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Titre

Les équations différentielles stochastiques rétrogrades et le contrôle optimal

Devant le jury

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Dédicace

Je dédie ce travail :

À la mémoire de mon père, Rahmatou Allah Alaihi,

ma chère mére,

À mon mari,

À mes frères et soeurs,

À tous ceux et celles qui m'ont encouragé, entouré de leur soutien durant les moments difficiles avec tant d'amour et de compréhension.

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ABSTRACT

This thesis presents two independent research topics. The first part of this dissertation deals with backward doubly stochastic differential equations (BDSDEs) with a superlinear growth generator and a square integrable terminal data. We introduce a new local condition on the generator, then we show that they ensure the existence and uniqueness as well as the stability of solutions. This work goes beyond the previous results on the subject. Although we are focused on multidimensional case, The uniqueness result is new for one dimensional BDSDEs. As application, we establish the existence and uniqueness of probabilistic solutions to some semilinear stochastic partial differential equations (SPDE's) with superlinear growth generator. By probabilistic solution, we mean a solution which is representable throughout a BDSDEs.

The second part of this PhD thesis is concerned with the stochastic control problems where the system is governed by backward stochastic differential equations (BSDE's). We are interested with existence of optimal relaxed controls for this kind of systems. Instead of proving this problem with the help of Skorokhod's representation theorem, our techniques are based on construction of the optimal control on an extended probability space, using Young measures.

Key Words. Backward doubly stochastic differential equation, Superlinear growth condition, Localization, Stochastic partial differential equation, Sobolev weak solution, Backward stochastic differential equation, Stochastic control, Relaxed control, Young measures,Tightness, Jakubowski's topology S.

Contents

Introduction

The main objective of this thesis is to study on the hand the existence and uniqueness as well as the stability of backward doubly stochastic differential equations (BDSDE's) with a superlinear growth generator and a square integrable terminal datum. On the other hand we also extablishe the problem of existence of optimal relaxed controls for system driven by backward stochastic differential equations (BSDE's).

Let $\{(W_t)\ , 0 \leq t \leq T\}$ be a d-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

Let $\mathcal{F}_t^W := \sigma(W_s; 0 \leq s \leq t)$ denote the natural filtration generated by (W_t) such that $\mathcal{F}_0$ contains all P-null sets of $\mathcal{F}$ and ξ be an $\mathcal{F}_T$ measurable d-dimensional random variable. The backward stochastic differential equation under consideration is;

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\ ds - \int_t^T Z_s dW_s \tag{1}$$

for $0 \leq t \leq T$, where f is called the coefficient or generator. This kind of stochastic equation are introduced in 1973 by J.M. Bismut [21] in his study of stochastic Pontryagin Maximum Principle, in which the adjoint equation is a backward stochastic differential equation with linear generator f.

In the first chapter we introduce the general nonlinear backward stochastic differential equation under Lipschitz conditions and square integrable terminal

data ξ, this pioneering work were first investigated in 1990 by E. Pardoux and S. Peng [49], the authors proved existence and uniqueness of $\mathcal{F}_t^W$-adapted solutions $\{Y_t, Z_t\}_{t\in[0,T]}$, this pair of process satisfying the BSDE (1) and some integrability assumptions. For the existence result, the martingale representation theorem, Picard iteration technique and a fixed-point theorem play a key role.

After this seminal paper the theory of BSDE's became very popular, and is an important

field of research due to its connection with stochastic control, the mathematical finance, partial differential equations and homogenization.
The Lipschitz condition on the generator of BSDE's was weakened by many authors: Lepeltier and San Martin [43] proved the existence of BSDE for continuous coefficients with linear growth condition $|f(t, y, z) \leq C(1 + |y| + \|z\|)|$, but the uniqueness of solution failed to be proved since the comparison theorem cannot be used under non-Lipschitz condition. In 2001 Bahlali [5] studied the multidimensional BSDE's with locally Lipschitz coefficients.
M. Kobylansky [39] and [40] considred in her PhD thesis an extension of the notion of BSDE to the cas where the dependence of the genrator in variable z has quadratic growth i.e.,$|f(t, \omega, y, z)| \leq C\left(1 + |y| + |z|^2\right)$, in this paper the existence and uniqueness result is given. Many authors have worked on this kind of backward stochastic differential equation, and then by Briand and Hu see [24] and [25] for unbounded terminal conditions, and Morlais [47] for continuous martingale drivers.
In 1991 Peng [52] was the first to introduce the probabilistic interpretation (Feynman-Kac formula) of a certain class of parabolic partial differential equation in terms of the solution of the corresponding backward stochastic differential equations.
Now let us introduce a new class of stochastic differential equation with terminal datum, called Backwrd doubly stochastic differential equations (in short BDSDE's) this kind of equations, first studied in 1994 by E. Padoux and S. Peng [50], has the following form:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)dB_s - \int_t^T Z_s dW_s \ , \qquad 0 \leq t \leq T$$

where $\{B_t, 0 \leq t \leq T\}$ and $\{W_t, 0 \leq t \leq T\}$ are two independent standard Brownian motions defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, dB is a backward Itô stochastic integral and dW is the usual Itô forward stochastic integral.
The authors proved existence and uniqueness of adapted solution for these kind of stochastic equation under the assumption that f and g are uniformly Lipschitz with respect to y and z and the Lipschitz constant of g with respect to z is less than one, they studied this problem in order to give a stochastic representation for solution of semilinear parabolic stochastic partial differential equations (in short SPDE's).
In 2001 Bally and Matoussi [20] applied the probabilistic interpretation of weak solution in

Sobolev space of parabolic semilinear (SPDE's) associated with the corresponding backward doubly stochastic differential equations with Lipschitz coefficients. This link was later developed in many papers (see e. g. [2, 23, 27, 28, 29, 36, 46, 57]) and have motivated many efforts in order to establish existence and uniqueness of solutions under more general conditions than the global Lipschitz one (see for instance [44, 55, 57, 58, 59]).
In 2008 Q. Zhang and H. Zhao [60] considred BDSDE's on finite and infinite horizon with linear growth without assuming Lipschitz conditions and related their solutions with the stationary solutions of certain SPDE's.
In one dimensional case, the comparison methods were mainly used to derive the existence of solutions for BDSDEs with continuous generator, (see e. g [34, 55, 59]). In multidimensional case, the problem is more delicate since the comparison methods do not work. Note also that as in classical BSDEs, the localization by stopping times, is ineffective in BDSDEs. Consequently, the most previous papers have considered BDSDEe with global assumption on the generator, like global Lipschitz or global monotony. Recently, the existence and uniqueness of solutions were established In [57], for BDSDEs under some local assumptions on the generator. More precisely, in [57] the generator f is μ_N-locally monotone in its y-variable and a L_N-locally Lipschitz in its z-variable on the ball of radius N. Supplementary conditions were also imposed on the behavior of μ_N and L_N when N tends to infinity. Note that these conditions, which were firstly introduced in the paper [11] for classical BSDEs, do not allow to cover BDSDEs with superlinear growth generator. Indeed, the generator remains with strictly sublinear growth in the variables (y, z) in both [11]and [57].

The second part of this PhD thesis concerned with the stochastic control problem that is a mathematical description of how to act optimally to minimize a cost or maximize a gain function, over the class $\mathcal{U}_{ad}$ of admissible controls, that is, $\mathcal{F}_t$-adapted processes with values in a compact metric space A. The existence of optimal strict control can be proved under some Roxin convexity hypotheses, since no convexity assumptions are made the problem reformulated in the larger or relaxed space, by replace the A-valued process u_t with $P(A)$-valued process (q_t), where $P(A)$ is the space of probability measures equipped with the topology of weak convergence. We denote by $\mathbb{V}$ the set of probability measures on $[0, T] \times A$, whose projection on $[0, T]$ coincide with the Lebesgue measure dt. Stable convergence is required for bounded measurable function $\Phi(t, a)$ such that for each fixed

$t \in [0, T]$ $\Phi(t, a)$ is continuous.

Equipped with the topology of stable convergence of measures, the set $\mathbb{V}$ of relaxed controls is compact metrizable.

Many authors turned to studiying the relaxed control problem Fleming [35] for systems driven by SDEs with uncontrolled diffusion coefficient and by El-Karoui et al.[32] with controlled diffusion coefficient using the compactification methods, and studing this problem for systems driven by BSDEs and FBSDEs (see e. g [3, 13, 14, 16, 26, 31, 51]). In the paper [13] the authors are studied the existence of optimal strict controls for linear backward stochastic differential equations, where the control domain is convex and compact, and the optimal control is adapted to the original filtration of the Brownian motion, then they continued and studied in [14] the problem of existence of optimal relaxed controls as well as strict optimal controls for systems governed by non linear (FBSDE's) where the driver of (BSDE) is supposed not to depend on z, the approach is based on tightness results and weak convergence of minimising controls and applying the Skorokhod representation theorem endowed with the topologie S of Jakhubowski.

In [15] S. Bahlali and B. Gerbal introduced a stochastic control problem where the system is governed by a nonlinear BDSDE and established necessary and sufficient optimality conditions in the form of stochastic maximum principle for this kind of systems.

In our work we introduce the existence of optimal relaxed contrôl, the system is governed by BSDEs where the generator f is explicitly depends on the process z.

Chapter 1

Backward stochastic differential equations BSDE's

Let $\{W_t, 0 \leq t \leq T\}$ be a d-dimensional Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we denote by $\mathbb{F} = (\mathcal{F}_t)_{0\leq t\leq T}$ the natural filtration generated by (W_t). Throught this chapter, our interest is on the BSDE :

$$dY_t = -\ f\left(t, Y_t, Z_t\right)\ dt + Z_t\ dW_t \qquad \text{and} \qquad Y_T = \xi;\ \mathbb{P}\text{-a.s.}$$

where ξ is some given $\mathcal{F}_T$-measurable random variable with values in $\mathbb{R}^k$, and $\{f\left(t, y, z\right)\}_{0\leq t\leq T}$ is a progressively measurable processes, the map f is called the generator and ξ the terminal datum.

We denote by $\mathcal{S}^2\left([0,T], \mathbb{R}^k\right)$ the vectoriel space of progressively measurable processes $\{Y_t; t \in [0,T]\}$ which satisfy

$$\|Y\|^2_{\mathcal{S}^2} := E\left(\sup_{0\leq t\leq T} |Y_t|^2\right) < \infty,$$

and $\mathcal{S}^2_c\left([0,T], \mathbb{R}^k\right)$ is the sub-space formed by continuous processes.

And by $\mathcal{M}^2\left([0,T], \mathbb{R}^{k\times d}\right)$, the space of progressively measurable processes $\{Z_t; t \in [0,T]\}$, such that:

$$\|Z\|^2_{\mathcal{M}^2} := E\int_0^T |Z_s|^2\, dt < \infty.$$

if $z \in \mathbb{R}^{k\times d}$, $\|z\|^2 = Tr\left(z\ z^*\right)$, and $M^2\left(\mathbb{R}^{k\times d}\right)$ is a set of equivalence classes of $\mathcal{M}^2\left([0,T], \mathbb{R}^{k\times d}\right)$.

We consider $\mathcal{B}$ the set of $\mathbb{R}^k \times \mathbb{R}^{k\times d}$-valued processes which are $\mathcal{F}_t$-adapted such that:

$$\|(U,V)\|_0 := \sqrt{E\left(\sup_{0\leq t\leq T} |U_t|^2\right) + E\left(\int_0^T \|V_s\|^2 \, ds\right)}.$$

The couple $(\mathcal{B}, \|.\|_0)$ is then a Banach space.

1.1 Formulation of the problem

We would like to solve the differential equation:

$$\frac{-dY_t}{dt} = f(Y_t), \quad t \in [0,T], \quad \text{with,} \;\; Y_T = \xi,$$

Y be a $\mathbb{F}$-adapted processe. when the generator $f \equiv 0$, the solution is $Y_t = \xi$, here the adaptation condition of processes Y gives that ξ must be determinist, the BSDE problem reduces to the martingale representation theorem in the present Brownian filtration, there is a unique process Z progressive measurable and square integrable such that:

$$\begin{aligned} Y_t := E(\xi/\mathcal{F}_t) &= E(\xi) + \textstyle\int_0^T Z_s dW_s \\ &= \xi - \textstyle\int_0^T Z_s dW_s \end{aligned} \tag{1.1}$$

in general, when the generator f depend on Z, BSDE (1.1) will be:

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)\; ds - \int_t^T Z_s dW_s \qquad t \leq T, \qquad \mathbb{P}\text{-a.s.}$$

Definition 1.1.1 *A solution of BSDE* (1) *is a couple of processes* $\{(Y_t, Z_t)\}_{0\leq t\leq T}$ *which belongs to the progressively measurable space* $\mathcal{S}^2\left([0,T], \mathbb{R}^k\right) \times \mathcal{M}^2\left([0,T], \mathbb{R}^{k\times d}\right)$ *and satisfies* (1)*.*

1.2 The main existence and uniqueness result

We consider the following assumptions:

(A_1) There exists a constant C such that for every t, y , y', z , z,$'$

$$|f(t,y,z) - f(t,y',z')| \leq C(|y-y'| + \|z-z'\|);$$

(A_2) The integrability condition

$$E\left[|\xi|^2 + \int_0^T |f(s,0,0)|^2 \, ds\right] < \infty.$$

We begin with a simple case, where generator f doesn't depend on (y,z), i.e ξ be square integrable and the processes $\{f(t)\}_{t\in[0,T]}$ belongs to $M^2(\mathbb{R}^k)$.

Lemma 1.2.1 *Let $\xi \in L^2(\mathcal{F}_T)$ and $\{f(t)\}_{t\in[0,T]} \in M^2(\mathbb{R}^k)$. The BSDE*

$$Y_t = \xi + \int_t^T f(s)\, ds - \int_t^T Z_s dW_s, \qquad 0 \leq t \leq T.$$

has a unique solution (Y,Z) such that $Z \in M^2$.

Proof. Let

$$Y_t = E\left(\xi + \int_t^T f(s)\, ds / \mathcal{F}_t\right),$$

we have f is progressively measurable, then from the Fubini theorem $\int_0^t f(t)\, dt$ is $\mathbb{F}$-adapted processes f is square integrable then f is belongs to $\mathcal{S}_c^2$, we get $\forall t \in [0,T]$,

$$Y_t = E\left(\xi + \int_0^T f(s)\, ds / \mathcal{F}_t\right) - \int_0^t f(s)\, ds := M_t - \int_0^t f(s)\, ds.$$

M is a Brownian martingale, then from the martingale representation theorem, there exists a unique process Z in M^2 such that

$$Y_t = M_t - \int_0^t f(s)\, ds = M_0 + \int_0^t Z_s dW_s - \int_0^t f(s)\, ds.$$

we obtain

$$\begin{aligned} Y_t - \xi &= M_0 + \int_0^t Z_s dW_s - \int_0^t f(s)\, ds - \left(M_0 + \int_0^T Z_s dW_s - \int_0^T f(s)\, ds\right) \\ &= \int_t^T f(s)\, ds - \int_t^T Z_s dW_s. \end{aligned}$$

■

The following result was proved by E. Prdoux and S. Peng [49].

Theoreme 1.2.1 *Under assumptions* (A_1), (A_2), *the BSDE* (1) *has a unique solution.*

Proof. We give a proof based on a fixed point argument on the Banach space $\mathcal{B}^2$, let us consider the operator Ψ:
$\forall (U,V) \in \mathcal{B}^2$, $(Y,Z) = \Psi(U,V)$ as a solution of BSDE:

$$Y_t = \xi + \int_t^T f(s, U_s, V_s)\, ds - \int_t^T Z_s dW_s, \quad 0 \leq t \leq T. \tag{1.2}$$

Since $|f(s,U_s,V_s)| \leq |f(s,0,0)| + K|U_s| + K\|V_s\|$, we see that the process $\{f(s,U_s,V_s),\ s \leq T\}$ is in $M^2(\mathbb{R}^k)$, then from lamma 1.2.1 the BSDE (1.2) has a unique solution (Y,Z) such that $Z \in M^2(\mathbb{R}^{k\times d})$ it remain to prove that $Y \in \mathcal{S}^2$. This easily obtained:

$$E\left[\sup_{t\leq T}|Y_t|^2\right] \leq C\left(|Y_0|^2 + E\left[\int_0^T |f(t,U_t,V_t)|^2\, dt\right] + E\left[\sup_{t\leq T}\left|\int_0^t Z_s.dW_s\right|^2\right]\right),$$

by using the Lipschitz property of the generator and Doob's inequality. Hence, Ψ is a well defined function from $\mathcal{B}^2$ into it self. we then see that (Y,Z) is a solution to the BSDE (1) if and only if it is a fixed point of Ψ.
Now, Let (U,V) and (U',V') are two elements belonging to $\mathcal{B}^2$, and $(Y,Z) = \Psi(U,V)$, $(Y',Z') = \Psi(U',V')$.
Puttig $y = Y - Y'$, $z = Z - Z'$, $u = U - U'$ and $v = V - V'$. we have $y_T = 0$ and

$$dy_t = -\{f(t,U_t,V_t) - f(t,U'_t,V'_t)\}\, dt + z_t.dW_t.$$

By applying Ito's formula to $e^{\alpha t}|y_t|^2$ we get that:

$$\begin{aligned} e^{\alpha t}|y_t|^2 + \int_t^T e^{\alpha s}\|z_s\|^2\, ds = &\int_t^T e^{\alpha s}\left(-\alpha|y_s|^2 + 2y_s.\{f(t,U_t,V_t) - f(t,U'_t,V'_t)\}\right) ds \\ &- \int_t^T 2e^{\alpha s} y_s.z_s dW_s, \end{aligned}$$

Observe that the Lipschitz condition, also implies:

$$e^{\alpha t} |y_t|^2 + \int_t^T e^{\alpha s} \|z_s\|^2 ds \le \int_t^T e^{\alpha s} \left(-\alpha |y_s|^2 + 2C |y_s| |u_s| + 2C |y_s| \|v_s\|\right) ds$$
$$- 2\int_t^T e^{\alpha s} y_s.z_s dW_s,$$

Using that, $\forall \varepsilon > 0$, $2ab \le a^2/\varepsilon + \varepsilon b^2$, we get

$$e^{\alpha t} |y_t|^2 + \int_t^T e^{\alpha s} \|z_s\|^2 ds \le \int_t^T e^{\alpha s} \left(-\alpha + 2C^2/\varepsilon\right) |y_s|^2 ds - \int_t^T 2e^{\alpha s} y_s.z_s dW_s$$
$$+ \varepsilon \int_t^T e^{\alpha s} \left(|u_s|^2 + \|v_s\|^2\right) ds$$

taking $\quad \alpha = 2C^2/\varepsilon \quad$ and $\quad R_\varepsilon = \varepsilon \int_0^T e^{\alpha s} \left(|u_s|^2 + \|v_s\|^2\right) ds$;

$$\forall t \in [0, T], \quad e^{\alpha t} |y_t|^2 + \int_t^T e^{\alpha s} \|z_s\|^2 ds \le R_\varepsilon - 2\int_t^T e^{\alpha s} y_s.z_s dW_s. \tag{1.3}$$

we prove that

$$M_. := \int_0^. e^{\alpha s} y_s.z_s dW_s \quad \text{is a uniformly integrable martingale.} \tag{1.4}$$

to prove (1.4), we verify that $\sup_{t\le T} |M_t| \in \mathbb{L}^1$.Indeed, by the Burkholder-Davis-Gundy inequality, we have:

$$\begin{aligned} E\left[\sup_{t\le T} |M_t|\right] &\le CE\left[\left(\int_0^T e^{2\alpha s} |y_s|^2 |z_s|^2 ds\right)^{1/2}\right] \\ &\le C'E\left[\sup_{t\le T} |y_s| \left(\int_0^T |z_s|^2 ds\right)^{1/2}\right] \\ &\le \frac{C'}{2}\left(\left[\sup_{t\le T} |y_s|^2\right] + E\left[\int_0^T |z_s|^2 ds\right]\right) < \infty. \end{aligned}$$

so we deduce from inequality (1.3) that

$$E\left[\int_0^T e^{\alpha s} \|z_s\|^2 ds\right] \le E\left[R_\varepsilon\right] \tag{1.5}$$

Taking the expectation, the inequality (1.3) and the BDG inequalities, we have:

$$E\left[\sup_{0\leq t\leq T} e^{\alpha t}\left|y_t\right|^2\right] \leq E\left[R_\varepsilon\right]+C'E\left[\left(\int_0^T e^{2\alpha s}\left|y_s\right|^2\left\|z_s\right\|^2 ds\right)^{1/2}\right]$$
$$\leq E\left[R_\varepsilon\right]+C'E\left[\sup_{0\leq t\leq T} e^{\alpha t/2}\left|y_s\right|\left(\int_0^T e^{\alpha s}\left\|z_s\right\|^2 ds\right)^{1/2}\right],$$

we use the inequality $ab\leq a^2/2+b^2/2$, we obtain:

$$E\left[\sup_{0\leq t\leq T} e^{\alpha t}\left|y_t\right|^2\right] \leq E\left[R_\varepsilon\right]+\frac{1}{2}E\left[\sup_{0\leq t\leq T} e^{\alpha t}\left|y_t\right|^2\right]+\frac{C'^2}{2}E\left[\int_0^T e^{\alpha s}\left\|z_s\right\|^2 ds\right].$$

which, combined with the estimate (1.5):

$$E\left[\sup_{0\leq t\leq T} e^{\alpha t}\left|y_t\right|^2+\int_0^T e^{\alpha s}\left\|z_s\right\|^2 ds\right] \leq \left(3+C'^2\right)E\left[R_\varepsilon\right]$$

returning to the definition of R_ε we get

$$E\left[\sup_{0\leq t\leq T} e^{\alpha t}\left|y_t\right|^2+\int_0^T e^{\alpha s}\left\|z_s\right\|^2 ds\right] \leq \varepsilon\left(3+C'^2\right)(1\vee T)\,E\left[\sup_{0\leq t\leq T} e^{\alpha t}\left|u_t\right|^2+\int_0^T e^{\alpha s}\left\|v_s\right\|^2 ds\right].$$

choosing ε such that $\varepsilon\left(3+C'^2\right)(1\vee T)=1/2$, this shows thati Ψ is a strict contraction on the Banach space $\mathcal{B}^2$endowed with the norm

$$\left\|(U,V)\right\|_\alpha := \sqrt{E\left(\sup_{0\leq t\leq T} e^{\alpha t}\left|U_t\right|^2\right)+E\left(\int_0^T e^{\alpha s}\left\|V_s\right\|^2 ds\right)}.$$

We conclude that Ψ admits a unique fixed point, which is the solution to the BSDE (1).

■

Proposition 1.2.1 *Given a pair (ξ,f) satisfying (A_1) and (A_2) .Let (Y,Z) be the solution of BSDE(1)such that $Z\in M^2$. Then, there exists a universal constant C_u such that.*

$$E\left[\sup_{0\leq t\leq T} e^{\beta t}\left|Y_t\right|^2+\int_0^T e^{\beta s}\left\|Z_s\right\|^2 ds\right] \leq C_u E\left[e^{\beta T}\left|\xi\right|^2+\int_0^T e^{\beta t}\left|f(t,0,0)\right|^2 dt\right],$$

with $\beta=1+2K+2K^2$.

