A NEW RECTANGULAR FINITE ELEMENT FOR PLANE ELASTICITY ANALYSIS

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ABSTRACT

A simple rectangular element having 2d.o.f/node at each node is developed. This element is based on the strain approach. From some numerical examples, by using the concept of static condensation it is concluded that the exact solutions can be obtained. This element is nonconforming but satisfies the patch test and produces results which are acceptable within practical engineering accuracy even when few elements are employed.

ملخص

نعرض في هذا البحث عنصرا محدودا مستطيلا ذو درجتي حرية (2d.o.f/node) في كل عقدة، تم تطويره اعتمادا على مبدأ التشوه. من خلال بعض الأمثلة العددية، وباستعمال مبدأ التكثيف الستاتيكي توصلنا إلى إمكانية الحصول على نتائج دقيقة. إن هذا العنصر لا يتوفر على شروط التطابق لكنه يجتاز اختبار (Patch Test) بنجاح، كما أنه يعطي نتائج جد مقبولة في الهندسة التطبيقية باستعمال عدد محدود من العناصر.

KEY WORDS: Rectangular element, strain approach, static condensation, elasticity problems.

1 INTRODUCTION

The development of finite elements for general plane elasticity problems occupied a prominent position in the early work on the matrix displacement method of analysis. Attention was therefore focused on the development of more sophisticated elements based on the strain element Sabir et al [1, 2, 3, and 4]. Several models such as rectangular elements were developed, among them the elements of Sabir et al [4] SBRIE and SBRIE2. The first element is based on linear variation of direct strains and constant shearing strain. The second is based on linear variation of all three strain components. These elements produce rapid convergence of deflections as well as stresses. A Further progress into the development of plane stress elements based on the strain approach is due to Belarbi [5, 6, 7 and 8].

In This paper the shape function for a rectangular element having two degrees of freedom at each of the four corner nodes is developed using the strain approach. However any singularity is eliminated by the use of local axes optimally oriented. This element is nonconforming but satisfies the patch test and produces results which are acceptable within practical engineering accuracy even when few elements are employed.

2 CONSTRUTING THE COMPONENT STIFNESS MATRICES

The above consideration will lead to an element requiring ten d.o.f, each of the four corner nodes has the two essential d.o.f, in addition, an internal node is also used fig.1

The assumed strains are:

$$\begin{cases} \varepsilon_x = a_4 + a_5 y - a_7 x - \alpha a_{10} x \\ \varepsilon_y = -a_5 y + a_6 + a_7 x - \alpha a_9 y \\ \gamma_{xy} = a_8 + a_9 x + a_{10} y \end{cases} \tag{1}$$

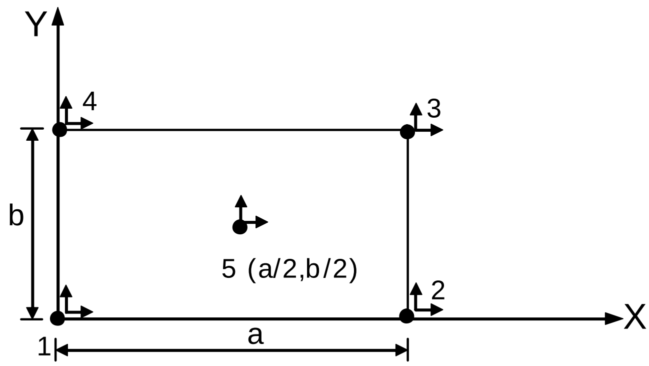


Figure 1: Co-ordinates and nodal points for the rectangular R4SB2 element

By integrating equations (1) and adding the condition of rigid body movement **(R4SB2**) we obtain:

$$\begin{cases} U = a_1 - a_3 y + a_4 x + a_5 xy - a_7 \dfrac{1}{2}(x^2+y^2) + a_8 \dfrac{y}{2} - a_{10} \dfrac{1}{2}(\alpha x^2 - y^2) \\ V = a_2 + a_3 x - a_5 \dfrac{1}{2}(x^2+y^2) + a_6 y + a_7 xy + a_8 \dfrac{x}{2} - a_9 \dfrac{1}{2}(\alpha y^2 - x^2) \end{cases} \quad (2)$$

α coefficient of condensation**,** in our case **α = 150**

Where U and V are the displacements in X and Y direction respectively, rigid body movement is represented by the terms associated with the constants a_1, a_2 and a_3 while the straining of the element is represented by the remaining constants. This element correctly represents the **R4SB2** and constant strain states.

