PEOPLE'S DEMOCRATIC REPUBLIC OF ALGERIA
MINISTRY OF HIGHER EDUCATION AND SCIENTIFIC RESEARCH

University Mohamed Khider of Biskra

Faculty of Exact Sciences and Natural and Life Sciences

Third cycle doctoral thesis in Mathematics

Option: Applied mathematics

Titre

On optimal stochastic control problem of McKean-Vlasov type with some applications via the derivative with respect the law of probability

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College year 2020/2021

Acknowledgements

I would like to express my deepest gratitude to my advisor Mokhtar Hafayed, Professor, University of Biskra, not only because this work would have been not possible without his help, but above all because in this years he taught me with passion and patience the art of being a mathematician. I would like to adress my sincere thanks to Professor Khalil SAADI, University of M'sila, Professor Abdelmouman Tiaiba, University of M'sila and Professor Boulakhras Gherbal, University of Biskra because they agreed to spend their times for reading and evaluating my thesis. I thank all my colleagues of the Mathematics Department in University of Biskra.

♡

❖ *Lina*

Résumé

Cette thèse de doctorat s'inscrit dans le cadre de l'analyse stochastique et optimisation stochastique, dont le thème central est basé sur l'étude d'un problème de contrôle optimal stochastique de type McKean-Vlasov. On s'intéresse par l'obtention des conditions nécessaires et suffisantes sous forme du maximum stochastique de type McKean-Vlasov et ses applications. Les systèmes considérerés dans ce travail sont gouvernés par des équations différentielles stochastiques de type McKean-Vlasov.

Cette thèse s'articule autour de trois chapitres:
Dans le premier chapitre, on décrit brièvement les différentes méthodes de résolution d'un problème de contrôle stochastique, bien-connues, qui sont la méthode de programmation dynamique et le principe du maximum de Pontryagin. On s'intéresse aussi dans ce chapitre aux différentes classes de contrôle optimal stochastique.
Dans le deuxième chapitre, on établit les conditions nécessaires d'optimalité pour un système gouverné par EDS de type McKean-Vlasov Ces resultats ont été prouvé par Andersson D, Djehiche B. Voir [7].
Dans le troisième chapitre, on a prouvé les conditions nécessaires et suffisantes d'optimalité vérifiées par un contrôle optimal stochastique singulier, pour un système différentiel gouverné par des équations différentielles stochastiques EDS de type McKean-Vlasov où la variable de contrôle est une paire $(u(\cdot), \xi(\cdot))$ de processus mesurables $\mathbb{A}_1 \times \mathbb{A}_2-$estimé, $\mathcal{F}_t-$adaptés, tels que $\xi(\cdot)$ est de variation bornée, non décroissante continue sur la gauche avec des limites à droite et $\xi(0-) = 0$.

Puisque $d\xi(t)$ peut être singulier par rapport à la mesure de Lebesgue dt, on appelle

$\xi(\cdot)$ la partie singulière du contrôle et le processus $u(\cdot)$ sa partie absolument continue. Le domaine de contrôle stochastique est supposé convexe. La méthode utilisée est basée sur la dérivation par rapport à une mesure de probabilité.

Les résultats obtenus dans ce chapitre, sont nouveaux et font l'objet d'un premier article intitulé:

L Guenane, M. Hafayed, S. Meherrem, S. Abbas:

On optimal solutions of general continuous-singular stochastic control problem of McKean-Vlasov type,

Journal: *Mathematical Methods in the Applied Sciences,* Doi.org/10.1002/mma.6392., Volume **43**, Issue **10**,, Pages 6498-6516 (2020)

Abstract

In this thesis, we study the optimal stochastic control for systems governed by McKean-Vlasov stochastic differential equation. of mean-field type. The central theme is the necessary conditions in the form of the Pontryagin's stochastic maximum of the McKean-Vlasov type for optimality with some applications. Recently, the main purpose of this thesis is to derive a set of necessary conditions of optimality, where the differential system is governed by stochastic differential equations of the McKean-Vlasov type. This thesis is structured around three chapters:

In the first chapter, we have presented the different class of stochastic control, such as singular controls, relaxed controls, feedback controls, ergodic controls,..etc. . We briefly write the different the well-known methods of solving a stochastic control problem, which are the dynamic programming method and the Pontryagin maximum principle.

In the second chapter, we establish the maximum principle for the optimal control for EDS of McKean-Vlasov type.These results have been proved by Andersson D, Djehiche B, See [7].

In the third chapter, we study singular control problem, where control variable is a pair $(u(\cdot), \xi(\cdot))$ of measurable $\mathbb{A}_1 \times \mathbb{A}_2-$valued, $\mathcal{F}_t-$adapted processes, such that $\xi(\cdot)$ is of bounded variation, non-decreasing continuous on the left with right limits and $\xi(0-) = 0$. Since $d\xi(t)$ may be singular with respect to Lebesgue measure dt, we call $\xi(\cdot)$: the singular part of the control and the process $u(\cdot)$: its absolutely continuous part. In this chaptre, we established a new set of necessary conditions of optimal singular control, where the system is governed by stochastic differential equations EDSs.

In this work, the control domain is not assumed to be convex (i.e., the control domain is a general action space). The derivatives with respect to measure is applied to establish our new result. The results obtained in Chapter 4 are all new and are the subject of a first article entitled:

L. Guenane, & M. Hafayed, & S. Meherrem, & S. Abbas: On optimal solutions of general continuous-singular stochastic control problem of McKean-Vlasov type,

Journal: *Mathematical Methods in the Applied Sciences,* Doi.org/10.1002/mma.6392., Volume **43**, Issue **10**,, Pages 6498-6516 (2020).

Contents

Symbols and Acronyms

- $(\Omega, \mathcal{F}, \mathbb{P})$: probability space.
- $\{\mathcal{F}_t\}_{t\geq 0}$: filtration.
- $\Big(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P}\Big)$: filtered probability space.
- $\mathbb{R}$: Real numbers.
- $\mathbb{R}_+$: Non-negative real numbers.
- $\mathbb{N}$: Natural numbers.
- $\mathbb{L}^p(\mathcal{F},)$: the set of $\mathbb{R}^n$-valued $\mathcal{F}-$measurable random variables X such that

$$\mathbb{E}(\,|X|^p) < \infty.$$

- $\mathbb{L}^p_{\mathcal{G}}(\Omega, \mathbb{R}^n)$: the set of $\mathbb{R}^n$-valued $\mathcal{G}-$measurable random variables X such that

$$\mathbb{E}(\,|X|^p) < \infty.$$

- $\mathbb{L}^p_{\mathcal{F}}([0,T]\,, \mathbb{R}^n)$: the set of all $(\mathcal{F}_t)_{t\geq 0}$-adapted $\mathbb{R}^n$-valued processes X such that

$$\mathbb{E}\int_0^T |X(t)|^p \, dt < \infty.$$

- $\mathbb{L}^\infty_{\mathcal{F}}([0,T]\,, \mathbb{R}^n)$: the set of all $(\mathcal{F}_t)_{t\geq 0}$-adapted $\mathbb{R}^n$-valued processes X essentially bounded processes.
- *a.e.,:* almost everywhere.
- *a.s.,:* almost surely.
- *cadlag:* right continuous with left limits.

