Nonlinear Systems Stabilisation using Fuzzy Models and Switching Control

Mohamed Boumehraz and Khier Benmahammed

***Abstract*— A design approach is proposed for the stabilization of non linear systems using fuzzy Takagi-Sugeno models. The fuzzy model is represented as a set of uncertain linear systems where the local system uncertainty depends on the fulfillment degree of the corresponding rule. An optimization procedure is used to design the local controller such as to maximize the stability region of each closed loop local system. The local controller design is based on the resolution of a set of independent LMIs. The global control law is obtained by switching between local controllers. A simulation example is given to illustrate the efficiency of the proposed method.**

I. INTRODUCTION

Over the past several years, fuzzy systems have attracted considerable attention from scientists and engineers. Fuzzy modeling is an efficient method to represent complex non-linear systems by fuzzy sets and fuzzy reasoning. By using a Takagi-Sugeno fuzzy model, a non linear system can be expressed as a weighted sum of simple subsystems[1]-[3]. Recently, there have been appeared a number of systematic stability analysis and controller design results in fuzzy control literature. Tanaka et al. discussed the stability and the design of fuzzy control systems in [4]. They gave some checking conditions for stability, which can be used to design fuzzy control laws, several methods have been proposed to relax the stability conditions[5]-[6]. Unfortunately, the stability conditions require the existence of a common positive definite matrix for all the local linear models. However, this is a difficult problem to be solved in many cases, especially when the number of rules is large. Representation of fuzzy models by a set of linear uncertain systems has been suggested by Kim et al.[7], based on linear uncertain system theory several control design approaches has been proposed [7],[8],[10]. The drawback of the precedent approaches is that the LMIs or the algebraic Riccati equations used to check the stability may be infeasible. Based on the representation of Cao et al. [8]-[10] we propose, in this work, a switching control design approach. The proposed approach is based on the resolution of a set of LMIs. The uncertainty of each local model is represented in function of its fullfillement degree. To overcome the problem of infeasibility the fullfilment degree is incorporated in the LMIs. The rest of the paper is organized as follows. Section 2 introduces the fuzzy dynamic model. Section 3 presents the switching controller design approach for fuzzy dynamic models based on the maximisation of the stability region of each local model. To demonstrate the efficiency of the proposed approach, a simulation example is given in section 4. Finally, conclusions are given in section 5.

M. Boumehraz is with Automatic Control Department, Faculty of Engineering Sciences, University of Biskra, B. P. 145, Biskra 07000, Algeria medboumehraz@netcourrier.com

K. Benmahammed is with the Department of Electronics, Faculty of Engineering, University of Setif, Setif 19000, Algeria khierben@ieee.org

II. TAKAGI-SUGENO FUZZY MODEL

The continuous-time Takagi-Sugeno fuzzy dynamic model is a piecewise interpolation of several linear models through membership functions . The fuzzy model is described by a set of fuzzy if-then rules. The i^{th} rule of the fuzzy model take the form:

Rule i:

If $z_1(t)$ is $F_1^i, \cdots$, and $z_g(t)$ is F_g^i

$$\text{Then} \begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}_i\mathbf{x}(t) \end{cases} \tag{1}$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ denotes the state vector, $\mathbf{u}(t) \in \mathbb{R}^m$ the control vector, $\mathbf{y}(t) \in \mathbf{R}^p$ the output vector, F_j^i is the jth fuzzy set of the ith rule, $\mathbf{A}_i \in \mathbb{R}^{n\times n}, \mathbf{B}_i \in \mathbb{R}^{n\times m}$ and $\mathbf{C}_i \in \mathbb{R}^{p\times n}$ are the state matrix, the input matrix and the output matrix for the ith local model, r is the number of if-then rules, and $z_1(t), z_2(t) \cdots , z_g(t)$ are some measurable system variables. The final output of the fuzzy model can be expressed as:

$$\begin{cases} \dot{\mathbf{x}}(t) = \sum_{i=1}^{r} \alpha_i(\mathbf{z}(t))\{\mathbf{A}_i\mathbf{x}(t) + \mathbf{B}_i\mathbf{u}(t)\} \\ \mathbf{y}(t) = \sum_{i=1}^{r} \alpha_i(\mathbf{z}(t))\mathbf{C}_i\mathbf{x}(t) \end{cases}$$

Where

$$\alpha_i(\mathbf{z}(t)) = \frac{\omega_i(\mathbf{z}(t))}{\sum_{i=1}^{r} \omega_i(\mathbf{z}(t))} \tag{2}$$

The scalars $\alpha_i(\mathbf{z}(t))$ are characterized by:

$$0 \leq \alpha_i(\mathbf{z}(t)) \leq 1 \text{ and } \sum_{i=1}^{r} \alpha_i(\mathbf{z}(t)) = 1 \tag{3}$$

The T-S fuzzy model (2) has strong nonlinear interactions among its fuzzy rules which complicates the analysis and the control. In order to overcome these difficulties, the TS fuzzy model is represented as a set of uncertain linear systems[8]. The global state space $\Omega \subseteq \mathbb{R}^n$ is partitioned into r subspaces, each subspace is defined as :

$$\Omega_l = \{\Omega \mid \alpha_l(\mathbf{z}(t)) > 0\} \tag{4}$$

These subspaces are characterized by:

$$\bigcup_{i=1}^{r} \Omega_i = \Omega \tag{5}$$

If the rules i and j can be inferred in the same time then :

$$\Omega_i \cap \Omega_j \neq \phi \tag{6}$$

If the rules i and j can't be inferred in the same time then :

$$\Omega_i \cap \Omega_j = \phi \tag{7}$$

In each subspace the TS fuzzy model (2) can be represented as:

$$\begin{aligned}\dot{\mathbf{x}}(t) =& \{\mathbf{A}_l + \sum_{\substack{R_i \in \mathcal{R}_l \\ i \neq l}}^{r} \alpha_i(\mathbf{z}(t)) \left(\mathbf{A}_i - \mathbf{A}_l\right)\}\mathbf{x}(t) \\ &+ \{\mathbf{B}_l + \sum_{\substack{R_i \in \mathcal{R}_l \\ i \neq l}}^{r} \alpha_i(\mathbf{z}(t)) \left(\mathbf{B}_i - \mathbf{B}_l\right)\}\mathbf{u}(t) \\ \mathbf{y}(t) =& \{\mathbf{C}_l + \sum_{\substack{R_i \in \mathcal{R}_l \\ i \neq l}}^{r} \alpha_i(\mathbf{z}(t)) \left(\mathbf{C}_i - \mathbf{C}_l\right)\}\mathbf{x}(t)\end{aligned} \tag{8}$$

