THE USE OF ANGULAR VELOCITY VECTOR IN SHELL STRUCTURES

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ABSTRACT

In the following work, the principle of deformation is investigated using a new parameter called the angular velocity. The deformed element of surface is fully defined by this new parameter that is whenever the element of the reference surface undergoes a motion the three components of angular velocity define completely this motion.

1 INTRODUCTION

A number of different approaches carrying different points of view concerning the deformation and strain measures of surfaces have been suggested for the treatment of the subject of sells and plates. Basically two of them have to be distinguished namely, a derivation based on the general three-dimensional measures of strain and deformation, and a derivation based on three concept of oriented bodies founded by Duhem and adopted later to one and two-dimensional problems by the brothers Cosserat.

In a derivation based on the three-dimensional theory, exact measures of strain are usually either using deformation gradients or using the components of the displacements vector, after the manner of Love (1). Naghdi (2) argued that, the strains and deformation derived on the basis of the deformations gradients are not necessarily convenient measures. However, the use of the displacement components enables us in the application of boundary value problems to express the boundary conditions in terms of displacement components.

In the oriented bodies, on the other hand, the basic ingredients for obtaining the Kinematical quantities of the deformation are the vector functions r and d, which represent respectively the position vector of the surface and the single deformable director. These two vectors are assumed to be differentiable as many times as requested, with respect to t (time) and the surface coordinates θ^α

In the present work, however, the theory which represents a particular case of the Cosserat model, in which no director is assigned to the materiel points of the surface, is basically followed. However, the angular velocity of an element of surface is introduced to give Kinematical results which will facilitate the discussion of the boundary conditions of shells.

The deformation of the shell is that of the reference surface, just as it is assumed in the statics of shell. Therefore no assumption or approximation is made through the following work, except the shell being two-dimensional.

Special results from differential geometry and tensor notation are used here without proof. For further details see Green & Zerna (3), Williams (4) and Chebili (5).

2 RATE OF CHANGE OF SURFACE QUANTITIES

A point in the space of figure (1) is defined by the following relation

$$R = r + \theta^3 n \tag{1}$$

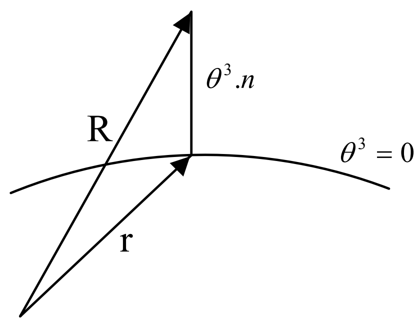

Figure 1 : position vector of a surface

In (1) **r** and **n** depend on the two coordinates (θ^1, θ^2), and the latter is vector of unit magnitude perpendicular to the reference surface.

As we are mainly concerned with a reference surface, then we put $\theta^3 = 0$.

The surface $\theta^3 = 0$ will be defined by the position vector $r(\theta^1,\theta^2)$. The position vector $r(\theta^1,\theta^2,t)$, will indicate Kinematically the position of the deformed surface. The variable t will define the position of the base vectors at any time t, during the process of deformation. it is to be noted that the surface geometry given in [2] and [3] , in which the first and second fundamental forms of the surface and some other quantities involving r , n and their derivatives remain valid, except that now these functions depend on the parameter t characterising time. In the forthcoming work, a dot over symbols indicates partial differentiation with respect to time.

Let the vector field v corresponds to the velocity of the surface, and denote.

$$v = v^i a_i = v_i a^i$$

$$v = \dot{r} = \frac{\partial r}{\partial t} = v^{\beta} a_{\beta} + vn = v_{\beta} a^{\beta} + vn \tag{2}$$

v^{β}, v_{β} and v are respectively, the contravariant, covariant and the normal components of the velocity vector V

The gradients of the velocity vector are its derivatives and are given by

$$v,_{\alpha} = \dot{r},_{\alpha} = \dot{a}_{\alpha} = v,^{\beta}_{\alpha} a_{\beta} + v^{\beta} a_{\beta,\alpha} + v,_{\alpha} n + vn,_{\alpha} \tag{3}$$

Using the formulate of Weingarten and Gauss, equations (3) becomes

$$\dot{a}_{\alpha} = v^{\beta},_{\alpha} a_{\beta} + v^{\beta}\Gamma^{\lambda}_{\beta\alpha} a_{\lambda} + v^{\beta} b^{\lambda}_{\beta\alpha} n + v,_{\alpha} n - vb^{\lambda}_{\alpha} a_{\lambda} \tag{4}$$

With little manipulation, and using the principle of the covariant differentiation, we get the rate of change of the base vectors written in the following manner

$$\dot{a}_{\alpha} = \left[v^{\beta} \Big| \alpha - vb^{\beta}_{\alpha} \right] a\beta + \left[v \Big|_{\alpha} + v^{\beta} b_{\beta\alpha} \right] n \tag{5}$$

Where the stroke in $v^{\beta}\big|_{\alpha}$ denote covariant differentiation with respect to the surface.