Proof. By Ito's formula to $e^{\beta t}\left|Y_t\right|^2$

$$e^{\beta t}\left|Y_t\right|^2+\int_t^T e^{\beta s}\left\|Z_s\right\|^2 ds=e^{\beta T}\left|\xi\right|^2+\int_t^T e^{\beta s}\left(-\beta\left|Y_s\right|^2+2Y_s.f\left(s,Y_s,Z_s\right)\right) ds$$
$$-\int_t^T 2e^{\beta s}Y_s.Z_s dW_s.$$

Assumption (A_1), and using that $2ab\leq\varepsilon a^2+b^2/\varepsilon$ for $\varepsilon=1$ then $\varepsilon=2$,

$$2y.f\left(t,y,z\right)\leq\left(1+2K+2K^2\right)\left|y\right|^2+\left|f\left(t,0,0\right)\right|^2+\left\|z\right\|^2/2.$$

if we take $\beta=1+2K+2K^2$ we obtain, $\forall t\in[0,T]$,

$$e^{\beta t}\left|Y_t\right|^2+\frac{1}{2}\int_t^T e^{\beta s}\left\|Z_s\right\|^2 ds\leq e^{\beta T}\left|\xi\right|^2+\int_0^T e^{\beta s}\left|f\left(s,0,0\right)\right|^2 ds-2\int_t^T e^{\beta s}Y_s.Z_s dW_s. \quad (1.6)$$

The locale martingale $\left\{\int_0^t e^{\beta s}Y_s.Z_s dW_s, t\in[0,T]\right\}$ is a uniformly integrable martingale using see (1.4), taking the expectation, we get for $t=0$,

$$E\left[\int_0^T e^{\beta s}\left\|Z_s\right\|^2 ds\right]\leq 2E\left[e^{\beta T}\left|\xi\right|^2+\int_0^T e^{\beta s}\left|f\left(s,0,0\right)\right|^2 ds\right].$$

Returning to the Inequality (1.6), the BDG inequalities

$$E\left[\sup_{0\leq t\leq T}e^{\beta t}\left|Y_t\right|^2\right]\leq E\left[e^{\beta T}\left|\xi\right|^2+\int_0^T e^{\beta s}\left|f\left(s,0,0\right)\right|^2 ds\right]+CE\left[\left(\int_0^T e^{2\beta s}\left|Y_s\right|^2\left\|Z_s\right\|^2 ds\right)^{1/2}\right].$$

On the other hand

$$CE\left[\left(\int_0^T e^{2\beta s}\left|Y_s\right|^2\left\|Z_s\right\|^2 ds\right)^{1/2}\right]\leq CE\left[\sup_{0\leq t\leq T}e^{\beta t/2}\left|Y_t\right|\left(\int_0^T e^{\beta s}\left\|Z_s\right\|^2 ds\right)^{1/2}\right]$$
$$\leq\frac{1}{2}E\left[\sup_{0\leq t\leq T}e^{\beta t}\left|Y_t\right|^2\right]+\frac{C^2}{2}E\left[\int_0^T e^{\beta s}\left\|Z_s\right\|^2 ds\right].$$

Then,

$$E\left[\sup_{0\leq t\leq T}e^{\beta t}\left|Y_t\right|^2\right]\leq 2E\left[e^{\beta T}\left|\xi\right|^2+\int_0^T e^{\beta s}\left|f\left(s,0,0\right)\right|^2 ds\right]+C^2E\left[\int_0^T e^{\beta s}\left\|Z_s\right\|^2 ds\right].$$

and finally, we obtain

$$E\left[\sup_{0\leq t\leq T} e^{\beta t}\left|Y_t\right|^2+\int_0^T e^{\beta s}\left\|Z_s\right\|^2 ds\right]\leq 2\left(2+C^2\right)E\left[e^{\beta T}\left|\xi\right|^2+\int_0^T e^{\beta s}\left|f\left(s,0,0\right)\right|^2 ds\right],$$

which complete the proof with $C_u=2\left(2+C^2\right)$. ■

1.2.1 BSDE's with linear generator

Proposition 1.2.2 *We consider a BSDEs* $(k=1)$ *with generator:*

$$f\left(t,y,z\right):=a_t\ y+b_t\ z+c_t,$$

where $\left\{\left(a_t,b_t\right)\right\}_{t\in[0,T]}$ *is* $\mathbb{F}$*-progressively measurable and bounded processes with values in* $\mathbb{R}\times\mathbb{R}^k$*, and* $\left\{c_t\right\}_{t\in[0,T]}$ *satisfies* $E\left[\int_0^T\left|c_t\right|^2 dt\right]<\infty$*. The linear BSDE:*

$$Y_t=\xi+\int_t^T\left(a_s\ Y_s+b_s\ Z_s+c_s\right)ds-\int_t^T Z_s dW_s;$$

has a unique solution satisfies:

$$\forall t\in\left[0,T\right],\quad Y_t=\Gamma_t^{-1}E\left(\xi\Gamma_T+\int_t^T c_s\Gamma_s ds/\mathcal{F}_t\right),\tag{1.7}$$

where, $\forall t\in\left[0,T\right]$,

$$\Gamma_t=\exp\left\{\int_0^t b_s.dW_s-\frac{1}{2}\int_0^t\left|b_s\right|^2 ds+\int_0^t a_s ds\right\}.$$

Proof. By integration by parts to $\Gamma_t Y_t$, we get

$$d\left(\Gamma_t Y_t\right)=\Gamma_t dY_t+Y_t d\Gamma_t+d<\Gamma,Y>_t=-\Gamma_t c_t dt+\Gamma_t\left(Z_t+Y_t b_t\right).dW_t,$$

and so

$$\Gamma_t Y_t+\int_0^t\Gamma_s c_s ds=Y_0+\int_0^t\Gamma_s\left(Z_s+Y_s b_s\right).dW_s.\tag{1.8}$$

Since a and b are bounded, the Doob's inequality gives that $E\left[\sup_{t\leq T}|\Gamma_t|^2\right] < \infty$, and by denoting by b_∞ the upper-bound of b, we have

$$E\left[\left(\int_0^T \Gamma_s^2 |Z_s + Y_s b_s|^2 ds\right)^{1/2}\right] \leq \frac{1}{2}E\left[\sup_t |\Gamma_t|^2 + 2\int_0^T |Z_t|^2 dt + 2b_\infty^2 \int_0^T |Y_t|^2 dt\right]$$

From the Burkhlder-Davis-Gundy inquality, this shows that the local martingale in (1.8) is a uniformly integrable martingale.By taking the expectation, we obtain:

$$\begin{aligned}\Gamma_t Y_t + \int_0^t c_s \Gamma_s ds &= E\left(\Gamma_T Y_T + \int_0^T c_s \Gamma_s ds / \mathcal{F}_t\right)\\ &= E\left(\Gamma_T \xi + \int_0^T c_s \Gamma_s ds / \mathcal{F}_t\right),\end{aligned}$$

which gives the expression (1.7) . ■

1.3 The comparison theorem of BSDE's

Theoreme 1.3.1 *Let $n = 1$, and let (Y^i, Z^i) be the solution of BSDE (f^i, ξ^i) for som pair (ξ^i, f^i) satisfying the condition of Theorem* (3.2.1)*, $i = 0, 1$. Assume that*

$$\xi^0 \leq \xi^1 \quad \textit{and} \quad f_t^0\left(Y_t^0, Z_t^0\right) \leq f_t^1\left(Y_t^0, Z_t^0\right), \quad dt \otimes d\mathbb{P}-a.s. \tag{1.9}$$

Then $Y_t^0 \leq Y_t^1$, $t \in [0, T]$, $\mathbb{P}-a.s.$

Proof. We denote

$$\delta Y := Y' - Y,\ \delta Z := Z' - Z,\ \delta f := f'(Y, Z) - f'(Y, Z) \quad \text{and } \zeta := \xi' - \xi,$$

and we compute that

$$\delta Y_t = \zeta + \int_t^T (a_s.\delta Y_s + b_s.\delta Z_s + \delta f_s)\, ds - \int_t^T \delta Z_s.dW_s,$$

where

$$a_s := \frac{f'(s, Y_s', Z_s') - f'(s, Y_s, Z_s')}{\delta Y_s}$$

and for $i = 1, ..., d$,

$$b_s := \frac{f'\left(s, Y_s, Z_s^{(i-1)}\right) - f'\left(s, Y_s, Z_s^{(i)}\right)}{\delta Z_s^i}$$

where δZ_s^i denote the $i-$th component of δZ,

Since f' is Lipschitz-continuous, the processes a and b are bounded. Solving the linear BSDE as in proposition(1.2.2), we get:

$$\delta Y_t = \Gamma_t^{-1} E\left(\zeta \Gamma_T + \int_t^T \delta f_s \Gamma_s ds / \mathcal{F}_t\right), \quad t \leq T,$$

where the process Γ_t is defined as in equation (1.7) with $(\alpha, \beta, \delta f)$ substituted to (a, b, c). Then condition (1.9) implies that $\delta Y \geq 0$, $\mathbb{P}-$a.s. ■

Chapter 2

Backward doubly stochastic differential equations (BDSDE's)

In this chapter we introduce the theory of BDSDE's under uniformly Lipschitz coefficients see E. Pardoux, E., Peng, S [50]

2.1 Notation and assumptions

Let $\{W_t, 0 \leq t \leq T\}$ and $\{B_t, 0 \leq t \leq T\}$ be two independent standard Brownian motion defined on the completed probability space $(\Omega, \mathcal{F}, P)$, with values in $\mathbb{R}^d$ and $\mathbb{R}^l$ respectively. we put

$$\mathcal{F}_t := \mathcal{F}_t^W \vee \mathcal{F}_{t,T}^B$$

where $\mathcal{F}_t^W := \sigma(W_s; 0 \leq s \leq t)$, $\mathcal{F}_{t,T}^B := \sigma(B_s - B_t; t \leq s \leq T)$ completed with P-null sets and $\mathcal{F}_T^B = \mathcal{F}_{0,T}^B$. Note that $\{\mathcal{F}_t, 0 \leq t \leq T\}$ is neither increasing nor decreasing family of sub $\sigma-$fields, and it is not a filtration.

Let $\mathcal{M}^2(0, T, \mathbb{R}^n)$ denote the set of $n-$ dimensional, $\mathcal{F}_t-$ adapted stochastic processes $\{\varphi_t; t \in [0,T]\}$, such that $E\int_0^T |\varphi_t|^2 dt < \infty$.

We denote by $\mathcal{S}^2([0,T], \mathbb{R}^n)$, the set of continuous and $\mathcal{F}_t-$ adapted stochastic processes $\{\varphi_t; t \in [0,T]\}$, which satisfy $E(\sup_{0\leq t\leq T} |\varphi_t|^2) < \infty$.

Let

$$f : \Omega \times [0,T] \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \longmapsto \mathbb{R}^k$$
$$g : \Omega \times [0,T] \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \longmapsto \mathbb{R}^{k\times l}$$

be measurable functions such that, for every $(y,\ z) \in \mathbb{R}^k \times \mathbb{R}^{k\times d}$,

$$f(.,y,z) \in \mathcal{M}^2(0,T,\mathbb{R}^k)$$
$$g(.,y,z) \in \mathcal{M}^2(0,T,\mathbb{R}^{k\times l}).$$

We consider the following BDSDE

$$Y_t = \xi + \int_t^T f(s,Y_s,Z_s)ds + \int_t^T g(s,Y_s,Z_s)dB_s - \int_t^T Z_s dW_s\ , \qquad 0 \le t \le T \quad (E^{f,g})$$

where the dW is a Forward Itô integral and dB is a Backward Itô integral.
We assume the following hypotheses:

$(A._1)$ There exists a constants $C > 0$ and $0 < \alpha < 1$ such that for every $(\omega,t) \in \Omega \times [0,T]\,, (y_1,y_2)\ , (z_1,z_2) \in \mathbb{R}^k \times \mathbb{R}^{k\times d}$

$$|f(t,y_1,z_1) - f(t,y_2,z_2)|^2 \le C\left(|y_1-y_2|^2 + \|z_1-z_2\|^2\right)$$
$$\|g(t,y_1,z_1) - g(t,y_2,z_2)\|^2 \le C\left(|y_1-y_2|^2 + \|z_1-z_2\|^2\right).$$

$(A._2)$

$$\begin{cases} \text{il existe } c \text{ telle que pour tout } (t,y,z) \in [0,T] \times \mathbb{R}^k \times \mathbb{R}^{k\times d}, \\ gg^*(t,y,z) \le zz^* + c\left(\|g(t,0,0)\| + |y|^2\right) I. \end{cases}$$

Definition 2.1.1 *A solution of equation $(E^{f,g})$ is a couple $(Y^{f,g}, Z^{f,g})$ which belongs to the space $\mathcal{S}^2([0,\ T],\mathbb{R}^k) \times \mathcal{M}^2(0,T,\mathbb{R}^{k\times d})$ and satisfies $(E^{f,g})$.*

2.2 The main existence and uniqueness result

Theoreme 2.2.1 *Let ξ be a square integrable random variable. Under assumption $(A._1)$, BDSDE $(E^{f,g,\xi})$ has unique solution*

We shall begin with a simple case where the coefficients f and g do not depend on Y and Z in the following result:

Proposition 2.2.1 *Let $\xi \in L^2\left(\mathcal{F}_T, \mathbb{R}^k\right)$, $\{f(t)\}_{t\in[0,T]} \in \mathcal{M}^2\left(0, T, \mathbb{R}^k\right)$ and $\{g(t)\}_{t\in[0,T]} \in \mathcal{M}^2\left(0, T, \mathbb{R}^{k\times l}\right)$. The SDE*

$$Y_t = \xi + \int_t^T f(s)\,ds + \int_t^T g(s)\,dB_s - \int_t^T Z_s dW_s, \qquad 0 \le t \le T. \tag{2.1}$$

has a unique solution $(Y, Z) \in .\mathcal{S}^2([0,\, T], \mathbb{R}^k) \times \mathcal{M}^2(0, T, \mathbb{R}^{k\times d})$.

Proof. Uniqueness is immediate, since if $\left(\bar{Y}, \bar{Z}\right)$ is the difference of two solutions, we have

$$\bar{Y}_t + \int_t^T \bar{Z}_s dW_s = 0, \qquad 0 \le t \le T.$$

Hence by orthogonality

$$E\left(\left|\bar{Y}_t\right|^2\right) + E\int_t^T Tr\left[\bar{Z}_s \bar{Z}_s^*\right] ds = 0,$$

and $\bar{Y}_t \equiv 0$ P a.s., $\bar{Z}_t \equiv 0$ $dtdP$ a.e.
Now we turn to prove the existence. We consider the filtration $(\mathcal{G}_t)_{0\le t\le T}$:

$$\mathcal{G}_t = \mathcal{F}_t^W \vee \mathcal{F}_T^B,$$

from the assumptions, the process M_t defined by

$$M_t = E\left[\xi + \int_0^T f(s)\,ds + \int_0^T g(s)\,dB_s/\mathcal{G}_t\right], \tag{2.2}$$

is $\mathcal{G}_t$-square integrable martingale. An extension of Itô's martingale representation theorem yields the existence of a $\mathcal{G}_t$-progressively measurable process (Z_t) in $\mathcal{M}^2\left(0, T, \mathbb{R}^{k\times d}\right)$ such that

$$M_T = M_t + \int_0^T Z_s dW_s.$$

Subtracting $\int_0^t f(s)\,ds + \int_0^t g(s)\,dB_s$ from both sides of the equality (2.2) yields

$$Y_t = \xi + \int_t^T f(s)\,ds + \int_t^T g(s)\,dB_s - \int_t^T Z_s dW_s,$$

where

$$Y_t = E\left[\xi + \int_t^T f(s)\,ds + \int_t^T g(s)\,dB_s/\mathcal{G}_t\right].$$

It remains to show that (Y_t) and (Z_t) are in fact $\mathcal{F}_t$-adapted. For Y_t, this is obvious since for each t,

$$Y_t = E\left(\Theta/\mathcal{F}_t \vee \mathcal{F}_t^B\right),$$

where Θ is $\mathcal{F}_T^W \vee \mathcal{F}_{t,T}^B$ measurable. Hence $\mathcal{F}_t^B$ is independent of $\mathcal{F}_t \vee \sigma(\Theta)$, and

$$Y_t = E\left(\Theta/\mathcal{F}_t\right).$$

Now

$$\int_t^T Z_s dW_s = \xi + \int_t^T f(s)\,ds + \int_t^T g(s)\,dB_s - Y_t,$$

and the right side is $\mathcal{F}_T^W \vee \mathcal{F}_{t,T}^B$ measurable.

Hence, from Itô's martingale representation theorem, $(Z_s)_{s\in[0,T]}$ is $\mathcal{F}_s^W \vee \mathcal{F}_{t,T}^B$ adapted. Consequently Z_s is $\mathcal{F}_s^W \vee \mathcal{F}_{t,T}^B$ measurable, for any $t < s$, so it is $\mathcal{F}_s^W \vee \mathcal{F}_{s,T}^B$ measurable.

■

We shall need the following generalized Itô's formula.

Lemma 2.2.1 *Let* $\alpha \in \mathcal{S}^2([0,T],\mathbb{R}^k)$, $\beta \in \mathcal{M}^2(0,T,\mathbb{R}^k)$, $\gamma \in \mathcal{M}^2(0,T,\mathbb{R}^{k\times l})$, $\delta \in \mathcal{M}^2(0,T,\mathbb{R}^{k\times d})$ *be such that:*

$$\alpha_t = \alpha_0 + \int_0^t \beta_s ds + \int_0^t \gamma_s dB_s + \int_0^t \delta_s dW_s, \qquad 0 \le t \le T.$$

Then

$$\begin{aligned}|\alpha_t|^2 = |\alpha_0|^2 &+ 2\int_0^t \langle \alpha_s, \beta_s\rangle ds + 2\int_0^t \langle \alpha_s, \gamma_s dB_s\rangle + 2\int_0^t \langle \alpha_s, \delta_s dW_s\rangle \\ &- \int_0^t \|\gamma_s\|^2 ds + \int_0^t \|\delta_s\|^2 ds.\end{aligned}$$

More generally, if $\phi \in C^2\left(\mathbb{R}^k\right)$,

$$\phi\left(\alpha_t\right) = \phi\left(\alpha_0\right) + \int_0^t \langle \phi'\left(\alpha_s\right), \beta_s \rangle ds + \int_0^t \langle \phi'\left(\alpha_s\right), \gamma_s dB_s \rangle + \int_0^t \langle \phi'\left(\alpha_s\right), \delta_s dW_s \rangle$$
$$- \frac{1}{2}\int_0^t Tr\left[\phi''\left(\alpha_s\right)\gamma_s\gamma_s^*\right] ds + \frac{1}{2}\int_0^t Tr\left[\phi''\left(\alpha_s\right)\delta_s\delta_s^*\right] ds.$$

Proof. The first identity is a combination of Itô's forward and backward formula, applied to the process $\{\alpha_t\}$ and the function $x \to |x|^2$.
Let $0 = t_0 < t_1 < ... < t_n = t$.

$$\begin{aligned}
\left|\alpha_{t_{i+1}}\right|^2 - \left|\alpha_{t_i}\right|^2 &= 2\left(\alpha_{t_{i+1}} - \alpha_{t_i}, \alpha_{t_i}\right) + \left|\alpha_{t_{i+1}} - \alpha_{t_i}\right|^2 \\
&= 2\left(\int_{t_i}^{t_{i+1}} \beta_s ds, \alpha_{t_i}\right) + 2\left(\int_{t_i}^{t_{i+1}} \gamma_s dB_s, \alpha_{t_{i+1}}\right) + 2\left(\int_{t_i}^{t_{i+1}} \delta_s dW_s, \alpha_{t_i}\right) \\
&- 2\left(\int_{t_i}^{t_{i+1}} \gamma_s dB_s, \alpha_{t_{i+1}} - \alpha_{t_i}\right) = wa + \left|\alpha_{t_{i+1}} - \alpha_{t_i}\right|^2 \\
&= 2\int_{t_i}^{t_{i+1}} \left(\alpha_{t_i}, \beta_s\right) ds + 2\int_{t_i}^{t_{i+1}} \left(\alpha_{t_{i+1}}, \gamma_s dB_s\right) + 2\int_{t_i}^{t_{i+1}} \left(\alpha_{t_i}, \delta_s dW_s\right) \\
&- \left|\int_{t_i}^{t_{i+1}} \gamma_s dB_s\right|^2 + \left|\int_{t_i}^{t_{i+1}} \delta_s dW_s\right|^2 + \rho_i,
\end{aligned}$$

where $\sum_{i=0}^{n-1} \rho_i \to 0$ in probability, as $\sup\limits_i t_{i+1} - t_i \to 0$.
The second identify follows from the first. ■

Now we give the proof of theorem 2.2.1.

Proof. of theorem (2.2.1) **Uniqueness:**
Let (Y_t^1, Z_t^1) and (Y_t^2, Z_t^2) be two solutions. Define

$$\bar{Y}_t = Y_t^1 - Y_t^2, \quad \bar{Z}_t = Z_t^1 - Z_t^2, \quad 0 \leq t \leq T.$$

Then

$$\bar{Y}_t = \int_t^T \left[f\left(s, Y_s^1, Z_s^1\right) - f\left(s, Y_s^2, Z_s^2\right)\right] ds + \int_t^T \left[g\left(s, Y_s^1, Z_s^1\right) - g\left(s, Y_s^2, Z_s^2\right)\right] dB_s$$
$$- \int_t^T \bar{Z}_s dW_s.$$

Using Itô's formula to $\left|\bar{Y}_t\right|^2$, we get:

$$E\left(\left|\bar{Y}_t\right|^2\right)+E\int_t^T\left\|\bar{Z}_s\right\|^2ds=2E\int_t^T\langle\bar{Y}_s,f\left(s,Y_s^1,Z_s^1\right)-f\left(s,Y_s^2,Z_s^2\right)\rangle ds$$
$$+E\int_t^T\left\|g\left(s,Y_s^1,Z_s^1\right)-g\left(s,Y_s^2,Z_s^2\right)\right\|^2ds.$$

Hence from $(A._1)$ and the inequality $ab\leq\frac{1}{2(1-\alpha)}a^2+\frac{1-\alpha}{2}b^2$,

$$E\left(\left|\bar{Y}_t\right|^2\right)+E\int_t^T\left\|\bar{Z}_s\right\|^2ds\leq c\left(\alpha\right)E\int_t^T\left|\bar{Y}_s\right|^2ds+\frac{1-\alpha}{2}E\int_t^T\left\|\bar{Z}_s\right\|^2ds$$
$$+\alpha E\int_t^T\left\|\bar{Z}_s\right\|^2ds,$$

where $0<\alpha<1$ is the constant appearing in $(A._1)$. Consequently

$$E\left(\left|\bar{Y}_t\right|^2\right)+\frac{1-\alpha}{2}E\int_t^T\left\|\bar{Z}_s\right\|^2ds\leq c\left(\alpha\right)E\int_t^T\left|\bar{Y}_s\right|^2ds.$$

From Gronwall's lemma, $E\left(\left|\bar{Y}_t\right|^2\right)=0$, $0\leq t\leq T$, and hence $E\int_0^T\left\|\bar{Z}_s\right\|^2ds=0$.