$\mathcal{R}_l$ is a rule subset containing rules that can be inferred in the same time as rule l.

$$\mathcal{R}_l = \{R_i, \exists t, \alpha_l(\mathbf{z}(t))\alpha_i(\mathbf{z}(t)) \neq 0\} \tag{9}$$

Since

$$\sum_{\substack{R_i \in \mathcal{R}_l \\ i \neq l}}^{r} \alpha_i(\mathbf{z}(t)) = 1 - \alpha_l(\mathbf{z}(t)) \tag{10}$$

The TS fuzzy model can be written as:

$$\begin{cases} \dot{\mathbf{x}}(t) = \tilde{\mathbf{A}}_l(\mathbf{z}(t))\mathbf{x}(t) + \tilde{\mathbf{B}}_l(\mathbf{z}(t))\mathbf{u}(t) \\ \mathbf{y}(t) = \tilde{\mathbf{C}}_l(\mathbf{z}(t))\mathbf{x}(t) \end{cases} \tag{11}$$

Where

$$\tilde{\mathbf{A}}_l(\mathbf{z}(t)) = \mathbf{A}_l + (1 - \alpha_l(\mathbf{z}(t)))\Delta\mathbf{A}_l(\alpha'(\mathbf{z}(t))) \tag{12}$$
$$\tilde{\mathbf{B}}_l(\mathbf{z}(t)) = \mathbf{B}_l + (1 - \alpha_l(\mathbf{z}(t)))\Delta\mathbf{B}_l(\alpha'(\mathbf{z}(t))) \tag{13}$$
$$\tilde{\mathbf{C}}_l(\mathbf{z}(t)) = \mathbf{C}_l + (1 - \alpha_l(\mathbf{z}(t)))\Delta\mathbf{C}_l(\alpha'(\mathbf{z}(t))) \tag{14}$$

$$\begin{aligned}\Delta\mathbf{A}_l(\alpha'(\mathbf{z}(t))) &= \sum_{\substack{R_i \in \mathcal{R}_l \\ i \neq l}}^{r} \alpha'_i(\mathbf{z}(t))(\mathbf{A}_i - \mathbf{A}_l) \\ \Delta\mathbf{B}_l(\alpha'(\mathbf{z}(t))) &= \sum_{\substack{R_i \in \mathcal{R}_l \\ i \neq l}}^{r} \alpha'_i(\mathbf{z}(t))(\mathbf{B}_i - \mathbf{B}_l) \\ \Delta\mathbf{C}_l(\alpha'(\mathbf{z}(t))) &= \sum_{\substack{R_i \in \mathcal{R}_l \\ i \neq l}}^{r} \alpha'_i(\mathbf{z}(t))(\mathbf{C}_i - \mathbf{C}_l)\end{aligned} \tag{15}$$

and

$$\alpha'_i(\mathbf{z}(t)) = \frac{\alpha_i(\mathbf{z}(t))}{1 - \alpha_l(\mathbf{z}(t))} \tag{16}$$

If $\alpha_l(\mathbf{z}(t)) = 1$ then the fuzzy system can be represented by the corresponding linear local model. In each subspace, the fuzzy model consists of a dominant nominal system $(\mathbf{A}_l, \mathbf{B}_l, \mathbf{C}_l)$ and a set of interacting systems representing the effect of other active rules. In this paper we suppose that the state vector is measurable and the stabilization is accomplished via a full state feedback. If $\mathbf{y}(t) = \mathbf{x}(t)$ the fuzzy system can be simplified to:

$$\dot{\mathbf{x}}(t) = \tilde{\mathbf{A}}_l(\alpha'(\mathbf{z}(t)))\mathbf{x}(t) + \tilde{\mathbf{B}}_l(\alpha'(\mathbf{z}(t)))\mathbf{u}(t) \tag{17}$$

with

$$\begin{aligned}\tilde{\mathbf{A}}_l(\alpha'(\mathbf{z}(t))) &= \mathbf{A}_l + (1 - \alpha_l(\mathbf{z}(t)))\Delta\mathbf{A}_l(\alpha'(\mathbf{z}(t))) \\ \tilde{\mathbf{B}}_l(\alpha'(\mathbf{z}(t))) &= \mathbf{B}_l + (1 - \alpha_l(\mathbf{z}(t)))\Delta\mathbf{B}_l(\alpha'(\mathbf{z}(t)))\end{aligned} \tag{18}$$

Suppose that the matrices $\mathbf{A}_i - \mathbf{A}_l$ and $\mathbf{B}_i - \mathbf{B}_l$ can be written as :

$$\mathbf{A}_i - \mathbf{A}_l = \mathbf{D}^A_{li} \cdot \mathbf{E}^A_{li}, \quad \mathbf{B}_i - \mathbf{B}_l = \mathbf{D}^B_{li} \cdot \mathbf{E}^B_{li} \tag{19}$$

Then $\Delta\mathbf{A}_l(\alpha'(\mathbf{z}(t)))$ and $\Delta\mathbf{B}_l(\alpha'(\mathbf{z}(t)))$ can be expressed as:

$$\begin{aligned}\Delta\mathbf{A}_l(\alpha'(\mathbf{z}(t))) &= \mathbf{D}_{A_l} \cdot \mathbf{F}_{A_l}(\alpha'(\mathbf{z}(t))) \cdot \mathbf{E}_{A_l} \\ \Delta\mathbf{B}_l(\alpha'(\mathbf{z}(t))) &= \mathbf{D}_{B_l} \cdot \mathbf{F}_{B_l}(\alpha'(\mathbf{z}(t))) \cdot \mathbf{E}_{B_l}\end{aligned} \tag{20}$$

