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Title

# Quadratic BSDEs and Applications

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# Dedication

First and above all, praise is to Allah who gives us the courage, the power and the patience to finish this modest work.

I dedicate my work to

The source of tenderness and life that have not stopped encouraging me and praying for me "**My parents",** I thank you for all support and love you do to me since my childhood, and I hope your blessing always accompanies me.

My dear grandparents thank you a lot for supporting me throughout my education your presence by my side helped me all the time.

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# Abstract

The aim of this work is to study a class of quadratic BSDEs of the following form

$$Y_t = \xi + \int_t^T \left(l\left(s\right) + f\left(Y_s\right)\left|Z_s\right|^2\right) \mathrm{d}s + \int_t^T Z_s \mathrm{d}W_s,$$

where the terminal data is assumed to be square-integrable, $l, f$ are two measurable functions. We study the existence, uniqueness, and comparison theorem to such equations. The main tool in the proofs is the so-called "Zvonkin" transformation which will be used to eliminate the generator or a part of it, so that we transform the original QBSDEs to a standard BSDE without a quadratic part. As an application, we provide the connection between the quadratic BSDE and the risk-sensitive control problem.

**Keywords:** Backward stochastic differential equations, Quadratic backward stochastic differential equations, exponential utility function, risk-sensitive, existence, and uniqueness of the solution.

# Symbols and Abbreviations

The following notation is frequently used in this thesis

$a, e$: almost everywhere.

$e.g$ : for example.

$a, s$: almost surely.

$\mathbb{R}$ : real numbers.

$\mathbb{R}^d$ : $d$-dimmensional real Euclidean space

$\mathbb{R}^{d\times d}$ : the set of all $(d \times d)$ real matrices.

$(\Omega, \mathcal{F}, \mathbb{P})$ : complete probability space.

$\{\mathcal{F}_t\}_{t\geq 0}$ : filtration.

$\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ : filtered probability space.

$\mathcal{N}$ : the totality of the $\mathbb{P}$-negligible sets.

$\mathbb{E}[x]$ : the expectation of the random variable $x$.

$\mathbb{E}[x \mid \mathcal{G}]$ : conditional expectation.

$W = (W)_{t\in[0,T]}$ : Brownian motion.

$SDE :=$ stochastic differential equations.

$BSDEs$ : Backward stochastic differential equations.

$QBSDEs :=$ Quadratic Backward stochastic differential equations.

$\mathbb{L}^2 :=$ the space of $\mathcal{F}_t$-adapted processes $\varphi$ satisfying $\int_0^T \mid \varphi_s \mid^2 \mathrm{d}s < +\infty$ $\mathbb{P}-$a.s.

$\mathbb{S}^2 :=$ the space of continuous and $\mathcal{F}_t-$adapted processes $\varphi$ such that : $\mathbb{E}\left[\sup_{0\leq t\leq T} \mid \varphi \mid^2\right] < +\infty$.

$\mathbb{M}^2 :=$ the space of $\mathcal{F}_t-$adapted processes $\varphi$ satisfying $\mathbb{E}\int_0^T \mid \varphi \mid^2 \mathrm{d}s < +\infty$.

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# General Introduction

The linear backward stochastic differential equations were first introduced in 1973 in the work of J.Bismut [2] when he was studying the adjoint equation associated with the stochastic maximum principle in optimal control. However, the first general result concerning the BSDEs dates only from 1990 and is due to E.Pardoux and S.Peng [12] who introduced a new form:

$$\begin{cases} -dY_t = g(t, Y_t, Z_t) - Z_t dW_t, 0 \leq t < T. \\ Y_T = \xi. \end{cases} \tag{1}$$

where $\xi$ is the terminal value and the coefficient $g$ is the generator of the BSDE which is a non-linear function that satisfies the globally Lipschitz condition with respect to the state variables. The solution of this equation is a couple of processes $(Y, Z)$. Indeed, as the boundary condition is given at the terminal instant $T$, the presence of the process $Z$ via the martingale representation theorem, ensures that $Y$ is adapted with respect of the filtration generated by the Brownian motion $(W_t)_{t\in[0T]}$. The theory of $BSDEs$ has been widely used in stochastic control especially in mathematical finance, namely the adjoint process can be written in terms of linear BSDEs, or non-linear $BSDEs$, for more information and examples about this subject we refer the reader to the seminal paper of El Karoui, Peng and Quenez [3].

Another direction which has attracted many works in this area, especially in connection of applications, is how to improve the existence, uniqueness conditions of a solution for $(1)$.

There where many articles weaken the Lipschitz condition on the generator of BSDE(1) and proved the existence of a solution for such kind of equations. Basically in those papers it is assumed that the generator $g$ is just continuous and satisfies a linear or a quadratic growth condition. Among them we can quote Hamadene [10], Lepeltier and San Martin [6], Kobylanski [9] and so on.

Since the early nineties, there has been an increasing interest for backward stochastic differential equations. These equations have a wide range of applications in stochastic control and finance differential equation theory. A particular class of BSDE has been studied for a few years: BSDEs with generators of quadratic growth with respect to the variable $z$, take the following form

$$Y_t = \xi + \int_t^T f(Y_s) \|Z_s\|^2 \, \mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s, t \leq T. \tag{2}$$

Existence and uniqeness of solution for the QBSDE (2) has been first proved by Kobylanski [9]. Since then many authors worked on this equation and they tried to reach the same result in simple ways. for example Bahlali *et al [1]* used the so-called" Zvonkin trasformation" to prove the existence and uniqueness.of solution to such type of equations.

It is worth mentioning that a particularly important class of BSDEs their generators have quadratic growth have a powerful tool in n stochastic finance, and more generally in stochastic control theory. More precisely they arise in the context of utility optimization problems with an exponential utility function, or alternatively in questions related to risk minimization for the entropic risk measure. As an illustration of the theoretical results, we provide the relation between the expected exponential utility and the quadratic backward stochastic differential equations.

This thesis consists of three chapters,

**Chapter 1 (Stochastic calculus and Preliminaries):** This chapter is essentially a kind of introduction, we will present a lots of definitions, properties and theorems made without demonstrations.

**Chapter 2 (Quadratic Backward Stochastic Differential Equations):** The objective of this chapter is to present briefly the result of Pardoux and Peng then by focusing on the QBSDEs and its properties.

**Chapter 3 (Application to a Risk sensitive Control Problem):** In this chapter we demonstrate the relationship between the expected exponential utility and the quadratic backward stochastic differential equations.

# Chapter 1

# Stochastic Calculus and Preliminaries

Stochastic calculus is an extension of differential calculus and classic integration, in which the processes on the continuous-time replace the functions and the martingales play the role of the constants. This chapter is essentially a kind of introduction, to expose the basic notions and the most important definitions properties and theorems that will be used throughout this thesis and they will be provided without demonstrations. For more details on stochastic calculus, we refer the reader to the following important references [5, 8, 11].

In the following $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ is a filtered probability space, $\mathbb{I}$ is a set of indices (which can be $(\mathbb{N}, \mathbb{Z}, \mathbb{R})$) or a part of $\mathbb{R}$ $(e.g : [0, T])$.

## 1.1 Tribe

$\Omega$: is an abstract set whose elements are noted $\omega$.

**Definition 1.1.1** *A tribe ($\sigma-$algebra) on $\Omega$ is a family of parts of $\Omega$, containing the empty set, stable by passing to the complementary, countable union and countable intersection.*

A tribe therefore contains $\Omega$.

A measurable space is a space provided with a tribe, *e.g* : $(\Omega, \mathcal{F})$ , such that $\mathcal{F}$ is a tribe on $\Omega$.

### 1.1.1 Measurability

**Definition 1.1.2** *Let $(\Omega, \mathcal{F})$ and $(E, \xi)$ be two measurable spaces. An application $f$ from $\Omega$ to $E$ is said to be $(\mathcal{F}, \xi)-$measurable if $f^{-1}(A) \in \mathcal{F}$, $\forall A \in \mathcal{F}$, where*

$$f^{-1}(A) := \{\omega \in \Omega \mid f(\omega) \in A\}.$$

### 1.1.2 Generated Tribe

**Definition 1.1.3** *The tribe generated by a family of sets $A$ is the smallest tribe containing this family, we denote it $\sigma(A)$. It is the intersection of all the tribes containing $A$.*

If $\mathcal{F}_1$ and $\mathcal{F}_2$ are two tribes, we denote by $\mathcal{F}_1 \vee \mathcal{F}_2$ the tribe generated by $\mathcal{F}_1 \cup \mathcal{F}_2$ it is the smallest tribe containing the two tribes $\mathcal{F}_1$ and $\mathcal{F}_2$.

