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Variable Hardy Spaces

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DÉDICACE

To my father

To my mother

To my dear brother and sisters

To my dear friends

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Introduction

The variable Lebesgue spaces, originally introduced by Orlicz [19] via replacing p with an exponent function $p(\cdot) : \mathbb{R}^n \to (0, \infty)$, are a generalization of the classical Lebesgue spaces. Many properties in classical Lebesgue spaces have been generalized to the variable Lebesgue spaces. In 1931's, Kováčik and Rákosník [18] proved some elementary properties for this kind of spaces. These spaces have been extensively studied by many researchers due to their wide use in different fields such as harmonic analysis and partial differential equations, see for example [3, 10, 17, 7]. The real variable theory of Hardy spaces $H^p(\mathbb{R}^n)$ is another generalisation of the classical Lebesgue spaces. It was introduced by Stein and Weiss in [8]. This theory was regularly developed by Fefferman and Stein in [3] for the case $p \in (0, 1]$. The Hardy spaces are a good replacement of the classical Lebesgue spaces especially in the study of the boundedness of various operators and the Reisz transforms are a typical example, they are bounded on the Hardy spaces $H^p(\mathbb{R}^n)$, however, they miss this property in the $L^p(\mathbb{R}^n)$ spaces. Additionally, the different characterization of Hardy spaces allow them to be more important and useful in many problems and play a considerable role in various fields of analysis such as harmonic analysis and partial differential equations (see for example [9, 16, 3] and their references). Nakai and Sawano [10] introduced and studied the variable Hardy space $H^{p(\cdot)}(\mathbb{R}^n)$ and investigated their dual spaces.In this dissertation, we give the study the variable Hardy space $H^{p(\cdot)}(\mathbb{R}^n)$ and it maximal function characterization.

We end this introduction by describing the layout of this thesis.

In **Chaptre 01**, we introduce some preliminaries and recall the definition of the variable Lebesgue space and its well-known properties.

In **Chaptre 02,** we study the variable Hardy space and its different characterization. $H^{p(\cdot)}(\mathbb{R}^n)$.

In **Chaptre 03,** we introduce the boundedness of the Singular operators on $H^{p(\cdot)}(\mathbb{R}^n)$.

Chapitre 1

Preliminaries

In this Chapter, we present some preliminaries and recall the definition of the variable Lebesgue space $L^{p(\cdot)}(\mathbb{R}^n)$ and some of its known and elementary properties.

1.1 Variable exponent and modular

Given an open set $\Omega \subset \mathbb{R}^n$. We put

$$\mathcal{P}_0(\Omega) := \{p \text{ mesurable} : p(.) : \Omega \to [c, \infty[\text{ for some } c > 0\} .$$

The elements of $\mathcal{P}_0(\Omega)$ are called exponent functions or simply exponents. In order to distinguish between variable and constant exponents, we will always denote exponent functions by $p(.)$.

Next, we give an example of exponent functions presented in see[4].

Example 1.1.1 *[2] Some examples of exponent functions on : $\Omega = \mathbb{R}$ include $p(x) = p$ for some constant p, $1 \leq p \leq \infty$, or $p(x) = 2 + \sin(x)$. Exponent functions can be unbounded : for instance, if $\Omega = (1, \infty)$, let $p(x) = x$ and if $\Omega = (0, 1)$, let $p(x) = 1/x$.*

Notation 1.1.1 *[11]* *We denote by*

$$\mathcal{P}(\Omega) := \{p \text{ mesurable} : p(.) : \Omega \subset \mathbb{R}^n \to [1, \infty[\}.$$

Given $p \in \mathcal{P}_0$ *and a set* $E \subseteq \Omega$, *let*

$$p_-(E) = ess \inf_{x \in E} p(x) \text{ and } p_+(E) = ess \sup_{x \in E} p(x).$$

If the domain $E = \Omega = \mathbb{R}^n$ *we will simply write*

$$p_- = p_-(\Omega) \text{ and } p_+ = p_+(\Omega).$$

We define three canonical subsets of Ω

$$\Omega_\infty^{p(.)} = \{x \in \Omega : p(x) = \infty\},$$

$$\Omega_1^{p(.)} = \{x \in \Omega : p(x) = 1\},$$

$$\Omega_*^{p(.)} = \{x \in \Omega : 1 < p(x) < \infty\}.$$

$$0 < p_- \leq p_+ < \infty$$

Remark 1.1.1 *1.* *Given* $p(.) \in \mathcal{P}(\Omega)$, *define the conjugate exponent function* $p^{'}(.)$ *by the formula*

$$\frac{1}{p(x)} + \frac{1}{p^{'}(x)} = 1, \qquad x \in \Omega,$$

with the convention that $\frac{1}{\infty} = 0$.

2. $(p^{'}(.))_+ = (p_-)^{'}, \quad (p^{'}(.))_- = (p_+)^{'}$.

Modular and properties of the modular

Given $p \in \mathcal{P}(\Omega)$, we define the variable Lebesgue space $L^{p(\cdot)}(\Omega)$ as the set of all measurable functions f such that

$$\int_{\Omega} |f(x)|^{p(x)}\, dx < \infty.$$

There are problems with this approach, the most obvious being that it does not work when Ω_∞ has positive measure. To remedy them, we begin with the following definition.

Definition 1.1.1 *[2]* *Given Ω, $p(.) \in \mathcal{P}(\Omega)$ and a Lebesgue measurable function f, define the modular functional (or simply the modular) associated with $p(.)$ by*

$$\rho_{p(.),\Omega}(f) = \int_{\Omega\backslash\Omega_\infty} |f(x)|^{p(x)}\, dx + \|f\|_{L^\infty(\Omega_\infty)}.$$

- *If f is unbounded on Ω_∞ or if $f(.)^{p(.)} \notin L^1(\Omega\backslash\Omega_\infty)$ we define $\rho_{p(.),\Omega}(f) = +\infty$. When $|\Omega_\infty| = 0$,*
- *in particular when $p_+ < \infty$, we let $\|f\|_{L^\infty(\Omega_\infty)} = 0$, when $|\Omega\backslash\Omega_\infty| = 0$, then $\rho_{p(.),\Omega}(f) = \|f\|_{L^\infty(\Omega_\infty)}$.*

  If there is no ambiguity, we will write simply $\rho(f)$.

Remark 1.1.2 *There are two other definitions of the modular in the literature. One immediate alternative is to define it as*

$$\rho(f) = \max\left(\int_{\Omega\backslash\Omega_\infty} |f(x)|^{p(x)}\, dx, \|f\|_{L^\infty(\Omega_\infty)}\right).$$

The modular has the following properties

Proposition 1.1.1 *[2]* *Given Ω, and $p(.) \in \mathcal{P}(\Omega)$, then :*

1. *for all f, $\rho(f) \geq 0$ and $\rho(|f|) = \rho(f)$.*
2. *$\rho(f) = 0$ if and only if $f(x) = 0$ for almost every $x \in \Omega$.*
3. *if $\rho(f) < \infty$, then $f(x) < \infty$ for almost every $x \in \Omega$.*
4. *ρ is convex : given $\alpha, \beta \geq 0$, $\alpha + \beta = 1$,*

$$\rho(\alpha f + \beta g) \leq \alpha \rho(f) + \beta \rho(g).$$

5. *if $|f(x)| \geq |g(x)|$ a.e., then $\rho(f) \geq \rho(g)$.*
6. *if for some $\Lambda > 0, \rho(f \ /\Lambda) < \infty$, then the function $\lambda \to \rho(f \ /\Lambda)$ is continuous and decreasing on $[\Lambda, \infty)$. Further, $\rho(f \ /\Lambda) \to 0$ as $\lambda \to \infty$.*

   An immediate consequence of the convexity of ρ is that if $\alpha > 1$, then $\alpha\rho(f) \leq \rho(\alpha f)$, and if $0 < \alpha < 1$, then $\rho(\alpha f) \leq \alpha \rho(f)$. We will often invoke this property by referring to the convexity of the modular.

Definition 1.1.2 *[11]* *Given Ω and $p(.) \in \mathcal{P}\ (\Omega)$, the variable Lebesgue space $L^{p(\cdot)}(\Omega)$ to be the set of all measurable functions f such that $\rho_{p(.)}(f/\lambda) < \infty$ for some $\lambda > 0$.*

$$L^{p(\cdot)}(\Omega) = \left\{ f \ \textit{measurable} :\exists\ \lambda > 0 : \rho_{p(.)}(f/\lambda) = \int_{\Omega} \left| \frac{f(x)}{\lambda} \right|^{p(.)} \leq 1 \right\},$$

equipped with the following quasi-norm

$$\|f\|_{L^{p(\cdot)}(\Omega)} := \inf \left\{ \lambda > 0 : \rho_{p(.)}(f/\lambda) \leq 1 \right\}.$$

If the set on the right-hand side is empty we define $\|f\|_{L^{p(\cdot)}(\Omega)} = \infty$. If $\Omega = \mathbb{R}^n$, we will often write $\|f\|_{p(.)}$ instead of $\|f\|_{L^{p(\cdot)}(\mathbb{R}^n)}$.