- *caglad:* left continuous with right limits.
- **cf**.: compare (abbreviation of Latin confer).
- **e.g**.: for example (abbreviation of Latin exempli gratia).
- **i.e,**. that is (abbreviation of Latin id est).
- **HJB:** *The Hamilton-Jacobi-Bellman equation.*
- **SDE**: Stochastic differential equations.
- **BSDE:** Backward stochastic differential equation.
- **FBSDEs**: Forward-backward stochastic differential equations.
- **FBSDEJs**: Forward-Backward stochastic differential equations with jumps.
- **PDE**: Partial differential equation.
- **ODE:** Ordinary differential equation.
- $\dfrac{\partial f}{\partial x}, f_x$: The derivatives with respect to x.
- $\mathbb{P}\otimes dt$: The product measure of $\mathbb{P}$ with the Lebesgue measure dt on $[0,T]$.
- $P_{X(\cdot)}$: the law of the random variable $X(\cdot)$.

  : the probability measure induced by the random variable $X(\cdot)$.
- $\mathbb{E}(\cdot)$: Expectation.
- $\mathbb{E}(\cdot \mid G)$: Conditional expectation.
- $\sigma(A)$: $\sigma-$algebra generated by A.
- 1_A : Indicator function of the set A.
- $\mathcal{F}^X$: The filtration generated by the process X.

- $B(\cdot)$: Brownian motions.

- $\mathcal{F}_t^B$: the natural filtration generated by the brownian motion $B(\cdot)$.

- $\mathcal{F}_1 \vee \mathcal{F}_2$ denotes the σ-field generated by $\mathcal{F}_1 \cup \mathcal{F}_2$.

- $(u(\cdot), \xi(\cdot))$: continous-singular control.

- $\partial_\mu g$: the derivatives with respect to measur μ.

- $\mathcal{D}_\zeta g(\mu_0)$: the *Fréchet-derivative* of g at μ_0 in the direction ξ.

Introduction

It is well-known that control theory was founded by *N. Wiener in 1948.* After that, this theory was greatly extended to various complicated settings and widely used in sciences and technologies. Clearly, *control* means a suitable manner for people to change the dynamics of a system. The modern optimal control theory has been well developed since early 1960s, when Pontryagin et al., published their work on the maximum principle and Bellman [12] put forward the dynamic programming method. Peng [60] obtained the optimality stochastic maximum principle for the general case.

In this work, we consider the optimal stochastic control for systems governed by stochastic differential equations of McKean-Vlasov type. The coefficients of the system depend on the state of the solution process as well as of its probability law and the control variable. We study continuous-singular control problem, where control variable is a pair $(u(\cdot), \eta(\cdot))$ of measurable $\mathbb{A}_1 \times \mathbb{A}_2-$valued, $\mathcal{F}_t-$adapted processes, such that $\eta(\cdot)$ is of bounded variation, non-decreasing continuous on the left with right limits and $\eta(0-) = 0$.

Since $d\eta(t)$ may be singular with respect to Lebesgue measure dt, we call $\eta(\cdot)$: the singular part of the control and the process $u(\cdot)$: its absolutely continuous part. More precisely, we established a set of necessary conditions in the form of the Pontryagin's stochastic maximum of the McKean-Vlasov type for optimal continuous-singular control.

Recently, the main purpose of this thesis is to derive the necessary conditions for optimal continuous-singular control, where the differential system is governed by stochastic differential equations of the McKean-Vlasov type of the form:

$$\begin{cases} dX^{u,\eta}(t) = f\left(t, X^{u,\eta}(t), P_{X^{u,\eta}(t)}, u(t)\right) dt + \sigma\left(t, X^{u,\eta}(t), P_{X^{u,\eta}(t)}, u(t)\right) \mathrm{d}B(t) \\ \qquad\qquad + G(t)\mathrm{d}\eta(t), \\ X^{u,\eta}(0) = x_0, \end{cases} \tag{0.1}$$

The cost functional to be minimized over the class of admissible controls is also of McKean-Vlasov type, which has the form

$$J\left(u(\cdot), \eta(\cdot)\right) = \mathbb{E}\left[\int_0^T l(t, X^{u,\eta}(t), P_{X^{u,\eta}(t)}, u(t))\mathrm{d}t + h(X^{u,\eta}(T), P_{X^{u,\eta}(t)}) \right. \tag{0.2}$$
$$\left. + \int_{[0,T]} M(t)\mathrm{d}\eta(t)\right].$$

where $B(\cdot)$ is a Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, and $P_{X^{u,\eta}(t)}$ denotes the law of the random variable $X^{u,\eta}$.

The stochastic differential equations of McKean-Vlasov is very general, in the meaning that the dependence of the coefficient on the law of the solution $P_{X^{u,\eta}(t)}$ could be genuinely nonlinear as an element of the space of probability measures. This kind of equations was studied by Kac [53] as a stochastic model for the Vlasov-kinetic equation of plasma and the study of which was initiated by McKean [54] to provide a rigorous treatment of special nonlinear partial differential equations. We note that the Vlasov equation describes the evolution of the system of particles in the force field $F(t, x, p)$, which depends on time t, position x, and momentum p.

Recently, the main purpose of this thesis is to prove the general McKean-Vlasov necessary conditions of the optimal continuous-singular control without the convexity assumption. Finally, we extend the maximum principle of Buckdahn et al., [11] to singular control problems.

In this work, the control domain is not assumed to be convex (i.e., the control domain is a general action space). The derivatives with respect to probability measure is applied to derive our new maximum principle. This method was introduced by Lions [63]. The main idea is to identify a distribution $\mu \in Q_2(\mathbb{R}^n)$ with a random variable $X \in \mathbb{L}^2(\mathcal{F}, \mathbb{R}^n)$ so that $\mu = P_X$.The results obtained are all new and are the subject of a first article by Guenane et al:

- On optimal solutions of general continuous-singular stochastic control problem of McKean-Vlasov type",
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- Volume **43**, Issue **10**, Pages 6498-6516 (2020).
- Site: https://onlinelibrary.wiley.com/doi/10.1002/mma.6392

We note that, our work distinguishes itself from the existing ones in the following aspects.

1. We consider the more general controlled nonlinear McKean-Vlasov type stochastic system, where the coefficients of the equation depend on the state of the solution process as well as of its probability measures.
2. We apply the first and second-order derivatives with respect to probability measures to establish our Peng's type necessary optimality conditions.
3. We study the general continuous-singular control problem, where the control domain is not assumed to be convex.
4. The second-order derivative with respect to probability measures in *Wasserstein space* is applied to establish our result without convexity conditions.
5. Our McKean-Vlasov control problem occur naturally in the probabilistic analysis of financial optimization problems. Moreover, the above mathematical McKean-Vlasov approaches play an important role in different fields of economics, finance, physics, chemistry and game theory.
6. We extend the stochastic maximum principle of Buckdahn et al., [11] to continuous-singular control problems.

Chapter 1

Stochastic optimal control problems

1.1 Formulation of the control problem

It is well-known that control theory was founded by *N. Wiener in 1948*. After that, this theory was greatly extended to various complicated settings and widely used in sciences and technologies. Clearly, *control* means a suitable manner for people to change the dynamics of a system under consideration.

