People's Democratic Republic of Algeria

Ministry of Higher Education and Scientific Research

MOHAMED KHIDER UNIVERSITY, BISKRA

FACULTY of EXACT SCIENCES, SIENCE of NATURE and LIFE

DEPARTMENT of MATHEMATICS

A thesis submitted for the fulfillment of the requirements of :

The Doctorate Degree in Mathematics

Option : **Probability**

By

Hanane BEN GHERBAL

Title :

Some contributions to the problems of stochastic control of diffusions with jumps

Members of the jury :

Mokhtar	Hafayed	Pr.	University of	Biskra	President
Brahim	Mezerdi	Pr.	University of	Biskra	Supervisor
Khaled	Bahlali	Dr.	University of	Toulon	Examiner
Nabil	Khelfallah	Dr.	University of	Biskra	Examiner
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To my family., especially to my little angel Lina

Abstract

This thesis studies optimal control of systems driven by stochastic differential equations (SDEs), with jump processes, where the control variable appears in the drift and the jump term. We study the relaxed problem for which admissible controls are measure-valued processes and the state variable is governed by an SDE driven by a counting measure valued process which we call relaxed Poisson measure such that the compensator is a product measure. Under some conditions on the coefficients, we prove that every diffusion process associated to a relaxed control is a limit of a sequence of diffusion processes associated to strict controls. And we show that the strict and the relaxed control problems have the same value function. The existence of an optimal relaxed control is a consequence of the development. Moreover we derive a maximum principle for this type of relaxed problem. In second step, we study optimal control problem of the same type of SDEs defined in the first one, but the control variable has two components, the first being absolutely continuous and the second singular. Our goal is to establish a stochastic maximum principle for relaxed controls for this type of relaxed problem, using strong perturbation on the absolutely continuous part of the control and a convex perturbation on the singular one. The proofs are based on the strict maximum principle, Ekeland's variational principle, and some stability properties of the trajectories and adjoint processes with respect to the control variable.

Résumé

Cette thèse étudie un contrôle optimal des systèmes gouvernés par des équations différentielles stochastiques (EDSs), avec des processus de saut, où la variable de contrôle apparaîsse dans le drift et le terme de saut. On étude les problèmes relaxés pour lesquels les contrôles admissibles sont des processus à valeurs mesures et la variable d'état est gouverné par une EDS conduite par un processus dont ces valeurs sont des mesures de comptage, ce qu'on appelle mesure de Poisson relaxé de telle sorte que le compensateur est une mesure produit. Sous certaines conditions sur les coefficients, on prouve que tous les processus de diffusion associés à un contrôle relaxé est une limite d'une suite des processus de diffusion associés à une suite des contrôles stricts. On montre que le problème de contrôle strict et le problème de contrôle relaxé ont la même fonction de valeur. L'existence d'un contrôle optimal relaxé est une conséquence de développement. En outre, on démontre un principe de maximum pour ce type de problème relaxé.

Dans la deuxième étape, nous étudions un problème de contrôle optimal du même type de SDEs définis dans la première, mais la variable de contrôle comporte deux composants, le premier étant absolument continu et le second singulier. Notre objectif est d'établir un principe du maximum pour ce type de problème relaxé, en utilisant une forte perturbation sur la partie absolument continue du contrôle et une perturbation convexe sur le singulier. Les preuves sont basées sur le principe du maximum strict, le principe variationnel d'Ekeland et certaines propriétés de stabilité des trajectoires et des processus adjoints par rapport à la variable de contrôle.

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Symbols and Abbreviations

The different symbols and abbreviations used in this thesis.

$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$	:	A filtered probability space.
SDE	:	A stochastic differential equation.
N	:	A Poisson random measure.
$\upsilon(d\theta)dt$	:	The compensator of N.
$\widetilde{N}$	:	The compensated Poisson measure.
$\delta_{\cdot}(da)$	:	The Dirac measure.
μ	:	The relaxed control.
$N^{\mu}(t, \theta, a)$	:	The relaxed Poisson random measure.
$\mu_t \otimes \upsilon(da, d\theta)$	:	The compensator of relaxed Poisson random measure N^{μ}.
$\widetilde{N}^{\mu}$	:	The compensated relaxed Poisson measure.
A	:	The set of values taken by the strict control u.
U	:	The set of admissible strict controls.
V	:	The space of positive Radon measures on $[0;1] \times A$
$\overline{V}$	:	$\begin{cases} \text{the smallest } \sigma-\text{field such that the mappings} \\ \mu \to \int\limits_0^1 \int\limits_A \phi(t,u)\mu_t(du)dt \text{ are measurable,where } \phi \text{ is a bounded} \\ \text{measurable function which is continuous in } a. \end{cases}$

$(\overline{V}_t)$	:	The filtration generated by $\left\{1_{[0;t]}\mu,\ \mu \in V\right\}$.
$P(A)$	:	The space of probability measures equipped with the topology of weak convergence.
$J(.)$	:	The cost function.
$J(.,.)$	:	The cost function associated with the singular problem.
u^*	:	Optimal strict control.
μ^*	:	Optimal relaxed control.
ζ	:	Singular control.
(u^*, ζ^*)	:	Optimal strict-singular control.
(μ^*, ζ^*)	:	Optimal relaxed-singular control.
H	:	The Hamiltonian.
(p, q, r)	:	Adjoint processes.
$\mathcal{R}$	:	The set of relaxed controls.
S	:	The set of rapidly decreasing functions.
S'	:	The topological dual of the Schwartz space of S.
$D_{S'}$	:	The space of all mappings càdlàg from $[0;T]$ with values in S'.
$(p^{\mu^*}, q^{\mu^*}, r^{\mu^*})$	:	The adjoint processes associated with the relaxed control problem.
$A_1 \times A_2$	:	The set of values taken by the strict-singular controls (u, ζ).
$\mathcal{U} = U_1 \times U_2$	:	The set of admissible strict-singular controls.
$\mathcal{R}^s = \mathcal{R}_1 \times U_2$	:	The set of admissible relaxed-singular controls.

Contents

General introduction

We consider a control problem where the state variable is a solution of a stochastic differential equation (SDE), in which the control enters the drift and the jump term. More precisely the system evolves according to the SDE

$$\begin{cases} dx_t & = b(t, x_t, u_t)dt + \sigma(t, x_t)dB_t + \int\limits_{\Gamma} f(t, x_{t^-}, \theta, u_t)\widetilde{N}(dt, d\theta) \\ x_0 & = 0, \end{cases}$$

on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$, where b, σ, and f are given deterministic functions, $(\mathcal{F}_t)_{t\geq 0}$ is the filtration governed by a standard Brownian motion B and an independent Poisson random measure N, whose compensator is given by $\upsilon(d\theta)dt$ and u stands for the control variable.

The expected cost to be minimized over the class of admissible controls is defined by

$$J(u) = E\left[g(x_T) + \int\limits_0^T h(t, x_t, u_t)dt\right].$$

A control process that solves this problem is called optimal. The strict control problem may fail to have an optimal solution, if we don't impose some kind of convexity assumption. In this case, we must embed the space of strict controls into a larger space that has nice properties of compactness and convexity. This space is that of probability measures on A, where A is the set of values taken by the strict control. These measure valued processes are called relaxed controls. The first existence result of an optimal relaxed control is proved by Fleming [14], for the SDEs with uncontrolled diffusion coefficient and no jump term. For such systems of SDEs a maximum principle has been established in [2, 3, 26]. For mean-field systems one can refer to [4, 5, 6]. The case where the control variable appears in the diffusion coefficient has been solved in [13]. The existence of an optimal relaxed control of SDEs, where the control variable enters in the jump term was derived by Kushner [23].

In this thesis, we first show that under a continuity condition of the coefficients, each relaxed diffusion process with controlled jump is a strong limit of a sequence of diffusion processes associated with strict controls. The proof of this approximation result is based on Skorokhod selection theorem, and the tightness of the processes. Consequently, we show that the strict and the relaxed control problems have the same value function. Using the same techniques, we give another proof of the existence of an optimal relaxed control, based on the Skorokhod selection theorem.

The second main goal of this part is to establish a Pontriagin maximum principle for the relaxed control problem. More precisely we derive necessary conditions for optimality satisfied by an optimal control. The proof is based on Pontriagin's maximum principle for nearly optimal strict controls and some stability results of trajectories and adjoint processes with respect to the control variable.

In second step, we consider mixed relaxed-singular stochastic control problems of systems governed by stochastic differential equations of the same type of SDEs defined in the forth chapter, but the control variable has two components, the first being measure valued process and the second singular. More precisely the system evolves according to the SDE

$$\begin{cases} dx_t^\mu &= \int\limits_{A_1} b(t,x_t^\mu,a)\mu_t(da)dt + \sigma(t,x_t^\mu)dB_t + \int\limits_{A_1}\int\limits_{\Gamma} f(t,x_{t^-}^\mu,\theta,a)\widetilde{N}^\mu(dt,d\theta,da) + G_t d\zeta_t \\ x_0^\mu &= 0, \end{cases} \tag{1}$$

on $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},P)$, such that $\mathcal{F}_0$ contains the $P-$null sets, We assume that $(\mathcal{F}_t)_{t\geq 0}$ is generated by a standard Brownian motion B and an independent Poisson random measure $\widetilde{N}^{\mu}$ defined in first section of chapter 4, which its compensator has the form $\mu_t \otimes \upsilon(da, d\theta)$, where μ is the relaxed control and υ is the compensator of Poisson measure N. The control variable is (μ, ζ), where μ is a $P(A_1)-$valued process, progressively measurable with respect to $(\mathcal{F}_t)_{t\geq 0}$ and $\zeta : [0; T] \times \Omega \longrightarrow A_2$ is of bounded variation, nondecreasing left-continuous with right limits and $\zeta_0 = 0$.

The expected cost to be minimized over the class of admissible controls has the form

$$J(\mu, \zeta) = E\left[g(x_T^{\mu}) + \int_{A_1}\int_0^T h(t, x_t^{\mu}, a)\mu_t(da)dt + \int_0^T k_t d\zeta_t\right].$$

A control process that solves this problem is called optimal.

Singular control problems without jump have been studied by many authors including Beněs, Sheep, and Witsenhausen [7], Chow, Menaldi, and Robin [11], Karatzas and Shreve [21], Davis and Norman [12], and Haussmann and Suo [15],[16],[17]. The approaches used in these papers are mainly based on dynamic programming. The first version of the stochastic maximum principle that covers singular control problems was obtained by Cadenillas and Haussman [10] for linear systems. Second order necessary conditions for optimality for nonlinear SDEs with a controlled diffusions matrix were obtained by S. Bahlali and B. Mezerdi [26], extending the Peng maximum principle to singular control problems. The stochastic maximum principle for relaxed-singular control problem is studied by S. Bahlali, B. Djehiche, and B. Mezerdi [2], where the proofs are based on the strict maximum principle, Ekeland's variational principle, and some stability properties of the trajectories and adjoint process with respect to the control variable.

Our main goal is to extended the result of S. Bahlali, B. Djehiche, and B. Mezerdi [2] to the problem where the system evolves according to the SDE (1), by the same techniques that used in the previous chapters, and using a strong perturbation of the absolutely continuous part of the control and a convex perturbation of the singular part.

In the first chapter, we introduce the general notions of stochastic calculus with jump diffusion.

Second chapter contains the relaxed control problem.

The third chapter gives the stochastic maximum principle of controlled jump diffusion.

In the forth chapter we will introduce the relaxed control problem of our system, and we will establish the stochastic maximum principle of our problem.

Finally, in the fifth chapter we will state and prove a stochastic maximum principle of our relaxed-singular control problem.

Chapter 1

Stochastic calculus with jump diffusion

In this chapter, we present the basic concepts and results needed for the applied calculus of jump diffusion, and we refer to the two books [27] and [28] for more information and more detailed proofs.

1.1 The Poisson process

Definition 1.1 (Counting process) *The process $N = (N_t)_{t\geq 0}$ defined by*

$$N_t = \sum_{n\geq 1} \mathbf{1}_{\{T_n \leq t\}},$$

with values in $\mathbb{N} \cup \{\infty\}$, where $(T_n)_{n\geq 0}$ is a strictly increasing sequence of positive random variables (with $T_0 = 0$), is called counting process associated to the sequence $(T_n)_{n\geq 1}$.

Remark 1.1 *Set $T = \sup_n T_n$, if $T = \infty$ a.s, then N is a counting process without explosions. Indeed,*

$$[T_n; \infty) = \{N \geq n\} = \{(t, \omega) \ : N_t(\omega) \geq n\},$$

as well as

$$[T_n; T_{n+1}) = \{N = n\}\,,$$

and

$$[T; \infty) = \{N = \infty\}\,.$$

Theorem 1.1 *A counting process N is adapted if and only if the associated random variables $(T_n)_{n\geq 1}$ are stopping times.*

Proof. If N is adapted, then $\{T_n \leq t\} = \{N_t \geq n\} \in \mathcal{F}_t$ for each t, then T_n is a stopping time. Mutually, if $(T_n)_{n\geq 0}$ are stopping times with $T_0 = 0$ $a.s$, then

$$\{N_t = n\} = \{\omega : T_n(\omega) \leq t < T_{n+1}(\omega)\}\,,$$

then,

$$\{N_t = n\} = \{\omega : T_n(\omega) \leq t\} \cap \{\omega : T_{n+1}(\omega) \leq t\}^c\,,$$

as we have $\{\omega : T_n(\omega) \leq t\} \in \mathcal{F}_t$ and $\{\omega : T_{n+1}(\omega) \leq t\}^c \in \mathcal{F}_t$, then $\{N_t = n\} \in \mathcal{F}_t$ for each n, and therefore N is adapted. ■

Definition 1.2 (Poisson process) *An adapted counting process N is a Poisson process if :*

1. *For any s, t, $0 \leq s < t < \infty$, $N_t - N_s$ is independent of $\mathcal{F}_s$,*

2. *For any s, t, u, v, $0 \leq s < t < \infty$; $0 \leq u < v < \infty$, $t - s = u - v$, then the distribution of $N_t - N_s$ is the same as that of $N_u - N_v$.*

Theorem 1.2 *Let N be a Poisson process. Then, the random variable N_t has the Poisson distribution with parameter λt, for some $\lambda \geq 0$. That is*

$$P(N_t = n) = \frac{(\lambda t)^n}{n!} \exp(-\lambda t);\ \ n \in \mathbb{N},$$

λ is called the intensity of N.

Proof. See [28] ■

Corollary 1.1 *A Poisson process N with intensity λ satisfies*

$$E[N_t] = Var[N_t] = \lambda t.$$

Theorem 1.3 *Let N be a Poisson process with intensity λ. Then, $(N_t - \lambda t)_{t\geq 0}$ and $\left((N_t - \lambda t)^2 - \lambda t\right)_{t\geq 0}$ are martingales.*

Proof.

- Since λt is non-random, then

  $$E[N_t - \lambda t] = E[N_t] - \lambda t = 0,$$

  and

  $$E[(N_t - \lambda t) - (N_s - \lambda s) \mid \mathcal{F}_s] = E[N_t - N_s \mid \mathcal{F}_s] - \lambda t + \lambda s.$$

  Since N has an independent increments, we have for $0 \leq s < t < \infty$,

  $$E[N_t - N_s \mid \mathcal{F}_s] - \lambda t + \lambda s = E[N_t - N_s] - \lambda t + \lambda s = 0,$$

  which implies that $(N_t - \lambda t)_{t\geq 0}$ is a martingale.

- The analogous statement holds for $\left((N_t - \lambda t)^2 - \lambda t\right)_{t\geq 0}$.

■

1.2 Lévy process

Definition 1.3 (Lévy process) *The process π defined on a filtered probability space*

$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ *is called a Lévy process if*

1. $\pi_0 = 0$ *a.s,*

2. π *has stationary and independent increments,*

3. π *is continuous in probability :*

$$\lim_{h\to 0} P\left(|\pi_{t+h} - \pi_t| > \varepsilon\right) = 0.$$

Exemple 1.1 1. *It is clear that the Poisson process introduced in the previous section is a Lévy process.*

2. *We know that the Brownian motion initiated from the origin, and has an independent and stationary increments, moreover has right continuous paths with left limits. Then it is a Lévy process.*

Remark 1.2 *These two processes are different because Brownian motion has continuous paths, whereas a Poisson process does not. And a Poisson process is a non-decreasing process, and thus has paths of bounded variation over finite time horizons, whereas a Brownian motion does not have monotone paths and its paths are of unbounded variation over finite horizons.*

Theorem 1.4 *Let* π *be a Lévy process, then* π *has a càdlàg version which is also a Lévy process.*

Proof. See [28] ■

Now, consider only the Lévy processes càdlàg

Definition 1.4 (jump of Lévy process) *If* π *is a Lévy process, we define the jump process of* π *by* $\Delta\pi = (\Delta\pi_t)_{t\geq 0}$ *with* $\Delta\pi_t = \pi_t - \pi_{t^-}$.

Definition 1.5 *Let $\mathcal{A}_0$ be the family of Borel sets $A \subset \mathbb{R}$ whose closure $\overline{A}$ does not contain* 0*, Fix $A \in \mathcal{A}_0$, we define*

$$N(\omega, t, A) = N(t, A) = \sum_{0 \leq s \leq t} \mathbf{1}_A(\Delta\pi_s) = \sum_{n \geq 1} \mathbf{1}_{\left\{T_A^n \leq t\right\}},$$

is a counting process without explosion.

Theorem 1.5 *1. The set function $A \to N(t, A, \omega)$ defines a $\sigma-$finite measure on $\mathcal{A}_0$ for each fixed (t, ω).*

2. The set function

$$\upsilon(A) = E\left[N(1, A)\right] = \sum_{0 \leq s \leq 1} \mathbf{1}_A(\Delta\pi_s)$$

also defines a $\sigma-$finite measure on $\mathcal{A}_0$ called Lévy measure of π.

3. Fix $A \in \mathcal{A}_0$, then the process $(N(t, A, \omega))_{t \geq 0}$ is a Poisson process of intensity $\upsilon(A)$.

Proof.

1. The set function $A \to N(t, A, \omega)$ is a counting measure because it represent the number of jumps of size $\Delta\pi_s \in A$, which occur before or at t.

2. By the proof of (1), it is clear that υ is also a measure.

3. For $0 \leq s < t < \infty$, $N(t, A) - N(s, A) \in \sigma\left\{\pi_u - \pi_v;\ s \leq v < u \leq t\right\}$, then $N(t, A) - N(s, A)$ is independent of $\mathcal{F}_s$; That is $N(., A)$ has an independent increments. And we have

   $$N(t, A) - N(s, A) = \sum_{0 \leq u \leq t-s} 1_A(\pi_{s+u} - \pi_s);$$

   by the stationarity of the distributions of π, we can conclude that $N(t, A) - N(s, A)$ has the same distribution as $N(t - s, A)$.

   Therefore $N(., A)$ is a counting process with stationary and independent increments; then, $N(., A)$ is a Poisson process.

■

Definition 1.6 (Poisson random measure) *Let* $\mathcal{A}_0$ *be the family of Borel sets* $A \subset \mathbb{R}$ *whose closure* $\overline{A}$ *does not contain* 0. *Fix* $A \in \mathcal{A}_0$, *we define*

$$A \to N(\omega, t, A) = N(t, A) = \sum_{0 \leq s \leq t} \mathbf{1}_A(\Delta \pi_s),$$

which is represent the number of jumps size $\Delta \pi_s \in A$, *is called Poisson random measure of* π. *The differential form of this measure is written as* $N(dt, d\theta)$.

1.2.1 Stochastic integral with respect to N

Theorem 1.6 *Let* $A \in \mathcal{A}_0$, *and let* f *be Borel and finite on* A. *Then*

$$\int_A f(\theta) N(t, d\theta) = \sum_{0 \leq s \leq t} f(\Delta \pi_s) \mathbf{1}_A(\Delta \pi_s)$$

is a Lévy process.