**Definition 1.1.4** *The tribe generated by a random variable $X$ defined on $(\Omega, \mathcal{F})$ is the set of parts of $\Omega$ which are written $X^{-1}(A)$ where $A \in \mathcal{B}_{\mathbb{R}}$. We denote this tribe $\sigma(X)$.*

The tribe $\sigma(X)$ is contained in $\mathcal{F}$. It is the smallest tribe on $\Omega$ making $X$ measurable.

**Definition 1.1.5** *The tribe generated by a family random variables $(X_t, t \in \mathbb{I})$ is the smallest tribe containing the sets $\left\{X_t^{-1}(A), \text{for all } t \in \mathbb{I} \text{ and } A \in \mathcal{B}(\mathbb{R})\right\}$. We denote it by $\sigma(X_t, t \in \mathbb{I})$.*

### 1.1.3 Random Variable

**Definition 1.1.6** *An application: $X : \Omega \to \mathbb{R}$ is called a random variable if $X$ is measurable as an application from $(\Omega, \mathcal{F})$ to $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ where $\mathcal{B}(\mathbb{R})$ is the Borel $\sigma-$algebra in $\mathbb{R}$ generated by the set all open intervals in $\mathbb{R}$.*

## 1.2 Probability

**Definition 1.2.1** *A probability on* $(\Omega, \mathcal{F})$ *is an application* $\mathbb{P}$*of* $\mathcal{F}$*in* $[0,1]$ *such that :*

**i)** $\mathbb{P}(\Omega) = 0$.

**ii)** $\mathbb{P}(\cup_{n=0}^{+\infty} A_n) = \sum_{n=0}^{+\infty} \mathbb{P}(A_n)$ such that $\forall n \in \mathbb{N}$ $A_n$ belonging to $\mathcal{F}$ two by two disjoint.

### 1.2.1 Negligible sets

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

**Definition 1.2.2** *A set is said to be negligible set if it has zero probability. We said also that a set* $G$ *is a negligible set if* $\exists M \subset \mathcal{F}$ *such that* $G \subset M$ *and* $\mathbb{P}(M) = 0$.

**Remark 1.2.1**

**i)** A negligible set is not necessary a measurable set.

**ii)** All subset of a negligible set is negligible.

A space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be complete if $\mathcal{F}$ contains all the negligible sets.

## 1.3 Law of probability

**Definition 1.3.1** *Let* $X$ *be a random variable defined on* $(\Omega, \mathcal{F}, \mathbb{P})$. *The law of* $X$ *is the probability* $\mathbb{P}_X$ *on* $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ *defined by* $\mathbb{P}_X(A) = \mathbb{P}\{\omega; X(\omega) \in A\} = \mathbb{P}(X \in A)$, $\forall A \in \mathcal{B}_{\mathbb{R}}$.

### 1.3.1 Expectation

**Definition 1.3.2** *The expectation of a random variable* $X$ *is by definition the quantity* $\int_\Omega X \mathrm{d}P$ *which we denote by* $\mathbb{E}(X)$ *or* $\mathbb{E}_{\mathbb{P}}(X)$ *if we wish to specify what the probability measure used on* $\Omega$.

**Proposition 1.3.1 (Properties of expectation)**

**a)** The expectation is linear, ie: $\mathbb{E}(aX + bY) = a\mathbb{E}(X) + b\mathbb{E}(Y)$, $a$, $b$ being real numbers and $X, Y$ are random variables.

**b)** The expectation is increasing: if $X < Y$ $(a.s)$, we have $\mathbb{E}(X) < \mathbb{E}(Y)$.

**c)** Jensen's inequality: if $\Phi$ is a convex function, such that $\Phi(X)$ is integrable, $\mathbb{E}(\Phi(X)) \geq \Phi(\mathbb{E}(X))$.

## 1.4 Stochastic Process

The notion of stochastic process models natural phenomenons where experiences whose evolution over time depends on chance. It is the equivalence of the notion of random variable for fixed time problems.

**Definition 1.4.1 (Filtration)** *A filtration is an increasing family of sub-tribes of $\mathcal{F}$, that is $\mathcal{F}_t \subset \mathcal{F}_s$, for all $s, t \in \mathbb{I}$ and $t < s$.*

We speak of usual hypotheses if :

- The negligible sets are contained in $\mathcal{F}_0$.

- The filtration is continuous on the right in the sense where $\mathcal{F}_t := \cap_{s>t}\mathcal{F}_s$.

**Definition 1.4.2 (Stochastic Process)** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A stochastic process $X_t = (X_t)_{t\in\mathbb{I}}$ is a family of random variables $X_t$ indexed by a set $\mathbb{I}$. In general $\mathbb{I} = \mathbb{R}^+$ and we consider that the process is indexed with the time $t$.*

**1)** *If the set $\mathbb{I}$ is finite, the process is a random vector.*

**2)** *If $\mathbb{I} = \mathbb{N}$ or $\mathbb{Z}$, the process is a sequence of random variables. In this case, we say that the process is discrete.*

**Remark 1.4.1** **(i)** *The value of the random variable $X_t$ describes the state of the process at time $t$.*

**(ii)** *The time set $\mathbb{I}$ can be defined in other ways as well. For example, $\mathbb{I} := [0, \infty)$ is sufficient time set for a process that has no terminal time.*

**(iii)** *For all $t \in \mathbb{I}$ fixed, $\omega \in \Omega \to X_t(\omega)$ is a random variable on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.*

**(iv)** *For $\omega \in \Omega$ fixed, $t \in \mathbb{I} \to X_t(\omega)$ is a real valued function, called process trajectories.*

**Definition 1.4.3** *A stochastic process $X = (X_t, t > 0)$ is said to be adapted (with respect to a filtration $\mathcal{F} = (\mathcal{F}_t)_{t \in \mathbb{I}}$) if $X_t$ is $\mathcal{F}_t-$measurable for all $t \in \mathbb{I}$*

**Remark 1.4.2**

**a)** We say that the process has continuous trajectories (or is continuous) if the applications $t \to X_t(\omega)$ are continuous for almost all $\omega$.

**b)** To a stochastic process $X$ we associate its natural filtration $\mathcal{F}_t^X$, ie: the growing family of tribes $\mathcal{F}_t^X := \sigma\{X_s, s \leq t\}$ which is the minimum choice for the process to be adapted.

**Definition 1.4.4** *Let $X = (X_t, t \in \mathbb{I})$, $X_t : \Omega \to \mathbb{R}$ be a stochastic process in $(\Omega, \mathcal{F}, \mathbb{P})$ and let $(\mathcal{F}_t)_{t \in \mathbb{I}}$ be a filtration.*

**1)** *The process $X$ is said measurable if the function $X_t : \Omega \times \mathbb{I} \to \mathbb{R}$, $(\omega, t) \to X_t(\omega)$ is $(\mathcal{F} \otimes \mathcal{B}(\mathbb{R}) - \mathcal{B}(\mathbb{R}))-$measurable.*

**2)** *The process $X$ is progressively measurable compared to the filtration $(\mathcal{F}_t)_{t \in \mathbb{I}}$ if $\forall s \in \mathbb{I}$ the function $X_t : \Omega \times \mathbb{I} \to \mathbb{R}$, $(\omega, t) \to X_t(\omega)$ is $(\mathcal{F} \otimes \mathcal{B}([0, s]) - \mathcal{B}(\mathbb{R}))-$measurable, such that $[0, s] \subset \mathbb{I}$.*

**Definition 1.4.5** *Let $X = (X_t)_{t\in\mathbb{I}}$ and $Y = (Y_t)_{t\in\mathbb{I}}$ be stochastic processes. The processes $X$ and $Y$ are*

**i) indistinguishable:** if $P(X_t = Y_t, \forall t \in \mathbb{I}) = 1$, and

**ii) modifications:** of each other if $P(X_t = Y_t) = 1$, for all $t \in \mathbb{I}$. We note that if $X_t$ is a modification of $Y_t$ then $X_t$ and $Y_t$ have the same finite-dimensional distribution.

## 1.5 Conditional Expectation

Let $X$ be a random variable (integrable) defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{G}$ a sub-tribe of $\mathcal{F}$.

**Definition 1.5.1 (conditional expectation with respect to a tribe)** *The conditional expectation $\mathbb{E}(X \mid \mathcal{G})$ of $X$ is the unique $\mathcal{G}-$measurable random variable, such that:*

$$\int_A \mathbb{E}(X \mid \mathcal{G})\mathrm{d}\mathbb{P} = \int_A X\mathrm{d}\mathbb{P}, \forall A \in G.$$

### 1.5.1 Properties of Conditional Expectation

**a)** Let $a$ and $b$ be two constants and $X$,$Y$ be two random variables. Then, $\mathbb{E}(aX + bY \mid \mathcal{G}) = a\mathbb{E}(X \mid \mathcal{G}) + b\mathbb{E}(Y \mid \mathcal{G})$.