The stiffness matrix is derived without using any tricks, which implies that it is obtained using exact and not reduced integration.

$$[K_e] = [A^{-1}]^T [K_0] [A^{-1}] \quad (3a)$$

$$[K_0] = \iint_S [Q]^T [D][Q] dx.dy \quad (3b)$$

With

$$[Q] = \begin{bmatrix} 0 & 0 & 0 & 1 & y & 0 & -x & 0 & 0 & -\alpha x \\ 0 & 0 & 0 & 0 & -y & 1 & x & 0 & -\alpha y & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & x & y \end{bmatrix}$$

And $[D] = \begin{bmatrix} D11 & D12 & 0 \\ D12 & D22 & 0 \\ 0 & 0 & D33 \end{bmatrix}$ the usual constitutive matrix

Where:

$$D11 = D22 = \frac{E}{(1-\nu^2)}; \quad D12 = \frac{\nu.E}{(1-\nu^2)}; \quad D33 = \frac{E}{2(1+\nu)};$$

For [A] and $[K_0]$ see the appendix.

3 PATCH - TESTS

3.1 Study of a simple element: dilatation of the element (R4SB2)

This element is subject to an imposed displacement (Fig.2).

Table 1: Dilatation of the element, results

		R4SB2		Theory of	Plane	Elasticity
Node	U	V		U		
2	0,01	$0.121.10^{-8}$		0,01	0	0
3	0,01	-0.001		0,01	-0.001	0
		Stresses			*Stresses*	
Node	σ_x	σ_y	τ_{xy}	σ_x	σ_y	τ_{xy}
1	10000	$0.843.10^{-3}$	$0.325.10^{-3}$	10000	0	0
4	10000	$0.809.10^{-3}$	$0.325.10^{-3}$	10000	0	0

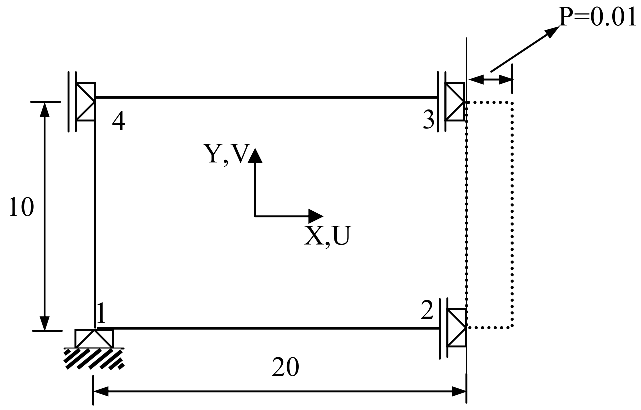


Figure 2: Dilatation of the element (R4SB2)
Data: E=210^7, ν = 0.2, t = 1

The results given by the element are perfectly analogous to the exact solution.

3.2 Dilatation of the element

3.2.1 Dilatation of the element in X direction

The objective of this test is to check the rigid body movement and the dilation of the element in X direction (Fig 3).

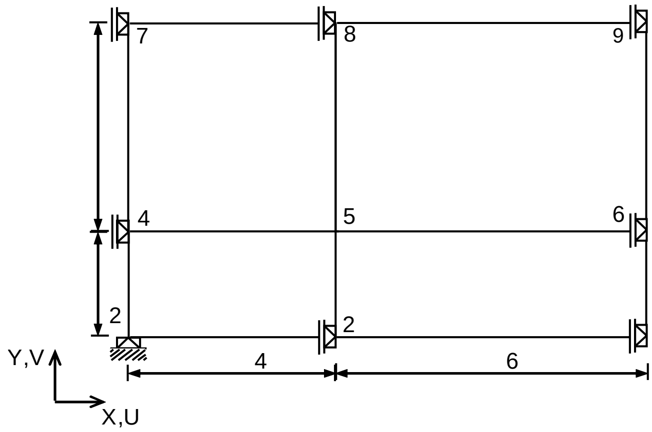


Figure 3: Patch Test Dilatation in X direction.
Data : E=1500, ν = 0.2, t =1

The displacement of the corresponding force (translation or dilation) is imposed on all nodes except the node n°5. By comparing the displacement provided by the element with that of the theory of plane elasticity, we can check easily if the element passes this patch test.

Two loading cases were considered:

A: 1st loading case: the rigid body movement, the displacement U = 10; for all nodes except the node n°5.