Definition 1.1.3 *[11]* *Given Ω and $p(.) \in \mathcal{P}\ (\Omega)$, Define $L^{p(.)}_{loc}(\Omega)$ dy*

$$L^{p(.)}_{loc}(\Omega) := \left\{ f \ \textit{measurable} : f \in L^{p(\cdot)}(K) \ \textit{for every compact set}\ K \subset \Omega \right\}.$$

Proposition 1.1.2 *[2]* *Given Ω and $p(.) \in \mathcal{P}(\Omega)$, if $p_+ < \infty$, then $f \in L^{p(.)}(\Omega)$ if and only if*

$$\rho(f) = \int_\Omega |f(x)|^{p(.)}\, dx < \infty.$$

Proof. Since $p_+ < \infty$, we can drop the L^∞ term in the modular. Clearly, if $\rho(f) < \infty$ then $f \in L^{p(.)}$. Conversely, by **Property (5)** in Proposition (1.1.1) , we have that $\rho(f/\lambda) < \infty$ for some $\lambda > 1$. But then

$$\rho(f) = \int_\Omega \left(\frac{|f(x)|\,\lambda}{\lambda}\right)^{p(.)} dx \le \lambda^{p_+(\Omega)} \rho(f/\lambda) < \infty.$$

■

Theorem 1.1.1 *Given Ω and $p(.) \in \mathcal{P}(\Omega)$, $L^{p(.)}(\Omega)$ is a vector space.*

Theorem 1.1.2 *Given Ω and $p(.) \in \mathcal{P}(\Omega)$, the function $\|.\|_{L^{p(.)}(\Omega)}$ defines a norm on $L^{p(.)}(\Omega)$.*

Remark 1.1.3 *[2]* *Let $p(.) \in \mathcal{P}(\Omega)$, $p_- \in [1, \infty)$, then $L^{p(.)}(\Omega)$ is a Banach space.*

Proposition 1.1.3 *[1]Let $p \in \mathcal{P}_0(\mathbb{R}^n)$ with $p_+ < \infty$ and $s > 0$ be such that $1/p_- \le s < \infty$. Then*

$$\||f|^s\|_{p(.)} = \|f\|^s_{sp(.)}\,.$$

Proof. This follows at once from the definition of the norm : since $|\Omega_\infty| = 0$, if we let $\mu = \lambda^{1/s}$

$$\begin{aligned}
\||f\,|^s\|_{p(.)} &= \inf\left\{\lambda > 0 \ : \int_\Omega \left(\frac{|f(x)|^s}{\lambda}\right)^{p(x)} dx \ \le 1\right\} \\
&= \inf\left\{\mu^s > 0 \ : \int_\Omega \left(\frac{|f(x)|}{\mu}\right)^{s\,p(x)} dx \le 1\right\} = \|f\|^s_{sp(.)}\,.
\end{aligned}$$

■

Example 1.1.2 *Let $\Omega = (1, \infty)$ and $p(x) = x$. Then there exists a function $f \in L^{p(x)}$ such that $\rho(f \Big/ \|f\|_{p(.)}) < 1$.*

Proof. We will construct a function f such that $\rho(f) < 1$ but for any $\lambda < 1$, $\rho(f/\lambda) = \infty$. Then $\|f\|_{p(.)} = 1$ and $\rho(f \Big/ \|f\|_{p(.)}) = \rho(f) < 1$. For $k \geq 2$ let $I_K = [k, k + k^{-2}]$ and define the function f by

$$f(x) = \sum_{k=2}^{\infty} \chi_{I_K(x)},$$

then

$$\rho(f) = \sum_{k=2}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6} - 1 < 1.$$

On the other hand, for any $\lambda < 1$

$$\rho(f/\lambda) = \sum_{k=2}^{\infty} \int_{k}^{k+k^{-2}} \lambda^{-x} dx \geq \sum_{k=2}^{\infty} \frac{1}{\lambda^k k^2} = \infty.$$

This example can be adapted to any space such that $p_+(\Omega \backslash \Omega_\infty) = \infty$; otherwise, equality must hold. ■

Lemma 1.1.1 *$p(.) \in \mathcal{P}_0(\Omega)$ and $p_- \leq 1$, $f, g \in L^{p(.)}$*

$$\|f + g\|_{L^{p(.)}} \leq \|f\|_{L^{p(.)}} + \|g\|_{L^{p(.)}} .$$

Proof. [1] Since $p/p_- \in \mathcal{P}$, by Lemma(1.1.3) , convexity and Minkowski's inequality, for the variable Lebesgue spaces,

$$\begin{aligned} \|f + g\|_{p(.)}^{p_-} &= \||f + g|^{p_-}\|_{p(.)/p_-}^{p_-} \leq \left\||f|^{p_-} + |g|^{p_-}\right\|_{p(.)/p_-}^{p_-} \\ &\leq \||f|^{p_-}\|_{p(.)/p_-}^{p_-} + \left\||g|^{p_-}\right\|_{p(.)/p_-}^{p_-} = \|f\|_{p(.)}^{p_-} \|g\|_{p(.)}^{p_-} . \end{aligned}$$

■

1.1.1 Lebesgue spaces with variable exponents

Let $\Omega \subset \mathbb{R}^n$. We recall that $L^p(\Omega)$ is the set of all measurable functions for which the norm

$$\|f\|_{L^p} = \left(\int_{\Omega} |f(x)|^p \, dx\right)^{\frac{1}{p}} < \infty$$

Here and below we consider Lebesgue spaces with variable exponent, which is the

heart of this paper. We are placing ourselves in the setting where the value of p above varies according to the position of $x \in \Omega$. The simplest case is as follows : suppose we are given a measurable partition $\Omega = \Omega_1 \cup \Omega_2$ of Ω. Consider the norm $\|f\|_z$ given by

$$\|f\|_z = \left(\int_{\Omega_1} |f(x)|^{p_1} \, dx\right)^{\frac{1}{p_1}} + \left(\int_{\Omega_2} |f(x)|^{p_2} \, dx\right)^{\frac{1}{p_2}},$$

so,if we set $p(.) = p_1\chi_{\Omega_1} + p_2\chi_{\Omega_2}$,

then we are led to the space $L^{p(.)}(\Omega)$. What happens if the measurable function $p(.)$ assumes infinitely many different values ? The answer can be given by way of modulars.

Lebesgue spaces with variable exponents have been studied intensively for these two

decades right after some basic properties were established by Kováčik and Rákosník[18].

Theorem 1.1.3 ***(Minkowski's inequality)***. *Let $1 \leq p \leq \infty$. Then, we have*

$$\|f + g\|_{L^p(\Omega)} \leq \|f\|_{L^p(\Omega)} + \|g\|_{L^p(\Omega)}.$$

for all $f, g \in L^p(\Omega)$.

1.1.2 Elementary properties

Given a variable exponent $p(.)$, we define the following :

a. $p_- = \sup \{a : \ p(x) \geq a, \ x \in \Omega\}$,

b. $p_+ = \inf \{a : \ p(x) \leq a, \ x \in \Omega\}$,

c. $\Omega_0 = p^{-1}((1,\infty)) = \Omega/\Omega_1 \cup \Omega_\infty$,

d. $\Omega_1 = p^{-1}(1)$,

e. $\Omega_\infty = p^{-1}(\infty)$,

f. the conjugate exponent $p^{'}(.)$

$$p^{'}(x) := \begin{cases} \infty & (x \in \Omega_1) \\ \frac{p(x)}{p(x)-1} & (x \in \Omega_0) \\ 1 & (x \in \Omega_\infty) \end{cases},$$

namely $\frac{1}{p(x)} + \frac{1}{p^{'}(x)} = 1$. see[5]

Remark 1.1.4 *[14]*

$$L^{p(.)}(\Omega) = L^{p_0}(\Omega), \quad \textit{and} \quad \|f\|_{L^{p(.)}(\Omega)} = \|f\|_{L^{p_0}(\Omega)},$$

if $p(.)$ *equals to a constant* $p_0 \in [1,\infty]$. *We will prove that* $\rho_p(f)$ *is a modular and that* $\|.\|_{L^{p(.)}(\Omega)}$ *is a norm in the above.*

Lemma 1.1.2 *[14]If* $x, y \in \mathbb{R}^n$ *and* $0 \leq t \leq 1 \leq r < \infty$, *then the following inequality holds :*

$$|tx + (1-t)y|^r \leq t\,|x|^r + (1-t)\,|y|^r,$$

Theorem 1.1.4 *Let* $p(.) : \Omega \to [1,\infty]$ *be a variable exponent. Then* $\rho_p(.)$ *is a modular. If* $p(.)$ *additionally satisfies* $\widetilde{p_+} := ess\ \sup_{x\in\Omega\setminus\Omega_\infty} p(x) < \infty$, *then* $\rho_p(.)$ *is a continuous modular.*

Lemma 1.1.3 *[14]Assume that* $f \in L^0(\Omega)$ *satisfies* $0 < \|f\|_{L^{p(.)}(\Omega)} < \infty$.

1. $\rho_p(\frac{f}{\|f\|_{L^{p(.)}(\Omega)}}) \leq 1$,

2. if , $\widetilde{p_+} = ess\ \sup_{x\in\Omega\setminus\Omega_\infty} p(x) < \infty$, *then* $\rho_p(\frac{f}{\|f\|_{L^{p(.)}(\Omega)}}) = 1$ *holds.*

Lemma 1.1.4 *Let* $p(.) : \Omega \to [1,\infty]$*be a variable exponent and* $f \in L^0(\Omega)$

1. *if* $\|f\|_{L^{p(\cdot)}(\Omega)} \leq 1$ *then we have* $\rho_p(f) \leq \|f\|_{L^{p(\cdot)}(\Omega)} \leq 1$.

2. *Conversely if* $\rho_p(f) \leq 1$ *, then* $\|f\|_{L^{p(\cdot)}(\Omega)} \leq 1$ *holds.*

3. *Assume in addition that* $1 \leq \widetilde{p_+} := ess\ \sup_{x\in\Omega\setminus\Omega_\infty} p(x) < \infty$ *and that* $\rho_p(f) \leq 1$ *holds.* *Then* $\|f\|_{L^{p(\cdot)}(\Omega)} \leq \rho_p(f)^{\frac{1}{\widetilde{p_+}}} \leq 1$.