**b)** Let $X$ and $Y$ be two random variables such that $X < Y$, then $\mathbb{E}(X \mid \mathcal{G}) < \mathbb{E}(Y \mid \mathcal{G})$.

**c)** $\mathbb{E}[\mathbb{E}(X \mid \mathcal{G})] = \mathbb{E}(X)$.

**d)** If $X$ is $\mathcal{G}-$measurable, $\mathbb{E}(X \mid \mathcal{G}) = X$.

**e)** If $Y$ is $\mathcal{G}-$measurable, $\mathbb{E}(XY \mid \mathcal{G}) = Y\mathbb{E}(X \mid \mathcal{G})$.

**f)** If $X$ is independent of $\mathcal{G}$, $\mathbb{E}(X \mid \mathcal{G}) = \mathbb{E}(X)$.

**g)** If $\mathcal{H}$ and $\mathcal{G}$ are two sub-tribes of $\mathcal{F}$ such that $\mathcal{H} \subset \mathcal{G}$ :

$$\mathbb{E}\left[X \mid \mathcal{H}\right] = \mathbb{E}\left[\mathbb{E}\left(X \mid \mathcal{H}\right) \mid \mathcal{G}\right] = \mathbb{E}\left[\mathbb{E}\left(X \mid \mathcal{G}\right) \mid \mathcal{H}\right].$$

## 1.6 Brownian Motion

**Definition 1.6.1** *A stochastic process $W = (W_t)_{t\in\mathbb{R}^+}$ is called a standard* ***Brownian Motion*** *provided that:*

**i)** $W_0(\omega) = 0$ for all $\omega \in \Omega$.

**ii)** $W$ is continuous.

**iii)** The random variables, $W_{t_n} - W_{t_{n-1}}, ..., W_{t_1} - W_{t_0}$ are independent for all $n \in \mathbb{N}$, $0 \leq t_0 \leq t_1 \leq ... \leq t_n \leq T, T \geq 0$.

**iv)** $W_t - W_s \sim N(0, t-s)$ for all $0 \leq s \leq t \leq T, T \geq 0$.

## 1.7 Martingale

**Definition 1.7.1** *A stochastic process $X = (X_t)_{t\in\mathbb{I}}$ defined in the filtred probabillity space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{I}}, \mathbb{P}\right)$ is called a martingale provided that*

**i)** *$X$ is adapted with respect to $(\mathcal{F}_t)_{t\in\mathbb{I}}$.*

**ii)** *$X$ is integrable ,*

**iii)** *$\mathbb{E}(X_t \mid \mathcal{F}_s) = X_s$ a.s. for all $s, t \in \mathbb{I}$ such that $0 \leq s \leq t$.*

**Lemma 1.7.1** *The standard* ***Brownian Motion*** *$(W_t)_{t\in\mathbb{R}^+}$ is a martingale with respect to its natural filtration $\mathcal{F}_t^X := \sigma\{X_s, s \leq t\}$.*

**Proof. 1.** By Cauchy Schwartz, we have

$$\mathbb{E}\left[|W_t|\right] \leq \sqrt[2]{\mathbb{E}\left[|W_t|^2\right]} = \sqrt[2]{t}.$$

**2.** $\forall 0 \leq s \leq t$, we have

$$\mathbb{E}\left[W_t \mid \mathcal{F}_t^X\right] = \mathbb{E}\left[W_t + W_s - W_s \mid \mathcal{F}_t^X\right] = \mathbb{E}\left[W_t - W_s \mid \mathcal{F}_t^X\right] + \mathbb{E}\left[W_s \mid \mathcal{F}_t^X\right]$$
$$= \mathbb{E}\left[W_s - W_s\right] + W_s = W_s.$$

■

**Example 1.7.1** *iI $X$ is a square integrable radom variable then the process $(X_t)_{t\in\mathbb{I}}$ defined by $X_t = \mathbb{E}(X \mid \mathcal{F}_t)$ is a square integrable martingale.*

**i)** $\mathbb{E}(|X_t|^2) = \mathbb{E}[| \mathbb{E}(X \mid \mathcal{F}_t) |^2] \leq \mathbb{E}[\mathbb{E}(| X |^2 | \mathcal{F}_t)] = \mathbb{E}(| X |^2) < \infty.$

**ii)** $X_t = \mathbb{E}(X \mid \mathcal{F}_t)$ is a random variable $\mathcal{F}_t-$measurable, for all $t \in \mathbb{I}$, according to the definition of the conditional expectation.

**iii)** $\mathbb{E}(X_t \mid \mathcal{F}_s) = \mathbb{E}(\mathbb{E}(X \mid \mathcal{F}_t) \mid \mathcal{F}_s) = \mathbb{E}(\mathbb{E}(X \mid \mathcal{F}_s) \mid \mathcal{F}_t) = \mathbb{E}(X \mid \mathcal{F}_s) = X_s$, for all $s, t \in \mathbb{I}$ such that $0 \leq s \leq t$.

## 1.8 Stochastic Integration

Let $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}\right)$ be a filtered probability space, where $\{\mathcal{F}_t\}_{t\geq 0}$ is a filtration of $\mathcal{F}$ satisfying the usual hypotheses, and $\{W_t, t \geq 0\}$ is a Brownian Motion defined on this probability space.

### 1.8.1 Itô Process

**Definition 1.8.1** *A process $X = (X_t)_{t\in\mathbb{I}}$ is an Itô process if $X_t := X_0 + \int_0^t b_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}W_s$, where $b = (b_s)_{s\in\mathbb{I}}$ is an adapted process and satisfies $\int_0^t |b_s| \,\mathrm{d}s < \infty$ a.s, $\forall t \geq 0$ and $\sigma = (\sigma_s)_{s\in\mathbb{I}} \in \mathbb{L}^2$.*

**Proposition 1.8.1** *The quadratic variation on $\mathbb{I}$ of an Itô process $X$ is given by*

$$\langle X, X\rangle_t = \left\langle \int_0^\cdot \sigma_s \mathrm{d}W_s, \int_0^\cdot \sigma_s \mathrm{d}W_s \right\rangle_t = \int_0^t \sigma_s^2 \mathrm{d}s.$$

The quadratic variation between the following two Itô's processes $X$ and $Y$ :

$$X_t := X_0 + \int_0^t b_s \mathrm{d}s + \int_0^t \sigma_s \mathrm{d}W_s,$$

$$Y_t := Y_0 + \int_0^t b'_s \mathrm{d}s + \int_0^t \sigma'_s \mathrm{d}W_s$$

given by: $\langle X, Y\rangle_t = \int_0^t \sigma_s \sigma'_s \mathrm{d}s$.

### 1.8.2 Itô's Formula

**Theorem 1.8.1**

**a) First Itô's formula:** Let $X$ be an Itô process and $f : \mathbb{R} \to \mathbb{R}$ a function belonging to $C^2$ bounded derivative, then

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)\mathrm{d}X_s + \frac{1}{2}\int_0^t f''(X_s)\sigma_s^2 \mathrm{d}s, \forall t \le T. \tag{1.1}$$

**b) Second Itô's formula:** Let $f$ be a function defined on $\mathbb{R}_+ \times \mathbb{R}$ twice differentiable in $x$ and one time differentiable in $t$ and $X$ be an Itô process

$$f(t, X_t) = f(0, X_0) + \int_0^t f'_x(s, X_s)\mathrm{d}X_s + \int_0^t f'_s(s, X_s)\mathrm{d}s + \frac{1}{2}\int_0^t f''_{xx}(s, X_s) d\langle X, X\rangle_s .$$

## 1.9 Useful results

In the following $W = (W_t)_{t\in\mathbb{I}}$ is a Brownian Motion, and for any $t \in \mathbb{I}$, $\mathcal{F}_t = \sigma(W_s, s \in \mathbb{I}, s < t)$

**Theorem 1.9.1 (Representation Martingal Brownian)** *If $(Y_t)_{t\in\mathbb{I}}$ is a square integrable martingale and $\mathcal{F}_t-$adapted then : $\exists!$ $(Z_t)_{t\in\mathbb{I}}$ a square integrable processes such that:*

$$Y_t = Y_0 + \int_0^t Z_s \mathrm{d}W_s. \tag{1.2}$$

**Theorem 1.9.2 (Burkholder-David-Gundy "BDG" inequality)**

There exist two positive constants $c_p$ and $C_p$ and $p > 0$ such that, for all continuous martingale $X = (X_t)_{t\in\mathbb{I}}$, vanish at $0$ :

$$c_p\mathbb{E}\left[\langle X, X\rangle_\infty^{\frac{p}{2}}\right] \leq \mathbb{E}\left[\sup_{t\geq 0} |X_t|^p\right] \leq C_p\mathbb{E}\left[\langle X, X\rangle_\infty^{\frac{p}{2}}\right].$$

In particular, if $T \geq 0$, we have

$$c_p\mathbb{E}\left[\langle X, X\rangle_T^{\frac{p}{2}}\right] \leq \mathbb{E}\left[\sup_{0\leq t\leq T} |X_t|^p\right] \leq C_p\mathbb{E}\left[\langle X, X\rangle_T^{\frac{p}{2}}\right].$$

**Theorem 1.9.3 (Itô's isometry)** *Let $\theta = (\theta_t)_{t\in\mathbb{R}^+}$ be a stochastic process,*

$$\mathbb{E}\left[\left(\int_0^t \theta_s \mathrm{d}W_s\right)^2\right] = \mathbb{E}\left[\int_0^t \theta^2(u)\mathrm{d}u\right]. \tag{1.3}$$

# Chapter 2

# Quadratic Backward Stochastic Differential Equations

Our concern in this work is to study quadratic BSDEs with and their related to Risk sensitive control problems when the generator is merely continuous and integrable and the terminal condition is square integrable. The main tool is to use the phase space transformation (known as Zvonkin transformation [13]) to eliminate the drift or its quadratic part only. We also provide a comprison theorem between the solutions.