B: 2nd loading case: U=0.01 displacement of nodes 3, 6 and 9; displacement

U=0.004 for nodes 2 and 8 (U = 0 for nodes 1, 4 and 7)

Table 2: Patch-Test. Loading Case A

C.C			R4SB2	Theory of	Plane elasticity
A	node	U	V	U	V
	5	10	$-0.4008.10^{-6}$	10	0

Table 3: Patch-Test. Loading Case B

C.C			R4SB2	Theory of	Plane elasticity
B	node	U	V	U	V
	5	0.004	$-4.0.10^{-4}$	0.004	-4.010^{-4}

3.2.2 Dilatation of the element Y direction (Fig. 4)

A: 1st loading case: The rigid body movement V=10 displacement for all nodes except the node n°5.

B: 2nd loading case: V=0.01 displacement of nodes 7, 8 and 9; displacement V=0.00333333 for nodes 4 and 6 (V=0 for nodes 1, 2 and 3)

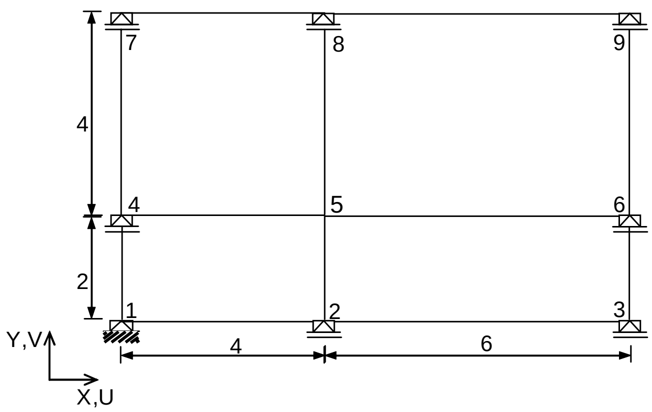


Figure 4: Patch-Test. Dilatation in Y direction
Data : E=1500, ν = 0.2, t =1

Table 4: Patch-Test. Loading Case A

C.C		R4SB2		Theory	Of plane elasticity
A	node	U	V	U	V
	5	$-0.7738.10^{-6}$	10	0	10

Table 5: Patch-Test. Loading Case B

C.C		R4SB2		Theory	Of plane elasticity
B	node	U	V	U	V
	5	-0.0013333	0.0033333	-0.0013333	0.0033333

In these tests, element (R4SB2) show the same properties as in test (3.1) like for dilations and the rigid body movement. The node n°5 gives exactly the solution of the plane elasticity theory. Hence; the element passes perfectly the patch-test.

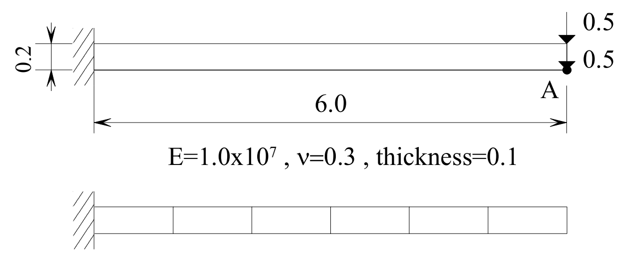
4 NUMERICAL EXAMPLES

The numerical results of several quadrilateral plane elements is used and compared with those obtained from the present **R4SB2** element and they are listed as follows:

- SBRIE: the strain based rectangular in-plane element [4].
- SBRIE2: the strain based rectangular in-plane element with an internal node [4].
- Q4: the standard four-node isoperimetric element.
- Most of the examples dealt with have been proposed at various stages in open literature to validate element performance. It will be seen that the SBRIE and the SBRIE2 versions show the same results for all cases.

4.1 An elongated thin cantilever beam subjected to end shear

An elongated thin cantilever beam subjected to end shear is a standard problem to test finite element accuracy. Young's modulus and Poisson's ratio are denoted by E and ν. These parameters and the mesh division are shown in Fig.5, while the results are presented in Table 6, it should be noted that the **R4SB2** element gives the most accurate results.



1x6 elements

Figure 5: An elongated thin cantilever beam subjected to end shear

Table 6: Normalized deflection at point A, of a thin cantilever beam under shear

	Normalised tip deflection
Mesh	1x 6
SBRIE	0.903
Q4	0.093
R4SB2	**1.000**
SBRIE2	0.682
Analyt.	1.000 (0.1081)

4.2 An elongated thin cantilever beam subjected to end pure bending

The tip deflection of an elongated thin cantilever beam under pure bending is compared using the present element **R4SB2**. The geometry, parameters and mesh discretesation of the beam are shown in Fig.6. Using four different mesh divisions, the normalised tip deflections of the **R4SB2** are computed and compared with those obtained by other elements in Table 7. A pertinent point to note is that exact solution can be obtained for the **R4SB2** element. The accuracy of the SBRIE and SBRIE2 is quite high.