Remark 1.1.5 *let*

$$\begin{aligned}
\rho_p^{(0)}(f) &= \int_\Omega |f(x)|^{p(x)}\, dx \quad \text{and} \\
\|f(x)\|_{L^{p(\cdot)}(\Omega)}^{(0)} &= \inf\left\{\lambda > 0 : \rho_p^{(0)}(f\ /\lambda) \leq 1\right\} \\
&= \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\lambda}\right)^{p(x)} dx \leq 1\right\},
\end{aligned}$$

where it is understood that

$$r^\infty = \begin{cases} 0 & 0 \leq r \leq 1 \\ \infty & r > 1 \end{cases},$$

then $\rho_p^{(0)}$ *is a semimodular and* $\|f(x)\|_{L^{p(\cdot)}(\Omega)}^{(0)}$ *is a norm. If* $p_+ < \infty$*, then* $\rho_p^{(0)}$ *clearly coincides with* ρ_p *and it is continuous.*

1.2 Hardy space

1.2.1 Definition and basic properties

Definition 1.2.1 *For* $0 < p < \infty$ *and* $0 < r < 1$*, for a function* f *defined on* $\mathbb{D}$ $(\mathbb{D} = \{x, y \in \mathbb{R},\ z = x + iy \in \mathbb{C} : |z| < 1\})$ *we set*

$$M_p(f, r) = \left(\frac{1}{2\pi}\int_0^{2\pi} \left|f(re^{i\theta})\right|^p d\theta\right)^{1/p},$$

We define the Hardy space $H^p = H^p(\mathbb{D})$ *as*

$$H^p = \left\{ f \in H(\mathbb{D}) : \sup_{0<r<1} M_p(f,r) < \infty \right\},$$

and for $f \in H^p$ *we set*

$$\|f\|_{H^p} = \sup_{0<r<1} M_p(f,r),$$

Furthermore, we define H^∞ *as the space of holomorphic functions that are bounded on the unit disc, endowed with the sup-norm.*

For the special case $p = \infty$, *we require that*

$$\|f\|_\infty = M_\infty(f,r) = \sup_{0\leq\theta\leq 2\pi} \left| f(re^{i\theta}) \right| < \infty,$$

and we write $f \in H^\infty$.

We mention in passing that, also when $0 < p < 1$ *we set*

$$\|f\|_{L^p} = \left(\frac{1}{2\pi} \int_0^{2\pi} \left| f(e^{i\theta}) \right|^p d\theta \right)^{1/p},$$

and, with an abuse of language, we call it the L^p*-norm. However, setting* $d(f,g) = \|f-g\|_{L^p}^p$, L^p *becomes a complete metric space. Notice that* $\|f+g\|_{L^p} \leq 2^{(1-p)/p}(\|f\|_{L^p} + \|g\|_{L^p})$.

Corollary 1.2.1

1. *For* $0 < p < \infty$ *we have*

$$H^p = \left\{ f \in H(\mathbb{D}) : \lim_{r\to 1-} M_p(f,r) < \infty \right\},$$

2. *If* $f \in H^p$ *for* $0 < p \leq \infty$ *we notice* $\|f\|_{H^p} = \lim_{r\to 1-} M_p(f,r)$,

Remark 1.2.1 *Aside from the case* $p = \infty$, *also the space* H^2 *can be described at once. For, if* f *is holomorphic on* $\mathbb{D}$, *then it admits power series expansion* $f(z) = \sum_{n=0}^{\infty} a_n z^n, z = re^{i\theta}$.

Using the uniform convergence on compact subsets of $\mathbb{D}$ *we have*

$$\begin{aligned} M_2(f,r)^2 &= \frac{1}{2\pi}\int_0^{2\pi} \left|f(re^{it})\right|^2 dt \\ &= \frac{1}{2\pi}\int_0^{2\pi} \sum_{n,m=0}^{+\infty} a_n\overline{a_m}r^{n+m}e^{it(n-m)}dt \\ &= \sum_{n=0}^{+\infty} r^{2n}\left|a_n\right|^2, \end{aligned}$$

Therefore,

$$\sup_{0<r<1} M_2(f,r) = \left(\sum_{n=0}^{+\infty}\left|a_n\right|^2\right)^{1/2},$$

that is $f \in H^2$ *if and only if* $\sum_{n=0}^{+\infty}\left|a_n\right|^2$ *is finite.In particular, it follows that* H^2 *is a Hilbert space. Observe that, in particular, a function* f *in* H^2 *can be extended to the boundary* $\partial\mathbb{D} = \mathbb{T} = \left\{\zeta \in C : \zeta = e^{i\theta}\right\}$, *(unit circle), having as "boundary values" the function* $\tilde{f} \in L^2(\partial\mathbb{D})$ *given by*

$$\tilde{f}(e^{it}) = \sum_{n=0}^{+\infty} a_n e^{int} = \lim_{r\to 1-} f(re^{it}),$$

Moreover, by ***Lemma (1.2.1)*** $\mathcal{P}(\tilde{f})(r.) = (P_r * \tilde{f}) = \sum_{n=0}^{\infty} a_n r^n e^{in(.)} = f(r.)$. *We can call* $\tilde{f}$ *"boundary values" since* $f(r.) = P_r * \tilde{f} \to \tilde{f}$ *in* $L^2(\mathbb{T})$ *as* $r \to 1-$

Remark 1.2.2 *Given* f *holomorphic in* $\mathbb{D}$, *the function* $M_p(f,r)$ *is increasing in* r, $0 < r < 1$. *This statement holds true for the full range* $0 < p \le \infty$, *but its proof is elementary only in the case* $p \ge 1$ *and we will restrict to this case.*

Lemma 1.2.1 *Let* $g \in L^1(\partial\mathbb{D})$. *Then for every* $0 < r < 1$ *we have that*

$$(P_r * g)(e^{i\eta}) = \sum_{k=-\infty}^{+\infty} \hat{g}(k)r^{|k|}e^{ik\eta},$$

in which

$$P_r(e^{i\eta}) = \sum_{k=-\infty}^{+\infty} r^{|k|}e^{ik\eta} = \frac{1-r^2}{\left|1-re^{i\eta}\right|^2} = P(re^{i\eta}).$$

Remark 1.2.3 *For $0 < \varrho < 1$and a function f defined in $\mathbb{D}$ we write $f_\varrho := f(\varrho.)$ to denote a function on the unit circle. For $0 < r; \varrho < 1$, given a function f holomorphic in $\mathbb{D}$ notice that f_r is holomorphic in a ngbh of $\overline{\mathbb{D}}$ so that*

$$f(r\varrho e^{i\eta}) = (f_r * P_\varrho)(e^{i\eta}),$$

Hence, we have

$$\begin{aligned} M_p(f, r\varrho) &= \|f_r * P_\varrho\|_{L^p(\mathbb{T})} \\ &\leq \|f_r\|_{L^p(\mathbb{T})} \|P_\varrho\|_{L^1(\mathbb{T})} = M_p(f, r), \end{aligned}$$

that is, $M_p(f, r)$ is increasing in r and

$$\sup_{0<r<1} M_p(f, r) = \lim_{r \to 1-} M_p(f, r),$$

Proposition 1.2.1 *For $1 \leq p \leq \infty$ H^p is a Banach space.*

Proof. It is clear that $\|.\|_{H^p}$ is a norm, so we only need to prove that it is complete. Let $0 < r$,$\varrho < 1$ and notice that for f holomorphic on $\mathbb{D}$ we have

$$\begin{aligned} \left| f_r * P_\varrho(e^{it}) \right| &\leq \|f_r\|_{L^p(\mathbb{T})} \|P_\varrho\|_{L^{p'}(\mathbb{T})} \\ &\leq C_\varrho \|f\|_{H^p} \ , \end{aligned}$$

This shows that

$$\sup_{|z| \leq \varrho} |f(z)| \leq C_\varrho \|f\|_{H^p} \ ,$$

and therefore the convergence in the H^p-norm implies the uniform convergence on compact subsets.Thus, let $\{f_n\}$ be a Cauchy sequence in the H^p-norm, and let f be the function uniform limit on compact subsets of $\mathbb{D}$. Then f is holomorphic and for $0 < r < 1$ fixed, using the uniform convergence, for $p < \infty$ we have

$$\begin{aligned} M_p(f_n - f, r)^p &= \frac{1}{2\pi}\int_0^{2\pi} \left|f_n(re^{it}) - f(re^{it})\right|^p dt \\ &= \lim_{m\to+\infty} \frac{1}{2\pi}\int_0^{2\pi} \left|f_n(re^{it}) - f_m(re^{it})\right|^p dt \\ &= \lim_{m\to+\infty} M_p(f_n - f_m, r)^p \\ &\leq \lim_{m\to+\infty} \|f_n - f_m\|_{H^p}^p , \end{aligned}$$

Therefore,

$$\|f_n - f\|_{H^p} \leq \lim_{m\to+\infty} \|f_n - f_m\|_{H^p} < \varepsilon.$$

for n sufficiently large. The case $p = \infty$ is similar ■

Proposition 1.2.2 *If $1 \leq p < q \leq \infty$,then $H^q(\mathbb{D}) \subset H^p(\mathbb{D})$ and for $f \in H^q(\mathbb{D})$*

$$\|f\|_{H^p} \leq \|f\|_{H^q} .$$

Proposition 1.2.3 *The function $f(z) = (1-z)^{-1}$ in is H^p for every $0 < p < 1$, but is not in H^1 and thus not in any H^p space for any $p \geq 1$.*

1.2.2 Atomic H^p spaces

A function $a \in L^\infty(\mathbb{R}^n)$ is called an atom if there exists a ball B such that

(i) *$supp(a) \subset B$;*

(ii) $\|a\|_{L^\infty} \leq |B|^{\frac{1}{q}}$;

(iii) $\int_{\mathbb{R}^n} a(x)x^\alpha dx = 0$ for all α with $|\alpha| \leq n(p^{-1} - 1)$.