Proof. It is a consequence of the fact that $N(., A)$ has an independent and stationary increments. ■

Definition 1.7 (Jump process) *For a given set* $A \in \mathcal{A}_0$, *we define the associated jump process by*

$$J(t, A) = \sum_{0 \leq s \leq t} \Delta \pi_s \mathbf{1}_A(\Delta \pi_s) = \int_A \theta N(t, d\theta).$$

Remark 1.3 $J(t, A)$ *is a Lévy process itself.*

Theorem 1.7 *Given a set* $A \in \mathcal{A}_0$, *the process* $(\pi_t - J(t, A))_{t \geq 0}$ *is a Lévy process.*

Proof. See [28] ■

Theorem 1.8 *Let $A \in \mathcal{A}_0$, and υ be the Lévy measure of π, and $f\mathbf{1}_A \in L^2(d\upsilon)$. Then,*

$$E\left[\int_A f(\theta)N(t,d\theta)\right] = t\int_A f(\theta)\upsilon(d\theta),$$

and

$$E\left[\int_A f(\theta)N(t,d\theta) - t\int_A f(\theta)\upsilon(d\theta)\right]^2 = t\int_A f^2(\theta)\upsilon(d\theta).$$

Proof. [28] First let f be a simple function, that is $f = \sum_i a_i 1_{A_i}$. Then

$$E\left[\sum_i a_i N(t,A_i)\right] = \sum_i a_i E\left(N(t,A_i)\right),$$

since $N(t,A_i)$ is a Poisson process with parameter $\upsilon(A_i)$, then

$$E\left[\sum_i a_i N(t,A_i)\right] = t\sum_i a_i \upsilon(A_i),$$

hence the first equality follows. For the second one, let $\widetilde{N}_t^i = N(t,A_i) - t\upsilon(A_i)$. The $\widetilde{N}_t^i$ are L^p martingales, for all $p \geq 1$, see theorem 34 in [28]. Moreover, $E\left[\widetilde{N}_t^i\right] = 0$. Suppose A_i, A_j are disjoint, we have

$$E\left[\widetilde{N}_t^i\widetilde{N}_t^j\right] = E\left[\sum_k\left(\widetilde{N}_{t_{k+1}}^i - \widetilde{N}_{t_k}^i\right)\sum_l\left(\widetilde{N}_{t_{l+1}}^j - \widetilde{N}_{t_l}^j\right)\right],$$

for any partition $0 = t_0 < t_1 < ... < t_n = t$. Using the martingale property, we have

$$E\left[\widetilde{N}_t^i\widetilde{N}_t^j\right] = E\left[\sum_k\left(\widetilde{N}_{t_{k+1}}^i - \widetilde{N}_{t_k}^i\right)\left(\widetilde{N}_{t_{k+1}}^j - \widetilde{N}_{t_k}^j\right)\right],$$

then, by the inequality $|ab| \leq a^2 + b^2$, we have

$$E\left[\widetilde{N}_t^i\widetilde{N}_t^j\right] \leq \sum_k\left(\widetilde{N}_{t_{k+1}}^i - \widetilde{N}_{t_k}^i\right)^2 + \sum_k\left(\widetilde{N}_{t_{k+1}}^j - \widetilde{N}_{t_k}^j\right)^2.$$

However, $\sum_k \left(\widetilde{N}^i_{t_{k+1}} - \widetilde{N}^i_{t_k}\right)^2 \leq N^2(t, A_i) + t^2\upsilon^2(A_i)$; therefore the sums are dominated by an integrable random variable. Since $\widetilde{N}^i_t$ and $\widetilde{N}^j_t$ have paths of finite variation on $[0;t]$ it is easy to deduce that if we take a sequence $(t_n)_{n\geq 1}$ of partitions where the mesh tends to 0 we have

$$\lim_{n\to\infty} \sum_{t_k, t_{k+1} \in (t_n)_{n\geq 1}} \left(\widetilde{N}^i_{t_{k+1}} - \widetilde{N}^i_{t_k}\right)\left(\widetilde{N}^j_{t_{k+1}} - \widetilde{N}^j_{t_k}\right) = \sum_{0<s\leq t} \Delta\widetilde{N}^i_s \Delta\widetilde{N}^j_s.$$

Now, using the Lebesgue's dominated convergence theorem, and since A_i, A_j are disjoint; this implies that $\widetilde{N}^i$ and $\widetilde{N}^j$ jump at different times. Then, we can conclude that

$$E\left[\widetilde{N}^i_t \widetilde{N}^j_t\right] = E\left[\sum_{0<s\leq t} \Delta\widetilde{N}^i_s \Delta\widetilde{N}^j_s\right] = 0. \tag{1.1}$$

Hence,

$$\begin{aligned} Var\left(\int_A f(\theta)N(t,d\theta)\right) &= E\left(\int_A f(\theta)N(t,d\theta) - t\int_A f(\theta)\upsilon(d\theta)\right)^2 \\ &= E\left[\sum_i a_i \widetilde{N}(t, A_i)\right]^2 = \sum_i a_i^2 E(\widetilde{N}^i_t)^2 + \sum_i a_i a_j E(\widetilde{N}^i_t \widetilde{N}^j_t), \end{aligned}$$

by the equality (1.1), we can deduce that

$$E\left[\sum_i a_i \widetilde{N}(t, A_i)\right]^2 = \sum_i a_i^2 t\upsilon_i(d\theta).$$

Then, the second equality is verify for simple functions.

For general f, let f_n be a sequence of simple functions such that $f_n\, 1_A$ converges to f in $L^2(d\upsilon)$, and the result follows. ■

Corollary 1.2 *For a set $A \in \mathcal{A}_0$, the process $\left(N(t, A) = \int_A N(t, d\theta)\right)_{t\geq 0}$ is a Poisson process with parameter $\upsilon(A)$, and $\left(\widetilde{N}(t, d\theta) = N(t, d\theta) - t\upsilon(d\theta)\right)_{t\geq 0}$ is a martingale, is called compensated Poisson process.*

1.2.2 Itô-Lévy process

Theorem 1.9 (Lévy decomposition) *Let $\pi = (\pi_t)_{t\geq 0}$ be a Lévy process. Then π has the decomposition*

$$\pi_t = bt + \sigma B_t + \int_{|\theta|<R} \theta \widetilde{N}(t, d\theta) + \int_{|\theta|\geq R} \theta N(t, d\theta) \tag{1.2}$$

for some $b, \sigma \in \mathbb{R}$, and where B is a Brownian motion independent of $N(t, A)$.

Definition 1.8 (Itô-Lévy process) *As a consequence of the decomposition (1.2), the integral with respect to $d\pi_t$ can be split into integrals with respect to ds, dB_s, $\widetilde{N}(ds, d\theta)$ and $N(ds, d\theta)$. That is*

$$x_t = x_0 + \int_0^t b(s)ds + \int_0^t \sigma(s)dB_s + \int_0^t \int_{|\theta|<R} f(s,\theta)\widetilde{N}(ds, d\theta) + \int_0^t \int_{|\theta|\geq R} f(s,\theta)N(ds, d\theta).$$

The differential notation of processes x is

$$dx_t = b(s)ds + \sigma(s)dB_s + \int_{|\theta|<R} f(s,\theta)\widetilde{N}(ds, d\theta) + \int_{|\theta|\geq R} f(s,\theta)N(ds, d\theta). \tag{1.3}$$

We call such processes Itô-Lévy processes.

1.3 Itô's formula for Itô-Lévy processes

In this section, we introduce the important Itô's formula for Itô-Lévy processes. For more information see [27].

Theorem 1.10 (Itô's formula) *Let $x = (x_t)_{t\geq 0}$ be a process of the form (1.3). Define* *the process* $y_t = g(t, x_t)$*, with* $g \in \mathbb{C}^{1,2}(\mathbb{R}^2)$*, then* $(y_t)_{t\geq 0}$ *is an Itô-Lévy process, and*

$$\begin{aligned} dy_t &= \tfrac{\partial g}{\partial t}(t,x_t)dt + \tfrac{\partial g}{\partial x}(t,x_t)\left[b(t)dt + \sigma(t)dB_t\right] + \tfrac{1}{2}\sigma^2(t)\tfrac{\partial^2 g}{\partial x^2}(t,x_t)dt \\ &+ \int\limits_{|\theta|<R} \left\{g(t,x_{t^-} + f(s,\theta)) - g(t,x_{t^-}) - \tfrac{\partial g}{\partial x}(t,x_{t^-})f(s,\theta)\right\} \upsilon(d\theta)dt \\ &+ \int\limits_{|\theta|<R} \left\{g\left(t,x_{t^-} + f(s,\theta)\right) - g(t,x_{t^-})\right\} \widetilde{N}(dt,d\theta) \\ &+ \int\limits_{|\theta|\geq R} \left\{g\left(t,x_{t^-} + f(s,\theta)\right) - g(t,x_{t^-})\right\} N(dt,d\theta). \end{aligned}$$

1.4 Stochastic differential equation driven by a Lévy process

Consider the following Lévy SDE in $\mathbb{R}^n$,

$$\begin{cases} dx_t &= b(s,x_t)ds + \sigma(s,x_t)dB_s + \int\limits_{\mathbb{R}^n} f(s,x_{t^-},\theta)\widetilde{N}(ds,d\theta) \\ x_0 &= 0. \end{cases}$$

Where

$$b : [0;T] \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

$$\sigma : [0;T] \times \mathbb{R}^n \longrightarrow \mathbb{R}^{n+m}$$

$$f : [0;T] \times \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{R}^{n+l}.$$

Theorem 1.11 (Existence and uniqueness of solutions of Lévy SDEs) *[30] Assume that* $b, \sigma,$ *and* f *satisfying*

1. *There exist a constant* $C_1 < \infty$ *such that*

$$\|\sigma(t,x)\|^2 + |b(t,x)|^2 + \int\limits_{\mathbb{R}} \sum_{i=1}^{l} |f_i(t,x,\theta)|^2 \upsilon_i(d\theta_i) \leq C_1\left(1 + |\theta|^2\right), \; \textit{for all } \theta \in \mathbb{R}^n.$$

2. *There exist a constant $C_2 < \infty$ such that*

$$\|\sigma(t,x)-\sigma(t,y)\|^2+|b(t,x)-b(t,y)|^2+\sum_{i=1}^{l}\int_{\mathbb{R}}|f_i(t,x,\theta_i)-f_i(t,y,\theta_i)|^2\,\upsilon_i(d\theta_i)\le C_2\,|x-y|^2,$$

for all $x,y\in\mathbb{R}^n$. Then, there exist a unique càdlàg adapted solution $(x_t)_{t\ge 0}$ such that $E\,|x_t|^2<\infty$, for all t.

Proof. See [30] ■

Exemple 1.2 (The geometric process) *[27] Consider the following stochastic differential equation*

$$dx_t = x_{t^-}\left[\alpha dt+\beta dB_t+\int_{|\theta|<R} f(t,\theta)\widetilde{N}(dt,d\theta)+\int_{|\theta|\ge R} f(t,\theta)N(dt,d\theta)\right],$$

where $\alpha,\beta\in\mathbb{R}$ and $f(t,\theta)\ge -1$. To find the solution, we define $y_t=\ln(x_t)$, then by Itô's formula (see theorem (1.10)), we get

$$\begin{aligned} dy_t &= \left(\alpha-\tfrac{1}{2}\beta^2\right)dt+\beta dB_t+\int_{|\theta|<R}\{\ln[1+f(t,\theta)]-f(t,\theta)\}\,\upsilon(d\theta)dt \\ &\quad +\int_{|\theta|<R}\ln[1+f(t,\theta)]\,\widetilde{N}(dt,d\theta)+\int_{|\theta|\ge R}\ln[1+f(t,\theta)]\,N(dt,d\theta), \end{aligned}$$

hence

$$\begin{aligned} y_t &= y_0+\left(\alpha-\tfrac{1}{2}\beta^2\right)t+\beta B_t+\int_0^t\int_{|\theta|<R}\{\ln[1+f(t,\theta)]-f(t,\theta)\}\,\upsilon(d\theta)dt \\ &\quad +\int_0^t\int_{|\theta|<R}\ln[1+f(t,\theta)]\,\widetilde{N}(dt,d\theta)+\int_0^t\int_{|\theta|\ge R}\ln[1+f(t,\theta)]\,N(dt,d\theta), \end{aligned}$$

this gives the solution

$$\begin{aligned} x_t &= x_0 \exp\left[\left(\alpha - \tfrac{1}{2}\beta^2\right) t + \beta B_t + \int_0^t \int_{|\theta|<R} \left\{\ln\left[1 + f(t,\theta)\right] - f(t,\theta)\right\} \upsilon(d\theta)dt \right. \\ &\left. + \int_0^t \int_{|\theta|<R} \ln\left[1 + f(t,\theta)\right] \widetilde{N}(dt,d\theta) + \int_0^t \int_{|\theta|\geq R} \ln\left[1 + f(t,\theta)\right] N(dt,d\theta)\right]. \end{aligned}$$

1.5 Relaxed control problem

We know that in stochastic control theory, and in the absence of additional hypotheses of convexity on the coefficients, the optimal stochastic control problem does not have a solution. For that, we should inject the space of strict controls in a wider space that has good properties of compactness and convexity. This space is that of probability measures on A, where A is the set of values taken by the strict control. In this new space, controls called relaxed controls. For more details see ([25]) and ([26]), Before defining the notion of relaxed stochastic control, we begin with an example for which an optimal solution in the strict control space does not exist.

Exemple 1.3 *Let a system that evolves according to the following SDE*

$$\begin{cases} dx_t^u = u_t dt \\ x_0 = 0. \end{cases}$$

where $u : [0;1] \rightarrow \{-1, 1\}$, *and the cost*

$$J(u) = \int_0^1 (x_t^u)^2 dt.$$

If we consider $u_t^n = (-1)^k$, *with* $\frac{k}{n} \leq t \leq \frac{k+1}{n}$; $0 \leq k \leq n-1$. *Then,* $\forall t \in [0;1]$, $|x_t^{u_n}| \leq \frac{1}{n}$, *then* $J(u^n) \leq \frac{1}{n^2}$ *which implies that* $\inf\limits_u J(u) = 0$. *But there is not u such that* $J(u) = 0$, *because* $x_t^u = 0, \forall t \in [0;1]$ *if and only if* $u_t = 0$ *which is impossible. The trouble is the fact that the sequence* $(u^n)_n$ *has not a limit in the space of strict controls. So we look* *for a space in which this limit exists. If we identify* u_t^n *with the Dirac measure, then* $\delta_{u_t^n}(du) = \mu_n(t, du)$ *is a sequence of measures over the space* $[0;1] \times U$. *converges weakly to* $\mu(t, du) = \frac{1}{2}\left[\delta_1 + \delta_{-1}\right] du$. *Indeed, if we take a continuous function* f *on* $[0;1] \times U$, *one has*

$$\int\limits_{[0;1]\times U} f(t,u)\mu_n(dt,du) = \sum_{i=0}^{n-1} \int\limits_{\frac{i}{n}}^{\frac{i+1}{n}} f(t,(-1)^i)dt.$$

Suppose first that is even number , that is $n = 2m$. *Let* $\varepsilon > 0$, *there is an* $M > 0$ *such that* $\forall m \geq M$,

$$|f(t,u) - f(s,u)| < \varepsilon \text{ if } |t-s| < \frac{1}{m},$$

where u *is either* 1 *and* -1. *Fix* $m \geq M$, *then for every* $j = 0, ..., m-1$,

$$\left| \int\limits_{\frac{2j}{2m}}^{\frac{2j+1}{2m}} f(t,u)dt - \int\limits_{\frac{2j+1}{2m}}^{\frac{2j+2}{2m}} f(t,u)dt \right| < \frac{\varepsilon}{2m},$$

then

$$\left| \sum_{j=0}^{m-1} \int\limits_{\frac{2j}{2m}}^{\frac{2j+1}{2m}} f(t,u)dt - \sum_{j=0}^{m-1} \int\limits_{\frac{2j+1}{2m}}^{\frac{2j+2}{2m}} f(t,u)dt \right| < \frac{\varepsilon}{2},$$

therefore,

$$\left| \sum_{j=0}^{m-1} \int\limits_{\frac{2j}{2m}}^{\frac{2j+1}{2m}} f(t,u)dt - \frac{1}{2}\int\limits_0^1 f(t,u)dt \right| < \frac{\varepsilon}{2},$$

and

$$\left| \sum_{j=0}^{m-1} \int\limits_{\frac{2j+1}{2m}}^{\frac{2j+2}{2m}} f(t,u)dt - \frac{1}{2}\int\limits_0^1 f(t,u)dt \right| < \frac{\varepsilon}{2},$$

because

$$\sum_{j=0}^{m-1} \int\limits_{\frac{2j}{2m}}^{\frac{2j+1}{2m}} f(t,u)dt + \sum_{j=0}^{m-1} \int\limits_{\frac{2j+1}{2m}}^{\frac{2j+2}{2m}} f(t,u)dt = \int\limits_0^1 f(t,u)dt.$$

So,

$$\left|\sum_{i=0}^{2m-1}\int_{\frac{i}{n}}^{\frac{i+1}{n}} f(t,(-1)^i)dt - \frac{1}{2}\left[\int_0^1 f(t,1)dt - \int_0^1 f(t,-1)dt\right]\right| < \varepsilon.$$

That is

$$\left|\int_{[0;1]\times U} f(t,u)\mu_n(dt,du) - \int_{[0;1]\times U} f(t,u)\frac{1}{2}\left[\delta_1(du)+\delta_{-1}(du)\right]dt\right| < \varepsilon.$$

The case where n is odd is treated in the same way.

Now, we can define a new control problem that generalize the strict one, which is associated to such a measure $\mu(dt,du)=\delta_{u_t}(du)dt$, which is called a relaxed control problem.

Consider the SDE

$$x_t^\mu = x_0^\mu + \int_0^t\int_A u_s\mu(ds,du),$$

and the cost is

$$J(u) = \int_0^1 (x_t^\mu)^2 dt.$$

If $\mu^*(dt,du) = \frac{1}{2}\left[\delta_1(du)+\delta_{-1}(du)\right]dt$, we have $J(\mu^*)=0$, so the new problem has μ^* as an optimal solution. μ is called a relaxed control.

Let V be the space of positive Radon measures on $[0;1]\times A$, whose projections on $[0;1]$ coincide with Lebesgue measure, and let the Borel $\sigma-$field $\overline{V}$ as the smallest $\sigma-$field such that the mappings $\mu \to \int_0^1\int_A \phi(t,u)\mu_t(du)dt$ are measurable, where ϕ is a bounded measurable function which is continuous in a. Let us also introduce the filtration $(\overline{V}_t)$ on V, where $\overline{V}_t$ is generated by $\left\{1_{[0;t]}\mu,\ \mu\in V\right\}$.

Definition 1.9 *A relaxed control on the filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},P)$ is a random variable μ with values in V such that $\mu(\omega,t,da)$ is progressively measurable with respect to $(\mathcal{F}_t)_{t\geq 0}$ and such that for each t, $1_{[0;t]}\mu$ is $\mathcal{F}_t-$measurable.*

Chapter 2

Stochastic maximum principle of controlled jump diffusions

In this chapter, we will give a detailed demonstration of the maximum principle for optimal control of systems driven by stochastic differential equations with jump processes, where the control variable appear in the drift and the jump term, we also study the maximum principle for near optimal controls. This result is based on Ekeland's variational principle. The maximum principle for near optimal controls has great utility to establish the relaxed maximum principle in the next chapter.