Existence:

the existence is proved with the Picard iterations, we define recursively a sequence $(Y_t^n,Z_t^n)_{n=0,1,\ldots}$as follows. Let $Y_t^0\equiv0$, $Z_t^0\equiv0$. Given (Y_t^n,Z_t^n), $\left(Y_t^{n+1},Z_t^{n+1}\right)$ is the unique solution, constructed as in proposition 2.2.1, of the following equation:

$$Y_t^{n+1}=\xi+\int_t^Tf\left(s,Y_s^n,Z_s^n\right)ds+\int_t^Tg\left(s,Y_s^n,Z_s^n\right)dB_s-\int_t^TZ_s^{n+1}dW_s. \tag{2.3}$$

Let

$$\bar{Y}_t^{n+1}:=Y_t^{n+1}-Y_t^n,\bar{Z}_t^{n+1}:=Z_t^{n+1}-Z_t^n,0\leq t\leq T.$$

The same computation as in the proof of uniqueness yield:

$$E\left(\left|\bar{Y}_t^{n+1}\right|^2\right)+E\int_t^T\left\|\bar{Z}_t^{n+1}\right\|^2ds=2E\int_t^T\langle f\left(s,Y_s^n,Z_s^n\right)-f\left(s,Y_s^{n-1},Z_s^{n-1}\right),\bar{Y}_t^{n+1}\rangle ds$$
$$+E\int_t^T\left\|g\left(s,Y_s^n,Z_s^n\right)-g\left(s,Y_s^{n-1},Z_s^{n-1}\right)\right\|^2ds.$$

Let $\beta \in \mathbb{R}$. By integration by parts, we deduce

$$
\begin{aligned}
&E\left(\left|\bar{Y}_t^{n+1}\right|^2\right)e^{\beta t}+\beta E\int_t^T\left|\bar{Y}_s^{n+1}\right|^2e^{\beta s}ds+E\int_t^T\left\|\bar{Z}_t^{n+1}\right\|^2e^{\beta s}ds\\
&=2E\int_t^T\langle f\left(s,Y_s^n,Z_s^n\right)-f\left(s,Y_s^{n-1},Z_s^{n-1}\right),\bar{Y}_t^{n+1}\rangle e^{\beta s}ds\\
&\quad+E\int_t^T\left\|g\left(s,Y_s^n,Z_s^n\right)-g\left(s,Y_s^{n-1},Z_s^{n-1}\right)\right\|^2e^{\beta s}ds.
\end{aligned}
$$

There exists c, $\gamma>0$ such that

$$
\begin{aligned}
&E\left(\left|\bar{Y}_t^{n+1}\right|^2\right)e^{\beta t}+(\beta-\gamma)E\int_t^T\left|\bar{Y}_s^{n+1}\right|^2e^{\beta s}ds+E\int_t^T\left\|\bar{Z}_t^{n+1}\right\|^2e^{\beta s}ds\\
&\le E\int_t^T\left(c\left|\bar{Y}_s^n\right|^2+\tfrac{1+\alpha}{2}\left\|\bar{Z}_s^n\right\|^2\right)e^{\beta s}ds.
\end{aligned}
$$

Now choose $\beta=\gamma+\frac{2c}{1+\alpha}$, and define $\bar{c}=\frac{2c}{1+\alpha}$.

$$
\begin{aligned}
&E\left(\left|\bar{Y}_t^{n+1}\right|^2\right)e^{\beta t}+E\int_t^T\left(\bar{c}\left|\bar{Y}_s^{n+1}\right|^2+\left\|\bar{Z}_t^{n+1}\right\|^2\right)e^{\beta s}ds\\
&\le\tfrac{1+\alpha}{2}E\int_t^T\left(c\left|\bar{Y}_s^n\right|^2+\left\|\bar{Z}_s^n\right\|^2\right)e^{\beta s}ds.
\end{aligned}
$$

It follows immediately that

$$
\begin{aligned}
&E\int_t^T\left(\bar{c}\left|\bar{Y}_s^{n+1}\right|^2+\left\|\bar{Z}_t^{n+1}\right\|^2\right)e^{\beta s}ds\\
&\le\left(\tfrac{1+\alpha}{2}\right)^nE\int_t^T\left(c\left|\bar{Y}_s^n\right|^2+\left\|\bar{Z}_s^n\right\|^2\right)e^{\beta s}ds.
\end{aligned}
$$

and since $\frac{1+\alpha}{2}<1$, $\left(Y_t^n,Z_t^n\right)_{n=0,1,\ldots}$ is a cauchy sequence in $\mathcal{S}^2\left(0,T,\mathbb{R}^k\right)\times\mathcal{M}^2\left(0,T,\mathbb{R}^{k\times d}\right)$. It is then easy to conclude that $\left(Y_t^n\right)_{n=0,1,\ldots}$ is also Cauchy in $\mathcal{S}^2\left(0,T,\mathbb{R}^k\right)$, and that:

$$
\left(Y_t,Z_t\right)=\lim_{n\to\infty}\left(Y_t^n,Z_t^n\right).
$$

solves equation $\left(E^{f,g,\xi}\right)$. ■

Now, we establish a priori estimations for the solution of BDSDE $\left(E^{f,g,\xi}\right)$. For that we need to use assumption $(A._2)$ on g.

Theoreme 2.2.2 *Let $\xi \in L^p\left(\mathcal{F}_T, \mathbb{R}^k\right)$. Assume, in addition to $(A._1)$and $(A._2)$, that:*

$$E\int_0^T \left(|f(t,0,0)|^p + \|g(t,0,0)\|^p\right) dt < \infty.$$

holds, then

$$E\left(\sup_{0\leq t\leq T} |Y_t|^p + \left(\int_0^T \|Z_t\|^2\right)^{p/2}\right) < \infty.$$

Proof. By Ito's formula to $|Y_t|^p$ we obtain

$$\begin{aligned}
&|Y_t|^p + \tfrac{p}{2}\int_t^T |Y_s|^{p-2}\|Z_t\|^2 ds + \tfrac{p}{2}(p-2)\int_t^T |Y_s|^{p-4}\langle Z_s Z_s^* Y_s, Y_s\rangle ds \\
&\quad = |\xi|^p + p\int_t^T |Y_s|^{p-2}\langle f(s,Y_s,Z_s), Y_s\rangle ds + p\int_t^T |Y_s|^{p-2}\langle Y_s, g(s,Y_s,Z_s)\rangle dB_s \\
&\quad + \tfrac{p}{2}\int_t^T |Y_s|^{p-2}\|g(s,Y_s,Z_s)\|^2 ds \\
&\quad + \tfrac{p}{2}(p-2)\int_t^T |Y_s|^{p-4}\langle gg^*(s,Y_s,Z_s)Y_s, Y_s\rangle ds - p\int_t^T |Y_s|^{p-2}\langle Y_s, Z_s\rangle dW_s.
\end{aligned}$$

Taking the expectation, we get

$$\begin{aligned}
&E(|Y_t|^p) + \tfrac{p}{2}E\int_t^T |Y_s|^{p-2}\|Z_t\|^2 ds + \tfrac{p}{2}(p-2)E\int_t^T |Y_s|^{p-4}\langle Z_s Z_s^* Y_s, Y_s\rangle ds \\
&\quad \leq E(|\xi|^p) + p\int_t^T |Y_s|^{p-2}\langle f(s,Y_s,Z_s), Y_s\rangle ds + \tfrac{p}{2}E\int_t^T |Y_s|^{p-2}\|g(s,Y_s,Z_s)\|^2 ds \\
&\quad + \tfrac{p}{2}(p-2)\int_t^T |Y_s|^{p-4}\langle gg^*(s,Y_s,Z_s)Y_s, Y_s\rangle ds.
\end{aligned}$$

Note that we can conclude from $(H._1)$ that for any $\alpha < \alpha' < 1$, there exists $c(\alpha')$ such that

$$\|g(t,y,z)\|^2 \leq c(\alpha')\left(|y|^2 + \|g(t,0,0)\|^2\right) + \alpha'\|z\|^2. \tag{2.4}$$

Using (2.4), $(H._1)$, $(H._2)$ and using Hölder's an Young's inequalities, we deduce that there exists a positif constant θ and c such that

$$\begin{aligned}
&E(|Y_t|^p) + \theta E\int_t^T |Y_s|^{p-2}\|Z_s\|^2 ds \\
&\quad \leq E(|\xi|^p) + cE\int_t^T \left(|Y_s|^p + |f(s,0,0)|^p + \|g(s,0,0)\|^p\right) ds.
\end{aligned}$$

Using Gronwall's lemma, we obtain

$$\sup_{0\leq t\leq T} E\left(|Y_t|^p\right) + E\int_0^T |Y_t|^{p-2}\|Z_t\|^2\,dt < \infty.$$

Applying the same inequalities we have already used to the first identify of the proof, we get

$$\begin{aligned}|Y_t|^p &\leq |\xi|^p + c\int_t^T \left(|Y_s|^p + |f(s,0,0)|^p + \|g(s,0,0)\|^p\right)ds\\ &+p\int_t^T |Y_s|^{p-2}\langle Y_s, g(s,Y_s,Z_s)\rangle dB_s - p\int_t^T |Y_s|^{p-2}\langle Y_s, Z_s\rangle dW_s.\end{aligned}$$

then from the Burkholder-Davis-Gundy inequality

$$\begin{aligned}E\left(\sup_{0\leq t\leq T}|Y_t|^p\right) &\leq E\left(|\xi|^p\right) + cE\int_0^T \left(|Y_t|^p + |f(s,0,0)|^p + \|g(s,0,0)\|^p\right)dt\\ &+ cE\sqrt{\int_0^T |Y_t|^{2p-4}\langle gg^*(t,Y_t,Z_t)\,Y_t, Y_t\rangle dt}\\ &+ cE\sqrt{\int_0^T |Y_t|^{2p-4}\langle Z_tZ_t^*Y_t, Y_t\rangle dt}.\end{aligned}$$

The last term estimate as follows:

$$\begin{aligned}E\sqrt{\int_0^T |Y_t|^{2p-4}\langle Z_tZ_t^*Y_t, Y_t\rangle dt} &\leq E\left(Y_t^{p/2}\sqrt{\int_0^T |Y_t|^{p-2}\|Z_t\|^2\,dt}\right)\\ &\leq \frac{1}{3}E\left(\sup_{0\leq t\leq T}|Y_t|^p\right) + \frac{1}{4}E\int_0^T |Y_t|^{p-2}\|Z_t\|^2\,dt.\end{aligned}$$

We deduce that

$$E\left(\sup_{0\leq t\leq T}|Y_t|^p\right) < \infty.$$

Now we have

$$\begin{aligned}\int_0^T \|Z_t\|^2\,dt &= |\xi|^2 + |Y_0|^2 + 2\int_0^T \langle f(t,Y_t,Z_t), Y_t\rangle dt + 2\int_0^T \langle Y_t, g(t,Y_t,Z_t)\rangle dB_t\\ &+ \textstyle\int_0^T \|g(t,Y_t,Z_t)\|^2\,dt - 2\displaystyle\int_0^T \langle Y_t, Z_t\rangle dW_t.\end{aligned}$$

Hence for any $\delta > 0$

$$\begin{aligned}\left(\int_0^T \|Z_t\|^2 dt\right)^{p/2} &\leq (1+\delta)\left(\int_0^T \|g(t,Y_t,Z_t)\|^2 dt\right)^{p/2} \\ &+ c(\delta,p)\left[|\xi|^p + |Y_0|^p + \left|\int_0^T \langle f(t,Y_t,Z_t), Y_t\rangle dt\right|^{p/2}\right. \\ &\left. + \left|\int_0^T \langle Y_t, g(t,Y_t,Z_t)\rangle dB_t\right|^{p/2} + \left|\int_0^T \langle Y_t, Z_t\rangle dW_t\right|^{p/2}\right]\end{aligned}$$

Passing to the expectation

$$\begin{aligned}E\left(\int_0^T \|Z_t\|^2 dt\right)^{p/2} &\leq (1+\delta)^2 \alpha E\left[\left(\int_0^T \|Z_t\|^2 dt\right)^{p/2}\right] + c'(\delta,p) \\ &+ c(\delta,p)\, E\left[\left(\int_0^T |Y_t| \;\; \|Z_t\| \, dt\right)^{p/2}\right] \\ &+ c(\delta,p)\, E\left[\left(\int_0^T |Y_t|^2 \;\; \|Z_t\|^2 dt\right)^{p/4}\right] \\ &\leq (1+\delta)^2 \alpha E\left[\left(\int_0^T \|Z_t\|^2 dt\right)^{p/2}\right] + c'(\delta,p) \\ &+ c(\delta,p)\, E\left\{\left(\sup_{0\leq t\leq T} |Y_t|^{p/2}\right)\left[\left(\int_0^T \|Z_t\| \, dt\right)^{p/2} + \left(\int_0^T \|Z_t\|^2 dt\right)^{p/4}\right]\right\} \\ &\leq \left[\left(1+\delta^2\right)\alpha + (1+\delta)\right] E\left[\left(\int_0^T \|Z_t\|^2 dt\right)^{p/2}\right] + c''(\delta,p).\end{aligned}$$

The second part of the result follows, if we choose $\delta > 0$ small enough such that

$$(1+\delta)^2 \alpha + (1+\delta) < 1$$

■

2.3 The comparison theorem of BDSDE's

We consider one-dimensional BDSDE's

Theoreme 2.3.1 *Let $k = 1$, and let (Y^i, Z^i) be the solution of BSDE(f^i, ξ^i) for som pair*

(ξ^i, f^i) *satisfying the condition of Theorem* 2.2.1*,* $i = 0, 1$*. Assume that*

$$\xi^0 \geq \xi^1 \quad \textit{and} \quad f_t^0(Y_t, Z_t) \geq f_t^1(Y_t, Z_t), \quad dt \otimes d\mathbb{P}-a.s. \tag{2.5}$$

Then $Y_t^0 \geq Y_t^1,\ t \in [0, T], \mathbb{P}-a.s.$

Proof. We put

$$\delta Y := Y^0 - Y^1,\ \delta Z := Z^0 - Z^1,\ \delta f = f^0 - f^1 \quad \text{and } \zeta = \xi^0 - \xi^1,$$

The paire $(\delta Y_t, \delta Z_t)$ is solution of the BDSDE:

$$\begin{aligned}\delta Y_t = \zeta &+ \int_t^T \left(f^0\left(s, Y_s^0, Z_s^0\right) - f^1\left(s, Y_s^1, Z_s^1\right)\right) ds \\ &+ \int_t^T \left(g\left(s, Y_s^0, Z_s^0\right) - g\left(s, Y_s^1, Z_s^1\right)\right) dB_s \\ &- \int_t^T \delta Z_s dW_s,\end{aligned}$$

Using Itô's formula to $\left|(\delta Y_t)^-\right|^2$, we obtaine

$$\begin{aligned}\left|(\delta Y_t)^-\right|^2 = \left|\zeta^-\right|^2 &- 2\int_t^T (\delta Y_t)^- \left(f^0\left(s, Y_s^0, Z_s^0\right) - f^1\left(s, Y_s^1, Z_s^1\right)\right) ds \\ &- 2\int_t^T (\delta Y_t)^- \left(g\left(s, Y_s^0, Z_s^0\right) - g\left(s, Y_s^1, Z_s^1\right)\right) dB_s \\ &+ 2\int_t^T (\delta Y_t)^- \ \delta Z_s dW_s + \int_t^T \mathbf{1}_{\{\delta Y_s < 0\}} \left|g\left(s, Y_s^0, Z_s^0\right) - g\left(s, Y^1, Z_s^1\right)\right|^2 ds \\ &- \int_t^T \mathbf{1}_{\{\delta Y_s < 0\}} \left|\delta Z_s\right|^2 ds.\end{aligned} \tag{2.6}$$

We have from assumptions (2.5) $\zeta \geq 0$, then

$$\left|\zeta^-\right|^2 = 0.$$

The pairs (Y^0, Z^0) and (Y^1, Z^1) are in the space $\mathcal{S}^2([0,\, T], \mathbb{R}^k) \times \mathcal{M}^2(0, T, \mathbb{R}^{k\times d})$ then

$$E\left(\int_t^T (\delta Y_t)^- \ \delta Z_s dW_s\right) = 0,$$
$$E\left(\int_t^T (\delta Y_t)^- \left(g\left(s, Y_s^0, Z_s^0\right) - g\left(s, Y_s^1, Z_s^1\right)\right) dB_s\right) = 0.$$

Let

$$\begin{aligned} B &= -2\int_t^T (\delta Y_t)^- \left(f^0\left(s, Y_s^0, Z_s^0\right) - f^1\left(s, Y_s^1, Z_s^1\right)\right) ds \\ &= -2\int_t^T (\delta Y_t)^- \left(f^0\left(s, Y_s^0, Z_s^0\right) - f^0\left(s, Y_s^1, Z_s^1\right)\right) ds \\ &\quad - 2\int_t^T (\delta Y_t)^- \left(f^0\left(s, Y_s^1, Z_s^1\right) - f^1\left(s, Y_s^1, Z_s^1\right)\right) ds \\ &= B_1 + B_2. \end{aligned}$$

where

$$B_1 = -2\int_t^T (\delta Y_t)^- \left(f^0\left(s, Y_s^0, Z_s^0\right) - f^0\left(s, Y_s^1, Z_s^1\right)\right) ds$$
$$B_2 = -2\int_t^T (\delta Y_t)^- \left(f^0\left(s, Y_s^1, Z_s^1\right) - f^1\left(s, Y_s^1, Z_s^1\right)\right) ds$$

we have from conditions (2.5) $B_2 \leq 0$. and then $B \leq B_1$.
condition $(A._1)$ and Young's inequality, it follows

$$\begin{aligned} B \leq B_1 &\leq 2C\int_t^T (\delta Y_s)^- \left(|\delta Y_s| + \|\delta Z_s\|\right) ds \\ &\leq \left(2C + \frac{C^2}{1-\alpha}\right)\int_t^T |\delta Y_s|^2 \, ds + (1-\alpha)\int_t^T \mathbf{1}_{\{\delta Y_s \leq 0\}} \|\delta Z_s\|^2 \, ds, \end{aligned}$$

Using again condition $(A._1)$ we get

$$\begin{aligned} \int_t^T \mathbf{1}_{\{\delta Y_s \leq 0\}} \left|g\left(s, Y_s^0, Z_s^0\right) - g\left(s, Y^1, Z_s^1\right)\right|^2 ds &\leq \int_t^T \mathbf{1}_{\{\delta Y_s \leq 0\}} \left[C\, |\delta Y_s|^2 + \alpha\, \|\delta Z_s\|^2\right] \\ &= C\int_t^T \left|(\delta Y_s)^-\right|^2 ds \\ &\quad + \alpha\int_t^T \mathbf{1}_{\{\delta Y_s \leq 0\}} \|\delta Z_s\|^2 \, ds. \end{aligned}$$

Taking the expectation to (2.6) , we get

$$E\left|(\delta Y_t)^-\right|^2 \leq CE\int_t^T \left|(\delta Y_s)^-\right|^2 ds$$

Using Gronwall's lemma:

$$E\left|(\delta Y_t)^-\right|^2 = 0$$

Then, $Y_t^0 \geq Y_t^1$, $a.s., \forall t \in [0, T]$. ■

Chapter 3

Backward Doubly SDEs and SPDEs with superlinear growth generators

We deal with multidimensional backward doubly stochastic differential equations (BDSDEs) with a superlinear growth generator and square integrable terminal datum. We introduce new local conditions on the generator then we show that they ensure the existence and uniqueness as well as the stability of solutions. Although we are focused on multidimensional case, the uniqueness result we establish is new in one dimensional too. As application, we establish the existence and uniqueness of probabilistic solution to some semilinear stochastic partial differantial equations (SPDEs) with superlinear growth generator. By probabilistic solution, we mean a solution which is representable throughout a BDSDEs.

3.1 Introduction

Let $g(s, y, z)$ be a suitable function (for instance, uniformly Lipschitz in (y, z) and with strictly sublinear growth). Consider the following simple example which is covered by the present chapter

$$Y_t = \xi - \int_t^T Y_s \log|Y_s| ds + \int_t^T g(s, Y_s, Z_s) d\overleftarrow{B}_s - \int_t^T Z_s dW_s \qquad 0 \leq t \leq T \tag{3.1}$$

The generator of the previous equation is not of sublinear growth in the y-variable. It is also neither locally monotone nor uniformly continuous. We will see that equation (3.1) is covered by our work. Therefore, the so-called logarithmic nonlinearity $y\log|y|$ which appears in (3.1) is interesting in itself. In our knowledge, when the dimension is greater than one, there is no results in the literature which cover this interesting example. The same question is also posed for the nonlinearity $h(y)z\sqrt{\log|z|}$ where h is some function. the present chapter extends the results [7, 9, 10] to BDSDEs. We establish the existence and uniqueness as well as the $\mathbb{L}^p$-stability ($p<2$) of solutions for the BDSDE when the generator f is of superlinear growth in y, z. Moreover, the terminal datum still remains merely square integrable. We introduce a new local assumption on the generator which cover the previous ones on multidimensional BDSDEs and go beyond. Compared for instance with [57] , the generator we consider here can be neither locally monotone in the y-variable nor locally Lipschitz in the z-variable. We cover in particular the logarithmic nonlinearities $y\log(|y|)$ as well as $h(y)\ z\sqrt{|\log(|z|)|}$ where h is a suitable function. Actually, we allow the strictly sub-quadratic growth to the generator: $|f(t,y,z)| \quad \leq \quad \eta_t+|y|^\alpha+|z|^\alpha$ for some $\alpha\in\,]0,\ 2[$ and some $\mathbb{L}^2-$integrable process η. It is worth noting that our conditions on the generator f are local in the three variables y, z and ω. This allows us to cover BDSDEs with stochastic Lipschitz conditions.

Due to the local assumptions on the generator, the usual techniques of BDSDEs do not work in our situation. In the other hand, due to the superlinear growth of the generator, the method used in [4, 5, 11, 57] no longer work and, in particular, neither Gronwall's inequality nor Bihari's Lemma can be used in our situation. We develop here the method initiated in [7]. Indeed, we approximate f by a suitable sequence $(f_n)_{n>1}$ of Lipschitz functions then we use an appropriate localization to identify the limit as a solution of equation $(E^{f,g})$. The idea mainly consists to establish that the sequence of solutions (Y^n,Z^n), associated to the data (ξ,f_n,g) converges weakly in $\mathbb{L}^2$ and strongly in $\mathbb{L}^1$. To this end, we apply Itô's formula to $(|Y^n-Y^m|^2+\varepsilon)^\beta$ for some $0<\beta<1$ and $\varepsilon>0$, instead of $|Y^n-Y^m|\,|^2$ as is usually done. This allows us to treat multidimensional BDSDEs having a superlinear growth generator in its two variables y and z. We first prove the existence of solutions for a small time duration and then use the continuation procedure to extend the result to an arbitrarily prescribed time duration. The uniqueness and stability of solutions is established by similar arguments.

The chapter is organized as follows. The main results as well as some examples are given in section 2. Section 3 is devoted to the proofs of main results. As application, we consider, in section 4, semilinear SPDEs with superlinear generator for which establish the existence and uniqueness of Sobolev solutions for semilinear SPDEs.

3.2 Existence and uniqueness of solutions

Let us first repeat some notations from previous chapter. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space. For $T > 0$, let $\{W_t, 0 \leq t \leq T\}$ and $\{B_t, 0 \leq t \leq T\}$ be two independent standard Brownian motion defined on $(\Omega, \mathcal{F}, P)$ with values in $\mathbb{R}^d$ and $\mathbb{R}^l$ respectively. Let $\mathcal{F}_t^W := \sigma(W_s; 0 \leq s \leq t)$ and $\mathcal{F}_{t,T}^B := \sigma(B_s - B_t; t \leq s \leq T)$, completed with P-null sets. We define $\mathcal{F}_t := \mathcal{F}_t^W \vee \mathcal{F}_{t,T}^B$. It should be noted that the collection $\{\mathcal{F}_t, 0 \leq t \leq T\}$ is neither increasing nor decreasing family of sub $\sigma-$fields, and hence it is not a filtration. We consider the following BDSDE $(E^{f,g,\xi})$,

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s)ds + \int_t^T g(s, Y_s, Z_s)dB_s - \int_t^T Z_s dW_s \ , \qquad 0 \leq t \leq T,$$

where

$$f : \Omega \times [0, T] \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \longmapsto \mathbb{R}^k$$
$$g : \Omega \times [0, T] \times \mathbb{R}^k \times \mathbb{R}^{k\times d} \longmapsto \mathbb{R}^{k\times l}$$

be measurable function such that, for every $(y, \ z) \in \mathbb{R}^k \times \mathbb{R}^{k\times d}$,

$$f(., y, z) \in M^2(0, T, \mathbb{R}^k)$$
$$g(., y, z) \in M^2(0, T, \mathbb{R}^{k\times l}).$$

For $N \in \mathbb{N}^*$, we define $\rho_N(f) := E\int_0^T \sup_{|y|,|z|\leq N} |f(s, y, z)|\, ds$

We consider the following assumptions:

(H.1) f is continuous in (y, z) for $a.e.$ (t, ω).

(H.2) There exist $K > 0$, $M > 0$, and $\eta \in \mathbb{L}^1(\Omega; \mathbb{L}^1([0, T]))$ such that for every $(y, z) \in \mathbb{R}^k \times \mathbb{R}^{k\times d}$,

$$\langle y, f(t, \omega, y, z)\rangle \leq \eta_t + M|y|^2 + K\,|y|\,|z| \quad P-a.s.,\ a.e.\, t \in [0, T].$$

(H.3) g is continuous in $(y,\, z)$ for almost all (t,ω), and there exist $L>0,\ 0<\lambda<1,$ $0<\alpha_1<1,$ and $\eta'_t,\ 0\leq t\leq T$ verify $E\int_0^T|\eta'_s|^{\frac{2}{\alpha_1}}ds<\infty$ such that for every $(t,\omega,y,y',z,z')\in[0,T]\times\Omega\times\left(\mathbb{R}^k\right)^2\times\left(\mathbb{R}^{k\times d}\right)^2$,

$$(i)\ \|g(t,y,z)-g(t,y',z')\|^2\leq L|y-y'|^2-\lambda\|z-z'\|^2$$

and

$$(ii)\ \|g(t,y,z)\|\leq\eta'_t+L|y|^{\alpha_1}+\lambda\|z\|^{\alpha_1}.$$

(H.4) There exist $M_1>0,\ 0\leq\alpha<2,\ \alpha'>1$ and $\bar{\eta}\in\mathbb{L}^{\alpha'}([0,T]\times\Omega)$ such that for every $(t,\omega,y,z)\in[0,T]\times\Omega\times\mathbb{R}^k\times\mathbb{R}^{k\times d}$,

$$|f\,(t,\omega,y,z)|\leq\bar{\eta}_t+M_1\left(|y|^\alpha+|z|^\alpha\right).$$

(H.5) There exists $v\in\mathbb{L}^2\left(\Omega;\,\mathbb{L}^2([0,T])\right)$, a real valued sequence $(A_N)_{N>1}$ and constants $M_2>1,\ r>0$ such that:

(i) $\forall N>1,\quad 1<A_N\leq N^r$.