In this section, we present two mathematical formulations (strong and weak formulations) of stochastic optimal control problems in the following two subsections, respectively.

Let $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P}\right)$ be a given filtered probability space.

1.1.1 Stochastic process

Let $\mathbb{T}$ be a nonempty index set and $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space. A family $\{X(t) : t \in \mathbb{T}\}$ of random variables from $(\Omega, \mathcal{F}, \mathbb{P})$ to $\mathbb{R}^n$ is called a stochastic process. For any $w \in \Omega$ the map $t \mapsto X\,(t, w)$ is called a sample path.

1.1.2 Natural fitration

Let $X = (X_t, t \geq 0)$ a stochastic process defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The natural filtration of X , denoted by $\mathcal{F}_t^X$, is defined by $\mathcal{F}_t^X = \sigma\,(X_s, 0 \leq s \leq t)$. Also, we called the filtaration generated by X.

1.1.3 Brownian motion

The stochastic process $(B(t),\, t \geq 0)$ is a brownian motion (standard) if:

1. $\mathbb{P}\,[B(0) = 0] = 1$.

2. $t \to B(t,w)$ is continuous.$\mathbb{P}-p.s.$

3. $\forall s \leq t$, $B(t) - B(s)$ is normally distributed; center with variation $(t-s)$ i.e $B(t) - B(s) \sim \mathcal{N}(0, t-s)$.

4. $\forall n$, $\forall$ $0 \leq t_0 \leq t_1 \leq ... \leq t_n$, the variables $\left(B_{t_n} - B_{t_{n-1}}, ..., B_{t_1 -} B_{t_0}, B_{t_0}\right)$ are independents. The following result gives special case of the Itô formula for jump diffusions.

1.1.4 Integration by parts formula

Suppose that the processes $x_i(t)$ are given by: for $i = 1, 2$, $t \in [0,T]$:

$$\begin{cases} dx_i(t) = f\left(t, x_i(t), u(t)\right) dt + \sigma\left(t, x_i(t), u(t)\right) dB(t) \\ x_i(0) = 0. \end{cases}$$

Then we get

$$\begin{aligned} \mathbb{E}\left(x_1(T)x_2(T)\right) = \mathbb{E}\left[\int_0^T x_1(t)dx_2(t) + \int_0^T x_2(t)dx_1(t)\right] \\ + \mathbb{E}\int_0^T \sigma^\intercal\left(t, x_1(t), u(t)\right)\sigma\left(t, x_2(t), u(t)\right)dt. \end{aligned}$$

1.1.5 Strong formulation

Let $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P}\right)$ be a given filtered probability space satisfying the usual condition, on which an d-dimensional standard Brownian motion $B(\cdot)$ is defined, denote by A the separable metric space, and $T \in (0, +\infty)$ being fixed.

Consider the following controlled stochastic differential equation:

$$\begin{cases} dx(t) &= f(t, x(t), u(t))dt + \sigma(t, x(t), u(t))dB(t), \\ x(0) &= x_0 \in \mathbb{R}^n, \end{cases} \tag{1.1}$$

where

$$f : [0,T] \times \mathbb{R}^n \times U \longrightarrow \mathbb{R}^n,$$
$$\sigma : [0,T] \times \mathbb{R}^n \times U \longrightarrow \mathbb{R}^{n\times d},$$

and $x(\cdot)$ is the variable of state.

The function $u(\cdot)$ is called the control representing the action of the decision-makers (controller). At any time instant the controller has some information (as specified by the information field $\{\mathcal{F}_t\}_{t\in[0,T]}$) of what has happened up to that moment, but not able to foretell what is going to happen afterwards due to the uncertainty of the system (as a consequence, for any t the controller cannot exercise his/her decision $u(t)$ before the time t really comes), This nonanticipative restriction in mathematical terms can be expressed as "$u(\cdot)$ is $\{\mathcal{F}_t\}_{t\in[0,T]}-$adapted".

The control $u(\cdot)$ is an element of the set

$$\mathcal{U}[0,T] = \{u : [0,T] \times \Omega \longrightarrow U \text{ such that } u(\cdot) \text{ is } \{\mathcal{F}_t\}_{t\in[0,T]} - \text{adapted}\}.$$

We define the cost functional

$$J(u(\cdot)) = \mathbb{E}\left[\int_0^T l(t, x(t), u(t))dt + h(x(T))\right], \tag{1.2}$$

where

$$l : [0,T] \times \mathbb{R}^n \times U \longrightarrow \mathbb{R},$$

$$h : \mathbb{R}^n \longrightarrow \mathbb{R}.$$

Definition 1.1

Let $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P}\right)$ be given satisfying the usual conditions and let $B(\cdot)$ be a given d-dimensional standard $\{\mathcal{F}_t\}_{t\in[0,T]}$-Brownian motion. A control $u(\cdot)$ is called an admissible control, and $(x(\cdot), u(\cdot))$ an admissible pair, if

1. $u(\cdot) \in \mathcal{U}([0,T])$;

2. $x(\cdot)$ *is the unique solution of equation* (1.1)

   $l(\cdot, x(\cdot), u(\cdot)) \in L^1_{\mathcal{F}}([0,T];\mathbb{R})$ *and* $h(x(T)) \in L^1_{\mathcal{F}}(\Omega;\mathbb{R})$.

We denote by $U[0;T]$ the set of all admissible controls.

Our stochastic optimal control problem under strong formulation can be stated as follows:

Problem 1.1

(SS) *Minimize (1.2) over* $\mathcal{U}([0,T])$. *The goal is to find* $u^*(\cdot) \in \mathcal{U}([0,T])$ *(if it ever exists), such that*

$$J(u^*(\cdot)) = \inf_{u(\cdot)\in\mathcal{U}([0,T])} J(u(\cdot)). \tag{1.3}$$

For any $u^*(\cdot) \in \mathcal{U}([0,T])$ satisfying (1.3) is called an strong optimal control.

The corresponding state process $x^*(\cdot)$ and the state control pair $(x^*(\cdot), u^*(\cdot))$ are called an strong optimal state process and an strong optimal pair, respectively.

In stochastic control problems, there exists for the optimal control problem another formulation of a more mathematical aspect, it is the weak formulation of the stochastic optimal control problem. Unlike in the strong formulation the filtered probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P}\right)$ on which we define the Brownian motion $B(\cdot)$ are all fixed, but it is not the case in the weak formulation, where we consider them as a parts of the control.

1.1.6 Weak formulation

The strong formulation is the one that stems from the practical world, whereas the weak formulation sometimes serves as an auxiliary but effective mathematical model aiming at ultimately solving problems with the strong formulation.