2.1 Formulation of the problem

We consider in this subsection a stochastic control problem of systems governed by stochastic differential equations on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$, such that $\mathcal{F}_0$ contains the $P-$null sets, We assume that $(\mathcal{F}_t)_{t\geq 0}$ is generated by a standard Brownian motion B and an independent Poisson measure N, and assume that the compensator of N has the form $\upsilon(d\theta)dt$, where the jumps are confined to a compact set Γ. And set

$$\widetilde{N}(dt, d\theta) = N(dt, d\theta) - \upsilon(d\theta)dt.$$

Consider the following set A is a nonempty subset of $\mathbb{R}^k$ and let U the class of measurable, adapted processes $u : [0;T] \times \Omega \longrightarrow A$. For any $u \in U$, we consider the following stochastic differential equation (SDE)

$$\begin{cases} dx_t &= b(t, x_t, u_t)dt + \sigma(t, x_t)dB_t + \int_\Gamma f(t, x_{t^-}, \theta, u_t)\widetilde{N}(dt, d\theta) \\ x_0 &= 0, \end{cases} \tag{2.1}$$

where

$$b : [0;T] \times \mathbb{R}^n \times A \longrightarrow \mathbb{R}^n$$
$$\sigma : [0;T] \times \mathbb{R}^n \longrightarrow \mathcal{M}_{n\times d}(\mathbb{R})$$
$$f : [0;T] \times \mathbb{R}^n \times \Gamma \times A \longrightarrow \mathbb{R}^n$$

are bounded, measurable and continuous functions.

The expected cost is given by :

$$J(u) = E\left[g(x_T) + \int_0^T h(t, x_t, u_t)dt\right], \tag{2.2}$$

where

$$g : \mathbb{R}^n \longrightarrow \mathbb{R}$$
$$h : [0;T] \times \mathbb{R}^n \times A \longrightarrow \mathbb{R}$$

be bounded and continuous functions.

The strict optimal control problem is to minimize the functional $J(.)$ over U. A control that solves this problem is called optimal.

2.2 The maximum principle for strict control

2.2.1 Using convex perturbations

The following assumptions will be in force throughout this subsection

(H_1) The maps b, σ, f and h are continuously differentiable with respect to (x,u), and g is continuously differentiable in x .

(H_2) The derivatives b_x b_u σ_x f_x, f_u and h_x, h_u are continuous in (x,u) and uniformly bounded, and g is continuous in x and bounded.

(H_3) b, σ, f and h are bounded by $K(1+|x|+|u|)$, and g is bounded by $K(1+|x|)$, for some $K>0$.

Under the above hypothesis, and since the probability space and Brownian motion do not change with the control, then (2.1) has a unique strong solution and the cost functional (2.2) is well defined from U into $\mathbb{R}$.

Suppose that there exist an optimal strict control u^*, which minimizing the cost functional $J(.)$ over U, and denote by x^* the corresponding trajectory. To derive optimality necessary conditions, satisfied by the optimal strict control u^*, we use the convex perturbations of the optimal control u^*, which defined by

$$u^h = u^* + h(u-u^*)$$

for some $u \in U$.

Lemma 2.1 *Under assumptions (H_1)-(H_3), one has*

$$\lim_{h\to 0} E\left[\sup_{t\in[0;T]} \left|x_t^h - x_t^*\right|^2\right] = 0.$$

Proof. See proof of lemma 2.3 ■

Since u^* is optimal, then

$$J(u^*) \leq J(u^h) = J(u^*) + h\frac{dJ(u^h)}{dh}\bigg|_{h=0} + \circ(h).$$

Thus a necessary condition for optimality is that

$$\frac{dJ(u^h)}{dh}\bigg|_{h=0} \geq 0.$$

Let us take care to compute this derivative :

$$\begin{aligned}\frac{dJ(u^h)}{dh}\bigg|_{h=0} &= \tfrac{1}{h}E\left[\left(g(x_T^h) - g(x_T^*)\right) + \int_0^T \left(h(t,x_t^h,u_t^h) - h(t,x_t^*,u_t^h)\right)dt\right. \\ &\left. + \int_0^T \left(h(t,x_t^*,u_t^h) - h(t,x_t^*,u_t^*)\right)dt\right],\end{aligned}$$

then

$$\begin{aligned}\frac{dJ(u^h)}{dh}\bigg|_{h=0} &= \tfrac{1}{h}E\left[g_x(x_T^*)(x_T^h - x_T^*) + \int_0^T h_x(t,x_t^*,u_t^h)(x_t^h - x_t^*)dt\right. \\ &\left. + \int_0^T h_u(t,x_t^*,u_t^*)(u_t^h - u_t^*)dt\right].\end{aligned}$$

If we denote by $z_t = \frac{dx_t^h}{dh}\Big|_{h=0}$, we get the following corollary

Corollary 2.1 *Under assumptions (H_1)-(H_3), one has*

$$\begin{aligned}\frac{dJ(u^h)}{dh}\bigg|_{h=0} &= E\left[\int_0^T \left\{h_x(t,x_t^*,u_t^*)z_t + h_u(t,x_t^*,u_t^*)u_t\right\}dt\right. \\ &\left. + g_x(x_T^*)z_T\right],\end{aligned} \tag{2.3}$$

where the process z is the solution of the linear SDE

$$\begin{cases} dz_t = \left(b_x(t,x_t^*,u_t^*)z_t + b_u(t,x_t^*,u_t^*)u_t\right)dt + \sigma_x(t,x_t^*)z_t dB_t \\ +\int\limits_{\Gamma}\left(f_x(t,x_{t-}^*,\theta,u_{t-}^*)z_{t-} + f_u(t,x_{t-}^*,\theta,u_{t-}^*)u_{t-}\right)\widetilde{N}(dt,d\theta) \\ z_0 = 0. \end{cases} \tag{2.4}$$

From (H_2) the variational equation (2.4) has a unique solution.

To prove the corollary (2.1) we need the following estimate.

Lemma 2.2 *Under assumptions (H_1)-(H_3), it holds that*

$$\lim_{h\to 0} E\left[\left|\frac{x_t^h - x_t^*}{h} - z_t\right|^2\right] = 0.$$

Proof. Let

$$y_t^h = \frac{x_t^h - x_t^*}{h} - z_t.$$

We denote $x_t^{h,\lambda} = x_t^* + \lambda h(y_t^h + z_t)$, and $u_t^{h,\lambda} = u_t^* + \lambda h u_t$, then y_t^h satisfies the following SDE

$$\begin{aligned} dy_t^h &= \tfrac{1}{h}\left[b(t,x_t^h,u_t^h) - b(t,x_t^*,u_t^*)\right]dt + \tfrac{1}{h}\left[\sigma(t,x_t^h) - \sigma(t,x_t^*)\right]dB_t \\ &+\tfrac{1}{h}\int\limits_{\Gamma}\left[f(t,x_{t-}^h,\theta,u_{t-}^h) - f(t,x_{t-}^*,\theta,u_{t-}^*)\right]\widetilde{N}(dt,d\theta) \\ &-\left[b_x(t,x_t^*,u_t^*)z_t + b_u(t,x_t^*,u_t^*)u_t\right]dt - \sigma_x(t,x_t^*)z_t dB_t \\ &-\int\limits_{\Gamma}\left[f_x(t,x_{t-}^*,\theta,u_{t-}^*)z_{t-} + f_u(t,x_{t-}^*,\theta,u_{t-}^*)u_{t-}\right]\widetilde{N}(dt,d\theta), \end{aligned}$$

then

$$\begin{aligned} y_t^h &= \int_0^t\int_0^1 b_x(s,x_s^{h,\lambda},u_s^h)y_s^h d\lambda ds + \int_0^t\int_0^1 \sigma_x(s,x_s^{h,\lambda})y_s^h d\lambda dB_s \\ &+\int_0^t\int_0^1\int\limits_{\Gamma} f_x(s,x_{s-}^{h,\lambda},\theta,u_{s-}^h)y_{s-}^h d\lambda\widetilde{N}(ds,d\theta) + \rho_t^h, \end{aligned}$$

where

$$\begin{aligned}
\rho_t^h &= \int_0^t\int_0^1 b_x(s,x_s^{h,\lambda},u_s^h)z_s d\lambda ds + \int_0^t\int_0^1 \sigma_x(s,x_s^{h,\lambda})z_s d\lambda dB_s \\
&+\int_0^t\int_0^1\int_\Gamma f_x(s,x_{s^-}^{h,\lambda},\theta,u_{s^-}^h)z_{s^-}d\lambda\widetilde{N}(ds,d\theta)+\frac{1}{h}\int_0^t \left[b(s,x_s^*,u_s^h)-b(s,x_s^*,u_s^*)\right]ds \\
&+\frac{1}{h}\int_0^t\int_\Gamma \left[f(s,x_{s^-}^*,\theta,u_{s^-}^h)-f(s,x_{s^-}^*,\theta,u_{s^-}^*)\right]\widetilde{N}(ds,d\theta)-\int_0^t\int_\Gamma f_x(s,x_{s^-}^*,\theta,u_{s^-}^*)z_s\widetilde{N}(ds,d\theta) \\
&+\int_0^t\int_\Gamma f_u(s,x_{s^-}^*,\theta,u_{s^-}^*)u_{s^-}\widetilde{N}(ds,d\theta)-\int_0^t b_x(s,x_s^*,u_s^*)z_s ds \\
&-\int_0^t b_u(s,x_s^*,u_s^*)u_s ds-\int_0^t \sigma_x(s,x_s^*)z_s dB_s,
\end{aligned}$$

hence

$$\begin{aligned}
E\left|y_t^h\right|^2 &\le KE\int_0^t\left|\int_0^1 b_x(s,x_s^{h,\lambda},u_s^h)y_s^h d\lambda\right|^2 ds + KE\int_0^t\left|\int_0^1 \sigma_x(s,x_s^{h,\lambda})y_s^h d\lambda\right|^2 ds \\
&+KE\int_0^t\int_\Gamma\left|\int_0^1 f_x(s,x_{s^-}^{h,\lambda},\theta,u_{s^-}^h)y_{s^-}^h d\lambda\right|^2 \upsilon(d\theta)ds + KE\sup_{t_0+h\le t\le T}\left|\rho_t^h\right|^2.
\end{aligned}$$

Since b_x, σ_x, and f_x are bounded, then

$$E\left|y_t^h\right|^2 \le CE\int_0^t \left|y_s^h\right|^2 ds + KE\sup_{t_0+h\le t\le T}\left|\rho_t^h\right|^2.$$

We conclude by the boundedness and continuity of b_x, σ_x f_x , b_u, and f_u and the dominated convergence that $\lim_{h\to 0} E\sup_{t_0+h\le t\le T}\left|\rho_t^h\right|^2 = 0$. Hence by the Gronwall lemma, we get

$$\lim_{h\to 0} E\left|y_t^h\right|^2 = 0.$$

■

We use the same notations as in the proof of lemma (2.2), we can prove the corollary (2.1).

Proof of corollary. We have by the definition of J that

$$\begin{aligned}\tfrac{1}{h}\left[J(u^h)-J(u^*)\right] &= \tfrac{1}{h}\Bigg[E\left[g(x_T^h)-g(x_T^*)\right]+\int_0^T\left[h(t,x_t^h,u_t^h)-h(t,x_t^*,u_t^h)\right]dt \\ &\quad+\int_0^T\left[h(t,x_t^*,u_t^h)-h(t,x_t^*,u_t^*)\right]dt\Bigg],\end{aligned}$$

then,

$$\begin{aligned}\tfrac{1}{h}\left[J(u^h)-J(u^*)\right] &= \Bigg[\int_0^T g_x(x_T^{h,\lambda})(y_T^h+z_T)d\lambda+\int_0^T\int_0^1 h_x(t,x_t^{h,\lambda},u_t^h)(y_t^h+z_t)d\lambda dt \\ &\quad+\int_0^T\int_0^1 h_u(t,x_t^*,u_t^{h,\lambda})u_t d\lambda dt\Bigg],\end{aligned}$$

hence

$$\begin{aligned}\tfrac{1}{h}\left[J(u^h)-J(u^*)\right] &= \Bigg[\int_0^1 g_x(x_T^{h,\lambda})z_T d\lambda+\int_0^T\int_0^1 h_x(t,x_t^{h,\lambda},u_t^h)z_t d\lambda dt \\ &\quad+\int_0^T\int_0^1 h_u(t,x_t^*,u_t^{h,\lambda})u_t d\lambda dt\Bigg]+\alpha_t^h,\end{aligned}$$

where

$$\alpha_t^h = E\left[\int_0^1 g_x(x_T^{h,\lambda})y_T^h d\lambda+\int_0^T\int_0^1 h_x(t,x_t^{h,\lambda},u_t^h)y_t^h d\lambda dt\right].$$

By the Cauchy-Schwartz inequality and boundedness of g_x and h_x, using the lemma (2.2) we can easily prove that

$$\lim_{h\to 0}\alpha_t^h = 0,$$

hence, the result follows by letting h go to 0 in the above equality. ∎

By the integration by parts formula [27], we can see that the solution of dz_t is given by

$z_t = \varphi_t \eta_t$ where

$$\begin{cases} d\varphi(t,\tau) = b_x(t,x_t^*,u_t^*)\varphi(t,\tau) + \sigma_x(t,x_t^*)\varphi(t,\tau)dB_t \\ \qquad + \int_\Gamma f_x(t,x_{t-}^*,\theta,u_t^*)\varphi(t^-,\tau)\widetilde{N}(dt,d\theta) & 0 \leq \tau \leq t \leq T, \\ \varphi(\tau,\tau) = I_d \end{cases}$$

and

$$\begin{cases} d\eta_t = \psi_t \left(b_u(t,x_t^*,u_t^*)u_t - \int_\Gamma f_u(t,x_{t-}^*,\theta,u_t^*)u_t\upsilon(d\theta)dt \right) \\ -\psi_{t^-} \int_\Gamma \left(f_x(t,x_{t-}^*,\theta,u_t^*) + I_d \right)^{-1} f_u(t,x_{t-}^*,\theta,u_t^*)u_t N(dt,d\theta) \\ \eta_0 = 0, \end{cases}$$

with ψ_t is the inverse of φ satisfying suitable integrability conditions, and it is the solution of the following equation

$$\begin{cases} d\psi(t,\tau) = (\sigma_x(t,x_t^*)\psi(t,\tau)\sigma_x(t,x_t^*) - b_x(t,x_t^*,u_t^*)\psi(t,\tau) \\ -\int_\Gamma f_x(t,x_{t-}^*,\theta,u_t^*)\psi(t^-,\tau)\upsilon(d\theta)dt \Big) \\ -\sigma_x(t,x_t^*)\psi(t,\tau)dB_t & 0 \leq \tau \leq t \leq T \\ -\psi(t^-,\tau)\int_\Gamma \left(f_x(t,x_{t-}^*,\theta,u_t^*) + I_d \right)^{-1} f_x(t,x_{t-}^*,\theta,u_t^*)N(dt,d\theta) \\ \psi(\tau,\tau) = I_d. \end{cases}$$

Remark 2.1 *1. From Itô's formula, we can easily check that* $d\left(\varphi(t,\tau)\psi(t,\tau)\right) = 0$, *and* $\varphi(\tau,\tau)\psi(\tau,\tau) = I_d$.

2. If $\tau = 0$, *we simply write* $\varphi(t,0) = \varphi_t$ *and* $\psi(t,0) = \psi_t$.

Then the equality (2.3) will become

$$\left.\frac{dJ(u^h)}{dh}\right|_{h=0} = E\left[\int_0^T \{h_x(t,x_t^*,u_t^*)\varphi_t\eta_t + h_u(t,x_t^*,u_t^*)u_t\}\,dt \right.$$
$$\left. + g_x(x_T^*)\varphi_T\eta_T\right].$$

Set

$$X = \int_0^T h_x(t,x_t^*,u_t^*)\varphi_t^* dt + g_x(x_T^*)\varphi_T^*$$
$$y_t = E\left[X \mid \mathcal{F}_t\right] - \int_0^t h_x(s,x_s^*,u_s^*)\varphi_s^* ds,$$

then, we have

$$y_T = E\left[X \mid \mathcal{F}_t\right] - \int_0^T h_x(s,x_s^*,u_s^*)\varphi_s^* ds = X - \int_0^T h_x(s,x_s^*,u_s^*)\varphi_s^* ds = g_x(x_T^*)\varphi_T^*, \tag{2.5}$$

replacing (2.5) in (2.18), we obtain

$$\left.\frac{dJ(u^h)}{dh}\right|_{h=0} = E\left[\int_0^T \{h_x(t,x_t^*,u_t^*)\varphi_t^*\eta_t + h_u(t,x_t^*,u_t^*)u_t\}\,dt + y_T\eta_T\right]. \tag{2.6}$$

By the Itô representation theorem [19], there exist two processes $Q \in \mathcal{M}^2$ and $R \in \mathcal{L}^2$ satisfying

$$E\left[X \mid \mathcal{F}_t\right] = E\left[X\right] + \int_0^t Q_s dB_s + \int_0^t\int_\Gamma R_s(\theta)\widetilde{N}(ds,d\theta),$$

hence,

$$y_t = E\left[X\right] - \int_0^t h_x(s,x_s^*,u_s^*)\varphi_s ds + \int_0^t Q_s dB_s + \int_0^t\int_\Gamma R_s(\theta)\widetilde{N}(ds,d\theta).$$

Now, let us calculate $E\left[y_T\eta_T\right]$, we have

$$dy_t = -h_x(t,x_t^*,u_t^*)\varphi_t dt + Q_t dB_t + \int_\Gamma R_t(\theta)\widetilde{N}(dt,d\theta),$$

by the integration by parts formula we get

$$\begin{aligned}
d(y_t\eta_t) &= y_t d\eta_t + \eta_t dy_t + \int_\Gamma \rho^1\rho^2\upsilon(d\theta)dt \\
&= y_t\psi_t\left[b_u(t,x_t^*,u_t^*)u_t - \int_\Gamma f_u(t,x_s^*,\theta,u_s^*)u_t\upsilon(d\theta)\right]dt \\
&-y_t\psi_{t^-}\int_\Gamma \left(f_x+Id\right)^{-1} f_u u_t N(dt,d\theta) - \eta_t\varphi_t^* h_x dt \\
&+\eta_t Q_t dB_t + \int_\Gamma \eta_t R_t(\theta)\widetilde{N}(dt,d\theta) \\
&+\int_\Gamma R_t(\theta)\psi_t\left(f_x+Id\right)^{-1} f_u u_t\upsilon(d\theta)dt.
\end{aligned}$$

If we define the adjoint process by : $p_t = y_t\psi_t$, then

$$\begin{aligned}
d(y_t\eta_t) &= p_t b_u u dt - p_t\int_\Gamma f_u u\upsilon(d\theta)dt - p_t\int_\Gamma \left(f_x+Id\right)^{-1} f_u u_t\widetilde{N}(dt,d\theta) \\
&-p_t\int_\Gamma \left(f_x+Id\right)^{-1} f_u u_t\upsilon(d\theta)dt - \eta_t\varphi_t^* h_x dt + \eta_t Q_t dB_t \\
&+\int_\Gamma \eta_t R_t(\theta)\widetilde{N}(dt,d\theta) + \int_\Gamma R_t(\theta)\psi_t\left(f_x+Id\right)^{-1} f_u u_t\upsilon(d\theta)dt
\end{aligned}$$

hence

$$\begin{aligned}
y_T\eta_T &= \int_0^T p_t b_u u dt - \int_0^T\int_\Gamma p_t f_u u\upsilon(d\theta)dt - \int_0^T\int_\Gamma p_t\left(f_x+Id\right)^{-1} f_u u_t\widetilde{N}(dt,d\theta) \\
&-\int_0^T\int_\Gamma p_t\left(f_x+Id\right)^{-1} f_u u_t\upsilon(d\theta)dt - \int_0^T \eta_t\varphi_t^* h_x dt + \int_0^T \eta_t Q_t dB_t \\
&+\int_0^T\int_\Gamma \eta_t R_t(\theta)\widetilde{N}(dt,d\theta) + \int_0^T\int_\Gamma R_t(\theta)\psi_t\left(f_x+Id\right)^{-1} f_u u_t\upsilon(d\theta)dt,
\end{aligned}$$

take the expectation, we obtain

$$\begin{aligned} E\left[y_T \eta_T\right] &= E\left[\int_0^T \left(p_t b_u u + \int_\Gamma R_t(\theta)\psi_t \left(f_x + Id\right)^{-1} - p_t\left(\left(f_x + Id\right) + Id\right) f_u u_t \upsilon(d\theta)\right) dt \right. \\ & \left. - \int_0^T \eta_t \varphi_t^* h_x dt\right]. \end{aligned}$$

We define the adjoint process r by

$$r_t(\theta) = R_t(\theta)\psi_t \left(f_x + Id\right)^{-1} - p_t\left(\left(f_x + Id\right) + Id\right),$$

hence

$$E\left[y_T \eta_T\right] = E\left[\int_0^T \left\{p_t b_u u + \int_\Gamma r_t(\theta) f_u u_t \upsilon(d\theta)\right\} dt - \int_0^T \eta_t \varphi_t^* h_x dt\right].$$

By the replacing in (2.6), we get

$$\left.\frac{dJ(u^h)}{dh}\right|_{h=0} = E\left[\int_0^T \left\{h_u(s, x_s^*, u_s^*) + p_s b_u(s, x_s^*, u_s^*) + \int_\Gamma r_s(\theta) f_u(s, x_s^*, \theta, u_s^*) \upsilon(d\theta)\right\} u_s ds\right] \geq 0. \tag{2.7}$$

Finally, if we define the Hamiltonian H from $[0;T] \times \mathbb{R}^n \times A \times \mathbb{R}^n \times \mathbb{R}^{n\times m} \times L_m^2$ into $\mathbb{R}$ by

$$\begin{aligned} H(t, x, u, p, q, r(.)) &= h(t, x_t, u_t) + pb(t, x_t, u_t) + q\sigma(t, x_t) \\ &+ \int_\Gamma r_t(\theta) f(s, x_t, \theta, u_t) \upsilon(d\theta) \end{aligned}$$

we get from (2.7) the next theorem which is the result of this subsection.