The base vectors scalar and products continue to hold as the surface deforms, i.e

$$a_{\alpha}.a_3 = 0$$

Then differentiating with respect to time and using (5) we write

$$\dot{n} = -\left[\; v\Big|_{\alpha} + v_{\beta} b^{\beta}_{\alpha} \; \right] a^{\alpha} \tag{6}$$

Equation (5) and (6) are respectively the rate of change of the surface base vectors and the unit normal to the surface.

The rate of change of the metric tensor will be

$$\dot{a}_{\alpha\beta} = \dot{a}_{\alpha}.a_{\beta} + a_{\alpha}.\dot{a}_{\beta} = v,_{\alpha}.a_{\beta} + a_{\alpha}.v,_{\beta} \tag{7}$$

$$\dot{a}_{\alpha\beta} = v_{\beta}\Big|_{\alpha} + v_{\alpha}\Big|_{\beta} - 2.vb_{\alpha\beta}. \tag{8}$$

The Kroneccker delta is constant, then

$$\dot{a}_{\rho\gamma} a^{\gamma\alpha} + a_{\rho\gamma} \dot{a}^{\gamma\alpha} = 0 \tag{9}$$

Equation (9) with (8) together give the rate of change of the contravariant metric tensor of the surface as:

$$\dot{a}^{\gamma\rho} = -\dot{a}_{\alpha\beta} a^{\alpha\gamma} a^{\beta\rho} = -\left[v_{\beta}\big|_{\alpha} + v_{\alpha}\big|_{\beta} - 2vb_{\alpha\beta} \right] a^{\alpha\gamma} a.^{\beta\rho} \tag{10}$$

Raising and lowering of indices in tensors will be modified when differentiation with respect to time is considered hence:

$$\begin{aligned} \dot{A}^{\alpha}_{\beta} &= \dot{a}^{\alpha\gamma} A_{\beta\gamma} + a^{\alpha\gamma} \dot{A}_{\beta\gamma} \quad \text{Raising} \\ \dot{A}^{\alpha}_{\beta} &= \dot{a}_{\gamma\beta} A^{\gamma\alpha} + a_{\gamma\beta} \dot{A}^{\gamma\alpha} \quad \text{lowering} \end{aligned} \tag{11}$$

The rate of change of the determinant (a) will be

$$\dot{a} = \dot{a}_{11} a_{22} + a_{11} \dot{a}_{22} - 2a_{12} \dot{a}_{12}. \tag{12}$$

Introducing the values of the rate of change of metric tensors from equations (8) into (12), we end up the following expressions,

$$\dot{a} = 2aa^{\alpha\beta} \left[v_{\alpha}\Big|_{\beta} - vb_{\alpha\beta} \right] = 2a \left[v^{\alpha}\Big|_{\alpha} - vb^{\alpha}_{\alpha} \right] \tag{13}$$

$$\dot{a}^{\gamma} = a^{\gamma\lambda} \left\{ \left[v\Big|_{\lambda} + v_{\beta} b^{\beta}_{\lambda} \right] n - \left[v_{\lambda}\Big|_{\beta} - vb_{\beta\lambda} \right] a^{\beta} \right\} \tag{14}$$

Lastly, the rate of change of an element of area dS is:

$$d\dot{S} = \frac{1}{2\sqrt{a}} \dot{a} d\vartheta^1 d\vartheta^2 = \sqrt{a} \left[v^{\alpha}\Big|_{\alpha} - vb^{\alpha}_{\alpha} \right] dS \tag{15}$$

3 RATE OF MEMBRANE STRAIN TENSOR

The surface r when subject to deformation may undergoes elongation or contraction in its plane. Differentiation of both sides with respect to time of the first fundamental form of the surface, gives

$$2\delta s\delta\dot{s} = \dot{a}_{\alpha\beta}\delta\theta^{\alpha}\delta\theta^{\beta} \tag{16}$$

Using (8), we write

$$\delta s\delta\dot{s} = \left[\frac{V_{\beta}\big|_{\alpha} + V_{\alpha}\big|_{\beta}}{2} - Vb_{\alpha\beta} \right] \delta\theta^{\alpha}\delta\theta = \gamma_{\alpha\beta}\delta\theta^{\alpha}\delta\theta^{\beta} \tag{17}$$

Where

$$\gamma_{\alpha\beta} = \gamma_{\beta\alpha} = \frac{\dot{a}_{\alpha\beta}}{2} = \left[\frac{v_\beta\big|_\alpha + v_\alpha\big|_\beta}{2} - v b_{\alpha\beta}\right] \quad (18)$$

Equation (18) is the rate of the membrane strain tensor, which has three independent components,

$$\gamma_{11}, \gamma_{12} = \gamma_{21} \text{ and } \gamma_{22}$$

Dividing the second equation of (17) by δs^2

$$\frac{\delta\dot{s}}{\delta s} = \frac{\gamma_{\alpha\beta}\delta\theta^\alpha\delta\theta^\beta}{a_{\lambda\rho}\delta\theta^\lambda\delta\theta^\rho}. \quad (19)$$

Let us imagine three adjacent points A, B and C which lie on and move with the surface. We will further imagine that the line CA is instantaneously perpendicular to AB and has the same length as AB at the same time at which we examine the surface. We will now find the rate of change of the angle, α, between AB and AC which is equal to π/ 2 at the instant we are considering, fig .2.

