\kappa s}(q^2(s) + \int_{\mathbb{R}_0} r^2\,(s,a)\,\nu(da))ds] < \infty,$$

for some constant $\kappa > 0$.

The unknown process $\lambda(t)$ is the adjoint process corresponding to the backward system $(Y(t), Z(t), K(t,\cdot))$ and the triple unknown $(p(t), q(t), r(t,\cdot))$ is the adjoint process corresponding to the forward system $X(t)$.

We show that in this infinite horizon setting the appropriate terminal conditions for the BSDEs for $(Y(t), Z(t), K(t,\cdot))$ and $(p(t), q(t), r(t,\cdot))$ should be replaced by asymptotic transversality conditions. See (H_3) and (H_6) below.

4.3 Sufficient maximum principle for partial information

We will prove in this section that under some assumptions the maximization of the Hamiltonian leads to an optimal control.

Theoreme 4.3.1 *Let* $\hat{u} \in \mathcal{A}$ *with corresponding solutions* $\hat{\mathbf{X}}(t), \hat{Y}(t), \hat{Z}(t), \hat{K}(t,\cdot), \hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot)$ *and* $\hat{\lambda}(t)$ *of equations* (4.1), (4.2), (4.10) *and* (4.11)*. Suppose that*

(H_1) *(Concavity)*

The functions $x \to h(x)$ *and*

$$(\boldsymbol{x}, y, z, k(\cdot), u) \to H(t, \boldsymbol{x}, y, z, k(\cdot), u, \hat{\lambda}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot))$$

are concave, for all $t \in [0, \infty)$.

(H_2) *(The conditional maximum principle)*

$$\begin{aligned}&\max_{v \in \mathcal{U}} E[H(t, \hat{\mathbf{X}}(t), \hat{Y}(t), \hat{Z}(t), \hat{K}(t,\cdot), v, \hat{\lambda}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot)) \mid \mathcal{E}_t] \\ &= E[H(t, \hat{\mathbf{X}}(t), \hat{Y}(t), \hat{Z}(t), \hat{K}(t,\cdot), \hat{u}(t), \hat{\lambda}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot)) \mid \mathcal{E}_t].\end{aligned}$$

Moreover, suppose that for any $u \in \mathcal{A}$ *with corresponding solutions* $\mathbf{X}(t), Y(t), Z(t), K(t,\cdot), p(t), q(t), r(t,\cdot)$ *and* $\lambda(t)$ *we have:*

(H_3) *(Transversality conditions)*

$$\underline{\lim}_{T \to \infty} E[\ \hat{p}(T) \triangle \hat{X}(T)] \leq 0$$

and

$$\overline{\lim}_{T \to \infty} E\ [\hat{\lambda}(T) \triangle \hat{Y}(T)] \geq 0.$$

where $\triangle \hat{X}(T) = \hat{X}(T) - X(T), \ \ \triangle \hat{Y}(T) = \hat{Y}(T) - Y(T).$

(H_4) *(Growth conditions I) Suppose that for all $T < \infty$ the following holds:*

$$\begin{aligned}
&E[\int_0^T \{(\triangle\hat{Y}(t))^2\{(\tfrac{\partial\hat{H}}{\partial y}(t))^2 + \int_{\mathbb{R}_0} \left\|\nabla_k\hat{H}(t,a)\right\|^2 \nu(da)\} \\
&+ \hat{\lambda}^2(t)\{(\triangle\hat{Z}(t))^2 + \int_{\mathbb{R}_0}(\triangle\hat{K}(t,a))^2\nu(da)\} \\
&+ (\triangle\hat{X}(t))^2\{\hat{q}^2(t) + \int_{\mathbb{R}_0}\hat{r}^2(t,a)\nu(da)\} \\
&+ \hat{p}^2(t)\{(\triangle\hat{\sigma}(t))^2 + \int_{\mathbb{R}_0}(\triangle\hat{\theta}(t,a))^2\nu(da)\}\}dt] < \infty.
\end{aligned} \tag{4.13}$$

(H_5) *(Growth conditions II) Suppose that*

$$\begin{aligned}
&E[\int_0^T\{\left|\hat{\lambda}(t)\triangle\hat{g}(t)\right| + \left|\triangle\hat{Y}(t)\tfrac{\partial\hat{H}}{\partial y}(t)\right| + \left|\triangle\hat{Z}(t)\tfrac{\partial\hat{H}}{\partial z}(t)\right| \\
&+\int_{\mathbb{R}_0}\left|\nabla_k\hat{H}(t,a)\triangle\hat{K}(t,a)\right|\nu(da) + \left|\triangle\hat{H}(t)\right| + \left|\triangle\hat{b}(t)\hat{p}(t)\right| \\
&+|\triangle\hat{\sigma}(t)\hat{q}(t)| + \int_{\mathbb{R}_0}\left|\triangle\hat{\theta}(t,a)\hat{r}(t,a)\right|\nu(da) \\
&+\left|\triangle\hat{X}(t)\tfrac{\partial\hat{H}}{\partial x}(t)\right| + \left|\triangle\hat{u}(t)\tfrac{\partial\hat{H}}{\partial u}(t)\right|\}dt] < \infty.
\end{aligned} \tag{4.14}$$

where $\sigma(t) = \sigma(t, \mathbf{X}(t), u(t))$, $\hat{\sigma}(t) = \sigma(t, \hat{\mathbf{X}}(t), \hat{u}(t))$ *etc.*
Then $\hat{u}$ is an optimal control for (4.8), *i.e.*

$$J(\hat{u}) = \sup_{u\in\mathcal{A}} J(u).$$

Proof. Assume that $u \in \mathcal{A}$. We want to prove that $J(\hat{u}) - J(u) \geq 0$, i.e. $\hat{u}$ is an optimal control. We put

$$J(\hat{u}) - J(u) = I_1 + I_2, \tag{4.15}$$

where

$$I_1 = E[\int_0^\infty \{\hat{f}(t) - f(t)\}\, dt],$$

and

$$I_2 = E[h(\hat{Y}(0)) - h(Y(0))].$$

By the definition of H, we have

$$I_1 = E[\int_0^\infty \{\triangle\hat{H}(t) - \triangle\hat{g}(t)\hat{\lambda}(t) - \triangle\hat{b}(t)\hat{p}(t) - \triangle\hat{\sigma}(t)\hat{q}(t) - \int_{\mathbb{R}_0} \triangle\hat{\theta}(t,a)\hat{r}(t,a)\nu(da)\}dt], \quad (4.16)$$

where we have used the simplified notation

$$\begin{aligned}\hat{H}(t) &= H(t, \hat{\mathbf{X}}(t), \hat{Y}(t), \hat{Z}(t), \hat{K}(t,\cdot), \hat{u}(t), \hat{\lambda}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot))\\ H(t) &= H(t, \mathbf{X}(t), Y(t), Z(t), K(t,\cdot), u(t), \hat{\lambda}(t), \hat{p}(t), \hat{q}(t), \hat{r}(t,\cdot)) \text{ etc.}\end{aligned}$$