Theorem 2.1 (maximum principle for strict control) *Let u^* be the optimal strict control minimizing the cost $J(.)$ over U, and denote by x^* the corresponding optimal* *trajectory, then the following inequality holds*

$$E\left[\int_0^T H_u(t,x_t^*,u_t^*,p_t,q_t,r_t(.))(u_t-u_t^*)ds\right]\geq 0,$$

where the Hamiltonian H is defined by (2.20).

2.2.2 Using strong perturbations

The following assumptions will be in force throughout this subsection

(H_1) The maps b, σ, f and h are continuously differentiable with respect to (x,u), and g is continuously differentiable in x.

(H_2) The derivatives σ_x f_x, and g_x are bounded, and b_x, h_x are uniformly bounded in u.

(H_3) b, σ, f and h are bounded by $K(1+|x|+|u|)$, and g is bounded by $K(1+|x|)$, for some $K>0$.

Under the above hypothesis, the SDE (2.1) has a unique strong solution and the cost functional (2.2) is well defined from U into $\mathbb{R}$.

The purpose of this subsection is to derive optimality necessary conditions, satisfied by an optimal strict control. The proof is based on the strong perturbation of the optimal control u^*, which defined by :

$$u^h=\begin{cases}\nu & \text{if } \ t\in[t_0;t_0+h]\\ u^* & \text{otherwise,}\end{cases}$$

where $0\leq t_0<T$ is fixed, h is sufficiently small, and ν is an arbitrary $A-$valued $\mathcal{F}_{t_0}-$measurable random variable such that $E\left|\nu\right|^2<\infty$. Let x_t^h denotes the trajectory associated with u^h, then

$$\begin{cases} x_t^h = x_t^* \quad ; \ t \leq t_0 \\ dx_t^h = b(t, x_t^h, \nu)dt + \sigma(t, x_t^h)dB_t + \int\limits_\Gamma f(t, x_{t-}^h, \theta, \nu)\widetilde{N}(dt, d\theta) \quad ; t_0 < t < t_0 + h \\ dx_t^h = b(t, x_t^h, u^*)dt + \sigma(t, x_t^h)dB_t + \int\limits_\Gamma f(t, x_{t-}^h, \theta, u^*)\widetilde{N}(dt, d\theta) \quad ; t_0 + h < t < T \end{cases}$$

We first have

Lemma 2.3 *Under assumptions (H_1)-(H_3), one has*

$$\lim_{h \to 0} E\left[\sup_{t \in [t_0;T]} \left|x_t^h - x_t^*\right|^2\right] = 0. \tag{2.8}$$

Proof. For $t \in [t_0; t_0 + h]$, we have

$$\begin{aligned} x_t^h - x_t^* = &\int\limits_{t_0}^t b(s, x_s^h, \nu) - b(s, x_s^*, u_s^*)ds \\ &+ \int\limits_{t_0}^t \sigma(s, x_s^h) - \sigma(s, x_s^*)ds \\ &+ \int\limits_{t_0}^t \int\limits_\Gamma \left[f(s, x_{s-}^h, \theta, u_s^h) - f(s, x_s^*, \theta, u_s^*)\right] \widetilde{N}(ds, d\theta). \end{aligned} \tag{2.9}$$

We can deduce by the standard arguments that,

$$E \int\limits_{t_0}^{t_0+h} \left[\left|x_s^h - x_s^*\right|^2\right] ds \leq K \int\limits_{t_0}^{t_0+h} E\left|\nu - u_s^*\right|^2 ds,$$

and by the martingale inequalities, we deduce

$$E\left[\sup_{t \in [t_0;t_0+h]} \left|x_t^h - x_t^*\right|^2\right] \leq K \int\limits_{t_0}^{t_0+h} E\left|\nu - u_s^*\right|^2 ds. \tag{2.10}$$

We next have for $t \in [t_0 + h; T]$,

$$\begin{aligned} x_t^h - x_t^* &= \left[x_{t_0+h}^h - x_{t_0+h}^* \right] + \int_{t_0+h}^{t} b(s, x_s^h, \nu) - b(s, x_s^*, u_s^*) ds \\ &+ \int_{t_0+h}^{t} \sigma(s, x_s^h) - \sigma(s, x_s^*) ds \\ &+ \int_{t_0+h}^{t} \int_{\Gamma} \left[f(s, x_{s^-}^h, \theta, u_s^h) - f(s, x_s^*, \theta, u_s^*) \right] \widetilde{N}(ds, d\theta) \end{aligned}$$

from which we deduce successively

$$E \int_{t_0+h}^{T} \left| x_s^h - x_s^* \right|^2 ds \leq KE \left| x_{t_0+h}^h - x_{t_0+h}^* \right|^2,$$

$$E \left[\sup_{t \in [t_0+h; T]} \left| x_t^h - x_t^* \right|^2 \right] \leq KE \left| x_{t_0+h}^h - x_{t_0+h}^* \right|^2. \tag{2.11}$$

From (2.10) and (2.11), letting h tend to 0, we obtain (2.8). ■

Since u^* is optimal, then

$$J(u^*) \leq J(u^h) = J(u^*) + h \left. \frac{dJ(u^h)}{dh} \right|_{h=0} + \circ(h).$$

Thus a necessary condition for optimality is that

$$\left. \frac{dJ(u^h)}{dh} \right|_{h=0} \geq 0.$$

Let us take care to compute this derivative :

Note that the following properties holds, because $b(t, x, u)$, $h(t, x, u)$ and $f(t, x_{t^-}, \theta, u)$ are sufficiently integrable

$$\frac{1}{h}\int_{t}^{t+h} E\left[\left|k(s,x_s,u_s)-k(t,x_t,u_t)\right|^2\right] \underrightarrow{h\to 0}\ 0\ dt-a.e \tag{2.12}$$

$$\frac{1}{h}\int_{\Gamma}\int_{t}^{t+h} E\left[\left|f(s,x_{s^-},\theta,u_s)-f(t,x_{t^-},\theta,u_t)\right|^2\right]\upsilon(d\theta)\ \underrightarrow{h\to 0}\ 0\ dt-a.e, \tag{2.13}$$

where k stands for b or h.

Choose t_0 such that (2.12) and (2.13) holds, then we have

Corollary 2.2 *Under assumptions (H_1)-(H_3), one has*

$$\left.\frac{dJ(u^h)}{dh}\right|_{h=0} = E\left[g_x(x_T^*)z_T+\varsigma_T\right], \tag{2.14}$$

where

$$\begin{cases} d\varsigma_t = h_x(t,x_t^*,u_t^*)z_t dt & t_0\le t\le T \\ \varsigma_{t_0} = h(t_0,x_{t_0}^*,\nu)-h(t_0,x_{t_0}^*,u_{t_0}^*) \end{cases}$$

and the process z is the solution of the linear SDE

$$\begin{cases} dz_t = b_x(t,x_t^*,u_t^*)z_t dt+\sigma_x(t,x_t^*)z_t dB_t+\int_{\Gamma} f_x(t,x_{t^-}^*,\theta,u_t^*)z_{t^-}\widetilde{N}(dt,d\theta);\ \ t_0\le t\le T \\ z_{t_0} = \left[b(t_0,x_{t_0}^*,\nu)-b(t_0,x_{t_0}^*,u_{t_0}^*)\right]. \end{cases} \tag{2.15}$$

From (H_2) the variational equation (2.4) has a unique solution.

To prove the corollary (2.1) we need the following estimates.

Lemma 2.4 *Under assumptions (H_1)-(H_3), it holds that*

$$\lim_{h\to 0} E\left[\left|\frac{x_t^h-x_t^*}{h}-z_t\right|^2\right]=0.$$

$$\lim_{h\to 0} E\left[\left|\frac{1}{h}\int_{t_0}^{T}\left[(h(t,x_t^*,u_t^h)-(h(t,x_t^*,u_t^*)\right]-\varsigma_T\right|^2\right]=0.$$

Proof. Let

$$y_t^h=\frac{x_t^h-x_t^*}{h}-z_t.$$

Then, we have for $t\in[t_0;t_0+h]$

$$\begin{cases} dy_t^h &= \frac{1}{h}\left[b(t,x_t^*+h(y_t^h+z_t),\nu)-b(t,x_t^*,u_t^*)-hb_x(t,x_t^*,u_t^*)z_t\right]dt \\ &+\frac{1}{h}\left[\sigma(t,x_t^*+h(y_t^h+z_t))-\sigma(t,x_t^*)-h\sigma_x(t,x_t^*)z_t\right]dB_t \\ &+\frac{1}{h}\int_{\Gamma}\left[f(t,x_{t^-}^*+h(y_{t^-}^h+z_{t^-}),\theta,\nu)-f(t,x_{t^-}^*,\theta,u_t^*)-hf_x(t,x_{t^-}^*,\theta,u_t^*)z_{t^-}\right]\widetilde{N}(dt,d\theta) \\ y_{t_0}^h &= -\left[b(t_0,x_{t_0}^*,\nu)-b(t_0,x_{t_0}^*,u_{t_0}^*)\right]. \end{cases}$$

Hence

$$\begin{aligned} y_{t_0+h}^h &= \frac{1}{h}\int_{t_0}^{t_0+h}\left[b(t,x_t^*+h(y_t^h+z_t),\nu)-b(t,x_t^*,\nu)\right]dt+\frac{1}{h}\int_{t_0}^{t_0+h}\left[b(t,x_t^*,\nu)-b(t,x_{t_0}^*,\nu)\right]dt \\ &+\frac{1}{h}\int_{t_0}^{t_0+h}\left[b(t,x_{t_0}^*,\nu)-b(t_0,x_{t_0}^*,\nu)\right]dt+\frac{1}{h}\int_{t_0}^{t_0+h}\left[b(t_0,x_{t_0}^*,u_{t_0}^*)-b(t,x_t^*,u_t^*)\right]dt \\ &+\frac{1}{h}\int_{t_0}^{t_0+h}\left[\sigma(t,x_t^*+h(y_t^h+z_t))-\sigma(t,x_t^*)\right]dB_t \\ &+\frac{1}{h}\int_{t_0}^{t_0+h}\int_{\Gamma}\left[f(t,x_{t^-}^*+h(y_{t^-}^h+z_{t^-}),\theta,\nu)-f(t,x_{t^-}^*,\theta,\nu)\right]\widetilde{N}(dt,d\theta) \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{h}\int_{t_0}^{t_0+h}\int_{\Gamma}\left[f(t,x^*_{t^-},\theta,\nu)-f(t,x^*_{t_0^-},\theta,\nu)\right]\widetilde{N}(dt,d\theta)\\
&+\frac{1}{h}\int_{t_0}^{t_0+h}\int_{\Gamma}\left[f(t,x^*_{t_0^-},\theta,\nu)-f(t_0,x^*_{t_0^-},\theta,\nu)\right]\widetilde{N}(dt,d\theta)\\
&+\frac{1}{h}\int_{t_0}^{t_0+h}\int_{\Gamma}\left[f(t_0,x^*_{t_0^-},\theta,\nu)-f(t_0,x^*_{t_0^-},\theta,u^*_{t_0})\right]\widetilde{N}(dt,d\theta)\\
&+\frac{1}{h}\int_{t_0}^{t_0+h}\int_{\Gamma}\left[f(t_0,x^*_{t_0^-},\theta,u^*_{t_0})-f(t,x^*_{t^-},\theta,u^*_{t})\right]\widetilde{N}(dt,d\theta)\\
&-\int_{t_0}^{t_0+h} b_x(t,x^*_t,u^*_t)z_t dt-\int_{t_0}^{t_0+h}\sigma_x(t,x^*_t)z_t dB_t-\int_{t_0}^{t_0+h}\int_{\Gamma} f_x(t,x^*_{t^-},\theta,u^*_t)z_t\widetilde{N}(dt,d\theta).
\end{aligned}
$$

Then

$$
\begin{aligned}
E\left|y^h_{t_0+h}\right|^2 \quad &\leq C[E\sup_{t_0\leq t\leq t_0+h}\left|x^h_t-x^*_t\right|^2+\sup_{t_0\leq t\leq t_0+h}E\left|b(t,x^*_{t_0},\nu)-b(t_0,x^*_{t_0},\nu)\right|^2 dt\\
&+\frac{1}{h}E\int_{t_0}^{t_0+h}\left|b(t_0,x^*_{t_0},u^*_{t_0})-b(t,x^*_t,u^*_t)\right|^2 dt+E\sup_{t_0\leq t\leq t_0+h}\left|x^*_t-x^*_{t_0}\right|^2\\
&+E\int_{t_0}^{t_0+h}\int_{\Gamma}\left|\nu-u^*_{t_0}\right|^2\upsilon(d\theta)dt+E\int_{t_0}^{t_0+h}\left|z_t\right|^2 dt\\
&+\sup_{t_0\leq t\leq t_0+h}E\int_{\Gamma}\left|f(t,x^*_{t_0^-},\theta,\nu)-f(t_0,x^*_{t_0^-},\theta,\nu)\right|^2\upsilon(d\theta)\\
&+\frac{1}{h}E\int_{t_0}^{t_0+h}\int_{\Gamma}\left|f(t_0,x^*_{t_0^-},\theta,u^*_{t_0})-f(t,x^*_{t^-},\theta,u^*_t)\right|^2\upsilon(d\theta)dt.
\end{aligned}
\tag{2.16}
$$

By Lemma (2.3), and the properties (2.12) and (2.13), it is easy to see that $E\left|y^h_{t_0+h}\right|^2$ tends to 0 as $h\to 0$.

For $t \in [t_0 + h; T]$, we denote $x_t^{h,\lambda} = x_t^* + \lambda h(y_t^h + z_t)$, then y_t^h satisfies the following SDE

$$\begin{aligned} dy_t^h &= \tfrac{1}{h}\left[b(t, x_t^* + h(y_t^h + z_t), u_t^*) - b(t, x_t^*, u_t^*)\right] dt + \tfrac{1}{h}\left[\sigma(t, x_t^* + h(y_t^h + z_t)) - \sigma(t, x_t^*)\right] dB_t \\ &+ \tfrac{1}{h}\int_\Gamma \left[f(t, x_{t-}^* + h(y_{t-}^h + z_{t-}), \theta, u_t^*) - f(t, x_{t-}^*, \theta, u_t^*)\right] \widetilde{N}(dt, d\theta) \\ &- b_x(t, x_t^*, u_t^*) z_t dt - \sigma_x(t, x_t^*) z_t dB_t \\ &- \int_\Gamma f_x(t, x_{t-}^*, \theta, u_t^*) z_t \widetilde{N}(dt, d\theta), \end{aligned}$$

then

$$\begin{aligned} y_t^h &= y_{t_0+h}^h + \int_{t_0+h}^t \int_0^1 b_x(s, x_s^{h,\lambda}, u_s^*) y_s^h d\lambda ds + \int_{t_0+h}^t \int_0^1 \sigma_x(s, x_s^{h,\lambda}) y_s^h d\lambda dB_s \\ &+ \int_0^1 \int_{t_0+h}^t \int_\Gamma f_x(s, x_s^{h,\lambda}, \theta, u_s^*) y_s^h d\lambda \widetilde{N}(ds, d\theta) + \rho_t^h, \end{aligned}$$

where

$$\begin{aligned} \rho_t^h &= \int_{t_0+h}^t \int_0^1 b_x(s, x_s^{h,\lambda}, u_s^*) z_s d\lambda ds + \int_{t_0+h}^t \int_0^1 \sigma_x(s, x_s^{h,\lambda}) z_s d\lambda dB_s + \int_{t_0+h}^t \int_0^1 \int_\Gamma f_x(s, x_s^{h,\lambda}, \theta, u_s^*) z_s d\lambda \widetilde{N}(ds, d\theta) \\ &- \int_{t_0+h}^t b_x(s, x_s^*, u_s^*) z_s ds - \int_{t_0+h}^t \sigma_x(s, x_s^*) z_s dB_s - \int_{t_0+h}^t \int_\Gamma f_x(s, x_{s-}^*, \theta, u_s^*) z_s \widetilde{N}(ds, d\theta). \end{aligned}$$

Hence

$$\begin{aligned} E\left|y_t^h\right|^2 &\leq E\left|y_{t_0+h}^h\right|^2 + KE\int_{t_0+h}^t \left|\int_0^1 b_x(s, x_s^{h,\lambda}, u_s^h) y_s^h d\lambda\right|^2 ds + KE\int_{t_0+h}^t \left|\int_0^1 \sigma_x(s, x_s^{h,\lambda}) y_s^h d\lambda\right|^2 ds \\ &+ KE\int_{t_0+h}^t \int_\Gamma \left|\int_0^1 f_x(s, x_s^{h,\lambda}, \theta, u_s^h) y_s^h d\lambda\right|^2 \upsilon(d\theta) ds + KE \sup_{t_0+h \leq t \leq T} \left|\rho_t^h\right|^2. \end{aligned}$$

Since b_x, σ_x, and f_x are bounded, then

$$E\left|y_t^h\right|^2 \leq E\left|y_{t_0+h}^h\right|^2 + CE\int_0^t \left|y_s^h\right|^2 ds + KE \sup_{t_0+h \leq t \leq T} \left|\rho_t^h\right|^2.$$

We conclude by the continuity of b_x, σ_x and f_x, and the dominated convergence that $\lim_{h\to 0} E\sup_{t_0+h\le t\le T}\left|\rho_t^h\right|^2 = 0$. Hence by the Gronwall lemma, and (2.16) we get

$$\lim_{h\to 0} E\left|y_t^h\right|^2 = 0.$$

The second estimate is proved in a similar way. ■

We use the same notations as in the proof of Lemma (2.4), we can prove the corollary (2.2).

Proof. We have by the definition of J that

$$\frac{1}{h}\left[J(u^h) - J(u^*)\right] = \frac{1}{h}\left[E\left[g(x_T^h) - g(x_T^*)\right] + \int_{t_0}^{T}\left[h(t,x_t^h,u_t^h) - h(t,x_t^*,u_t^*)\right]dt\right],$$

then,

$$\frac{1}{h}\left[J(u^h) - J(u^*)\right] = E\left[\int_0^1 g_x(x_T^{h,\lambda})\left(\frac{x_T^h - x_T^*}{h}\right)d\lambda + \frac{1}{h}\int_{t_0}^{T}\left[h(t,x_t^h,u_t^h) - h(t,x_t^*,u_t^*)\right]dt\right].$$

From Lemma (2.4), we obtain (2.3) by letting h tend to 0. ■

Let us introduce the adjoint process. We proceed as in [9] and [27].