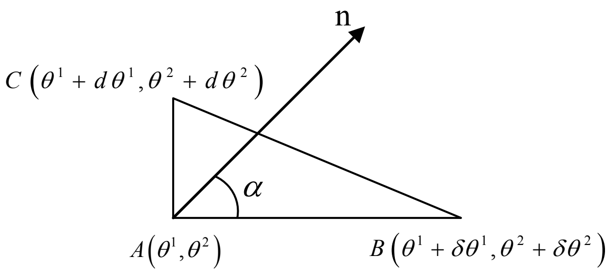

Figure 2

From the base vectors scalar rule, we write;

AC = n x AB

$$\begin{aligned} a_\beta d\theta^\beta &= n x a_\rho \delta\theta^\rho \\ d\theta^\beta &= a^{\lambda\beta}\varepsilon_{\rho\lambda}\delta\theta^\rho \end{aligned} \quad (20)$$

And the angle, α, between these two lines is given as:

$$\cos\alpha = \frac{a_{\alpha\beta}\delta\theta^\alpha d\theta^\beta}{\sqrt{\left[a_{\lambda\upsilon}\delta\theta^\lambda\delta\theta^\upsilon\right]}\sqrt{\left[a_{\eta\psi}d\theta^\eta d\theta^\psi\right]}} \quad (21)$$

In differentiating (21) with respect to time $\delta\theta^\alpha$ and $d\theta^\beta$ are taken as constants since the points are convected, that is move with the surface. Thus since $a_{\alpha\beta}\delta\theta^\alpha d\theta^\beta$ is instantaneously equal to zero,

$$-\left(\frac{\dot{\alpha}}{2}\right) = \frac{\gamma_{\alpha\beta}a^{\lambda\beta}\varepsilon_{\rho\lambda}\delta\theta^\rho\delta\theta^\alpha}{a_{\lambda u}\delta\theta^\lambda\delta\theta^u} \quad (22)$$

Using (18) and (20), equation (22) becomes

$$-\left(\frac{\dot{\alpha}}{2}\right) = \frac{\dot{a}_{\alpha\beta}\delta\theta^\alpha d\theta^\beta}{\left[a_{\lambda\upsilon}\delta\theta^\lambda\delta\theta^\upsilon\right]\left[a_{\eta\psi}d\theta^\eta d\theta^\psi\right]} \quad (23)$$

Equations (19) and (23) have the same structure as the equations which represent the normal and twist curvatures. They are the rates of direct strain as shear strains respectively, they both depend on the tensor $\gamma_{\alpha\beta}$ in the same way as the curvature and twist depend on the second order tensor $b_{\alpha\beta}$.

4 MEMBRANE STRAIN TENSOR

By analogy to the rate of the membrane strain tensor, which is found to be one half the rate of change of the metric tensor, the membrane strain tensor will be given by

$$G_{\alpha\beta} = \frac{1}{2}\left[a_{\alpha\beta} - A_{\alpha\beta}\right] \quad (24)$$

Where $a_{\alpha\beta}$ is the deformed metric tensor (final state of the deformed surface at some fixed time) which is function of θ^α and *t*. where as $A_{\alpha\beta}$ is the value of the metric tensor in some reference configuration (undeformed metric tensor) which is independent of *t*. Also, it is to be noted that

$$\dot{G}_{\alpha\beta} = \gamma_{\alpha\beta} = \frac{\dot{a}_{\alpha\beta}}{2} \quad (25)$$

5 THE CONCEPT OF ANGULAR VELOCITY

The deformation of surface induces not only stretches but also rotations, and as we used velocities of instead of simple displacement, let us introduce the concept of angular velocities.

