Applied Mathematics E-Notes, 15(2015), 173-186 © ISSN 1607-2510
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Common Fixed Point Theorems For Weakly Subsequentially Continuous Generalized Contractions With Applications*

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Received 03 January 2015

Abstract

The object of this paper is to prove some fixed point theorems for two pairs of self mappings satisfying generalized contractive condition by using the weak subsequential continuity with compatibility of type (E) in metric spaces. We illustrate two examples to support the main result. Some applications concerning the existence of a solution for systems of integral equations and systems of functional equations are given.

1 Introduction.

Jungck [7] introduced the notion of commuting mappings to prove a common fixed point theorem, two self mappings A and S of a metric space (X, d) are commuting if $ASx = SAx$ for all $x \in X$. Later, Sessa [19] defined that A and S are said to be weakly commuting if for all $x \in X$, $d(ASx, SAx) \leq d(Sx, Ax)$. Jungck [8] gave a generalization to the last notions as follows: A and S are said to be compatible if $\lim_{n\to\infty} d(ASx_n, SAx_n) = 0$, where $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$.

It is well known that "commuting" implies "weakly commuting" implies compatible. Jungck et al. [9] defined A and S are compatible mappings of type (A) if

$$\lim_{n\to\infty} d(ASx_n, S^2x_n) = 0 \text{ and } \lim_{n\to\infty} d(SAx_n, A^2x_n) = 0,$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$. Pathak et al. [16, 17, 18] respectively defined two self mappings S and T to be

- compatible of type (B) if

$$\lim_{n\to\infty} d(SAx_n, A^2x_n) \leq \frac{1}{2}\left[\lim_{n\to\infty} d(St, SAx_n) + \lim_{n\to\infty} d(St, S^2x_n)\right],$$

*Mathematics Subject Classifications: 47H10, 54H25.

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$$\lim_{n\to\infty} d(ASx_n, S^2x_n) \leq \frac{1}{2}\left[\lim_{n\to\infty} d(ASx_n, At) + \lim_{n\to\infty} d(At, A^2x_n)\right],$$

- compatible mappings of type (P) if $\lim_{n\to\infty} d(A^2x_n, S^2x_n) = 0$,
- compatible mappings of type (C) if

$$\lim_{n\to\infty} d(ASx_n, S^2x_n) \leq \frac{1}{3}[\lim_{n\to\infty} d(ASx_n, At)+\lim_{n\to\infty} d(At, S^2x_n)+\lim_{n\to\infty} d(At, A^2x_n)],$$

and

$$\lim_{n\to\infty} d(SAx_n, A^2x_n) \leq \frac{1}{3}[\lim_{n\to\infty} d(SAx_n, St)+\lim_{n\to\infty} d(St, S^2x_n)+\lim_{n\to\infty} d(St, S^2x_n)],$$

whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$, for some $t \in X$. Notice that, compatibility of type (A) implies compatibility of type (C), however compatibility (compatibility of type (A), compatibility of type (B) and compatibility of type (C)) are equivalent under the continuity of A and S.

Jungck and Rhoades [10] defined two self mappings A, S of space metric (X, d) to be weakly compatibility if they commute at their coincidence points; i.e., if $Au = Su$ for some $u \in X$, then $ASu = SAu$.

2 Preliminaries

Singh et al. [21, 22] introduced the notion of compatibility of type (E) as follows.

DEFINITION 1. Two self mappings A and S of a metric space (X, d) are said to be compatibility of type (E), if $\lim_{n\to\infty} S^2x_n = \lim_{n\to\infty} SAx_n = At$ and $\lim_{n\to\infty} A^2x_n = \lim_{n\to+\infty} ASx_n = St$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$, for some $t \in X$.

REMARK 1. If $At = St$, then compatibility of type (E) implies compatibility (compatibility of type (A), compatibility of type (B), compatibility of type (C), compatibility of type (P)), however the converse may be not true. Generally if $\{A, S\}$ is compatibility of type (E) implies compatibility of type (B).