Let $\varphi(t,\tau)$ be the solution of the linear equation

$$\begin{cases} d\varphi(t,\tau) = b_x(t,x_t^*,u_t^*)\varphi(t,\tau) + \sigma_x(t,x_t^*)\varphi(t,\tau)dB_t \\ \qquad\qquad + \displaystyle\int_\Gamma f_x(t,x_{t^-}^*,\theta,u_t^*)\varphi(t^-,\tau)\widetilde{N}(dt,d\theta) & 0\le\tau\le t\le T. \\ \varphi(\tau,\tau) = I_d \end{cases} \tag{2.17}$$

This equation is linear with bounded coefficients. Hence it admits a unique strong solution. Moreover, the process φ is invertible, with an inverse ψ satisfying suitable integrability conditions.

From Itô's formula, we can easily check that $d(\varphi(t,\tau)\psi(t,\tau)) = 0$, and $\varphi(\tau,\tau)\psi(\tau,\tau) = I_d$, where ψ is the solution of the following equation

$$\begin{cases} d\psi(t,\tau) = \Big(\sigma_x(t,x_t^*)\psi(t,\tau)\sigma_x(t,x_t^*) - b_x(t,x_t^*,u_t^*)\psi(t,\tau) \\ -\int\limits_\Gamma f_x(t,x_{t^-}^*,\theta,u_t^*)\psi(t^-,\tau)\upsilon(d\theta)dt\Big) \\ -\sigma_x(t,x_t^*)\psi(t,\tau)dB_t & 0 \le \tau \le t \le T \\ -\psi(t^-,\tau)\int\limits_\Gamma \left(f_x(t,x_{t^-}^*,\theta,u_t^*) + I_d\right)^{-1} f_x(t,x_{t^-}^*,\theta,u_t^*)N(dt,d\theta) \\ \psi(\tau,\tau) = I_d. \end{cases}$$

If $\tau = 0$ we simply write $\varphi(t,0) = \varphi_t$ and $\psi(t,0) = \psi_t$.

By the uniqueness property, it is easy to check that

$$z_t = \varphi(t,t_0)\left[b(t_0,x_{t_0}^*,\nu) - b(t_0,x_{t_0}^*,u_{t_0}^*)\right],$$

then, (2.3) will become

$$\begin{aligned} \left.\frac{dJ(u^h)}{dh}\right|_{h=0} &= E\left[\int\limits_{t_0}^{T} h_x(t,x_t^*,u_t^*)\varphi(t,t_0)\left[b(t_0,x_{t_0}^*,\nu) - b(t_0,x_{t_0}^*,u_{t_0}^*)\right]dt\right. \\ &+g_x(x_T^*)\varphi(T,t_0)\left[b(t_0,x_{t_0}^*,\nu) - b(t_0,x_{t_0}^*,u_{t_0}^*)\right] \\ &\left.+\left[h(t_0,x_{t_0}^*,\nu) - h(t_0,x_{t_0}^*,u_{t_0}^*)\right]\right]. \end{aligned} \tag{2.18}$$

Now, if we define the adjoint process by

$$p_t = y_t\psi_t^*, \tag{2.19}$$

where

$$\begin{aligned} y_t &= E\left[g_x(x_T^*)\varphi_T^* + \int\limits_t^T h_x(s,x_s^*,u_s^*)\varphi_s^* dt \mid \mathcal{F}_t\right] \\ &= E\left[X \mid \mathcal{F}_t\right] - \int\limits_0^t h_x(s,x_s^*,u_s^*)\varphi_s^* dt, \end{aligned}$$

with

$$X = g_x(x_T^*)\varphi_T^* + \int_0^T h_x(s, x_s^*, u_s^*)\varphi_s^* dt.$$

It follows that

$$\left.\frac{dJ(u^h)}{dh}\right|_{h=0} = E\left[p_t\left[b(t_0, x_{t_0}^*, \nu) - b(t_0, x_{t_0}^*, u_{t_0}^*)\right] + \left[h(t_0, x_{t_0}^*, \nu) - h(t_0, x_{t_0}^*, u_{t_0}^*)\right]\right].$$

Define the Hamiltonian H from $[0;T] \times \mathbb{R}^n \times A \times \mathbb{R}^n$ into $\mathbb{R}$ by

$$H(t, x, u, p) = h(t, x_t, u_t) + pb(t, x_t, u_t), \tag{2.20}$$

we get from optimality of u^*

$$E\left[H(t_0, x_{t_0}, \nu, p_{t_0}) - H(t_0, x_{t_0}, u_{t_0}^*, p_{t_0})\right] \geq 0.dt_0 - a.e.$$

By the Itô representation theorem [19], there exist two processes $Q \in \mathcal{M}^2$ and $R \in \mathcal{L}^2$ satisfying

$$E\left[X \mid \mathcal{F}_t\right] = E\left[X\right] + \int_0^t Q_s dB_s + \int_0^t \int_\Gamma R_s(\theta)\widetilde{N}(ds, d\theta),$$

hence,

$$y_t = E\left[X\right] - \int_0^t h_x(s, x_s^*, u_s^*)\varphi_s ds + \int_0^t Q_s dB_s + \int_0^t \int_\Gamma R_s(\theta)\widetilde{N}(ds, d\theta).$$

Let

$$\begin{aligned} q_t &= Q_t\psi_t - p_t\sigma_x(t, x_t^*), \\ r_t(\theta) &= R_t(\theta)\psi_t\left(f_x(t, x_{t-}^*, \theta, u_t^*) + I_d\right)^{-1} + p_t\left[\left(f_x(t, x_{t-}^*, \theta, u_t^*) + I_d\right) - I_d\right]. \end{aligned}$$

The above discussion will allow us to introduce the next theorem which is the main result of this subsection.

Theorem 2.2 (maximum principle for strict control) *Let u^* be the optimal strict* *control minimizing the cost $J(.)$ over U, and denote by x^* the corresponding optimal trajectory, then there exist a unique triple of square integrable adapted processes (p,q,r) which is the solution of the backward SDE*

$$\begin{cases} dp_t = -\left[h_x(t,x_t^*,u_t^*) + p_t b_x(t,x_t^*,u_t^*) + q_t\sigma_x(t,x_t^*) \right. \\ \left. + \int\limits_\Gamma r_t(\theta) f(t,x_{t-}^*,\theta,u_t^*)\upsilon(d\theta)\right] dt. \\ +q_t dB_t + \int\limits_\Gamma r_t(\theta)\widetilde{N}(dt,d\theta) \\ p_T = g_x(x_T^*). \end{cases}$$

such that for all $\nu \in U$ the following inequality holds

$$E\left[H(t,x_t^*,\nu,p_t) - H(t,x_t^*,u_t^*,p_t)\right] \geq 0.dt - a.e.$$

where the Hamiltonian H is defined by (2.20).

2.3 The maximum principle for near optimal controls

We know that there is always a near optimal control. It is interesting to know what kind of necessary conditions are verified by these controls. For this we need to introduce the Ekeland's variational principle

Lemma 2.5 (Ekeland's variational principle) *Let (E,d) be a complete metric space and $f : E \to \overline{\mathbb{R}}$ be lower semicontinuous and bounded from below. Given $\varepsilon > 0$, suppose $u^\varepsilon \in E$ satisfies $f(u^\varepsilon) \leq \inf(f) + \varepsilon$. Then for any $\lambda > 0$, there exists $\nu \in E$ such that*

- $f(\nu) \leq f(u^\varepsilon)$
- $d(u^\varepsilon, \nu) \leq \lambda$
- $f(\nu) \leq f(\omega) + \frac{\varepsilon}{\lambda} d(\omega,\nu)$ *for all $\omega \neq \nu$.*

To apply Ekeland's variational principle, we have to endow the set U of strict controls with an appropriate metric. For any u and $\nu \in U$, we set

$$d(u, \nu) = P \otimes dt \left\{ (\omega, t) \in \Omega \times [0;T] ,\ u(\omega, t) \neq \nu(\omega, t) \right\}.$$

A suitable version of Ekeland's variational principle implies that, given any $\varepsilon_n > 0$, there exist $u^n \in U$ such that

$$J(u^n) \leq J(u) + \varepsilon_n d(u^n, u),\ \forall u \in U. \tag{2.21}$$

Let us define the perturbation, $\forall u \in U, S \in \mathcal{F}_{t_0}$

$$u^{n,h} = \begin{cases} u; \quad (t, \omega) \in [t_0; t_0 + h] \times S \\ u^n \text{....otherwise.} \end{cases}$$

from (2.21), we have

$$0 \leq J(u^{n,h}) - J(u^n) + \varepsilon_n d(u^{n,h}, u^n),$$

by the definition of d, it holds that

$$J(u^n) \leq J(u^{n,h}) + \varepsilon_n h C,$$

where C is a positive constant.

Finally, we use the same method as in the previous chapter, we can prove the next theorem which is the main result of this subsection

Theorem 2.3 *For each $\varepsilon_n > 0$,there exists $(u^n) \in U$ such that there exist a unique triple of square integrable adapted processes (p^n, q^n, r^n) which is the solution of the backward*

SDE

$$\begin{cases} dp_t^n = -\left[h_x(t, x_t^n, u_t^n) + p_t^n b_x(t, x_t^n, u_t^n) + q_t^n \sigma_x(t, x_t^n) \right. \\ \left. + \int\limits_\Gamma r_t^n(\theta) f(t, x_{t^-}^n, \theta, u_t^n) \upsilon(d\theta) \right] dt. \\ +q_t^n dB_t + \int\limits_\Gamma r_t^n(\theta) \widetilde{N}(dt, d\theta) \\ p_T^n = g_x(x_T^n). \end{cases} \tag{2.22}$$

such that for all $u \in U$

$$E\left[H(t, x_t^n, u, p_t^n) - H(t, x_t^n, u_t^n, p_t^n)\right] + C\varepsilon_n \geq 0, \tag{2.23}$$

where C *is a positive constant.*

Chapter 3

The relaxed maximum principle of controlled jump diffusions

We know that in theory of stochastic control of diffusions and in the absence of additional convexity assumptions on the coefficients of the control problem defined in the second chapter has no optimal solution. For that, we should inject the space of strict controls in a wider space that has good properties of compactness and convexity. This space is that of probability measures on A where A is the set of values taken by the strict control. In this new space, controls called relaxed controls. The first existence result of an optimal relaxed control is proved by Fleming [14], for the SDEs with uncontrolled diffusion coefficient and no jump term. For such systems of SDEs a maximum principle has been established in [2, 3, 26]. The case where the control variable appears in the diffusion coefficient has been solved in [13]. The existence of an optimal relaxed control of SDEs, where the control variable enters in the jump term was derived by Kushner [23]. Our main goal in this chapter is show that under a continuity condition of the coefficients, each relaxed diffusion process with controlled jump is a strong limit of a sequence of diffusion processes associated with strict controls. The proof of this approximation result is based on Skorokhod selection theorem, and the tightness of the processes. Consequently, we show that the strict and the relaxed control problems have the same value function. Using the same techniques, we give another proof of the existence of an optimal relaxed control, based on the Skorokhod selection theorem. After that we establish a Pontriagin maximum principle for the relaxed control problem. More precisely we derive necessary conditions for optimality satisfied by an optimal control. The proof is based on Pontriagin's maximum principle for nearly optimal strict controls and some stability results of trajectories and adjoint processes with respect to the control variable.

3.1 Formulation of the relaxed control problem

If μ^n is a sequence of admissible relaxed controls with corresponding solution x^n, then there might be a weakly convergent subsequence of (x^n, μ^n) whose limit does not satisfy (2.1) for some Brownian motion, Poisson measure and admissible control u, because in the relaxed control framework, the way of representing the limit controlled jump terms not clear. To get the desired closure or compactness, it is necessary to enlarge the model, for that we need to introduce an extension of the Poisson measure, which we call the relaxed Poisson measure, to do this we follow closely [22].

Let us begin with a simple example. Suppose that the admissible control u takes the two values a_1 and a_2 such that

$$u^\rho(t) = \begin{cases} a_1, & t \in [k\rho; k\rho + \beta_1\rho] \\ a_2, & t \in [k\rho + \beta_1\rho; k\rho + \rho] \end{cases} \qquad k = 1, 2, ...$$

where $\rho > 0$, and $\beta_1 + \beta_2 = 1$.

Let x^ρ denote the associated solution to (2.1), if we define $1_i^\rho(s)$ by

$$1_i^\rho(s) = \begin{cases} 1, & u^\rho(s) = a_i(s) \\ 0, & \text{otherwise,} \end{cases}$$

then the SDE (2.1) takes the form

$$\begin{cases} dx_t^\rho & = \sum\limits_{i=1}^{2} 1_i^\rho(t) b(t, x_t^\rho, a_i(t))dt + \sigma(t, x_t^\rho)dB_t + \sum\limits_{i=1}^{2} \int\limits_\Gamma 1_i^\rho(t) f(t, x_{t-}^\rho, \theta, a_i(t)) N(dt, d\theta) \\ x_0^\rho & = 0. \end{cases}$$

Let μ^ρ denote the relaxed version of the control u^ρ, that is $\mu_t^\rho(da_i)dt = \delta_{u^\rho(t)}(da_i)dt$. It is easy to see that $1_i^\rho(s) = \mu_t^\rho(a_i)$ which converges weakly to $\mu_t(a_i) = \beta_i$, when $\rho \to 0$, indeed

$$\begin{aligned} \int\limits_{\mathbb{R}} \int\limits_A \varphi(a_i)\delta_{u^\rho(t)}(da_i)dt &= \int\limits_{\mathbb{R}} \varphi(u^\rho(t))dt \\ &= \int\limits_{\mathbb{R}} \varphi(a_1) 1_{[k\rho;k\rho+\beta_1\rho]}(t)dt + \int\limits_{\mathbb{R}} \varphi(a_2) 1_{[k\rho+\beta_1\rho;k\rho+\rho]}(t)dt \\ &= \varphi(a_1)\beta_1\rho + \varphi(a_2)\beta_2\rho \\ &= \int\limits_{\mathbb{R}} \sum_{i=1}^{2} \varphi(a_i)\beta_i 1_{[k\rho;k\rho+\rho]}(t)dt \\ &= \int\limits_{\mathbb{R}} \varphi(a_1)\beta_1 1_{[k\rho;k\rho+\beta_1\rho]}(t)dt + \int\limits_{\mathbb{R}} \varphi(a_2)\beta_2 1_{[k\rho+\beta_1\rho;k\rho+\rho]}(t)dt \\ &= \int\limits_{\mathbb{R}} \int\limits_A \varphi(a_i)\beta_i dt. \end{aligned}$$

By the tightness of the set of jumps, we can fix a weakly convergent subsequence of the jumps, such that the limit satisfies the following SDE

$$\begin{cases} dx_t & = \int\limits_A b(t, x_t, a_t)\mu_t(da)dt + \sigma(t, x_t)dB_t + \sum\limits_{i=1}^{2} \int\limits_\Gamma f(t, x_{t-}, \theta, a_i(t)) \overline{N_i}(dt, d\theta) \\ x_0 & = 0, \end{cases}$$

where $\overline{N_i}, i = 1, 2$ are independent Poisson measures with compensator $\upsilon(d\theta)\beta_i dt$.

Remark 3.1 *Note that the previous type of approximation can be adapted to the case where the fractions of the intervals on which the a_i are used are time dependent in a* *nonanticipative way. in this case the compensator of* $\overline{N_i}, i = 1, 2,$ *is the random and time varying quantity* $\upsilon(d\theta)\mu_t(da_i)dt$. *Moreover, the* $\overline{N_i}, i = 1, 2,$ *would not be independent, but the martingales defined by*

$$\int_0^t \int_\Gamma 1_i^\rho(s) f(s, x_{s^-}^\rho, \theta, a_i) N(ds, d\theta) - \int_0^t \int_\Gamma f(s, x_{s^-}^\rho, \theta, a_i) \upsilon(d\theta) \mu_s^\rho(da_i) ds$$

converge weakly to the processes

$$\int_0^t \int_\Gamma f(s, x_{s^-}, \theta, a_i) \overline{N_i}(dt, d\theta) - \int_0^t \int_\Gamma f(s, x_{s^-}, \theta, a_i) \upsilon(d\theta) \mu_s(da_i) ds$$

which are orthogonal $\mathcal{F}_t-$martingales.

General case

The above discussion suggests a generalization of the concept of Poisson measure. For that, let μ be the relaxed representation of an admissible control u, and let $A_0 \in \mathcal{B}(A)$ and $\Gamma_0 \in \mathcal{B}(\Gamma)$. Then define

$$N^\mu([0; t], A_0, \Gamma_0) \equiv N^\mu(t, A_0, \Gamma_0) = \int_0^t \int_{\Gamma_0} 1_{A_0}(u(s)) N(ds, d\theta),$$

the number of jumps of $\int_0^{\cdot} \int_{\Gamma_0} \theta N(ds, d\theta)$ on $[0; t]$ with values in Γ_0 and where $u(s) \in A_0$ at the jump times s.

Since $1_{A_0}(u(s)) = \mu_s(A_0)$, then the compensator of the counting measure valued process N^μ is $\upsilon(d\theta)\mu_t(da)dt = \mu_t \otimes \upsilon(da, d\theta)dt$. Moreover, for bounded and measurable real-valued functions $\varphi(.)$, the process

$$\int_0^t \int_\Gamma \int_A \varphi(s, x_{s^-}, \theta, a) N^\mu(dt, d\theta, da) - \int_0^t \int_\Gamma \int_A \varphi(s, x_{s^-}, \theta, a) \upsilon(d\theta) \mu_s(da) ds$$

is also an $\mathcal{F}_t-$martingale.

Definition 3.1 *A relaxed Poisson measure N^μ is a counting measure valued process such that its compensator is the product measure of the relaxed control μ with the compensator υ of N. Which have the property that for any Borel set $\Gamma_0 \subset \Gamma$ and $A_0 \subset A$, the processes*

$$\widetilde{N}^\mu(t, A_0, \Gamma_0) = N^\mu(t, A_0, \Gamma_0) - \mu(t, A_0)\upsilon(\Gamma_0)$$

are $\mathcal{F}_t-$martingales and are orthogonal for disjoint $\Gamma_0 \times A_0$.

Now, we can write the stochastic differential equation with controlled jumps in terms of relaxed Poisson measure as follows

$$\begin{cases} dx_t^\mu & = \int\limits_A b(t, x_t^\mu, a)\mu_t(da)dt + \sigma(t, x_t^\mu)dB_t + \int\limits_A \int\limits_\Gamma f(t, x_{t^-}^\mu, \theta, a)\widetilde{N}^\mu(dt, d\theta, da) \\ x_0^\mu & = 0. \end{cases} \tag{3.1}$$

The expected cost associated to a relaxed control is defined as

$$J(\mu) = E\left[g(x_T^\mu) + \int\limits_A \int\limits_0^T h(t, x_t^\mu, a)\mu_t(da)dt\right].$$

Consider a sequence of random predictable measures $(\mu_s^n \otimes \nu)_n$ converging weakly to $\mu_s \otimes \nu$ on $[0;T] \times A \times \Gamma$ $P-$almost surely, then there exists a sequence of orthogonal martingale measures $\widetilde{N}^n$ defined on $\Omega \times [0;T] \times A \times \Gamma$ with compensator $\mu_s^n \otimes \nu(da, d\theta)ds$, such that for each bounded function φ

$$\int\limits_0^t \int\limits_A \int\limits_\Gamma \varphi(s, x_{s^-}^\mu, \theta, a)\widetilde{N}^n(ds, d\theta, da) \text{ converges to } \int\limits_0^t \int\limits_A \int\limits_\Gamma \varphi(s, x_{s^-}^\mu, \theta, a)\widetilde{N}^\mu(ds, d\theta, da).$$

3.2 Approximations and existence of relaxed control

3.2.1 Approximation of trajectories

In order for the relaxed control problem to be truly an extension of the strict one, the infimum of the expected cost for the relaxed controls must be equal to the infimum for the strict controls. This result is based on the approximation of a relaxed control by a sequence of strict controls, given by the next lemma, which called chattering lemma.