Where

$$\begin{aligned}&\mathbf{D}_{A_l} = \begin{bmatrix} \mathbf{D}^A_{l1} & \cdots & \mathbf{D}^A_{lr} \end{bmatrix}, \quad \mathbf{D}_{B_l} = \begin{bmatrix} \mathbf{D}^B_{l1} & \cdots & \mathbf{D}^B_{lr} \end{bmatrix} \\ &\mathbf{E}_{A_l} = \begin{bmatrix} \mathbf{E}^A_{l1} \\ \vdots \\ \mathbf{E}^A_{lr} \end{bmatrix}, \quad \mathbf{E}_{B_l} = \begin{bmatrix} \mathbf{E}^B_{l1} \\ \vdots \\ \mathbf{E}^B_{lr} \end{bmatrix} \\ &\mathbf{F}_{A_l}(\alpha'(\mathbf{z}(t))) = \begin{bmatrix} \alpha'_1(\mathbf{z}(t))\mathbf{I}_{q_l} & 0 & \cdots & 0 \\ 0 & \ddots & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \alpha'_r(\mathbf{z}(t))\mathbf{I}_{q_r} \end{bmatrix} \\ &\mathbf{F}_{B_l}(\alpha'(\mathbf{z}(t))) = \begin{bmatrix} \alpha'_1(\mathbf{z}(t))\mathbf{I}_{p_l} & 0 & \cdots & 0 \\ 0 & \ddots & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha'_r(\mathbf{z}(t))\mathbf{I}_{p_r} \end{bmatrix}\end{aligned} \tag{21}$$

$$0 \leq \alpha'_i(\mathbf{z}(t)) \leq 1 \quad \Longrightarrow \quad \begin{cases} \mathbf{F}_{A_l}(\cdot)\mathbf{F}^T_{A_l}(\cdot) \leq \mathbf{I} \\ \mathbf{F}_{B_l}(\cdot)\mathbf{F}^T_{B_l}(\cdot) \leq \mathbf{I} \end{cases} \tag{22}$$

III. Controller Design

We assume that the fuzzy system (2) is locally controllable, that is, the pairs $(\mathbf{A}_l, \mathbf{B}_l)$, $l = 1, \ldots, r$, are controllable. The basic idea is to design local feedback controllers that maximize the stability region of each closed loop local model. The switching controller consists of r linear state feedback controllers that will be switched from one to another to control the system. The switching controller can be described by:

$$\mathbf{u}(t) = \sum_{l=1}^{r} \zeta_l(\mathbf{x}(t))\mathbf{u}_l(t) \tag{23}$$

with:

$$\mathbf{u}_l(t) = \mathbf{K}_l\mathbf{x}(t) \tag{24}$$

and:

$$\zeta_l(\mathbf{x}(t)) = \begin{cases} 1 & \mathbf{x}(t) \in \Omega_l^c \\ 0 & \text{otherwize} \end{cases} \tag{25}$$

$\Omega_l^c \subseteq \Omega_l$ is the subregion in which the command is generated using the local state feedback $\mathbf{K}_l$ to be designed. It can be seen that (23) is a linear combination of r_c linear state feedback controllers, the number of controllers r_c may be different of the number of rules r. At each moment, only one of the linear state feedback controllers is chosen to generate the control signal.

Theorem 1: If there exist symmetric positive definite matrix $\mathbf{P}_l$ and positive scalars, $\varepsilon_l^A > 0, \varepsilon_l^B > 0, 0 \leq \underline{\alpha}_l < 1$ such that the following LMI holds:

$$\begin{bmatrix} \Psi_l & \varepsilon_l^B\mathbf{D}_{B_l} & \varepsilon_l^A\mathbf{D}_{A_l} & \mathbf{Y}_l^T\mathbf{E}_{B_l}^T & \mathbf{X}_l\mathbf{E}_{A_l}^T \\ \varepsilon_l^B\mathbf{D}_{B_l}^T & -\frac{\varepsilon_l^B}{1-\underline{\alpha}_l}\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \varepsilon_l^A\mathbf{D}_{A_l}^T & \mathbf{0} & -\frac{\varepsilon_l^A}{1-\underline{\alpha}_l}\mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{E}_{B_l}\mathbf{Y}_l & \mathbf{0} & \mathbf{0} & -\frac{\varepsilon_l^B}{1-\underline{\alpha}_l}\mathbf{I} & \mathbf{0} \\ \mathbf{E}_{A_l}\mathbf{X}_l & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\frac{\varepsilon_l^A}{1-\underline{\alpha}_l}\mathbf{I} \end{bmatrix} < \mathbf{0} \tag{26}$$

where :

$$\Psi_l = \mathbf{X}_l\mathbf{A}_l^T + \mathbf{A}_l\mathbf{X}_l + \mathbf{Y}_l^T\mathbf{B}_l^T + \mathbf{B}_l\mathbf{Y}_l \tag{27}$$

and

$$\mathbf{X}_l = \mathbf{P}_l^{-1}, \quad \mathbf{Y}_l = \mathbf{K}_l\mathbf{P}_l^{-1} \tag{28}$$

then the fuzzy subsystem (17) is quadratically stable for the values of $\alpha_l(\mathbf{z}(t))$ such that :

$$\alpha_l(\mathbf{z}(t)) \geq \underline{\alpha}_l \tag{29}$$

Proof: Consider the following Lyapunouv function candidate:

$$V_l(t) = \mathbf{x}^T(t)\mathbf{P}_l\mathbf{x}(t) \tag{30}$$

where $\mathbf{P}_l$ is a symmetric positive definite matrix. The time derivative of $V_l(t)$ along the trajectory of the fuzzy system is given by:

$$\begin{aligned} \dot{V}_l(t) =& \dot{\mathbf{x}}^T(t)\mathbf{P}_l\mathbf{x}(t) + \mathbf{x}^T(t)\mathbf{P}_l\dot{\mathbf{x}}(t) \\ =& \mathbf{x}^T(t)\tilde{\mathbf{A}}_l^T(\alpha')\mathbf{P}_l\mathbf{x}(t) + \mathbf{x}^T(t)\mathbf{P}_l\tilde{\mathbf{A}}_l(\alpha')\mathbf{x}(t) \\ &+ \mathbf{u}^T(t)\tilde{\mathbf{B}}_l^T(\alpha')\mathbf{P}_l\mathbf{x}(t) + \mathbf{x}^T(t)\mathbf{P}_l\tilde{\mathbf{B}}_l(\alpha')\mathbf{u}(t) \\ =& \mathbf{x}^T\left\{\mathcal{L}_1(\mathbf{P}_l) + (1-\alpha_l)\left[\mathcal{L}_A(\mathbf{P}_l) + \mathcal{L}_B(\mathbf{P}_l)\right]\right\}\mathbf{x}. \end{aligned}$$

where :

$$\begin{aligned} \mathcal{L}_1(\mathbf{P}_l) =& \mathbf{A}_l^T\mathbf{P}_l + \mathbf{P}_l\mathbf{A}_l + \mathbf{K}_l^T\mathbf{B}_l^T\mathbf{P}_l + \mathbf{P}_l\mathbf{B}_l\mathbf{K}_l \\ \mathcal{L}_A(\mathbf{P}_l) =& \mathbf{E}_{A_l}^T\mathbf{F}_{A_l}^T\mathbf{D}_{A_l}^T\mathbf{P}_l + \mathbf{P}_l\mathbf{D}_{A_l}\mathbf{F}_{A_l}\mathbf{E}_{A_l} \\ \mathcal{L}_B(\mathbf{P}_l) =& \mathbf{K}_l^T\mathbf{E}_{B_l}^T\mathbf{F}_{B_l}^T\mathbf{D}_{B_l}^T\mathbf{P}_l + \mathbf{P}_l\mathbf{D}_{B_l}^T\mathbf{F}_{B_l}\mathbf{E}_{B_l}\mathbf{K}_l \end{aligned}$$