DEFINITION 2. Two self mappings A and S of a metric space (X, d) are S-compatibility (A-compatible) of type (E), if $\lim_{n\to\infty} S^2x_n = \lim_{n\to\infty} SAx_n = At$ (respectively $\lim_{n\to\infty} A^2x_n = \lim_{n\to\infty} ASx_n = St$) for some $t \in X$.

Notice that if A and S are compatible of type (E), then they are A-compatible and S-compatible of type (E), but the converse is not true.

EXAMPLE 1. Consider $X = \mathbb{R}_+$ endowed with the Euclidian metric, we define A, S as follows

$$Ax = \begin{cases} x & \text{for } 0 \leq x \leq 2, \\ \frac{x+2}{2} & \text{for } x > 2, \end{cases} \quad \text{and} \quad Sx = \begin{cases} 4-x & \text{for } 0 \leq x \leq 2, \\ 1 & \text{for } x > 2. \end{cases}$$

Consider a sequence $\{x_n\}$ such that for each $n \geq 1$, $x_n = 2 + e^{-n}$. Clearly that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = 2.$$

Also, we have

$$\lim_{n\to\infty} ASx_n = \lim_{n\to\infty} A(2+e^{-n}) = 2 = A(2),$$

$$\lim_{n\to\infty} A^2x_n = \lim_{n\to\infty} A(2-e^{-n}) = 2 = S(2).$$

Then $\{A, S\}$ is S-compatible of type (E). On other hand we have

$$\lim_{n\to\infty} SAx_n = \lim_{n\to\infty} S(2+e^{-n}) = 1 \neq A(2),$$

which implies that $\{A, S\}$ is never compatible of type (E).

Pant [14] introduced the notion of reciprocal continuity for a pair of mappings in metric spaces.

DEFINITION 3. Two self mappings A and S of a metric space (X, d) are said to be reciprocally continuous, if $\lim_{n\to\infty} ASx_n = At$ and $\lim_{n\to\infty} SAx_n = St$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$, for some $t \in X$.

In 2009, Bouhadjera and Godet Thobie [5] introduced the concept of subsequential continuity as follows

DEFINITION 4. Two self maps A and S of a metric space (X, d) is called to be subsequentially continuous if there exists a sequence $\{x_n\}$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$, $\lim_{n\to\infty} ASx_n = At$ and $\lim_{n\to\infty} SAx_n) = St$.

If A and S are continuous or reciprocally continuous, then they are subsequentially continuous.

Motivated by Definition 4, we give the following definitions

DEFINITION 5. Let f and S to be two self mappings of a metric space (X, d), the pair $\{A, S\}$ is said to be weakly subsequentially continuous (wsc for short) if there exists a sequence $\{x_n\}$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$, for some $t \in X$ and $\lim_{n\to\infty} ASx_n = At$, or $\lim_{n\to\infty} SAx_n = St$.

DEFINITION 6. A pair $\{A, S\}$ of mappings is said to be S-subsequentially continuous if there exists a sequence $\{x_n\}$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$ and $\lim_{n\to\infty} SAx_n = St$.

DEFINITION 7. The pair $\{A, S\}$ is said to be A-subsequentially continuous if there exists a sequence $\{x_n\}$ such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$ for some $t \in X$ and $\lim_{n\to\infty} ASx_n = At$.

If the pair $\{A, S\}$ is A-subsequentially continuous (or S-subsequentially continuous), then it is wsc.

EXAMPLE 3. Let $X = [0, 2]$ and d is the euclidian metric, we define A, S as follows

$$Ax = \begin{cases} 1 & \text{for } 0 \le x \le 1, \\ \frac{x}{2} & \text{for } 1 < x \le 2, \end{cases} \quad \text{and} \quad Sx = \begin{cases} 2 - x, & \text{for } 0 \le x \le 1, \\ x + \frac{1}{2}, & \text{for } 1 < x \le 2. \end{cases}$$

Consider a sequence $\{x_n\}$ such that for each $n \ge 1$, $x_n = 1 - \frac{1}{n}$. Clearly that

$$\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = 1.$$

Also we have

$$\lim_{n\to\infty} ASx_n = \lim_{n\to\infty} A(1 + \frac{1}{n}) = \frac{1}{2} \text{ and } \lim_{n\to\infty} SAx_n = \lim_{n\to\infty} S(1) = 1.$$

Then $\{A, S\}$ is S-subsequentially continuous so it is wsc, but and since $A(1) \ne \frac{1}{2}$ it is never subsequentially continuous.