(ii) $\lim_{N\to\infty}A_N=\infty$.

(iii) For every $N\in\mathbb{N}^*$ and every $(t,\omega,y,y',z,z')\in[0,T]\times\Omega\times\left(\mathbb{R}^k\right)^2\times\left(\mathbb{R}^{k\times d}\right)^2$, such that $|y|,\ |y'|,\ |z|,\ |z'|\leq N$, we have

$$\begin{aligned}\langle y-y',f(t,y,z)-f(t,y',z')\rangle\mathbf{1}_{\{v_s(\omega)\leq N\}}\leq\ & M_2|y-y'|^2\log A_N\\ &+M_2|y-y'||z-z'|\sqrt{\log A_N}+M_2A_N^{-1}\end{aligned}$$

Theoreme 3.2.1 *Let ξ be a square integrable random variable. Assume that **(H.1)**–**(H.5)** are satisfied. Then equation $(E^{f,g})$ has a unique solution.*

The second main result of this section gives the stability of solutions in any $\mathbb{L}^q$ such that $q<2$. Let (f_n) be a sequence of processes which are $\mathcal{F}_t$-progressively measurable for each n. Let (ξ_n) be a sequence of square integrable random variables which are moreover $\mathcal{F}_T$-measurable for each n. We assume that for each n, the BDSDE (E^{f_n,g,ξ_n}) corresponding to the data (f_n,g,ξ_n) has a (not necessarily unique) solution. Each solution to equation (E^{f_n,g,ξ_n}) is denoted by (Y^n,Z^n). Let $(Y,\,Z)$ be the unique solution of the BDSDE $E^{(f,g,\xi)}$. We also assume that

(H.6) $\quad F$or every N, $\rho_N(f_n - f) \longrightarrow 0$ as $n \to \infty$.

(H.7) $\quad E(|\xi_n - \xi|^2) \longrightarrow 0$ as $n \to \infty$.

(H.8) There exist $K > 0$, $M > 0$ and $\eta \in \mathbb{L}^1(\Omega; \mathbb{L}^1([0,T]))$ such that for every $(y,z) \in \mathbb{R}^k \times \mathbb{R}^{k\times d}$,

$$\sup_n \langle y, f_n(t,\omega,y,z)\rangle \leq \eta_t + M|y|^2 + K|y||z| \quad P-a.s.,\, a.e.\, t \in [0,T].$$

(H.9) There exist $M_1 > 0$, $0 \leq \alpha < 2$, $\alpha' > 1$ and $\bar{\eta} \in \mathbb{L}^{\alpha'}([0,T]\times\Omega)$ such that for every $(y,z) \in \mathbb{R}^k \times \mathbb{R}^{k\times d}$,

$$\sup_n |f_n(t,\omega,y,z)| \leq \bar{\eta}_t + M_1(|y|^\alpha + |z|^\alpha).$$

Theoreme 3.2.2 *(Stability of solutions) Let f, g and ξ be as in Theorem 3.2.1. Assume that **(H.1)-(H.9)** are satisfied. Then, for every $q < 2$,*

$$\lim_{n\to+\infty}\left(\mathbb{E}\sup_{0\leq t\leq T}|Y_t^n - Y_t|^q + \mathbb{E}\int_0^T |Z_s^n - Z_s|^q ds\right) = 0.$$

Remark 3.2.1 *In a different context and for one dimensional BSDEs, it is shown in [53] that for $q < 2$,*

$$\lim_{n\to+\infty}\mathbb{E}\int_0^T |Z_s^n - Z_s|^q ds\bigg) = 0.$$

However, in [53] the component Y^n converges to Y only weakly, while here we have a strong convergence of Y^n to Y.
We point out that in [53] the BSDE is one dimensional and the generator is uniformly Lipschitz in (y,z), while here the BDSDE is multidimensional and the generator could be neither locally monotone in y nor locally Lipschitz in z.

Although the solution (Y,Z) of equation $(E^{f,g,\xi})$ belongs to $\mathcal{S}^2 \times \mathcal{M}^2$, the convergence given in Theorem 3.2.2 holds only in $\mathcal{S}^q \times \mathcal{M}^q$ for each $q < 2$. We unfortunately do not succeed to show the convergence in $\mathcal{S}^2 \times \mathcal{M}^2$ under our assumptions.

3.3 Some observations and examples

In order to clarify our results and assumptions, we present some remarks and examples. To the best of our knowledge, these examples are not covered by the previous works on BDSDEs. The first one deal with BDSDEs with locally generator.

- Assumption **(H.2)** can be replaced by

  (H'.2) There exist $M > 0$, $\eta \in \mathbb{L}^1(\Omega; \mathbb{L}^1([0,T]))$ and $\gamma > 0$ satisfying $2\gamma + \lambda < 1$ such that,

  $$\langle y, f(t, \omega, y, z)\rangle \leq \eta_t + M|y|^2 + \gamma|z|^2 \quad P - a.s., \, a.e. \, t \in [0,T].$$

  where λ is the constant defined in assumption **(H.3)**.

  In this case, minor modifications are needed in the proofs.

- The BDSDEs as well as the SPDEs we consider here are interesting in their own since the nonlinear part $f(t, y, z)$ can be neither locally Lipschitz in z nor locally monotone in y. Moreover, since f is of a super linear growth in y and z, then it cannot be uniformly continuous.

- It was established in [6] that the BSDEs with logarithmic growth $K|z|\sqrt{|\log|z||}$ also appear in some stochastic control problems. We hope that the present paper will be used in order stochastic control of SDE-BDSDEs.

- Since the system of SPDEs associated to the Markovian version of the BDSDE $(E^{(\xi,f,g)})$ can be degenerate, then our result also covers certain systems of first order SPDEs.

- Assumption **(H.2)** expresses the fact that the generator f can have a superlinear growth on $(y,\ z)$.

- The parameter α_1 appearing in assumption **(H.3)** has a role in the construction of solutions. More precisely, it allows to identify the backward stochastic integral driven by B. We think that in assumption **(H3)**-(ii), the condition $\alpha_1 < 1$ can be replaced by the weaker one : $\alpha_1 = 1$. However, we have not managed to do this.

- The term $\mathbf{1}_{\{v_s(\omega) \leq N\}}$ appearing in assumption **(H.5)** can be interpreted as a localization on ω. It allows in particular to cover generators with stochastic Lipschitz condition. It also enables to consider SDE-BDSDEs and systems of SPDEs with superlinear growth in the x-variable.

- Assumption **(H.5)** is related to the uniqueness and the stability of solutions. We think that it is possible to establish the existence of solutions without this condition. In particular, for one dimensional BDSDEs.

The following example treats a BDSDE with locally Lipschitz generator.

Example 3.3.1 *Assume that ξ is square integrable and g satisfies assumption **(H.3)**. Let f satisfies **(H.1)**, **(H.2)** and **(H.4)**. Assume moreover that there exists a positive constant C such that, for every $N > 0$ and every $|y|$, $|z|$ $|y'|$, $|z'| \leq N$,*

$$|f(t,y,z) - f(t,y',z)| \ \leq \ C(\log N)|y - y'|$$

and

$$|f(t,y,z) - f(t,y,z')| \ \leq \ C(\sqrt{\log N})|z - z'|$$

Then the BDSDE $(E^{f,g})$ has a unique solution which is stable in the sense of Theorem 3.2.2.

Proof. **(H.5)** is satisfied with $A_N = N$.

■

In the following example, we consider a situation where the generator is of logarithmic growth in its y-varaiable, typically $y \log |y|$ in dimension $d > 1$. Our interest to generators of type $y \log |y|$ arises from the papers [4, 5, 11, 7, 8] which are devoted classical BSDEs. Equation (3.1) is a natural continuation of BSDEs proposed in [4, 5, 11]. Indeed, consider the BDSDE $(E^{\xi,f,g})$ and assume for simplicity that the generator f does not depend on the variable z. Let f be L_N-locally Lipschitz and with at most linear growth. We know from the previous example that if moreover L_N behaves as $\log N$, then the BDSDE $(E^{\xi,f,g})$ has a unique solution. Now, if we drop the linear growth condition on f, then the assumption $L_N \sim \log N$ implies that $|f(y)| \leq K(1 + |y| \log |y|)$ for some positive constant K. Hence, it is natural to ask:
Could the BDSDEs of type (3.1) have solutions ? If yes, what happens about the uniqueness and the stability of solutions? The following example gives a positive answer to these questions. Note that the present chapter go beyond BDSDEs of type (3.1).

Example 3.3.2 *Assume that ξ is square integrable and g satisfies assumption **(H.3)**. Then, the BDSDE (3.1) has a unique solution. Moreover, this solution is stable in the sense of Theorem 3.2.2.*

Proof. Since $\langle y, f(y)\rangle \leq 1$ and $|f(y)| \leq 1 + \frac{1}{\varepsilon}|y|^{1+\varepsilon}$ for all $\varepsilon > 0$, we then deduce that f satisfies **(H.1)**, **(H.2)** and **(H.4)**. We shall check **(H.5)**. Thanks to triangular inequality, it is sufficient to treat separately the two cases: $0 \leq |y|, |y'| \leq \dfrac{1}{N}$ and $\dfrac{1}{N} \leq |y|, |y'| \leq N$.
The case $0 \leq |y|, |y'| \leq \dfrac{1}{N}$. Since the map $x \longmapsto -x\log x$ increases in the interval $]0,\ e^{-1}]$, then for $N > e$

$$
\begin{aligned}
|f(y) - f(y')| &\leq |f(y)| + |f(y')| \\
&\leq 2\frac{\log N}{N}
\end{aligned}
$$

The case $\dfrac{1}{N} \leq |y|, |y'| \leq N$. The finite increments theorem applied to f shows that

$$
|f(y) - f(y')| \leq (1 + \log N)|y - y'|.
$$

Therefore, **(H.5)** is satisfied for every $N > e$ with $v_s = 0$ and $A_N = N$.

■

In the following example, the generator is of superlinear growth in its z-varaiable.

Example 3.3.3 *Assume that ξ is square integrable and g satisfies assumption **(H.3)**. Let $\varepsilon_0 \in]0,\ 1[$. Let $l(y) := y\log\dfrac{|y|}{1+|y|}$ and $h \in \mathcal{C}(\mathbb{R}^{dr}; \mathbb{R}_+) \cap \mathcal{C}^1(\mathbb{R}^{dr} - \{0\};\ \mathbb{R}_+)$ be such that*

$$
h(z) = \begin{cases} |z|\sqrt{-\log|z|} & \text{if } |z| < 1 - \varepsilon_0 \\ |z|\sqrt{\log|z|} & \text{if } |z| > 1 + \varepsilon_0 \end{cases}
$$

Then, the BDSDE $(E^{(f,g,\xi)})$, where $f(y,z) := l(y)h(z)$, has a unique solution which is stable in the sense of Theorem 3.2.2.

Proof. It is not difficult to see that f satisfies **(H.1)**. We shall prove that f satisfies **(H.2)**, **(H.4)** and **(H.5)**. Since $\langle y, f(y,z)\rangle = \langle y, l(y)\rangle h(z) \leq 0$, then **(H.2)** is satisfied. We shall check **(H.4)**.

(i) Since l is continuous, $l(0) = 0$ and $|l(y)|$ tends to 1 as $|y|$ tends to ∞, we deduce that l is bounded. Moreover, l satisfies $\langle y - y', l(y) - l(y')\rangle \leq 0$. Indeed, in one dimensional case it is not difficult to show that l is a decreasing function. Since, $-\langle y, y'\rangle \log \frac{|y|}{1+|y|} \leq -|y||y'| \log \frac{|y|}{1+|y|}$ (because $\log \frac{|y|}{1+|y|} \leq 0$), then the problem is reduced to one dimension case by using the following computation:

$$\begin{aligned}
&\langle y - y', l(y) - l(y'^2 \log \frac{|y|}{1+|y|} + |y'^2 \log \frac{|y'|}{1+|y'|} - |y||y'|(\log \frac{|y|}{1+|y|} + \log \frac{|y'|}{1+|y'|}) \\
&= (|y| - |y'|)(|y| \ \log \frac{|y|}{1+|y|} - |y'| \ \log \frac{|y'|}{1+|y'|}) \\
&\leq \ 0.
\end{aligned}$$

(ii) Since $\sqrt{2\varepsilon \log |z|} = \sqrt{\log |z|^{2\varepsilon}} \leq |z|^{\varepsilon}$ for each $\varepsilon > 0$ and $|z| > 1$, then the function h satisfies for every $\varepsilon > 0$

$$0 \leq h(z) \leq M + \frac{1}{\sqrt{2\varepsilon}} |z|^{1+\varepsilon}, \qquad \text{where} \quad M = \sup_{|z| \leq 1+\varepsilon_0} |h(z)|.$$

Assumption **(H.4)** follows now directly from the previous observations (i) and (ii). To check **(H.5)**, it is enough to show that there exists a positive constant c such that for every z, z' satisfying $|z|, |z'| \leq N$ we have, for N large enough,

$$|h(z) - h(z')| \leq c \left(\sqrt{\log N} |z - z'| + \frac{\log N}{N} \right) \tag{3.2}$$

The previous inequality can be established by considering separately the following five cases, $0 \leq |z|, |z'| \leq \frac{1}{N}$, $\frac{1}{N} \leq |z|, |z'| \leq 1 - \varepsilon_0$, $1 - \varepsilon_0 \leq |z|, |z'| \leq 1 + \varepsilon_0$ and $1 + \varepsilon_0 \leq |z|, |z'| \leq N$.

The case $0 \leq |z|, |z'| \leq \frac{1}{N}$. Since the map $x \longmapsto x\sqrt{-\log x}$ increases in the interval $[0, \ \frac{1}{\sqrt{e}}]$, then for every $N > \sqrt{e}$,

$$\begin{aligned}
|h(z) - h(z')| &\leq |h(z)| + |h(z')| \\
&\leq 2 \frac{1}{N} \sqrt{-\log \frac{1}{N}} \\
&\leq \frac{2 \log N}{N}.
\end{aligned}$$

The other cases can be proved by using the finite increments theorem.

■

The following example treats the situation when the generator satisfies a stochastic Lipschitz (or monotone) condition.

Example 3.3.4 *Assume that ξ is square integrable and g satisfies assumption **(H.3)**. Let f satisfies **(H.1)**, **(H.2)**, **(H.4)** and*

$$
\textbf{(H}'\textbf{.5)}\begin{cases}
\text{There are a positive process } C \text{ satisfying } \mathbb{E}\displaystyle\int_0^T e^{q'C_s}ds<\infty \text{ (for some } q'>0) \text{ and } K'\in\mathbb{R}_+ \\
\text{such that for every } (t,\,\omega,\,y,\,y',\,z,\,z')\in[0,\ T]\times\Omega\times(\mathbb{R}^k)^2\times(\mathbb{R}^{k\times d})^2, \\
\\
\langle y-y', f(t,\omega,y,z)-f(t,\omega,y',z')\rangle \le K'\,|y-y'|^2\,\{C_t(\omega)+|\log|y-y'||\} \\
\\
\qquad\qquad\qquad\qquad +K'\,|y-y'|\,|z-z'|\sqrt{C_t(\omega)+|\log|z-z'||}.
\end{cases}
$$

Then, the BDSDE $(E^{f,g,\xi})$ has a unique solution which is stable in the sense of Theorem 3.2.2.

Proof. To check **(H.5)**, it is enough to show that there exists a positive constant c such that whenever $v_s := e^{C_s}\le N$ and $|y|$, $|y'|$, $|z|$, $|z'|\le N$ we have

$$\langle y-y', f(t,y,z)-f(t,y',z)\rangle \le c\log N\left(|y-y'^2+\frac{1}{N}\right)$$

and

$$|f(t,y,z)-f(t,y,z')|\le c\sqrt{\log N}\left(|z-z'|+\frac{1}{N}\right)$$

These two inequalities can be respectively proved by considering the following cases:

$$|y-y'|\le\frac{1}{2N},\qquad \frac{1}{2N}\le|y-y'|\le 2N,$$

and

$$|z-z'|\le\frac{1}{2N},\qquad \frac{1}{2N}\le|z-z'|\le 2N.$$

In particular, one can show that for every z,z',

$$|f(t,\omega,y,z)-f(t,\omega,y,z')|\le K''|\sqrt{C_t(\omega)+|\log|z-z'||}. \tag{3.3}$$

■

Remark 3.3.1 *If we put $C_t = 0$ in the previous example, it then follows from inequality (3.3) that we also cover generators satisfying the so-called Log-Lipschitz condition in the variable z.*

3.4 Proofs

3.4.1 Proofs of Theorem 3.2.1

Lemma 3.4.1 *(Approximation of f) Let f satisfy **(H.1)**– **(H.5)**. Then, there exists a sequence (f_n) such that,*

(a) For each n, f_n is bounded and globally Lipschitz in (y,z) for a.e. (t,ω).

There exists $M' > 0$, such that:

(b) $\sup_n |f_n(t,\omega,y,z)| \leq \bar{\eta} + M' + M_1(|y|^\alpha + |z|^\alpha)$ for a.e. (t,ω),

(c) $\sup\limits_n < y, f_n(t,\omega,y,z) > \leq \eta_t + M' + M|y|^2 + K|z||z|$ for a.e. (t,ω),

(d) For every N, $\rho_N(f_n - f) \longrightarrow 0$ as $n \longrightarrow \infty$.

Proof. . Let $\bar{\rho}_n : \mathbb{R}^d \times \mathbb{R}^{d\times r} \longrightarrow \mathbb{R}_+$ be a sequence of smooth functions with compact support which approximate the Dirac measure at 0 and which satisfy $\int \bar{\rho}_n(u)du = 1$. Let $\varphi_n : \mathbb{R}^d \longrightarrow \mathbb{R}_+$ be a sequence of smooth functions such that $0 \leq \varphi_n \leq 1$, $\varphi_n(u) = 1$ for $|u| \leq n$ and $\varphi_n(u) = 0$ for $|u| \geq n+1$. Likewise we define the sequence ψ_n from $\mathbb{R}^{d\times r}$ to $\mathbb{R}_+$. We put, $f_{q,n}(t,y,z) = 1\!\!1_{\{\bar{\eta}\leq q\}} \int f(t,(y,z)-u)\bar{\rho}_q(u)du\varphi_n(y)\psi_n(z)$. For $n \in \mathbb{N}^*$, let $q(n)$ be an integer such that $q(n) \geq n + n^\alpha$. The sequence (f_n), defined for each n by $f_n := f_{q(n),n}$, satisfies then assertions (a)-(d).

■

Estimate of solutions of BDSDE $(E^{f_n,g,\xi})$

Lemma 3.4.2 *Let f, g and ξ be as in Theorem 3.2.1. Let (f_n) be the sequence of functions associated to f by Lemma 3.4.1. For every integer n, we denote by (Y^n, Z^n) the unique solution of BDSDE $(E^{f_n,g,\xi})$. Then, there exits a universal constant ℓ such that*

a) $\mathbb{E}\displaystyle\int_0^T e^{2Ms}|Z_s^n|^2 ds \leq \frac{K_0}{(1-\lambda)}\left[e^{2MT}\mathbb{E}|\xi|^2 + 2\mathbb{E}\int_0^T e^{2Ms}(\eta_s + M' + (\eta_s')^2)ds\right] = K_1,$

b) $\mathbb{E}\sup\limits_{0\leq t\leq T}(e^{2Mt}|Y_t^n|^2) \leq \ell K_1 = K_2,$

c) $\mathbb{E}\int_0^T e^{2Ms}|f_n(s,Y_s^n,Z_s^n)|^{\bar{\alpha}}ds \leq 4^{\bar{\alpha}-1}\left[\mathbb{E}\int_0^T e^{2Ms}((\bar{\eta}_s+M')^{\bar{\alpha}}+4)ds + M_1^{\bar{\alpha}}K_1 + TM_1^{\bar{\alpha}}K_2\right]$
$= K_3,$

where $\bar{\alpha} = \min(\alpha', \frac{2}{\alpha})$.

d) $\mathbb{E}\int_0^T e^{2Ms}|g(s,Y_s^n,Z_s^n)|^2ds \leq K_4$.

Proof. Using Itô's formula and lemma 3.1 (c), we show that for every $t \in [0,\ T]$

$$\begin{aligned}
e^{2Mt}\left|Y_t^n\right|^2 + 2M\int_t^T e^{2Ms}|Y_s^n|^2ds \leq e^{2MT}\left|\xi\right|^2 &+ 2\int_t^T e^{2Ms}\left(\eta_s + M' + M\left|Y_s^n\right|^2 + K\left|Y_s^n\right|\left|Z_s^n\right|\right)ds \\
&+ 2\int_t^T e^{2Ms}\left(Y_s^n, Z_s^n\right)dW_s \\
&+ 2\int_t^T e^{2Ms}\langle Y_s^n, g(s,Y_s^n,Z_s^n)\rangle d\overleftarrow{B}_s \\
&+ \int_t^T e^{2Ms}(L\left|Y_s^n\right|^2 + \lambda\left|Z_s^n\right|^2 + (\eta_s')^2)ds \\
&- \int_t^T e^{2Ms}\left|Z_s^n\right|^2 ds.
\end{aligned}$$

Using the inequality $K|y||z| \leq \frac{K^2}{\varepsilon}|y|^2 + \varepsilon|z|^2$ then taking expectation it follows that,

$$\begin{aligned}
\mathbb{E}e^{2Mt}\left|Y_t^n\right|^2 + (1-\varepsilon-\lambda)\mathbb{E}\int_t^T e^{2Ms}\left|Z_s^n\right|^2ds \ \leq\ & e^{2MT}\mathbb{E}\left|\xi\right|^2 + 2\mathbb{E}\int_t^T e^{2Ms}(\eta_s + M' + (\eta_s')^2)ds \\
&+ 2\mathbb{E}\int_t^T e^{2Ms}(M + L + \frac{K^2}{\varepsilon})\left|Y_s^n\right|^2ds.
\end{aligned}$$

Taking $\varepsilon = \frac{1-\lambda}{4}$ then using Gronwall's lemma, we get assertion *a*). Using the Burkhölder-Davis-Gundy inequality, we get *b*). Assertions *c*) and *d*) follow from Lemma 3.1 *b*) and assumption (H.4). Using assumption (H.3) and assertions *a*), *b*), we get *e*).

■

After extracting a subsequence, we have

Corollary 3.4.1 *There are* $Y \in \mathbb{L}^2(\Omega, L^\infty[0,T])$, $Z \in \mathbb{L}^2(\Omega\times[0,T])$, $F \in \mathbb{L}^{\bar{\alpha}}(\Omega\times[0,T])$,

$\bar{g} \in \mathbb{L}^2(\Omega \times [0,T])$ *such that*

$$\begin{aligned} &Y^n \rightharpoonup Y, \text{ weakly star in } \mathbb{L}^2(\Omega, L^\infty[0,T]) \\ &Z^n \rightharpoonup Z, \text{ weakly in } \mathbb{L}^2(\Omega \times [0,T]) \\ &f_n(.,Y^n,Z^n) \rightharpoonup \bar{f} \text{ weakly in } \mathbb{L}^{\bar{\alpha}}(\Omega \times [0,T]), \\ &g_n(.,Y^n,Z^n) \rightharpoonup \bar{g} \text{ weakly in } \mathbb{L}^2(\Omega \times [0,T]) \end{aligned}$$

Moreover,

$$Y_t = \xi + \int_t^T \bar{f}(s)ds + \int_t^T \bar{g}(s)d\overleftarrow{B}_s - \int_t^T Z_s dW_s, \ \forall t \in [0,T].$$

The following technical Lemma is a direct consequence of Hölder's and Schwarz's inequalities.