Since for any positive scalar $\rho > 0$ and real matrices $\mathbf{Y}$ and $\mathbf{Z}$ we have [11]:

$$\mathbf{Z}\mathbf{Y}^T + \mathbf{Y}\mathbf{Z}^T \leq \rho\mathbf{Y}\mathbf{Y}^T + \frac{1}{\rho}\mathbf{Z}\mathbf{Z}^T \tag{31}$$

It follows that:

$$\mathcal{L}_A(\mathbf{P}_l) \leq \bar{\mathcal{L}}_A(\mathbf{P}_l), \quad \mathcal{L}_B(\mathbf{P}_l) \leq \bar{\mathcal{L}}_B(\mathbf{P}_l)$$

where

$$\begin{aligned} \bar{\mathcal{L}}_A(\mathbf{P}_l) &= \varepsilon_l^A\mathbf{P}_l\mathbf{D}_{A_l}\mathbf{D}_{A_l}^T\mathbf{P}_l + \frac{1}{\varepsilon_l^A}\mathbf{E}_{A_l}^T\mathbf{E}_{A_l} \\ \bar{\mathcal{L}}_B(\mathbf{P}_l) &= \varepsilon_l^B\mathbf{P}_l\mathbf{D}_{B_l}\mathbf{D}_{B_l}^T\mathbf{P}_l + \frac{1}{\varepsilon_l^B}\mathbf{K}_l^T\mathbf{E}_{B_l}^T\mathbf{E}_{B_l}\mathbf{K}_l \end{aligned}$$

$$\dot{V}_l(t) \leq \mathbf{x}^T(t)\left\{\mathcal{L}_1(\mathbf{P}_l) + (1-\alpha_l)\left[\bar{\mathcal{L}}_A(\mathbf{P}_l) + \bar{\mathcal{L}}_B(\mathbf{P}_l)\right]\right\}\mathbf{x}$$

Since

$$\underline{\alpha}_l \leq \alpha_l \Rightarrow 1 - \alpha_l \leq 1 - \underline{\alpha}_l \tag{32}$$

it yields

$$(1-\alpha_l)\left[\bar{\mathcal{L}}_A(\cdot) + \bar{\mathcal{L}}_B(\cdot)\right] \leq (1-\underline{\alpha}_l)\left[\bar{\mathcal{L}}_A(\cdot) + \bar{\mathcal{L}}_B(\cdot)\right]$$

$$\dot{V}_l(t) \leq \mathbf{x}^T(t)\left\{\mathcal{L}_1(\mathbf{P}_l) + (1-\underline{\alpha}_l)\left[\bar{\mathcal{L}}_A(\mathbf{P}_l) + \bar{\mathcal{L}}_B(\mathbf{P}_l)\right]\right\}\mathbf{x}(t)$$

$$\mathcal{L}_1(\mathbf{P}_l) + (1-\underline{\alpha}_l)\left[\bar{\mathcal{L}}_A(\mathbf{P}_l) + \bar{\mathcal{L}}_B(\mathbf{P}_l)\right] < 0 \quad \Rightarrow \quad \dot{V}_l(t) < 0$$

Let $\mathbf{X}_l = \mathbf{P}_l^{-1}$ and $\mathbf{Y}_l = \mathbf{K}_l\mathbf{X}_l$,by right and left multiplying by $\mathbf{X}_l$:

$$\begin{aligned} &\mathcal{L}_1(\mathbf{P}_l) + (1-\underline{\alpha}_l)\left[\bar{\mathcal{L}}_A(\mathbf{P}_l) + \bar{\mathcal{L}}_B(\mathbf{P}_l)\right] < 0 \Leftrightarrow \\ &\mathbf{X}_l\mathbf{A}_l^T + \mathbf{A}_l\mathbf{X}_l + \mathbf{Y}_l^T\mathbf{B}_l^T + \mathbf{B}_l\mathbf{Y}_l \\ &+ (1-\underline{\alpha}_l)\left(\varepsilon_l^A\mathbf{D}_{A_l}\mathbf{D}_{A_l}^T + \varepsilon_l^B\mathbf{D}_{B_l}\mathbf{D}_{B_l}^T\right) \\ &+(1-\underline{\alpha}_l)\left(\frac{1}{\varepsilon_l^A}\mathbf{X}_l\mathbf{E}_{A_l}^T\mathbf{E}_{A_l}\mathbf{X}_l + \frac{1}{\varepsilon_l^B}\mathbf{Y}_l^T\mathbf{E}_{B_l}^T\mathbf{E}_{B_l}\mathbf{Y}_l\right) < 0 \end{aligned}$$

using Schur complement [12] we get:

$$\begin{bmatrix} \Psi_l & \varepsilon_l^B\mathbf{D}_{B_l} & \varepsilon_l^A\mathbf{D}_{A_l} & \mathbf{Y}_l^T\mathbf{E}_{B_l}^T & \mathbf{X}_l\mathbf{E}_{A_l}^T \\ \varepsilon_l^B\mathbf{D}_{B_l}^T & -\frac{\varepsilon_l^B}{1-\underline{\alpha}_l}\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \varepsilon_l^A\mathbf{D}_{A_l}^T & \mathbf{0} & -\frac{\varepsilon_l^A}{1-\underline{\alpha}_l}\mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{E}_{B_l}\mathbf{Y}_l & \mathbf{0} & \mathbf{0} & -\frac{\varepsilon_l^B}{1-\underline{\alpha}_l}\mathbf{I} & \mathbf{0} \\ \mathbf{E}_{A_l}\mathbf{X}_l & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\frac{\varepsilon_l^A}{1-\underline{\alpha}_l}\mathbf{I} \end{bmatrix} < \mathbf{0}$$

with $\Psi_l = \mathbf{X}_l\mathbf{A}_l^T + \mathbf{A}_l\mathbf{X}_l + \mathbf{Y}_l^T\mathbf{B}_l^T + \mathbf{B}_l\mathbf{Y}_l$

which is an LIM where the variables are : $\mathbf{X}_l = \mathbf{P}_l^{-1}, \mathbf{Y}_l = \mathbf{K}_l\mathbf{P}_l^{-1}, \varepsilon_l^B$ and ε_l^A. ■