Let Ψ be a set of all non-decreasing continuous functions $\psi\mathbb{R}_+ \to \mathbb{R}_+$, such $\psi(x) = 0$ and let Φ be a set of all lower semicontinuous functions $\phi\mathbb{R}_+ \to \mathbb{R}_+$ such $\phi(x) = 0$ if and only if $x = 0$. The aim of this paper is to prove some common fixed point theorems for generalized weak contractions and generalized contractions in metric spaces by using the weak subsequential continuity and compatibility of type (E). Two examples are also furnished to illustrate our results with two useful applications in solvability of systems of integral equations and of functional equations.

3 Main Results

THEOREM 1. Let (X, d) be a metric space, $A, B, S, TX \to X$ are four self mappings such for all $x, y \in X$ we have

$$\psi(d(Sx, Ty)) \le \psi(M(x, y)) - \phi(M(x, y)), \tag{1}$$

where

$$M(x, y) = \max\left\{d(Ax, By), d(Ax, Sx), d(By, Ty), \frac{d(Ax, Ty) + d(By, Sx)}{2}\right\}$$

and $\psi \in \Psi, \phi \in \Phi$. If the two pairs $\{A, S\}$ and $\{B, T\}$ are weakly subsequentially continuous (wsc) and compatible of type (E), then A, B, S and T have a unique common fixed point in X.

PROOF. Since $\{A, S\}$ is wsc (suppose that it is A-subsequentially continuous), there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$ and $\lim_{n\to\infty} ASx_n = Az$, again A and S are compatible of type (E), so

$$\lim_{n\to\infty} A^2x_n = \lim_{n\to\infty} ASx_n = Sz \text{ and } \lim_{n\to\infty} S^2x_n = \lim_{n\to\infty} SAx_n = Az,$$

which implies that $Az = Sz$. Also, for B and T and since $\{B, T\}$ is wsc, there is a sequence $\{y_n\}$ in X such that

$$\lim_{n\to\infty} By_n = \lim_{n\to\infty} Ty_n = t \text{ and } \lim_{n\to\infty} BTy_n = Bt.$$

The pair $\{B, T\}$ is compatible of type (E), then so

$$\lim_{n\to\infty} B^2 y_n = \lim_{n\to\infty} BTy_n = Tt \text{ and } \lim_{n\to\infty} T^2 y_n = \lim_{n\to\infty} TBy_n = Bt,$$

which implies that $Bt = Tt$.

We prove $Az = Bt$. If not by using (1) we get

$$\begin{aligned}
\psi(d(Az, Bt)) &= \psi(d(Sz, Tt)) \le \psi(M(x, y)) - \phi(M(x, y)) \\
&\le \psi(\max(d(Az, Bt), 0, 0, d(Az, Bt), d(Az, Bt))) \\
&\quad -\phi(\max(d(Az, Bt), 0, 0, \frac{1}{2}(d(Az, Bt) + d(Az, Bt))) \\
&\le \psi(d(Az, Bt)) - \phi(d(Az, Bt)) < \psi(d(Az, Bt)),
\end{aligned}$$

which is a contradiction. So $Az = Sz = Bt = Tt$.

Next, we will show that $z = Az$. If not by using (1) we get

$$\psi(d(Sx_n, Tt)) \le \psi(M(x_n, t)) - \phi(M(x_n, t)).$$

Letting $n \to \infty$, we obtain

$$\begin{aligned}
\psi(z, Az) &= \psi(d(z, Tt)) \\
&\le \psi(\max(d(z, Az), 0, 0, d(z, Az)) - \phi(\max(d(z, Az), 0, 0, d(z, Az)) \\
&< \psi(d(z, Az)),
\end{aligned}$$

which is a contradiction, then $z = Az = Sz$.