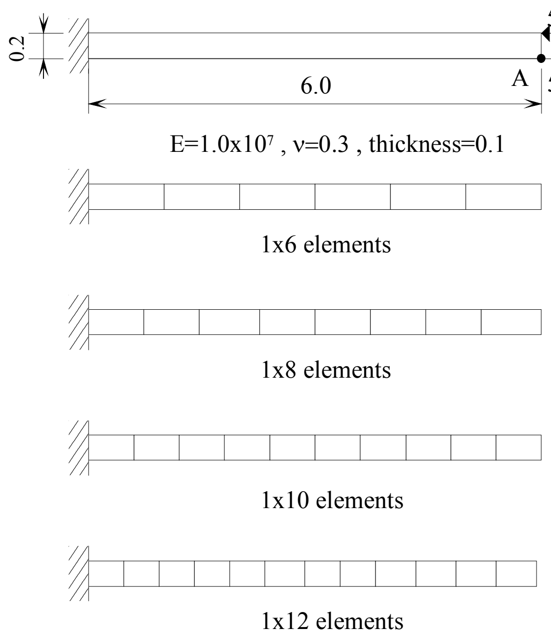


Figure 6: An elongated thin cantilever beam subjected to end pure bending

Table 7: Normalised deflection at point A, of a thin cantilever beam under pure bending

	Normalised tip deflection			
Mesh	1x 6	1x 8	1x 10	1x 12
SBRIE	0.91	0.91	0.91	0.91
Q4	0.093	0.153	0.219	0.285
R4SB2	**0.993**	**0.997**	**0.9978**	**0.998**
SBRIE2	0.678	0.713	0.75	0.75
Analyt.	1.000 (0.270)			

4.3 Simply supported beam loaded at mid-span

The deep simply supported beam whose details are given in fig.7 has been used in the finite element literature. It is also used here to test the performance of the present element R4SB2, and a comparison is made with the existing results given by the use of elements sited above.

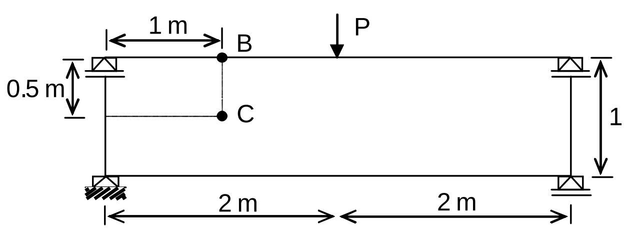
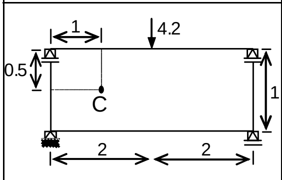


Figure 7: simply supported beam loaded at mid-span
Data : E=2x10^7 kn/m, ν = 0.2, t =0.5 m, P= 4.2 kn

Tables 8 and 9 show the results obtained for shearing stress at C and the bending stress at B respectively. These tables show that the **R4SB2** element gives better results than all the other elements. Even for the coarse mesh this element produces results which are acceptable within practical engineering accuracy.

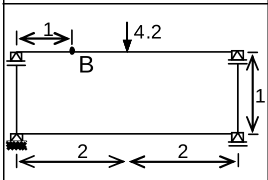
◘ Stresses σ_{xy} at point C:

Table 8: Simply supported beam, Shear stress at point C

E=2.10^7, ν=0.2 , t=0.5	Mesh	SBRIE 2	Q4	R4SB2
	12x4	6.0160	5.2582	6.0344
	12x6	6.0745	5.4544	6.0810
	16x8	6.1606	5.8177	6.1645
	20x10	6.2143	5.9922	6.2172
Exact Solution [2]	**σ_{xy}(c)=6.3**			

◘ Stresses σ_{xx} at point B:

Table 9: Simply supported beam, Normal Stress at point B

E=2.10^7 , ν=0.2 , t=0.5	Mesh	SBRIE 2	Q4	R4SB2
	12x4	28.7085	30.6545	28.7425
	12x6	24.6435	23.8300	24.8365
	16x8	24.2135	24.3630	24.2730
	20x10	24.5055	24.5860	24.5575
Exact Solution [2]	**σ_{xx}(B) = 25.2**			

5 CONCLUSION

From the previous examples, the developed element appears to be more accurate and versatile than the standard displacement based element. The robustness of the present element **R4SB2** *via* the patch test has also been shown. The numerical tests demonstrate that satisfactory finite element solutions can be obtained for beam bending without the use of large number of elements.

