A Switching Controller for Nonlinear Systems via TS Fuzzy Models

M. BOUMEHRAZ, K. BENMAHAMMED
Department of Computer Science
INSA Rennes
20 Av. Des Buttes de Coesmes, 35043 RENNES Cedex
FRANCE
mohamed.boumehraz@ens.insa-rennes.fr http://www.insa-rennes.fr

Abstract: - A design approach is proposed for the stabilization of non linear systems using fuzzy Takagi-Sugeno models. The fuzzy model is represented as a set of uncertain linear systems where the uncertainty depends on the fulfillment degree of each rule. An optimization procedure is used to maximize the stability region of each closed loop local system. The local controller design is based on the resolution of a set of independent algebraic Ricatti equations. The global control law is obtained by a switching between local controllers. A simulation example is given to illustrate the efficiency of the proposed method.

Key-Words: TS fuzzy model, Uncertain system, Switching control, Stability covering.

1 Introduction

During the last few years, the analysis and design of fuzzy logic controllers based on the Takagi-Sugeno fuzzy model have been a popular research topic in control community . Tanaka et al. discussed the stability and the design of fuzzy control systems in [1, 2]. They gave some checking conditions for stability, which can be used to design fuzzy control laws. Unfortunately, the stability conditions require the existence of a common positive definite matrix for all the local linear models. However, this is a difficult problem to be solved in many cases, especially when the number of rules is large. Representation of fuzzy models by a set of linear uncertain systems has been suggested by Cao et al. [3, 4], and based on linear uncertain system theory several control design approaches has been proposed. The drawback of the precedent approaches is that the LMIs or the algebraic Riccati equations used to check the stability can be infeasible. Based on the representation of Cao et al. [3] we propose, in this work, a switching control design approach. The proposed approach is based on the resolution of a set of independent algebraic Ricatti equation. The fulfillment degree of each rule is incorporated in the algebraic Riccati equation to overcome the problem of infeasibility. The rest of the paper is organized as follows. Section 2 introduces the fuzzy dynamic model. Section 3 presents the switching controller design approach for fuzzy dynamic models based on algebraic Ricatti equations. To demonstrate the efficiency of the proposed approach, a simulation example is given in section 4. Finally, conclusions are given in section 5.

2 Takagi-Sugeno Fuzzy Model

Many physical systems are very complex in practice so that rigorous mathematical models can be very difficult to obtain, if not impossible. However, many of these systems can be expressed in some form of mathematical models and Takagi-Sugeno fuzzy models has been largely used to model complex non linear systems[5]. The continuous-time Takagi-Sugeno fuzzy dynamic model is a piecewise interpolation of several linear models through membership functions. The fuzzy model is described by a set of fuzzy if-then rules. The *i*th rule of the fuzzy model for the non linear system take the form:

$$\textit{if } z_1(t) \textit{ is } F_1^i \textit{ and} \cdots z_g(t) \textit{ is } F_g^i \textit{ then}$$

$$\begin{cases} \dot{x}(t) = A_i x(t) + B_i u(t) \\ y(t) = C_i x(t) \end{cases} \quad i = 1,2,\ldots,r \tag{1}$$

where $x(t) \in R^n$ denotes the state vector, $u(t) \in R^m$ the control vector, $y(t) \in R^p$ the output vector, F_j^i is the jth fuzzy set of the *i*th rule, $A_i \in R^{n\times n}$, $B_i \in R^{n\times m}$ and $C_i \in R^{p\times n}$ are the state matrix, the input matrix and the output matrix for the ith local model. r is the number of if-then rules, and $z_1(t), z_2(t), \ldots, z_g(t)$ are some measurable system variables, i.e. the premise variables.