Now we claim $z = t$, if not by using (1) we get

$$\psi(d(Sx_n, Ty_n)) \le \psi(M(x_n, y_n)) - \phi(M(x_n, y_n)).$$

Letting $n \to \infty$, we obtain

$$\begin{aligned}
\psi(d(z, t)) &\le \psi(\max(d(z, t), 0, 0, d(z, t))) - \phi \max(d(z, t), 0, 0, d(z, t)) \\
&< \psi(d(z, t)),
\end{aligned}$$

which is a contradiction, then $z = t$ so z is a common fixed point for A, B, S and T.

For the uniqueness, suppose there exists another fixed point w, by using (1) we get

$$\begin{aligned}
\psi(d(z, w)) &= \psi(d(Sz, Tw)) \le \psi(M(z, w)) - \phi(M(z, w)) \\
&\le \psi(d(z, w)) - \phi(d(z, w)) < \psi(d(z, w)),
\end{aligned}$$

which is a contradiction, then z is unique.

If $\psi(t) = t$, we obtain the following corollary

COROLLARY 1. Let (X,d) be a metric space and let A,B,S and T be self mappings such for all $x,y\in X$ we have

$$d(Sx,Ty)\leq M(x,y)-\phi(M(x,y)),$$

where $\phi\in\Phi$ and

$$M(x,y)=\max(d(Ax,By),d(Ax,Sx),d(By,Ty),\frac{1}{2}(d(Ax,Ty)+d(By,Sx))).$$

Assume that the two pairs $\{A,S\}$ and $\{B,T\}$ are wsc and compatible of type (E), then A,B,S and T have a unique common fixed point in X.

If $A=B$ and $S=T$, we get the following Corollary 2.

COROLLARY 2. For two self mappings A and S of a metric space (X,d) such that for all $x,y\in X$ we have

$$\begin{aligned}\psi(d(Sx,Sy)) \quad\leq\quad & \psi(\max(d(Ax,Ay),d(Ax,Sx),d(Ay,Sy),\frac{1}{2}(d(Ax,Sy)+d(Ay,Sx)))\\ & -\phi(\max(d(Ax,Ay),d(Ax,Sx),d((Ay,Sy),\frac{1}{2}(d(Ax,Sy)+d(Ay,Sx))).\end{aligned}$$

If $\{A,S\}$ is wsc and compatible of type (E), then A and S have a unique common fixed point.

THEOREM 2. Let A,B,S and T be self mappings of a metric space (X,d) into itself satisfying (1). Assume that

(i) $\{A,S\}$ is A-subsequentially continuous and A-compatible of type (E).

(ii) $\{B,T\}$ is$B$-subsequentially continuous and B-compatible of type (E).

Then A,B,S and T have a unique common fixed point in X.

Nextly, we will obtain the same results by using a generalized contractive condition.

THEOREM 3. Let A,B,S and T be mappings from a metric space (X,d) into itself such that for all $x,y\in X$,

$$d(Sx,Ty))\leq\varphi(N(x,y)),\tag{2}$$

where

$$N(x,y)=\max(d(Ax,By),d(Ax,Sx),d(By,Ty),d(Ax,Ty),d(By,Sx)),$$

$\varphi:\mathbb{R}^+\to\mathbb{R}^+$ is non decreasing and upper semicontinuous function such $\varphi(t)=0$ if and only if $t=0$ and for all $t>0$, $\varphi(t)<t$. If the two pairs $\{A,S\}$ and $\{B,T\}$ are wsc and compatible of type (E), then A,B,S and T have a common fixed point in X.

PROOF. As in proof of Theorem 1 z is a coincidence point for A and S, also the point t is a coincidence one for B and T, then we have

$$N(z,t) = \max(d(Az,Bt),0,0,d(Az,Tt),d(Bt,Sz)) = d(Az,Bt),$$

We claim $Az = Bt$. If not by using (2) we get

$$d(Az,Bt) = d(Sz,Tt) \leq \varphi(N(z,t)) < d(Az,Bt),$$

which is a contradiction, then $Az = Sz = Bt = Tt$.