Lemma 3.1 *Let μ be a predictable process with values in the space $\mathcal{P}(A)$. Then there exist a sequence of predictable processes (u^n) with values in A such that*

$$\mu_t^n(da)dt = \delta_{u_t^n}(da)dt \longrightarrow \mu_t(da)dt \qquad \textit{weakly..}$$

Proof. see [14] ■

The next theorem which is our main result in this section gives the stability of the stochastic differential equations with respect to the control variable, and that the two problems has the same infimum of the expected costs.

Theorem 3.1 *Let μ be a relaxed control, and let x^μ be the corresponding trajectory. Then there exist a sequence (u^n) of strict controls such that*

$$\lim_{n\to\infty} E\left[\sup_{0\le t\le T} |x_t^n - x_t^\mu|^2\right] = 0$$

and

$$\lim_{n\to\infty} J(u^n) = J(\mu). \tag{3.2}$$

where x^n denote the trajectory associated with (u^n).

To prove theorem (**??**) and theorem (3.1), we need some results on the tightness of the processes.

Lemma 3.2 *The family of relaxed controls* $((\mu^n)_{n\geq 0}, \mu)$ *is tight in* $\mathcal{R}$ *the space of probability measures on* $[0;T] \times A$.

Proof. see [3]. ■

Lemma 3.3 *The family of martingale measures* $((\widetilde{N}^n)_{n\geq 0}, \widetilde{N}^\mu)$ *is tight in the space* $D_{S'}([0;T])$ *of all mappings càdlàg from* $[0;T]$ *with values in* S' *the topological dual of the Schwartz space* S *of rapidly decreasing functions.*

Proof. If we denote

$$Y_t^n = \int_0^t \int_{A\times\Gamma} \psi(t, x_{t^-}^n, \theta, a)\widetilde{N}^n(dt, d\theta, da)$$

and

$$Y_t = \int_0^t \int_{A\times\Gamma} \psi(t, x_{t^-}^\mu, \theta, a)\widetilde{N}^\mu(dt, d\theta, da),$$

and let S, T two stopping times, such that $S \leq T \leq S + \theta$, then we have $\forall n \in \mathbb{N}$, $\epsilon > 0$, $\exists m$ and $k > 0$, such that

$$n \geq m \qquad P(\sup_{t\leq n} |Y_t^n| > k) \leq \frac{E\,|Y_t^n|^2}{k^2} \leq \varepsilon,$$

and $\forall n \in \mathbb{N}$, $\epsilon > 0$, by the proposition (4.1) (see appendix), we have

$$P(\sup_{t\in[S;T]} |Y_S^n - Y_T^n| \geq \eta) \leq \frac{\varepsilon}{\eta^2} + P(< Y^n >_T - < Y^n >_S \geq k).$$

Since $< Y^n >_T - < Y^n >_S \leq \omega(< Y^n >, \delta) = \sup_{|T-S|<\delta} |< Y^n >_T - < Y^n >_S|$, because $|T - S| \leq \delta$. This implies that

$$P(\sup_{t\in[S;T]} |Y_S^n - Y_T^n| \geq \eta) \leq \frac{\varepsilon}{\eta^2} + P(\omega(< Y^n >, \delta) \geq k).$$

By the $\mathcal{C}-$tightness of $< Y^n >$, we have

$$P(\omega(< Y^n >, \delta) \geq k) \leq \varepsilon.$$

Finally, we conclude that

$$\lim_{\delta \to 0} \limsup_{n} \sup_{S \leq T \leq S+\theta} P(\sup_{t \in [S;T]} |Y_S^n - Y_T^n| \geq \eta) = 0,$$

that is the Aldous conditions is fulfilled (see appendix). Hence the sequence $(Y_t^n)_{n \geq 0}$ is tight. By the same method we can prove the tightness of (Y_t) . ■

Lemma 3.4 *if x^n and x are the solutions of (3.1) associated with μ^n and μ respectively, then the family of processes (x^n, x) is tight in $D([0;T], \mathbb{R}^d)$.*

Proof. By the same method in the proof of lemma (3.3). ■

Proof of theorem 4.2. 1- Let μ be a relaxed control, then by the Lemma (3.1), there exists a sequence (μ^n) such that $\mu_t^n(da)dt = \delta_{u_t^n}(da)dt \longrightarrow \mu_t(da)dt$ in $\mathcal{R}$, $P-a.s$, Let x^n, and x are the solutions of (3.1) associated with μ^n and μ, respectively, Suppose that the result of theorem (3.1) is false, then there exists $\gamma > 0$ such that

$$\inf_n E\left[|x_t^n - x_t^\mu|^2 \right] \geq \gamma. \tag{3.3}$$

According to lemmas (3.2), (3.3) and (3.4), the family of processes

$$\beta^n = (\mu^n, \mu, x^n, x, \widetilde{N}^n, \widetilde{N}^\mu)$$

is tight in the space

$$(\mathcal{R} \times \mathcal{R}) \times (D \times D) \times (D_{S'} \times D_{S'}).$$

Then, by the Skorokhod selection theorem, there exist a probability space $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{P})$ and a sequence $\widehat{\beta^n} = (\widehat{\mu^n}, \widehat{\upsilon^n}, \widehat{x^n}, \widehat{y^n}, \widehat{\widetilde{N}^n}, \widehat{\widetilde{M}^n})$ defined on it such that

a- For each $n \in \mathbb{N}$, the laws of β^n and $\widehat{\beta^n}$ coincides,

b- there exists a subsequence $(\widehat{\beta^{n_k}})$ of $(\widehat{\beta^n})$ which converges to $\widehat{\beta}$, $\widehat{P}-a.s$ on the space $(\mathcal{R} \times \mathcal{R}) \times (D \times D) \times (D_{S'} \times D_{S'})$, where $\widehat{\beta} = (\widehat{\mu}, \widehat{\upsilon}, \widehat{x}, \widehat{y}, \widehat{\widetilde{N}^\mu}, \widehat{\widetilde{M}^\mu})$.

By the uniform integrability, we have

$$\gamma \leq \liminf_{n} E\left[\sup_{0\leq t\leq T}\left|x_t^n - x_t^{\mu}\right|^2\right] = \liminf_{n} \widehat{E}\left[\sup_{0\leq t\leq T}\left|\widehat{x_t^n} - \widehat{y_t^n}\right|^2\right] = \widehat{E}\left[\sup_{0\leq t\leq T}\left|\widehat{x}_t - \widehat{y}_t\right|^2\right],$$

where $\widehat{E}$ is the expectation with respect to $\widehat{P}$. We see that $\widehat{x_t^n}$ and $\widehat{y_t^n}$ satisfy the following equations

$$\begin{cases} d\widehat{x_t^n} &= \int\limits_A b(s, \widehat{x_t^n}, a)\widehat{\mu_s^n}(da)ds + \sigma(s, \widehat{x_t^n})dB_s + \int\limits_A\int\limits_\Gamma f(s, \widehat{x_{t-}^n}, \theta, a)\widehat{\widetilde{N}^n}(ds, d\theta, da) \\ \widehat{x_0^n} &= 0, \end{cases}$$

$$\begin{cases} d\widehat{y_t^n} &= \int\limits_A b(s, \widehat{y_t^n}, a)\widehat{\upsilon_s^n}(da)ds + \sigma(s, \widehat{y_t^n})dB_s + \int\limits_A\int\limits_\Gamma f(s, \widehat{y_{t-}^n}, \theta, a)\widehat{\widetilde{M}^n}(ds, d\theta, da) \\ \widehat{y_0^n} &= 0, \end{cases}$$

using the fact that $(\widehat{\beta^n})$ converges to $\widehat{\beta}$, $\widehat{P} - a.s$, it holds that $(\widehat{x_t^n})$ and $(\widehat{y_t^n})$ converge respectively to $\widehat{x}_t$ and $\widehat{y}_t$, which satisfy

$$\begin{cases} d\widehat{x}_t &= \int\limits_A b(t, \widehat{x}_t, a)\widehat{\mu t}(da)dt + \sigma(t, \widehat{x}_t)dB_t + \int\limits_A\int\limits_\Gamma f(t, \widehat{x_{t^-}}, \theta, a)\widehat{\widetilde{N}^{\mu}}(dt, d\theta, da) \\ \widehat{x_0} &= 0, \end{cases}$$

$$\begin{cases} d\widehat{y}_t &= \int\limits_A b(t, \widehat{y}_t, a)\widehat{\upsilon}_t(da)dt + \sigma(t, \widehat{y}_t)dB_t + \int\limits_A\int\limits_\Gamma f(t, \widehat{y_{t^-}}, \theta, a)\widehat{\widetilde{M}^{\mu}}(dt, d\theta, da) \\ \widehat{y}_0 &= 0. \end{cases}$$

By the lemma (3.1), the sequence (μ^n, μ) converges to (μ, μ) in $\mathcal{R}^2$. Moreover

$$law(\mu^n, \mu) = law(\widehat{\mu^n}, \widehat{\upsilon^n}),$$

$$(\widehat{\mu^n}, \widehat{\upsilon^n}) \longrightarrow (\widehat{\mu}, \widehat{\upsilon}),\ \widehat{P} - a.s \ \ in\ \mathcal{R}^2,$$

if n tends to ∞.

Hence, $law(\widehat{\mu}, \widehat{v}) = law(\mu, \mu)$, then $\widehat{\mu} = \widehat{v}$, $\widehat{P} - a.s.$ By the same method we can prove that $\widehat{\widetilde{N}^{\mu}}(ds, d\theta, da) = \widehat{\widetilde{M}^{\mu}}(ds, d\theta, da)$, $\widehat{P} - a.s.$ It follows that $\widehat{x_t} = \widehat{y_t}$, $\widehat{P} - a.s$, by the uniqueness of solution. Which is a contradiction (3.3).

2- By using the Cauchy-Schawrz inequality, we get

$$\begin{aligned} |J(u^n) - J(\mu)| &\leq C \left(E \left| g(x_T^n) - g(x_T^{\mu}|^2 \right) \right)^{\frac{1}{2}} \\ &+ CE \left| \int_0^t \int_A h(s, x_s^n, a) \mu_s^n(da) ds - \int_0^t \int_A h(s, x_s^n, a) \mu_s(da) ds \right| \\ &+ C \int_0^t \left(E \left| h(s, x_s^n, u) - h(s, x_s^{\mu}, u) \right|^2 \right)^{\frac{1}{2}} ds. \end{aligned}$$

The first and the third terms in the right hand side converge to 0 because g and h are bounded and continuous functions in x, and the fact that

$$\lim_{n \to \infty} E \left[|x_t^n - x_t^{\mu}|^2 \right] = 0.$$

Since h is bounded and continuous in a, an application of the dominated convergence theorem allows us to conclude that the second term in the right hand side tends to 0 . ■

3.2.2 Existence of an optimal relaxed control

We show in this section that there exist an optimal solution for the relaxed control problem, the proof is based on Skorokhod selection theorem and some results of tightness.

Theorem 3.2 *Under assumptions on the coefficients b, σ, f, g, and h the relaxed control problem admits an optimal relaxed control.*

proof . Let (x^n, μ^n) be a minimizing sequence for the cost function $J(\mu)$, that is

$$\lim_{n \to \infty} J(\mu^n) = \inf_{\mu \in \mathcal{R}} J(\mu),$$

where x^n is the solution of (3.1) corresponding to μ^n.

According to lemmas (3.2), (3.3) and (3.4), the family of processes $\beta^n = (\mu^n, x^n, \widetilde{N^n})$ is tight in the space $(\mathcal{R}, D, D_{S'})$, by the Skorokhod selection theorem, there exist a probability space $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{P})$ and a sequence $\widehat{\beta^n} = (\widehat{\mu^n}, \widehat{x^n}, \widehat{\widetilde{N^n}})$ defined on it such that

1. For each $n \in \mathbb{N}$, the laws of β^n and $\widehat{\beta^n}$ coincides,

2. there exists a subsequence $(\widehat{\beta^{n_k}})$ of $(\widehat{\beta^n})$ which converges to $\widehat{\beta}$, $\widehat{P} - a.s$ on the space $\mathcal{R} \times D \times D_{S'}$, where $\widehat{\beta} = (\widehat{\mu}, \widehat{x}, \widehat{\widetilde{N}})$ it holds that $\widehat{x^n} \underrightarrow{proba} \widehat{x}$ then, we have

$$\begin{aligned}
\left|J(\widehat{\mu^{n_k}}) - J(\widehat{\mu})\right| &\leq E\left|g(\widehat{x_T^{n_k}}) - g(\widehat{x_T})\right| \\
&+E\left|\int_0^T\int_A h(t, \widehat{x_t^{n_k}}, a)\widehat{\mu_t^{n_k}}(da)dt - \int_0^T\int_A h(t, \widehat{x}_t, a)\widehat{\mu_t^{n_k}}(da)dt\right| \\
&+E\left|\int_0^T\int_A h(t, \widehat{x}_t, a)\widehat{\mu_t^{n_k}}(da)dt - \int_0^T\int_A h(t, \widehat{x}_t, a)\widehat{\mu}_t(da)dt\right|,
\end{aligned}$$

then

$$\begin{aligned}
\left|J(\widehat{\mu^{n_k}}) - J(\widehat{\mu})\right| &\leq E\left|g(\widehat{x_T^{n_k}}) - g(\widehat{x_T})\right| \\
&+E\int_0^T \left|h(t, \widehat{x_t^{n_k}}, u_t^{n_k}) - h(t, \widehat{x}_t, u_t^{n_k})\right| dt \\
&+E\left|\int_0^T\int_A h(t, \widehat{x}_t, a)\widehat{\mu_t^{n_k}}(da)dt - \int_0^T\int_A h(t, \widehat{x}_t, a)\widehat{\mu}_t(da)dt\right|.
\end{aligned}$$

The first and second terms in the right-hand side converge to 0, because h and g are bounded and continuous functions with respect to x. Using the convergence of $(\widehat{\mu_t^{n_k}})_n$ to $\widehat{\mu}_t$, and the dominated convergence theorem to conclude that the last term tends to 0. Hence

$$\inf_{\mu \in \mathcal{R}} J(\mu) = \lim_{n\to\infty} J(\mu^n) = \lim_{n\to\infty} J(\widehat{\mu^n}) = \lim_{n\to\infty} J(\widehat{\mu^{n_k}}) = J(\widehat{\mu}),$$

then $\widehat{\mu}$ is an optimal control. ■

Remark 3.2 *From the previous results, we see that the relaxed model is a true extension of the strict one, because the infimum of the two cost functions are equal, and the relaxed model have an optimal solution.*

3.3 Maximum principle for relaxed control problems

Now, we can introduce the next theorem, which is the main result of this section.

Theorem 3.3 (The relaxed stochastic maximum principle) *Let μ^* be an optimal relaxed control minimizing the functional J over $\mathcal{R}$, and let $x_t^{\mu^*}$ be the corresponding optimal trajectory. Then there exist a unique triple of square integrable and adapted processes $(p^{\mu^*}, q^{\mu^*}, r^{\mu^*})$ which is the solution of the backward SDE*

$$\begin{cases} dp_t^{\mu^*} = -\left[\int_A h_x(t, x_t^{\mu^*}, a)\mu_t^*(da) + \int_A p_t^{\mu^*} b_x(t, x_t^{\mu^*}, a)\mu_t^*(da) + q_t^{\mu^*}\sigma_x(t, x_t^{\mu^*}) \right. \\ \left. +\int_A\int_\Gamma r_t^{\mu^*}(\theta) f(t, x_{t-}^{\mu^*}, \theta, a)\mu_t^* \otimes v(da, d\theta).\right] dt \\ +q_t^{\mu^*} dB_t + \int_\Gamma r_t^{\mu^*}(\theta)\widetilde{N}^{\mu^*}(dt, d\theta, da) \\ p_T^{\mu_t^*} = g_x(x_T^{\mu_t^*}), \end{cases} \tag{3.4}$$

such that for all $u \in U$

$$E\int_0^T \left[H(t, x_t^{\mu^*}, u_t, p_t^{\mu^*}, q^{\mu^*}, r_t^{\mu^*}(.)) - \int_\Gamma H(t, x_t^{\mu^*}, a, p_t^{\mu^*}, q^{\mu^*}, r_t^{\mu^*}(.))\mu_t^*(da) \right] dt \geq 0. \tag{3.5}$$

The proof of this theorem is based on the following lemma.

Lemma 3.5 *Let (p^n, q^n, r^n) and $(p^{\mu^*}, q^{\mu^*}, r^{\mu^*})$, be the solutions of (2.22) and (3.4),* *respectively. Then we have*

$$lim_{n\to\infty}\left[E\left|p^{n}-p^{\mu^{*}}\right|^{2}+E\int_{t}^{T}\left|q^{n}-q^{\mu^{*}}\right|^{2}ds+E\int_{t}^{T}\int_{\Gamma}\left|r^{n}-r^{\mu^{*}}\right|^{2}\upsilon(d\theta)ds\right]=0.$$

To prove the lemma (3.5), we need to state and prove the stability theorem of BSDEs with jump. Note that this theorem is proved by Hu and Peng [18] in the case without jump.

Stability theorem for BSDE's with jump

Let us denote by $M^2(0,T;\mathbb{R}^m)$ the subset of $L^2(\Omega\times[0;T],dP\times dt;\mathbb{R}^m)$ consisting of $\mathcal{F}_t-$progressively measurable processes. Consider the following BSDE's with jump depending on a parameter n.

$$p_t^n=p_T^n+\int_t^T F^n(s,p_s^n,q_s^n,r_s^n)ds-\int_t^T q_s^n dB_s-\int_t^T\int_\Gamma r_s^n(\theta)N^n(ds,d\theta)\qquad t\in[0;T].$$

We assume that :

1. For any n, $(p,q,r)\in\mathbb{R}^m\times\mathbb{R}^{m\times d}\times\mathbb{R}$, $F^n(.,p,q,r)\in M^2(0,T;\mathbb{R}^m)$ and $p_T^n\in L^2(\Omega,\mathcal{F}_t,P,\mathbb{R}^m)$,

2. There exists a constant $C_0>0$ such that

$$\left|F^n(s,p_1,q_1,r_1)-F^n(s,p_2,q_2,r_2)\right|\ \le C_0\left(\left|p_1-p_2\right|+\left|q_2-q_2\right|+\int_\Gamma\left|r_1-r_2\right|\upsilon(d\theta)\right)$$

$$P.a.s\quad a.e\quad t\in[0;T],$$

3. $E\left(\left|p_T^n-p_T^*\right|^2\right)\overrightarrow{n\to\infty}\ 0,$

4. $\forall t \in [0;T]$,

$$\lim_{n\to\infty} E\left[\left|\int_t^T \left(F^n(s,p_s^*,q_s^*,r_s^*) - F^*(s,p_s^*,q_s^*,r_s^*)\right) ds\right|^2\right] = 0.$$

Theorem 3.4 (Stability theorem for BSDE's with jump) *Let* $(p^{\ n}, q^{\ n}, r^{\ n})$ *and* $(p^{\ *}, q^{\ *}, r^{\ *})$, *be the solutions of (2.22) and (3.4), respectively. Then we have*

$$lim_{n\to\infty} E\left[|p^n - p^*|^2 + \int_t^T |q^n - q^*|^2\, ds + \int_t^T\int_\Gamma |r^n - r^*|^2\, \upsilon(d\theta) ds\right] = 0.$$

Proof. We proceed as in [18]. Let $\widehat{p^n} = p^n - p^*$, $\widehat{q^n} = q^n - q^*$, $\widehat{r^n} = r^n - r^*$ and $\widehat{p^n_T} = p^n_T - p^*_T$, then

$$\begin{aligned}\widehat{p^n_t} + \int_t^T \widehat{q^n_s} dB_s + \int_t^T\int_\Gamma \widehat{r^n_s}\widetilde{N}(dt,d\theta) &= \widehat{p^n_T} + \int_t^T \left[F^n(s,p_s^n,q_s^n,r_s^n) - F^n(s,p_s^*,q_s^*,r_s^*)\right] ds \\ &+ \int_t^T \left[F^n(s,p_s^*,q_s^*,r_s^*) - F^*(s,p_s^*,q_s^*,r_s^*)\right] ds.\end{aligned}$$