Since h is concave, we have

$$h(\hat{Y}(0)) - h(Y(0)) \geq h^{'}(\hat{Y}(0)) \triangle \hat{Y}(0) = \hat{\lambda}(0) \triangle \hat{Y}(0).$$

By Itô's formula, (H_4), (4.2) and (4.10), we have for all T

$$\begin{aligned}&E[\hat{\lambda}(0) \triangle \hat{Y}(0)] = E[\hat{\lambda}(T) \triangle \hat{Y}(T) - \int_0^T \hat{\lambda}(t)d(\triangle\hat{Y}(t)) - \int_0^T \triangle\hat{Y}(t)d\hat{\lambda}(t)\\ &- \int_0^T \triangle\hat{Z}(t)\tfrac{\partial\hat{H}}{\partial z}(t)dt - \int_0^T \int_{\mathbb{R}_0} \nabla_k\hat{H}(t,a) \triangle \hat{K}(t,a)\nu(da)dt].\end{aligned} \quad (4.17)$$

By (H_4) all the local martingales involved in (4.17) are martingales up to time T, for all $T < \infty$.

Therefore, letting $T \to \infty$, we obtain by (4.14)

$$\begin{aligned}&E[\hat{\lambda}(0) \triangle \hat{Y}(0)] = \lim_{T\to\infty} E\ [\hat{\lambda}(T) \triangle \hat{Y}(T)] - E[\int_0^\infty \{-\hat{\lambda}(t) \triangle \hat{g}(t)\\ &+ \triangle \hat{Y}(t)\tfrac{\partial\hat{H}}{\partial y}(t) + \triangle\hat{Z}(t)\tfrac{\partial\hat{H}}{\partial z}(t) + \int_{\mathbb{R}_0} \nabla_k\hat{H}(t,a) \triangle \hat{K}(t,a)\nu(da)\}dt].\end{aligned} \quad (4.18)$$

Combining (4.16) – (4.18), we obtain

$$\begin{aligned}&J(\hat{u}) - J(u) \geq \lim_{T\to\infty} E\ [\hat{\lambda}(T) \triangle \hat{Y}(T)] + E[\int_0^\infty \{\triangle\hat{H}(t) - \triangle\hat{b}(t)\hat{p}(t) - \triangle\hat{\sigma}(t)\hat{q}(t)\\ &- \int_{\mathbb{R}_0} \triangle \hat{\theta}(t,a)\hat{r}(t,a)\nu(da) - \triangle\hat{Y}(t)\tfrac{\partial\hat{H}}{\partial y}(t) - \triangle\hat{Z}(t)\tfrac{\partial\hat{H}}{\partial z}(t) - \int_{\mathbb{R}_0} \nabla_k\hat{H}(t,a) \triangle \hat{K}(t,a)\nu(da)\}dt].\end{aligned}$$

Since H is concave, we have

$$J(\hat{u}) - J(u) \geq \lim_{T\to\infty} E\ [\hat{\lambda}(T) \triangle \hat{Y}(T)] + E[\int_0^\infty \{\triangle \hat{X}(t) \tfrac{\partial \hat{H}}{\partial x}(t) + \triangle \hat{X}_1(t) \tfrac{\partial \hat{H}}{\partial x_1}(t)$$
$$+ \triangle \hat{X}_2(t) \tfrac{\partial \hat{H}}{\partial x_2}(t) + \triangle \hat{u}(t) \tfrac{\partial \hat{H}}{\partial u}(t) - \triangle \hat{b}(t)\hat{p}(t) - \triangle \hat{\sigma}(t)\hat{q}(t) - \int_{\mathbb{R}_0} \triangle \hat{\theta}(t,a)\hat{r}(t,a)\nu(da)\}dt]. \tag{4.19}$$

Applying now (H_1), (H_4) and (H_5) together with the Itô formula to $\hat{p}(t) \triangle \hat{X}(t)$, we get

$$\begin{aligned}
0 &\geq \lim_{T\to\infty} E\ [\ \hat{p}(T) \triangle \hat{X}(T)] \\
&= E[\int_0^\infty \{\triangle \hat{b}(t)\hat{p}(t) - \triangle \hat{X}(t) E[\hat{\mu}(t) \mid \mathcal{F}_t] + \triangle \hat{\sigma}(t)\hat{q}(t) + \int_{\mathbb{R}_0} \triangle \hat{\theta}(t,a)\hat{r}(t,a)\nu(da)\}dt] \\
&= E[\int_0^\infty \{\triangle \hat{b}(t)\hat{p}(t) - \triangle \hat{X}(t)\hat{\mu}(t) + \triangle \hat{\sigma}(t)\hat{q}(t) + \int_{\mathbb{R}_0} \triangle \hat{\theta}(t,a)\hat{r}(t,a)\nu(da)\}dt].
\end{aligned} \tag{4.20}$$

By the definition (4.12) of $\hat{\mu}$, we have

$$\begin{aligned}
&E[\int_0^\infty \triangle \hat{X}(t)\hat{\mu}(t)dt] = \lim_{T\to\infty} E[\int_\delta^{T+\delta} \triangle \hat{X}(t-\delta)\hat{\mu}(t-\delta)dt)] \\
&= \lim_{T\to\infty} E[-\int_\delta^{T+\delta} \tfrac{\partial \hat{H}}{\partial x}(t-\delta) \triangle \hat{X}(t-\delta)dt - \int_\delta^{T+\delta} \tfrac{\partial \hat{H}}{\partial x_1}(t) \triangle \hat{X}_1(t)dt \\
&- \int_\delta^{T+\delta} (\int_{t-\delta}^{t} \tfrac{\partial \hat{H}}{\partial x_2}(s)\, e^{-\rho s} ds) e^{\rho(t-\delta)} \triangle \hat{X}(t-\delta)dt].
\end{aligned} \tag{4.21}$$

Using Fubini's theorem and the definition of X_2, we obtain

$$\int_0^T \frac{\partial \hat{H}}{\partial x_2}(s) \triangle \hat{X}_2(s)ds = \int_\delta^{T+\delta} \left(\int_{t-\delta}^{t} \frac{\partial \hat{H}}{\partial x_2}(s)\, e^{-\rho s} ds \right) e^{\rho(t-\delta)} \triangle \hat{X}(t-\delta)dt. \tag{4.22}$$