Now, we will prove $z = Az$. If not by using (2) we get

$$(d(Sx_n,Tt) \leq \varphi(N(x_n,t)).$$

Letting $n \to \infty$, we obtain

$$d(z,Az) \leq \varphi(\max(d(z,Az),0,0,d(z,Az),d(z,Az)) < d(z,Az),$$

which is a contradiction, then $z = Az = Sz$.

We prove $z = t$. If not by using (2) we get

$$d(Sx_n,Ty_n)) \leq \varphi(N(x_n,y_n)).$$

Letting $n \to \infty$, we obtain

$$d(z,t) \leq \varphi(\max(d(z,t),0,0,d(z,t),d(z,t)))) < d(z,t),$$

which is a contradiction, then z is a common fixed point for A, B, S and T.

For the uniqueness, it is similar as in Theorem 1.

If $A = B$ and $S = T$, we obtain the following Corollary 3.

COROLLARY 3. Let (X,d) be a metric space and let A and S be two self mappings satisfying for all $x, y \in X$

$$d(Sx,Sy) \leq \varphi(\max\{d(Ax,Ay),d(Ax,Sx),d(Ay,Sy),d(Ax,Sy),d(Ay,Sx)\}.$$

If the pair $\{A,S\}$ is wsc and compatible of type (E), then A and S have a unique common fixed point.

COROLLARY 4. For the self mappings A, B, S and T of a metric space (X,d) such for all $x, y \in X$,

$$d(Sx,Ty) \leq \alpha \max(d(Ax,By),d(Ax,Sx),d(Ty,By),d(Ax,Ty),d(Ty,Ax))),$$

where $0 \leq \alpha < 1$. If the pair $\{A,S\}$ is A-subsequentially continuous and A-compatible of type (E), also $\{B,T\}$ is B-subsequentially continuous and B-compatible of type (E), then A, B, S and T have a unique fixed point in X.

Corollary 4 improves and generalizes corollary 1 in paper [22] to four self mappings of metric space.

THEOREM 4. Let A, B, S and T be mappings from a metric space (X,d) into itself satisfying (1) or (2). If one of the following assumptions is satisfied:

(i) the pair $\{A, S\}$ is S-subsequentially continuous and S-compatible of type (E), again $\{B, T\}$ is T-subsequentially continuous and T-compatible of type (E),

(ii) $\{A, S\}, \{B, T\}$ are compatible of type (E) and S-subsequentially continuous (B-subsequentially continuous resp),

(iii) $\{A, S\}, \{B, T\}$ are A-subsequentially continuous (T-subsequentially continuous resp) and A-compatible (Tcompatible resp) of type (E),

(iv) $\{A, S\}, \{B, T\}$ are subsequentially continuous and S or A-compatible (T or B-compatible resp) of type (E),

then A, B, S and T have a common fixed point in X.

EXAMPLE 3. Let $X = \mathbb{R}_+$ with the euclidian metric, define A, B, S and T by

$$Ax = Bx = \begin{cases} 2 - x & \text{for } 0 \leq x \leq 1, \\ 2x - 1 & \text{for } x > 1, \end{cases} \quad \text{and} \quad Sx = Tx = \begin{cases} \frac{x+1}{2} & \text{for } 0 \leq x \leq 1, \\ 0 & \text{for } x > 1. \end{cases}$$

Consider a sequence $\{x_n\}$ such for each $n \geq 1$ we have

$$x_n = 1 - \frac{1}{n}.$$

It is clair that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = 1$, also we have

$$\lim_{n\to\infty} ASx_n = \lim_{n\to\infty} A(1 - \frac{1}{2n}) = 1 = A(1),$$

$$\lim_{n\to\infty} A^2 x_n = \lim_{n\to\infty} A(1 + \frac{1}{n}) = 1 = S(1),$$

then $\{A, S\}$ is A-subsequentially continuous and A-compatible of type (E), but not subsequentially continuous since

$$\lim_{n\to\infty} SAx_n = \lim_{n\to\infty} S(1 + \frac{1}{2n}) = 0 \neq S(1).$$

We will apply corollary 1 with $\phi(t) = \frac{1}{5}t$, so for the inequality (1), we get the following.