The final output of the fuzzy model can be expressed as:

$$\begin{cases} x(t) = \sum_{i=}^{r} \alpha_i(z(t))\{A_i x(t) + B_i u(t)\} \\ y(t) = \sum_{i=1}^{r} \alpha_i(z(t)) C_i x(t) \end{cases} \tag{2}$$

where

$$\alpha_i(z(t)) = \frac{\omega_i(z(t))}{\sum_{i=1}^{r}\omega_i(z(t))}, \quad \omega_i(z(t)) = \prod_{j=1}^{g} F_j^i(z_j(t)) \quad (3)$$

The scalars $\alpha_i(z(t))$ are characterized by:

$$0 \le \alpha_i(z(t)) \le 1 \text{ and } \sum_{i=1}^{r}\alpha_i(z(t)) = 1 \quad (4)$$

The T-S fuzzy model (2) has strong nonlinear interactions among its fuzzy rules which complicates the analysis and the control. In order to overcome these difficulties, the TS fuzzy model is represented as a set of uncertain linear systems [3]. The global state space $\Omega \subseteq R^n$ is partitioned into r subspaces, each subspace is defined as :

$$\Omega_l = \{\Omega | \alpha_l(z(t)) > 0\}, i = 1,2\ldots,r \quad (5)$$

Each subspace Ω_l is the union of two subsets:

$$\Omega_l = \overline{\Omega}_l \cap \Delta\Omega_l \quad (6)$$

where

$$\overline{\Omega}_l = \{\Omega_l \mid \alpha_l(z(t)) = 1\}, \ \Delta\Omega_l = \{\Omega_l \mid 0 < \alpha_l(z(t)) < 1\} \quad (7)$$

These subspaces are characterized by:

$$\bigcup_{i=}^{r}\Omega_l = \Omega \quad (8)$$

If the rules i and j can be inferred in the same time than $\Omega_i \cap \Omega_j \ne \phi$

$\Omega_i \cap \Omega_j = \Delta\Omega_i \cap \Delta\Omega_j \subseteq \Delta\Omega_i, \Omega_i \cap \Omega_j \subseteq \Delta\Omega_j$

If the rules i and j can't be inferred in the same time than $\Omega_i \cap \Omega_j = \phi$

In each subspace the TS fuzzy model (2) can be represented as:

$$\begin{cases} \dot{x}(t) = \left(A_l + \sum_{\substack{i=1 \\ i\ne l}}^{r}\alpha_i(z(t))A_{il}\right)x(t) + \left(B_l + \sum_{\substack{i=1 \\ i\ne l}}^{r}\alpha_i(z(t))B_{il}\right)u(t) \\ y(t) = \left(C_l + \sum_{\substack{i=1 \\ i\ne l}}^{r}\alpha_i(z(t))C_{il}\right)x(t) \end{cases} \quad (9)$$

where

$$A_{il} = A_i - A_l, B_{il} = B_i - B_l, \quad C_{il} = C_i - C_l \quad (10)$$

Since $\sum_{\substack{i=1 \\ i\ne l}}^{r}\alpha_i(z(t)) = 1 - \alpha_l(z(t))$

The TS fuzzy model can be written as:

$$\begin{cases} \dot{x}(t) = (A_l + (1-\alpha_l(z(t)))\Delta A_l(z(t)))x(t) + \\ \quad (B_l + (1-\alpha_l(z(t)))\Delta B_l(z(t)))u(t) \\ y(t) = (C_l + (1-\alpha_l(z(t)))\Delta C_l(z(t)))x(t) \end{cases} \quad (11)$$

where

$$\Delta A_l(z(t)) = \sum_{\substack{i=1 \\ i\ne l}}^{r}\alpha_i'(z(t))(A_i - A_l)$$

$$\Delta B_l(z(t)) = \sum_{\substack{i=1 \\ i\ne l}}^{r}\alpha_i'(z(t))(B_i - B_l)$$

$$\Delta C_l(z(t)) = \sum_{\substack{i=1 \\ i\ne l}}^{r}\alpha_i'(z(t))(C_i - C_l) \quad (12)$$

and

$$\alpha_i'(z(t)) = \frac{\alpha_i(z(t))}{1-\alpha_l(z(t))} \quad (13)$$

If $\alpha_l(z(t)) = 1$ then the fuzzy model can be represented by the corresponding linear local model.

In each subspace, the fuzzy model consists of a dominant nominal system (A_l, B_l, C_l) and a set of interacting systems representing the effect of other active rules.

In this paper we suppose that the state space is measurable and $y(t) = x(t)$.