## 2.1 The case of a Lipschitz generator:

Let $T$ be a positive real number, $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P}\right)$ be a filtered probability space in which we define a d-dimensional a Brownian motion. we assume that $\mathcal{F}_t = \sigma\{W_s, s \leq t\}$ is the natural filtration of the Brownian motion $(W_t)_{t\in[0,T]}$.

We will study the one-dimensional BSDE of the following type:

$$Y_t = \xi + \int_t^T g(s, X_s, Z_s)\,\mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s \; t \leq T, \qquad \mathbb{P}-a.s, \tag{2.1}$$

or equivalently in its differential form:

$$\begin{cases} \mathrm{d}Y_s = -g(s, Y_s, Z_s)\mathrm{d}s + Z_s \mathrm{d}W_s \\ Y_T = \xi, \end{cases}$$

where $\xi$ is a square integrable and $\mathcal{F}_T-$measurable random variable called the terminal condition and $g$: $\Omega \times [0,T] \times \mathbb{R} \times \mathbb{R}^{d*d}$ is a given measurable function called the generator

**Definition 2.1.1** *A solution of the equation* (2.1) *is a pair of adapted processes* $(Y_t, Z_t)_{t\in[0,T]}$*, typically in* $\mathbb{S}^2 \times \mathbb{M}^2$ *with values in* $\mathbb{R} \times \mathbb{R}^{d\times d}$.

$Y_t = \mathbb{E}(\xi \mid \mathcal{F}_t)$ and $Z$ from the representation theorem see (1.9.1).

**Assumption(2.1) :**

**i)** $g$ is globally Lipschitz in $(y_t, z_t)_{t\in[0,T]}$, that is: $\exists k > 0$, such that

$$|g(t, y, z) - g(t, y', z')| \leq k(|y - y'| + |z - z'|).$$

**ii)** Integrability condition:

$$E[|\xi|^2 + \int_0^T |g(r, 0, 0)|^2 \,\mathrm{d}r] < \infty.$$

**Theorem 2.1.1 (E.Pardoux, S.Peng)** *Let assumption (2.1) holds. Then,* (2.1) *has a unique solution* $(Y_t, Z_t)_{t\in[0,T]}$ *which belonging to* $\mathbb{S}^2 \times \mathbb{M}^2$.

**Proof.** For the detailed proof we refer the reader to the excellent reference [12]. ■

## 2.2 Quadratic BSDEs with a non-constant $f$:

In this section we want to study the case where : $g(s, Y_s, Z_s) = f(Y_s)\,|Z_s|^2$, where $f$ is supposed to be a continuous function defined from $\mathbb{R}$ to $\mathbb{R}$.

$$Y_t = \xi + \int_t^T f(Y_s)\,|Z_s|^2\,\mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s,\ t \leq T \tag{2.2}$$

**Definition 2.2.1** *A BSDE is called quadratic if its generator has at most a quadratic growth in the random variable $Z$.*

**Lemma 2.2.1** *The function $F$ defined for every $x \in \mathbb{R}$, by*

$$F(x) = \int_0^x \exp\left(2\int_0^y f(t)\mathrm{d}t\right)\mathrm{d}y \tag{2.3}$$

*enjoys the following properties:*

**i)** $F''(x) - 2f(x)F'(x) = 0$, for $a.e.\ x \in \mathbb{R}$

**ii)** $F$ and $F^{-1}$ are quasi–isometry, that is for any $x,\ y \in \mathbb{R}$ and $|f|_1 = \int_{\mathbb{R}} |f(x)|\,\mathrm{d}x$

$$\begin{aligned} e^{-2|f|_1}\,|x-y| \leq |F(x) - F(y)| \leq e^{2|f|_1}\,|x-y|\,, \\ e^{-2|f|_1}\,|x-y| \leq |F^{-1}(x) - F^{-1}(y)| \leq e^{2|f|_1}\,|x-y|\,. \end{aligned} \tag{2.4}$$

**Proof.** By definition the functions $F$ and its inverse $F^{-1}$ are continuous, one to one, strictly increasing functions.

**i)** We have $F(x) = \int_0^x \exp\left(2\int_0^y f(t)\mathrm{d}t\right)\mathrm{d}y$, then $F'(x) = \exp(2\int_0^x f(t)\mathrm{d}t)$ and $F''(x) = 2f(x)F'(x)$,hence $F''(x) - 2f(x)F'(x) = 0$ for $a.e.\ x \in \mathbb{R}$.

**ii)** We have,

$$\left|\left(F^{-1}\right)'(x)\right| = \frac{1}{|F'(x)|} \leq \frac{1}{\exp(-2\int_0^x f(t)\mathrm{d}t)} = \exp(2\int_0^x f(t)\mathrm{d}t) \tag{2.5}$$

then

$$\forall x \in \mathbb{R}, m =: e^{-2|f|_1} \leq (F^{-1})'(x) \leq e^{2|f|_1} := M.$$

From (2.5) we have that

$$|F'(x)| = \frac{1}{\left|(F^{-1})'(x)\right|} = \exp(-2\int_0^x f(t)\mathrm{d}t)$$

Hence

$$\forall x \in \mathbb{R},\ m =:\ e^{-2|f|_1} \leq F'(x) \leq e^{2|f|_1} := M\ .$$

■

**Remark 2.2.1** *1)Let $f : \mathbb{R} \to \mathbb{R}$ be a given bounded and continuous function, and set $M := \sup_{y\in\mathbb{R}} |f(y)|$, the BSDE (2.2) is then of quadratic growth since $f(Y_s)|Z_r|^2 \leq M|Z_r|^2$.*
*2) in the sequale we will denote by $\mathrm{Eq}(\xi, H_f)$ the quadratic BSDE with the generator $H_f(t, y, z) = f(y)|z|^2$ and the terminal condition $\xi$.*

**Theorem 2.2.1** *Let $\xi$ be an $\mathcal{F}_T$-measurable and square integrable random variable. If $f$ is a bounded and continuous function, then $(Y_t, Z_t)_{0\leq t\leq T}$ is a solution to $\mathrm{Eq}(\xi, H_f)$ if and only if $(y_t, z_t)_{0\leq t\leq T}$ is a solution to $\mathrm{Eq}(F(\xi), 0)$.*

**Proof.** If $(Y_t, Z_t)_{t\in[0,T]}$ is a solution of (2.2),then Itô's formula see (1.8.1) applied to $F(Y_t)$ shows that

$$\begin{aligned}
\mathrm{d}F(Y_t) &= F'(Y_t)\mathrm{d}Y_t + \frac{1}{2}F''(Y_t)\mathrm{d}\langle Y_t\rangle \\
&= -F'(Y_t)f(Y_t)|Z_t|^2\,\mathrm{d}t + F'(Y_t)Z_t\mathrm{d}W_t + \frac{1}{2}F''(Y_t)|Z_t|^2\,\mathrm{d}t \\
&= F'(Y_t)Z_t\mathrm{d}W_t + \left[-F'(Y_t)f(Y_t) + \frac{1}{2}F''(Y_t)\right]|Z_t|^2\,\mathrm{d}t \\
&= F'(Y_t)Z_t\mathrm{d}W_t.
\end{aligned}$$

Since $-F'(Y_t)f(Y_t) + \frac{1}{2}F''(Y_t) = 0$, we have

$$y_t = F(Y_t) = F(\xi) - \int_t^T F'(Y_s)Z_s \mathrm{d}W_s, \tag{2.6}$$

by taking the conditional expectation in both sides

$$y_t = \mathbb{E}\left[F(\xi) \mid \mathcal{F}_t\right],$$

If $\xi$ is a square integrable random variable it is easy to see that $F(\xi)$ is also a square integrable random variable,then $y_t$ is a square integrable $\mathcal{F}_t-$martingale. Then according to the representation theorem see (1.9.1) $\exists!$ $(z_t)_{t\in[0,T]}$ a square integrable process such that:

$$y_t = y_0 + \int_0^t z_s \mathrm{d}W_s,$$

so we have

$$y_t = \mathbb{E}\left[F(\xi)\right] + \int_0^t z_s \mathrm{d}W_s,$$

and

$$y_T = \mathbb{E}\left[F(\xi)\right] + \int_0^T z_s \mathrm{d}W_s,$$

so

$$y_T - y_t = \mathbb{E}\left[F(\xi)\right] + \int_0^T z_s \mathrm{d}W_s - \mathbb{E}\left[F(\xi)\right] - \int_0^t z_s \mathrm{d}W_s$$

Hence

$$y_t = \xi - \int_t^T z_s \mathrm{d}W_s \tag{2.7}$$

By matching between (2.6) and (2.7) we remark that $F'(Y_s)Z_s = z_s$.