Lemma 3.4.3 *For every* $\beta \in]1,2[, \ \ 0 < \lambda < 1, \ \ A > 0, \ \ (y)_{i=1..d} \subset \mathbb{R}$ *and* $(z)_{i=1..d,j=1..r} \subset \mathbb{R}$ *we have*

$$A\left[\sum_{i=1}^d y_i^2\right]^{\frac{1}{2}} \left[\sum_{i=1}^d \sum_{j=1}^r z_{ij}^2\right]^{\frac{1}{2}} - \frac{1-\lambda}{2}\sum_{i=1}^d \sum_{j=1}^r z_{ij}^2 + \frac{(1-\lambda)(2-\beta)}{2}\left[\sum_{i=1}^d y_i^2\right]^{-1} \sum_{j=1}^r \left[\sum_{i=1}^d y_i \ z_{ij}\right]^2$$
$$\leq \frac{1}{(1-\lambda)(\beta-1)} \ A^2 \ \sum_{i=1}^d y_i^2 \ - \frac{(1-\lambda)(\beta-1)}{4}\sum_{i=1}^d \ \sum_{j=1}^r z_{ij}^2.$$

Proof. Using the inequality $ab \leq \frac{\alpha^2}{2}a^2 + \frac{1}{2\alpha^2}b^2$ we have

$$A\left[\sum_{i=1}^d y_i^2\right]^{\frac{1}{2}} \left[\sum_{i=1}^d \sum_{j=1}^r z_{ij}^2\right]^{\frac{1}{2}} - \frac{1-\lambda}{2}\sum_{i=1}^d \sum_{j=1}^r z_{ij}^2 + \frac{(1-\lambda)(2-\beta)}{2}\left[\sum_{i=1}^d y_i^2\right]^{-1} \sum_{j=1}^r \left[\sum_{i=1}^d y_i \ z_{ij}\right]^2$$
$$\leq \frac{\alpha^2}{2}A^2\sum_{i=1}^d y_i^2 + \frac{1}{2\alpha^2}\sum_{i=1}^d \ \sum_{j=1}^r z_{ij}^2 - \frac{1-\lambda}{2}\sum_{i=1}^d \ \sum_{j=1}^r z_{ij}^2 + \frac{(1-\lambda)(2-\beta)}{2}\left[\sum_{i=1}^d y_i^2\right]^{-1} \sum_{j=1}^r \left[\sum_{i=1}^d y_i \ z_{ij}\right]^2.$$

By H¨older inequality we have $\sum_{i=1}^{d} y_i z_{ij} \leq (\sum_{i=1}^{d} y_i^2)^{\frac{1}{2}} (\sum_{i=1}^{d} z_{ij}^2)^{\frac{1}{2}}$. Hence

$$A\left[\sum_{i=1}^{d} y_i^2\right]^{\frac{1}{2}}\left[\sum_{i=1}^{d}\sum_{j=1}^{r} z_{ij}^2\right]^{\frac{1}{2}} - \frac{1-\lambda}{2}\sum_{i=1}^{d}\sum_{j=1}^{r} z_{ij}^2 + \frac{(1-\lambda)(2-\beta)}{2}\left[\sum_{i=1}^{d} y_i^2\right]^{-1}\sum_{j=1}^{r}\left[\sum_{i=1}^{d} y_i \quad z_{ij}\right]^2$$
$$\leq \frac{\alpha^2}{2}A^2\sum_{i=1}^{d} y_i^2 + \frac{1}{2\alpha^2}\sum_{i=1}^{d}\sum_{j=1}^{r} z_{ij}^2 - \frac{1-\lambda}{2}\sum_{i=1}^{d}\sum_{j=1}^{r} z_{ij}^2 + \frac{(1-\lambda)(2-\beta)}{2}\sum_{j=1}^{r}\sum_{i=1}^{d} \quad z_{ij}^2.$$

To complete the proof, we choose $\alpha^2 = \frac{2}{(1-\lambda)(\beta-1)}$. Lemma 3.4.3 is proved.

■

The following simple proposition is very useful in BSDEs and BDSDEs with superlinear growth generator.

Proposition 3.4.1 *Let (Y, Z) be an element of $\mathcal{S}^2 \times \mathcal{M}^2$. Assume **(H.4)** be satisfied. Let $\bar{\alpha} := \frac{2}{\alpha} \wedge \alpha'$. Then, there exists a positive constant $K(\bar{\alpha}, T, M_1) := K$, such that*

$$E\int_0^T |f(s, Y_s, Z_s)|^{\bar{\alpha}} ds \leq K\left[1 + E\int_0^T \bar{\eta}_s^{\bar{\alpha}} ds + E\left(\sup_{0\leq s\leq T}|Y_s|^2\right) + E\int_0^T |Z_s|^2 ds\right].$$

Proof. Using assumption **(H.4)**, we get

$$\begin{aligned}
E\int_0^T |f(s, Y_s, Z_s)|^{\bar{\alpha}} ds &\leq E\int_0^T (\bar{\eta}_s + M_1|Y_s|^{\alpha} + M_1|Z_s|^{\alpha})^{\bar{\alpha}} ds \\
&\leq 3^{\bar{\alpha}} M_1^{\bar{\alpha}} E\int_0^T \left(\bar{\eta}_s^{\bar{\alpha}} + |Y_s|^{\alpha\bar{\alpha}} + |Z_s|^{\alpha\bar{\alpha}}\right) ds \\
&\leq 3^{\bar{\alpha}} M_1^{\bar{\alpha}} E\int_0^T \left((1+\bar{\eta}_s)^{\bar{\alpha}} + (1+|Y_s|)^{\alpha\bar{\alpha}} + (1+|Z_s|)^{\alpha\bar{\alpha}}\right) ds \\
&\leq K\left[1 + E\int_0^T \bar{\eta}_s^{\bar{\alpha}} ds + E\sup_s|Y_s|^2 + E\int_0^T |Z_s|^2 ds\right].
\end{aligned}$$

■

Estimate between two approximating solutions

The key estimate is given by:

Proposition 3.4.2 *Let f, g and ξ be as in Theorem 3.2.1. Let (f_n) be the sequence of functions associated to f by Lemma 3.4.1. For any integer n, we denote by (Y^n, Z^n) the*

unique solution of the BDSDE $(E^{f_n,g,\xi})$. *Let* $\nu_R := \sup\{(A_N \log A_N)^{-1}, N \geq R\}$ *and* $C_N := \frac{\beta L}{2} + \frac{M_2^2 \beta \log A_N}{(1-\lambda)(\beta-1)} + \beta M_2 \log A_N$.
Then, for every $R \in \mathbb{N}$, $\beta \in \left]1,\ \min\left(3 - \frac{2}{\bar{\alpha}}\ ,\ 2\right)\right[$, $\varepsilon > 0$ *and*
$\delta' < (1-\lambda)(\beta-1)\min\left(\dfrac{1}{2M_2^2 + 2M_2(1-\lambda)(\beta-1)}\ ,\ \dfrac{\kappa}{rM_2^2\beta + r\beta M_2(1-\lambda)(\beta-1)}\right)$,
there exists $N_0 > R$ *such that for every* $N > N_0$ *and* $T' \leq T$,

$$\begin{aligned}\limsup_{n,m\to+\infty} E \sup_{(T'-\delta')^+ \leq t \leq T'} |Y_t^n - Y_t^m|^\beta + E\int_{(T'-\delta')^+}^{T'} \frac{|Z_s^n - Z_s^m|^2}{\left(|Y_s^n - Y_s^m|^2 + \nu_R\right)^{\frac{2-\beta}{2}}} ds \\ \leq \varepsilon + \frac{\ell}{(1-\lambda)(\beta-1)} e^{C_N \delta'} \limsup_{n,m\to+\infty} E|Y_{T'}^n - Y_{T'}^m|^\beta. \qquad (3.4)\end{aligned}$$

where ℓ *is the universal positive constant.*

Before giving the proof of this Proposition, let us give some explanation about it. Due to the superlinear growth of the generator, the techniques used in [4, 5] no longer work to give an estimate between two approximating solutions. In particular, neither Gronwall's inequality nor Bihari's Lemma can be used in our situation. Furthermore, in our situation, one cannot estimate quantities of type $|Y_t^n - Y_t^m|^2$ or of type $e^{at}|Y_t^n - Y_t^m|^2$ as usually done. Here, only estimates of type (3.4) are possible. That is one can only estimate quantities of type $|Y_t^n - Y_t^m|^\beta$ for $\beta < 2$. Otherwise the integrals can be infinite. However, estimate (3.4) is sufficient for our purpose. Note also that, this estimate allows to establish the stability of solutions in $\mathcal{S}^q \times \mathcal{M}^q$ for any $q < 2$. We think that the stability in $\mathcal{S}^2 \times \mathcal{M}^2$ is probably false under our assumptions. The proof of Proposition 3.4.2 necessitate numerous length computations. We will divide it into several steps presented as lemmas.

We summarize the idea we use. According to Lemma 3.4.2, the sequence (Y^n, Z^n) converges weakly in $\mathbb{L}^2$. In order to show that (Y^n, Z^n) converges strongly in $\mathbb{L}^1$, we define a new process for $N \in \mathbb{N}^\star$ by:

$$\Delta_t := |Y_t^n - Y_t^m|^2 + (A_N \log A_N)^{-1}. \qquad (3.5)$$

We will apply Itô's formula to $e^{Ct}\Delta_t^{\frac{\beta}{2}}$ for some $0 < \beta < 2$ then we pass to the limit successively on n and N to show the existence of solutions for a small time duration. We finally use a continuation procedure to extend the result to an arbitrarily prescribed time duration. The use of the process $\Delta_t^{\frac{\beta}{2}}$ allows us to treat BDSDEs having a superlinear growth generator in its variables y and z.

Lemma 3.4.4 *Let assumptions of Proposition 3.4.2 be satisfied. Then, for any $C > 0$ we have*

$$
\begin{aligned}
e^{Ct}\Delta_t^{\frac{\beta}{2}} + C\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}}ds \ \leq\ & e^{CT'}\Delta_{T'}^{\frac{\beta}{2}} + I_1 + I_2 + I_3 \\
& + \beta\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-1}\langle Y_s^n - Y_s^m, \quad g(s, Y_s^n, Z_s^n) - g(s, Y_s^m, Z_s^m)\rangle d\overleftarrow{B}_s \\
& - \beta\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-1}\langle Y_s^n - Y_s^m, (Z_s^n - Z_s^m)\, dW_s\rangle,
\end{aligned}
$$

where

$$
\begin{aligned}
I_1 := 2\beta e^{CT'}[2N^2 + \nu_1]^{\frac{\beta-1}{2}}\Bigg[&\int_t^{T'} \sup_{|y|,|z|\leq N} |f_n(s,y,z) - f(s,y,z)|ds \\
&+ \int_t^{T'} \sup_{|y|,|z|\leq N} |f_m(s,y,z) - f(s,y,z)|ds\Bigg],
\end{aligned}
$$

$$
I_2 := \beta\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-1}\langle Y_s^n - Y_s^m, \quad f_n(s, Y_s^n, Z_s^n) - f_m(s, Y_s^m, Z_s^m)\rangle 1\!\!1_{\{\Phi(s)>N\}}ds,
$$

$$
\text{with} \qquad \Phi(s) := |Y_s^n| + |Y_s^m| + |Z_s^n| + |Z_s^m| + v_s,
$$

$$
\begin{aligned}
I_3 := \quad & \left(\frac{\beta L}{2} + \frac{\beta M_2^2\log A_N}{(1-\lambda)(\beta-1)} + \beta M_2 \log A_N\right)\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}}ds \\
& - \beta\frac{(1-\lambda)(\beta-1)}{4}\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-2}|Z_s^n - Z_s^m|^2 ds.
\end{aligned}
$$

Proof. Using Itô's formula (applied to the process $e^{Ct}\Delta_t^{\frac{\beta}{2}}$) and assumption **(H3)**-(i), we show that for every $t \le T'$,

$$\begin{aligned}
e^{Ct}\Delta_t^{\frac{\beta}{2}} + C\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}}ds \;\le\; & e^{CT'}\Delta_{T'}^{\frac{\beta}{2}} \;+\; I_1' \;+\; I_1'' \;+\; I_2 \\
& - \frac{\beta}{2}\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-1}\,|Z_s^n - Z_s^m|^2\,ds \\
& + \frac{\beta}{2}\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-1}(L|Y_s^n - Y_s^m|^2 + \lambda|Z_s^n - Z_s^m|^2)ds \\
& - \beta(\frac{\beta}{2}-1)\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-2}\sum_{j=1}^{r}\left(\sum_{i=1}^{d}\left(Y_{i,s}^n - Y_{i,s}^m\right)\left(Z_{i,j,s}^n - Z_{i,j,s}^m\right)\right)^2 ds \\
& + \beta(\frac{\beta}{2}-1)\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-2}|Y_s^n - Y_s^m|(L|Y_s^n - Y_s^m| + \lambda|Z_s^n - Z_s^m|)ds \\
& + \beta\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-1}\langle Y_s^n - Y_s^m, \quad g(s,Y_s^n,Z_s^n) - g(s,Y_s^m,Z_s^m)\rangle d\overleftarrow{B}_s \\
& - \beta\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-1}\langle Y_s^n - Y_s^m, (Z_s^n - Z_s^m)\,dW_s\rangle,
\end{aligned}$$

where

$$\begin{aligned}
I_1' := \beta\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-1}\Big[&\langle Y_s^n - Y_s^m, \quad f_n(s,Y_s^n,Z_s^n) - f(s,Y_s^n,Z_s^n)\rangle \\
&+ \langle Y_s^n - Y_s^m, \quad f(s,Y_s^m,Z_s^m) - f_m(s,Y_s^m,Z_s^m)\rangle\Big]\mathbb{1}_{\{\Phi(s)\le N\}}ds,
\end{aligned}$$

$$I_1'' := \beta\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-1}\langle Y_s^n - Y_s^m, \quad f(s,Y_s^n,Z_s^n) - f(s,Y_s^m,Z_s^m)\rangle\mathbb{1}_{\{\Phi(s)\le N\}}ds.$$

Since $|Y_s^n - Y_s^m| \le \Delta_s^{\frac{1}{2}}$, it follows that

$$I_1' \le I_1.$$

Observe that $\mathbb{1}_{\{\Phi(s)\leq N\}} \leq \mathbb{1}_{\{v_s\leq N\}}$ then use assumption **(H.5)** to get

$$\begin{aligned}
\bar{I}_2'' &\leq \beta M_2 \int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-1}\Big[|Y_s^n - Y_s^m|^2 \log A_N \\
&\qquad\qquad + A_N^{-1} + |Y_s^n - Y_s^m||Z_s^n - Z_s^m|\sqrt{\log A_N}\Big]\mathbb{1}_{\{\Phi(s)\leq N\}}ds \\
&\leq \beta M_2 \int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-1}\Big[\Delta_s \log A_N + |Y_s^n - Y_s^m||Z_s^n - Z_s^m|\sqrt{\log A_N}\Big]\mathbb{1}_{\{\Phi(s)\leq N\}}ds. \qquad (3.6)
\end{aligned}$$

We put

$$\begin{aligned}
J \;:=\; & \bar{I}_1'' - \frac{\beta}{2}\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-1}\,|Z_s^n - Z_s^m|^2\,ds \\
&+ \frac{\beta}{2}\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-1}(L|Y_s^n - Y_s^m|^2 + \lambda|Z_s^n - Z_s^m|^2)ds \\
&- \beta(\frac{\beta}{2}-1)\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-2}\sum_{j=1}^{r}\left(\sum_{i=1}^{d}\left(Y_{i,s}^n - Y_{i,s}^m\right)\left(Z_{i,j,s}^n - Z_{i,j,s}^m\right)\right)^2 ds \\
&+ \beta(\frac{\beta}{2}-1)\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-2}|Y_s^n - Y_s^m|(L|Y_s^n - Y_s^m| + \lambda|Z_s^n - Z_s^m|)ds.
\end{aligned}$$

Using Lemma 3.4.3, we show that

$$J \;\leq\; I_3$$

Lemma 3.4.4 is proved. ■

The following Lemma allows to see how we use assumption **(H.5)**.

Lemma 3.4.5 *Let* $\gamma := \frac{M_2^2\delta'\beta}{(1-\lambda)(\beta-1)} + \delta'\beta M_2$. *Let assumptions of Proposition 3.4.2 be satisfied. Then, there exists a universal constant* ℓ *such that,*

$$\begin{aligned}
&\mathbb{E}\sup_{(T'-\delta')^+\leq t\leq T'}|Y_t^n - Y_t^m|^\beta + \mathbb{E}\int_{(T'-\delta')^+}^{T'}\frac{|Z_s^n - Z_s^m|^2}{\left(|Y_s^n - Y_s^m|^2 + \nu_R\right)^{\frac{2-\beta}{2}}}ds \\
&\leq \frac{\ell}{(1-\lambda)(\beta-1)}\Bigg(e^{C_N\delta'}\mathbb{E}|Y_{T'}^n - Y_{T'}^m|^\beta + \frac{A_N^\gamma}{(A_N\log A_N)^{\frac{\beta}{2}}} \\
&\qquad + 2\beta K_3^{\frac{1}{\alpha}}\left(4TK_2 + T\ell\right)^{\frac{\beta-1}{2}}\left(8TK_2 + 8K_1\right)^{\frac{\kappa}{2}}\frac{A_N^\gamma}{(A_N)^{\frac{\kappa}{r}}} \\
&\qquad + 2\beta e^{C_N\delta'}[2N^2 + \nu_1]^{\frac{\beta-1}{2}}\left[\rho_N(f_n - f) + \rho_N(f_m - f)\right]\Bigg).
\end{aligned}$$

Proof.

We choose $C = C_N := \frac{\beta L}{2} + \frac{M_2^2 \beta \log A_N}{(1-\lambda)(\beta-1)} + \beta M_2 \log A_N$ in Lemma 3.4.4 to get

$$
\begin{aligned}
& e^{C_N t}\Delta_t^{\frac{\beta}{2}} + \beta\frac{(1-\lambda)(\beta-1)}{4}\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-1}\,|Z_s^n - Z_s^m|^2\,ds \\
&\quad \le e^{C_N T'}\Delta_{T'}^{\frac{\beta}{2}} \; + \; I' \; + \; I'' + \; I_2 \\
&\qquad + \beta\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-1}\langle Y_s^n - Y_s^m, \quad g(s, Y_s^n, Z_s^n) - g(s, Y_s^m, Z_s^m)\rangle d\overleftarrow{B}_s \\
&\qquad - \beta\int_t^{T'} e^{Cs}\Delta_s^{\frac{\beta}{2}-1}\langle Y_s^n - Y_s^m, \quad (Z_s^n - Z_s^m)\,dW_s\rangle.
\end{aligned}
$$

Using estimate (3.6), Burkhölder-Davis-Gundy's inequality and H¨older's inequality (since $\frac{(\beta-1)}{2} + \frac{\kappa}{2} + \frac{1}{\bar\alpha} = 1$), we show that there exists a universal constant $\ell > 0$ such that for every $\delta' > 0$,

$$
\begin{aligned}
& \mathbb{E}\sup_{(T'-\delta')^+ \le t \le T'}\left[e^{C_N t}\Delta_t^{\frac{\beta}{2}}\right] + \mathbb{E}\int_{(T'-\delta')^+}^{T'} e^{C_N s}\Delta_s^{\frac{\beta}{2}-1}\,|Z_s^n - Z_s^m|^2\,ds \\
& \le \frac{\ell}{(1-\lambda)(\beta-1)} e^{C_N T'}\Bigg\{\mathbb{E}\left[\Delta_{T'}^{\frac{\beta}{2}}\right] + \frac{\beta}{N^\kappa}\left[\mathbb{E}\int_0^T \Delta_s ds\right]^{\frac{\beta-1}{2}}\left[\mathbb{E}\int_0^T \Phi(s)^2 ds\right]^{\frac{\kappa}{2}} \\
& \quad \times \left[\mathbb{E}\int_0^T |f_n(s, Y_s^n, Z_s^n) - f_m(s, Y_s^m, Z_s^m|^{\bar\alpha} ds\right]^{\frac{1}{\bar\alpha}} \\
& \quad + \beta[2N^2 + \nu_1]^{\frac{\beta-1}{2}}\mathbb{E}\Bigg[\int_0^T \sup_{|y|,|z| \le N} |f_n(s, y, z) - f(s, y, z)|ds \\
& \quad + \int_0^T \sup_{|y|,|z| \le N} |f_m(s, y, z) - f(s, y, z)|ds\Bigg]\Bigg\}.
\end{aligned}
$$

Using assumption **(H.5)** and Lemma 3.2, we get, for every $N > R$,

$$\begin{aligned}
&\mathbb{E} \sup_{(T'-\delta')^+ \leq t \leq T'} |Y_t^n - Y_t^m|^\beta + \mathbb{E}\int_{(T'-\delta')^+}^{T'} \frac{|Z_s^n - Z_s^m|^2}{(|Y_s^n - Y_s^m|^2 + \nu_R)^{\frac{2-\beta}{2}}} ds \\
&\leq \frac{\ell}{(1-\lambda)(\beta-1)} e^{C_N \delta'} \Big\{ (A_N \log A_N)^{\frac{-\beta}{2}} + \beta \frac{2K_3^{\frac{1}{\bar\alpha}}}{N^\kappa} (4TK_2 + T\ell)^{\frac{\beta-1}{2}} (8TK_2 + 8K_1)^{\frac{\kappa}{2}} \\
&\qquad\qquad + \mathbb{E}|Y_{T'}^n - Y_{T'}^m|^\beta + \beta[2N^2 + \nu_1]^{\frac{\beta-1}{2}} \big[\rho_N(f_n - f) + \rho_N(f_m - f)\big] \Big\} \\
&\leq \frac{\ell}{(1-\lambda)(\beta-1)} e^{C_N \delta'} \mathbb{E}|Y_{T'}^n - Y_{T'}^m|^\beta + \frac{\ell}{(1-\lambda)(\beta-1)} \frac{A_N^\gamma}{(A_N \log A_N)^{\frac{\beta}{2}}} \\
&\quad + \frac{2\ell}{(1-\lambda)(\beta-1)} \beta K_3^{\frac{1}{\bar\alpha}} (4TK_2 + T\ell)^{\frac{\beta-1}{2}} (8TK_2 + 8K_1)^{\frac{\kappa}{2}} \frac{A_N^\gamma}{(A_N)^{\frac{\kappa}{r}}} \\
&\quad + \frac{2\ell}{(1-\lambda)(\beta-1)} e^{C_N \delta'} \beta[2N^2 + \nu_1]^{\frac{\beta-1}{2}} \big[\rho_N(f_n - f) + \rho_N(f_m - f)\big].
\end{aligned}$$

Lemma 3.4.5 is proved.

■

Proof of Proposition 3.4.2 Taking $\delta' < (1-\lambda)(\beta-1) \min\left(\frac{1}{2M_2^2 + 2M_2(1-\lambda)(\beta-1)}\,,\ \frac{\kappa}{rM_2^2\beta + r\beta M_2(1-\lambda)(\beta-1)}\right)$, we show that

$$\frac{A_N^\gamma}{(A_N \log A_N)^{\frac{\beta}{2}}} \longrightarrow 0, \quad \text{as } N \to \infty$$

and

$$\frac{A_N^\gamma}{(A_N)^{\frac{\kappa}{r}}} \longrightarrow 0, \quad \text{as } N \to \infty.$$

We conclude the proof of Proposition 3.4.2 by using assertion (d) of Lemma 3.4.1.

We are now in position to prove our main results.