From (18) we have

$$v b_{\alpha\beta} = -\gamma_{\alpha\beta} + \frac{v_\beta\big|_\alpha + v_\alpha\big|_\beta}{2} \quad (26)$$

Substituting the above quality in the second equation of (5) we get the following expression for the derivative of the velocity

$$v,_\alpha = \gamma_{\alpha\beta}a^\beta + \left[\frac{v_\beta\big|_\alpha - v_\alpha\big|_\beta}{\,}\right]a^\beta + \left[v\big|_\alpha + v_\beta b^\beta_\alpha\right]n. \quad (27)$$

We next introduce a new scalar quantity based on the derivatives of the velocity i.e the velocity gradient $v,_{\alpha}$

$$\Omega = -\frac{\varepsilon^{\rho\alpha} a_{\rho}.v,_{\alpha}}{2}. \tag{28}$$

Substituting the value of $v,_{\alpha}$ from (5) into the above expression we get

$$\Omega = -\frac{\varepsilon^{\rho\alpha} a_{\rho}\left[\left[v_{\beta}\big|_{\alpha} - v b_{\alpha\beta}\right] a^{\beta} + \left[v\big|_{\alpha} + v_{\beta} b_{\alpha}^{\beta}\right] n\right]}{2} \tag{29}$$

With $b_{\alpha\beta} = b_{\beta\alpha}$, then (29) becomes :

$$\Omega = -\frac{\varepsilon^{\rho\alpha} v_{\rho}\big|_{\alpha}}{2}. \tag{30}$$

Also we introduce a second pair of quantities, Ω^{β} defined as

$$\Omega^{\beta} = -\varepsilon^{\alpha\beta} n . v,_{\alpha}. \tag{31}$$

Again substituting the value of $v,_{\alpha}$ from (5) into (31), we write

$$\Omega^{\beta} = -\varepsilon^{\alpha\beta}\left[v\big|_{\alpha} + v_{\rho} b_{\alpha}^{\rho}\ \right] \tag{32}$$

Equation (30) and (32) will be written as follows:

$$\Omega\varepsilon_{\alpha\beta} = \frac{\left[v_{\beta}\big|_{\alpha} - v_{\alpha}\big|_{\beta}\right]}{2} \tag{33}$$

$$\Omega^{\beta}\varepsilon_{\alpha\beta} = -\left[v_{\beta} b_{\alpha}^{\beta} + v\big|_{\alpha}\right] \tag{34}$$

Then,

$$v,_{\alpha} = \gamma_{\alpha\beta} a^{\beta} + \Omega\varepsilon_{\alpha\beta} a^{\beta} - \Omega^{\beta}\varepsilon_{\alpha\beta} n \tag{35}$$

$$= \gamma_{\alpha\beta} a^{\beta} + \Omega\left(n x a_{\alpha}\right) - \Omega^{\beta}\left(a_{\alpha} x a_{\beta}\right) = \gamma_{\alpha\beta} a^{\beta} + \left[\Omega^{\beta} a_{\beta} + \Omega n\right] x a_{\alpha} \tag{36}$$

The quantity in bracket represents a space vector $\overline{\Omega}, i.e$

$$\overline{\Omega} = \Omega^{\beta} a_{\beta} + \Omega\ \ n. \tag{37}$$

Differentiation of (37) with respect to α gives

$$\overline{\Omega},_{\alpha} = \Omega^{\beta},_{\alpha} a_{\beta} + \Omega^{\beta} a_{\beta,\alpha} + \Omega,_{\alpha} n + \Omega\ \ n,_{\alpha}. \tag{38}$$

We make use know of the two formulate of Gauss and Weingarten. We note then, the two following special results for future convenience

$$\overline{\Omega},_{\alpha} . a^{\beta} = \Omega^{\beta}\big|_{\alpha} - \Omega\ \ b_{\alpha}^{\beta} \tag{39}$$

$$\overline{\Omega},_{\alpha} . n = \Omega^{\beta} b_{\beta\alpha} + \Omega\big|_{\alpha}. \tag{40}$$

Equations (36) and (37) together form

$$v,_{\alpha} = \gamma_{\alpha\beta} a^{\beta} + \overline{\Omega} x a_{\alpha} \tag{41}$$

Where the right hand side of the velocity gradient is composed from two parts.$\gamma_{\alpha\beta}$ a^{β} represents the rate of membrane strain and $\overline{\Omega} x a_{\alpha}$, due to the vector $\overline{\Omega}$ which represents an angular velocity of the surface, fig. (5.1)

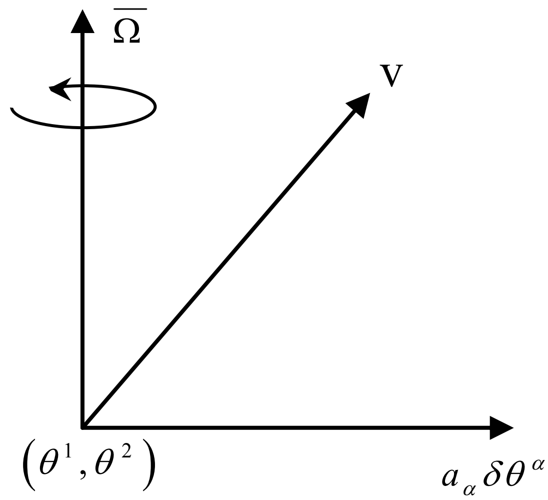

Figure 3

The meaning of equation (41) and particularly the term of the angular velocity can be better explained by the following arguments. Let us imagine the location of two adjacent points A and B on the surface with coordinates θ^{α} and $\theta^{\alpha} + \delta\theta^{\alpha}$. The line AB between these points is perpendicular to the surface normal n and as the surface deforms both lines AB rotate but remain perpendicular to each other. The rate of change of the unit normal was expressed in (6) on the basis of