In order to maximize the region of stability of each subregion Ω_l^s, the minimal value that guarantee the stability is obtained by solving the following minimization program:

$$\underset{\mathbf{X}_l,\mathbf{Y}_l,\varepsilon_l^A,\varepsilon_l^B}{\text{Minimize}} \ \underline{\alpha}_l$$

Subject to $0 \leq \underline{\alpha}_l < 1, \mathbf{X}_l = \mathbf{X}_l^T > 0, \varepsilon_l^A > 0, \varepsilon_l^B > 0$

$$\begin{bmatrix} \Psi_l & \varepsilon_l^B\mathbf{D}_{B_l} & \varepsilon_l^A\mathbf{D}_{A_l} & \mathbf{Y}_l^T\mathbf{E}_{B_l}^T & \mathbf{X}_l\mathbf{E}_{A_l}^T \\ \varepsilon_l^B\mathbf{D}_{B_l}^T & -\frac{\varepsilon_l^B}{1-\underline{\alpha}_l}\mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \varepsilon_l^A\mathbf{D}_{A_l}^T & \mathbf{0} & -\frac{\varepsilon_l^A}{1-\underline{\alpha}_l}\mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{E}_{B_l}\mathbf{Y}_l & \mathbf{0} & \mathbf{0} & -\frac{\varepsilon_l^B}{1-\underline{\alpha}_l}\mathbf{I} & \mathbf{0} \\ \mathbf{E}_{A_l}\mathbf{X}_l & \mathbf{0} & \mathbf{0} & \mathbf{0} & -\frac{\varepsilon_l^A}{1-\underline{\alpha}_l}\mathbf{I} \end{bmatrix} < \mathbf{0}$$

$$\Psi_l = \mathbf{X}_l\mathbf{A}_l^T + \mathbf{A}_l\mathbf{X}_l + \mathbf{Y}_l^T\mathbf{B}_l^T + \mathbf{B}_l\mathbf{Y}_l \tag{33}$$

Note that this minimization program has always a solution $\underline{\alpha}_l < 1$, since we assume that the local systems are controllable.

Remark 1: $\underline{\alpha}_l$ is the lth rule minimal degree that garanties the quadratic stability of the fuzzy system (2) using the local model $(\mathbf{A}_l, \mathbf{B}_l)$ as nominal model and $\mathbf{K}_l$ as state feedback gain. Another rule will be used as nominal model to generate the control signal for $\alpha_l < \underline{\alpha}_l$.

Definition 1: We say that the state feedback gains, $\mathbf{K}_l, l = 1, 2, \ldots, r$ satisfy the *stability covering* condition [13] if:

$$\bigcup_{l=1}^{r} \Omega_l^s = \Omega \tag{34}$$

Lemma 1: If there exists, at each moment t, at least one integer $k \in \{1, 2, \ldots, r\}$ so that :

$$\alpha_k(\mathbf{z}(t)) \geq \underline{\alpha}_k \tag{35}$$

then stability covering condition (34) is satisfied.

Proof:

$$\forall t, \exists k, \alpha_k(\mathbf{z}(t)) \geq \underline{\alpha}_k \Leftrightarrow \forall t, \exists k, \mathbf{x}(t) \in \Omega_k^s \tag{36}$$

$$\forall t, \exists k, \mathbf{x}(t) \in \Omega_k^s \Leftrightarrow \bigcup_{k=1}^{r} \Omega_k^s = \Omega \tag{37}$$

■

Since several rules may satisfy the condition (35) in common subregions, in this case the control can inferred by selecting the control of the dominant system whose membership degree is of maximum distance from its guaranteed stability boundary:

$$\mathbf{u}(t) = \mathbf{K}_l \mathbf{x}(t), \quad l = arg \max_{i=1,r} \left(\alpha_i(\mathbf{z}(t)) - \underline{\alpha}_i\right) \tag{38}$$

Each state feedback $\mathbf{K}_l, r = 1, 2, \ldots, r_c$ is applied in the local region $\Omega_l^c \subseteq \Omega_l^s$ defined as:

$$\Omega_l^c = \{\Omega_l^s \mid \quad l = arg \max_{i=1,r} \left(\alpha_i(\mathbf{z}(t)) - \underline{\alpha}_i\right)\} \tag{39}$$

Let α_l^c the rule degree corresponding to boundary of the subregion Ω_l^c.
The resolution of the r independent minimization programs (33) leads to three possible cases as shown in figure 1:

Case 1: Several or all $\underline{\alpha}_l = 0$, $l = 1, 2, \ldots, r$, figure 1.a, a local controller can be used to stabilize the fuzzy system in its own local subregion and in adjacent subregions and the number of controllers can be reduced. The number of controllers is inferior to the number of rules $(r_c < r)$. In figure 1.a, the state feedback gain $\mathbf{K}_1$ is sufficient to control the fuzzy system.

Case 2: If the number of controllers can't be reduced and the condition (37) is fulfilled then the number of controllers is equal to the number of rules $(r_c = r)$, figure 1.b.

Case 3: If the condition (37) is not fulfilled, the global system may be instable. To solve this problem, we can add new rules to the model since we know exactly in which region, in the state space, we need new ones. Or we can add new controllers $(r_c > r)$, $\mathbf{K}_4$ and $\mathbf{K}_5$ in figure 1.c, without changing the model by using new nominal local systems, which is equivalent to the addition of new rules to the model.

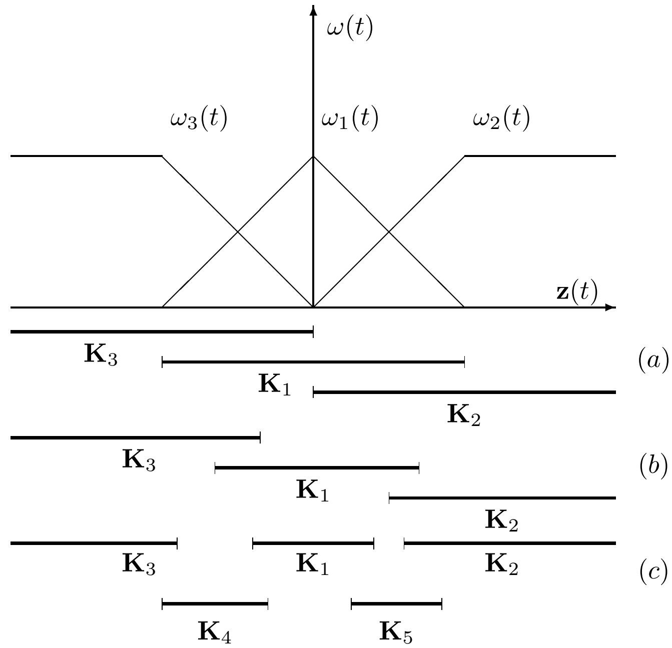

Fig. 1. Possible cases

Let $\tau_i, i = 1, 2, \ldots, N$ the i^{th} time instant at which the state meets the boundary of a subregion $\Omega_j^c, j = 1, 2, \ldots, r_c$. We assume that the state $\mathbf{x}(t)$ does not jump at the transition time τ_i, that is [10]

$$\mathbf{x}(\tau_i^-) = \mathbf{x}(\tau_i) = \mathbf{x}(\tau_i^+), \quad i = 1, 2, \ldots, N \tag{40}$$

Lemma 2: The fuzzy system (2) is globally stable if the transition time instant are finite $(N < \infty)$ and the stability covering condition (35) is verified.