Since $\xi$ is a square integrable random variable and the generator is vanish(Lipschitz) then according to (2.1.1) the equation (2.7) admits a unique solution $(y_t, z_t)_{t\in[0,T]} \in \mathbb{S}^2\times\mathbb{M}^2$.

Reciprocally by applying Ito's formula to $F^{-1}(y_t)$ we find that

$$\mathrm{d}F^{-1}(y_t) = (F^{-1})'(y_t)\mathrm{d}y_t + \frac{1}{2}(F^{-1})''(y_t)\mathrm{d}\left\langle y_t \right\rangle,$$

but we have $(F^{-1})'(y_t) = \frac{1}{F'(F^{-1}(y_t))}$ and $(F^{-1})''(y_t) = \frac{-F''(F^{-1}(y_t))}{(F'(F^{-1}(y_t)))^3}$ so

$$\mathrm{d}F^{-1}(y_t) = \frac{1}{F'(F^{-1}(y_t))} z_t \mathrm{d}W_t - \frac{1}{2}(\frac{F''(F^{-1}(y_t))}{(F'(F^{-1}(y_t)))^3} \left|z_t\right|^2 \mathrm{d}t),$$

since $F(Y_t) = y_t$ and $z_t = F'(Y_t)Z_t$, we have that

$$\begin{aligned}
\mathrm{d}Y_t &= \mathrm{d}F^{-1}(F(Y_t)) \\
&= \frac{1}{F'(F^{-1}(F(Y_t)))} F'(Y_t)Z_t\mathrm{d}W_t - \frac{1}{2}(\frac{F''(F^{-1}(F(Y_t)))}{(F'(F^{-1}(F(Y_t))))^3}(F'(Y_t))^2 \left|Z_t\right|^2 \mathrm{d}t) \\
&= \frac{1}{F'(Y_t)} F'(Y_t)Z_t\mathrm{d}W_t - \frac{1}{2}(\frac{F''(Y_t)}{(F'(Y_t))^3}(F'(Y_t))^2 \left|Z_t\right|^2 \mathrm{d}t) \\
&= Z_t\mathrm{d}W_t - \frac{1}{2}\left(\frac{F''(Y_t)}{F'(Y_t)}\right)\left|Z_t\right|^2 \mathrm{d}t \\
&= Z_t\mathrm{d}W_t - \frac{1}{2}(2f(Y_t))\left|Z_t\right|^2 \mathrm{d}t \\
&= Z_t\mathrm{d}W_t - f(Y_t)\left|Z_t\right|^2 \mathrm{d}t,
\end{aligned}$$

and

$$Y_T = F^{-1}(y_T) = F^{-1}(\zeta') = F^{-1}(F(\zeta)) = \xi.$$

Hence

$$Y_t = \xi + \int_t^T f(Y_s)\left|Z_s\right|^2 \mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s.$$

■

## 2.3 Quadratic BSDEs with a constant $f$:

We want to solve the following quadratic BSDE

$$Y_t = \xi + \int_t^T \left( l\left(t\right) + \frac{\gamma}{2} \left|Z_s\right|^2 \right) \mathrm{d}s - \int_t^T Z_s \mathrm{d}W_s,$$

where $l : [0, T] \to \mathbb{R}$ is a bounded function,

**Theorem 2.3.1** *Let $\xi$ be an $\mathcal{F}_T$-measurable and square integrable random variable. Then $(Y_t, Z_t)_{0 \le t \le T}$ is a solution to* $\mathrm{Eq}(\xi, l\left(t\right) + \frac{\gamma}{2} \left|Z_s\right|^2)$ *if and only if $(y_t, z_t)_{0 \le t \le T}$ is a solution to* $\mathrm{Eq}(\exp(\gamma \xi), 0)$.

**Proof.** By applying Itô's formula see (1.8.1) to $y_t = \exp(\gamma Y_t)$:

$$\begin{aligned}
\mathrm{d}y_t &= \mathrm{d}\left(\exp(\gamma Y_t)\right) \\
&= \gamma \exp(\gamma Y_t) \mathrm{d}Y_t + \frac{\gamma^2}{2} \exp(\gamma Y_t) \left|Z_t\right|^2 \mathrm{d}t \\
&= \gamma \exp(\gamma Y_t)(Z_t \mathrm{d}W_t - \left( l(t) + \frac{\gamma}{2} \left|Z_t\right| \right) \mathrm{d}t) + \frac{\gamma^2}{2} \exp(\gamma Y_t) \left|Z_t\right|^2 \mathrm{d}t \\
&= \gamma \exp(\gamma Y_t) Z_t \mathrm{d}W_t - \frac{\gamma^2}{2} \exp(\gamma Y_t) \left|Z_t\right|^2 \mathrm{d}t - \gamma \exp(\gamma Y_t) l(t) \mathrm{d}t + \frac{\gamma^2}{2} \exp(\gamma Y_t) \left|Z_t\right|^2 \mathrm{d}t \\
&= -\gamma \exp(\gamma Y_t) l(t) \mathrm{d}t + \gamma \exp(\gamma Y_t) Z_t \mathrm{d}W_t,
\end{aligned}$$

So we have

$$y_t = \exp(\gamma \xi) + \int_t^T \gamma y_s l(s) \mathrm{d}s - \int_t^T \gamma y_s Z_s \mathrm{d}W_s.$$

We put $z_t = \gamma y_s Z_s$

$$y_t = \exp(\gamma \xi) + \int_t^T \gamma y_s l(s) \mathrm{d}s - \int_t^T z_s \mathrm{d}W_s, \tag{2.8}$$

since $\xi$ is a square integrable random variable then $\exp(\gamma\xi)$,and $l$ is bounded ie: the generator of $BSDE$ (2.8) is Lipschitz in $y$. Hence according to (2.1.1) the $BSDE$ (2.8) has one and only one solution $(y_t, z_t)_{t \in [0,T]}$ .

Then, (2.8) admits a unique solution $(y_t, z_t)_{t \in [0,T]}$ because it has a Lipschitz generator.

Reciprocally, by applying Ito's formula see (1.8.1) to $F^{-1}(y_t) = \frac{1}{\gamma}\ln(y_t)$ we find that:

$$\begin{aligned}
\mathrm{d}Y_t &= \mathrm{d}(\frac{1}{\gamma}\ln(y_t)) \\
&= (\frac{1}{\gamma}\ln(y_t))'\mathrm{d}y_t + \frac{1}{2}(\frac{\gamma}{2}\ln(y_t))''\mathrm{d}\left\langle y_t\right\rangle \\
&= \frac{1}{\gamma y_t}(-\gamma y_t l(t)\mathrm{d}t + z_t\mathrm{d}W_t) + \frac{1}{2}(\frac{-\gamma}{\gamma^2 y_t^2}\left|z_t\right|^2)\mathrm{d}t \\
&= \frac{1}{\gamma y_t}(-\gamma y_t l(t)\mathrm{d}t + \gamma y_t Z_t\mathrm{d}W_t) + \frac{1}{2}(\frac{-\gamma}{\gamma^2 y_t^2}\left|\gamma y_t Z_t\right|^2)\mathrm{d}t \\
&= -l(t)\mathrm{d}t + Z_t\mathrm{d}W_t - \frac{\gamma}{2}\left|Z_t\right|^2\mathrm{d}t \\
&= -\left(l(t) + \frac{\gamma}{2}\left|Z_t\right|^2\right)\mathrm{d}t + Z_t\mathrm{d}W_t.
\end{aligned}$$

Because we have that $z_t = \gamma y_t Z_t$,

$$\begin{aligned}
\mathrm{d}Y_t &= -\left(l(t) + \frac{\gamma}{2}\left|Z_t\right|^2\right)\mathrm{d}t + Z_t\mathrm{d}W_t \\
Y_T &= \frac{1}{\gamma}\ln y_T = \frac{1}{\gamma}\ln(\exp(\gamma\xi)) = \xi
\end{aligned}$$

Then

$$Y_t = \xi + \int_t^T \left(l\left(t\right) + \frac{\gamma}{2}\left|Z_s\right|^2\right)\mathrm{d}s - \int_t^T Z_s\mathrm{d}W_s. \tag{2.9}$$

According to (2.1.1), the BSDE (2.9) admits a unique solution $(Y_t, Z_t)_{t\in[0,T]}$ . ■

## 2.4 Comparison Theorem

The following proposition allows to compare the solutions for QBSDEs of type $\mathrm{Eq}(\xi, f)$. The novelty is that the comparison of solutions holds whenever we can compare the generators *a.e.* in the $y$–variable. Moreover, both the generators can be non–Lipschitz.