Taking the square and the expectation, we get

$$\begin{aligned}E\left[\left|\widehat{p^n_t}\right|^2 + \int_t^T \left|\widehat{q^n_s}\right| 2ds + \int_t^T\int_\Gamma \left|\widehat{r^n_s}\right|^2 \upsilon(d\theta)ds\right] &\leq 2E\left|\alpha_t^n\right|^2 \\ &+ 2E\left(\int_t^T \left[F^n(s,p_s^n,q_s^n,r_s^n) - F^n(s,p_s^*,q_s^*,r_s^*)\right] ds\right)^2 \\ &\leq 2E\left|\alpha_t^n\right|^2 \\ &+ 2(T-t)E\int_t^T \left|F^n(s,p_s^n,q_s^n,r_s^n) - F^n(s,p_s^*,q_s^*,r_s^*)\right|^2 ds,\end{aligned}$$

with

$$\alpha_t^n = \widehat{p_T^n} + \int_t^T \left[F^n(s, p_s^*, q_s^*, r_s^*) - F^*(s, p_s^*, q_s^*, r_s^*) \right] ds.$$

Because of the assumption 2,

$$\begin{aligned} E\left|\widehat{p_t^n}\right|^2 + E\int_t^T \left|\widehat{q_s^n}\right|^2 ds + E\int_t^T\int_\Gamma \left|\widehat{r_s^n}\right|^2 \upsilon(d\theta)ds \quad &\leq 2E\left|\alpha_t^n\right|^2 + 2(T-t)C_0 E\left|\widehat{p_t^n}\right|^2 \\ &+2(T-t)C_0 E\left[\int_t^T \left|\widehat{q_s^n}\right|^2 ds + \int_t^T\int_\Gamma \left|\widehat{r_s^n}\right|^2 \upsilon(d\theta)ds\right]. \end{aligned}$$

For $t \in [T-\varepsilon; T]$ with $\varepsilon = \frac{1}{4C_0}$

$$\begin{aligned} E\left|\widehat{p_t^n}\right|^2 + \int_t^T \left|\widehat{q_s^n}\right|^2 ds + \int_t^T\int_\Gamma \left|\widehat{r_s^n}\right|^2 \upsilon(d\theta)ds \quad &\leq 2E\left|\alpha_t^n\right|^2 \\ &+\tfrac{1}{2}E\int_t^T \left[\left|\widehat{p_s^n}\right|^2 + \left|\widehat{q_s^n}\right|^2 + \int_\Gamma \left|\widehat{r_s^n}\right|^2 \upsilon(d\theta)\right] ds, \end{aligned}$$

hence

$$E\left|\widehat{p_t^n}\right|^2 + \tfrac{1}{2}E\int_t^T \left|\widehat{q_s^n}\right|^2 ds + \tfrac{1}{2}E\int_t^T\int_\Gamma \left|\widehat{r_s^n}\right|^2 \upsilon(d\theta)ds \quad \leq 2E\left|\alpha_t^n\right|^2 + \tfrac{1}{2}\int_t^T E\left|\widehat{p_s^n}\right|^2 ds.$$

Then, we have

$$\begin{aligned} E\left|\widehat{p_t^n}\right|^2 \quad &\leq \tfrac{2}{3}E\left|\alpha_t^n\right|^2 + \tfrac{1}{6}\int_t^T E\left|\widehat{p_s^n}\right|^2 ds, \\ E\int_t^T \left|\widehat{q_s^n}\right|^2 ds \quad &\leq \tfrac{4}{3}E\left|\alpha_t^n\right|^2 + \tfrac{1}{3}\int_t^T E\left|\widehat{p_s^n}\right|^2 ds, \\ E\int_t^T\int_\Gamma \left|\widehat{r_s^n}\right|^2 \upsilon(d\theta)ds \quad &\leq \tfrac{4}{3}E\left|\alpha_t^n\right|^2 + \tfrac{1}{3}\int_t^T E\left|\widehat{p_s^n}\right|^2 ds. \end{aligned}$$

Now, for apply the Gronwall lemma we need to prove that $lim_{n\to\infty}E\left|\alpha_t^n\right|^2=0$. We have

$$E\left|\alpha_t^n\right|^2\leq 2E\left|p_T^n-p_T^*\right|^2+2(T-t)C_0E\int_t^T\left|F^n(s,p_s^*,q_s^*,r_s^*)-F^*(s,p_s^*,q_s^*,r_s^*)\right|^2ds,$$

by the assumptions 3 and 4, we deduce that $lim_{n\to\infty}E\left|\alpha_t^n\right|^2=0$.

By the Gronwall lemma, we can deduce that $lim_{n\to\infty}E\left|\widehat{p_t^n}\right|^2=0$, hence $lim_{n\to\infty}E\int_t^T\left|\widehat{q_s^n}\right|^2ds=0$ and $im_{n\to\infty}E\int_t^T\int_\Gamma\left|\widehat{r_s^n}\right|^2\upsilon(d\theta)ds=0$.

We can use the same argument to prove that the above convergence is hold on $[T-2\delta;T-\delta]$, $[T-3\delta;T-2\delta]$ ....This complete the proof. ■

To prove the lemma (3.5), it is sufficient to show that the coefficients of our BSDE verify the assumptions of stability theorem (3.4) :

Proof of lemma 4.5. By the continuity of the derivatives of the coefficients, and the fact that $lim_{n\to\infty}E\left|x_T^n-x_T^{\mu^*}\right|^2=0$, we can deduce that

$$\lim_{n\to\infty}E\left[\left|\int_t^T\left(F^n(s,p_s^{\mu^*},q_s^{\mu^*},r_s^{\mu^*})-F^{\mu^*}(s,p_s^{\mu^*},q_s^{\mu^*},r_s^{\mu^*})\right)ds\right|^2\right]=0$$

and

$$\lim_{n\to\infty}E\left(\left|p_T^n-p_T^{\mu^*}\right|^2\right)=0.$$

And, by the boundedness of b_x,σ_x, and f, we can easily check that there exists a constant $C_0>0$ such that

$$\left|F^n(s,p_s^n,q_s^n,r_s^n)-F^n(s,p_s^{\mu^*},q_s^{\mu^*},r_s^{\mu^*})\right|\leq C_0\left(\left|p_s^n-p_s^{\mu^*}\right|+\left|q_s^n-q_s^{\mu^*}\right|+\int_\Gamma\left|r_s^n-r_s^{\mu^*}\right|\upsilon(d\theta)\right)$$

$$P.a.s\quad a.e\quad t\in[0;T],$$

where

$$F^{n}(s,X,Y,Z)=h_{x}+Xb_{x}+Y\sigma_{x}+\int\limits_{\Gamma}Zf\upsilon(d\theta),$$

and

$$F^{\mu^{*}}(s,X,Y,Z)=\int\limits_{A}h_{x}(a)\mu_{t}^{*}(da)+\int\limits_{A}Xb_{x}(a)\mu_{t}^{*}(da)+Y\sigma_{x}+\int\limits_{\Gamma}Zf(\theta,a)\mu_{t}^{*}\otimes\upsilon(da,d\theta).$$

This complete the proof. ■

Proof of theorem 4.4. The result is proved by passing to the limit in inequality (2.23), and using lemma (3.5), we get easily the inequality (3.5). ■

Chapter 4

The relaxed maximum principle in singular optimal control of controlled jump diffusions

In this chapter, we consider mixed relaxed-singular stochastic control problems of systems governed by stochastic differential equations of the same type of SDEs defined in the forth chapter, but the control variable has two components, the first being measure valued process and the second singular. Our main goal is to extend the result of S. Bahlali, B. Djehiche, and B. Mezerdi [2] to the problem where the system evolves according to SDE with jumps, by the same techniques that used in the previous chapters, and using a strong perturbation of the absolutely continuous part of the control and a convex perturbation of the singular part.

4.1 Formulation of the problem

4.1.1 Strict control problem

We consider in this subsection a stochastic control problem of systems governed by stochastic differential equations on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$, such that $\mathcal{F}_0$ contains the $P-$null sets, We assume that $(\mathcal{F}_t)_{t\geq 0}$ is generated by a standard Brownian motion B and an independent Poisson measure N, and assume that the compensator of N has the form $\upsilon(d\theta)dt$, where the jumps are confined to a compact set Γ. And set

$$\widetilde{N}(dt, d\theta) = N(dt, d\theta) - \upsilon(d\theta)dt.$$

Consider the following sets A_1, is a nonempty subset of $\mathbb{R}^k$ and $A_2 = ([0;\infty))^m$, let U_1 the class of measurable, adapted processes $u : [0;T] \times \Omega \longrightarrow A_1$, and U_2 the class of measurable, adapted processes $\zeta : [0;T] \times \Omega \longrightarrow A_2$.

Definition 4.1 *An admissible strict control is a pair (u, ζ) of $(A_1 \times A_2)-$valued measurable, $\mathcal{F}_t-$adapted processes, such that*

- ζ is of bounded variation, nondecreasing left-continuous with right limits and $\zeta_0 = 0$
- $$E\left[\sup_{t\in[0;T]} |u_t|^2 + |\zeta_T|^2\right] < \infty.$$

We denote by $\mathcal{U} = U_1 \times U_2$ the set of admissible strict controls.

For any $(u, \zeta) \in \mathcal{U}$, we consider the following stochastic differential equation (SDE)

$$\begin{cases} dx_t & = b(t, x_t, u_t)dt + \sigma(t, x_t)dB_t + \int_\Gamma f(t, x_{t^-}, \theta, u_t)\widetilde{N}(dt, d\theta) + G_t d\zeta_t \\ x(0) & = x_0, \end{cases} \tag{4.1}$$

where

$$
\begin{aligned}
b &: [0;T] \times \mathbb{R}^n \times A \longrightarrow \mathbb{R}^n \\
\sigma &: [0;T] \times \mathbb{R}^n \longrightarrow \mathcal{M}_{n\times d}(\mathbb{R}) \\
f &: [0;T] \times \mathbb{R}^n \times \Gamma \times A \longrightarrow \mathbb{R}^n \\
G &: [0;T] \longrightarrow \mathcal{M}_{n\times m}(\mathbb{R})
\end{aligned}
$$

are bounded, measurable and continuous functions.

The expected cost is given by:

$$
J(u,\zeta) = E\left[g(x_T) + \int_0^T h(t, x_t, u_t)dt + \int_0^T k_t d\zeta_t \right], \tag{4.2}
$$

where

$$
\begin{aligned}
g &: \mathbb{R}^n \longrightarrow \mathbb{R} \\
h &: [0;T] \times \mathbb{R}^n \times A \longrightarrow \mathbb{R} \\
k &: [0;T] \longrightarrow ([0;\infty))^m,
\end{aligned}
$$

be bounded and continuous functions.

The strict optimal control problem is to minimize the functional $J(.,.)$ over $\mathcal{U}$. A control that solves this problem is called optimal.

The following assumptions will be in force throughout this chapter :

(H_1) The maps b, σ, f and h are continuously differentiable with respect to x. They and their derivatives b_x σ_x f_x and h_x are continuous in (x,u).

(H_2) The derivatives σ_x, $f_x$and g_x are bounded and b_x and h_x are uniformly bounded in u.

(H_3) b, σ and f are bounded by $K(1+|x|+|u|)$, and g is bounded by $K(1+|x|)$, for some

$K > 0$.

(H_4) G and k are continuous and bounded.

4.1.2 Relaxed-Singular control problem

In this subsection, we introduce relaxed controls of our systems of SDE. The idea of relaxed control is to replace the absolutely continuous part of the control u with a $P(A_1)-$valued process μ, where $P(A_1)$ denotes the space of probability measures equipped with the topology of weak convergence. Consequently, the state variable is governed by a counting measure valued process called the relaxed Poisson measure, as described in the previous chapter in particular the first section.

Definition 4.2 *A relaxed-singular control is a pair (μ, ζ) of processes such that*

1. μ is a $P(A_1)-$valued process, progressively measurable with respect to $(\mathcal{F}_t)_{t\geq 0}$,
2. $\zeta \in U_2$.

We denote by $\mathcal{R}^s = \mathcal{R}_1 \times U_2$ the set of relaxed-singular controls.

For any $(\mu, \zeta) \in \mathcal{R}$, write the stochastic differential equation with controlled jumps in terms of relaxed Poisson measure as follows

$$\begin{cases} dx_t^\mu &= \int\limits_{A_1} b(t, x_t^\mu, a)\mu_t(da)dt + \sigma(t, x_t^\mu)dB_t + \int\limits_{A_1}\int\limits_{\Gamma} f(t, x_{t^-}^\mu, \theta, a)\widetilde{N}^\mu(dt, d\theta, da) + G_t d\zeta_t \\ x_0^\mu &= 0. \end{cases} \tag{4.3}$$

The expected cost associated to a relaxed-singular control is defined as

$$J(\mu, \zeta) = E\left[g(x_T^\mu) + \int\limits_{A_1}\int\limits_0^T h(t, x_t^\mu, a)\mu_t(da)dt + \int\limits_0^T k_t d\zeta_t \right].$$

4.2 Approximation of trajectories

In order for the relaxed-singular control problem to be truly an extension of the strict one, the infimum of the expected cost for the relaxed-singular controls must be equal to the infimum of the expected cost for the strict ones. This result is based on the approximation of a relaxed control by a sequence of strict controls, and the convergence of the relaxed Poisson measures corresponding with them, given by the chattering lemma (3.1). The next theorem which is our main result in this section gives the stability of the stochastic differential equations with respect to the control variable, and that the two problems has the same infimum of the expected costs.

Theorem 4.1 *Let (μ, ζ) be a relaxed-singular control, and let x^μ be the corresponding trajectory. Then there exist a sequence (u^n, ζ) of strict controls such that*

$$\lim_{n\to\infty} E\left[\sup_{0\leq t\leq T} |x_t^n - x_t^\mu|^2\right] = 0,$$

and

$$\lim_{n\to\infty} J(u^n, \zeta) = J(\mu, \zeta), \tag{4.4}$$

where x^n denote the trajectory associated with (u^n, ζ).

Proof. Using the same method of the proof of theorem 5 in chapter 4. ■

4.3 Maximum principle for relaxed control problems

Our main goal in this section is to establish optimality necessary conditions for relaxed-singular control problems, where the system is described by a SDE driven by a relaxed Poisson measure which is a martingale measure, of the form (4.3) and the admissible controls are measure-valued processes which are called relaxed controls. The proof is based on the chattering lemma (3.1), and using Ekeland's variational principle (2.5), we derive necessary conditions of near optimality satisfied by a sequence of strict controls. By using stability properties of the state equations and adjoint processes, we obtain the maximum principle for our relaxed problem.

4.3.1 The maximum principle for strict control

The purpose of this subsection is to derive optimality necessary conditions, satisfied by an optimal strict control. The proof is based on strong perturbations for the absolutely continuous part, and the convex perturbations for the singular components of the optimal control (u^*, ζ^*), which defined by :

$$(u^h, \zeta^*) = \begin{cases} (\nu, \zeta^*) & \text{if } \ t \in [t_0; t_0 + h] \\ (u^*, \zeta^*) & \quad \text{otherwise,} \end{cases} \tag{4.5}$$

$$(u^*, \zeta^h) = (u^*, \zeta^* + h(\xi - \zeta^*), \tag{4.6}$$

for some $(\nu, \xi) \in \mathcal{U}$.

The first variational inequality

To obtain the first variational inequality in the stochastic maximum principle, we use the strong perturbations (4.5), the first variational inequality is derived from the fact that

$$\left. \frac{dJ(u^h, \zeta^*)}{dh} \right|_{h=0} \geq 0.$$

Note that the singular part is not affected by the perturbation (4.5). So, we have

$$\lim_{h \to 0} E \left[\sup_{t \in [0;T]} \left| x_t^{(u^h, \zeta^*)} - x_t^* \right|^2 \right] = 0, \tag{4.7}$$

where

$$\begin{cases} x_t^{(u^h,\zeta^*)} = x_t^* \quad ; t \leq t_0 \\ dx_t^{(u^h,\zeta^*)} = b(t, x_t^{(u^h,\zeta^*)}, \nu)dt + \sigma(t, x_t^{(u^h,\zeta^*)})dB_t + \int\limits_\Gamma f(t, x_{t-}^{(u^h,\zeta^*)}, \theta, \nu)\widetilde{N}(dt, d\theta) \\ +G_t d\zeta_t^* \quad ; t_0 < t < t_0 + h \\ dx_t^{(u^h,\zeta^*)} = b(t, x_t^{(u^h,\zeta^*)}, u^*)dt + \sigma(t, x_t^{(u^h,\zeta^*)})dB_t + \int\limits_\Gamma f(t, x_{t-}^{(u^h,\zeta^*)}, \theta, u^*)\widetilde{N}(dt, d\theta) \\ +G_t d\zeta_t^* \quad ; t_0 + h < t < T. \end{cases}$$

For more detail about the proof see lemma (2.3) chapter 3.

Choose t_0 such that (2.12) and (2.13) holds, then we have

Corollary 4.1 *Under assumptions (H_1)-(H_3), one has*

$$\left.\frac{dJ(u^h, \zeta^*)}{dh}\right|_{h=0} = E\left[g_x(x_T^*)z_T + \varsigma_T\right], \tag{4.8}$$

where

$$\begin{cases} d\varsigma_t = h_x(t, x_t^*, u_t^*)z_t dt \qquad t_0 \leq t \leq T \\ \varsigma_{t_0} = h(t_0, x_{t_0}^*, \nu) - h(t_0, x_{t_0}^*, u_{t_0}^*), \end{cases}$$

and the process z is the solution of the linear SDE

$$\begin{cases} dz_t &= b_x(t, x_t^*, u_t^*)z_t dt + \sigma_x(t, x_t^*)z_t dB_t + \int\limits_\Gamma f_x(t, x_{t-}^*, \theta, u_t^*)z_{t-}\widetilde{N}(dt, d\theta); \ t_0 \leq t \leq T \\ z_{t_0} &= \left[b(t_0, x_{t_0}^*, \nu) - b(t_0, x_{t_0}^*, u_{t_0}^*)\right]. \end{cases} \tag{4.9}$$

From (H_2) the variational equation (4.9) has a unique solution.

To prove the corollary (4.1) we need the following estimates.

Lemma 4.1 *Under assumptions (H_1)-(H_3), it holds that*

$$\lim_{h\to 0} E\left[\left|\frac{x_t^{(u^h,\zeta^*)} - x_t^*}{h} - z_t\right|^2\right] = 0.$$

$$\lim_{h\to 0} E\left[\left|\frac{1}{h}\int_{t_0}^{T} \left[(h(t,x_t^*,u_t^h) - (h(t,x_t^*,u_t^*)\right] - \varsigma_T\right|^2\right] = 0.$$

Proof. Since $x_t^{(u^h,\zeta^*)} - x_t^*$ does not depend on the singular part, the proof follows that of lemma (2.4), chapter 3. ■

By the same method of section (3.2.2) of chapter 3, we can get the first variational inequality

$$E\left[H(t,x_t^*,\nu,p_t) - H(t,x_t^*,u_t^*,p_t)\right] \geq 0.dt - a.e,$$

where the Hamiltonian H is defined by (2.20).

The second variational principle

To obtain the second variational inequality of the stochastic maximum principle, we use the perturbations (4.6) of the second parts of the optimal control. Since (u^*,ζ^*) is optimal control, then we have

$$J(u^*,\zeta^h) - J(u^*,\zeta^*) \geq 0. \tag{4.10}$$

From this inequality, we will derive the second variational inequality.