Combining (4.19) with (4.20) – (4.22), we deduce that

$$\begin{aligned}
J(\hat{u}) - J(u) &\geq \lim_{T\to\infty} E\ [\hat{\lambda}(T) \triangle \hat{Y}(T)] - \lim_{T\to\infty} E[\hat{p}(T) \triangle \hat{X}(T)] + E[\int_0^\infty \triangle \hat{u}(t) \tfrac{\partial \hat{H}}{\partial u}(t)dt] \\
&= \lim_{T\to\infty} E[\ \hat{\lambda}(T) \triangle \hat{Y}(T)] - \lim_{T\to\infty} E\ [\hat{p}(T) \triangle \hat{X}(T)] + E[\int_0^\infty E\{\triangle \hat{u}(t) \tfrac{\partial \hat{H}}{\partial u}(t) \mid \mathcal{E}_t\}dt].
\end{aligned}$$

Then

$$J(\hat{u}) - J(u) \geq \lim_{T\to\infty} E[\hat{\lambda}(T)(\hat{Y}(T) - Y(T))] - \lim_{T\to\infty} E\ [\hat{p}(T) \triangle \hat{X}(T)] + E[\int_0^\infty E\{\tfrac{\partial \hat{H}}{\partial u}(t) \mid \mathcal{E}_t\} \triangle \hat{u}(t)dt].$$

By assumptions (H_1) and (H_3), we conclude $J(\hat{u}) - J(u) \geq 0,$ i.e. $\hat{u}$ is an optimal control.
■

4.4 Necessary conditions of optimality for partial information

A drawback of the previous section is that the concavity condition is not always satisfied in applications. In view of this, it is of interest to obtain conditions for the existence of an optimal control with partial information where concavity is not needed. We assume the following:

(A_1) For all $u \in \mathcal{A}$ and all $\beta \in \mathcal{A}$ bounded, there exists $\epsilon > 0$ such that

$$u + s\beta \in \mathcal{A} \qquad \text{for all } s \in (-\epsilon, \epsilon).$$

This implies in particular that the corresponding solution $X^{u+s\beta}(t)$ of $(4.1) - (4.5)$ exists.

(A_2) For all $t_0 > 0$, $h > 0$ and all bounded $\mathcal{E}_{t_0}$-measurable random variables α, the control process $\beta(t)$ defined by

$$\beta(t) = \alpha 1_{[t_0, t_0+h)}(t) \tag{4.23}$$

belongs to $\mathcal{A}$.

(A_3) The following derivative processes exist

$$\xi(t) := \tfrac{d}{ds} X^{u+s\beta}(t) \mid_{s=0} \tag{4.24}$$

$$\phi(t) := \tfrac{d}{ds} Y^{u+s\beta}(t) \mid_{s=0} \tag{4.25}$$

$$\eta(t) := \tfrac{d}{ds} Z^{u+s\beta}(t) \mid_{s=0} \tag{4.26}$$

$$\psi(t,a) := \tfrac{d}{ds} K^{u+s\beta}(t,a) \mid_{s=0} \tag{4.27}$$

(A_4) We also assume that

$$\begin{aligned} E[\int_0^\infty \{ \left|\tfrac{\partial f}{\partial x}(t)\xi(t)\right| + \left|\tfrac{\partial f}{\partial x_1}(t)\xi(t-\delta)\right| + \left|\tfrac{\partial f}{\partial x_2}(t) \int_{t-\delta}^{t} e^{-\rho(t-r)}\xi(r)dr\right| + \left|\tfrac{\partial f}{\partial y}(t)\phi(t)\right| \\ + \left|\tfrac{\partial f}{\partial z}(t)\eta(t)\right| + \left|\tfrac{\partial f}{\partial u}(t)\beta(t)\right| + \int_{\mathbb{R}_0} \left|\nabla_k f(t,a)\psi(t,a)\right| \nu(da)\} dt] < \infty. \end{aligned} \tag{4.28}$$

We can see that

$$\tfrac{d}{ds} X_1^{u+s\beta}(t) \mid_{s=0} = \xi(t-\delta)$$

and

$$\frac{d}{ds} X_2^{u+s\beta}(t) \mid_{s=0} = \int_{t-\delta}^{t} e^{-\rho(t-r)}\xi(r)dr.$$

Note that

$$\xi(t) = 0 \text{ for } t \in [-\delta, 0].$$

Theoreme 4.4.1 *Assume that* $(A_1)-(A_4)$ *hold. Suppose that* $\hat{u} \in \mathcal{A}$ *with corresponding solutions* $\hat{\mathbf{X}}(t), \hat{Y}(t), \hat{Z}(t), \hat{K}(t,\cdot), \hat{\lambda}(t), \hat{p}(t), \hat{q}(t)$ *and* $\hat{r}(t,\cdot)$ *of equations* (4.1), (4.2), (4.10) *and* (4.11).
Assume that (4.13) *and the following transversality conditions hold:*

(H_6)

$$\lim_{T\to\infty} E[\ \hat{p}(T)\xi(T)] = 0,$$
$$\lim_{T\to\infty} E[\hat{\lambda}(T)\phi(T)] = 0.$$

(H_7) *Moreover, assume that the following growth condition holds*

$$E[\int_0^T\{\hat{\lambda}^2(t)(\eta^2(t)+\int_{\mathbb{R}_0}\psi^2(t,a)\nu(da))+\phi^2(t)((\tfrac{\partial\hat{H}}{\partial z})^2(t)+\int_{\mathbb{R}_0}\nabla_k\hat{H}^2(t,a)\nu(da))$$
$$+\hat{p}^2(t)(\tfrac{\partial\sigma}{\partial x}(t)\xi(t)+\tfrac{\partial\sigma}{\partial x_1}(t)\xi(t-\delta)+\tfrac{\partial\sigma}{\partial x_2}(t)\int_{t-\delta}^t e^{-\rho(t-r)}\xi(r)dr+\tfrac{\partial\sigma}{\partial u}(t)\beta(t))^2$$
$$+\hat{p}^2(t)(\int_{\mathbb{R}_0}\{\tfrac{\partial\theta}{\partial x}(t,a)\xi(t)+\tfrac{\partial\theta}{\partial x_1}(t,a)\xi(t-\delta)+\tfrac{\partial\theta}{\partial x_2}(t,a)\int_{t-\delta}^t e^{-\rho(t-r)}\xi(r)dr$$
$$+\tfrac{\partial\theta}{\partial u}(t,a)\beta(t)\}^2\nu(da))\}dt]<\infty,\ \textit{for all}\ \ T<\infty.$$

Then the following assertions are equivalent.