The atomic Hardy space H_A^p, $0 < p \leq 1$, is defined as the set of all distributions $f \in S'$ that can be represented in the form

$$f = \sum_{j=1}^{\infty} \lambda_j a_j, \quad \text{where} \sum_{j=1}^{\infty} |\lambda_j|^p < \infty,$$

1.2.3 Atomic decomposition of H_A^p spaces

Definition 1.2.2 *For any $0 < p \leq 1$ the continuous embedding $H^p \subset H_A^p$ is valid, that is, if $f \in H^p$, then $f \in H_A^p$ and*

$$\|f\|_{H_A^p} \leq c \|f\|_{H^p},$$

where $c > 0$ is a constant depending only on p, n.

1.3 The Hardy-Littlewood maximal operator

1.3.1 Basic properties

Given a function $f \in L^1_{loc}(\mathbb{R}^n)$, we define the maximal function of f, is defined for any $x \in \mathbb{R}^n$ by :

$$Mf(x) = \sup_{Q \ni x} \fint_Q |f(y)| \, dy,$$

where $\fint_Q g dy = |Q|^{-1} \int_Q g dy,$ and the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ that contain x and whose sides are parallel to the coordinate axes.

Proposition 1.3.1 *[2] The Hardy-Littlewood maximal operator has the following properties :*

1. *M is sublinear : $M(f+g)(x) \leq Mf(x) + Mg(x)$, and for all $\alpha \in \mathbb{R}$, $M(\alpha f)(x) = |\alpha| \, Mf(x)$.*
2. *If f is not identically zero, then on any bounded set Ω there exists $\epsilon > 0$ such that*

$$Mf(x) \geq \epsilon, \quad x \in \Omega.$$

3. *If f is not equal to 0, then $Mf(x) \notin L^1(\mathbb{R}^n)$.*
4. *If $f \in L^\infty(\mathbb{R}^n)$, then $Mf \in L^\infty(\mathbb{R}^n)$ and $\|Mf\|_\infty = \|f\|_\infty$.*

Proposition 1.3.2 *[2]* *Given a locally integrable function f, then for a. e. $x \in \mathbb{R}^n$,*

$$|f(x)| \leq Mf(x)$$

Definition 1.3.1 *[1]* *Given $p(.) \in \mathcal{P}_0$, we say $p(.) \in M\mathcal{P}_0$ if $p_- > 0$ and there exists p_0, $0 < p_0 < p_-$, such that $\|Mf\|_{p(.)/p_0} \leq C\ (n, p(.), p_0)\ \|f\|_{p(.)/p_0}$.*

A useful sufficient condition for the boundedness of the maximal operator is log-Hölder continuity : for a proof, see [6, 15]

Lemma 1.3.1 *[1]* *Given $p(\cdot) \in \mathcal{P}$, such that $1 < p_- \leq p_+ < \infty$, suppose that $p(.)$satisfies the log-Hölder continuity condition locally,*

$$|p(x) - p(y)| \leq \frac{C_0}{-\log(|x-y|)}, \quad |x-y| < 1/2. \tag{1.1}$$

and at infinity : there exists p_∞ such that

$$|p(x) - p_\infty| \leq \frac{C_\infty}{\log(e+|x|)}. \tag{1.2}$$

Then $\|Mf\|_{p(.)} \leq C\ (n, p(.))\ \|f\|_{p(.)}$.

Lemma 1.3.2 *Let $p(.) \in \mathcal{P}(\mathbb{R}^n)$. If the maximal operator M is bounded on $L^{p(\cdot)}(\mathbb{R}^n)$ then for all $s \in (1, \infty)$, M is also bounded on $L^{sp(\cdot)}$.*

Proof. This follows at once from Hölder's inequality and **proposition(1.1.3)**

$$\|Mf\|_{sp(.)} = \|(Mf)^s\|_{p(.)}^{1/s} \leq \|(M\,|f|)^s\|_{p(.)}^{1/s} \leq C^{1/s}\,\||f|^s\|_{p(.)}^{1/s} = C^{1/s}\,\|f\|_{sp(.)}.$$

■

Lemma 1.3.3 *[1]* *Given $p(.) \in \mathcal{P}$, if the maximal operator is bounded on $L^{p(\cdot)}$, then for every ball $B \subset \mathbb{R}^n$*

$$\|\chi_B\|_{p(.)}\,\|\chi_B\|_{p'(.)} \leq C\ |B|.$$

The maximal operator also satisfies a vector-valued inequality.

Lemma 1.3.4 *Given $p(.) \in \mathcal{P}$ such that $p_+ < \infty$, if the maximal operator is bounded on $L^{p(.)}$, then for any r, $1 < r < \infty$*

$$\left\| \left(\sum_k (Mf_k)^r \right)^{1/r} \right\|_{p(.)} \leq C(n, p(.), r) \left\| \left(\sum_k |f_k|^r \right)^{1/r} \right\|_{p(.)}.$$

Lemma 1.3.5 *[1] Given $p(\cdot) \in \mathcal{P}$ such that $1 < p_- \leq p_+ < \infty$, the maximal operator is bounded on $L^{p(\cdot)}$ if and only if it is bounded on $L^{p'(\cdot)}$.*

Chapitre 2

Variable Hardy spaces

In this chapter, let $p(\cdot)$ be a measurable function on $\mathbb{R}^n$ satisfying $0 < p_- := \operatorname{ess\,inf}_{x\in\mathbb{R}^n} p(x) \leq \operatorname{ess\,sup}_{x\in\mathbb{R}^n} p(x) =: p_+ < \infty$ and the globally log-Hölder continuity condition, and $q \in (0, \infty)$. We study the variable Hardy spaces and its different characterizations.

2.1 Definition and atomic decomposition

In this section, we define the variable Hardy spaces and give equivalent characterizations in terms of maximal operators. We need a few definitions.

Let S be the **space of Schwartz functions** and let $S^{'}$ denote the **space of tempered distributions**. We will say that a tempered distribution f is bounded if $f * \Phi \in L^\infty$ for every $\Phi \in S$. For complete information on distributions. Define the family of semi-norms on $\|.\|_{\alpha,\beta}$, α and β multi-indices, on S by

$$\|f\|_{\alpha,\beta} = \sup_{x\in\mathbb{R}^n} \left|x^\alpha D^\beta f(x)\right|.$$

and for each integer $N > 0$ let

$$S_N = \left\{f \in S : \|f\|_{\alpha,\beta} \leq 1,\ \ |\alpha|, |\beta| \leq N\right\}.$$

Given Φ and $t > 0$, let $\Phi_t(x) = t^{-n}\Phi(x/t) = t^{-n}\Phi(t^{-1}x)$. We define three maximal operators : given $\Phi \in S$ and $f \in S'$, define the radial maximal operator

$$M_{\Phi,0}f = \sup_{t>0} |f * \Phi_t(x)| .$$

and for each N (large)> 0 the grand maximal operator

$$\mathcal{M}_N f(x) = \sup_{\Phi \in S_N} M_{\Phi,0}f(x).$$

Finally, define the non-tangential maximal operator

$$\mathcal{N}f(x) = \sup_{|x-y|<t} |P_t * f(y)| .$$

where P is the Poisson kernel

$$P(x) = \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \frac{1}{(1+|x|^2)^{\frac{n+1}{2}}}.$$

Theorem 2.1.1 *[1]* *Given $p(.) \in M\mathcal{P}_0$, for every $f \in S'$ the following are equivalent :*

1. *there exists $\Phi \in S$, $\int \Phi(x)dx \neq 0$, such that $M_{\Phi,0}f \in L^{p(\cdot)}$.*
2. *for all $N > n/p_0 + n + 1$, $\mathcal{M}_N f \in L^{p(\cdot)}$.*
3. *f is a bounded distribution and $\mathcal{N}f \in L^{p(\cdot)}$.*

Definition 2.1.1 ***(Variable hardy space)*** *Let $p(.) \in M\mathcal{P}_0$ for $N > n/p_0 + n + 1$, define the space $H^{p(\cdot)}$ to be the collection of $f \in S'$ such that $\|f\|_{H^{p(\cdot)}} = \|\mathcal{M}_N f\|_{p(.)} < \infty$.*

Remark 2.1.1 *[1]*

The spaces $H^{p(\cdot)}(\mathbb{R}^n)$ are independent of the choice of $N > n/p_0 + n + 1$.

2.1.1 Atomic decomposition

THE ATOMIC DECOMPOSITION (p(·), ∞) ATOMS

Definition 2.1.2 *[1]* *Given $p(.) \in M\mathcal{P}_0$, and q, $1 < q < \infty$ a function $a(.)$is a $(p(\cdot), q)$atom if $supp(a) \subset B = B(x_0, r) = \{y \in \mathbb{R}^n : |x_0 - y| < r\}$ for some $x_0 \in \mathbb{R}^n$, $r > 0$ and it satisfies*

(i) $\|a\|_q \leq |B|^{\frac{1}{q}} \|\chi_B\|_{p(.)}^{-1}$.

(ii) $\int a(x)x^\alpha dx = 0$ *for all* $|\alpha| \leq \lfloor n(p_0^{-1} - 1\rfloor$.

In (i) we interpret $1/\infty = 0$. These two conditions are called the size and vanishing moments conditions of atoms.