Definition 1.2

We define a set of admissible control denoted by $\mathcal{U}^B([0,T])$ *the set of 6-tuple* $\pi = \left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P}, B(\cdot), u(\cdot)\right)$ *satisfying the following conditions:*

i) $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ *is a filtered probability space satisfying the usual conditions;*

ii) $B(\cdot)$ *is an* m*-dimensional standard Brownian motion defined on* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$;

iii) $u(\cdot)$ *is an* $\{\mathcal{F}_t\}_{t\geq 0}$*-adapted process on* $(\Omega, \mathcal{F}, \mathbb{P})$*taking values in* U*;*

iv) $x(\cdot)$ *is the unique solution of equation (1.1) on* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ *under* $u(\cdot)$ *and some prescribed state constraints are satisfied;*

v) $l(\cdot, x(\cdot), u(\cdot)) \in L^1_{\mathcal{F}}([0,T];\mathbb{R})$ *and* $h(x(T) \in L^1_{\mathcal{F}_T}(\Omega;\mathbb{R})$. $L^1_{\mathcal{F}}([0,T];\mathbb{R})$ *and* $L^1_{\mathcal{F}_T}(\Omega;\mathbb{R})$ *are defined on the given filtered probability space* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ *associated* *with the 6-tuple π is called weak-admissible control and $(x(\cdot), u(\cdot))$ an weak admissible pair, if the set of all weak admissible controls is denoted by $\mathcal{U}^B([0,T])$. Sometimes, might write $u(\cdot)) \in \mathcal{U}^B([0,T])$ instead of $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P}, B(\cdot), u(\cdot)\right) \in \mathcal{U}^B([0,T])$.*

Our stochastic optimal control problem under weak formulation can be formulated as follows:

Problem 1.2

(WS) *The objective is to minimize the cost functional given by equation (1.2) over the of admissible controls $\mathcal{U}^B([0,T])$. Namely, one seeks $\pi^*(\cdot) \in \mathcal{U}^B([0,T])$ such that*

$$J(\pi^*(\cdot)) = \inf_{\pi(\cdot)\in\mathcal{U}^B([0,T])} J(\pi(\cdot)).$$

1.2 Methods to solving optimal control problem

Two major tools for study an optimal control are Bellman's dynamic programming method and Pontryagin's maximum principle.

1.2.1 Dynamic Programming Method

In this section, we are researching a valuable approach to solve optimal control problems, utilizing the dynamic programming technique that R.Bellman originated in the early 1950s.

It is a mathematical technique for making a series of interrelated decisions that can be extended to many optimization problems, namely optimal control problems. The basic principle of this approach used for optimal control is to consider the family of optimal control problems with distinct initial terms and states,to define relationships between these problems through the so-called Hamilton-Jacobi-Bellman equation (HJB, for short) which is a nonlinear first order in the deterministic case or second order in the stochastic case of partial differential equation. If the HJB equation is resolvable (either analytically or numerically), an optimal feedback control can be obtained by maximizing / minimizing

Hamiltonian or generalized Hamiltonian involved in the HJB equation. It's the so-called authentication technique. Notice that, in truth, this approach offers solutions to the entire family of problems (with distinct initial terms and states).

However, there was a big downside in the classical dynamic programming approach: it allowed the HJB equation to admit classical solutions, meaning that the solutions must be smooth enough (to the order of the derivatives involved in the equation). In the early 1980s, Crandall and Lions implemented the so-called viscosity methods to address this challenge.This latest paradigm is a kind of non-smooth approach to partial differential equations, the core function of which is to substitute traditional derivatives with (set-valued) super-/sub differentials while preserving the uniqueness of solutions under very mild conditions. In the stochastic case where diffusion can degenerate, the HJB equation may not necessarily have any classical solutions either.

The Bellman principle: Let $T > 0$ be given and let U be a metric space. For any $(s;x) \in [0;T) \times \mathbb{R}^n$, consider the state equation

$$dx(s) \quad = \quad f(s, x(s), u(s))ds + \sigma(s, x(s), u(s))dB(s), \ \mathrm{s} \in [0, T] \ . \tag{1.4}$$

To ensure the existence of the solution to SDE-(1.4), the Borelian functions

$$f : [0, T] \times \mathbb{R}^n \times U \longrightarrow \mathbb{R}^n$$
$$\sigma : [0, T] \times \mathbb{R}^n \times U \longrightarrow \mathbb{R}^{n \times d}$$

satisfy the following conditions:

$$|f(t, x, u) - f(t, y, u)| + |\sigma(t, x, u) - \sigma(t, y, u)| \leq C |x - y| ,$$
$$|f(t, x, u)| + |\sigma(t, x, u)| \leq C [1 + |x|] ,$$

for some constant $C > 0$.

The cost functional associated with (1.4) is the following:

$$J(t, x, u) \quad = \quad \mathbb{E}\left[\int_t^T l(s, x(s), u(s))ds + h(x(T))\right], \tag{1.5}$$

where

$$l : [0, T] \times \mathbb{R}^n \times U \longrightarrow \mathbb{R},$$
$$h : \mathbb{R}^n \longrightarrow \mathbb{R},$$

be given functions.We have to impose integrability conditions on l and h in order for the above expectation to be well-defined, e.g. a lower boundedness or quadratic growth condition.

We denote by $U\,[s;T]$ the set of all 5-tuples $(\Omega, \mathcal{F}, \mathbb{P}, B(\cdot), u(\cdot))$ satisfying the following :

a) $(\Omega, \mathcal{F}, \mathbb{P})$ is complete probability space;

b) $(B_t)_{t\geq s}$ is an m-dimensional standard Brownian motion defined on $(\Omega, \mathcal{F}, \mathbb{P})$ over $[s;T]$ (with $B(s) = 0$ a:s:) ; and $\mathcal{F}_t^s = \sigma\{B_t : s \leq r \leq t\}$ augmented by all P-nul sets in $\mathcal{F}$;

c) $u : [s;T] \times \Omega \to U$ is an $(\mathcal{F}_t^s)_{t\geq s}$-adapted process on $(\Omega, \mathcal{F}, \mathbb{P})$;

d) under $u(.)$, for any $x \in \mathbb{R}^n$equation (1.4) admits a unique solution;

e) $l(., x(.), u(.)) \in L^1_{\mathcal{F}}(0, T; \mathbb{R})$ and $h(x(T)) \in L^1_{\mathcal{F}}(\Omega; \mathbb{R})$ are defined on $\left(\Omega, \mathcal{F}, (\mathcal{F}_t^s)_{t\geq s}, \mathbb{P}\right)$;

The objective is to maximize the gain function therefore we introduce the so-called value function.

Definition 1.3

We define the value function of the original Problem

$$\begin{cases} V(s,y) &= \inf\limits_{u(.)\in\mathcal{U}([s,T])} J(s,y,u(.)), \quad \forall\ (s,y) \in [0;T] \times \mathbb{R}^n, \\ V(t,y) &= h(y), \ \forall y \in \mathbb{R}^n. \end{cases}$$

Note that the value function to be well defined $J(s, y, u(.))$ must be defined, i.e the function f, σ, l and h satisfy the following condition ***(S1)***

i) f, σ, l and h are uniformly continuous;

ii) there exists a constant $L > 0$ such that for :$\varphi(t, x, u) = f(t, x, u), \sigma(t, x, u), l(t, x, u), h(x)$; in addition the following condition ***(S2)***

1. (U, d) is polish space (complete separable metric space).