(i) *For all bounded* $\beta\in\mathcal{A}$,

$$\frac{d}{ds}J(\hat{u}+s\beta)\mid_{s=0}=0.$$

(ii) *For all* $t\in[0,\infty)$,

$$E[\frac{\partial}{\partial u}H(t,\hat{\mathbf{X}}(t),\hat{Y}(t),\hat{Z}(t),\hat{K}(t,\cdot),u,\hat{\lambda}(t),\hat{p}(t),\hat{q}(t),\hat{r}(t,\cdot))\mid\mathcal{E}_t]_{u=\hat{u}(t)}=0.$$

Proof. $(i)\Longrightarrow(ii)$:
In the following we use the short- hand notation $\frac{\partial b}{\partial x_i}(t)=\frac{\partial}{\partial x_i}b(t,\mathbf{x},u(t))_{\mathbf{x}=\mathbf{X}(t)}$ etc; $i=1,2,3$. It follows from (4.1) that

$$d\xi(t)=\{\tfrac{\partial b}{\partial x}(t)\xi(t)+\tfrac{\partial b}{\partial x_1}(t)\xi(t-\delta)+\tfrac{\partial b}{\partial x_2}(t)\int_{t-\delta}^t e^{-\rho(t-r)}\xi(r)dr+\tfrac{\partial b}{\partial u}(t)\beta(t)\}dt$$
$$+\{\tfrac{\partial\sigma}{\partial x}(t)\xi(t)+\tfrac{\partial\sigma}{\partial x_1}(t)\xi(t-\delta)+\tfrac{\partial\sigma}{\partial x_2}(t)\int_{t-\delta}^t e^{-\rho(t-r)}\xi(r)dr+\tfrac{\partial\sigma}{\partial u}(t)\beta(t)\}dB(t)$$
$$+\int_{\mathbb{R}_0}\{\tfrac{\partial\theta}{\partial x}(t,a)\xi(t)+\tfrac{\partial\theta}{\partial x_1}(t,a)\xi(t-\delta)+\tfrac{\partial\theta}{\partial x_2}(t,a)\int_{t-\delta}^t e^{-\rho(t-r)}\xi(r)dr+\tfrac{\partial\theta}{\partial u}(t,a)\beta(t)\}\tilde{N}(dt,da),$$

and

$$d\phi(t)=\{-\tfrac{\partial g}{\partial x}(t)\xi(t)-\tfrac{\partial g}{\partial x_1}(t)\xi(t-\delta)-\tfrac{\partial g}{\partial x_2}(t)\int_{t-\delta}^t e^{-\rho(t-r)}\xi(r)dr-\tfrac{\partial g}{\partial y}(t)\phi(t)$$
$$-\tfrac{\partial g}{\partial u}(t)\beta(t)-\tfrac{\partial g}{\partial z}(t)\eta(t)\}dt+\eta(t)dB(t)+\int_{\mathbb{R}_0}\psi(t,a)\tilde{N}(dt,da),$$

Suppose that assertion (i) holds. Then by (A_4) and dominated convergence

$$\begin{aligned}
&0 = \tfrac{d}{ds}J(\hat{u}+s\beta)\mid_{s=0} \\
&= E[\int_0^\infty\{\tfrac{\partial f}{\partial x}(t)\xi(t) + \tfrac{\partial f}{\partial x_1}(t)\xi(t-\delta) + \tfrac{\partial f}{\partial x_2}(t)\int_{t-\delta}^{t} e^{-\rho(t-r)}\xi(r)dr + \tfrac{\partial f}{\partial y}(t)\phi(t) \\
&+\tfrac{\partial f}{\partial z}(t)\eta(t) + \tfrac{\partial f}{\partial u}(t)\beta(t) + \int_{\mathbb{R}_0}\nabla_k f(t,a)\psi(t,a)\nu(da)\}dt + h'(\hat{Y}(0))\phi(0)].
\end{aligned} \tag{4.29}$$

We know by the definition of H that

$$\frac{\partial f}{\partial x}(t) = \frac{\partial H}{\partial x}(t) - \frac{\partial g}{\partial x}(t)\lambda(t) - \frac{\partial b}{\partial x}(t)p(t) - \frac{\partial \sigma}{\partial x}(t)q(t) - \int_{\mathbb{R}_0}\frac{\partial \theta}{\partial x}(t,a)r(t,a)\nu(da)$$

and similarly for $\frac{\partial f}{\partial x_1}(t)$, $\frac{\partial f}{\partial x_2}(t)$, $\frac{\partial f}{\partial u}(t)$, $\frac{\partial f}{\partial y}(t)$, $\frac{\partial f}{\partial z}(t)$ and $\nabla_k f(t,a)$.
By the Itô formula and (H_7), the local martingales which appear after integration by parts of the process $\hat{\lambda}(t)\phi(t)$ are martingales, and we get

$$\begin{aligned}
&E[h'(\hat{Y}(0)\phi(0))] = E[\hat{\lambda}(0)\phi(0)] \\
&= \lim_{T\to\infty}E[\hat{\lambda}(T)\phi(T)] \\
&-\lim_{T\to\infty}E[\int_0^T\{\hat{\lambda}(t)(-\tfrac{\partial g}{\partial x}(t)\xi(t) - \tfrac{\partial g}{\partial x_1}(t)\xi(t-\delta) - \tfrac{\partial g}{\partial x_2}(t)\int_{t-\delta}^{t} e^{-\rho(t-r)}\xi(r)dr - \tfrac{\partial g}{\partial y}(t)\phi(t) \\
&-\tfrac{\partial g}{\partial z}(t)\eta(t) - \tfrac{\partial g}{\partial u}(t)\beta(t)) + \phi(t)\tfrac{\partial H}{\partial y}(t) + \eta(t)\tfrac{\partial H}{\partial z}(t) + \int_{\mathbb{R}_0}\nabla_k H(t,a)\psi(t,a)\nu(da)\}dt].
\end{aligned} \tag{4.30}$$