**Proposition 2.4.1 (Comparison)** : *Let $\xi_1$, $\xi_2$ be $\mathcal{F}_T$–measurable and square integrable random variables. Let $f_1$, $f_2$ be elements of $\mathbb{L}^1(\mathbb{R})$. Let $(Y^{f_1}, Z^{f_1})$, $(Y^{f_2}, Z^{f_2})$ be respectively the solution of* $\mathrm{Eq}(\xi_1, H_{f_1})$ *and* $\mathrm{Eq}(\xi_2, H_{f_2})$. *Assume that $\xi_1 \le \xi_2$ a.s, and $f_1 \le f_2$ a.e.*

*Then* $Y_t^{f_1} \leq Y_t^{f_2}$ $\mathbb{P}$*–a.s.*

**Proof.** Notice that the solutions $(Y^{f_1}, Z^{f_1})$ and $(Y^{f_2}, Z^{f_2})$ belong to $\mathbb{S}^2 \times \mathbb{M}^2$ For a given function $h$, we put,

$$F_h(x) := \int_0^x \exp\left(2\int_0^y h(t)\mathrm{d}t\right)\mathrm{d}y.$$

We first apply Itô's formula see (1.8.1) to $F_{f_1}(Y_t^{f_2})$, to obtain:

$$\begin{aligned}
F_{f_1}(Y_T^{f_2}) &= F_{f_1}(Y_t^{f_2}) + \int_t^T F'_{f_1}(Y_s^{f_2})\mathrm{d}Y_s^{f_2} + \frac{1}{2}\int_t^T F''_{f_1}(Y^{f_2})\mathrm{d}\left\langle Y_t^{f_2}\right\rangle \\
&= F_{f_1}(Y_t^{f_2}) - \int_t^T F'_{f_1}(Y_s^{f_2}) f_2(Y_s^{f_2})|Z_s^{f_2}|^2\mathrm{d}s + \int_t^T F'_{f_1}(Y_s^{f_2})Z_s^{f_2}\mathrm{d}W_s \\
&+ \frac{1}{2}\int_t^T F''_{f_1}(Y_s^{f_2})\left|Z_s^{f_2}\right|^2\mathrm{d}s \\
&= F_{f_1}(Y_t^{f_2}) - \int_t^T F'_{f_1}(Y_s^{f_2}) f_2(Y_s^{f_2})|Z_s^{f_2}|^2\mathrm{d}s \\
&+ (M_T - M_t) + \frac{1}{2}\int_t^T F''_{f_1}(Y_s^{f_2})\left|Z_s^{f_2}\right|^2\mathrm{d}s
\end{aligned}$$

where

$$M_t = \int_0^t F'_{f_1}(Y_s^{f_2})Z_s^{f_2}\mathrm{d}W_s,$$

is an $\mathcal{F}_t$–martingale.

According to Lemma (2.2.1) we obtain

$$F_{f_1}(Y_T^{f_2}) = F_{f_1}(Y_t^{f_2}) + (M_T - M_t) - \int_t^T \left(F'_{f_1}(Y_{s-}^{f_2})\left(f_2(Y_s^{f_2}) - f_1(Y_s^{f_2})\right)|Z_s^{f_2}|^2\right)\mathrm{d}s.$$

Since the term $\int_t^T F'_{f_1}(Y_{s-}^{f_2})(f_2(Y_s^{f_2}) - f_1(Y_s^{f_2}))|Z_s^{f_2}|^2\mathrm{d}s$ is positive, then

$$F_{f_1}(Y_t^{f_2}) \geq F_{f_1}(Y_T^{f_2}) - (M_T - M_t).$$

Passing to conditional expectation and using the fact that $F_{f_1}$ is an increasing function and $\xi_2 \geq \xi_1$, we get

$$\begin{aligned} F_{f_1}(Y_t^{f_2}) &\geq \mathbb{E}\left[F_{f_1}(Y_T^{f_2})\mathcal{F}_t\right] = \mathbb{E}\left[F_{f_1}(\xi_2)\mathcal{F}_t\right] \\ &\geq \mathbb{E}\left[F_{f_1}(\xi_1)\mathcal{F}_t\right] = F_{f_1}(Y_t^{f_1}). \end{aligned}$$

Taking $F_{f_1}^{-1}$ in both sides, we conclude $Y_t^{f_2} \geq Y_t^{f_1}$. Proposition (2.4.1) is proved. ■

## 2.5 A priori estimates

**Lemma 2.5.1** *Let* $\xi \in L^2(\Omega)$*. If* $(Y, Z)$ *satisfies the* $\mathrm{Eq}(\xi, H_f)$*, then we have:*

**(i)** $(z_r)_{0\leq r\leq T}$ , $(Z_r)_{0\leq r\leq T} \in \mathbb{M}^2$ ,

**(ii)** $(y_r)_{0\leq r\leq T}$ , $(Y_r)_{0\leq r\leq T} \in \mathbb{S}^2$,

**(iii)** $\mathbb{E}\left|\int_0^T f(Y_r)\left|Z_r\right|^2 \mathrm{d}r\right|^2$ is finite.

**Proof. (i):** From Itô's formula (1.8.1) we have

$$F(Y_t) = F(\xi) - \int_t^T F'(Y_s)Z_s \mathrm{d}W_s, \tag{2.10}$$

since $F$ satisfies (2.2.1). For $t = 0$ we get

$$\int_0^T F'(Y_s)Z_s \mathrm{d}W_s = F(\xi) - F(Y_0). \tag{2.11}$$

Take the square of the $\mathbb{L}^2(\Omega)$ norm in (2.11), (2.4), we get

$$\begin{aligned} m^2\left(\int_0^T \mathbb{E}\left[|Z_s|^2\right] \mathrm{d}s\right) &\le \mathbb{E}\left|\int_0^T F'(Y_s)Z_s \mathrm{d}W_s\right|^2 \\ &= \mathbb{E}\left|\int_0^T z_s \mathrm{d}W_s\right|^2 \\ &\le \mathbb{E}\int_0^T |z_s|^2 \mathrm{d}s \\ &\le 2(\mathbb{E}[F^2(Y_0)] + \mathbb{E}\left[F^2(\xi)\right]) \le 2M^2(\mathbb{E}\,|Y_0|^2 + \mathbb{E}\,|\xi|^2) < \infty. \end{aligned}$$

This implies that $z,\ Z \in \mathbb{M}^2$.

**(ii):** From Itô's formula (1.8.1), we have

$$F(Y_t) = F(\xi) - \int_t^T F'(Y_s)Z_s \mathrm{d}W_s \tag{2.12}$$

Now, thanks to (2.4) and the fact that $F\,(0) = 0$.

$$\begin{aligned} m\,|Y_t| &\le |F(Y_t)| \\ &\le |F(\xi)| + \left|\int_t^T F'(Y_s)Z_s \mathrm{d}W_s\right| \\ &\le |F(\xi)| + \sup_{0\le t\le T}\left|\int_0^t z_s \mathrm{d}W_s\right|. \end{aligned}$$

Using convex inequality and taking the supremum over $[0, T]$ lead to

$$\begin{aligned} m^2 \sup_{0\le t\le T} |Y_t|^2 &\le \sup_{0\le t\le T} |y_t|^2 \\ &\le 2^2\left(M\,|\xi|^2 + \sup_{0\le t\le T}\left|\int_0^t z_s \mathrm{d}W_s\right|^2\right). \end{aligned}$$

Now, by taking the expectation and using BDG inequality see(1.9.2), we get

The right hand side of the above inequality is finite by (i).