Lemma 4.2 *Let $x_t^{(u^*,\zeta^h)}$ be the trajectory associated with (u^*,ζ^h), and x_t^* be the trajectory associated with (u^*,ζ^*), then the following estimate holds :*

$$\lim_{h\to 0} E\left[\sup_{t\in[0;T]}\left|x_t^{(u^*,\zeta^h)} - x_t^*\right|^2\right] = 0. \tag{4.11}$$

Proof. From the boundedness and continuity of b_x, σ_x, and f_x and by using the Burkholder-Davis-Gundy inequality for the martingale part, we get

$$\begin{aligned} E\left[\sup_{t\in[0;T]}\left|x_t^{(u^*,\zeta^h)}-x_t^*\right|^2\right] &\leq C_1\int_0^t E\left[\sup_{s\in[0;T]}\left|x_s^{(u^*,\zeta^h)}-x_s^*\right|^2\right]ds + C_2h^2 d\left|\xi-\zeta^*\right|^2 \\ &+C_3\int_0^t E\left[\sup_{s\in[0;T]}\int_\Gamma\left(\sup_{\theta\in\Gamma}\left|x_s^{(u^*,\zeta^h)}-x_s^*\right|^2\right)\upsilon(d\theta)ds\right], \end{aligned}$$

which implies that

$$\begin{aligned} E\left[\sup_{t\in[0;T]}\left|x_t^{(u^*,\zeta^h)}-x_t^*\right|^2\right] &\leq C_1\int_0^t E\left[\sup_{s\in[0;T]}\left|x_s^{(u^*,\zeta^h)}-x_s^*\right|^2\right]ds + C_2h^2 d\left|\xi-\zeta^*\right|^2 \\ &+C_3\upsilon(\Gamma)\int_0^t E\left[\sup_{s\in[0;T]}\left(\sup_{\theta\in\Gamma}\left|x_s^{(u^*,\zeta^h)}-x_s^*\right|^2\right)ds\right], \end{aligned}$$

then

$$E\left[\sup_{t\in[0;T]}\left|x_t^{(u^*,\zeta^h)}-x_t^*\right|^2\right] \leq (C_1+C_3\upsilon(\Gamma))\int_0^t E\left[\sup_{s\in[0;T]}\left|x_s^{(u^*,\zeta^h)}-x_s^*\right|^2\right]ds + C_2h^2 d\left|\xi-\zeta^*\right|^2.$$

Since $\upsilon(\Gamma)<\infty$, by the Gronwall inequality, the result follows immediately by letting h go to zero. ■

Lemma 4.3 *Under assumptions, it holds that*

$$\lim_{h\to 0} E\left[\left|\frac{x_t^{(u^*,\zeta^h)}-x_t^*}{h}-z_t\right|^2\right]=0, \tag{4.12}$$

where z_t is the solution of the following equation :

$$z_t=\int_0^t b_x(s,x_s^*,u_s^*)z_s ds+\int_0^t \sigma_x(s,x_s^*)z_s dB_s+\int_0^t\int_\Gamma f_x(s,x_{s-}^*,\theta,u_s^*)z_{s-}\widetilde{N}(ds,d\theta)+\int_0^t G_s d(\xi-\zeta^*)_s.$$

Proof. Let
$$y_t^h = \frac{x_t^{(u^*,\zeta^h)} - x_t^*}{h} - z_t,$$
then,

$$\begin{aligned}
dy_t^h &= \tfrac{1}{h}\left[b(t, x_t^{(u^*,\zeta^h)}, u_t^*) - b(t, x_t^*, u_t^*)\right] dt + \tfrac{1}{h}\left[\sigma(t, x_t^{(u^*,\zeta^h)}) - \sigma(t, x_t^*)\right] dB_t \\
&+ \tfrac{1}{h}\int_\Gamma \left[f(t, x_{t^-}^{(u^*,\zeta^h)}, \theta, u_{t-}^*,) - f(t, x_{t-}^*, \theta, u_{t-}^*)\right] \widetilde{N}(dt, d\theta) \\
&- b_x(t, x_t^*, u_t^*) z_t dt - \sigma_x(t, x_t^*) z_t dB_t \\
&- \int_\Gamma f_x(t, x_{t-}^*, \theta, u_{t-}^*) z_{t^-} \widetilde{N}(dt, d\theta),
\end{aligned}$$

hence

$$\begin{aligned}
y_t^h &= \int_0^t \int_0^1 b_x(s, x_t^* + \lambda(x_t^{(u^*,\zeta^h)} - x_t^*), u_s^*) y_s^h d\lambda ds + \int_0^t \int_0^1 \sigma_x(s, x_t^* + \lambda(x_t^{(u^*,\zeta^h)} - x_t^*)) y_s^h d\lambda dB_s \\
&+ \int_0^t \int_0^1 \int_\Gamma f_x(s, x_{s-}^* + \lambda(x_{s^-}^{(u^*,\zeta^h)} - x_{s-}^*), \theta, u_{s-}^*) y_{s^-}^h d\lambda \widetilde{N}(ds, d\theta) + \rho_t^h,
\end{aligned}$$

where

$$\begin{aligned}
\rho_t^h &= \int_0^t \int_0^1 b_x(s, x_t^* + \lambda(x_t^{(u^*,\zeta^h)} - x_t^*), u_s^*) z_s d\lambda ds + \int_0^t \int_0^1 \sigma_x(s, x_t^* + \lambda(x_t^{(u^*,\zeta^h)} - x_t^*)) z_s d\lambda dB_s \\
&+ \int_0^t \int_0^1 \int_\Gamma f_x(s, x_{s-}^* + \lambda(x_{s^-}^{(u^*,\zeta^h)} - x_{s-}^*), \theta, u_{s-}^*) z_{s^-} d\lambda \widetilde{N}(ds, d\theta) - \int_0^t \int_\Gamma f_x(s, x_{s-}^*, \theta, u_{s-}^*) z_{s^-} \widetilde{N}(ds, d\theta) \\
&- \int_0^t b_x(s, x_s^*, u_s^*) z_s ds - \int_0^t \sigma_x(s, x_s^*) z_s dB_s,
\end{aligned}$$

hence

$$
\begin{aligned}
E\left|y_t^h\right|^2 &\leq KE\int_0^t \left|\int_0^1 b_x(s, x_t^* + \lambda(x_t^{(u^*,\zeta^h)} - x_t^*), u_s^*) y_s^h d\lambda\right|^2 ds \\
&+KE\int_0^t \left|\int_0^1 \sigma_x(s, x_t^* + \lambda(x_t^{(u^*,\zeta^h)} - x_t^*)) y_s^h d\lambda\right|^2 ds \\
&+KE\int_0^t \int_\Gamma \left|\int_0^1 f_x(s, x_s^* + \lambda(x_s^{(u^*,\zeta^h)} - x_s^*), \theta, u_s^*) y_s^h d\lambda\right|^2 \upsilon(d\theta) ds + KE\left|\rho_t^h\right|^2.
\end{aligned}
$$

Since b_x, σ_x, and f_x are bounded, then

$$
E\left|y_t^h\right|^2 \leq CE\int_0^t \left|y_s^h\right|^2 ds + KE\left|\rho_t^h\right|^2.
$$

We conclude by the boundedness and continuity of b_x, σ_x, and f_x, and the dominated convergence theorem that $\lim_{h\to 0} E\left|\rho_t^h\right|^2 = 0$. Hence by the Gronwall lemma, we get

$$
\lim_{h\to 0} E\left|y_t^h\right|^2 = 0.
$$

■

Lemma 4.4 *The following inequality holds :*

$$
E\left[g_x(x_T^*) z_T + \int_0^T h_x(t, x_t^*, u_t^*) z_t dt + \int_0^T k_t d(\xi - \zeta^*)_t\right] \geq 0. \tag{4.13}
$$

Proof. From (4.10), we have

$$
\begin{aligned}
0 \quad &\leq \tfrac{1}{h}\left[E\left[g(x_T^{(u^*,\zeta^h)}) - g(x_T^*)\right] + E\int_0^T \left[h(t,x_t^{(u^*,\zeta^h)},u_t^*) - h(t,x_t^*,u_t^*)\right]dt\right] + E\int_0^T k_t d(\xi-\zeta^*)_t \\
&= E\int_0^1 \left[\left(\tfrac{x_T^{(u^*,\zeta^h)}-x_T^*}{h}\right) g_x\left[x_T^* + \lambda(x_T^{(u^*,\zeta^h)} - x_T^*)\right]d\lambda\right] \\
&+E\int_0^T\int_0^1 \left[\left(\tfrac{x_t^{(u^*,\zeta^h)}-x_t^*}{h}\right) h_x\left[x_t^* + \lambda(x_t^{(u^*,\zeta^h)} - x_t^*),u_t^*\right]\right]d\lambda dt + E\int_0^T k_t d(\xi-\zeta^*)_t.
\end{aligned}
$$

From the continuity and boundedness of g_x and h_x, by letting h go to zero, we can deduce the result from (4.11) and (4.12). ■

Now, we are able to derive the second variational inequality from (4.13). If $\varphi(t,s)$ denotes the solution of (2.17), it's easy to check that z_t is given by

$$z_t = \int_0^T \varphi(t,s) G_t d(\xi-\zeta^*)_t.$$

Replacing z_t with its value, we obtain the second variational inequality

$$E\left[\int_0^T (k_t + G_t^* p_t) d(\xi-\zeta^*)_t\right] \geq 0,$$

where p is the adjoint process defined in chapter 3 by (2.19).

The above discussion will allow us to introduce the next theorem which is the main result of this subsection.

Theorem 4.2 (the maximum principle for strict control problem) *Let (u^*,ζ^*) be the optimal strict control minimizing the cost $J(.,.)$ over $\mathcal{U}$, and denote by x^* the corre-*

sponding optimal trajectory, then the following inequalities holds

$$\begin{aligned} E\left[H(t,x_t^*,\nu,p_t)-H(t,x_t^*,u_t^*,p_t)\right] &\geq 0.dt-a.e, \\ \int_0^T \left\{k_t+G_t p_t\right\} d(\zeta-\zeta^*)_t &\geq 0, \end{aligned}$$

where the Hamiltonian H is defined by (2.20).

4.3.2 The maximum principle for near optimal controls

In this subsection, we establish necessary conditions of near optimality satisfied by a sequence of nearly optimal strict controls, this result is based on Ekeland's variational principle, which is given by the lemma (2.5).

To apply Ekeland's variational principle, we have to endow the set $\mathcal{U}$ of strict controls with an appropriate metric. For any (u,ζ) and $(\nu,\xi)\in\mathcal{U}$, we set

$$\begin{aligned} d_1(u,\nu) &= P\otimes dt\left\{(\omega,t)\in\Omega\times[0;T],\ u(\omega,t)\neq\nu(\omega,t)\right\}, \\ d_2(\zeta,\xi) &= E\left(\sup_{t\in[0;T]}|\zeta_t-\xi_t|^2\right)^{\frac{1}{2}}, \\ d\left[(u,\zeta),(\nu,\xi)\right] &= d_1(u,\nu)+d_2(\zeta,\xi). \end{aligned}$$

where $P\otimes dt$ is the product measure of P with the lebesgue measure dt.

Remark 4.1 *$(\mathcal{U},d)$ is a complete metric space, and it well known that the cost functional J is continuous from $\mathcal{U}$ into $\mathbb{R}$, for more detail see [24].*

Let $(\mu^*,\zeta^*)\in\mathcal{R}^s$ be an optimal relaxed-singular control and denote by x_t^* the trajectory of the system controlled by (μ^*,ζ^*), from lemma (3.1), there exist a sequence (u^n) of strict controls such that

$$\mu_t^n(da)dt=\delta_{u_t^n}(da)dt\longrightarrow\mu_t^*(da)dt \quad \text{weakly},$$

and

$$\lim_{n\to\infty} E\left[\left|x_t^n - x_t^{\mu^*}\right|^2\right] = 0,$$

where x^n is the solution of (4.3) corresponding to μ^n.

According to the optimality of μ^* and (2.5), there exist a sequence (ε_n) of positive numbers with $\lim_{n\to\infty} \varepsilon_n = 0$ such that

$$J(u^n, \zeta^*) = J(\mu^n, \zeta^*) \leq J(\mu^*, \zeta^*) + \varepsilon_n = \inf_{u\in U} J(u, \zeta) + \varepsilon_n,$$

a suitable version of Lemma (2.5) implies that, given any $\varepsilon_n > 0$, there exist $(u^n, \zeta^*) \in \mathcal{U}$ such that

$$J(u^n, \zeta^*) \leq J(\nu, \xi) + \varepsilon_n d\left[(u^n, \zeta^*), (\nu, \xi)\right], \forall (\nu, \xi) \in \mathcal{U}. \tag{4.14}$$

Let us define the perturbations

$$\begin{aligned} (u^{n,h}, \zeta^*) &= \begin{cases} (\nu, \zeta^*) & \text{if } \ t \in [t_0; t_0 + h] \\ (u^n, \zeta^*) & \text{otherwise,} \end{cases} \\ (u^n, \zeta^h) &= (u^n, \zeta^* + h(\xi - \zeta^*)). \end{aligned}$$

From (4.14) we have

$$\begin{aligned} 0 &\leq J(u^{n,h}, \zeta^*) - J(u^n, \zeta^*) + \varepsilon_n d\left[(u^{n,h}, \zeta^*), (u^n, \zeta^*)\right], \\ 0 &\leq J(u^n, \zeta^h) - J(u^n, \zeta^*) + \varepsilon_n d\left[(u^n, \zeta^h), (u^n, \zeta^*)\right]. \end{aligned}$$

Using the definition of d, it holds that

$$\begin{aligned} 0 &\leq J(u^{n,h}, \zeta^*) - J(u^n, \zeta^*) + \varepsilon_n d_1(u^{n,h}, u^n), \\ 0 &\leq J(u^n, \zeta^h) - J(u^n, \zeta^*) + \varepsilon_n d_2(\zeta^h, \zeta^*). \end{aligned}$$

Finally, using the definition of d_1 and d_2, we obtain

$$
\begin{aligned}
0 &\leq J(u^{n,h},\zeta^*) - J(u^n,\zeta^*) + \varepsilon_n C_1 h, \\
0 &\leq J(u^n,\zeta^h) - J(u^n,\zeta^*) + \varepsilon_n C_2 h,
\end{aligned} \tag{4.15}
$$

where C_i is a positive constant.

Now, we can introduce the next theorem which is the main result of this subsection.

Theorem 4.3 *For each $\varepsilon_n > 0$, there exists $(u^n,\zeta) \in \mathcal{U}$ such that there exist a unique triple of square integrable adapted processes (p^n,q^n,r^n) which is the solution of the backward SDE*

$$
\begin{cases}
dp_t^n = -\Big[h_x(t,x_t^n,u_t^n) + p_t^n b_x(t,x_t^n,u_t^n) + q_t^n \sigma_x(t,x_t^n) \\
\quad + \displaystyle\int_\Gamma r_t^n(\theta) f(t,x_{t-}^n,\theta,u_t^n)\upsilon(d\theta)\Big] dt. \\
\quad + q_t^n dB_t + \displaystyle\int_\Gamma r_t^n(\theta)\widetilde{N}(dt,d\theta) \\
p_T^n = g_x(x_T^n).
\end{cases}
$$

such that for all $(u,\zeta) \in \mathcal{U}$

$$
\begin{aligned}
E\int_0^T \left[\left[H(t,x_t^*,u,p_t) - H(t,x_t^*,u_t^*,p_t)\right] + C_1\varepsilon_n\right] dt \geq 0, \\
E\int_0^T \left[(k_t + G_t p_t^n) d(\zeta_t - \zeta_t^*) + C_2\varepsilon_n\right] \geq 0,
\end{aligned} \tag{4.16}
$$

where C_i is a positive constant.

Proof. From the inequality (4.15), we use the same method as in the previous subsection, we obtain (4.16). ■

4.3.3 The relaxed stochastic maximum principle

Now, we can introduce the next theorem, which is the main result of this section

Theorem 4.4 (The relaxed stochastic maximum principle) *Let* (μ^*, ζ^*) *be an optimal relaxed-singular control minimizing the functional* $J\,(.,.)$ *over* $\mathcal{R}^s$*, and let* $x_t^{\mu^*}$ *be the corresponding optimal trajectory. Then there exist a unique triple of square integrable and adapted processes* $(p^{\,\mu^*}, q^{\,\mu^*}, r^{\,\mu^*})$ *which is the solution of the backward SDE*

$$\begin{cases} dp_t^{\mu^*} = -\left[\int\limits_A h_x(t, x_t^{\mu^*}, a)\mu_t^*(da) + \int\limits_A p_t^{\mu^*} b_x(t, x_t^{\mu^*}, a)\mu_t^*(da) + q_t^{\mu^*}\sigma_x(t, x_t^{\mu^*}) \right. \\ \left. + \int\limits_A \int\limits_\Gamma r_t^{\mu^*}(\theta) f(t, x_{t-}^{\mu^*}, \theta, a)\mu_t^* \otimes \upsilon(da, d\theta). \right] dt \\ +q_t^{\mu^*} dB_t + \int\limits_\Gamma r_t^{\mu^*}(\theta)\widetilde{N}^{\mu^*}(dt, d\theta, da) \\ p_T^{\mu_t^*} = g_x(x_T^{\mu_t^*}), \end{cases}$$

such that for all $(u, \zeta) \in \mathcal{U}$

$$E\int\limits_0^T \left[H(t, x_t^{\mu^*}, u_t, p_t^{\mu^*}, q^{\mu^*}, r_t^{\mu^*}(.)) - \int\limits_{A_1} H(t, x_t^{\mu^*}, a, p_t^{\mu^*}, q^{\mu^*}, r_t^{\mu^*}(.))\mu_t^*(da) \right] dt \geq 0,$$

$$E\left[\int\limits_0^T (k_t + G_t p_t^{\mu^*}) d(\zeta_t - \zeta_t^*) \right] \geq 0.$$

Proof. Since the singular term does not affect the adjoint processes, so the proof is the same as the proof of theorem (3.3). ∎

4.4 Appendix

Lemma 4.5 (Skorokhod selection theorem) *[19] Let (E, ρ) be a complete separable metric space, and let P and P_n, $n = 1, 2....$ be probability measures on $(E, \mathcal{B}(E))$, such that (P_n) converges weakly to to P. Then, on a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P})$, there exist E-valued random variables $x^n, n = 1, 2....$, and x such that*

- $P = \widetilde{P}_x,$
- $P_n = \widetilde{P}_{x^n}, n = 1, 2....,$
- $x^n \longrightarrow_{n\to\infty} x,\ \widetilde{P} - a.s.$

Lemma 4.6 (Aldous criterion of tightness) *[1] Let (x^n) be a sequence of càdlàg processes, suppose that for each n, x^n is defined on a filtered probability space $(\Omega^n, \mathcal{F}^n, (\mathcal{F}_t^n)_t, P^n)$, and the two following conditions holds*

- (x_t^n) is tight on $\mathbb{R}, \forall t$
- $\forall \varepsilon > 0, \forall \eta > 0$, there exist $\delta > 0$ and $n_0 \in \mathbb{N}$, such that $\forall n \geq n_0$, for all stopping time S, T, such that $S < T < S + \delta$, we have

$$\lim_{\delta\to 0} \limsup_{n} \sup_{S<T<S+\delta} P(\sup_{t\leq n} |Y_S^n - Y_T^n| \geq \eta) = 0.$$

Then, (x^n) is tight on $D(\mathbb{R})$.

Proposition 4.1 *([20]) Let x be a càdlàg square integrable martingale and let $< x >$ its predictable " crochet ". If $S \leq T$ two finite stopping times, then*

$$P(\sup_{t\in[S;T]} |x_S^n - x_T^n| \geq \eta) \leq \frac{\varepsilon}{\eta^2} + P(< x^n >_T - < x^n >_S \geq k).$$

Conclusion

As a conclusion of this work, we note that the problem of relaxed control is interesting because it is a generalization of the problem of ordinary control, that we always have an optimal control relaxed which verifies conditions similar to the principle of the ordinary maximum. We should also note the importance of studying jump problems because they are closest to reality, because it is considered to be a model to many problems in many areas, for example in telecommunication.

Many questions remain unresolved and deserve closer consideration, including

- Generalization of the maximum relaxed principle

It is a question of seeking the necessary conditions of optimality in the case where the diffusion coefficient σ is controlled.

- The aim is to find the necessary conditions of optimality which are verified by a relaxed control without the need for a maximum principle for ordinary problem, and without using the Ekeland variational principle, i-e the objective is to establish a maximum principle by the perturbation of the relaxed control itself.

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