Substituting (4.30) into (4.29) we get

$$\begin{aligned}
&0 = \tfrac{d}{ds}J(\hat{u}+s\beta)\mid_{s=0} \\
&= E[\int_0^\infty\{\tfrac{\partial f}{\partial x}(t)\xi(t) + \tfrac{\partial f}{\partial x_1}(t)\xi(t-\delta) + \tfrac{\partial f}{\partial x_2}(t)\int_{t-\delta}^{t} e^{-\rho(t-r)}\xi(r)dr + \tfrac{\partial f}{\partial y}(t)\phi(t) \\
&+\tfrac{\partial f}{\partial z}(t)\eta(t) + \tfrac{\partial f}{\partial u}(t)\beta(t) + \int_{\mathbb{R}_0}\nabla_k f(t,a)\psi(t,a)\nu(da) \\
&-\hat{\lambda}(t)(-\tfrac{\partial g}{\partial x}(t)\xi(t) - \tfrac{\partial g}{\partial x_1}(t)\xi(t-\delta) - \tfrac{\partial g}{\partial x_2}(t)\int_{t-\delta}^{t} e^{-\rho(t-r)}\xi(r)dr - \tfrac{\partial g}{\partial y}(t)\phi(t) \\
&-\tfrac{\partial g}{\partial z}(t)\eta(t) - \tfrac{\partial g}{\partial u}(t)\beta(t)) + \phi(t)\tfrac{\partial H}{\partial y}(t) + \eta(t)\tfrac{\partial H}{\partial z}(t) + \int_{\mathbb{R}_0}\nabla_k H(t,a)\psi(t,a)\nu(da)\}dt].
\end{aligned} \tag{4.31}$$

Applying the Itô formula to the process $\hat{p}(t)\xi(t)$ and using (H_7), we get

$$\begin{aligned}
0 &= \lim_{T\to\infty} E[\hat{p}(T)\ \xi(T)] \\
&= E[\int_0^\infty \hat{p}(t)\ \{\tfrac{\partial b}{\partial x}(t)\xi(t) + \tfrac{\partial b}{\partial x_1}(t)\xi(t-\delta) + \tfrac{\partial b}{\partial x_2}(t) \int_{t-\delta}^{t} e^{-\rho(t-r)}\xi(r)dr + \tfrac{\partial b}{\partial u}(t)\beta(t)\}dt \\
&+ \int_0^\infty \xi(t)E[\mu(t) \mid \mathcal{F}_t]dt + \int_0^\infty \hat{q}(t)\{\tfrac{\partial \sigma}{\partial x}(t)\xi(t) + \tfrac{\partial \sigma}{\partial x_1}(t)\xi(t-\delta) + \tfrac{\partial \sigma}{\partial x_2}(t) \int_{t-\delta}^{t} e^{-\rho(t-r)}\xi(r)dr + \tfrac{\partial \sigma}{\partial u}(t)\beta(t)\}dt \\
&+ \int_0^\infty \int_{\mathbb{R}_0} \hat{r}(t,a)\{\tfrac{\partial \theta}{\partial x}(t,a)\xi(t) + \tfrac{\partial \theta}{\partial x_1}(t,a)\xi(t-\delta) + \tfrac{\partial \theta}{\partial x_2}(t,a) \int_{t-\delta}^{t} e^{-\rho(t-r)}\xi(r)dr + \tfrac{\partial \theta}{\partial u}(t,a)\beta(t)\}\nu(da)dt] \\
&= -\tfrac{d}{ds}J(\hat{u}+s\beta)\mid_{s=0} + E[\int_0^\infty \tfrac{\partial H}{\partial u}(t)\beta(t)dt].
\end{aligned} \tag{4.32}$$

Adding (4.31) and (4.32) we obtain

$$E[\int_0^\infty \frac{\partial H}{\partial u}(t)\beta(t)dt] = 0.$$

Now apply this to

$$\beta(t) = \alpha 1_{[s,s+h)}(t),$$

where α is bounded and $\mathcal{E}_{t_0}$-mesurable, $s \geq t_0$. Then we get

$$E[\int_s^{s+h} \frac{\partial H}{\partial u}(s)ds\ \alpha] = 0.$$

Differentiating with respect to h at $h = 0$ we obtain

$$E[\frac{\partial H}{\partial u}(s)\ \alpha] = 0.$$

Since this holds for all $s \geq t_0$ and all α, we conclude

$$E[\frac{\partial H}{\partial u}(t_0) \mid \mathcal{E}_{t_0}] = 0.$$

This proves that (i) implies (ii).

$(ii) \Longrightarrow (i)$:

The argument above shows that

$$\frac{d}{ds}J(u+s\beta)\mid_{s=0}=E[\int_0^\infty \frac{\partial H}{\partial u}(t)\beta(t)dt],$$

for all u, $\beta \in \mathcal{A}$ with β bounded. So to complete the proof we use that every bounded $\beta \in \mathcal{A}$ can be approximated by linear combinations of controls β of the form (4.23). We omit the details. ■

4.5 Application to optimal consumption with respect to recursive utility

4.5.1 A general optimal recursive utility problem

Let $X(t) = X^{(c)}(t)$ be a cash flow modeled by

$$\begin{cases} dX(t) = X(t-\delta)[b_0(t)dt + \sigma_0(t)dB(t) + \int_{\mathbb{R}_0} \gamma(t,a)\tilde{N}(dt,da)] - c(t)dt; t \geq 0, \\ X(0) = x > 0, \end{cases} \tag{4.33}$$

where $b_0(t)$, $\sigma_0(t)$ and $\gamma(t,a)$ are given bounded $\mathbb{F}$-predictable processes, $\delta \geq 0$ is a fixed delay and $\gamma(t,a) > -1$ for all $(t,a) \in [0,\infty) \times \mathbb{R}$.
The process $u(t) = c(t) \geq 0$ is our control process, interpreted as our relative consumption rate such that $X^{(c)}(t) > 0$ for all $t \geq 0$. We let $\mathcal{A}$ denote the family of all $\mathbb{E}$-predictable relative consumption rates. To every $c \in \mathcal{A}$ we associate a recursive utility process $Y^{(c)}(t) = Y(t)$ defined as the solution of the infinite horizon BSDE

$$Y(t) = E[Y(T) + \int_t^T g\left(s, Y(s), c(s)\right) ds \mid \mathcal{F}_t] \text{ for all } t \leq T, \tag{4.34}$$

valid for all deterministic $T < \infty$. The number $Y^{(c)}(0)$ is called the recursive utility of consumption process $c(t)$; $t \geq 0$ (See e.g. Duffie & Epstein (1992), [14]).
Suppose the solution (Y, Z, K) of the infinite horizon BSDE (4.34) satisfies the condition (4.4) and let $c(s); s \geq 0$ be the consumption rate.