$$\left[n.a_{\alpha}\right]^{\bullet} = 0.$$

Then with the use of (34), it becomes

$$\dot{n} = \Omega^{\beta}\varepsilon_{\alpha\beta} a^{\alpha}. \tag{42}$$

As the normal retains its original length after deformation, then $\dot{n}$ must lie in the plane of the surface. Then the component of the angular velocity of the pair of lines, AB and n, in the plane of the surface is given by

$$n x \dot{n} = \Omega^{\beta}\varepsilon_{\alpha\beta} n x a^{\alpha} = \Omega^{\beta} a_{\beta}.$$

The normal component of the angular velocity of the pair lines, as and n, i.e. their rotation about the normal, is given by

$$\frac{\left[\ \left(a_{\alpha}\delta\theta^{\alpha}\right) x \left(v,_{\beta}\ \delta\theta^{\beta}\right)\right] .n}{a_{\lambda\gamma}\delta\theta^{\lambda}\delta\theta^{\gamma}} = \frac{\left(a_{\alpha} x v,_{\beta}\right).n\ \ \delta\theta^{\alpha}\delta\theta^{\beta}}{a_{\lambda\gamma}\delta\theta^{\lambda}\delta\theta^{\gamma}}\ .$$

Using the base vectors product rule, (28) and (33), we write the above expression after having simplified it as:

$$=\Omega+\frac{\varepsilon_{\alpha\mu}a^{\mu\rho}\gamma_{\beta\rho}\delta\theta^{\alpha}\delta\theta^{\beta}}{a_{\lambda\gamma}\delta\theta^{\lambda}\delta\theta^{\gamma}}$$

Which is the normal component of angular velocity plus the rate of shear strain given in equation (23).

Differentiation of (41) with respect to λ and then use of the principle of covariant differentiation, gives

$$v,_{\alpha\lambda}=\gamma_{\alpha\beta}\Big|_{\lambda}a^{\beta}+\gamma_{\alpha\beta}b_{\lambda}^{\beta}n+\overline{\Omega},_{\lambda}x\ a_{\alpha}+\overline{\Omega},xa_{\alpha,\lambda}. \quad (43)$$

Now interchanging α and λ in the above equation and subtracting, we write.

$$\varepsilon^{\alpha\lambda}\left[\gamma_{\alpha\beta}\Big|_{\lambda}a^{\beta}+\gamma_{\alpha\beta}b_{\lambda}^{\beta}n+\overline{\Omega},_{\lambda}xa_{\alpha}\right]=0. \quad (44)$$

By replacing the value of $\overline{\Omega}$ from (37) into (44), we obtain

$$\varepsilon^{\alpha\lambda}\left[\ \gamma_{\alpha\beta}\Big|_{\lambda}a^{\beta}+\gamma_{\alpha\beta}b_{\lambda}^{\beta}n+\left(\Omega^{\beta},_{\lambda}a_{\beta}+\Omega^{\beta}a_{\beta,\lambda}+\Omega,_{\lambda}n+\Omega n,_{\lambda}\right)xa_{\alpha}\right]=0. \quad (45)$$

Again, by using the formulae of Gauss and Weingarten, we write

$$=\varepsilon^{\alpha\lambda}\left[\gamma_{\alpha\beta}\big|_{\lambda}a^{\beta}+\gamma_{\alpha\beta}b_{\lambda}^{\beta}n+\left(\Omega^{\beta}\big|_{\lambda}a_{\beta}+\Omega^{\beta}b_{\beta\lambda}n+\Omega\big|_{\lambda}n-\Omega b_{\lambda}^{\beta}a_{\beta}\right)xa_{\alpha}\right]=0$$

$$=\varepsilon^{\alpha\lambda}\left\{\gamma_{\alpha\gamma}\big|_{\lambda}a^{\gamma}+\left[\Omega^{\beta}b_{\beta\lambda}+\Omega\big|_{\lambda}\right]\varepsilon_{\alpha\gamma}a^{\gamma}+\left[\Omega^{\beta}\big|_{\lambda}-\Omega b_{\lambda}^{\beta}\right]\varepsilon_{\beta\alpha}n+\gamma_{\alpha\beta}b_{\lambda}^{\beta}n\right\}=0 \quad (46)$$

Where $\Omega^{\beta}\Big|_{\lambda}=\Omega^{\beta},_{\lambda}+\Omega^{\rho}\Gamma_{\rho\lambda}^{\beta}$.