We say that $u^* \in \mathcal{U}([t,T])$ is an optimal control if

$$V(t,x) = J(t,x,u^*).$$

Theorem 1.1

*Let **(S1)-(S2)**hold. Then for any $(t,x) \in [0,T] \times \mathbb{R}^n$ be given. Then we have*

$$V(t,x) = \sup_{u \in \mathcal{U}([t,T])} \mathbb{E}\left[\int_t^{t+h} l(s,x(s),u(s))dt + V(t+h,x(t+h))\right], \text{ for } t \leq t+h \leq T. \tag{1.6}$$

Proof: The proof of the dynamic programming principle is technical and has been studied by different methods, we refer the reader to Yong and Zhou [89]. ■

The Hamilton-Jacobi-Bellman equation

Proposition 1.1

*Let **(S1)-(S2)** hold. Then the value function $V(s,y)$ satisfies the following:*

i) $|V(s,y)| \leq K(1+|y|), \quad \forall\ (t,x) \in [0,T] \times \mathbb{R}^n, K > 0.$

ii) $|V(s,y) - V(s^*,y^*)| \leq K\left\{|y-y^*| + (1+|y| \vee |y^*|)\,|s-s^*|^{\frac{1}{2}}\right\}, \forall\ s,s^* \in [0,T],$ $y,y^* \times \mathbb{R}^n, K > 0$ $(a \vee f = \max(a,f))$.

iii) *if $V \in C^{1,2}([0,T] \times \mathbb{R}^n)$. Then V is a solution of a second-order partial differential equation:*

$$\begin{cases} -v(t) + \sup\limits_{u \in U} G(t,x,u,-v_x,-v_{xx}) = 0, & (t,x) \in 0,T) \times \mathbb{R}^n, \\ v|_{t=T} = h(x), & x \in \mathbb{R}^n, \end{cases} \tag{1.7}$$

where

$$\begin{aligned} G(t,x,u,p,P) = {} & \frac{1}{2} tr(P\sigma(t,x,u)\sigma(t,x,u)^T) \\ & + (p, f(t,s,u)) - f(t,x,u), \\ & \forall (t,x,u,p,P) \in [0,T] \times \mathbb{R}^n \times U \times \mathbb{R}^n \times S^n. \end{aligned} \tag{1.8}$$

Equation (1.7) is called the Hamilton -Jacobi-Bellman equation, which is the infinitesimal version of the dynamic programming principle. The function $G(t,x,u,p,P)$ defined by (3.25) is called the generalized Hamiltonian.

Viscosity Solutions

The HJB equation generally does not admit regular solutions, i.e. which are not not in $C^{1,2}\left([0,T]\times\mathbb{R}^n\right)$. Viscosity solutions introduced by Crandall and Lions (1983) remedy this shortcoming.

Definition 1.4

1. *A function $v\in C([0,T]\times\mathbb{R}^n)$ is called a viscosity sub-solution of (1.7) if*

$$v(T,x)\leq h(x),\qquad \forall x\in\mathbb{R}^n,$$

and for $\varphi\in C^{1,2}\left([0,T]\times\mathbb{R}^n\right)$, whenever $v-\varphi$ attains a local maximum at $(t,x)\in[0,T]\times\mathbb{R}^n$, we have

$$-\varphi_t(t,x)+\sup_{u\epsilon U}G(t,x,u,-\varphi_x(t,x),-\varphi_{xx}(t,x))\leq 0.$$

2. *A function $v\in C([0,T]\times\mathbb{R}^n)$ is called a viscosity supersolution of (1.7) if*

$$v(T,x)\geq h(x),\qquad \forall x\in\mathbb{R}^n,$$

and for $\varphi\in C^{1,2}\left([0,T]\times\mathbb{R}^n\right)$, whenever $v-\varphi$ attains a local minimum at $(t,x)\in[0,T]\times\mathbb{R}^n$, we have

$$-\varphi_t(t,x)+\sup_{u\epsilon U}G(t,x,u,-\varphi_x(t,x),-\varphi_{xx}(t,x))\geq 0$$

3. *A function $v\in C([0,T]\times\mathbb{R}^n)$ is both a viscosity sub-solution and viscosity supe-rsolution of (1.7), then it is called a viscosity solution of (1.7).*

Theorem 1.2

*Let **(S1)-(S2)** hold. Then the value function V is a viscority solution of (1.7).*

The classical verification

The classical verification approach consists in finding a smooth solution to the HJB equation, and to check that this candidate, under suitable sufficient conditions, coincides with the value function. This result is usually called a verification theorem and provides as a byproduct an optimal control. It relies mainly on Itô's formula. The assertions of a verification theorem may slightly vary from problem to problem, depending on the required sufficient technical conditions. These conditions should actually be adapted to the context of the considered problem. In the above context, a verification theorem is roughly stated as follows:

Theorem 1.3

. Let ***(S1)-(S2)*** *hold. Then the value function V is a viscority solution of the HJB equation (1.8), so we have:*

$$v(s,y) \leq J(s,y,u(.)), \qquad \forall u(.) \in \mathcal{U}^B([s,T]),\ \forall\ (s,y) \in [0;T] \times \mathbb{R}^n.$$

An admissible pair $(x^*(\cdot), u^*(\cdot))$ is an optimal pair for $C_{s,y}$ if and only if

$$\begin{aligned} v(t,x^*(t)) &= \max G(t,x^*(t),u,-v_x(t,x^*(t)),-v_{xx}(t,x^*(t))) \\ &= G(t,x^*(t),u^*,-v_x(t,x^*(t)),-v_{xx}(t,x^*(t))). \end{aligned}$$

A proof of this verification theorem can be found in book, by Yong & Zhou [89].

1.2.2 Pontryagin's maximum principle

The seminal work on the stochastic maximum principle has been established by Kushner. Since then, a lot of work has been done on this subject, including, in particular, those by Bensoussan, Peng, and so on. The classic approach to optimization and control problems is to guarantee that the required conditions are satisfied by an optimal solution. The claim is to use an accurate calculus of the variations on the gain function $J(t,x,.)$ as far as the control variable is concerned, in order to derive the required condition of optimality. The Maximum Principle, proposed by Pontryagin in the 1960s, specifies that the optimal state trajectory must solve the Hamilton system along with the maximum condition of a function called the generalized Hamilton. In general, it should be easier to solve a Hamilton than to solve the original control problem.

The original version of the Pontryagin Maximum Principle was derived for deterministic concerns. (see Yong & Zhou [89]) As in the classic variance calculus, the basic concept is to perturb optimal control and to use some kind of Taylor expansion of state trajectory and objective functional around optimal control. By sending a perturbation to zero, one obtains any inequality, and by duality, the maximum principle is articulated in terms of an adjoint variable.In the 1970s, Bismut, Kushner, Bensoussan and Haussmann extensively developed the first version of the stochastic maximum principle. However, at that time, the results were basically obtained on the basis that there is no control on the coefficient of diffusion. For instance, see Haussmann [52] examined the maximum transformation principle of Girsanov and this limitation explains why this approach does not work with control-dependent and degenerate diffusion coefficients. See also [3, 4, 22, 35, 48].

But, the first version of the stochastic maximum principle, where the diffusion coefficient depends directly on the control variable and the control domain is not convex, was obtained by Peng who studied the second-order term in Taylor's expansion of the perturbation method resulting from the Itô integral. He then obtained the maximum principle for potentially degenerating and controlling-dependent diffusion, which, in addition to the first-order adjoint variable, includes the second-order adjoint variable. The adjoint variables are defined by what is known today as backward stochastic differential equations (BSDE). Bismut first suggested linear BSDE in 1973. We note Pardoux and Peng had the uniqueness and existence theorem for solutions of nonlinear BSDE driven by Brownian motion under the Lipschitz condition in 1990. Today, BSDE theory plays a vital role not only in dealing with stochastic optimal control problems, but also in mathematical science, especially in the hedging and non-linear pricing theory of imperfect markets.