We assume that the function $g(t, y, c) : \mathbb{R}^3_+ \to \mathbb{R}$ satisfies the following conditions:

1. $g(t, y, c)$ is concave with respect to y and c

2.
$$\int_0^T E\left[|g(s, Y(s), c(s))|\right] ds < \infty, \text{ for all } c \in \mathcal{A},\, T < \infty. \tag{4.35}$$

3. $\frac{\partial}{\partial c} g(t, y, c)$ has an inverse:

$$I(t, v, y) = \begin{cases} 0 & \text{if } v \geq v_0(t, y), \\ \left(\dfrac{\partial}{\partial c} g(t, y, c)\right)^{-1} (v) & \text{if } 0 \leq v \leq v_0(t, y), \end{cases}$$

where $v_0(t, y) = \frac{\partial}{\partial c} g(t, y, 0)$.
We want to maximize the recursive utility $Y^{(c)}(0)$, i.e. we want to find $c^* \in \mathcal{A}$ such that

$$\sup_{c \in \mathcal{A}} Y^{(c)}(0) = Y^{(c^*)}(0). \tag{4.36}$$

We call such a process c^* an optimal recursive utility consumption rate.

We see that the problem (4.36) is a special case of problem (4.8) with

$$J(u) = Y(0),$$

$f = 0$, $h(y) = y$, $u = c$ and

$$\begin{aligned} b(t, \boldsymbol{x}, c) &= x_1 b_0(t) - c, \\ \sigma(t, \boldsymbol{x}, u) &= x_1 \sigma_0(t), \\ \theta(t, \boldsymbol{x}, u, a) &= x_1 \gamma(t, a). \end{aligned}$$

In this case the Hamiltonian defined in (4.9) takes the form

$$\begin{aligned} H(t, \boldsymbol{x}, y, z, c, \lambda, p, q, r(\cdot)) &= \lambda g(t, y, c) + (x_1 b_0(t) - c)\, p \\ &\quad + x_1 \sigma_0(t) q + x_1 \int_{\mathbb{R}_0} \gamma(t, a) r(a) \nu(da). \end{aligned} \tag{4.37}$$

Maximizing $E[H \mid \mathcal{E}_t]$ as a function of c gives the first order condition

$$E[\lambda(t)\frac{\partial g}{\partial c}(t, Y(t), c(t)) \mid \mathcal{E}_t] = E[p(t) \mid \mathcal{E}_t], \tag{4.38}$$

for an optimal $c(t)$.

The pair of adjoint processes $(4.10) - (4.11)$ is given by

$$\begin{cases} d\lambda(t) = \lambda(t)\frac{\partial g}{\partial y}(t, Y(t), c(t))dt, \\ \lambda(0) = 1, \end{cases} \tag{4.39}$$

and

$$dp(t) = E[\mu(t) \mid \mathcal{F}_t]dt + q(t)dB(t) + \int_{\mathbb{R}_0} r(t,a)\tilde{N}(dt, da); t \in [0, \infty)\,, \tag{4.40}$$

where

$$\mu(t) = -[b_0(t+\delta)p(t+\delta) + \sigma_0(t+\delta)q(t+\delta) + \int_{\mathbb{R}_0} \gamma(t+\delta, a)r(t+\delta, a)\nu(da)]. \tag{4.41}$$

with terminal condition as in (4.4), i.e.

$$E[\sup_{t\geq 0}\ e^{\kappa t}p^2(t) + \int_0^\infty e^{\kappa s}(q^2(s) + \int_{\mathbb{R}_0} r^2\,(s,a)\,\nu(da))ds] < \infty,$$

for some constant $\kappa > 0$.

Equation (4.39) has the solution

$$\lambda(t) = \exp(\int_0^t \frac{\partial g}{\partial y}(s, Y(s), c(s))ds); t \geq 0 \tag{4.42}$$

which substituted into (4.38) gives

$$E[\tfrac{\partial g}{\partial c}(t, Y(t), c(t)) \exp(\int_0^t \tfrac{\partial g}{\partial y}(s, Y(s), c(s))ds) \mid \mathcal{E}_t] = E[p(t) \mid \mathcal{E}_t]. \tag{4.43}$$

We refer to Theorem 5.1 in [2] for a proof of the existence of the solution of the ABSDE (4.40).

4.5.2 A solvable special case

In order to get a solvable case we choose the driver g in (4.34) to be of the form

$$g(t, y, c) = -\alpha(t)y + \ln c, \tag{4.44}$$

where $\alpha(t) \geq \alpha > 0$ is an $\mathbb{F}$-adapted process.
We also choose

$$\delta = 0 \text{ and } \mathcal{E}_t = \mathcal{F}_t; t \geq 0, \tag{4.45}$$

and we represent the consumption rate $c(t)$ as

$$c(t) = \rho(t)X(t), \tag{4.46}$$

where $\rho(t) = \frac{c(t)}{X(t)} \geq 0$ is the relative consumption rate.
We restrict our attention to processes c such that the wealth process, solution of (4.33), is strictly positive and ρ is bounded away from 0. This set of controls ρ is denoted by $\mathcal{A}$.
The FBSDE system now has the form

$$\begin{cases} dX(t) = X(t^-)[(b_0(t) - \rho(t))dt + \sigma_0(t)dB(t) + \int_{\mathbb{R}_0} \gamma(t, a)\tilde{N}(dt, da)]; t \geq 0, \\ X(0) = x > 0, \end{cases} \tag{4.47}$$

and

$$Y(t) = Y^{(\rho)}(t) = E[Y(T) + \int_t^T (-\alpha(s)Y(s) + \ln \rho(s)X(s))\, ds \mid \mathcal{F}_t], \tag{4.48}$$

i.e.

$$dY(t) = -\left(-\alpha(t)Y(t) + \ln \rho(t) + \ln(X(t))\right) dt + Z(t)dB(t); t \geq 0. \tag{4.49}$$

We want to find $\rho^* \in \mathcal{A}$ such that

$$\sup_{\rho \in \mathcal{A}} Y^{(\rho)}(0) = Y^{(\rho^*)}(0). \tag{4.50}$$

In this case the Hamiltonian (4.9) gets the form

$$H(t,x,y,\rho,\lambda,p,q,r) = \lambda(-\alpha(t)y + \ln\rho + \ln x) + x\left(b_0(t) - \rho\right)p$$
$$+ x\sigma_0(t)q + x\int_{\mathbb{R}_0} \gamma(t,a)r(a)\nu(da). \tag{4.51}$$

Maximizing H with respect to ρ gives the first order equation

$$\lambda(t)\frac{1}{\rho(t)} = p(t)X(t), \tag{4.52}$$

where, by (4.10) – (4.11) $\lambda(t)$ and $(p(t), q(t), r(t,\cdot))$ satisfy the FBSDEs

$$\begin{cases} d\lambda(t) = -\alpha(t)\lambda(t)dt, \\ \lambda(0) = 1, \end{cases} \tag{4.53}$$

and

$$\begin{aligned} dp(t) &= -[\lambda(t)\tfrac{1}{X(t)} + \left(b_0(t) - \rho(t)\right)p(t) + \sigma_0(t)q(t) \\ &+ \int_{\mathbb{R}_0} \gamma(t,a)r(a)\nu(da)]dt + q(t)dB(t) + \int_{\mathbb{R}_0} r(t,a)\tilde{N}(dt,da), \end{aligned} \tag{4.54}$$

with terminal condition as in (4.4), i.e.

$$E[\sup_{t\geq 0}\ e^{\kappa t}p^2(t) + \int_0^\infty e^{\kappa s}(q^2(s) + \int_{\mathbb{R}_0} r^2\left(s,a\right)\nu(da))ds] < \infty, \tag{4.55}$$

for some constant $\kappa > 0$.