1. For $x, y \in [0, 1]$, we have

$$d(Sx, Ty) = \frac{1}{2}|x - y| \leq \frac{4}{5}|x - y| = \frac{4}{5}d(Ax, Ay).$$

For $x \in [0, 1]$ and $y \in (1, \infty)$, we have

$$d(Sx, Ty) = \frac{1}{2}|x + 1| \leq \frac{2}{5}|3 - x| = \frac{4}{5}d(Ax, Sx).$$

2. For $x > 1$ and $y \in [0, 1]$, we have

$$d(Sx, Ty) = \frac{1}{2}\left|\frac{y+1}{2}\right| \leq \frac{2}{5}|3 - y| = \frac{4}{5}d(Ay, Sy).$$

3. For $x, y \in (1, \infty)$, it is obviously sice $d(Sx, Ty) = 0$.

Consequently all hypotheses of corollary 1 are satisfied, and 1 is the unique common fixed for A, B, S and T.

EXAMPLE 4. Let $X = [0, 1]$ and d is the Euclidian metric. We define A, B, S and T by

$$Ax = \begin{cases} x & \text{for } 0 \leq x \leq \frac{1}{2}, \\ \frac{3}{4} & \text{for } \frac{1}{2} < x \leq 1, \end{cases} \qquad Bx = \begin{cases} 1 - x & \text{for } 0 \leq x \leq \frac{1}{2}, \\ 1 & \text{for } \frac{1}{2} < x \leq 1, \end{cases}$$

$$Sx = \begin{cases} \frac{2x+1}{4} & \text{for } 0 \leq x \leq \frac{1}{2}, \\ \frac{1}{4} & \text{for } \frac{1}{2} < x \leq 1, \end{cases} \quad \text{and} \quad Tx = \begin{cases} \frac{1}{2} & \text{for } 0 \leq x \leq \frac{1}{2}, \\ \frac{x}{4} & \text{for } \frac{1}{2} < x \leq 1. \end{cases}$$

We consider a sequence $\{x_n\}$ defined for each $n \geq 2$ by

$$x_n = \frac{1}{2} - \frac{1}{n}.$$

Clearly that $\lim_{n\to\infty} Ax_n = \frac{1}{2}$ and $\lim_{n\to\infty} Sx_n = \frac{1}{2}$, also we have

$$\lim_{n\to\infty} ASx_n = \lim_{n\to\infty} A(\frac{1}{2} - \frac{1}{2n}) = A(\frac{1}{2}) = S(\frac{1}{2}) = \frac{1}{2}.$$

Then $\{A, S\}$ is A-subsequentially continuous and A-compatible of type (E). On the other hand consider a sequence defined by $y_n = \frac{1}{2} - e^{-n}$ for all $n > 1$. It is clear that

$$\lim_{n\to\infty} By_n = \lim_{n\to\infty} Tx_n = \frac{1}{2} \text{ and } \lim_{n\to\infty} BTy_n = B(\frac{1}{2}) = T(\frac{1}{2}) = \frac{1}{2}.$$

This yields that $\{B, T\}$ is B-subsequentially continuous and B-compatible of type (E).

For the contractive condition, we have the following.

1. For $x, y \in [0, \frac{1}{2}]$, we have

$$d(Sx, Ty) = \frac{1}{4}|2x - 1| \leq \frac{1}{3}|2x - 1| = \frac{2}{3}d(Ax, Ty)$$

for $x \in [0, \frac{1}{2}]$ and $y \in (\frac{1}{2}, 1]$, we have

$$d(Sx, Ty) = \frac{1}{4}|2x - y + 1| \leq \frac{3}{8} \leq \frac{1}{6}|4 - y| = \frac{2}{3}d(By, Ty).$$

2. For $x \in (\frac{1}{2}, 1]$ and $y \in [0, \frac{1}{2}]$, we have

$$d(Sx, Ty) = \frac{1}{4} \leq \frac{1}{3} = \frac{2}{3}d(Ax, Sx).$$

3. For $x, y \in (\frac{1}{2}, 1]$, we have

$$d(Sx, Ty) = \frac{1}{4}|1 - y| \leq \frac{1}{2} = \frac{2}{3}d(By, Sx).$$

Consequently all hypotheses of corollary 4 with $\alpha = \frac{2}{3}$ are satisfied. Therefore $\frac{1}{2}$ is the unique common fixed for A, B, S and T.