The fuzzy system can be simplified to:

$$\dot{x}(t) = \tilde{A}_l(z(t))x(t) + \tilde{B}_l(z(t))u(t) \quad (14)$$

with

$$\tilde{A}_l(z(t)) = A_l + (1-\alpha_l(z(t)))\Delta A_l(z(t))$$

$$\tilde{B}_l(z(t)) = B_l + (1-\alpha_l(z(t)))\Delta B_l(z(t)) \quad (15)$$

The matrices $A_i - A_l$, $B_i - B_l$ can be written as:

$$A_i - A_l = M_{il}^A \cdot N_{il}^A, \ B_i - B_l = M_{il}^B \cdot N_{il}^B \quad (16)$$

Then $\Delta A_l(z(t))$ and $\Delta B_l(z(t))$ can be expressed as

$$\Delta A_l(z(t)) = M_l^A \cdot F_l^A(z(t)) \cdot N_l^A$$

$$\Delta B_l(z(t)) = M_l^B \cdot F_l^B(z(t)) \cdot N_l^B \quad (17)$$

where

$$M_{A_l} = [M_{1l}^A, M_{2l}^A, \ldots, M_{rl}^A], N_{A_l} = [N_{1l}^A, N_{2l}^A, \ldots, N_{rl}^A]$$

$$M_{B_l} = [M_{1l}^B, M_{2l}^B, \ldots, M_{rl}^B], N_{B_l} = [N_{1l}^B, N_{2l}^B, \ldots, N_{rl}^B]$$

$$F_{B_l}(z(t)) = \begin{bmatrix} \alpha_1'(z(t))I_{q1} & 0 & \cdots & 0 \\ 0 & \alpha_2'(z(t))I_{q2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_r'(z(t))I_{qr} \end{bmatrix}$$

$$F_{B_l}(z(t)) = \begin{bmatrix} \alpha_1'(z(t))I_{p1} & 0 & \cdots & 0 \\ 0 & \alpha_2'(z(t))I_{p2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_r'(z(t))I_{pr} \end{bmatrix} \quad (18)$$

$$0 \le \alpha_i'(z(t)) \le 1 \;\Rightarrow\; \begin{cases} F_{A_l}(z(t)) \cdot F_{A_l}^T(z(t)) \le I \\ F_{B_l}(z(t)) \cdot F_{B_l}^T(z(t)) \le I \end{cases} \quad (19)$$

3 Controller Design

We assume that the fuzzy system (2) is locally controllable, that is, the pairs (A_l, B_l), $l = 1,\ldots,r$ are controllable.

The basic idea is to design local feedback controllers that maximize the region of stability of each closed loop local model.

Theorem 1: If there exist positive definite matrices $R_l \in R^{m\times m}$, $Q_l \in R^{n\times n}$ scalars $\mu_l^1 > 0, \mu_l^2 > 0$ and $0 \le \underline{\alpha}_l < 1$ such that the following algebraic Ricatti equation:

$$A_l^T P_l + P_l A_l - P_l B_l \tilde{R}_l^{-1} B_l^T P_l + \tilde{Q}_l + P_l H_l P_l = 0 \quad (20)$$

has a solution $P_l = P_l^T > 0$ where

$$\tilde{Q}_l = Q_l + \frac{1}{\mu_l^1}(1 - \underline{\alpha}_l) N_{A_l}^T N_{A_l}$$

$$H_l = (1 - \underline{\alpha}_l)(\mu_l^1 M_{A_l} M_{A_l}^T + \mu_l^2 M_{B_l} M_{B_l}^T)$$

$$\tilde{R}_l = R_l + \frac{1}{\mu_l^2}(1 - \underline{\alpha}_l) N_{B_l}^T N_{B_l} \quad (21)$$

then the state feedback control law:

$$u_l(t) = -\tilde{R}_l^{-1} B_l^T P_l x(t) \quad (22)$$

quadratically stabilize the fuzzy system in the sub region

$$\Omega_l^s = \{\Omega \,|\, \alpha_l(z(t)) \ge \underline{\alpha}_l\} \quad (23)$$