Proof: Consider the following piecewise quadratic Lyapunouv function candidate:

$$V(t) = \sum_{l=1}^{r_c} \zeta_l(\mathbf{x}(t)) \mathbf{x}^T(t) \mathbf{P}_l \mathbf{x}(t) \tag{41}$$

Since the stability covering condition is verified:

$$\forall t \geq 0, \quad \exists l, \mathbf{x}(t))\mathbf{x}(t)) \in \Omega_l^s$$

if τ_i is the time instant at which the state leaves the subregion Ω_j^c and enters into the subregion Ω_k^c then:

$$V(\tau_i^-) = \mathbf{x}^T(\tau_i^-)\mathbf{P}_j\mathbf{x}(\tau_i^-) = \mathbf{x}^T(\tau)\mathbf{P}_j\mathbf{x}(\tau) \tag{42}$$

$$V(\tau_i^+) = \mathbf{x}^T(\tau_i^+)\mathbf{P}_k\mathbf{x}(\tau_i^+) = \mathbf{x}^T(\tau)\mathbf{P}_k\mathbf{x}(\tau) \tag{43}$$

The local symmetric positive matrices $\mathbf{P}_l, l = 1, 2, \ldots, r$, are determined so as to guarantee the local stability:

$$(26) \Rightarrow \exists \delta_l,\ \mathcal{L}_1(\mathbf{P}_l) + (1 - \alpha_l)\left[\mathcal{L}_A(\mathbf{P}_l) + \mathcal{L}_B(\mathbf{P}_l)\right] \leq -\delta_l \mathbf{I}$$

$$V(t) > 0, \quad \mathbf{x}(t) \neq 0, \Rightarrow \frac{\dot{V}(t)}{V(t)} \leq -\sigma_l,\ \sigma_l = \frac{\delta_l}{\lambda_{max}(\mathbf{P}_l)}$$
$$\mathbf{x}(t) \in \Omega_l^c, \quad \tau_i^+ < t < \tau_{i+1}^-, \quad i = 1, 2, \ldots, N$$

$$V(t) \leq V(\tau_i^+) e^{-\sigma_l (t-\tau_i^+)}$$
$$\tau_i^+ < t < \tau_{i+1}^-, \quad i = 1, 2, \ldots, N$$

Since :

$$\lambda_{min}(\mathbf{P}_l) \|\mathbf{x}(t)\|^2 \leq V(t) \leq \lambda_{max}(\mathbf{P}_l) \|\mathbf{x}(t)\|^2$$
$$\tau_i^+ < t < \tau_{i+1}^-, \quad i = 1, 2, \ldots, N$$

It follows that:

$$\|\mathbf{x}(t)\| \leq C_l \|\mathbf{x}(\tau_i)\| e^{-\frac{\sigma_l}{2}(t-\tau_i^+)}, C_l = \sqrt{\frac{\lambda_{max}(\mathbf{P}_l)}{\lambda_{min}(\mathbf{P}_l)}}$$
$$\tau_i^+ < t < \tau_{i+1}^-, \quad i = 1, 2, \ldots, N \quad (44)$$

Since the number of transition is finite, $N < \infty$ then :

$$\|\mathbf{x}(t)\| \leq C_{l_0} \|\mathbf{x}(\tau_N)\| e^{-\frac{\sigma_{l_0}}{2}(t-\tau_N^+)}, \quad t > \tau_N^+$$

At the N^{th} transition($t = \tau_N^+$) the state enters into the subregion $\Omega_{l_0}^c$ containing the origin and converges to the origin at $t \to \infty$.

$$\mathbf{x}(t) \in \Omega_{l_0}^c, \quad t > \tau_N^+ \qquad \|\mathbf{x}(t)\| \underset{t\to\infty}{\longrightarrow} 0$$

The fuzzy system is globally stable. ■

Lemma 3: If the state stays in each region Ω_l^c for a period of time $\Delta\tau$ such that :

$$\Delta\tau > \frac{\ln\left(\frac{\lambda_{max}(\mathbf{P}_l)}{\lambda_{min}(\mathbf{P}_l)}\right)}{\sigma_l} \quad (45)$$

then the fuzzy system (2) is globally stable.

Proof: From 44 it follows that when the state leaves the region Ω_l^c :

$$\|\mathbf{x}(\tau_{i+1}^-)\| \leq C_l \|\mathbf{x}(\tau_i)\| e^{-\frac{\sigma_l}{2}(\tau_{i+1}^- - \tau_i^+)}, C_l = \sqrt{\frac{\lambda_{max}(\mathbf{P}_l)}{\lambda_{min}(\mathbf{P}_l)}}$$
$$i = 1, 2, \ldots, N \quad (46)$$

since there is no jump in the state:

$$\frac{\|\mathbf{x}(\tau_{i+1})\|}{\|\mathbf{x}(\tau_i)\|} \leq C_l e^{-\frac{\sigma_l}{2}\Delta\tau}, C_l = \sqrt{\frac{\lambda_{max}(\mathbf{P}_l)}{\lambda_{min}(\mathbf{P}_l)}}$$
$$i = 1, 2, \ldots, N \quad (47)$$

If $\Delta\tau$ verify the condition (45) then:

$$\|\mathbf{x}(\tau_{i+1})\| < \|\mathbf{x}(\tau_i)\| < \cdots < \|\mathbf{x}(0)\| \quad (48)$$

and

$$\|\mathbf{x}(\tau_i)\| \underset{i\to\infty}{\longrightarrow} 0 \quad (49)$$

and the fuzzy system (2) is globally stable. ■

Theorem 2: If the the set $\alpha_l^c, l = 1, \ldots, r_c$ are such that the stability covering condition (34) is verified and the set of matrices $\mathbf{P}_l, l = 1, \ldots, r_c$ are such that:

$$\mathbf{P}_i \leq \mathbf{P}_j \quad \text{for all states } \mathbf{x}(\tau_k^-) \in \Omega_j^s \text{ and } \mathbf{x}(\tau_k^+) \in \Omega_i^s$$
$$k = 1, \ldots, N \quad (50)$$

then the fuzzy system is globally asymptotically stable.