Proof of Theorem 3.2.1. Taking successively $T' = T$, $T' = (T - \delta')^+$, $T' = (T - 2\delta')^+$... in Proposition 3.4.2 we show that, for every $\beta \in]1,\ \min(3 - \frac{2}{\bar\alpha}\ ,\ 2)[$,

$$\lim_{n,m \to +\infty} \left(\mathbb{E} \sup_{0 \leq t \leq T} |Y_t^n - Y_t^m|^\beta + \mathbb{E}\int_0^T \frac{|Z_s^n - Z_s^m|^2}{(|Y_s^n - Y_s^m|^2 + \nu_R)^{\frac{2-\beta}{2}}} ds \right) = 0.$$

We immediately deduce that,

$$\lim_{n,m\to+\infty} \mathbb{E} \sup_{0\le t\le T} |Y_t^n - Y_t^m|^\beta = 0$$

In the other hand, by Schwarz's inequality we have

$$\mathbb{E}\int_0^T |Z_s^n - Z_s^m| ds \le \left(\mathbb{E}\int_0^T \frac{|Z_s^n - Z_s^m|^2}{(|Y_s^n - Y_s^m|^2 + \nu_R)^{\frac{2-\beta}{2}}} ds\right)^{\frac{1}{2}} \left(\mathbb{E}\int_0^T \left(|Y_s^n - Y_s^m|^2 + \nu_R\right)^{\frac{2-\beta}{2}} ds\right)^{\frac{1}{2}}.$$

Using Lemma 3.4.2, it holds that, $\sup_{n,m}\left(\mathbb{E}\int_0^T \left(|Y_s^n - Y_s^m|^2 + \nu_R\right)^{\frac{2-\beta}{2}} ds\right)^{\frac{1}{2}} < \infty.$ It follows that

$$\lim_{n,m\to+\infty}\left(\mathbb{E}\sup_{0\le t\le T}|Y_t^n - Y_t^m|^\beta + \mathbb{E}\int_0^T |Z_s^n - Z_s^m| ds\right) = 0.$$

Hence, there exists a subsequence which we still denote $(Y^n,\, Z^n)$ such that

$$\lim_{n\to+\infty} (|Y_t^n - Y_t| + |Z_t^n - Z_t|) = 0 \quad a.e.\ (t,\omega).$$

Since (Y^n) and (Z^n) are square integrable, it follows that for every $q < 2$.

$$\lim_{n\to+\infty} E\int_0^T (|Y_t^n - Y_t|^q + |Z_t^n - Z_t|^q)\, ds = 0. \tag{3.7}$$

We shall show that $(Y,\, Z)$ satisfies the BDSDE $E^{f,g,\xi}$. We have

$$\begin{aligned}
&\mathbb{E}\int_0^T |f_n(s, Y_s^n, Z_s^n) - f(s, Y_s^n, Z_s^n)| ds \\
&\le \mathbb{E}\int_0^T |f_n(s, Y_s^n, Z_s^n) - f(s, Y_s^n, Z_s^n)| \mathbb{1}_{\{|Y_s^n| + |Z_s^n| \le N\}} ds \\
&+ \mathbb{E}\int_0^T |f_n(s, Y_s^n, Z_s^n) - f(s, Y_s^n, Z_s^n)| \frac{(|Y_s^n| + |Z_s^n|)^{(2-\frac{2}{\bar\alpha})}}{N^{(2-\frac{2}{\bar\alpha})}} \mathbb{1}_{\{|Y_s^n| + |Z_s^n| \ge N\}} ds \\
&\le \rho_N(f_n - f) + \frac{2K_3^{\frac{1}{\bar\alpha}}\left[TK_2 + K_1\right]^{1-\frac{1}{\bar\alpha}}}{N^{(2-\frac{2}{\bar\alpha})}}.
\end{aligned}$$

Passing to the limit, first on n and next on N, we get

$$\lim_n E\int_0^T |f_n(s,Y_s^n,Z_s^n)-f(s,Y_s^n,Z_s^n)|ds=0.$$

We use (**H**.**1**), Lemma 3.4.1 and Lemma 3.4.2 to show that,

$$\lim_n E\int_0^T |f_n(s,Y_s^n,Z_s^n)-f(s,Y_s,Z_s)|ds=0.$$

Since the constant α_1 in assumption **(H.3)** is strictly less than 1, we then use Hölder's inequality (because $\frac{\alpha_1}{2}+\frac{2-\alpha_1}{2}=1$), assumption **(H.3)** and Lemma 3.4.2 to show that

$$\begin{aligned}
&E\int_0^T |g(s,Y_s^n,Z_s^n)-g(s,Y_s,Z_s)|^2ds\\
&\le E\int_0^T (|g(s,Y_s^n,Z_s^n)|+|g(s,Y_s,Z_s)|)\ |g(s,Y_s^n,Z_s^n)-g(s,Y_s,Z_s)|ds\\
&\le \left(E\int_0^T (|g(s,Y_s^n,Z_s^n)|+|g(s,Y_s,Z_s)|^{\frac{2}{\alpha_1}})ds\right)^{\frac{\alpha_1}{2}}\left(E\int_0^T |g(s,Y_s^n,Z_s^n)-g(s,Y_s,Z_s)|^{\frac{2}{2-\alpha_1}}ds\right)^{\frac{2-\alpha_1}{2}}\\
&\le \bar{K}\left(E\int_0^T (|Y_s^n-Y_s|^{\frac{2}{2-\alpha_1}}+|Z_s^n-Z_s|^{\frac{2}{2-\alpha_1}})ds\right)^{\frac{2-\alpha_1}{2}},
\end{aligned}$$

where $\bar{K}:=\sup(L,\lambda)\sup_n\left(E\int_0^T (2(\eta_s')^{\frac{1}{\alpha_1}}+K|Y_s^n|+K|Z_s^n|+K|Y_s|+K|Z_s|)^2ds\right)^{\frac{2-\alpha_1}{2}}<\infty$.

Since $\frac{2}{2-\alpha_1}<2$, then according to (3.7) we get

$$\lim_{n\to\infty} E\int_0^T |g(s,Y_s^n,Z_s^n)-g(s,Y_s,Z_s)|^2ds=0$$

The existence of solutions is proved.

Uniqueness. Let (Y,Z) and (Y',Z') be two solutions of equation $(E^{f,g,\xi})$. Arguing as previously one can show that for every $R>2$, $\beta\in]1,\ \min(3-\frac{2}{\bar{\alpha}},2)[$, $\varepsilon>0$ and $\delta'<(1-\lambda)(\beta-1)\min\left(\frac{1}{2M_2^2+2M_2(1-\lambda)(\beta-1)}\ ,\ \frac{\kappa}{rM_2^2\beta+r\beta M_2(1-\lambda)(\beta-1)}\right)$, there exists $N_0>R$ such that for every $N > N_0$ and $T' \leq T$,

$$\mathbb{E} \sup_{(T'-\delta')^+ \leq t \leq T'} |Y_t - Y_t^{'}|^\beta + \mathbb{E} \int_{(T'-\delta')^+}^{T'} \frac{\left|Z_s - Z_s^{'}\right|^2}{\left(|Y_s - Y_s^{'}|^2 + \nu_R\right)^{\frac{2-\beta}{2}}} ds$$
$$\leq \varepsilon + \frac{\ell}{(1-\lambda)(\beta-1)} e^{C_N \delta'} \mathbb{E}|Y_{T'} - Y_{T'}^{'}|^\beta.$$

Taking successively $T' = T$, $T' = (T-\delta')^+$, $T' = (T-2\delta')^+$..., we get the uniqueness of solutions. Theorem 3.2.1 is proved.

3.4.2 Proof of Theorem 3.2.2

Let $(\bar{Y}^n,\, \bar{Z}^n)$ be solution to the BDSDE $E^{(f^n,g,\xi^n)}$. Let $(Y,\, Z)$ be the unique solution of the BDSDE $E^{(f,g,\xi)}$. Arguing as in the proof of Theorem 3.2.1, we show that, for every $R > 2$, $\varepsilon > 0$, $\beta \in]1, \min\left(3 - \frac{2}{\bar{\alpha}}, 2\right)[$ and $\delta' < (1-\lambda)(\beta-1) \min\left(\frac{1}{2M_2^2 + 2M_2(1-\lambda)(\beta-1)}\, ,\, \frac{\kappa}{rM_2^2\beta + r\beta M_2(1-\lambda)(\beta-1)}\right)$, there exists $N_0 > R$ such that for every $N > N_0$ and $T' \leq T$,

$$\limsup_{n\to+\infty} \mathbb{E} \sup_{(T'-\delta')^+ \leq t \leq T'} |\bar{Y}_t^n - Y_t|^\beta + \mathbb{E} \int_{(T'-\delta')^+}^{T'} \frac{\left|\bar{Z}_s^n - Z_s\right|^2}{\left(|\bar{Y}_s^n - Y_s|^2 + \nu_R\right)^{\frac{2-\beta}{2}}} ds$$
$$\leq \varepsilon + \frac{\ell}{\beta-1} e^{C_N \delta'} \limsup_{n\to+\infty} \mathbb{E}|\bar{Y}_{T'}^n - Y_{T'}|^\beta.$$

Again as in the proof of Theorem 3.2.1, taking successively $T' = T$, $T' = (T-\delta')^+$, $T' = (T-2\delta')^+$..., we establish the convergence in the whole interval $[0,T]$. In particular,

$$\lim_{n\to+\infty} \left(|\bar{Y}^n - Y| + |\bar{Z}^n - Z|\right) = 0 \quad \text{in } P \times dt \text{ measure.}$$

Since $(\bar{Y}^n)$ and $(\bar{Z}^n)$ are square integrable, the proof is completed by using a uniform integrability argument. Theorem 3.2.2 is proved.

3.5 Application to Sobolev solutions of SPDEs

This section is devoted to the study of the existence and uniqueness of Sobolev solutions to the SPDE associated with the following decoupled system of SDE-BDSDE.

$$X_s^{t,x} = x + \int_t^s b\left(X_r^{t,x}\right) dr + \int_t^s \sigma\left(X_r^{t,x}\right) dW_r \tag{3.8}$$

$$Y_s^{t,x} = H\left(X_T^{t,x}\right) + \int_s^T F\left(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}\right) dr + \int_s^T G\left(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x}\right) d\overleftarrow{B}_r - \int_s^T Z_r^{t,x} dW_r \tag{3.9}$$

where σ, b, F and G are given measurable functions defined as follows:

$\sigma : \mathbb{R}^k \longmapsto \mathbb{R}^{k\times d}, \qquad b : \mathbb{R}^k \longmapsto \mathbb{R}^k, \qquad H : \mathbb{R}^k \longmapsto \mathbb{R}^k,$

$F : [0,T] \times \mathbb{R}^k \times \mathbb{R}^d \times \mathbb{R}^{k\times d} \longmapsto \mathbb{R}^d \quad , \quad G : [0,T] \times \mathbb{R}^k \times \mathbb{R}^d \times \mathbb{R}^{k\times d} \longmapsto \mathbb{R}^{d\times l}.$

The SPDE associated to the previous system of SDE-BDSDE is given, for $t \le s \le T$, by:

$$\begin{aligned} u(s,x) = H(x) &+ \int_s^T \mathcal{L}u(r,x) + F\big(r, x, u(r,x), \sigma^*\nabla u(r,x)\big) dr \\ &+ \int_s^T G\left(r, x, u(r,x), \sigma^*\nabla u(r,x)\right) d\overleftarrow{B}_r \end{aligned} \tag{3.10}$$

where

$$\mathcal{L} := \frac{1}{2}\sum_{i,j}(a_{ij})\frac{\partial^2}{\partial x_i \partial x_j} + \sum_i b_i \frac{\partial}{\partial x_i}, \qquad \text{with } (a_{ij}) := \sigma\sigma^* \tag{3.11}$$

We assume throughout this section that

$$b \in \mathcal{C}_b^2\left(\mathbb{R}^k, \mathbb{R}^k\right) \quad \text{and} \quad \sigma \in \mathcal{C}_b^3\left(\mathbb{R}^k, \mathbb{R}^{k\times r}\right) \tag{3.12}$$

For $0 < q < 2$, let $\mathcal{H}$ be the set of random fields $\{u(t,x),\ 0 \le t \le T,\ x \in \mathbb{R}^k\}$ such that, for every (t,x), $u(t,x)$ is $\mathcal{F}_{t,T}^B$-measurable and

$$\|u\|_{\mathcal{H}}^q = E[\int_{\mathbb{R}^k}\int_0^T (|u(r,x)|^q + |(\sigma^*\nabla u)(r,x)|^q) dr e^{-\delta|x|} dx] < \infty$$

The couple $(\mathcal{H},\ \|.\|_{\mathcal{H}})$ is a Banach space.

We denote by $\mathcal{C}_c^{1,\infty}([0,T]\times\mathbb{R}^d)$ the set of compactly supported functions $\varphi(t,x)$ which are continuously derivable in the t-variable and infinitely continuously derivable in the x-variable.

Definition 3.5.1 *We say that u is a Sobolev solution to SPDE (3.10), if $u \in \mathcal{H}$ and for any $\varphi \in \mathcal{C}_c^{1,\infty}([0,T] \times \mathbb{R}^d)$,*

$$\begin{aligned}
&\int_{\mathbb{R}^k} \int_s^T u(r,x) \frac{\partial \varphi(r,x)}{\partial r}(r,x) dr dx + \int_{\mathbb{R}} u(r,x)\varphi(r,x) dx - \int_{\mathbb{R}^k} H(x)\varphi(T,x) dx \\
&\quad - \frac{1}{2} \int_{\mathbb{R}^k} \int_s^T \sigma^* u(r,x) \sigma^* \varphi(r,x) dr dx - \int_{\mathbb{R}^k} \int_s^T u div\big[(b-A)\varphi\big](r,x) dr dx \\
&= \int_{\mathbb{R}^k} \int_s^T F(r,x,u(r,x),\sigma^* \nabla u(r,x)) \varphi(r,x) dr dx + \int_{\mathbb{R}^k} \int_s^T G(r,x,u(r,x),\sigma^* \nabla u(r,x)) \varphi(r,x) d\overleftarrow{B}_r dx
\end{aligned}$$

where A is a d-dimensional vector whose coordinates are defined by $A_j := \frac{1}{2} \sum_{i=1}^d \frac{\partial a_{ij}}{\partial x_i}$.

Assumptions. We assume that there exist $\delta \geq 0$ such that

(H.10) H belongs to $\mathbb{L}^2(\mathbb{R}^k, e^{-\delta|x|}dx;\ \mathbb{R}^d)$, that is $\int_{\mathbb{R}^d} |H(x)|^2 e^{-\delta|x|} dx < \infty$.

(H.11) $F(t,x,.,.)$ is continuous for $a.e.$ (t,x)

(H.12) There exist $M > 0$, $K > 0$ and $\eta \in \mathbb{L}^1([0,T] \times \mathbb{R}^k, e^{-\delta|x|}dtdx;\ \mathbb{R}_+)$ such that,

$$\langle y, F(t,x,y,z) \rangle \leq \eta(t,x) + M|y|^2 + K|y||z| \qquad \mathbb{P}-a.s., a.e.t \in [0,T].$$

(H.13) $\displaystyle\int_{\mathbb{R}^k} \int_0^T |G(t,x,0,0)|^2 e^{-\delta|x|} dt dx < \infty$ and there exist $L > 0$, $0 < \lambda < 1$, $0 < \alpha_1 < 1$, and $\eta \in \mathbb{L}^{\frac{2}{\alpha_1}}([0,T] \times \mathbb{R}^k, e^{-\delta|x|}dtdx;\ \mathbb{R}_+)$ such that for every $(t,\ x,\ y,\ y',\ z,\ z') \in [0,T] \times \mathbb{R}^k \times (\mathbb{R}^d)^2 \times (\mathbb{R}^{d\times r})^2$,

$$(i) \qquad |G(t,x,y,z) - G(t,x,y',z')|^2 \leq L|y-y'|^2 + \lambda|z-z'|^2.$$

$$(ii) \qquad |G(t,x,y,z)| \leq \eta'(t,x) + L|y|^{\alpha_1} + \lambda|z|^{\alpha_1}$$

(H.14) There exists $M_1 > 0$, $0 \leq \alpha < 2$, $\alpha' > 1$ and $\bar{\eta} \in \mathbb{L}^{\alpha'}([0,T] \times \mathbb{R}^k, e^{-\delta|x|}dtdx;\ \mathbb{R}_+)$ such that

$$|F(t,x,y,z)| \leq \bar{\eta}(t,x) + M_1(|y|^\alpha + |z|^\alpha).$$

(H.15) There exist $r > 0$ and $M_2 > 0$ such that, for every $N \in \mathbb{N}^*$, every $(t,\ x,\ y,\ y',\ z,\ z')$

satisfying

$e^{r|x|},\ |y|, |y'|, |z|, |z'| \leq N$, we have

$$\langle y-y',\ F(t,x,y,z)-F(t,x,y',z')\rangle \leq M_2 \log N \left(\frac{1}{N} + |y-y'^2\right) + \sqrt{M_2 \log N}|y-y'||z-z'|.$$

It is well known that, under condition (3.12), the forward SDE (3.8) has a unique solution X which is $\mathcal{F}_t^W$-adapted. Therefore, according to Theorem 3.2.1, the BDSDE (3.9) has a unique solution which is $\mathcal{F}_t$-adapted. The main result of this section is given by the following theorem.

Theoreme 3.5.1 *Let **(H.10)–(H.15)** be satisfied. Then, SPDE (3.10) has a unique Sobolev solution u such that for every $t \in [0,\ T]$,*

$$u(s, X_s^{t,x}) = Y_s^{t,x} \quad \text{and} \quad \sigma^* \nabla u(s, X_s^{t,x}) = Z_s^{t,x} \qquad \text{for a.e. } (s,\omega,x) \text{ in } [t,\ T] \times \Omega \times \mathbb{R}^k \tag{3.13}$$

where $\{(X_s^{t,x}, Y_s^{t,x}, Z_s^{t,x}), t \leq s \leq T\}$ is the unique solution of the SDE-BDSDE (3.8)-(3.9).

To prove this theorem, we need some lemmas. The following one can be found for instance in [20, 41, 42].

Lemma 3.5.1 *Let $X_s^{t,x},\ 0 \leq s \leq T$ be the unique solution to SDE (3.8). Then there exists a constant $K_{\delta,T} > 1$, such that for any $\Phi \in L^1\left(\Omega \times \mathbb{R}^k, \mathbb{P}\otimes e^{-\delta|x|}dx\right)$*

$$K_{\delta,T}^{-1}\left[\int_{\mathbb{R}^k} |\Phi(x)| e^{-\delta|x|}dx\right] \leq E\left[\int_{\mathbb{R}^k} \left|\Phi\left(X_s^{t,x}\right)\right| e^{-\delta|x|}dx\right] \leq K_{\delta,T}\left[\int_{\mathbb{R}^k} |\Phi(x)| e^{-\delta|x|}dx\right]$$

and for any $\Psi \in L^1\left(\Omega \times [0,\ T] \times \mathbb{R}^k \times \ \mathbb{P}\otimes dt \otimes e^{-\delta|x|}dx\right)$

$$\begin{aligned} K_{\delta,T}^{-1}\left[\int_{\mathbb{R}^k}\int_t^T |\Psi(s,x)|\, ds\ e^{-\delta|x|}dx\right] &\leq E\left[\int_{\mathbb{R}^k}\int_t^T \left|\Psi\left(s, X_s^{t,x}\right)\right| ds\ e^{-\delta|x|}dx\right] \\ &\leq K_{\delta,T}\left[\int_{\mathbb{R}^k}\int_t^T |\Psi(s,x)|\, ds\ e^{-\delta|x|}dx\right]. \end{aligned}$$

The following lemma can be proved by using the arguments we developed in the proofs of Theorem 3.2.1 and Theorem 3.2.2.

Lemma 3.5.2 *Assume* ***(H.10)–(H.15)*** *be satisfied. Let* $(X^{t,x})$ *be the unique solution to SDE (3.8). Let* $(Y^{t,x},\ Z^{t,x})$ *be the unique solution to BDSDE (3.9). Let* (F^n) *be a sequence of functions associated to* F *as in Lemma 3.4.1. For a fixed* $n \in \mathbb{N}^*$*, we denote by* $(Y^{n,t,x},\ Z^{n,t,x})$ *the unique solution to the following BDSDE:*

$$\begin{aligned} Y_s^{n,t,x} &= H(X_T^{t,x}) + \int_s^T F^n(r, X_r^{t,x}, Y_r^{n,t,x}, Z_r^{n,t,x})dr \\ &\quad + \int_s^T G(r, X_r^{t,x}, Y_r^{n,t,x}, Z_r^{n,t,x})d\overleftarrow{B}_r - \int_t^T Z_r^{n,t,x} dW_r \end{aligned} \tag{3.14}$$

Then,

(i) there exists $K(T,t,.) \in \mathbb{L}^1(e^{-\delta|x|}dx)$ *such that*

$$\sup_n \mathbb{E}\left[\sup_{s\le T}|Y_s^{n,t,x}|^2 + \sup_{s\le T}|Y_s^{t,x}|^2 + \int_s^T |Z_s^{n,t,x}|^2 ds + \int_s^T |Z_s^{n,t,x}|^2 ds\right] \le K(T,t,x) \tag{3.15}$$

(ii) for every $q < 2$*,*

$$\lim_{n\to+\infty}\left(\mathbb{E}\sup_{0\le s\le T}|Y_s^{n,t,x} - Y_s^{t,x}|^q + \mathbb{E}\int_0^T |Z_s^{n,t,x} - Z_s^{t,x}|^q ds\right) = 0. \tag{3.16}$$

Proof of Theorem 3.5.1. The uniqueness of solutions follows from the uniqueness of BDSDE (3.9). The proof of existence will be divided into four steps. Our strategy is to build a sequence of SPDEs associated to the sequence (F^n). We know from [50] that for each n, the SPDE associated to F^n has a unique solution u^n. By passing to the limit, we show that $u := lim_{n\to\infty} u^n$ solves SPDE (3.10). To do this, we argue as in the proof of Theorem 3.2.1.

Step 1. *Approximation of SPDE (3.10).*

Let (F^n) be the sequence of functions defined in the previous Lemma. For $(t,x) \in [0,\ T] \times \mathbb{R}^k$ and $n \in \mathbb{N}^*$, we define the functions u^n and v^n by

$$u^n(t,x) := Y_t^{n,t,x} \qquad \text{and} \qquad v^n(t,x) := Z_t^{n,t,x}$$

where $(Y^{n,t,x},\ Z^{n,t,x})$ is the unique solution of BDSDE (3.14).

Thanks to Theorem 4.5 of [57] (we can also use Theorem 2.1, p. 253 in [42]), we have

$v^n(s,x) = (\sigma^\star \nabla u^n)(s,x)$ and u^n is a Sobolev solution to the following SPDE:

$$(\mathcal{P}^{(f^n,g)}) \qquad \begin{cases} u^n(s,x) = H(x) + \displaystyle\int_s^T \{\mathcal{L}u^n(r,x) + F^n(r,x,u^n(r,x),\sigma^*\nabla u^n(r,x)\}\, dr \\ \qquad + \displaystyle\int_s^T G(r,x,u^n(r,x),\sigma^*\nabla u^n(r,x))\, d\overleftarrow{B}_r, \quad t \le s \le T, \end{cases}$$

such that

$$\big(u_n(s,X_s^{t,x}),\ \sigma^*\nabla u_n(s,X_s^{t,x})\big) = \big(Y_s^{n,t,x},\ Z_s^{n,t,x}\big) \quad a.s.\ \omega,\ a.e.\ s \in [t,\ T],\ x \in \mathbb{R}^k \tag{3.17}$$

where $(Y^{n,t,x},\ Z^{n,t,x})$ is the unique solution of BDSDE (3.14).

The uniqueness of u^n follows from the uniqueness of BDSDE (3.14).

Step 2. *Convergence of the problem* $(P^{(f^n,g)})$.