Remark 2.1.2 *If $p_0 > 1$ (which can happen if $p_- > 1$), then $\lfloor n(p_0^{-1} - 1\rfloor < 0$, and we interpret this to mean that no vanishing moments are required.*

In the remainder of this section we consider the case $q = \infty$

Theorem 2.1.2 *Suppose $p(.) \in M\mathcal{P}_0$. Then a distribution f is in $H^{p(\cdot)}(\mathbb{R}^n)$ if and only if there exists a collection $\{a_j\}$ of $(p(\cdot), \infty)$ atoms supported on balls $\{B_j\}$, and non-negative coefficients $\{\lambda_j\}$ such that*

$$f = \sum_j \lambda_j a_j,$$

where the series converges in $H^{p(\cdot)}(\mathbb{R}^n)$. Moreover

$$\|f\|_{H^{p(.)}} \backsimeq \inf \left\{ \left\| \sum_j \lambda_j \frac{\chi_{B_j}}{\left\|\chi_{B_j}\right\|_{p(.)}} \right\|_{p(.)} : f = \sum_j \lambda_j a_j \right\}. \tag{2.1}$$

Lemma 2.1.1 *Given $p(.) \in M\mathcal{P}_0$, suppose $\{a_j\}$ is a sequence of $(p(\cdot),\infty)$ atoms, supported on $B_j = B(x_j, r_j)$, and $\{\lambda_j\}$ is a non-negative sequence that satisfies*

$$\left\| \sum_j \lambda_j \frac{\chi_{B_j}}{\left\|\chi_{B_j}\right\|_{p(.)}} \right\|_{p(.)} < \infty. \tag{2.2}$$

Then the series $f = \sum_j \lambda_j a_j$ *converges in $H^{p(.)}$, and*

$$\|f\|_{H^{p(.)}} \leq C(n, p(.), p_0) \left\| \sum_j \lambda_j \frac{\chi_{B_j}}{\left\|\chi_{B_j}\right\|_{p(.)}} \right\|_{p(.)}. \tag{2.3}$$

Proof. Fix $\Phi \in S$ such that $\int \Phi dx \neq 0$ and $supp(\Phi) \subset B(0,1)$. Fix atoms $\{a_j\}$ with support $\{B_j\}$ and coefficients $\{\lambda_j\}$ such that **2.2** holds.

Given $B = B(x_0, r)$, let $2B = B(x_0, 2r)$. We consider the case $p_- < 1$; if $p_- \geq 1$ the proof is essentially the same, omitting the exponent p_-

$$\begin{aligned} \|M_{\Phi,0}\|_{p(.)}^{p_-} &\lesssim \left\| \sum_j \lambda_j M_{\Phi,0}(a_j) \right\|_{p(.)}^{p_-} \\ &\leq \underbrace{\left\| \sum_j \lambda_j M_{\Phi,0}(a_j).\chi_{(2B_j)} \right\|_{p(.)}^{p_-}}_{I_1} + \underbrace{\left\| \sum_j \lambda_j M_{\Phi,0}(a_j).\chi_{\left(2B_j^c\right)} \right\|_{p(.)}^{p_-}}_{I_2}, \end{aligned}$$

We first estimate I_1 By the size condition on $(p(\cdot),\infty)$ atoms, we have that

$$M_{\Phi,0} a_j(x) \leq \|a_j\|_\infty \|\Phi\|_1 \leq c \left\|\chi_{B_j}\right\|_{p(.)}^{-1},$$

Define $g_j = (\left\|\chi_{B_j}\right\|_{p(.)}^{-1} \lambda_j)^{p_0} \chi_{B_j}$.

If $x \in \chi_{(2B_j)}$, then by the definition of the maximal operator

$$Mg_j(x) \geq (\left\|\chi_{B_j}\right\|_{p(.)}^{-1} \lambda_j)^{p_0} \chi_{B_j} \frac{1}{|2B_j|} \int_{2B_j} \chi_{B_j} dx = 2^{-n} (\left\|\chi_{B_j}\right\|_{p(.)}^{-1} \lambda_j)^{p_0}.$$

Then by **proposition(1.1.3)** and **lemma (1.3.4)**

$$\begin{aligned} I_1 &\leq C \left\| \sum_j \left\|\chi_{B_j}\right\|_{p(.)}^{-1} \lambda_j \chi_{2B_j} \right\|_{p(.)}^{p_-} \leq C \left\| \sum_j M(g_j)^{1/p_0} \right\|_{p(.)}^{p_-} \\ &= C \left\| \left(\sum_j M(g_j)^{1/p_0} \right)^{p_0} \right\|_{\frac{p(.)}{p_0}}^{\frac{p_-}{p_0}} \leq C \left\| \left(\sum_j (g_j)^{1/p_0} \right)^{p_0} \right\|_{\frac{p(.)}{p_0}}^{\frac{p_-}{p_0}} \\ &= C \left\| \sum_j \left\|\chi_{B_j}\right\|_{p(.)}^{-1} \lambda_j \chi_{B_j} \right\|_{p(.)}^{p_-}, \end{aligned}$$

To estimate I_2, let a be an atom supported on $B = B(x_0, r)$

$$\begin{aligned} M_{\Phi,0}a(x) &\leq c \left(\frac{r}{|x-x_0|} \right)^{n+1+d} \oint_B a(y)dy \\ &\leq \left(\frac{r}{|x-x_0|} \right)^{n+1+d} \|a\|_\infty \leq c \left(\frac{r}{|x-x_0|} \right)^{n\gamma} \|\chi_B\|_{p(.)}^{-1} \end{aligned}$$

we have for each j that

$$M_{\Phi,0}a_j(x) \leq c \left(\frac{r_j}{|x-x_j|} \right)^{n\gamma} \left\|\chi_{B_j}\right\|_{p(.)}^{-1} \leq c \left\|\chi_{B_j}\right\|_{p(.)}^{-1} M(\chi_{B_j})^\gamma,$$

We can now estimate I_2 : by **proposition(1.1.3)** and **lemma(1.3.4)**

$$\begin{aligned} I_2 &\leq \left\| \sum_j \lambda_j M_{\Phi,0}(a_j).\chi_{\left(2B_j^c\right)} \right\|_{p(.)}^{p_-} \leq c \left\| \sum_j \frac{\lambda_j}{\left\|\chi_{B_j}\right\|_{p(.)}} M(\chi_{B_j})^\gamma \right\|_{p(.)}^{p_-} \\ &= \left\| \left(M \sum_j \left(\frac{\lambda_j^{1/\gamma}}{\left\|\chi_{B_j}\right\|_{p(.)}^{1/\gamma}} \chi_{B_j} \right) \right)^{\gamma} \right)^{1/\gamma} \right\|_{\gamma p(.)}^{\gamma p_-} \leq C \left\| \left(\sum_j \left\|\chi_{B_j}\right\|_{p(.)}^{-1} \lambda_j \chi_{B_j} \right)^{1/\gamma} \right\|_{\gamma p(.)}^{\gamma p_-} \\ &= C \left\| \sum_j \left\|\chi_{B_j}\right\|_{p(.)}^{-1} \lambda_j \chi_{B_j} \right\|_{p(.)}^{p_-}. \end{aligned}$$

■

Lemma 2.1.2 *Let $p(.) \in M\mathcal{P}_0$. if $f \in H^{p(\cdot)}$, then there exist $(p(.), \infty)$ atoms $\{a_{k,j}\}$, supported on balls $B_{k,j}$, and non-negative coefficients $\{\lambda_{k,j}\}$ such that*

$$f = \sum_{k,j} \lambda_{k,j} a_{k,j},$$

Moreover

$$\left\| \sum_{k,j} \lambda_{k,j} \frac{\chi_{B_{k,j}}}{\left\| \chi_{B_{k,j}} \right\|_{p(.)}} \right\|_{p(.)} \leq C(n, p(.), p_0) \left\| f \right\|_{H^{p(\cdot)}}.$$

THE ATOMIC DECOMPOSITION $(p(\cdot), q)$ ATOMS :

Infinite atomic decomposition using $(p(\cdot), q)$ atoms We extend **Theorem (2.1.2)** by giving an atomic decomposition using $(p(\cdot), q)$ atoms.

Theorem 2.1.3 *[1] Suppose $p(\cdot) \in M\mathcal{P}_0$. Then a distribution f is in $H^{p(\cdot)}$ if and only if for $q > 1$ sufficiently large, there exists a collection $\{a_j\}$ of $(p(\cdot), q)$ atoms supported on balls $\{B_j\}$, and non-negative coefficients $\{\lambda_j\}$ such that*

$$f = \sum_j \lambda_j a_j,$$

where the series converges in $H^{p(\cdot)}$. Moreover

$$\|f\|_{H^{p(\cdot)}} \backsimeq \inf \left\{ \left\| \sum_j \lambda_j \frac{\chi_{B_j}}{\left\| \chi_{B_j} \right\|_{p(.)}} \right\|_{p(.)} : f = \sum_j \lambda_j a_j \right\}. \tag{2.4}$$

Remark 2.1.3 *Denote the norm of the maximal operator by $\|M\|_{(p(\cdot)/p_0)'}$. Then it suffices to take* $q > \max(1, p_+, p_0(1 + 2^{n+3} \|M\|_{(p(\cdot)/p_0)'}))$.