APPENDIX

Matrices [A] and [k_0] for element **R4SB2**: **with α =150**

$$[A]=\begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & a & 0 & 0 & \frac{-a^2}{2} & 0 & 0 & \frac{-\alpha a^2}{2}\\
0 & 1 & a & 0 & \frac{-a^2}{2} & 0 & 0 & \frac{a}{2} & \frac{a^2}{2} & 0\\
1 & 0 & -b & a & ab & 0 & \frac{-\left(a^2+b^2\right)}{2} & \frac{b}{2} & 0 & \frac{-\left(\alpha a^2-b^2\right)}{2}\\
0 & 1 & a & 0 & \frac{-\left(a^2+b^2\right)}{2} & b & ab & \frac{a}{2} & \frac{-\left(\alpha\, b^2-a^2\right)}{2} & 0\\
1 & 0 & -b & 0 & 0 & 0 & \frac{-b^2}{2} & \frac{b}{2} & 0 & \frac{b^2}{2}\\
0 & 1 & 0 & 0 & \frac{-b^2}{2} & b & 0 & 0 & \frac{-\alpha\, b^2}{2} & 0\\
1 & 0 & \frac{-b}{2} & \frac{a}{2} & \frac{ab}{4} & 0 & \frac{-\left(a^2+b^2\right)}{8} & \frac{b}{4} & 0 & \frac{-\left(\alpha a^2-b^2\right)}{8}\\
0 & 1 & \frac{a}{2} & 0 & \frac{-\left(a^2+b^2\right)}{8} & \frac{b}{2} & \frac{ab}{4} & \frac{a}{4} & \frac{-\left(\alpha\, b^2-a^2\right)}{8} & 0
\end{bmatrix}$$

$$
\left[K_0\right]=\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 & & & H_1 & H_2 & H_3 & H_4 & 0 & H_5 & H_6 \\
 & & & & H_7 & H_8 & H_9 & 0 & H_{10} & H_{11} \\
 & & & & & H_{12} & H_{13} & 0 & H_{14} & H_{15} \\
 & & & & & & H_{16} & 0 & H_{17} & H_{18} \\
 & & & & & & & H_{19} & H_{20} & H_{21} \\
 & & & & & & & & H_{22} & H_{23} \\
 & & & & & & & & & H_{24}
\end{bmatrix}
$$

$$H_1 = abD_{11}$$
$$H_2 = \frac{ab^2}{2}\left(D_{11}-D_{12}\right)$$
$$H_3 = abD_{12}$$
$$H_4 = \frac{ba^2}{2}\left(D_{12}-D_{11}\right)$$
$$H_5 = \frac{-\alpha ab^2 D_{12}}{2}$$
$$H_6 = \frac{-\alpha a^2 bD_{11}}{2}$$
$$H_7 = \frac{ab^3}{3}\left(D_{11}-2D_{12}+D_{22}\right)$$
$$H_8 = \frac{ab^2}{2}\left(D_{12}-D_{22}\right)$$
$$H_9 = \frac{a^2b^2}{4}\left(2D_{12}-D_{11}-D_{22}\right)$$
$$H_{10} = \frac{\alpha ab^3}{3}\left(D_{22}-D_{12}\right)$$
$$H_{11} = \frac{\alpha a^2b^2}{4}\left(D_{12}-D_{11}\right)$$
$$H_{12} = abD_{22}$$
$$H_{13} = \frac{ba^2}{2}\left(D_{22}-D_{12}\right)$$
$$H_{14} = \frac{-\alpha ab^2 D_{22}}{2}$$
$$H_{15} = \frac{-\alpha a^2 bD_{12}}{2}$$
$$H_{16} = \frac{ba^3}{3}\left(D_{22}-2D_{12}+D_{11}\right)$$
$$H_{17} = \frac{\alpha a^2b^2}{4}\left(D_{12}-D_{22}\right)$$
$$H_{18} = \frac{\alpha a^3 b}{3}\left(D_{11}-D_{12}\right)$$
$$H_{19} = abD_{33}$$
$$H_{20} = \frac{ba^2 D_{33}}{2}$$
$$H_{21} = \frac{ab^2 D_{33}}{2}$$
$$H_{22} = \frac{ab}{3}\left(\alpha^2 b^2 D_{22}+a^2 D_{33}\right)$$
$$H_{23} = \frac{a^2b^2}{4}\left(\alpha^2 D_{12}+D_{33}\right)$$
$$H_{24} = \frac{ab}{3}\left(\alpha^2 a^2 D_{11}+b^2 D_{33}\right)$$

Where:

$$D11 = D22 = \frac{E}{\left(1-\nu^2\right)} \quad D12 = \frac{\nu.E}{\left(1-\nu^2\right)} \quad D33 = \frac{E}{2\left(1+\nu\right)}$$

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