**(iii):** Since$(Y, Z)$ satisfies Eq$(\xi, f)$, thus

$$\int_0^T f(Y_s)\,|Z_s|^2\,\mathrm{d}s = \int_0^T Z_s \mathrm{d}W_s + Y_0 - \xi.$$

Now, using convex inequality and taking the expectation we obtain

$$\begin{aligned}
\mathbb{E}\left|\int_0^T f(Y_s)\,|Z_s|^2\,\mathrm{d}s\right|^2 &\leq 4(\mathbb{E}\left|\int_0^T Z_s \mathrm{d}W_s\right|^2 + |Y_0|^2 + \mathbb{E}\,|\xi|^2) \\
&\leq 4(\mathbb{E}\int_0^T |Z_s|^2\,\mathrm{d}s + |Y_0|^2 + \mathbb{E}\,|\xi|^2).
\end{aligned}$$

Finally $\mathbb{E}\left|\int_0^T f(Y_s)\,|Z_s|^2\,\mathrm{d}s\right|^2$ is finite thanks to (i). ■

# Chapter 3

# Application to a Risk sensitive Control Problem

In this chapter we will pay our attention to the application of the theoretical results that have been shown and proved in the previous chapter to a risk sensitive control problem. More precisely, we will prove the relationship between the $QBSDEs$ and the expected exponential utility function.

## 3.1 Problem formulation

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t^W)_{t\in[0,T]}, \mathbb{P})$ be a probability space satisfying the usual conditions, in which a one-dimensional Brownian motion $W = (W_t, 0 \leq t \leq T)$ is defined.
We assume that $\mathbb{F} :=(\mathcal{F}_t^W)_{t\in[0,T]}$ is defined by$\forall t \geq 0, \mathcal{F}_t^W = \sigma(W_s,$for any $s \in [0,T]) \vee \mathcal{N}$, where $\mathcal{N}$ denote the totality of $\mathbb{P}-$null sets.
Let $\mathbb{M}^2([0,T], \mathbb{R})$ denote the set of one-dimensional jointly measurable random, processes $\{\varphi_t, t \in [0,T]\}$ which satisfy the following conditions:

**(i)** : $\varphi \in \mathbb{M}^2([0,T], \mathbb{R})$.

**(ii)** : $\varphi_t$ is $\mathcal{F}_t^W-$measurable for any $t \in [0,T]$.

We denote similarly by $\mathbb{S}^2([0,T],\mathbb{R})$ the set of continuous one-dimensional random processes that satisfy the following conditions:

**(i)** : $\varphi \in \mathbb{S}^2([0,T],\mathbb{R})$.

**(ii)** : $\varphi_t$ is $\mathcal{F}_t^W-$measurable for any $t \in [0,T]$.

Let the process $v(\cdot)$ stand for the control variable, which assumed to be an $\mathbb{F}$-adapted process that takes values in a given non-empty subset $U$ of $\mathbb{R}$. We denote the set of all admissible controls by $\mathcal{U}_{ad}$.

## 3.2 Expected Exponential Utility

In this part, we want to prove the relationship between the expected exponential and the quadratic backward stochastic differential equation .

We require the following condition

$$A_{t,T}^{\theta} := \exp \theta \left\{ \Psi(y_0^{\vartheta}) + \int_t^T l(s, y_s^{\vartheta}) \mathrm{d}s \right\}, \tag{3.1}$$

where $l : [0,T] \times U \to \mathbb{R}, \Psi : \mathbb{R} \to \mathbb{R}$

We assume the following:

**(N1)**

**i)** $\Psi$ is continuously differentiable with respect to $(y^v, v)$.

**ii)** The derivative of $\Psi$ is bounded by $C(1 + |y^v|)$.

**iii)** $l$ is a bounded function.

**iv)** The derivative of $\Psi$ is bounded by $C(1 + |y^v|)$.

We denote by $l\,(t) = (t, v_t)$.

We set

$$Y_t^\theta = \Psi(y_0^v) + \int_t^T l(s)\mathrm{d}s$$

Then

$$\exp(\theta Y_t^\theta) = \mathbb{E}[A_{t,T}^\theta \mid \mathcal{F}_t^W], \tag{3.2}$$

where $\theta$ is the risk-sensitive index, the process $Y^\theta$ is the first component of the $\mathcal{F}_t^W-$adapted pair of processes $\left(Y_t^\theta, Z_t\right)_{t\in[0,T]}$ which is the unique solution of the following quadratic backward stochastic differential equation according to the result of the equation (2.9)

$$\begin{cases} \mathrm{d}Y_t^\theta = \left(l\,(t) - \frac{\theta}{2}\left|Z_t\right|^2 \mathrm{d}t\right) + Z_t \mathrm{d}W_t, \\ Y_T^\theta = \Psi(y_0^u), \end{cases}, \tag{3.3}$$

where $\mathbb{E}[\int_0^T \left|Z_t\right|^2 \mathrm{d}t] < \infty$.

We also assume the following

**(N2)**

**(i)** The process $Z = (Z_t)_{t\in[0,T]}$ is $\mathcal{F}_t^W-$measurable with value in $\mathbb{R}$ such that $\mathbb{E}[\int_0^T \left|Z_t\right|^2 \mathrm{d}t] < \infty$.

**(ii)** The process $(Y_t^\theta)_{t\geq 0}$ is $\mathbb{P}-$measurable uniformly bounded i.e. there exists a constant $C \geq 0$ such that $\mathbb{P} - a.s$, $\mathbb{E}[\sup_{0\leq t\leq T}\left|Y_t^\theta\right|] \leq C$.

The following lemma shows the relationship between the expected exponential utility and the quadratic backward stochastic differential equation.

**Lemma 3.2.1** *We assume that N1-N2 hold. The necessary and sufficient condition for the expected exponential utility* (3.1) *to be hold, is the quadratic backward stochastic differential equation* (3.3)*.*

**Proof.** We assume that (3.2) holds, then we have

$$\begin{aligned}\exp\left\{\theta Y_0^\theta\right\} &= \mathbb{E}[\exp\left(\theta\Psi(y_0^u)\right) \mid \mathcal{F}_t^W] \\ &= \mathbb{E}[\exp\theta\left\{\Psi(y_0^u) + \int_0^T l(s)\mathrm{d}s\right\} \mid \mathcal{F}_t^W] \\ &= \mathbb{E}[A_{0,T}^\theta \mid \mathcal{F}_t^W].\end{aligned}$$

By assumption (N1), we know that $A_{0,T}^\theta$ is a square integrable and $\mathbb{E}[A_{0,T}^\theta \mid \mathcal{F}_t^W]$ is a square integrable martingale, adapted to the Brownian filtration $\mathcal{F}_t^W = \sigma(W_s,\text{for any } s \in [0,T])$, then by using the martingale representation theorem, there exist a unique square process $\varphi$ such that

$$\mathbb{E}[A_{0,T}^\theta \mid \mathcal{F}_t^W] - \mathbb{E}[A_{0,T}^\theta \mid \mathcal{F}_0] = \int_0^t \varphi(s)\mathrm{d}W_s.$$

We have that $\mathbb{E}[A_{0,T}^\theta] = \mathbb{E}[A_{0,T}^\theta \mid \mathcal{F}_0] = \exp\theta\{Y_0^\theta\}$,so

$$\exp\theta\{Y_t^\theta + \int_0^t l(s)\mathrm{d}s\} - \exp\theta\{Y_0^\theta\} = \int_0^t \varphi(s)\mathrm{d}W_s.$$

By applying Itô's formula to $\exp\theta\{Y_t^\theta + \int_0^t l(s)\mathrm{d}s\}$, we obtain

$$\begin{aligned}&\mathrm{d}\left(\exp\theta\{Y_t^\theta + \int_0^t l(s)\mathrm{d}s\}\right) \\ &= \theta l(t)\exp\theta\{Y_t^\theta + \int_0^t l(s)\mathrm{d}s\}\mathrm{d}t + \theta\exp\theta\{Y_t^\theta + \int_0^t l(s)\mathrm{d}s\}\mathrm{d}Y_t^\theta + \frac{\theta^2}{2}\exp\theta\{Y_t^\theta + \int_0^t l(s)\mathrm{d}s\}\left\langle \mathrm{d}Y_t^\theta, \mathrm{d}Y_t^\theta\right\rangle \\ &= \varphi(t)\mathrm{d}W_t.\end{aligned}$$

Then

$$l(t)\mathrm{d}t + \mathrm{d}Y_t^\theta + \frac{\theta}{2}\left\langle \mathrm{d}Y_t^\theta, \mathrm{d}Y_t^\theta\right\rangle = \frac{1}{\theta}\varphi(t)\exp\theta\left\{-Y_t^\theta - \int_0^t l(s)\mathrm{d}s\right\}\mathrm{d}W_t. \tag{3.4}$$