Lemma 2.1.3 *Given $p(\cdot) \in M\mathcal{P}_0$, there exists* $q = q(p(\cdot), p_0, n) > \max(p_+, 1)$ *such that if $\{a_j\}$ is a sequence of $(p(\cdot), q)$ atoms supported on $B_j = B(x_j, r_j)$, and $\{\lambda_j\}$ is a* *non-negative sequence that satisfies*

$$\left\| \sum_j \lambda_j \frac{\chi_{B_j}}{\left\|\chi_{B_j}\right\|_{p(.)}} \right\|_{p(.)} < \infty, \tag{2.5}$$

then the series $f = \sum_j \lambda_j a_j$ *converges in* $H^{p(.)}$*, and*

$$\|f\|_{H^{p(.)}} \leq C(n, p(.), p_0, q) \left\| \sum_j \lambda_j \frac{\chi_{B_j}}{\left\|\chi_{B_j}\right\|_{p(.)}} \right\|_{p(.)}, \tag{2.6}$$

Lemma 2.1.4 *Given* $w \in A_1$*, then* $w \in RH_s$*, where* $s = 1 + (2^{n+2}\,[w]_{A_1})^{-1}$ *in which*

$$[w]_{A_1} = ess \sup_{x\in\mathbb{R}^n} \frac{Mw(x)}{w(x)} < \infty.$$

Remark 2.1.4 *Given a weight* $w \in A_1$ *and* $p_0 > 0$*, the weighted Hardy space* $H^{p_0}(w)$ *consists of all tempered distributions* f *such that*

$$\|f\|_{H^{p_0}(w)} = \|M_{\Phi,0} f\|_{L^{p_0}(w)} = \left(\int_{\mathbb{R}^n} M_{\Phi,0} f(x)^{p_0} w(x) dx \right)^{1/p_0} < \infty.$$

These spaces have an atomic decomposition.

Lemma 2.1.5 *Given* $p(\cdot) \in M\mathcal{P}_0$ *and* $q > \max(p_0, 1)$*, suppose* $\{a_j\}$ *is a sequence of* $(p(\cdot), q)$ *and* $\{\lambda_j\}$ *is a non-negative sequence and* $w \in A_1 \cap RH_{(p(.)/p_0)'}$*. if*

$$\left\| \sum_j \lambda_j \frac{\chi_{B_j}}{\left\|\chi_{B_j}\right\|_{p(.)}} \right\|_{L^{p_0}(w)} < \infty,$$

Then the series $f = \sum_j \lambda_j a_j$ *converges in* $H^{p_0}(w)$ *and*

$$\|f\|_{H^{p_0}(w)} \leq C(p(.), p_0, q, n, [w]_{A_1}, [w]_{RH_{(p(.)/p_0)'}}) \left\| \sum_j \lambda_j \frac{\chi_{B_j}}{\left\|\chi_{B_j}\right\|_{p(.)}} \right\|_{L^{p_0}(w)}.$$

Remark 2.1.5 *form the Rubio de Francia iteration algorithm with respect to* $L^{(p(.)/p_0)'}$ *Given a function* h, *define*

$$Rh = \sum_{i=0}^{\infty} \frac{M^i h}{2^i \|M\|_{(p(.)/p_0)'}},$$

where $M^0 h = |h|$ *and for* $i \geq 1$ $M^i h = M \circ M \circ ... \circ Mh$. *is* i *iterates of the maximal operator.*

Finite atomic decompositions : Given $q < \infty$, let $H^{p(.),q}_{fin}$ be the subspace of $H^{p(.)}$ consisting of all f that have decompositions as finite sums of $(p(\cdot), q)$ atoms.

Theorem 2.1.4 *Let* $p(\cdot) \in M\mathcal{P}_0$ *and fix* q *as in* ***Theorem (2.1.3)*** *For* $f \in H^{p(.),q}_{fin}(\mathbb{R}^n)$, *define*

$$\|f\|_{H^{p(.),q}_{fin}} = \inf \left\{ \left\| \sum_{j=1}^{k} \lambda_j \frac{\chi_{B_j}}{\|\chi_{B_j}\|_{p(.)}} \right\|_{p(.)} : f = \sum_{j=1}^{k} \lambda_j a_j \right\}, \tag{2.7}$$

where infimum is taken over all finite decompositions of f *using* $(p(\cdot), q)$ *atoms* a_j, *supported on balls* B_j. *Then*

$$\|f\|_{H^{p(.)}} \simeq \|f\|_{H^{p(.),q}_{fin}}.$$

Lemma 2.1.6 *Define the non-tangential grand maximal function* $\mathcal{M}_{N,1}$,*by*

$$\mathcal{M}_{N,1} f(x) = \sup_{\Phi \in S_N} \sup_{|y-x|<t} |\Phi_t * f(x)|,$$

Then for all $x \in \mathbb{R}^n$ *and tempered distributions* f

$$\mathcal{M}_{N,1} f(x) \approx \mathcal{M}_N f(x),$$

where the constants depend only on N

The second lemma is a decay estimate for the grand maximal operator.

Lemma 2.1.7 *Given $p(\cdot) \in M\mathcal{P}_0$, suppose $f \in H^{p(\cdot)}$ is such that* $supp(f) \subset B(0, R)$ *for*

some $R > 1$. Then for all $x \in B(0, 4R)^c$

$$\mathcal{M}_N f(x) \leq C(N, p(.), p_0) \left\| \chi_{B(0,R)} \right\|^{-1}.$$

Chapitre 3

Boundedness of Operators on variable Hardy spaces

In this Chapter, we show that convolution type Calderón-Zygmund singular integrals with sufficient regularity are bounded on $H^{p(\cdot)}$.

First we define the class of singular integrals we are interested in.

Definition 3.0.3 *[1] Let $K \in S'$ we say $Tf = K * f = \int_{\mathbb{R}^n} K(x-y)f(y)dy$ is a convolution-type singular integral operator with regularity of order k if the distribution K coincides with a function on $\mathbb{R}^n \setminus \{0\}$ and has the following properties :*

1. *$\widehat{K} \in L^\infty$;*
2. *for all multi-indices $0 \leq |\beta| \leq k+1$ and $x \neq 0$, $\left|\partial^\beta K(x)\right| \leq \frac{C}{|x|^{n+|\beta|}}$.*

Singular integrals that satisfy this definition are bounded on L^p, $1 < p < \infty$.

Lemma 3.0.8 *Let T be a convolution-type singular integral operator as defined above. Given $w \in A_1$ and $0 < p < 1$, then for every ball B,*

$$\int_B |Tf(x)|^p \, w(x)dx \leq C(T, n, p, [w]_{A_1}) w(B)^{1-p} \left(\int_{\mathbb{R}^n} |f(x)| \, w(x)dx \right)^p .$$

Theorem 3.0.5 *Given $p(\cdot) \in M\mathcal{P}_0$ and $q > 1$ sufficiently large (as in* ***Theorem*** *(2.1.3)), let T be a singular integral operator that has regularity of order $k \geq \lfloor n(\frac{1}{p_0} - 1) \rfloor$. then*

$$\|Tf\|_{p(.)} \leq C(T, p(.), p_0, q, n) \|f\|_{H^{p(.)}}.$$

Theorem 3.0.6 *Given $p(\cdot) \in M\mathcal{P}_0$ and $q > 1$ sufficiently large (as in* ***Theorem*** *(2.1.3)), let T be a singular integral operator that has regularity of order $k \geq \lfloor n(\frac{1}{p_0} - 1) \rfloor$. then*

$$\|Tf\|_{H^{p(.)}} \leq C(T, p(.), p_0, q, n) \|f\|_{H^{p(.)}}.$$

Theorem 3.0.7 *[1] Given $p(\cdot) \in M\mathcal{P}_0$ with $0 < p_0 < 1$ and $q > 1$ sufficiently large (as in* ***Theorem*** *(2.1.3)), suppose that T is a sublinear operator that is defined on $(p(\cdot), q)$ atoms. Then :*

(1). *If for all $w \in A_1 \cap RH_{(q/p_0)'}$ and every $(p(\cdot), q/p_0)$ atom $a(\cdot)$ with support B,*

$$\|Ta\|_{L^{p_0}(w)} \leq C(T, p(.), p_0, q, n, [w]_{A_1}, [w]_{RH_{(q/p_0)'}}) \frac{w(B)^{1/p_0}}{\|\chi_B\|_{p(.)}}, \tag{3.1}$$

then T has a unique, bounded extension $\widetilde{T} : H^{p(.)} \to L^{p(.)}$.

(2). *If for all $w \in A_1 \cap RH_{(q/p_0)'}$ and every $(p(\cdot), q/p_0)$ atom $a(\cdot)$ with support B,*

$$\|Ta\|_{H^{P_0}(w)} \leq C(T, p(.), p_0, q, n, [w]_{A_1}, [w]_{RH_{(q/p_0)'}}) \frac{w(B)^{1/p_0}}{\|\chi_B\|_{p(.)}}, \tag{3.2}$$

then T has a unique, bounded extension $\widetilde{T} : H^{p(.)} \to H^{p(.)}$

Remark 3.0.6 *The additional hypothesis that $0 < p_0 < 1$ is not a real restriction, since by* ***Lemma*** *(1.3.2) we may take p_0 as small as desired.*

Remark 3.0.7 *Note that when $p(\cdot)$ is constant and $w \equiv 1$, then conditions* ***3.1*** *and* ***3.2*** *reduce to showing that T is uniformly bounded on atoms, which is the condition used to prove singular integrals are bounded on classical Hardy spaces.*

Proof. First suppose that **3.1** holds.

Fix $f \in H_{fin}^{p(\cdot),q/p_0}$; by **Theorem**(2.1.3) this set is dense in $H^{p(\cdot)}$. Since T is well-defined on the elements of $H_{fin}^{p(\cdot),q/p_0}$, it will suffice to prove that

$$\|Tf\|_{L^{p(.)}} \leq C(T, p(.), p_0, q, n) \|f\|_{H^{p(.)}}. \tag{3.3}$$

For in this case by a standard density argument there exists a unique bounded extension $\widetilde{T}$ such that $\widetilde{T} : H^{p(.)} \to L^{p(.)}$.

To prove **3.3** we will use the extrapolation argument in **Lemma** (2.1.5) to reduce the variable norm estimate to a weighted norm estimate. Arguing as we did in that proof, we have that

$$\|Tf\|_{L^{p(.)}}^{p_0} \leq \sup \int |Tf(x)|^{p_0} \mathcal{R}g(x)dx,$$

with the supremum taken over all $g \in L^{(p(\cdot)/p_0)'}$ with $\|g\|_{L^{(p(\cdot)/p_0)'}} \leq 1$.