Proof: Let the Lyapounuv function candidate given by (41). Since each matrix $\mathbf{P}_j$ assures that $V(t)$ is decreasing inside each sub-region Ω_j^s then:

$$V(t) = V_j(t) < V(\tau_{k-1}^+), \tau_{k-1}^+ < t \leq \tau_k^- \quad (51)$$

It yields

$$V(\tau_k^-) < V(\tau_{k-1}^+), k = 1, \ldots, N \quad (52)$$

At $t = \tau_k$ the state leaves the subregion Ω_j^s and enters into the subregion Ω_i^s. Since we assume there is no jump in the state

$$(50) \Rightarrow V_i(\tau_k^+) \leq V_j(\tau_k^-) \Rightarrow V(\tau_k^+) \leq V(\tau_k^-) \quad (53)$$

and

$$V(\tau_k) < V(\tau_{k-1}) < \cdots < V(0) \quad (54)$$

$$\|\mathbf{x}(\tau_k)\| < \|\mathbf{x}(\tau_{k-1})\| < \cdots < \|\mathbf{x}(0)\| \quad (55)$$

If condition (50) holds then the fuzzy system (2) is asymptotically stable. ■

IV. SIMULATION EXAMPLE

To show the effectiveness of the proposed method, we consider the following problem of balancing an inverted pendulum on a cart. The motion of the pendulum can be described by the following equations[16]:

$$\begin{cases} \dot{x}_1(t) = x_2(t) \\ \dot{x}_2(t) = \frac{g\sin(x_1(t)) - \frac{1}{2}amlx_2^2\sin(2x_1(t)) - a\cos(x_1(t))u(t)}{\frac{4l}{3} - a\cos^2(x_1(t)} \end{cases} \quad (56)$$

where:

$$a = \frac{1}{M+m} \quad (57)$$

where x_1 denotes the angle of the pendulum from the vertical and x_2 is the angular velocity, $g = 9.8\ m/s^2$ is the gravity constant $m = 0.8\ kg$ is the mass of the pendulum, $M = 2.0\ kg$ is the mass of the cart, $l = 0.5\ m$ is the half length of the pendulum, and u is the force applied to the cart. The inverted pendulum can be described by the following TS fuzzy model:

R_1 : if $x_1(t)$ is close to 0 Then $\dot{\mathbf{x}}(t) = \mathbf{A}_1\mathbf{x}(t) + \mathbf{B}_1 u(t)$
R_2 : if $x_1(t)$ is close to $\pm\frac{\pi}{2}$ Then $\dot{\mathbf{x}}(t) = \mathbf{A}_2\mathbf{x}(t) + \mathbf{B}_2 u(t)$
where

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 1 \\ \frac{g}{\frac{4l}{3} - aml} & 0 \end{bmatrix}, \quad \mathbf{B}_1 = \begin{bmatrix} 0 \\ -\frac{a}{\frac{4l}{3} - aml} \end{bmatrix}$$
$$\mathbf{A}_2 = \begin{bmatrix} 0 & 1 \\ \frac{2g}{\pi(\frac{4l}{3} - amlb^2)} & 0 \end{bmatrix}, \quad \mathbf{B}_2 = \begin{bmatrix} 0 \\ -\frac{ab}{\frac{4l}{3} - amlb^2} \end{bmatrix} \quad (58)$$

and $b = cos(80°)$. The membership functions are given by:

$$\omega_1(x_1(t)) = 1 - \frac{2}{\pi}|x_1(t)|, \quad \omega_2(x_1(t)) = \frac{2}{\pi}|x_1(t)| \quad (59)$$

The TS fuzzy model can be decomposed into two subsystems:

- *Sub-system 1:*

$$\dot{\mathbf{x}}(t) = (\mathbf{A}_1 + (1 - \alpha_1(t))\mathbf{\Delta A}_1)\mathbf{x}(t) + (\mathbf{B}_1 + (1 - \alpha_1(t))\mathbf{\Delta B}_1)u(t)$$

$$\mathbf{\Delta A}_1 = \alpha_2'(t)(\mathbf{A}_2 - \mathbf{A}_1),\ \mathbf{\Delta B}_1 = \alpha_2'(t)(\mathbf{B}_2 - \mathbf{B}_1)$$

$$\mathbf{A}_1 = \begin{bmatrix} 0 & 1 \\ 18.7282 & 0 \end{bmatrix},\ \mathbf{B}_1 = \begin{bmatrix} 0 \\ -0.6818 \end{bmatrix}$$

$$\mathbf{\Delta A}_1 = \alpha_2'(t)\begin{bmatrix} 0 & 0 \\ -9.2994 & 0 \end{bmatrix}, \quad \mathbf{\Delta B}_1 = \alpha_2'(t)\begin{bmatrix} 0 \\ 0.5882 \end{bmatrix}$$

- *Sub-system 2:*

$$\dot{\mathbf{x}}(t) = (\mathbf{A}_2 + (1-\alpha_2(t))\mathbf{\Delta A}_2)\mathbf{x}(t) + (\mathbf{B}_2 + (1-\alpha_2(t))\mathbf{\Delta B}_2)u(t)$$

$$\mathbf{\Delta A}_2 = \alpha_1'(t)(\mathbf{A}_1 - \mathbf{A}_2), \quad \mathbf{\Delta B}_2 = \alpha_1'(t)(\mathbf{B}_1 - \mathbf{B}_2)$$

$$\mathbf{A}_2 = \begin{bmatrix} 0 & 1 \\ 9.4288 & 0 \end{bmatrix},\ \mathbf{B}_2 = \begin{bmatrix} 0 \\ -0.0936 \end{bmatrix}$$

$$\mathbf{\Delta A}_2 = \alpha_2'(t)\begin{bmatrix} 0 & 0 \\ 9.2994 & 0 \end{bmatrix},\ \mathbf{\Delta B}_2 = \alpha_2'(t)\begin{bmatrix} 0 \\ -0.5882 \end{bmatrix}$$