The limits which we will consider below hold along a subsequence. But for simplicity, this subsequence will be also indexed by n. From Lemma 3.5.1 and Lemma 3.5.2 we have

$$\begin{aligned} E\int_{\mathbb{R}^k}\int_t^T & \left(|u^n(s,x)|^2 + |v^n(s,x)|^2\right) ds\ \ e^{-\delta|x|}dx \\ & \le K_{\delta,T}^{-1}\ E\Big\{\Big[\int_{\mathbb{R}^k}\Big(\sup_t \left|Y_t^{n,t,x}\right|^2 + \int_t^T \left|Z_s^{n,t,x}\right|^2 ds\Big)\ e^{-\delta|x|}dx\Big] \\ & \le K_{\delta,T}^{-1}\int_{\mathbb{R}^k} K(T,t,x)\ e^{-\delta|x|}dx \\ & < \infty \end{aligned} \tag{3.18}$$

Using **(H.3)**, we get

$$|G(s,x,u^n(s,x),v^n(s,x))|^2 \le 2\,|G(s,x,0,0)|^2 + 2L\,|u^n(s,x)|^2 + 2\lambda\,|v^n(s,x)|^2$$

Thanks to inequality (3.18), we deduce that

$$E\Big[\int_{\mathbb{R}^k}\int_t^T \left(|F^n(s,x,u^n(s,x),v^n(s,x))|^{\bar{\alpha}} + |G(s,x,u^n(s,x),v^n(s,x))|^2\right) ds\ e^{-\delta|x|}dx\Big] < \infty \tag{3.19}$$

Using Lemma 3.5.1, Lemma 3.5.2-(ii) and the Lebesgue dominated convergence theorem, we show that for every $q < 2$,

$$\lim_{n,m} \int_0^T \int_{\mathbb{R}^k} |u^n(s,x) - u^m(s,x)|^q \, e^{-\delta|x|} dx = 0$$
$$\lim_{n,m} \int_0^T \int_{\mathbb{R}^k} |\sigma^* \nabla u^n(s,x) - \sigma^* \nabla u^m(s,x)|^q \, e^{-\delta|x|} dx = 0$$

By Lemma 3.5.1 and the fact that $\mathcal{H}$ is complete, it follows that there exists $u \in \mathcal{H}$ such that for every $q < 2$,

$$(i) \quad \lim_{n} \int_0^T \int_{\mathbb{R}^k} |u^n(s,x) - u(s,x)|^q \, e^{-\delta|x|} dx = 0$$

$$(ii) \quad \lim_{n} \int_0^T \int_{\mathbb{R}^k} |\sigma^* \nabla u^n(s,x) - \sigma^* \nabla u(s,x)|^q \, e^{-\delta|x|} dx = 0$$

$$(iii) \quad \sup_{0 \leq s \leq T} \int_{\mathbb{R}^k} |u(s,x)|^q \, e^{-\delta|x|} dx + \int_0^T \int_{\mathbb{R}^k} |\sigma^* \nabla u(s,x)|^q \, e^{-\delta|x|} dx < \infty$$

Step 3. *We show that* $u(s, X_s^{t,x}) = Y_s^{t,x}$ *and* $\sigma^* \nabla u(s, X_s^{t,x}) = Z_s^{t,x}$.

By triangular inequality, we have

$$\begin{aligned}\mathbb{E} \int_{\mathbb{R}^k} \int_0^T \big(|u(s\ \ X_s^{t,x}) - Y_s^{t,x}|\big) ds \ e^{-\delta|x|} dx &\leq \mathbb{E} \int_{\mathbb{R}^k} \int_0^T \big(|u(s\ \ X_s^{t,x}) - u^n(s\ \ X_s^{t,x})|\big) ds \ e^{-\delta|x|} dx \\ &+ \mathbb{E} \int_{\mathbb{R}^k} \int_0^T \big(|u^n(s\ \ X_s^{t,x}) - Y_s^{t,x}|\big) ds \ e^{-\delta|x|} dx\end{aligned}$$

Using Lemma 3.5.1 and the the previous assertion (i), we show that the first term, in the right hand side of the previous inequality, tends to 0 as n tends to ∞. Since $u^n(s\ \ X_s^{t,x}) = Y_s^{n,t,x}$, then we use Lemma 3.5.2-(ii) and the Lebesgue dominated convergence Theorem to prove that the second term also tends to 0 as n tends to ∞. This shows that $u(s, X_s^{t,x}) = Y_s^{t,x}$. We shall prove that $\sigma^* \nabla u(s, X_s^{t,x}) = Z_s^{t,x}$. Again by triangular inequality, we have

$$
\begin{aligned}
&\mathbb{E}\int_{\mathbb{R}^k}\int_0^T \big(|\sigma^*\nabla u(s\ \ X_s^{t,x}) - Z_s^{t,x}|\big)ds\ e^{-\delta|x|}dx \\
&\qquad\qquad \leq \mathbb{E}\int_{\mathbb{R}^k}\int_0^T \big(|\sigma^*\nabla u(s\ \ X_s^{t,x}) - \sigma^*\nabla u^n(s\ \ X_s^{t,x})|\big)ds\ e^{-\delta|x|}dx \\
&\qquad\qquad\quad + \mathbb{E}\int_{\mathbb{R}^k}\int_0^T \big(|\sigma^*\nabla u^n(s\ \ X_s^{t,x}) - Z_s^{t,x}|\big)ds\ e^{-\delta|x|}dx.
\end{aligned}
$$

Using Lemma 3.5.1 and the previous assertion (ii), we show that the first term, in the right hand side of the previous inequality, tends to 0 as n tends to ∞. Since $\sigma^*\nabla u^n(s\ \ X_s^{t,x}) = Z_s^{n,t,x}$, then using Lemma 3.5.2-(ii), one can prove that the second term tends also to 0 as n tends to ∞.

Step 4. *We prove that u is a Sobolev solution to SPDE (3.10).*

It was shown in step 2 that u belongs to $\mathcal{H}$. So, it remains to prove that u satisfies the definition 3.5.1. Let $\varphi \in \mathcal{C}_c^{1,\infty}([0,T]\times\mathbb{R}^d)$. Since, for every n, u^n is a Sobolev solution to the problem $(P^{(f^n,g)})$, we then have

$$
\begin{aligned}
&\int_{\mathbb{R}^k}\int_s^T u^n(r,x)\frac{\partial\varphi(r,x)}{\partial r}(r,x)drdx + \int_{\mathbb{R}} u^n(r,x)\varphi(r,x)dx - \int_{\mathbb{R}^k} H(x)\varphi(T,x)dx \\
&\qquad - \frac{1}{2}\int_{\mathbb{R}^k}\int_s^T \sigma^* u^n(r,x)\sigma^*\varphi(r,x)drdx - \int_{\mathbb{R}^k}\int_s^T u^n div\big[(b-A)\varphi\big](r,x)drdx \\
&= \int_{\mathbb{R}^k}\int_s^T F^n(r,x,u^n(r,x),\sigma^*\nabla u^n(r,x))\varphi(r,x)drdx \\
&\quad + \int_{\mathbb{R}^k}\int_s^T G(r,x,u^n(r,x),\sigma^*\nabla u^n(r,x))\varphi(r,x)d\overleftarrow{B}_r dx. \qquad (3.20)
\end{aligned}
$$

Since $\varphi \in \mathcal{C}_c^{1,\infty}([0,T]\times\mathbb{R}^d)$, $b \in \mathcal{C}_b^2\left(\mathbb{R}^k,\mathbb{R}^k\right)$ and $\sigma \in \mathcal{C}_b^3\left(\mathbb{R}^k,\mathbb{R}^{k\times r}\right)$, then clearly the left hand side of the previous equality tends to the following quantity, as n tends to ∞,

$$
\begin{aligned}
&\int_{\mathbb{R}^k}\int_s^T u(r,x)\frac{\partial\varphi(r,x)}{\partial r}(r,x)drdx + \int_{\mathbb{R}} u(r,x)\varphi(r,x)dx - \int_{\mathbb{R}^k} H(x)\varphi(T,x)dx \\
&\qquad - \frac{1}{2}\int_{\mathbb{R}^k}\int_s^T \sigma^* u(r,x)\sigma^*\varphi(r,x)drdx - \int_{\mathbb{R}^k}\int_s^T u\ div\big[(b-A)\varphi\big](r,x)drdx.
\end{aligned}
$$

We shall compute the limit of the the right hand side. Arguing as in Proposition 3.1, we show that

$$
\begin{aligned}
& E\int_0^T \left|F^n\left(s, X_s^{t,x},\ u^n\left(s, X_s^{t,x}\right), \sigma^*\nabla u^n(s, X_s^{t,x})\right)\right|^{\bar{\alpha}} ds \\
& \leq C\left[1+E\int_0^T \bar{\eta}_s^{\bar{\alpha}} ds + E\int_0^T \sup_{0\leq s\leq T}\left|Y_s^{n,t,x}\right|^2 + E\left(\int_0^T \left|Z_s^{n,t,x}\right|^2 ds\right)\right] \\
& \leq C'.
\end{aligned}
$$

where C' is some constant which depends from $\bar{\eta}$, M_1, α, $\eta^{'}$ and T.

Since $(u^n(s, X_s^{t,x}),\ \sigma^*\nabla u^n(s, X_s^{t,x})) = (Y_s^{n,t,x},\ Z_s^{n,t,x})$ and $(u(s, X_s^{t,x}),\ \sigma^*\nabla u(s, X_s^{t,x})) = (Y_s^{t,x},\ Z_s^{t,x})$, then using equality (3.16) and arguing as in the proof of Theorem 3.2.1, we get

$$
\lim_n E\int_0^T \left|F^n\left(s, X_s^{t,x},\ u^n\left(s, X_s^{t,x}\right), \sigma^*\nabla u^n(s, X_s^{t,x})\right) - F\left(s, X_s^{t,x},\ u\left(s, X_s^{t,x}\right), \sigma^*\nabla u(s, X_s^{t,x})\right)\right| ds = 0.
$$

Hence, according to the Lebesgue dominated convergence theorem, it follows that

$$
E\int_0^T\int_{\mathbb{R}^k} \left|F^n\left(s, X_s^{t,x},\ u^n\left(s, X_s^{t,x}\right), \sigma^*\nabla u^n(s, X_s^{t,x})\right) - F\left(s, X_s^{t,x},\ u\left(s, X_s^{t,x}\right), \sigma^*\nabla u(s, X_s^{t,x})\right)\right| ds\ e^{-\delta|x|}dx
$$

tends to 0 as n goes to infinity.

We use Lemma 3.5.1 to show that

$$
\lim_n E\int_0^T\int_{\mathbb{R}^k} \left|F^n\left(s, x, u^n(s,x), \sigma^*\nabla u^n(s,x)\right) - F\left(s, x, u(s,x), \sigma^*\nabla u(s,x)\right)\right| ds\ e^{-\delta|x|}dx = 0,
$$

which implies that the first term in the right hand side of equality (3.20) satisfies:

$$
\begin{aligned}
\lim_{n\to\infty}\int_{\mathbb{R}^k}\int_s^T & F^n(r, x, u^n(r,x), \sigma^*\nabla u^n(r,x))\varphi(r,x)drdx \\
& = \int_{\mathbb{R}^k}\int_s^T F(r, x, u(r,x), \sigma^*\nabla u(r,x))\varphi(r,x)drdx.
\end{aligned}
$$

It remains to compute the limit of the second term in the right hand side of equality (3.20).

Arguing as in the proof of Theorem 3.2.1, we show that

$$\int_0^T \big[G(s, X_s^{t,x}, u^n(s, X_s^{t,x}), \sigma^* \nabla u^n(s, X_s^{t,x})) - G(s, X_s^{t,x}, u(s, X_s^{t,x}), \sigma^* \nabla u(s, X_s^{t,x}))\big] d\overleftarrow{B}_s$$

tends to 0 *in probability.*

Clearly, the function

$$\Phi(x) := \int_s^T \big[G(r, x, u^n(r,x), \sigma^* \nabla u^n(r,x)) \ - \ G(r, x, u(r,x), \sigma^* \nabla u(r,x))\big] \varphi(r,x) e^{\delta|x|} d\overleftarrow{B}_r$$

belongs to $\mathbb{L}^1(\mathbb{R}^k, \ e^{-\delta|x|})$.

Hence, using Lemma 3.5.1 we get, for every $s \in [0, \ T]$,

$$\begin{aligned}
&\int_{\mathbb{R}^k} \mathbb{E}\Big| \int_s^T \big[G(r, x, u^n(r,x), \sigma^* \nabla u^n(r,x)) - G(r, x, u(r,x), \sigma^* \nabla u(r,x))\big] \varphi(r,x) e^{\delta|x|} d\overleftarrow{B}_r \Big| e^{-\delta|x|} dx \\
&\leq K \int_{\mathbb{R}^k} \mathbb{E}\Big| \int_s^T \big[G(r, X_r^{t,x}, u^n(r, X_r^{t,x}), \sigma^* \nabla u^n(r, X_r^{t,x})) - G(r, X_r^{t,x}, u(r, X_r^{t,x}), \sigma^* \nabla u(r, X_r^{t,x}))\big] \times \\
&\qquad\qquad \varphi(r, X_r^{t,x}) e^{\delta|X_r^{t,x}|} d\overleftarrow{B}_r \Big| e^{-\delta|x|} dx \\
&\leq K \int_{\mathbb{R}^k} \mathbb{E}\Big| \int_s^T \big[G(r, X_r^{t,x}, Y_r^{n,t,x}, Z_r^{n,t,x}) - G(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})\big] \varphi(r, X_r^{t,x}) e^{\delta|X_r^{t,x}|} d\overleftarrow{B}_r \Big| e^{-\delta|x|} dx \\
&:= I_n.
\end{aligned}$$

We shall show that I_n tends to 0 as n tends to ∞.

Since,

$\sup_n \mathbb{E}\Big| \int_s^T \big[G(r, X_r^{t,x}, Y_r^{n,t,x}), Z_r^{n,t,x}) - G(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})\big] \varphi(r, X_r^{t,x}) e^{\delta|X_r^{t,x}|} d\overleftarrow{B}_r \Big|^2 < \infty$ and $\int_s^T \big[G(r, X_r^{t,x}, Y_r^{n,t,x}), Z_r^{n,t,x}) \ - \ G(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})\big] \varphi(r, X_r^{t,x}) e^{\delta|X_r^{t,x}|} d\overleftarrow{B}_r$ tends to 0 *in probability*,

it follows that

$$\lim_n \mathbb{E}\Big| \int_s^T \big[G(r, X_r^{t,x}, Y_r^{n,t,x}), Z_r^{n,t,x}) - G(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})\big] \varphi(r, X_r^{t,x}) e^{\delta|X_r^{t,x}|} d\overleftarrow{B}_r \Big| = 0$$

Therefore, according to the Lebesgue dominated convergence theorem, we deduce that I_n tends to 0 as n tends to ∞. Theorem 3.5.1 is proved.

Chapter 4

Existence of optimal relaxed control for systems driven by backward stochastic differential equations (BSDE's)

Stochastic control problems is a mathematical description of how to act optimally to minimize a cost or maximize a gain function, the control process is a progressively measurable process on the space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P\right)$, A-valued where A is a compact metric space, the existence of optimal strict control can be proved under some convexity hypotheses, since no convexity assumptions are made, the problem reformulated in the larger or relaxed space, by replace the A-valued process u_t with $P(A)$ valued process (q_t), where $P(A)$ is the space of probability measures equipped with the the topology of weak convergence. We denote by $\mathbb{V}$ the set of probability measures on $[0,T]\times A$ whose projections on $[0,T]$ coincide with the Lebesgue measures. Stable convergence is required for bounded measurable function $\Phi(t,a)$ such that for each fixed $t\in[0,T]$ $\Phi(t,a)$ is continuous.
With this weak topology, $\mathbb{V}$ is compact and metrizable space.
In this chapter we aim to establish the existence of optimal relaxed controls for system driven by the following BSDEs

$$Y_t = \xi + \int_t^T \int_A f(s, X_s, Y_s, Z_s, a)(q_s)(da)\, ds - \int_t^T Z_s dW_s \tag{4.1}$$

where the generator $f(t,x,y,z)$ is affine with respect to z, and satisfies a sublinear growth condition the optimal control is defined on an extended probability space with the help of Young measures on the space of trajectories, Young measures were introduced in [19, 30, 56], here the solution satisfies

$$Y_t = \xi + \int_t^T \int_A f(s, X_s, Y_s, Z_s, a)(q_s)(da)\, ds - \int_t^T Z_s dW_s - (L_T - L_t)$$

where L is a martingale orthogonal to W.
The expected cost is of the form

$$J(q) = E\left(\int_0^T \int_A h(s, X_s, Y_s, Z_s, a)\, q_t(da)\, dt + l(Y_0) \right).$$

The chapter is organized as follows. In Section 1, we introduce the problem and some assumptions. Section 2 is devoted to the study of the approximating controls, then we prove the tightness results of the approximating sequence, then passing the limit using the prohorov cretirion for Young measures. Section 3 we give the proof of the main theorem.

4.1 The setting and its assumtions

For a given finite time horizon $[0,T]$, let $W = (W_t)_{t\in[0,T]}$ be a standard brownian motion with values in $\mathbb{R}^m$ defined on some complete probability space $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,T]}, P\right)$, $(\mathcal{F}_t)_{t\in[0,T]}$ is the augmented natural filtration of a Brownian motion W which satisfies the usual conditions, and $\mathcal{F} = \mathcal{F}_T$.In this chapter, we prove the existence of optimal relaxed controls to the following BSDE

$$Y_t = \xi + \int_t^T \int_A f(s, X_s, Y_s, Z_s, a)(q_s)(da)\, ds - \int_t^T Z_s dW_s$$

Here the process X_t is $(\mathcal{F}_t)$-adapted and continious with values in a separable metric space $\mathbb{M}$, Y and Z are square integrable adapted processes defined on $\mathbb{R}^d$ and $\mathbb{L}$ respectively, where $\mathbb{L}$ is the space of linear mappings from $\mathbb{R}^m$ to $\mathbb{R}^d$, we require the process (q_t) to be $\mathbb{P}(A)$-valued.

Definition 4.1.1 *A relxed control $\mathcal{C}$ is a term $(\Omega, \mathcal{F}, \mathcal{F}_t, P, q_t, W_t, X_t, Y_t, Z_t)$ such that:*
(1) $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ *is a probability space equipped with a filtration satisfing the usual coditions.*

(2) *q is a $\mathbb{V}$-valued random variable, adapted (i.e. for each t, $1_{]0,t]}q$ is $\mathcal{F}_t$-measurable.*

(3) *W_t is a $(\mathcal{F}_t, P)$-Brownian motion and (W_t, X_t, Y_t, Z_t) satisfies BSDE(1).*

Let us assume the following conditions:

(A_1) the measurable mapping

$$f : [0,T] \times \mathbb{M} \times \mathbb{R}^d \times \mathbb{L} \times \mathbb{A} \to \mathbb{R}^d$$

is continuous with respect to (x, y, u) and affine with respect to z ie: $f(t, x, y, z, u)$ has the form

$$f(s, x, y, z, u) = f_1(s, x, y, u) + f_2(s, x, y) z$$

where f_1 and f_2 are bounded and continuous in (x, y) and are lipschitz in y.

(A_2) the measurable mappings

$$h : [0,T] \times \mathbb{M} \times \mathbb{R}^d \times \mathbb{L} \times \mathbb{A} \to \mathbb{R}$$
$$l : \mathbb{R}^d \to \mathbb{R}$$

are continuous in (x, y, u) uniformly in (t, a) and affine with respect to z i.e: $h(t, x, y, z, u)$ has the form

$$h(s, x, y, z, u) = h_1(s, x, y, u) + h_2(s, x, y) z$$

where $h_1$and h_2 are bounded and continuous in (x, y) and are lipschitz in y uniformly in (t, a).

In the sequel

Let $\mathbb{D}_{\mathbb{R}^d}([0,T])$ be the Skorokhod space of càdlag functions from $[0,T]$ into $\mathbb{R}^d$.

Let $\mathbb{H} = L^2_{\mathbb{L}}([0,T])$ and let $\mathbb{H}_\sigma$ be the space $\mathbb{H}$ endowed with its weak topology.

4.2 Construction of a weak solution

Theoreme 4.2.1 *Assume that conditions (A_1) and (A_2) are satisfies. Then the BSDE (1) has an optimal relaxed control.*

Let $(q^n)_{n\geq 0}$ be a minimizing sequence for the cost function $J(q)$, that is:

$$\lim_{n\to\infty} J(q^n) = \inf\{J(q); q \in \mathcal{R}\}.$$

Let (X, Y^n, Z^n) be the unique solution of BSDE

$$Y_t^n = \xi + \int_t^T \int_A f(s, X_s, Y_s^n, Z_s^n, a)(q_s^n)(da)\, ds - \int_t^T Z_s^n dW_s \tag{4.2}$$

4.2.1 Tightness Results

The following lammas of the tightness will be useful to our future discussion.

Lemma 4.2.1 *Let (X, Y^n, Z^n) be the unique solution of BSDE 4.2. There exists a positive constant C such that*

$$\sup_n E\left(\sup_{0\leq t\leq T} |Y_t^n|^2 + \int_t^T |Z_s^n|^2\, ds\right) \leq C \tag{4.3}$$

Proof. Applying Ito's formula to the semi-martingale $|Y_t^n|^2$, we get

$$\begin{aligned}
|Y_t^n|^2 + \int_t^T |Z_s^n|^2\, ds &= |\xi|^2 + 2\int_t^T \int_A Y_s^n f(s, X_s, Y_s^n, Z_s^n, a)\, q_s^n(da)\, ds \\
&\quad - 2\int_t^T Y_s^n Z_s^n dW_s \\
&\leq |\xi|^2 + 2\int_t^T \int_A |Y_t^n| . |f(s, X_s, Y_s^n, Z_s^n, a)|\, q_s^n(da)\, ds \\
&\quad - 2\int_t^T Y_s^n Z_s^n dW_s
\end{aligned} \tag{4.4}$$

we have from (4.4)

$$\begin{aligned}
E\Big[\int_t^T |Z_s^n|^2\, ds \leq E\big[\,|\xi|^2 \\
&+ 2E\Big[\int_t^T \int_A |Y_t^n| . |f(s, X_s, Y_s^n, Z_s^n, a)|\, q_s^n(da)\, ds
\end{aligned}$$

observe that (A_2) also implies

$$2E\Big[\int_t^T \int_A |Y_t^n| . |f(s, X_s, Y_s^n, Z_s^n, a)|\, q_s^n(da)\, ds \leq 2K\; E\Big[\int_t^T |Y_t^n|\,(1 + |Z_s^n|)\, ds$$

Using that, $\forall a \geq 0, b \geq 0$ and $\varepsilon \neq 0$, $\quad 2ab \leq a^2\varepsilon^2 + b^2/\varepsilon^2$, we get

$$2\ K\ E\big[\int_t^T |Y_t^n|\,(1+|Z_s^n|)\,ds \leq \varepsilon^2 K E\big[\int_t^T |Y_t^n|^2\,ds + (T-t)\,K/\varepsilon^2 \\ + K/\varepsilon^2 E\big[\int_t^T |Z_s^n|^2\,ds \tag{4.5}$$

taking $\varepsilon^2 > 2K$ implies

$$\left(1 - K\ /\varepsilon^2\right) E\big[\int_t^T |Z_s^n|^2\,ds \leq E\big[\,|\xi|^2 + K\left(T/\varepsilon^2 + \varepsilon^2\int_t^T |Y_s^n|^2\,ds\right)$$

$$E\big[\int_t^T |Z_s^n|^2\,ds \leq C_1 + C_2 E\big[\int_t^T |Y_s^n|^2\,ds$$

where C_1 and C_2 are positive constants, for $t = 0$ we get

$$E\big[\int_0^T |Z_s^n|^2\,ds \leq C_1 + C_2 E\big[\int_0^T |Y_t^n|^2\,ds \tag{4.6}$$

The expression (4.4), (4.5) and the application of Burkholder-Davis-Gundy provides

$$\begin{aligned} E\big[\left(\sup_{0\leq t\leq T}|Y_t^n|^2\right) &\leq E\big[\,|\xi|^2 + \varepsilon^2 K\ E\big[\int_t^T |Y_s^n|^2\,ds + T\ K\ /\varepsilon^2 \\ &+ K\ /\varepsilon^2 E\big[\int_t^T |Z_s^n|^2\,ds + CE\big[\left(\int_0^T |Y_s^n|^2 . |Z_s^n|^2\,ds\right)^{\frac{1}{2}} \\ &\leq E\,|\xi|^2 + \varepsilon^2 K\ E\big[\int_0^T \sup_s |Y_s^n|^2\,ds + TK/\varepsilon^2 \\ &+ \frac{1}{2}E\big[\left(\sup_{0\leq t\leq T}|Y_t^n|^2\right) \\ &+ \left(\frac{C^2}{2} + K\ /\varepsilon^2\right) E\big[\int_t^T |Z_s^n|^2\,ds \end{aligned}$$

were we have used the inequality $ab \leq a^2/2 + b^2/2$,we obtain

$$\frac{1}{2}E\big[\left(\sup_{0\leq t\leq T}|Y_t^n|^2\right) \leq E\big[\,|\xi|^2 + \varepsilon^2 K\ E\big[\int_0^T \sup_t |Y_t^n|^2\,ds + TK/\varepsilon^2 \\ + \left(\frac{C^2}{2} + K\ /\varepsilon^2\right) E\big[\int_t^T |Z_s^n|^2\,ds$$

the proof follows by Gronwall's lamma and the expression (4.6)

$$E\Big[\left(\sup_t |Y_t^n|^2\right) \leq C'.$$

■

Lemma 4.2.2 *Let (X, Y^n, Z^n) be the unique solution of BSDE 4.2The sequence $\left(Y^n, \int_0^{\cdot} Z^n dW_s\right)$ is tight on the space $\mathbb{D}_{\mathbb{R}^d}([0,T]) \times \mathbb{D}_{\mathbb{R}^d}([0,T])$ endowed with the S-topology.*

Proof. We define the conditional variation by

$$V_t(Y^n) = \sup E\Big[\left(\sum_i \left| E\left(Y_{t_{i+1}}^n - Y_{t_i}^n / \mathcal{F}_{t_i}\right)\right|\right)$$

where the supremum is over all partitions of the interval $[0,T]$. Clearly

$$\begin{aligned}
V_t(Y^n) &\leq E\Big[\left[\int_0^T \int_A |f(s, X_s, Y_s^n, Z_s^n, a)| \, q_s^n(da)\, ds\right] \\
&\leq \sup_n E\Big[\left[\int_0^T \int_A |f(s, X_s, Y_s^n, Z_s^n, a)| \, q_s^n(da)\, ds\right] \\
&\leq \sup_n E\Big[\left(\int_0^T K(1 + |Z_s^n|)\, ds\right) \\
&\leq \sup_n K\left[T - T^2 E\Big[\left(\int_0^T |Z_s^n|^2\, ds\right)^{1/2}\right] \\
&< +\infty
\end{aligned}$$

and it follows from (4.3) that

$$\sup_n \left(V_t(Y^n) + \sup_{0\leq t\leq T} E\,|Y_t^n|^2 + \sup_{0\leq t\leq T} E\left|\int_0^t Z^n dW_s\right|\right) < \infty \tag{4.7}$$

hence the sequence $\left(Y^n, \int_0^{\cdot} Z^n dW_s\right)$ satisfy Meyer-Zheng tightness criterion for quasi-martingales [45].