The infinite horizon BSDE (4.54) – (4.55) has a unique solution, (see e.g. Theorem 3.1 in [19]).

Then, the solutions of (4.53) – (4.54) are respectively,

$$\lambda(t) = \exp(-\int_0^t \alpha(s)ds), \tag{4.56}$$

and, for all $0 \leq t \leq T$ and all $T < \infty$,

$$p(t)\Gamma(t) = E[p(T)\Gamma(T) + \int_t^T \lambda(s)\frac{\Gamma(s)}{X(s)}ds \mid \mathcal{F}_t], \tag{4.57}$$

where $\Gamma(t)$ is given by

$$\begin{cases} d\Gamma(t) = \Gamma(t^-)[(b_0(t) - \rho(t))\,dt + \sigma_0(t)dB(t) + \int_{\mathbb{R}_0} \gamma(t,a)\tilde{N}(dt,da)];\, t \geq 0, \\ \Gamma(0) = 1. \end{cases} \tag{4.58}$$

(See e.g. [35]).

This gives

$$\begin{aligned} \Gamma(t) = \exp\Big(& -\int_0^t \sigma_0(s)dB(s) + \int_0^t \{b_0(s) - \rho(s) - \tfrac{1}{2}\sigma_0^2(s)\}ds \\ & + \int_0^t \int_{\mathbb{R}_0} \{\ln(1+\gamma(s,a)) - \gamma(s,a)\}\nu(da)ds \\ & + \int_0^t \int_{\mathbb{R}_0} \ln(1+\gamma(s,a))\tilde{N}(ds,da) \Big);\, t \geq 0. \end{aligned} \tag{4.59}$$

Comparing with (4.47) we see that

$$X(t) = x\Gamma(t);\, t \geq 0. \tag{4.60}$$

Substituting this into (4.57) we obtain

$$p(t)X(t) = E[p(T)X(T) + \int_t^T \exp(-\int_0^s \alpha(r)dr)ds \mid \mathcal{F}_t]. \tag{4.61}$$

Since ρ is bounded away from 0 we deduce from (4.52) that

$$p(T)X(T) = \frac{\lambda(T)}{\rho(T)} = \frac{1}{\rho(T)} \exp(-\int_0^T \alpha(r)dr) \to 0 \text{ dominatedly as } T \to \infty. \tag{4.62}$$

Hence, by letting $T \to \infty$ in (4.61) we get

$$p(t)X(t) = E[\int_t^\infty \exp(-\int_0^s \alpha(r)dr)ds \mid \mathcal{F}_t]. \tag{4.63}$$

This implies that $p(t) > 0$ and hence $\rho(t)$ given by (4.52) is indeed a maximum point of H.

By (4.52) we therefore get the following candidate for the optimal relative consumption rate

$$\rho(t) = \rho^*(t) = \frac{\exp(-\int_0^t \alpha(r)dr)}{E[\int_t^\infty \exp(-\int_0^s \alpha(r)dr)ds \mid \mathcal{F}_t]}; t \geq 0, \tag{4.64}$$

If α is such that this expression for $\rho^*(t)$ is bounded away from 0, then ρ^* is optimal. Note that the corresponding optimal net cash flow $X^*(t)$ is given by

$$X^*(t) = x \exp(\int_0^t \sigma_0(s)dB(s) + \int_0^t \{b_0(s) - \rho(s) - \frac{1}{2}\sigma_0^2(s)\}ds); t \geq 0. \tag{4.65}$$

In particular, $X^*(t) > 0$ for all $t \geq 0$, as required.

In particular, if $\alpha(r) = \alpha > 0$ (constant) for all r, then

$$\rho^*(t) = \alpha; t \geq 0. \tag{4.66}$$

With this choice of ρ^* we see by (4.63), (4.56) and condition (4.4) for $Y(t)$ that the transversality conditions (H_3) and (H_6) hold, and we have proved:

Theoreme 4.5.1 *The optimal relative consumption rate $\rho^*(t)$ for problem (4.44) − (4.50) is given by (4.64), provided that $\rho^*(t)$ is bounded away from 0.*

In particular, if $\alpha(r) = \alpha > 0$ (constant) for all r, then $\rho^*(t) = \alpha$ for all t.

Bibliography

[1] N. Agram and B. Øksendal. Infinite horizon optimal control of forward-backward stochastic differential equations with delay. Computational and Applied Mathematics., 259, 336-349. (2014). http://dx.doi.org/10.1016/j.cam.2013.04.048.

[2] N. Agram, S. Haadem, B. Øksendal, and F. Proske. A maximum principle for infinite horizon delay equations. SIAM J. Math. Anal., 45(4), 2499–2522. (2013).

[3] A. Al-Hussein and B. Gherbal. Maximum principle for optimal control of forward-backward doubly stochastic differential equations with jumps. arXiv (2013).

[4] F. Antonelli. Backward-forward stochastic differential equations, Ann. Appl. Prob.,3 (1993), 777-793.

[5] F. Baghery and B. Øksendal. A maximum principle for stochastic control with partial information. Stochastic Analysis and Applications, 25(3), 705–717. (2007).

[6] K. Bahlali, B. Gherbal and B. Mezerdi. Existence of optimal controls for systems driven by FBSDEs. Systems & Control Letters (2011)344-349.

[7] S. Bahlali and B. Labed. Necessary and sufficient conditions of optimality for optimal control problem with initial and terminal costs, Random Oper. Stoch. Equ. 14, No. 3, 291-301 (2006).

[8] G. Barles, R. Buckdahn, and E. Pardoux. Backward stochastic differential equations and integral-partial diffFraerential equations. Stochastics and Stochastics Reports, 60:57–83, 2009.