Scalar multiplication of (46) by a_γ and respectively gives

$$\varepsilon^{\alpha\lambda}\gamma_{\alpha\gamma}\Big|_{\lambda}+\Omega^{\beta}b_{\beta\gamma}+\Omega\Big|_{\gamma}=0 \quad (47)$$

$$\left[\Omega^{\lambda}\Big|_{\lambda}-\Omega b_{\lambda}^{\lambda}\right]+\varepsilon^{\alpha\lambda}\gamma_{\alpha\beta}b_{\lambda}^{\beta}=0 \quad (48)$$

The above two equations (47), (48), become when comparing them to (39) and (40)

$$\overline{\Omega},_{\gamma}.n=-\varepsilon^{\alpha\lambda}\gamma_{\alpha\gamma}\Big|_{\lambda} \quad (49)$$

$$\overline{\Omega},_{\gamma}.a^{\gamma}=\varepsilon^{\alpha\lambda}\gamma_{\alpha\beta}b_{\lambda}^{\beta} \quad (50)$$

6 THE RATE OF BENDING TENSOR

Consider the rate of bending tensor, expressed as the following second order surface tensor

$$B^{\alpha\beta}=\varepsilon^{\alpha\gamma}\overline{\Omega},_{\gamma}a^{\beta}. \quad (51)$$

Using (39), equation (51) becomes

$$\beta^{\alpha\beta}=\varepsilon^{\alpha\gamma}\left[\ \Omega^{\beta}\Big|_{\gamma}-\Omega b_{\gamma}^{\beta}\right]. \quad (52)$$

Now if we multiply, (51) by $\varepsilon_{\alpha\beta}$ then, use (50), we take

$$\varepsilon_{\alpha\beta}\beta^{\alpha\beta}=\varepsilon_{\alpha\beta}\varepsilon^{\alpha\gamma}\overline{\Omega},_{\gamma}.a^{\beta}=\overline{\Omega},_{\beta}.a^{\beta} \quad (53)$$

$$\varepsilon_{\alpha\beta}\beta^{\alpha\beta}=\varepsilon^{\alpha\lambda}\gamma_{\alpha\beta}b_{\lambda}^{\beta} \quad (54)$$

Therefore,

$$B^{\alpha\beta}\neq B^{\beta\alpha} \quad (55)$$

We proceed to evaluate the rate of change of the coefficients of the second fundamental form of the surface

$$\dot{b}_{\alpha\lambda}=v,_{\alpha\lambda}.n+a_{\alpha,\lambda}.\dot{n}.$$

After substituting (42) and (43) into equation (55), we write

$$\dot{b}_{\alpha\lambda}=\gamma_{\alpha\beta}b_{\lambda}^{\beta}+\varepsilon_{\rho\alpha}\varepsilon_{\beta\lambda}B^{\beta\rho}. \quad (56)$$

$$\dot{b}_{\lambda}^{\alpha}=\dot{a}^{\alpha\beta}b_{\lambda\beta}+a^{\alpha\beta}\dot{b}_{\lambda\beta}. \quad (57)$$

Substituting (10) and (56) into (57), then

$$\dot{b}_{\lambda}^{\alpha}=-\gamma_{\gamma\lambda}b^{\gamma\alpha}+\varepsilon_{\rho\lambda}\varepsilon_{\gamma\beta}B^{\gamma\rho}a^{\alpha\beta}. \quad (58)$$

The rate of change of the mean and the Gaussian curvatures are:

$$[\dot{H}=\frac{\dot{b}_{\alpha}^{\alpha}}{2}=\frac{-\gamma_{\rho\alpha}b^{\rho\alpha}+\varepsilon_{\rho\alpha}\varepsilon_{\gamma\beta}B^{\gamma\rho}a^{\alpha\beta}}{2}. \quad (59)$$

$$\dot{K}=\gamma_{\beta\lambda}\left[b^{\lambda\rho}b_{\rho}^{\beta}-b^{\lambda\beta}b_{\rho}^{\rho}\right]+b_{\lambda\rho}B^{\lambda\rho} \quad (60)$$

6.1 The compatibility equations

The compatibility equations is the relations between the deformation of the reference surface and the overall displacements. The deformation of the reference surface has been expressed by the bending tensor $B^{\lambda\rho}$ and the membrane strain tensor $\gamma_{\beta\lambda}$. These tensors are also functions of displacements and rotations of the surface which involves the first and second fundamental forms of the surface that are the related by gauss and codazzi equations. Thus we also expect to find a relations between the membrane and bending tensors to ensure the continuity of deformation of the surface.

As the Gaussian curvature is a bending invariant, therefore any change in its final expression is due to the change of the lengths and angles corresponding to the intrinsic geometry of the surface.

Gauss's theorem permits writing the expression of Gaussian curvature in terms of the coefficients of the first fundamental form only thus, from equation (60) we expect to be able to find a relation between the rate of bending tensor and the rate of membrane strain tensor.