The maximum principle

We provide a sketch of how the maximum principle for a deterministic control problem is derived. In this setting, consider the following stochastic controlled system

$$\begin{cases} dx(t) &= f(t, x(t), u(t))dt, \ t \in [0, T], \\ x(0) &= x_0, \end{cases} \tag{1.9}$$

where

$$f : [0, T] \times \mathbb{R} \times \mathcal{U} \longrightarrow \mathbb{R},$$

and the action space $\mathcal{U}$ is some subset of $\mathbb{R}$. The objective is to minimize some cost function of the form:

$$J(u(\cdot)) = \int_0^T l(t, x(t), u(t)) + h(x(T)), \tag{1.10}$$

where

$$l : [0, T] \times \mathbb{R} \times \mathcal{U} \longrightarrow \mathbb{R},$$
$$h : \mathbb{R} \longrightarrow \mathbb{R}.$$

That is, the function l inflicts a running cost and the function h inflicts a terminal cost. Any $u^* \in U[0, T]$ satisfying

$$J(u^*(\cdot)) = \inf_u J(u(\cdot)),$$

is called an optimal control.

We now assume that there exists a control (t) which is optimal. And we are going to derive necessary conditions for optimality, for this we make small perturbation of the optimal control. Where u^ε is the spike variation of u^* defined as follows:

$$u^\varepsilon(t) = \begin{cases} v & \text{for } \tau - \varepsilon \leq t \leq \tau, \\ u^*(t) & \text{otherwise.} \end{cases} \tag{1.11}$$

We denote by $x^\varepsilon(t)$ the solution to (1.9) with the control $u^\varepsilon(t)$. We set that $x^*(t)$ and $x^\varepsilon(t)$ are equal up to $t = \tau - \varepsilon$ and that

$$\begin{aligned} x^\varepsilon(\tau) - x^*(\tau) &= (f(\tau, x^\varepsilon(\tau), v) - f(\tau, x^*(\tau), u^*(\tau)))\varepsilon + o(\varepsilon) \\ &= (f(\tau, x^*(\tau), v) - f(\tau, x^*(\tau), u^*(\tau)))\varepsilon + o(\varepsilon), \end{aligned} \tag{1.12}$$

where the second equality holds since $x^{\varepsilon}(\tau) - x^{*}(\tau)$ is of order ε. We look at the Taylor expansion of the state with respect to ε. Let

$$z(t) = \frac{\partial}{\partial \varepsilon} x^{\varepsilon}(t) \mid_{\varepsilon=0},$$

i.e. the Taylor expansion of $x^{\varepsilon}(t)$ is

$$x^{\varepsilon}(t) = x^{*}(t) + z(t)\varepsilon + o(\varepsilon). \tag{1.13}$$

Then, by (1.12)

$$z(\tau) = f(\tau, x^{*}(\tau), v) - f(\tau, x^{*}(\tau), u^{*}(\tau)). \tag{1.14}$$

Moreover, we can derive the following differential equation for $z(t)$.

$$\begin{aligned} dz(t) &= \frac{\partial}{\partial \varepsilon} dx^{\varepsilon}(t) \mid_{\varepsilon=0} \\ &= \frac{\partial}{\partial \varepsilon} f(t, x^{\varepsilon}(t), u^{\varepsilon}(t))dt \mid_{\varepsilon=0} \\ &= f_x(t, x^{\varepsilon}(t), u^{\varepsilon}(t)) \frac{\partial}{\partial \varepsilon} x^{\varepsilon}(t)dt \mid_{\varepsilon=0} \\ &= f_x(t, x^{*}(t), u^{*}(t))z(t)dt, \end{aligned}$$

where f_x denotes the derivative of f with respect to x. If we for the moment assume that $l = 0$, the optimality of $u^{*}(t)$ leads to the inequality

$$\begin{aligned} 0 \leq \frac{\partial}{\partial \varepsilon} J(u^{\varepsilon}) \bigg|_{\varepsilon=0} &= \frac{\partial}{\partial \varepsilon} h\left(x^{\varepsilon}(T)\right) \mid_{\varepsilon=0} \\ &= h_x\left(x^{\varepsilon}(T)\right) \frac{\partial}{\partial \varepsilon} x^{\varepsilon}(T) \mid_{\varepsilon=0} \\ &= h_x\left(x^{*}(T)\right) z(T). \end{aligned}$$

We shall use duality to obtain a more explicit necessary condition from this. To this end we introduce the adjoint equation:

$$\begin{cases} d\Psi(t) = -f_x(t, x^{*}(t), u^{*}(t))\Psi(t)dt, t \in [0, T], \\ \Psi(T) = h_x(x^{*}(T)). \end{cases}$$

Then it follows that

$$d(\Psi(t)z(t)) = 0,$$

i.e. $\Psi(t)z(t)) =$ constant. By the terminal condition for the adjoint equation we have

$$\Psi(t)z(t) = h_x(x^*(T))z(T) \geq 0, \text{ for all } 0 \leq t \leq T.$$

In particular, by (1.14)

$$\Psi(\tau)\left(f(\tau, x^*(\tau), v) - f(\tau, x^*(\tau), u^*(\tau))\right) \geq 0.$$

Since τ was chosen arbitrarily, this is equivalent to

$$\Psi(t)f(t, x^*(t), u^*(t)) = \inf_{v \in \mathcal{U}} \Psi(t)f(t, x^*(t), v), \text{ for all } 0 \leq t \leq T.$$

By repeating the calculations above for this two-dimensional system, one can derive the necessary condition

$$H(t, x^*(t), u^*(t), \Psi(t)) = \inf_v H(t, x^*(t), v, \Psi(t)) \text{ for all } 0 \leq t \leq T, \tag{1.15}$$

where H is the so-called Hamiltonian (sometimes defined with a minus sign which turns the minimum condition above into a maximum condition) :

$$H(x, u, \Psi) = l(x, u) + \Psi f(x, u),$$

and the adjoint equation is given by

$$\begin{cases} d\Psi(t) = -(l_x(t, x^*(t), u^*(t)) + f_x(t, x^*(t), u^*(t))\Psi(t))dt, \\ \Psi(T) = h_x(x^*(T)). \end{cases} \tag{1.16}$$

The minimum condition (1.15) together with the adjoint equation (1.16) specifies the Hamiltonian system for our control problem.

The stochastic maximum principle

Stochastic control is the extension of optimal control to problems where it is of importance to take into account some uncertainty in the system. One possibility is then to replace the differential equation by an SDE:

$$dx(t) = f(t, x(t), u(t))dt + \sigma(t, x(t))dB(t), t \in [0, T], \tag{1.17}$$

where f and σ are deterministic functions and the last term is an Itô integral with respect to a Brownian motion B defined on a probability space $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[0,T]}, \mathbb{P}\right)$.