And so

$$\mathrm{d}Y_t^\theta = -l(t)\mathrm{d}t - \frac{\theta}{2}\left\langle \mathrm{d}Y_t^\theta, \mathrm{d}Y_t^\theta\right\rangle + \frac{1}{\theta}\varphi(t)\exp\theta\left\{-Y_t^\theta - \int_0^t l(s)\mathrm{d}s\right\}\mathrm{d}W_t.$$

Hence,

$$\left\langle \mathrm{d}Y_t^\theta, \mathrm{d}Y_t^\theta \right\rangle = [\frac{1}{\theta}\varphi(t)\exp\theta\left\{-Y_t^\theta - \int_0^t l(s)\mathrm{d}s\right\}]^2\mathrm{d}t := |Z_t|^2\,\mathrm{d}t. \tag{3.5}$$

Then by replacing (3.5) in (3.4), we have the quadratic backward stochastic differential equation as the following expression

$$\begin{cases} \mathrm{d}Y_t^\theta = -\left(l(t) + \frac{\theta}{2}\left|Z_t\right|^2\right)\mathrm{d}t + Z_t\mathrm{d}W_t, \\ Y_T^\theta = \Psi(y_0^u), \end{cases}$$

where

$$Z_t = \frac{1}{\theta}\varphi(t)\exp\theta\left\{-Y_t^\theta - \int_0^t l(s)\mathrm{d}s\right\}.$$

On the other hand, we assume that (3.3) holds,and by applying Itô's formula to $\exp(\theta Y_t^\theta)$, we get

$$\mathrm{d}(\exp\theta\{Y_t^\theta\}) + \theta l(t)\exp\theta\{Y_t^\theta\}\mathrm{d}t = \theta Z_t\exp\theta\{Y_t^\theta\}\mathrm{d}W_t$$

Multiply with $\exp\theta\left\{\int_0^t l(s)\mathrm{d}s\right\}$ to both sides, we get

$$\begin{aligned}&\exp\theta\left\{\int_0^t l(s)\mathrm{d}s\right\}\mathrm{d}(\exp\theta\{Y_t^\theta\}) + \theta l(t)\exp\theta\left\{\int_0^t l(s)\mathrm{d}s\right\}\exp\theta\{Y_t^\theta\}\mathrm{d}t \\ &= \theta Z_t\exp\theta\left\{\int_0^t l(s)\mathrm{d}s\right\}\exp\theta\{Y_t^\theta\}\mathrm{d}W_t.\end{aligned}$$

The right side is the same as the $\mathrm{d}\left(\exp\theta\{Y_t^\theta + \int_0^t l(s)\mathrm{d}s\}\right)$ , then we have

$$\mathrm{d}\left(\exp\theta\{Y_t^\theta + \int_0^t l(s)\mathrm{d}s\}\right) = \theta Z_t\exp\theta\{Y_t^\theta + \int_0^t l(s)\mathrm{d}s\}\mathrm{d}W_t.$$

By taking the integral from $t$ to$T$ in both sides of the previous equality, we have

$$\int_t^T \mathrm{d}\left(\exp\theta\{Y_s^\theta + \int_0^s l(r)\mathrm{d}r\}\right) = \theta\int_t^T Z_s\exp\theta\{Y_s^\theta + \int_0^s l(r)\mathrm{d}r\}\mathrm{d}W_s.$$

Then

$$\exp\theta\{Y_T^\theta + \int_0^T l(r)\mathrm{d}r\} = \exp\theta\{Y_t^\theta + \int_0^t l(r)\mathrm{d}r\} + \theta\int_t^T Z_s \exp\theta\{Y_s^\theta + \int_0^s l(r)\mathrm{d}r\}\mathrm{d}W_s.$$

By taking conditional expectation in above equality, we have

$$\begin{gathered}\mathbb{E}[\exp\theta\{Y_T^\theta + \int_0^T l(r)\mathrm{d}r\} \mid \mathcal{F}_t^W] = \\ \mathbb{E}[\exp\theta\{Y_t^\theta + \int_0^t l(r)\mathrm{d}r\} \mid \mathcal{F}_t^W] + \theta\mathbb{E}[\int_t^T Z_s \exp\theta\{Y_s^\theta + \int_0^s l(r)\mathrm{d}r\}\mathrm{d}W_s \mid \mathcal{F}_t^W],\end{gathered}$$

such that $\mathbb{E}[\int_t^T Z_s \exp\theta\{Y_s^\theta + \int_0^s l(r)\mathrm{d}r\}\mathrm{d}W_s \mid \mathcal{F}_t^W] = 0$, then

$$\mathbb{E}[\exp\theta\{Y_T^\theta + \int_0^T l(r)\mathrm{d}r\} \mid \mathcal{F}_t^W] = \exp\theta\{Y_t^\theta + \int_0^t l(r)\mathrm{d}r\}.$$

As we know that $Y_T^\theta = \Psi(y_0^u)$, we can write

$$\mathbb{E}[\exp\theta\{\Psi(y_0^u) + \int_t^T l(s)\mathrm{d}s\} \mid \mathcal{F}_t^W] = \exp\theta\{Y_t^\theta\}.$$

■

# Chapter 4

# Conclusion

In this thesis we are interested into the quadratic backward stochastic differential equations. The main results provided within this work is the study the problem of Existence and uniqueness for a class of BSDEs their generators are quadratic with respect to Brownian component. In fact, we have studied tow different cases, in the first one, we have considered that the factor of the quadratic term is a general continuous and integrable function, while in the second one it is assumed to be constant which is not integrable even in the Riemann sense. Finally, we have established an application to a risk sensitive which explains the relationship between the QBSDEs and the expected exponential utility function which has proven in different method in the article of Hamdene and El karoui [10].

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# الملخص

*الهدف من هذا العمل هو دراسة صنف من المعادلات التفاضلية العشوائية التراجعية التربيعية من الشكل التالي :*

$$Y_t = \xi + \int_t^T \left( l(s) + f(Y_s) |Z_s|^2 \right) ds - \int_t^T Z_s dW_s,$$

*بحيث الشرط النهائي $\xi$ يفرض أن يكون تكامله مربع منته،f و l دالتان قابلتان للقياس، درسنا وجود و وحدانية الحل و نظرية المقارنة بالنسبة لهذه المعادلات. الأداة الرئيسية المستخدمة في البراهين هي تحويل زفانكن للتخلص من المولد أو جزء منه، بحيث حولنا المعادلات التفاضلية العشوائية التراجعية التربيعية إلى معادلات تفاضلية عشوائية تراجعية كلاسيكية بدون الجزء التربيعي ، كتطبيق قمنا بإثبات العلاقة بين المعادلات التفاضلية العشوائية التراجعية التربيعية و مشكلة السيطرة على الخطر الحساس .*

***الكلمات المفتاحية:*** *المعادلات التفاضلية العشوائية التراجعية، المعادلات التفاضلية العشوائية التراجعية التربيعية، الخطر الحساس ، وظيفة الأداة المساعدة الأسية، وجود و وحدانية الحل.*



# Résumé

*Le but de ce travail est d'étudier une classe des EDSR quadratique de la forme suivante :*

$$Y_t = \xi + \int_t^T \left( l(s) + f(Y_s) |Z_s|^2 \right) ds - \int_t^T Z_s dW_s,$$

*où la donnée terminale $\xi$ est une variable aléatoire de carrée intégrable, l, f sont deux fonctions mesurables, nous étudions l'existence, l'unicité et le principe de comparaison pour ce genre des équations. L'outil principal dans les preuves est ce qu'on appelle la transformation de Zvonkin qui sera utilisé pour éliminer le générateur ou une partie de celui-ci, de sorte que nous transformons l'EDSR quadratique originale à une EDSR standard son générateur ne contient pas la partie quadratique. Une application est également a été étudiée pour nous présenter e lien entre les EDSRs quadratiques et le problème du contrôle au risque sensible.*

***Mots clés***: *équations différentielles stochastiques rétrogrades, équations différentielles stochastiques rétrogrades quadratique, fonction d'utilité exponentielle, risque sensible, existence et unicité*



# Abstract

*The aim of this work is to study a class of quadratic BSDEs, of the following form:*

$$Y_t = \xi + \int_t^T \left( l(s) + f(Y_s) |Z_s|^2 \right) ds - \int_t^T Z_s dW_s,$$

*Where the terminal data is assumed to be a square integrable random variable, f and l are two measurable functions. We study the existence, uniqueness and comparison theorem to such equations. The main tool in the proofs is the so called Zvonkin transformation that will be used to eliminate the generator or a part of it, so that we transform the original QBSDEs to a standard BSDE without a quadratic part. As an application, we provide the connection between the quadratic BSDE and the risk sensitive control problem.*

***Key words***: *backward stochastic differential equations, Quadratic backward stochastic differential equations, exponential utility function, risk sensitive, existence and uniqueness of solution.*