Then, substituting the value of $\beta^{\alpha\beta}$ from (52) into the second term of equation (60), using (30) and (32), we write.

$$b_{\lambda\rho}B^{\lambda\rho} = -b_{\lambda\rho}\varepsilon^{\lambda\gamma}\left\{ \varepsilon^{\alpha\rho}\left[v\big|_{\alpha} + v_{\theta}b_{\alpha}^{\theta}\right]\Big|_{\gamma} - \frac{\varepsilon^{\theta\alpha}v_{\theta}\big|_{\alpha}}{2}b_{\gamma}^{\rho}\right\} \quad (61)$$

Taking the second covariant differentiation of (18) and applying the codazzi equations, we write

$$\varepsilon^{\alpha\upsilon}\varepsilon^{\lambda\beta}\gamma_{\alpha\beta}\big|_{\lambda\upsilon} = \varepsilon^{\alpha\upsilon}\varepsilon^{\lambda\beta}\left[\frac{v_{\alpha}\big|_{\beta} + v_{\beta}\big|_{\alpha}}{2}\right]\Big\|_{\lambda\upsilon} - \varepsilon^{\alpha\upsilon}\varepsilon^{\lambda\beta}b_{\alpha\beta}v\big|_{\lambda\upsilon} \quad (62)$$

Subtracting (62) from (61) and simplifying, we get

$$b_{\lambda\rho}B^{\lambda\rho} - \varepsilon^{\alpha\upsilon}\varepsilon^{\lambda\beta}\gamma_{\alpha\beta}\big|_{\lambda\upsilon} = -b_{\rho}^{\upsilon}a^{\gamma\eta}\varepsilon_{\upsilon\eta}\left\{\varepsilon^{\alpha\rho}\left[v_{\vartheta}b_{\alpha}^{\vartheta}\right]\Big\|_{\gamma} - \frac{\varepsilon^{\vartheta\alpha}v_{\vartheta}\big|_{\alpha}}{2}b_{\gamma}^{\rho}\right\} - \varepsilon^{\alpha\upsilon}\varepsilon^{\lambda\beta}\left[\frac{v_{\alpha}\big|_{\beta} + v_{\beta}\big|_{\alpha}}{2}\right]\Big\|_{\lambda\upsilon} \quad (63)$$

Having in mind the following relations,

$$\begin{aligned} \varepsilon^{\lambda\beta}v_{\alpha}\big|_{\beta\lambda} &= -v_{\rho}a^{\rho\lambda}\varepsilon_{\lambda\alpha}k \\ \varepsilon^{\alpha\upsilon}v_{\beta}\big|_{\alpha\upsilon} &= v_{\rho}a^{\rho\upsilon}\varepsilon_{\upsilon\beta}k \end{aligned} \quad (64)$$

And

$$v_{\beta}\big|_{\alpha\lambda\upsilon} - v_{\beta}\big|_{\alpha\upsilon\lambda} = R^{\rho}_{\cdot\beta\lambda\upsilon}v_{\rho}\big|_{\alpha} + R^{\rho}_{\cdot\alpha\lambda\upsilon}v_{\beta}\big|_{\rho} = a^{\rho\xi}\varepsilon_{\xi\beta}\varepsilon_{\lambda\upsilon}Kv_{\rho}\big|_{\alpha} + a^{\rho\xi}\varepsilon_{\xi\alpha}\varepsilon_{\lambda\upsilon}Kv_{\beta}\big|_{\rho}. \quad (65)$$

Then, (63) becomes

$$b_{\lambda\rho}B^{\lambda\rho} - \varepsilon^{\alpha\upsilon}\varepsilon^{\lambda\beta}\gamma_{\alpha\beta}\big|_{\lambda\upsilon} = 0 \quad (66)$$

Finally (6.10) takes the following form

$$\dot{K} = \gamma_{\beta\lambda}\left[b^{\lambda\rho}b_{\rho}^{\beta} - b^{\lambda\beta}b_{\rho}^{\rho}\right] + \varepsilon^{\alpha\upsilon}\varepsilon^{\lambda\beta}\gamma_{\alpha\beta}\big|_{\lambda\upsilon}. \quad (67)$$

If we take the covariant differentiation of equation (52) we write

$$B^{\alpha\beta}\big|_{\alpha} = \varepsilon^{\alpha\lambda}\left[\Omega^{\beta}\big|_{\lambda} - \Omega b_{\lambda}^{\beta}\right]_{\alpha} = \varepsilon^{\alpha\lambda}\Omega^{\beta}_{\lambda\alpha} - \varepsilon^{\alpha\lambda}\Omega\big|_{\alpha}b_{\lambda}^{\beta} - \varepsilon^{\alpha\lambda}\Omega b_{\lambda}^{\beta}\big|_{\alpha}. \quad (68)$$