More generally, the diffusion coefficient σ may has an explicit dependence on the control: $t \in [0,T]$.

$$dx(t) = f(t,x(t),u(t))dt + \sigma(t,x(t),u(t))dB(t), \tag{1.18}$$

The optimal control problem we are concerned with is to minimize the following cost functional over $U[0,T]$:

$$J(u(\cdot)) = \mathbb{E}\left[\int_0^T l(t,x(t),u(t)) + h(x(T))\right].$$

For the case (1.17) the adjoint equation is given by the following Backward SDE:

$$\begin{cases} -d\Psi(t) = \{f_x(t,x^*(t),u^*(t))\Psi(t) + \sigma_x(t,x^*(t))Q(t) \\ \qquad\qquad +(l_x(t,x^*(t),u^*(t))\}dt - Q(t)dB(t), \\ \Psi(T) = h_x(x^*(T)). \end{cases} \tag{1.19}$$

A solution to this backward SDE is a pair $(\Psi(t), Q(t))$ which fulfills (1.19). The Hamiltonian is

$$H(x,u,\Psi(t),Q(t)) = l(t,x,u) + \Psi(t)f(t,x,u) + Q(t)\sigma(t,x),$$

and the maximum principle reads for all $0 \leq t \leq T$,

$$H(t,x^*(t),u^*(t),\Psi(t),Q(t)) = \inf_{u\in\mathcal{U}} H(t,x^*(t),u,\Psi(t),Q(t)) \;\; \mathbb{P}-\text{a.s.} \tag{1.20}$$

Noting that there is also third case: if the state is given by (1.18) but the action space $\mathcal{U}$ is assumed to be convex, it is possible to derive the maximum principle in a local form. This is accomplished by using a convex perturbation of the control instead of a spike variation, see Bensoussan 1983 [9]. The necessary condition for optimality is then given by the following: for all $0 \leq t \leq T$

$$\mathbb{E}\int_0^T H_u(t,x^*(t),u^*(t),\Psi^*(t),Q^*(t))\,(u - u^*(t)) \geq 0.$$

1.3 Some classes of stochastic controls

Let $(\Omega, \mathcal{F}, \mathcal{F}_{t\geq 0}, P)$ be a complete filtred probability space.

1.3.1 Admissible control

An admissible control is $\mathcal{F}_t$-adapted process $u(t)$ with values in a borelian $A \subset \mathbb{R}^n$

$$\mathcal{U} := \{u(\cdot) : [0,T] \times \Omega \to A : u(t) \text{ is } \mathcal{F}_t\text{-adapted}\}. \tag{1.21}$$

1.3.2 Optimal control

The optimal control problem consists to minimize a cost functional $J(u)$ over the set of admissible control $\mathcal{U}$. We say that the control $u^*(\cdot)$ is an optimal control if

$$J(u^*(t)) \leq J(u(t)), \text{ for all } u(\cdot) \in \mathcal{U}.$$

1.3.3 Near-optimal control

Let $\varepsilon > 0$, a control is a near-optimal control (or ε-optimal) if for all control $u(\cdot) \in \mathcal{U}$ we have

$$J(u^\varepsilon(t)) \leq J(u(t)) + \varepsilon. \tag{1.22}$$

See Yong & Zhou [89].

1.3.4 Singular control

An admissible control is a pair $(u(\cdot), \xi(\cdot))$ of measurable $\mathbb{A}_1 \times \mathbb{A}_2-$valued, $\mathcal{F}_t-$adapted processes, such that $\xi(\cdot)$ is of bounded variation, non-decreasing continuous on the left with right limits and $\xi(0_-) = 0$. Since $d\xi(t)$ may be singular with respect to Lebesgue measure dt, we call $\xi(\cdot)$ the singular part of the control and the process $u(\cdot)$ its absolutely continuous part, see [3, 4, 22, 35, 48].

1.3.5 Feedback control

We say that $u(\cdot)$ is a feedback control if $u(\cdot)$ depends on the state variable $X(\cdot)$. If $\mathcal{F}_t^X$ the natural filtration generated by the process X, then $u(\cdot)$ is a feedback control if $u(\cdot)$ is $\mathcal{F}_t^X-$adapted.

1.3.6 Impulsive control

Impulse control: Here one is allowed to reset the trajectory at stopping times (τ_i) from $X_{\tau_{i-}}$ (the value immediately before i) to a new (non-anticipative) value X_{τ_i}, resp., with an associated cost $L\left(X_{\tau_{i-}}, X_{\tau_i}\right)$. The aim of the controlled is to minimizes the cost functional:

$$\begin{aligned}
&\mathbb{E}\int_0^T \exp\left[-\int_0^t C(X(s),u(s))ds\right] K(X(t),u(t)) \\
&+\sum_{\tau_i<T} \exp\left[-\int_0^{\tau_i} C(X(s),u(s))ds\right] h(X_\tau, X_{\tau_{i-}}) \\
&+\exp\left[-\int_0^{\tau_i} C(X(s),u(s))ds\right] h(X(T)).
\end{aligned}$$

1.3.7 Ergodic control

Some stochastic systems may exhibit over a long period a stationary behavior characterized by an invariant measure. This measure, if it does exists, is obtained by the average of the states over a long time. An ergodic control problem consists in optimizing over the long term some criterion taking into account this invariant measure. (See Pham [69], Borkar [16]). The cost functional is given by

$$\lim \sup_{T\to+\infty} \frac{1}{T}\mathbb{E}\int_0^T f(x(t),u(t))dt.$$

1.3.8 Robust control

In the above formulated problems, the dynamics of the control system are assumed to be known and fixed. Robust control theory is a method to measure the performance changes of a control system with changing system parameters. This is of course important in engineering systems, and it has recently been used in finance in relation with the theory of risk measure. Indeed, it is proved that a coherent risk measure for an uncertain payoff $x(T)$ at time T is represented by :

$$\rho(-X(t)) = \sup_{Q\in S} \mathbb{E}^Q(X(T)),$$

where S is a set of absolutely continuous probability measures with respect to the original probability P.

1.3.9 Partial observation control problem

It is assumed so far that the controller completely observes the state system. In many real applications, he is only able to observe partially the state via other variables and there is noise in the observation system. For example in financial models, one may observe the asset price but not completely its rate of return and/or its volatility, and the portfolio investment is based only on the asset price information. We are facing a partial observation control problem. This may be formulated in a general form as follows:
We have a controlled signal (unobserved) process governed by the following SDE:

$$dx\left(t\right) = f\left(t, x\left(t\right), y\left(t\right), u\left(t\right)\right) dt + \sigma\left(t, x\left(t\right), y\left(t\right), u\left(t\right)\right) dB\left(t\right),$$

and

$$dy\left(t\right) = h\left(t, x\left(t\right), y\left(t\right), u\left(t\right)\right) dt + h\left(t, x\left(t\right), y\left(t\right), u\left(t\right)\right) dB\left(t\right),$$

where $B\left(t\right)$ is another Brownian motion, eventually correlated with $B\left(t\right)$. The control $u(t)$ is adapted with respect to the filtration generated by the observation F_t^Y and the functional to optimize is :

$$J\left(u\left(\cdot\right)\right) = \mathbb{E}\left[h\left(x\left(T\right), y(T)\right) + \int_0^T h\left(t, x\left(t\right), y(t), u\left(t\right)\right) dt\right].$$

1.3.10 Random horizon

In classical problem, the time horizon is fixed until a deterministic terminal time T. In some real applications, the time horizon may be random, the cost functional is given by the following:

$$J\left(u\left(\cdot\right)\right) = \mathbb{E}\left[h\left(x\left(\tau\right)\right) + \int_0^\tau h\left(t, x\left(t\right), y(t), u\left(t\right)\right) dt\right],$$

where τ s a finite random time.