[9] Bordigoni, G., Matoussi, A., Schweizer, M.: A stochastic control approach to a robust utility maximization problem. In: Benth, F.E. et al (eds):Stochastic Analysis and Applications. The Abel Symposium 2005,Springer (2007), pp. 125-151.

[10] A. Bensoussan. Stochastic maximum principle for distributed parametre system, J Franklin Inst., 315 (1983), 387-406.

[11] R. Bellman. Dynamic Programming, Princeton University Press, Princeton, New Jersey, 1957.

[12] J. M. Bismut. An introduction approach to duality in optimal stochastic control. SIAM Rev. 20 (1978), 62-78.

[13] L. Chen and Z. Wu. Maximum principle for the stochastic optimal control problem with delay and application. Automatica, 46:1074–1080, 2010.

[14] D. Duffie and L.G. Epstein. Stochastic differential utility. Econometrica, volume 60, Issue 2 (1992), 353-394.

[15] G. Di Nunno, B. Øksendal and F. Proske. Malliavin Calculus for Lévy Processes with Application to Finance. Springer, first edition, (2009).I.

[16] Elsanosi, B. Øksendal, and A. Sulem. Some solvable stochastic control problems with delay. Stochastics and Stochastics Reports, 71:69–89, 2000.

[17] W. H. Fleming, and R. W. Rishel. Deterministic and Stochastic Optimal Control, Springer-Verlag, New York, New York, 1975.

[18] N. C. Framstad, B. Øksendal and A. Sulem. Sufficient Stochastic Maximum Principle for the Optimal Control of Jump Diffusions and Applications to Finance. J. Optimization Theory and Applications (2005) Volume 124, Issue 2, pp 511-512.

[19] S. Haadem, B. Øksendal and F. Proske. A maximum principle for jump diffusion processes with infinite horizon. Automatica, 49:2267–2275 (2013). http://dx.doi.org/10.1016/j.automatica.2013.04.011.

[20] H. Halkin. Necessary conditions for optimal control problems with infinite horizons. Econometrica, 42:267–272, 1974.

[21] U.G. Haussmann. General necessary conditions for optimal control of stochastic system. Mathematical Programming Study, 6, 30–48. (1976).

[22] H. Kushner, necessary conditions for continuous parameter stochastic optimisation problems. SIAM J. Control and Optimisation. 10 (1972), 550-565.J.

[23] Li and S. Peng. Stochastic optimization theory of backward stochastic differential equations with jumps and viscosity solutions of Hamilton-Jacobi-Bellman equations. Nonlinear Analysis, 70:1779–1796, 2009.

[24] X. Li. and S. Tang Necessary conditions for optimal control of stochastic systems with random jumps. SIAM J. Control and Optimisation Vol. 32, No. 5, pp. 1447-1475, September 1994.

[25] J. Ma, P. Protter and J. Yong. Solving forward-backward stochastic differential equations explicitly –a four step scheme, Prob. Theory & Related Fields, 98 (1994), 339-359.

[26] B. Maslowski, P. Veverka. Sufficient stochastic maximum principle for discounted control problem. http://arxiv.org/abs/1105.4737.

[27] O. Menoukeu-Pamen. Optimal control for stochastic delay system under model uncertainty. Manuscript 2012.

[28] B. Mezerdi. Necessary conditions for optimality for a diffusion with a non-smooth drift. Stochastics 24, 305-326 (1988).

[29] S. E. A. Mohammed. Stochastic differential equations with memory: Theory, examples and applications. Stochastic analysis and related topics VI. The Geilo Workshop, 1996, Progress in Probability, Birkhauser, 1998.

[30] S.-E. A. Mohammed and M. K. R. Scheutzow, Lyapunov exponents of linear stochastic functional differential equations driven by semimartingales, Part ii: Examples and case studies, Ann. Probab., 25 (1997), pp. 1210–1240.

[31] B. Øksendal and A. Sulem. A maximum principle for optimal control of stochastic systems with delay, with applications to finance: In J.M. Menaldi, E. Rofman and

A. Sulem (editors): Optimal Control and Partial Differential Equations - Innovations and Applications. IOS Press, Amsterdam, 2000.

[32] B. Øksendal, A. Sulem. Maximum principles for optimal control of forward–backward stochastic differential equations with jumps. SIAM J. Control Optim. 48 (5) (2010) 2945–2976.

[33] B. Øksendal, A. Sulem, T. Zhang. A maximum principle of optimal control of stochastic delay equations and time-advanced backward stochastic differential equations. Adv. Appl. Prob., 43 (2011), 572-596.

[34] B. Øksendal and A. Sulem. Forward-backward SDE games and stochastic control under model uncertainty. J. Optimazation Theory and Applications; DOI: 10. 1007/ s 10957-012-0166-7 (2012).

[35] B. Øksendal and A. Sulem. Applied Stochastic Control of Jump Diffusions. Springer, 2nd edition, 2007.

[36] E. Pardoux. Bsdes', weak convergence and homogenizations of semilinear pdes. In F.H. Clark and R.J. Stern, editors, Nonlinear Analysis, Differential Equations and Control, pages 503–549. Kluwer Academic, Dordrecht, 1999.

[37] E. Pardoux and S. Peng. Adapted solution of a backward stochastic differential equation. Systems & Control Lett. 14(1990), 55- 61.

[38] S. Peng. A general stochastic maximum principle for optimal control problems. SIAM J. Control and Optimization, 1990, 28(4), 966-979.

[39] S. Peng. Backward stochastic differential equations and its application to optimal control. Applied Mathematics and Optimization. 27(4), (1993) 125- 144.

[40] S. Peng, Y. Shi. Infinite horizon forward-backward stochastic differential equations. Stochastic Proc. and Their Appl., 85 (2000), 75-92.

[41] M. L. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze and E. F.Mischenko. The Mathematics Theory of Optimal Processes, Interscience, New York, New York, 1962.

[42] M. Royer. Backward stochastic dierential equations with jumps and related non-linear expectations. Stochastic Processes and Their Applications, 116:1358–1376, 2006.

[43] R. Situ. On solutions of backward stochastic differential equations with jumps and with non-Lipschitzian coefficients in Hilbert spaces and stochastic control. Statistics and Probability Letters, 60:279–288, 2002.

[44] W. S. Xu. Stochastic maximum principle for optimal control problem of forward and backward system. J. Australian. Math. Soc. Ser. B 37 (1995) 172 1785.

[45] J. Yin. On solutions of a class of infinite horizon fbsdes. Statistics and Probability Letters, 78:2412–2419, 2008.