For $\mathbf{D}_{A_1} = \mathbf{D}_{B_1} = \mathbf{D}_{A_2} = \mathbf{D}_{B_2} = \begin{bmatrix} 0 & 1.0000 \end{bmatrix}^T, \mathbf{E}_{A_1} = -\mathbf{E}_{A_1} = \begin{bmatrix} -9.2994 & 0 \end{bmatrix}$ and $\mathbf{E}_{B_1} = -\mathbf{E}_{B_2} = 0.5882$ the values obtained after the resolution of the minimization program (33) :

$$\underline{\alpha}_1 = 0,\ \varepsilon_1 = 26.7968,\ \mathbf{P}_1 = \begin{bmatrix} 18.78523 & 5.3854 \\ 5.3854 & 1.7371 \end{bmatrix}$$

$$\underline{\alpha}_2 = 0.85,\ \varepsilon_2 = 23.8166,\ \mathbf{P}_2 = \begin{bmatrix} 162.6934 & 57.7613 \\ 57.7613 & 20.5073 \end{bmatrix}$$

$$\mathbf{K}_1 = \begin{bmatrix} 290.7869 & 88.9461 \end{bmatrix}$$

$$\mathbf{K}_2 = \begin{bmatrix} 2476.4698 & 873.5949 \end{bmatrix}$$

Since the minimal value obtained are $\underline{\alpha}_1 = 0.0$ and $\underline{\alpha}_2 = 0.85$ the linear state feedback $\mathbf{u}(t) = \mathbf{K}_1\mathbf{x}(t)$ is sufficient to stabilize the inverted pendulum as shown in figure 2. It is possible to drive the inverted pendulum to its equilibrium position for initial angles $\theta(0) \in [-75°, 75°]$ using the linear state feedback $\mathbf{K}_1$.

V. Conclusion

In this paper an LMI approach has been proposed to design a fuzzy model based switching controller for non linear systems. The fuzzy model is represented as a set of uncertain linear systems. A local controller is designed such that the stability region of the corresponding local subsystem is maximized. Under some conditions this switching controller has the ability to stabilize the non linear system. The inverted pendulum stabilization problem has been used to demonstrate the effectiveness of this approach.

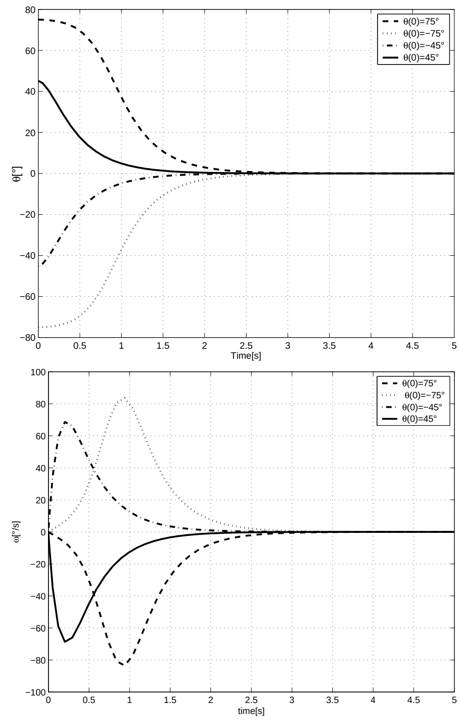

Fig. 2. The pendulum angle and angular velocity for $\theta(0) = \{-75, -45, 45, 75\}$

References

[1] T. Takagi and M. Sugeno, 'Fuzzy identification of systems and its application to modeling and control', *IEEE Transactions on Systems, Man and Cybernetics*, Vol.15, 1985, pp.116–132.

[2] M. Sugeno and G. T. Kang, 'Structure identification of fuzzy models', *Fuzzy Sets and Systems*, Vol. 28,1988, pp 15–33.

[3] S. G. Cao, N. W. Rees, and G. Feng, 'Analysis and design for a class of complex control systems, Part I: Fuzzy modeling and identification', *Automatica*, Vol. 33, 1997, pp. 1017–1028.

[4] K. Tanaka, T. Ikeda and H. Wang, 'Fuzzy regulators and fuzzy observers: Relaxed stability conditions and LMI based design', *IEEE Transactions on Fuzzy systems*, Vol. 6, 1998, pp. 250–265.

[5] E. Kim, H. Lee, 'New approaches to relaxed quadratic stability of fuzzy control systems', *IEEE Transactions on Fuzzy Systems*, Vol. 8, 2000, pp. 523–534.

[6] M. C. M. Teixeira and E.Assunçao and R. G. Avellar, "On Relaxed LMI-Based Designs For Fuzzy Regulators And Fuzzy observers", *IEEE Transactions on Fuzzy Systems*, Vol. 11, 2003, pp. 613–623.

[7] W. C. Kim, S. C. Ahn and W. H. Kwon, 'Stability analysis and stabilization of fuzzy state space models', *Fuzzy Sets and Systems*, Vol. 71, 1995, pp. 131-142.

[8] S. G. Cao, N. W. Rees and G. Feng, 'Stability analysis and design for a class of continuous-time fuzzy control systems', *International Journal of Control*, Vol.64, 1996, pp.1069–1089.

[9] S. G. Cao, N. W. Rees, and G. Feng, 'H_∞ Control of uncertain dynamical fuzzy discrete-time systems', *IEEE Transactions on Systems Man and Cybernetics*, Vol.31, 2001, pp.802–812.

[10] G. Feng, 'Approaches to quadratic stabilization of uncertain fuzzy dynamic systems', *IEEE Transactions on Circuits and Systems - I: Fundamental Theory and Applications* , Vol. 48, 2001, pp. 760–769.

[11] I. R. Petersen, 'A stabilization algorithm for a class of uncertain linear systems', *System Control Letters*, Vol. 8, 1987, pp. 351–357.

[12] S. Boyd, L. Ghaoui, E. Feron, and V. Balakrishnan, *Linear Matrix Inequalities in Systems and Control Theory*, Philadelphia, PA: SIAM, 1994.

[13] D. J. Stilwell and W. G. Rugh, Interpolation of observer state feedback controllers for gain scheduling, IEEE Transactions on Automatic Control, Vol. 44, 1999, 1225-1229.

[14] P. Gahinet, A. Nemirovski, A. Laub, and M. Chilali, *The LMI Control Toolbox*. Natick, MA: The Mathworks, Inc., 1995.

[15] H. J. Lee, J. B. Park, and G. Chen, 'Robust fuzzy control of nonlinear systems with parametric uncertainties', *IEEE Transactions on fuzzy systems*, Vol. 9, 2001, pp. 369–380.

[16] H. O. Wang, K. Tanaka, M. Griffin, "An approach to fuzzy control of nonlinear systems : stability and design issues", *IEEE Transactions on Fuzzy Systems*, Vol. 4, 1996, pp. 14–23.