4 Applications

4.1 Existence of A Solution of Systems of Integral Equations

We will utilize Theorem 3 to assert the existence of a solution of the following system of integral equations

$$x(t) = f(t) + \int_a^b K_i(t, s, x(s))ds \text{ for } i = 1, 2 \text{ and } \ t, s \in [a, b], \tag{3}$$

where $a, b \in \mathbb{R}$ and $K_i I = [a, b] \times I \times \mathbb{R} \to \mathbb{R}$, where $\mathcal{C}(I, \mathbb{R})$ is the set of continuous functions from I to $\mathbb{R}$. It is clear that the space $\mathcal{C}(I, \mathbb{R})$ endowed with the metric

$$\forall u, v \in \mathcal{C}(I, \mathbb{R}), d(u, v) = \max |u - v|,$$

is a complete metric space

THEOREM 5. Assume that

(i) there exists a functions $\theta I \times I \to \mathbb{R}_+$ such for all $x, y \in \mathcal{C}(I, \mathbb{R})$, we have

$$|K_1(t, s, x(s) - K_2(t, s, y(s))| \leq \theta(t, s)\varphi(|x - y|),$$

where φ is non decreasing and non negative function such $\varphi(x) = 0$ if and only if $x = 0$.

(ii) $\sup_{t \in I} \int_a^b \theta(t, s)ds \leq 1$.

Then the system (3) have a unique common solution in $\mathcal{C}(I, \mathbb{R})$.

PROOF. Consider the two mappings

$$Sx(t) = f(t) + \int_a^b K_1(t, s, x(s))ds, Tx(t) = f(t) + \int_a^b K_2(t, s, x(s))ds, \ \ s, t \in [a, b],$$

where $S, T\mathcal{C}(I, \mathbb{R}) \to \mathcal{C}(I, \mathbb{R})$, the system (3) has a common solution if and only if the self-mappings S and T have a common fixed point in $\mathcal{C}(I, \mathbb{R})$, since f, K_i are continuous so S, T are continuous, then $\{id_{\mathcal{C}(I,\mathbb{R})}, S\}, \{id_{\mathcal{C}(I,\mathbb{R})}, T\}$ (id the identity in the space $\mathcal{C}(I, \mathbb{R})$) are subsequentially continuous and compatible of type (E). Further we have

$$\begin{aligned}
|Sx(t) - Ty(t)| &= \left| \int_a^b |K_1(t, s, x(s)) - K_2(t, s, y(s))ds \right| \leq \int_a^b \theta(t, s)\varphi(|x - y|)ds \\
&\leq \varphi(|x - y|) \int_a^b \theta(t, s)ds \\
&\leq \varphi(\max(d(x, y), d(x, Sx), d(y, Ty), d(x, Ty), d(y, Sx)).
\end{aligned}$$

Consequently all the hypotheses of theorem 3 (with $A = B = id_{\mathcal{C}(I,\mathbb{R})}$) hold. Then S and T have a unique common fixed point and so the system (3) have a unique common solution.

4.2 Existence of A Solution of a System of Functional Equations

Let X, Y two Banach space and $W \subset X$, $D \subset Y$ are state and decision space respectively. $B(W)$ is the set of all bounded functions defined on W, endowed with the following metric

$$\forall f, g \in B(W),\ \ d(f,g) = \sup_{x\in W} |f(x) - g(x)|\,.$$

Consider the following systems of functional equations arising in dynamic programming

$$\begin{cases} F(x) = \sup_{x\in W}\{u(x,t) + H(x,y,F(\tau(x,y)))\}, \\ G(x) = \sup_{x\in W}\{u(x,t) + K(x,y,G(\tau(x,y)))\}. \end{cases} \tag{4}$$

Putting

$$\begin{cases} Sf(t) = \sup_{x\in W}\{u(x,t) + H(x,y,f(\tau(x,y)))\}, \\ Tg(t) = \sup_{x\in W}\{u(x,y) + K(x,y,g(\tau(x,y)))\}. \end{cases}$$

THEOREM 6. Assume that the following hypotheses hold.