Using (40) and (49), we get

$$B^{\alpha\beta}\big|_{\alpha} = -\varepsilon^{\alpha\lambda}b_{\lambda}^{\beta}\overline{\Omega}_{,\alpha}.n = a^{\alpha\rho}b^{\beta\lambda}\left[\gamma_{\rho\alpha}\big|_{\lambda} - \gamma_{\lambda\alpha}\big|_{\rho}\right] \quad (69)$$

Thus, we write finally the set of compatibility equations from (54), (66) and (69) as follows

$$\begin{aligned} \varepsilon_{\alpha\beta}B^{\alpha\beta} &= \varepsilon^{\alpha\lambda}\gamma_{\alpha\beta}b_{\lambda}^{\beta} \\ b_{\lambda\rho}B^{\lambda\rho} &= \varepsilon^{\alpha\upsilon}\varepsilon^{\lambda\beta}\gamma_{\alpha\beta}\big|_{\lambda\upsilon} \\ B^{\alpha\beta}\big|_{\alpha} &= a^{\alpha\rho}b^{\beta\lambda}\left[\gamma_{\rho\alpha}\big|_{\lambda} - \gamma_{\lambda\alpha}\big|_{\rho}\right] \end{aligned} \quad (70)$$

This set of equation comprises 4 equations with 7 unknowns, 4 components of bending strains 3 components of membrane strains equations (47) and (48) are also compatibility equations and Ω^{β} can be eliminated to form a single equation in the normal component of the angular velocity Ω. The procedure starts first by eliminating the tangential component of the angular velocity from (47). multiplication of (47) by $\varepsilon^{\xi\eta}\varepsilon^{\rho\gamma}b_{\xi\rho}$,

$$\varepsilon^{\varepsilon\eta}\varepsilon^{\rho\gamma}b_{\varepsilon\rho}\left[\varepsilon^{\alpha\lambda}\gamma_{\alpha\gamma}\big|_{\lambda} + \Omega\big|_{\gamma}\right] = -\delta_{\beta}^{\eta}k\Omega^{\beta} = -k\Omega^{\eta}. \quad (71)$$

Also

$$\varepsilon^{\xi\eta}\varepsilon^{\rho\gamma}b_{\xi\rho}b_{\beta\gamma} = \delta_{\beta}^{\eta}k \quad (72)$$

Hence, equation (61) becomes

$$\varepsilon^{\xi\eta}\varepsilon^{o\gamma}b_{\xi\rho}\left[\varepsilon^{\alpha\lambda}\gamma_{\alpha\gamma}\Big|_{\lambda} + \Omega\Big|_{\gamma}\right] = -\delta_{\beta}^{\eta}k\Omega^{\beta} = -k\Omega^{\eta}. \quad (73)$$

For surfaces, where K is different from zero i.e surface which are not developable, we write the following

$$\Omega^{\eta}\Big|_{\eta} = -\left[\varepsilon^{\varepsilon\eta}\varepsilon^{\rho\eta}b_{\varepsilon\rho}\frac{\left[\varepsilon^{\alpha\lambda}\gamma_{\alpha\gamma,\lambda} + \Omega\big|_{\gamma}\right]}{k}\right]\Big|_{\eta}. \quad (74)$$

Using the codazzi relations, equation (74) becomes

$$\Omega^{\lambda}\Big|_{\lambda} = -\varepsilon^{\varepsilon\lambda}\varepsilon^{\rho\gamma}b_{\varepsilon\rho}\left[\frac{\left[\varepsilon^{\alpha\eta}\gamma_{\alpha\alpha}\big|_{\eta} + \Omega\big|_{\gamma}\right]}{}\right]\Big|_{\lambda}. \quad (75)$$

Substituting (75) into (48), we obtain a second order partial differential equation in the normal component of the angular velocity

$$\varepsilon^{\varepsilon\lambda}\varepsilon^{\rho\gamma}b_{\varepsilon\rho}\left[\frac{\left[\varepsilon^{\alpha\eta}\gamma_{\alpha\gamma}\big|_{\eta} + \Omega\big|_{\gamma}\right]}{k}\right]\Big|_{\lambda} + \Omega b_{\lambda}^{\lambda} + \varepsilon^{\alpha\lambda}\gamma_{\alpha\beta}b_{\lambda}^{\beta} = 0 \quad (76)$$

The above equation is applicable provided K (the gaussian curvature) is different from zero. For developable surfaces a distinct procedure has to be followed. The solution of such equation depends strongly on the form of the surface of the shell, in particular on the sing of gaussian curvature.

7 CONCLUSION

The use of the angular velocity vector in the derivation of the shell's equations permits, to express all the deformed quantities on the shell surface, to write the deformed state in a single equation and hence discuss possible solutions using appropriate boundary conditions. It also leads to particular states of stress as the membrane theory and inextensional theory of deformation by simply omitting quantities like $\gamma_{\alpha\beta}$ from the general equations.

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