Suppose for the moment that we can prove that for all $f \in H_{fin}^{p(\cdot),q/p_0}$

$$\|Tf\|_{L^{p_0}(\mathcal{R}g)} \leq C(T, p(.), p_0, q, n) \|f\|_{H^{p_0}(\mathcal{R}g)}. \tag{3.4}$$

(In particular, the constant is independent of g) Then we can continue the argument as in the proof of **Lemma** (2.1.5) to get

$$\begin{aligned} \|Tf\|_{L^{p_0}(\mathcal{R}g)}^{p_0} &\leq C(T, p(.), p_0, q, n) \|f\|_{H^{p_0}(\mathcal{R}g)}^{p_0} \leq C \int \mathcal{M}_N f(x)^{p_0} \mathcal{R}g(x)dx \\ &\leq C \|(\mathcal{M}_N f)^{p_0}\|_{p(\cdot)/p_0} \|\mathcal{R}g\|_{(p(\cdot)/p_0)'} \leq C \|\mathcal{M}_N f\|_{p(\cdot)}^{p_0} \leq C \|f\|_{H^{p(.)}}^{p_0}. \end{aligned}$$

This gives us **3.3**.

To complete the proof we will show **3.4**.Recall that as sets, $H_{fin}^{p_0,q/p_0}(\mathcal{R}g) = H_{fin}^{p(\cdot),q/p_0}$.

Therefore, let

$$f = \sum_{j=1}^{k} \lambda_j a_j$$

be an arbitrary finite decomposition of f in terms of $(p(\cdot), q/p_0)$ atoms.

Since, $0 < p_0 < 1$, by the sublinearity of T, convexity and **3.1**,

$$\begin{aligned}\|Tf\|_{L^{p_0}(\mathcal{R}g)}^{p_0} &= \int |Tf(x)|^{p_0}\,\mathcal{R}g(x)dx \leq \sum_{j=1}^{k} \lambda_j^{p_0} \int_{B_j} |Ta_j(x)|^{p_0}\,\mathcal{R}g(x)dx \\ &\leq C \sum_{j=1}^{k} \lambda_j^{p_0} \frac{\mathcal{R}g(B_j)}{\|\chi_{B_j}\|_{p(.)}^{p_0}} = C \left\| \sum_{j=1}^{k} \lambda_j^{p_0} \frac{\chi_{B_j}}{\|\chi_{B_j}\|_{p(.)}^{p_0}} \right\|_{L^1(\mathcal{R}g)}.\end{aligned}$$

This is true for any such decomposition of f.

Therefore, since $\mathcal{R}g \in A_1 \cap L^{(p(\cdot)/p_0)'}$ by construction, by see **[1, Lemma 7.11]** we can take the infimum over all such decompositions to get

$\|Tf\|_{L^{p_0}(\mathcal{R}g)} \leq C\,\|f\|_{H^{P_0}(\mathcal{R}g)}$, where $C = C(T, p(\cdot), p_0, q, n)$. This proves **3.4** for all $f \in H_{fin}^{p(\cdot), q/p_0}$.

We now consider the case when condition **3.2** holds.

The proof is essentially the same as before, except instead of proving **3.4**, we need to prove that for all $f \in H_{fin}^{p(\cdot), q/p_0}$,

$$\|Tf\|_{H^{p_0}(\mathcal{R}g)} \leq C(T, p(.), p_0, q, n)\,\|f\|_{H^{p_0}(\mathcal{R}g)}. \tag{3.5}$$

■

Given this, we can then repeat the extrapolation argument as before.

To prove **3.5** we use the same argument used to prove **3.4**, replacing Tf with $M_{\Phi,0}(Tf)$ where $\Phi \in S$ with $\int \Phi dx = 1$, and using **3.2** instead of **3.1**.

Proof. of Theorem (3.0.5)

By Theorem (3.0.7) it will suffice to show that condition 3.1 holds for all $(p(\cdot), q/p_0)$ atoms and all $w \in A_1 \cap RH_{(q/p_0)'}$.

Fix such an atom $a(\cdot)$ with support $B = B(x_0, r)$. Let $2B = B(x_0, 2r)$ and write

$$\begin{aligned} \|Ta\|_{L^{p_0}(w)}^{p_0} &= \int |Ta(x)|^{p_0} w(x)dx \\ &= \underbrace{\int_{2B} |Ta(x)|^{p_0} w(x)dx}_{I_1} + \underbrace{\int_{(2B)^c} |Ta(x)|^{p_0} w(x)dx}_{I_2} \end{aligned}$$

We first **estimate** I_1 :

By Lemma (3.0.8) there exists a constant $C = C(T, n, p_0, [w]_{A_1})$ such that

$$\begin{aligned} I_1 &\leq Cw(B)^{1-p_0} \left(\int_{\mathbb{R}^n} |a(x)|\, w(x)dx \right)^{p_0} \\ &\leq Cw(B)^{1-p_0} |B|^{p_0} \left(\fint_B |a(x)|^{q/p_0} dx \right)^{1/q} \left(\fint_B w(x)^{(q/p_0)'} dx \right)^{p_0/(q/p_0)'} \end{aligned}$$

Since $a(\cdot)$ is a $(p(\cdot), q/p_0)$ atom and $w \in RH_{(q/p_0)'}$, we get that

$$I_1 \leq C\, [w]_{RH_{(q/p_0)'}}^{p_0} w(B)^{1-p_0} |B|^{p_0} \|\chi_B\|_{L^{p(.)}}^{-p_0} |B|^{-p_0} w(B)^{p_0} = C\, [w]_{RH_{(q/p_0)'}}^{p_0} w(B) \|\chi_B\|_{L^{p(.)}}^{-p_0}.$$

To **estimate** I_2

Let $d = \lfloor n(\frac{1}{p_0} - 1) \rfloor$, We claim that there exists a constant $C = C(T, n)$ such that for all $x \in (2B)^c$,

$$|Ta(x)| \leq C \frac{|B|^{1+\frac{d+1}{n}}}{\|\chi_B\|_{L^{p(.)}}} . \frac{1}{|x - x_0|^{n+d+1}}. \tag{3.6}$$

To prove this, let P_d be the Taylor polynomial of K of degree d centered at $x - x_0$. By our definition of d and our assumption on k, $d + 1 \leq k + 1$. Therefore, the remainder

$|K(x - y) - P_d(y)|$ can be estimated by **Condition (2)** in Definition(3.0.3) .

Hence, by the vanishing moment and size conditions on $a(\cdot)$ and Hölder's inequality

$$\begin{aligned}
|Ta(x)| &\leq \int |K(x-y) - P_d(y)|\, |a(y)|\, dy \\
&\leq \frac{C}{|x-x_0|^{n+d+1}} \int_{B(x_0,r)} |y-x_0|^{d+1}\, |a(y)|\, dy \\
&\leq C \frac{r^{d+1}\, |B|}{|x-x_0|^{n+d+1}} \fint_B a(y)dy \\
&\leq C \frac{|B|^{\frac{n+d+1}{n}}\, |B|^{-p_0/q}\, \|a\|_{q/p_0}}{|x-x_0|^{n+d+1}} \\
&\leq C \frac{|B|^{1+\frac{d+1}{n}}}{\|\chi_B\|_{p(.)}} . \frac{1}{|x-x_0|^{n+d+1}}.
\end{aligned}$$

Given 3.6 we have that

$$\int_{(2B)^c} |Ta(x)|^{p_0}\, w(x)dx \leq C \frac{|B|^{p_0(\frac{n+d+1}{n})}}{\|\chi_B\|_{p(.)}^{p_0}} \underbrace{\int_{(2B)^c} \frac{w(x)}{|x-x_0|^{p_0(n+d+1)}} dx}_{J}.$$

To complete the proof we will show that there exists a constant $C = C(n, p_0)$ such that

$$J \leq C \frac{[w]_{A_1}\, w(B)}{|B|^{p_0\left(\frac{n+d+1}{n}\right)}}. \tag{3.7}$$

we have $(2B)^c = \cup_{i=1}^{\infty} (2^{i+1}B/2^iB)$, for $x \in 2^{i+1}B/2^iB$, we have $|x - x_0| \backsimeq 2^i r \backsimeq 2^i\, |B|^{1/n}$.

Since $w \in A_1$ and $p_0(n+d+1) > n$, we can estimate as follows :

$$\begin{aligned}
J &= \sum_{i=1}^{\infty} \int_{2^{i+1}B/2^iB} \frac{w(x)}{|x-x_0|^{p_0(n+d+1)}} dx \\
&\leq \frac{C}{|B|^{p_0\left(\frac{n+d+1}{n}\right)}} \sum_{i=1}^{\infty} \frac{1}{2^{ip_0(n+d+1)}} \int_{2^{i+1}B/2^iB} w(x)dx \\
&= \frac{C}{|B|^{p_0\left(\frac{n+d+1}{n}\right)}} \sum_{i=1}^{\infty} \frac{2^{n(i+1)}|B|}{2^{ip_0(n+d+1)}} \fint_{2^{i+1}B} w(x)dx \\
&\leq \frac{C2^n [w]_{A_1}}{|B|^{p_0\left(\frac{n+d+1}{n}\right)}} \sum_{i=1}^{\infty} \frac{1}{2^{ip_0(n+d+1)-in}} (|B| \, ess_{x\in B} \inf w(x)) \\
&= C \frac{[w]_{A_1} w(B)}{|B|^{p_0\left(\frac{n+d+1}{n}\right)}}.
\end{aligned}$$

■

Proof. of Theorem (3.0.6)

Our argument is similar to the proof of Theorem (3.0.5) . By Theorem (3.0.7) it will suffice to show that condition 3.2 holds for an arbitrary $(p(\cdot), q/p_0)$ atom $a(\cdot)$ with support $B = B(x_0, r)$, and all $w \in A_1 \cap RH_{(q/p_0)'}$.