In order to maximize the region of stability, the minimal value that guarantee the stability is obtained by solving the following minimization program:

$$\underset{P_l, Q_l, R_l, \mu_l^1, \mu_l^2}{Minimize} \quad \alpha_l$$

$$subject\ to\ P_l > 0, Q_l > 0, R_l > 0, \mu_1^l > 0, \mu_l^2 > 0$$

$$A_l^T P_l + P_l A_l - P_l B_l \tilde{R}_l^{-1} B_l^T P_l + \tilde{Q}_l + P_l H_l P_l = 0 \quad (24)$$

Note that this minimization program has always a solution, $\underline{\alpha}_l < 1$, since the local systems are controllable.

If $\bigcup_{i=1}^{r} \Omega_i^s = \Omega$ then the local controllers K_i, $i = 1,2,\ldots,r$ satisfy the *stability covering* condition [6].

Corollary: The scalars $\underline{\alpha}_i$, $i = 1,2,\cdots,r$ satisfy the stability covering condition if there exist, at each time *t*, at least one integer $1 \le k \le r$ so that:

$$\alpha_k(z(t)) \ge \underline{\alpha}_k \quad (25)$$

Theorem 2. If there exists a common positive definite matrix $P = P_1 = P_2 = \ldots = P_r$ solution to the algebraic Ricatti equation (20) for $i = 1,2,\cdots,r$ and the scalars $\underline{\alpha}_i$ satisfy the stability covering condition than the fuzzy system (2) is globally stabilizable by the following switching control law:

$$u(t) = -K_l x(t) \quad (25)$$

with l satisfying : $\alpha_l(z(t)) \ge \underline{\alpha}_l$

Since several rules may satisfy this condition, the control law is given by:

$$u(t) = -K_l x(t), \quad l = \arg\max_{i=1,r} (\alpha_i(z(t)) - \underline{\alpha}_i) \quad (26)$$

It is difficult, if not impossible to find a common matrix P that satisfy r algebraic Ricatti equations in the same time.

A common matrix is not necessary if the switching between the local controllers is sufficiently slow [7].

Lemma: If the controllers satisfy the stability covering condition and the switching time is sufficiently slow, so as to allow the transient effects to dissipate after each switch, than the switching control law (26) globally stabilize the fuzzy system (2).

Remark: Even if the stability covering condition is not fulfilled the fuzzy system may be stable [7].

The resolution of the *r* independent minimization programs leads to three possible cases:

Case 1 : Several or all $\underline{\alpha}_i = 0$, $i = 1,2,\cdots,r$, the number of controllers can be reduced since a local controller can be used to stabilize its own local system and local systems of neighborhood regions. The number of controllers is inferior to the number of rules .

Case 2 : If the number of controllers can't be reduced and the stability covering condition is fulfilled than the number of controllers is equal to the number of rules.

Case 3: If the stability covering condition is not fulfilled than the global system may be instable. To solve this problem, we can add new rules to the model since we know exactly in which region in the state space we need new ones. Or we can add new controllers without changing the model by using new nominal systems which is equivalent to the addition of new rules in the model.

Design procedure

The design procedure of the switching controller can be summarized in the following steps:

- *Step 1* : Obtain the fuzzy plant model of the non linear plant by means of the methods in [5], or other suitable ways
- *Step 2*: Determine the subsystems matrices A_i and B_i, $i = 1,\ldots,r$
- *Step 3* : Choose the suitable matrices M_{A_l}, N_{A_l}, M_{B_l}, and N_{B_l} for each local model.
- *Step 4 :* For each subsystem, solve the minimization program (24).
- *Step 5 :* Check if the stability covering condition (25) is satisfied, otherwise go to *Step 3* and choose other values for the free design parameters or add new controllers until the covering condition will be fulfilled.

4 Simulation Example

To illustrate the controller synthesis approach, we consider the following problem of stabilizing the ball and beam system represented in Fig. 1. The motion of the ball and the beam can be described by the following differential equations [8]:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \\ \dot{x}_4 \end{bmatrix} = \begin{bmatrix} x_2 \\ B(x_1 x_4 - G\sin x_3) \\ x_4 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} u \qquad (27)$$

where $x_1 = r$ denotes the position of the ball, $x_2 = \dot{r}$ its velocity, $x_3 = \theta$ the angle of the beam and $x_4 = \dot{\theta}$ the angular velocity of the beam.