(a) H and K are bounded.

(b) For all $x, y \in W$ and $f, g \in B(W)$ there exists a non-decreasing and non negative function φ such $\varphi(x) = 0$ if and only if $x = 0$ and

$$\begin{aligned} & |H(x,y,f(\tau(x,y))) - K(x,y,g(\tau(x,y)))| \\ \leq\ & \varphi(\max(d(f,g), d(Sf,f), d(Tg,g), d(Sf,g), d(Tg,f))). \end{aligned}$$

(c) There exists two sequences $\{f_n\}$ in w and $f \in B(w)$ such that

$$\lim_{n\to\infty} \sup|Sf_n - f| = \lim_{n\to\infty} \sup|f_n - f| = 0 \text{ and } \lim_{n\to\infty} \sup|Sf_n - Sf| = 0.$$

(d) There exists a sequence $\{g_n\}$ in W and $g \in B(W)$ such that

$$\lim_{n\to\infty} \sup|Tg_n - g| = \lim_{n\to\infty} \sup|g_n - g| = 0 \text{ and } \lim_{n\to\infty} \sup|Tg_n - Tg| = 0.$$

Then the system (4) has a unique bounded solution.

PROOF. The system (4) have a unique solution if and only if the self mappings S, T have a common fixed point.

Firstly the condition (a) implies that S and T are two self mappings from the metric space $(B(W), d)$ into itself. For all $f, g \in B(W)$ and $\varepsilon > 0$, there exists $y, z \in W$ such that

$$Sf < u(x,y) + H(x,y,f(\tau(x,y))) + \varepsilon, \tag{5}$$

$$Tg < u(x,z) + K(x,z,g(\tau(x,z))) + \varepsilon. \tag{6}$$

Since

$$Sf \geq u(x,z) + H(x,z,f(\tau(x,z))) \tag{7}$$

and

$$Tg \geq u(x,y) + K(x,y,g(\tau(x,y))), \tag{8}$$

we see that by (5) and (8), we get

$$\begin{aligned} Sf - Tg &\leq H(x,y,f(\tau(x,y))) - K(x,y,g(\tau(x,y))) + \varepsilon \\ &\leq \varphi(\max(d(f,g), d(Sf,f), d(Tg,g), d(Sf,g), d(Tg,f))) + \varepsilon. \end{aligned} \tag{9}$$

On the other hand and from (6) and (7) we get

$$\begin{aligned} Sf - Tg &> H(x,y,f(\tau(x,y))) - K(x,y,g(\tau(x,y))) - \varepsilon \\ &\geq -\varphi(\max(d(f,g), d(Sf,f), d(Tg,g), d(Sf,g), d(Tg,f))) - \varepsilon. \end{aligned} \tag{10}$$

Consequently, (9) and (10) implies that

$$\begin{aligned} d(Sf,Tg) &= \sup |Sf - Tg| \leq |H(x,y,f(\tau(x,y))) - K(x,y,g(\tau(x,y)))| + \varepsilon \\ &\leq \varphi(\max(d(f,g), d(Sf,f), d(Tg,g), d(Sf,g), d(Tg,f))) + \varepsilon \end{aligned}$$

since the last inequality is true for any arbitrary $\varepsilon > 0$, we can write

$$d(Sf,Tg) \leq \varphi(\max(d(f,g), d(Sf,f), d(Tg,g), d(Sf,g), d(Tg,f))), \tag{11}$$

the condition (c) implies that $\{id_{B(W)}, S\}$ is S-subsequentially continuous and compatible of type (E), as well as the pair $\{id_{B(W)}, T\}$ is T-subsequentially continuous and compatible of type (E) from (d). Consequently all the conditions of theorem 3 (with $A = B = id_{B(W)}$) are satisfied, S, T have a unique common fixed point in $B(W)$ and this point is a solution of the system of functional equations (4).

Acknowledgment. The author thanks the editor and the referees for their remarks and suggestions to improve this paper.

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