Fix $\Phi \in S$ with $\int \Phi dx = 1$; then we can estimate $\|Ta\|_{H^{p_0}(w)}$ as follows :

$$\|Ta\|_{H^{p_0}(w)}^{p_0} \lesssim \underbrace{\int_{2B} M_{\Phi,0}(Ta)(x)^{p_0} w(x)dx}_{R_1} + \underbrace{\int_{(2B)^c} M_{\Phi,0}(Ta)(x)^{p_0} w(x)dx}_{R_2}$$

To estimate R_1 the we first use the fact that$M_{\Phi,0}(Ta) \leq cM(Ta)$. Moreover, we have that since $w \in A_1$,

$$R_1 \leq Cw(2B)^{1-p_0} (\underbrace{\int_{\mathbb{R}^n} |Ta(x)| \, w(x)dx}_{L})^{p_0}$$

To get the desired estimate for R_1 it will suffice to show that

$$L \leq \frac{w(B)}{\|\chi_B\|_{p(.)}}.$$

To prove this, we again split the integral :

$$L=\int_{\mathbb{R}^n}|Ta(x)|\,w(x)dx=\underbrace{\int_{2B}|Ta(x)|\,w(x)dx}_{L_1}+\underbrace{\int_{(2B)^c}|Ta(x)|\,w(x)dx}_{L_2}.$$

To estimate L_1 we apply Hölder's inequality, the boundedness of T on L^{q/p_0}, and the fact that $w\in RH_{(q/p_0)'}$ to get

$$\begin{aligned}L_1&\leq\left(\int_{2B}|Ta(x)|^{q/p_0}dx\right)^{p_0/q}\left(\int_{2B}w(x)^{(q/p_0)'}dx\right)^{1/(q/p_0)'}\\&\leq\|a\|_{L^{q/p_0}}.|2B|^{1/(q/p_0)'}\left(\fint_{2B}w(x)^{(q/p_0)'}dx\right)^{1/(q/p_0)'}\leq C(n,[w]_{A_1},[w]_{RH_{(q/p_0)'}})\frac{w(B)}{\|\chi_B\|_{p(.)}}.\end{aligned}$$

To estimate L_2

$$\begin{aligned}L_2&\leq C\frac{|B|^{\frac{n+d+1}{n}}}{\|\chi_B\|_{p(.)}}\left(\int_{(2B)^c}\frac{w(x)}{|x-x_0|^{n+d+1}}dx\right)\\&\leq C\frac{|B|^{\frac{n+d+1}{n}}}{\|\chi_B\|_{p(.)}}.\frac{w(B)\,[w]_{A_1}}{|B|^{\frac{n+d+1}{n}}}.\left(\sum_{i=0}^{\infty}\frac{2^{ni}}{2^{i(n+d+1)}}\right)\leq C\frac{w(B)}{\|\chi_B\|_{p(.)}}.\end{aligned}$$

To estimate R_2, we will prove a pointwise bound for $M_{\Phi,0}(Ta_j)(x)$ for $x\in(2B_j)^c$ similar to 3.6. Define $K^t=K*\Phi_t$; then $K^{(t)}$ satisfies condition **(3)** of Definition (3.0.3) uniformly for all $t>0$.

Moreover, for $x\in(2B)^c$, the integral for $K*a(x)$ converges absolutely, so

$$|\Phi_t*(K*a)(x)|=|\Phi_t*K(x)*a(x)|=\left|K^{(t)}*a(x)\right|.$$

Let $d=n\lfloor(\frac{1}{p_0}-1)\rfloor$ and fix $t>0$.

If P_d is the Taylor polynomial of $K^{(t)}$centered at $x-x_0$, we can argue exactly as we did to prove 3.6 to get

$$
\begin{aligned}
\left|K^{(t)} * a(x)\right| &= \left|\int \left[K^{(t)}(x-y) - P_d(y)\right] a(y)dy\right| \\
&\leq \frac{C}{|x-x_0|^{n+d+1}} \int_{B(x_0,r)} |y-x_0|^{d+1} |a(y)|\, dy \\
&\leq C \frac{|B|^{1+\frac{d+1}{n}} |B|^{-p_0/q}}{|x-x_0|^{n+d+1}} \|a\|_{L^{q/p_0}} \\
&\leq C \frac{|B|^{1+\frac{d+1}{n}}}{\|\chi_B\|_{L^{p(.)}}} \frac{1}{|x-x_0|^{n+d+1}}.
\end{aligned}
$$

The final constant is independent of t, an so we can take the supremum over all t to

$$
M_{\Phi,0}(Ta)(x) \leq C \frac{|B|^{1+\frac{d+1}{n}}}{\|\chi_B\|_{L^{p(.)}}} \frac{1}{|x-x_0|^{n+d+1}}.
$$

Then arguing as we did before, by 3.7 we have that

$$
J_2 \leq \frac{w(B)}{\|\chi_B\|_{p(.)}^{p_0}}.
$$

■

Bibliographie

[1] Cruz-Uribe,D.,& Wang, L.A.D.(2014). Variable Hardy Space. Indiana university mathematics journal, 447-493.

[2] Crus Uribe, D.V., & Fiorenza,A.(2013).Variable Lebesgue Spaces :foundations and harmonic analysis. Springer Science & Business Media.

[3] C. Fefferman and E. M. Stein, H^p spaces of several variables, Acta math, 129 (1972), n. 1, 137-193.

[4] D. Cruz-Uribe and A. Fiorenza. Variable Lebesgue Spaces : Foundations and Harmonic Analys is. Applied and Numerical Harmonic Analysis. Birkhäuser, Basel, 2013.

[5] Diening , L.,& Samko, s.(2007). Hardy inequality in variable exponent Lebesgue spaces. Fractional calculus and applied Analysic 10(1), 01p-18p.

[6] D. Cruz-Uribe, A. Fiorenza, J. M. Martell, and C. Pérez. The boundedness of classical operators on variable L^p spaces. Ann. Acad. Sci. Fenn. Math., 31(1) :239–264, 2006.

[7] D. Yang, W. Yuan and C. Zhuo, A survey on some variable function spaces, in : Function Spaces and Inequalities, 299-335, Springer Proc. Math. Stat. 206, Springer, Singapore, 2017.

[8] E. Stein and G. Weiss, On the theory of harmonic functions of several variables. I. the theory of H^p-spaces, Acta Math. **103** (1960), 25-62.

[9] E. M. Stein, Harmonic analysis : Real variable Methods, Orthogonality, and Oscillatory integrals, Princeton Univ. Press. Princeton, NJ, 1993.

[10] E. Nakai and Y. Sawano, Hardy spaces with variable exponents and generalized Campanato spaces, J. Funct. Anal. **262** (2012), no. 9, 3665-3748.

[11] Heraiz Rabah, Some properties of variable herz type besov spaces and applications, Submitted for the degree of doctorate in sciences 2017, Mathematics and Informatics, Functional Analysic, Mohamed BOUDIAF University – M'sila.

[12] H. Nakano, Topology of linear topological spaces, Maruzen Co., Ltd., Tokyo, 1951.

[13] H. Nakano, Modulared semi-ordered linear spaces, Maruzen Co., Ltd., Tokyo, 1950.

[14] Izuki, M., Nakai, E., & Sawano, Y. (2014) function spaces with variable exponenents -an introduction -. scientiae Mathematicae Japonicae, 77(2), 187-315.

[15] L. Diening, P. Harjulehto, P. Hästö, and M. Růžička. Lebesgue and Sobolev spaces with variable exponents, volume 2017 of Lecture Notes in Mathematics. Springer, Heidelberg, 2011.

[16] L. Grafakos, Modern Fourier Analysis, third ed, Graduate texts in mathematics, **250**, Springer, Verlag New Yorc Inc, 2014.

[17] M. Izuki, E. Nakai and Y. Sawano, Function spaces with variable exponents-an introduction, Sci. Math. Jpn. **77** (2014), no. 2, 187-315.

[18] O. Kováčik and J. Rákosník, On spaces $L^{p(\cdot)}$ and $W^{k,p(x)}$, Czechoslovak Math. **41** (116) (1991), 592–618.

[19] W. Orlicz,Über konjugierte Exponentenfolgen, Studia Math. **3** (1931), 200-211.

Annexe A :Abbreviations and Notations

The different abbreviations and notations used throughout this thesis are explained below :

Ω	:	open set in $\mathbb{R}^n$.
$\mathcal{P}(\Omega)$	:	set of variable exponents.
ρ	:	semimodular ;modular.
$\mathbb{R}^n$	:	Euclidean,n-dimensional space.
$\mathbb{D}$	:	is the open unit disc.
$\mathbb{T}$	:	is open unit circle.
H^p	:	Hardy space.
$p(.)$	:	exponent function.
$\mathcal{P}_0(\Omega)$	:	simply exponents.
$p^{'}(.)$	:	conjugate exponent function.
$\mathcal{S}^{'}$	:	space of tempered distributions.
$\mathcal{S}$	:	space of schwartz functions.
α, β	:	multi indices.
$P(x)$	:	poisson kernel.
$H^{p(.)}$	:	variable hardy space.
$\mathcal{N}f(x)$	:	non-tangential maximal operator.
$M_{\Phi,0}f(x)$	:	radial maximal operator.

$\|.\|$	:	norm.
$L^{p(\cdot)}(\Omega)$	:	variable lebesgue space.
Tf	:	convolution -type singular integral operator.
Mf	:	maximal function.
$\mathcal{M}_N f$	:	grand maximal operator.
$L^0(Q)$	:	The set of measurable functions on Q