Lemma 4.2.3 *Let (X, Y^n, Z^n) be the unique solution of BSDE 4.2. The sequence (Z^n) is tight in $\mathbb{H}_\sigma$.*

■

Proof. The closed balls are compact in $\mathbb{H}_\sigma$, and we have by lemma 4.2.1 that,

$$\sup_n P\left[\|Z^n\|_{\mathbb{H}} \geq R\right] \leq \sup_n \frac{1}{R^2} E\int_0^T \|Z_s^n\|^2_{L^2_{\mathbb{L}}([0,T])} ds$$
$$\to 0 \text{ when } R \to \infty \ .$$

■

Lemma 4.2.4 *The family of relaxed controls (q^n) is tight in $\mathbb{V}$.*

Proof. $[0,T]\times A$ being compact, then by Prokhorov's theorem, the space V of probability measures on $[0,T]\times A$ is then compact for the topology of weak convergence. The fact that $q^n, n \geq 0$ is a random variable with values in the compact set V yields that the family of distributions associated to $(q^n)_{n\geq 0}$ is tight. ■

Now let us construct the extended probability space:

Folloowing the terminology of [22]. Let $\mathcal{D}$ be the Borel $\sigma-$algebra of $\mathbb{D}$ and, for each $t \in [0,T]$, let $\mathcal{D}_t$ be the sub-σ-algebra of $\mathcal{D}$ generated by $\mathbb{D}_{\mathbb{R}^d}[0,T]$, let $\mathcal{H}$ denote the Borel $\sigma-$algebra of $\mathbb{H}_\sigma$ and, for each $t \in [0,T]$, let $\mathcal{H}_t$ be the sub-σ-algebra of $\mathcal{H}$ generated by $\mathbb{H}_\sigma$,let $\mathcal{V}$ be the Borel $\sigma-$algebra of $\mathbb{V}$ and, let $\mathcal{V}_t$ be the sub-σ-algebra of $\mathcal{V}$ generated by $\mathbb{V}$.

We define a stochastique basis $\left(\Omega, \hat{\mathcal{F}}, \left(\hat{\mathcal{F}}\right)_t, \mu\right)$ by

$$\hat{\Omega} = \Omega \times \mathbb{D} \times \mathbb{D} \times \mathbb{H} \times \mathbb{V},$$
$$\hat{\mathcal{F}} = \mathcal{F} \otimes \mathcal{D} \otimes \mathcal{D} \otimes \mathcal{H} \otimes \mathcal{V},$$
$$\hat{\mathcal{F}}_t = \mathcal{F}_t \otimes \mathcal{D}_t \otimes \mathcal{D}_t \otimes \mathcal{H}_t \otimes \mathcal{V}_t,$$

and a probability measure μ on $\left(\hat{\Omega}, \hat{\mathcal{F}}\right)$ is Young measure defined by:

$$\forall A \in \mathcal{F}, \mu\left(A \times \mathbb{D} \times \mathbb{D} \times \mathbb{H} \times \mathbb{V}\right) = P\left(A\right)$$

The space of Young measures with basis P is denoted by $\mathcal{Y}\left(\Omega, \mathcal{F}, P; \mathbb{D}_s \times \mathbb{D}_s \times \mathbb{H}_\sigma \times \mathbb{V}\right)$, thanks to previous lammas, we have that the sequence (Y^n, M^n, Z^n, q^n) is tight in $\mathbb{D}_s \times \mathbb{D}_s \times \mathbb{H}_\sigma \times \mathbb{V}$, then by Prohorov's compactness criterion for Young measures [[30] Theorem

4.3.5], we can extract a subsequence of (Y^n, M^n, Z^n, q^n), we denote this extracted sequence by (Y^n, M^n, Z^n, q^n) which converges stably to a Young measure $\mu \in \mathcal{Y}$, that is, for every measurable bounded mapping $\Theta : \Omega \times \mathbb{D}_s \times \mathbb{D}_s \times \mathbb{H}_\sigma \times \mathbb{V} \to \mathbb{R}$ such that $\Theta(\omega, ., ., ., .)$ is continuous for all ω, we have

$$\begin{aligned} &\lim_{n\to\infty} \int_\Omega \Theta(\omega, Y^n(\omega), M^n(\omega), Z^n(\omega), q^n(\omega))\, dP(\omega) \\ &= \int_\Omega \int_{\mathbb{D}_s\times\mathbb{D}_s\times\mathbb{H}_\sigma\times\mathbb{V}} \Theta(\omega, y, m, z, q)\, d\mu_\omega(y, m, z, q)\, dP(\omega) \end{aligned} \tag{4.8}$$

We define the process $\left(\hat{Y}, \hat{M}, \hat{Z}, \hat{q}\right)$ on $\hat{\Omega}$ by

$$\begin{aligned} \hat{Y}(\omega, y, m, z, q) &= y, \quad & \hat{M}(\omega, y, m, z, q) &= m \\ \hat{Z}(\omega, y, m, z, q) &= z, \quad & \hat{q}(\omega, y, m, z, q) &= q. \end{aligned}$$

we have $\left(\hat{Y}, \hat{M}, \hat{Z}, \hat{q}\right)$ is $\left(\hat{\mathcal{F}}_t\right)$-adapted. The random variables (Y^n, M^n, Z^n, q^n) can be seen as random elements defined on $\hat{\Omega}$, using the notations, for $n \geq 1$:

$$\begin{aligned} Y^n(\omega, y, m, z, q) &:= Y^n(\omega), \\ M^n(\omega, y, m, z, q) &:= M^n(\omega), \\ Z^n(\omega, y, m, z, q) &:= Z^n(\omega), \\ q^n(\omega, y, v, z, q) &:= q^n(\omega), \end{aligned}$$

Furthermore, (Y^n, M^n, Z^n, q^n) is $\left(\hat{\mathcal{F}}_t\right)$-adapted for each n.
We denote $M_t^n = \displaystyle\int_0^t Z_s^n dW_s$.

Lemma 4.2.5 *The process $\hat{M}$ is a martingale with respect to $\left(\hat{\Omega}, \hat{\mathcal{F}}, \left(\hat{\mathcal{F}}_t\right)_t, \mu\right)$.*

Proof. in order to prove that $E\left(\hat{M}_{t+s}/\hat{\mathcal{F}}_t\right) = \hat{M}_s$, we only need to show that, for each bounded $\hat{\mathcal{F}}_t$-measurable $\Phi : \hat{\Omega} \to \mathbb{R}$ such that $\Phi(\omega, ., ., ., .)$ is continuous for all $\omega \in \Omega$, we have

$$E\left(\Phi \times \hat{M}_{t+s}\right) = E\left(\Phi \times \hat{M}_t\right)$$

the mapping $g_r : \mathbb{D} \to \mathbb{R}^d$ defined by $g_r(m) = m(r)$ is not continuous for the topology S, but $g_{r,\delta}(m) = \frac{1}{\delta}\int_r^{r+\delta} m(r)\,ds$ is S-continuous and $\lim\limits_{\delta\to 0} g_{r,\delta}(m) = g_r(m)$, and we define

$$\Theta(\omega, y, m, z, a) = \Phi(\omega, y, m, z, a)\left(g_{t+s,\delta}(m) - g_{t,\delta}(m)\right).$$

by lamma 4.2.1, the sequence $(\Theta(\omega, Y^n, M^n, Z^n, a))$ is bounded in $L^2_{\mathbb{R}^d}(\Omega)$, then it is uniformly integrable. By lamma 3.12 in [22] to the integrand Θ we get

$$\begin{aligned}
&E\left(\Phi \times \left(\frac{1}{\delta}\int_{t+s}^{t+s+\delta} \hat{M}_u du - \frac{1}{\delta}\int_t^{t+\delta} \hat{M}_u du\right)\right)\\
&= \int_\Omega\int_{\mathbb{D}\times\mathbb{D}\times\mathbb{H}\times\mathbb{V}} \Phi(\omega, y, m, z, q)\left(g_{t+s,\delta} - g_{t,\delta}\right)(m)\, d\mu_\omega(y, m, z, q)\, dP(\omega)\\
&= \lim_{n\to\infty}\int_\Omega \Phi(\omega, Y^n(\omega), M^n(\omega), Z^n(\omega), q^n(\omega))\, \tfrac{1}{\delta}\int_t^{t+\delta} \left(M^n_{u+s}(\omega) - M^n_u(\omega)\right) du dP(\omega)\\
&= \lim_{n\to\infty}\int_\Omega \Phi(\omega, Y^n(\omega), M^n(\omega), Z^n(\omega), a)\, E^{\mathcal{F}_t}\left(\tfrac{1}{\delta}\int_t^{t+\delta} \left(M^n_{u+s} - M^n_u\right) du\right) dP\\
&= 0.
\end{aligned}$$

then

$$\begin{aligned}
&E\left(\Phi \times \left(\hat{M}_{t+s} - \hat{M}_t\right)\right)\\
&= \lim_{\delta\to 0} E\left(\Phi \times \left(\frac{1}{\delta}\int_{t+s}^{t+s+\delta} \hat{M}_u du - \frac{1}{\delta}\int_t^{t+\delta} \hat{M}_u du\right)\right) = 0
\end{aligned}$$

by boundness of $\left(\hat{M}_s\right)_{0\le s\le T}$ in $L^2_{\mathbb{R}^d}\left(\hat{\Omega}, \hat{\mathcal{F}}, \mu\right)$.

Lemma 4.2.6 *The process W is an $\left(\hat{\mathcal{F}}_t\right)$-standard Brownian motion under the probability μ.*

■

Proof. We set $W(\omega, y, m, z, q) = W(\omega)$. By Balder's result on K-convergence [17, 18], which is valid for Hausdorff spaces whith metrizable compact subsets [[30], Lemma 4.5.4], each subsequence of $\left(Y^{(n)}, M^{(n)}, Z^{(n)}, q^{(n)}\right)$ contains a further subsequence $\left(Y^{(n_k)}, M^{(n_k)}, Z^{(n_k)}, q^{(n_k)}\right)$ which K-converges to μ, that is, for each subsequence $\left(Y^{\left(n'_k\right)}, M^{\left(n'_k\right)}, Z^{\left(n'_k\right)}, q^{\left(n'_k\right)}\right)$ of $\left(Y^{(n_k)}, M^{(n_k)}, Z^{(n_k)}, q^{(n_k)}\right)$, we have

$$\lim_n \frac{1}{n}\sum_{k=1}^n \delta_{\left(Y^{\left(n'_k\right)}(\omega), M^{\left(n'_k\right)}(\omega), Z^{\left(n'_k\right)}(\omega), q^{\left(n'_k\right)}(\omega)\right)} = \mu_\omega \quad \text{a.e.}$$

where $\delta_{(y,m,z,q)}$ denotes the Dirac measure on (y,m,z,q), we thus have for every $B \in \mathcal{D}_t \otimes \mathcal{D}_t \otimes \mathcal{H}_t \otimes \mathcal{V}_t$ the mapping $\omega \longmapsto \mu_\omega(B)$ is $\mathcal{F}_t$-measurable. Now we check that W has independent increments under μ. Let $t \in [0,T]$, and let $s > 0$ such that $t+s \in [0,T]$ .Let us prove that, for any $A \in \hat{\mathcal{F}}_t$ and any Borel subset C of $\mathbb{R}^m$, we have

$$\mu\left(A \cap \{W_{t+s} - W_t \in C\}\right) = \mu(A)\, \mu\{W_{t+s} - W_t \in C\}.$$

Let $B = \{\omega \in \Omega; W_{t+s}(\omega) - W_t(\omega) \in C\}$. We have

$$\begin{aligned}
&\mu\left(A \cap (B \times \mathbb{D} \times \mathbb{D} \times \mathbb{H} \times \mathbb{V})\right) \\
&= \int_{\Omega \times \mathbb{D} \times \mathbb{D} \times \mathbb{H} \times \mathbb{V}} 1_A(\omega, y, m, z, q)\, 1_B(\omega)\, d\mu(\omega, y, m, z, q) \\
&= \int_\Omega \mu_\omega\left(1_A(\omega, ., ., ., .)\right) 1_B(\omega)\, d\mu(\omega) \\
&= \int_\Omega \mu_\omega\left(1_A(\omega, ., ., ., .)\right) dP(\omega)\, P(B) \\
&= \mu(A)\, \mu\left(B \times \mathbb{D} \times \mathbb{D} \times \mathbb{H} \times \mathbb{V}\right),
\end{aligned}$$

which proves. Thus $W_{t+s} - W_t$ is independent of $\left(\hat{\mathcal{F}}_t\right)$. ■

Lemma 4.2.7 *Let $\mathbb{L}$ be the space of linear mapping from $\mathbb{R}^d$ to $\mathbb{R}^l$ for some $l \geq 1$.Let $k : [0,T] \to \mathbb{L}$ be a continuous function. For each $t \in [0,T]$, the mapping*

$$\Psi : \begin{cases} \mathbb{D}_s \times \mathbb{D}_s \times \mathbb{H}_\sigma \times \mathbb{V} \to \mathbb{R}^l \\ (y,z,a) \mapsto \displaystyle\int_0^t \int_A k(s)\,.f\left(s, x(s), y(s), z(s), a\right) q_s(da)\, ds \end{cases}$$

is sequentially continuous.

Proof. We have y_n converges to y in $\mathbb{D}_s$ in particular $y_n(s)$ converges to $y_n(s)$ for a.e.$s \in [0,T]$

z_n converges z in $\mathbb{H}_\sigma$ and $\sup_n \|z_n\|_{\mathbb{H}} < +\infty$ and q_n converges stabky to q in $\mathbb{V}$,

$$\begin{aligned}
&|\Psi(y_n, z_n, q_n) - \Psi(y, z, q)| \\
&= \left| \int_0^t \int_A k(s)\ f(s, x(s), y_n(s), z_n(s), a)\ q_s^n(da)\, ds \right. \\
&\left. - \int_0^t \int_A k(s)\ f(s, x(s), y(s), z(s), a)\ q_s(da)\, ds \right| \\
&\leq \left| \int_0^t \int_A k(s)\ \left(f_1(s, x(s), y_n(s), a)\ q_s^n(da)\, ds + f_2(s, x(s), y_n(s))\, z_n(s)\right) q_s^n(da)\, ds \right. \\
&\left. - \int_0^t \int_A k(s)\ \left(f_1(s, x(s), y(s), a)\ q_s(da)\, ds - f_2(s, x(s), y(s))\, z(s)\right) q_s(da)\, ds \right| \\
&\leq I_1(n) + I_2(n) + I_3(n).
\end{aligned}$$

where

$$\begin{aligned}
I_1(n) &= \left| \int_0^t \int_A k(s)\ f_1(s, x(s), y_n(s), a)\ q_s^n(da)\, ds - \int_0^t \int_A k(s)\ f_1(s, x(s), y(s), a)\ q_s^n(da)\, ds \right| \\
I_2(n) &= \left| \int_0^t \int_A k(s)\ f_1(s, x(s), y(s), a)\ q_s^n(da)\, ds - \int_0^t \int_A k(s)\ f_1(s, x(s), y(s), a)\ q_s(da)\, ds \right| \\
I_3(n) &= \left| \int_0^t k(s)\ f_2(s, x(s), y_n(s))\, z_n(s)\, ds - \int_0^t k(s)\ f_2(s, x(s), y(s))\, z(s)\, ds \right|.
\end{aligned}$$

The Cauchy-Schartz inequality gives

$$I_1(n) \leq \left(\int_0^t |k(s)|^2\, ds \right) \left(\int_0^t \int_A |\ f_1(s, x(s), y_n(s), a) - f_1(s, x(s), y(s), a)|^2\, q_s^n(da)\, ds \right)^{1/2}.$$

f_1 is Lipschitz in y

$$I_1(n) \leq \left(\int_0^t |k(s)|^2\, ds \right)^{1/2} . \left(\int_0^t |y_n(s) - y(s)|\, ds \right)^{1/2}.$$

there exists a subsequence $y_{n_k}(s)$ of $y_n(s)$, still denoted $y_n(s)$, which converges to $y(s)$ $\mu.p.s$ and (4.7) allows us to show that $I_1(n)$ converges to 0 by the dominated convergence theorem.

Now we shall prove that $I_2(n)$ converges to 0,

The function $(s, a) \longmapsto k(s)\, f_1(s, x(s), y(s), a)$ is bounded measurable in (s, a) and continuousin a, then from the stable convergence of q_t^n to q_t, we get $I_2(n)$ tends to 0.

It remaints to prove that $I_3(n)$ tends to 0 as n tends to ∞.

$$\begin{aligned}
I_3(n) &= \left| \int_0^t k(s).\left(f_2(s,x(s),y_n(s)) - f_2(s,x(s),y(s))\right) z_n(s)\, ds \right.\\
&\quad + \left. \int_0^t k(s).f_2(s,x(s),y(s))\left(z_n(s) - z(s)\right) ds \right| \\
&\leq \sup_n \|z_n\|_{\mathbb{H}} \left(\int_0^t |k(s)| \quad |f_2(s,x(s),y_n(s)) - f_2(s,x(s),y(s))|^2\, ds \right)^{1/2} \\
&\quad + \left| \int_0^t k(s).f_2(s,x(s),y(s))\left(z_n(s) - z(s)\right) ds \right|
\end{aligned}$$

which converges to 0.

Lemma 4.2.8 *The sequence* $\left(\int_t^T \int_A f(s,X_s,Y_s^n,Z_s^n,a)\, q_s^n(da)\, ds \right)$ *converges in law to* $\left(\int_t^T \int_A f\left(s,\hat{X}_s,\hat{Y}_s,\hat{Z}_s,a\right) \hat{q}_s(da)\, ds \right)$.

Proof. we consider the bounded continuous funtion $\Psi : \mathbb{R}^d \to \mathbb{R}$ and we define Θ by:

$$\Theta : \begin{cases} \mathbb{D}_s \times \mathbb{D}_s \times \mathbb{H}_\sigma \times \mathbb{V} & \to \qquad \mathbb{R} \\ (x,y,z,a) & \to \quad \Psi\left(\int_.^T \int_A f(s,x_s,y_s,z_s,a)\, q_s(da)\, ds \right) \end{cases}$$

the function Θ is sequentially continuous, then from the $\mathcal{F}$-stable convergence of (X,Y^n,Z^n,q^n),

$$\begin{aligned}
\lim_n E\left[\Theta(X,Y^n,Z^n,q^n)\right] &= \lim_n E\Psi \int_.^T \int_A f(s,X_s,Y_s^n,Z_s^n,a)\, q_s^n(da)\, ds \\
&= E\Psi \int_.^T \int_A f\left(s,X_s,\hat{Y}_s,\hat{Z}_s,a\right) \hat{q}_s(da)\, ds \\
&= \mu(\Theta).
\end{aligned}$$

■

Now we are ready to give the proof of the main result

4.3 Proof of the main result: theorem (4.2.1)

Proof. Let $\alpha = \inf \{J(q); q \in \mathcal{R}\}$, where

$$J(q) = E\left(\int_0^T \int_A h(s, X_s, Y_s, Z_s, a)\, q_s(da)\, ds + l(Y_0)\right)$$

Let $(q^n)_{n\geq 0}$ be a minimising sequence for the cost fuction $J(q)$, that is, $\lim\limits_{n\to+\infty} J(q^n) = \alpha$, where (X, Y^n, Z^n) is the unique solution of BSDE (4.2), from lamma 4.2.5 $\hat{M}$ is a martingale with respect to $\left(\hat{\Omega}, \hat{\mathcal{F}}, \left(\hat{\mathcal{F}}_t\right)_t, \mu\right)$. Therefore by the martingale decomposition theorem, there exist a process $\hat{Z} \in L^2(t, T, \mathbb{L})$ such that

$$\hat{M}_t = \int_0^t \hat{Z}_s dW_s + \hat{L}_t$$

where $\hat{L}$ is martingale orthogonal to W.

Using previous lemmas of tightness and passing to the limit, we can show that the BSDE (4.2) converge in law in $\mathbb{D}_s$ to

$$\hat{Y}_t = \xi + \int_t^T \int_A f\left(s, X_s, \hat{Y}_s, \hat{Z}_s, a\right) \hat{q}_s(da)\, ds - \int_t^T \hat{Z}_s dW_s + \hat{L}_t - \hat{L}_T$$

Now we prove that $\hat{q}$ is an optimal control.

we consider the bounded continuous funtion $\Psi : \mathbb{R}^d \to \mathbb{R}$ and we define Θ by:

$$\Theta : \begin{cases} \mathbb{D}_s \times \mathbb{D}_s \times \mathbb{H}_\sigma \times \mathbb{V} & \to \qquad \mathbb{R} \\ (x, y, z, a) & \to \quad \Psi\left(\int_.^T \int_A h(s, X_s, Y_s, Z_s, a)\, q_s(da)\, ds\right) \end{cases}$$

According to assumption (A_2) and from the $\mathcal{F}$-stable convergence of (Y^n, Z^n, q^n) to

$\left(\hat{Y},\hat{Z},\hat{q}\right)$ we have

$$
\begin{aligned}
\inf_{q\in\mathcal{R}} J(q) &= \lim_{n\to\infty} J(q^n), \\
&= \lim_{n\to\infty} E\left[\Psi\int_0^T\int_A h(s,X_s,Y_s^n,Z_s^n,a)\,q_s^n(da)\,ds + l(Y_0^n)\right] \\
&= \lim_{n\to\infty}\int_\Omega\left[\Psi\int_0^T\int_A h(\omega,s,X_s(\omega),Y_s^n(\omega),Z_s^n(\omega),a)\,q_s^n(\omega)(da)\,ds\right. \\
&\left.+l(Y_0^n(\omega))\right]dP(\omega) \\
&= \int_\Omega\int_{\mathbb{D}\times\mathbb{D}\times\mathbb{H}\times\mathbb{V}}\left[\Psi\int_0^T\int_A h(\omega,s,x(s),y(s),z(s),a)\,q_s(da)\,ds\right. \\
&\left.+l(y_0)\right]d\mu(\omega,x,y,z,q) \\
&= E\left[\Psi\int_0^T\int_A h\left(s,X_s,\hat{Y}_s,\hat{Z}_s,a\right)\hat{q}_s(da)\,ds + l\left(\hat{Y}_0\right)\right].
\end{aligned}
$$

which implies that $\hat{q}$ is a relaxed optimal control. The proof is now complete. ■

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