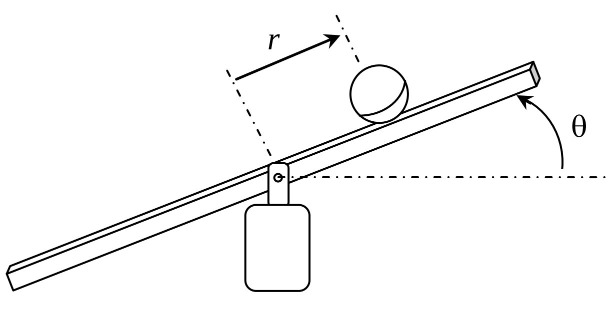

Fig. 1 Ball and beam system

The control u equals the angular acceleration, B and G are parameters reflecting the mass of the ball and the beam. In our simulations below, we choose $B = 0.7143$ and $G = 9.81$. The goal is to determine the control u such that the ball will converge to its stability position. Assuming that x_3 is about 0 and $x_1 x_4 \in [-d, d]$, where $d = 5$, the ball and beam system can be represented by the following TS fuzzy model [8]:

R^1 : If x_3 is about 0 and $x_1 x_4$ is about 0,

Then $\dot{x} = A_1 x + B_1 u$

R^2 : If x_3 is about 0 and $x_1 x_4$ is about d,

Then $\dot{x} = A_2 x + B_2 u$

R^3 : If x_3 is about 0 and $x_1 x_4$ is about $-d$,

Then $\dot{x} = A_3 x + B_3 u$

where

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -BG & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -BG & Bd \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -BG & -Bd \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, B_1 = B_2 = B_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \qquad (28)$$

and the membership functions, shown in Fig. 2, are :

$$\alpha_1(x) = 1 - \alpha_2(x) - \alpha_3(x),$$

$$\alpha_2(x) = \begin{cases} 1, & x_1 x_4 \geq d \\ \dfrac{x_1 x_4}{d}, & 0 < x_1 x_4 < d \\ 0, & x_1 x_4 \leq 0 \end{cases},$$

$$\alpha_2(x) = \begin{cases} 1, & x_1 x_4 \geq d \\ \dfrac{x_1 x_4}{d}, & 0 < x_1 x_4 < d \\ 0, & x_1 x_4 \leq 0 \end{cases} \qquad (29)$$

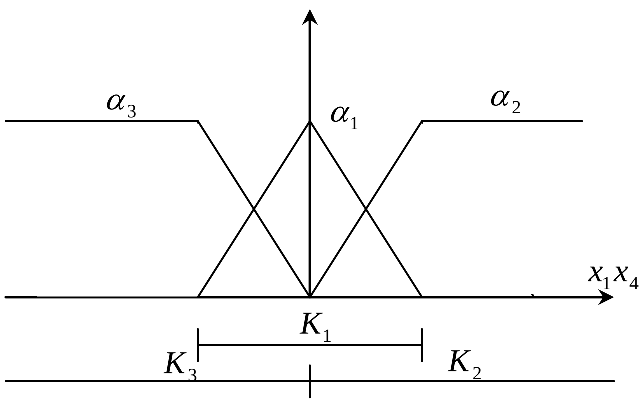

Fig. 2, Membership functions and local controllers

The values obtained after the resolution of the three minimization programs:

Subsystem 1:

$$Q_1 = 2.0 I_4, R_1 = 1.0,\ \mu_1^1 = 0.005$$

$$\underline{\alpha}_1 = 0$$

$$K_1 = \begin{bmatrix} 10.55 & 30.920 & -260.817 & -44.568 \end{bmatrix}$$

$$P_1 = \begin{bmatrix} 20.886 & 52.677 & -389.952 & -10.055 \\ 52.677 & 154.158 & -1197.779 & -30.920 \\ -389.952 & -1197.779 & 10084.775 & 260.817 \\ -10.055 & -30.920 & 260.817 & 44.568 \end{bmatrix}$$

Subsystem 2 :

$$Q_2 = 2.0 I_4, R_2 = 1.0,\ \mu_2^1 = 0.01$$

$$\underline{\alpha}_2 = 0$$

$$K_2 = \begin{bmatrix} 6.196 & 17.853 & -158.054 & -23.645 \end{bmatrix}$$

$$P_2 = \begin{bmatrix} 14.687 & 31.922 & -240.166 & -6.196 \\ 31.922 & 84.147 & -662.821 & -17.853 \\ -240.166 & -662.821 & 5556.816 & 158.054 \\ -6.196 & -17.853 & 158.054 & 23.645 \end{bmatrix}$$

Subsystem 3:

$$Q_3 = 2.0 I_4, R_3 = 1.0,\ \mu_3^1 = 0.01$$

$$\underline{\alpha}_3 = 0$$

$$K_3 = \begin{bmatrix} 1.738 & 4.089 & -26.449 & -21.020 \end{bmatrix}$$

$$P_3 = \begin{bmatrix} 5.077 & 5.342 & -16.665 & -1.738 \\ 5.342 & 10.635 & -44.685 & -4.089 \\ -16.665 & -44.685 & 361.184 & 26.449 \\ -1.738 & -4.089 & 26.449 & 21.020 \end{bmatrix}$$

The minimal values found are $\underline{\alpha}_1 = \underline{\alpha}_2 = \underline{\alpha}_3 = 0$ which means, as shown in Fig. 2, that the state feedback $u(t) = -K_1 x(t)$ is sufficient to stabilize the Ball-and-Beam system and the controller is simplified. To illustrate the controller performance, the position of the bal is shown in Fig. 3, and the angle of the beam in Fig. 4, for the following initial conditions [0.5 m, 0, 30.0°, 0], [0.5 m, 0, 60.0°, 0] and [-0.5 m, 0, -45.0°, 0]. The switching controller is reduced to a simple linear state feedback, it is one advantage of this approach.

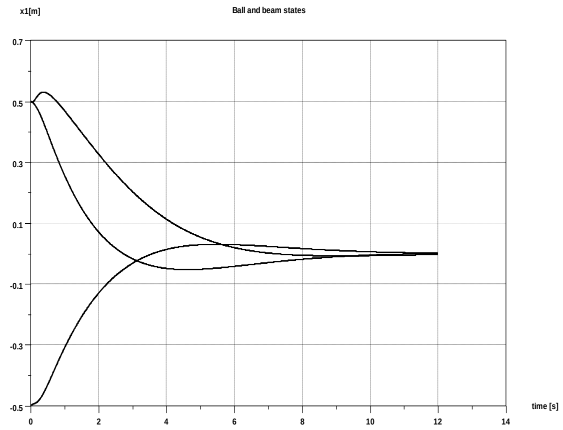

Fig. 3, Position of the ball

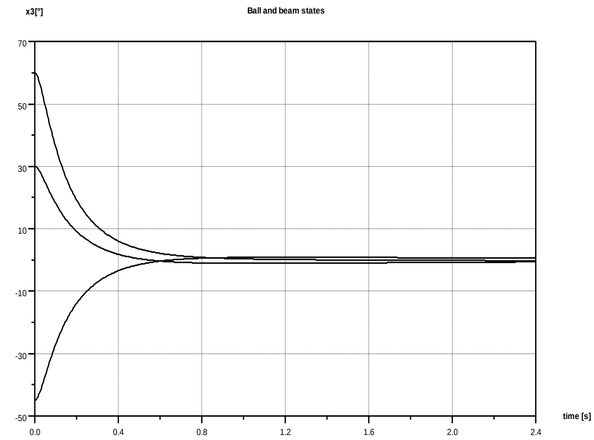

Fig. 4, Angle of the beam

4 Conclusion

This work presents a switching control design approach for the stabilization of non linear systems represented by fuzzy models. The basic idea of this approach is to decompose the global non linear design control problem into a number of simple linear state feedback local controllers. The maximization of the stability region of each local controller permit the minimization of the number of controllers. However, the problem of global stability still unsolved in the case of fast switching between local controllers.

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