1.3.11 Relaxed control

The idea is then to compact the space of controls $\mathcal{U}$ by extending the definition of controls to include the space of probability measures on U. The set of relaxed controls $\mu_t\,(du)\,dt$, where μ_t is a probability measure, is the closure under weak* topology of the measures $\delta_{u(t)}(du)dt$ corresponding to usual, or strict, controls. This notion of relaxed control is introduced for deterministic optimal control problems in Young (*Young, L.C. Lectures on the calculus of variations and optimal control theory, W.B. Saunders Co., 1969.*) (See Borkar [16]).

Chapter 2

Maximum Principle for SDE of mean-field type

In this chapter, we presente the stochastic maximum principle for optimal control, where the system is governed by stochastic differential equations SDEs of mean-field type. the control domain is assumed to be convex. this result was introduced by Anderson & Djehiche [7].

2.1 Formulation of the Problem

Let $T > 0$ be a fixed time horizon and $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ be a filtered probability space satisfying the usual conditions, on which a standard Brownian motion $B = (B_t)_{t\geq 0}$ is defined. We assume that $(\mathcal{F}_t)_{t\geq 0}$ is the natural filtration of B augmented by $\mathbb{P}$-null sets of $\mathcal{F}$.

We consider the following stochastic differential equation

$$\begin{cases} dx(t) = f\left(t, x(t), \mathbb{E}\Psi\left(x(t)\right), u(t)\right) dt + \sigma\left(t, x(t), \mathbb{E}\Phi\left(x(t)\right), u(t)\right) dB_t, \\ \\ x\left(0\right) = x_0, \end{cases} \tag{2.1}$$

where,

$$\begin{aligned} &f : [0, T] \times \mathbb{R} \times \mathbb{R} \times U \to \mathbb{R}, \\ &\Psi : \mathbb{R} \to \mathbb{R}, \\ &\sigma : [0, T] \times \mathbb{R} \times \mathbb{R} \times U \to \mathbb{R}, \\ &\Phi : \mathbb{R} \to \mathbb{R}. \end{aligned}$$

The action space, U, is a non-empty, closed and convex subset of $\mathbb{R}$, and U is the class of measurable, $\mathcal{F}_t$ -adapted and square integrable processes $u : [0, T] \times \Omega \to U$.

We denote the set of all admissible controls by $U[0,T]$.

The optimal control problem we are concerned with is to minimize the following cost functional over $U[0,T]$:

$$J(u)=\mathbb{E}\left(\int_0^T l\left(t,x(t),\mathbb{E}\varphi\left(x(t)\right),u(t)\right)dt+h\left(x_T,\mathbb{E}\chi\left(T\right)\right)\right), \tag{2.2}$$

where,

$$\begin{aligned}
h &: \mathbb{R}\times\mathbb{R}\to\mathbb{R},\\
\chi &: \mathbb{R}\to\mathbb{R},\\
l &: [0,T]\times\mathbb{R}\times\mathbb{R}\times U\to\mathbb{R},\\
\varphi &: \mathbb{R}\to\mathbb{R}.
\end{aligned}$$

The following assumptions will be in force throughout this chapter.

$(\mathbf{H}_1)$ Ψ,Φ,χ and φ are continuously differentiable. h is continuously differentiable with respect to (x,y). f,σ,l are continuously differentiable with respect to (x,y,v).

$(\mathbf{H}_2)$ All the derivatives are Lipschitz continuous and bounded.

$(\mathbf{H}_3)$ The function h is convex in(x,y) .

$(\mathbf{H}_4)$ The Hamiltonian is convex in (x,y,v) .

$(\mathbf{H}_5)$ The functions Ψ,Φ,φ and χ are convex.

$(\mathbf{H}_6)$ The functions f_y,σ_y,l_y and h_y are nont-negative.

We note that x denotes the state variable, y the 'expected value', and v the control variable.

Under the above assumptions $(\mathbf{H}_1),(\mathbf{H}_2)$, the SDE (2.1) has a unique strong solution.

The optimal control problem is to minimize the functional $J(\cdot)$ over U.

A control that solves this problem is called optimal. We denote for any process φ_t, such that

$$|\varphi|_T^{*,2}=\sup_{t\in[0,T]}|\varphi(t)|^2.$$

f_x,f_y,f_v denotes the derivative of f with respect to the state trajectory, the '*expected value*' and the control variable, respectively, and similarly for the other functions.

Finally, we denote by $x(t)$ and $u^*(t)$ the optimal trajectory and control, respectively.

2.2 Necessary conditions for optimality

In this section, we state the conditions necessary for optimality in the form of a maximum principle, we can use the convex perturbation method of optimal control.

2.2.1 Taylor Expansions

Let $x(t)$ denote the state trajectory corresponding to the following perturbation :

$$u^{\theta}(t) = u^{*}(t) + \theta \upsilon(t), \quad \upsilon_t \in U.$$

We denote by

$$\widehat{f}\,(t) = f\left(t, x(t), \mathbb{E}\widehat{\Psi}\,(t)\,, u^{*}(t)\right),$$
$$\widehat{\Psi}\,(t) = \Psi\,(x(t))\,,$$

and similarly for the other functions and their derivatives.

The objective of this section is to determine the Gateaux derivative of the cost functional in terms of the Taylor expansion of the state process.

Lemma 2.1

Let

$$\begin{cases} dz(t) = \left(\widehat{f}_x\,(t)\,z(t) + \widehat{f}_y\,(t)\,\mathbb{E}\left(\widehat{\Psi}_x\,(t)\,z(t)\right) + \widehat{f}_\upsilon\,(t)\,\upsilon(t)\right)dt \\ \qquad + \left(\widehat{\sigma}_x\,(t)\,z(t) + \widehat{\sigma}_y\,(t)\,\mathbb{E}\left(\widehat{\Phi}_t\,(t)\,z(t)\right) + \widehat{\sigma}_\upsilon\,(t)\,\upsilon(t)\right)dB_t, \\ z_0 = 0. \end{cases} \tag{2.3}$$

Then,it holds that,

$$\lim_{\theta \to 0} \mathbb{E}\left|\frac{x(t) - \widehat{x}(t)}{\theta} - z(t)\right|_T^{*,2} = 0.$$

Proof: Since the coefficients in (2.3) are bounded, it follows from Proposition (3.53) in [[73]], that there exists a unique solution such that

$$\mathbb{E}\left(\sup_{t\in[0,T]} \mid z(t) \mid^p\right) < \infty. \qquad \forall p \in \mathbb{N}_+. \tag{2.4}$$

we define $y^{\theta}(t) = \dfrac{x(t) - \widehat{x}(t)}{\theta} - z(t)$ and noting (2.4), it is also clear that

$$\mathbb{E}\left(\sup_{t\in[0,T]} \mid y^{\theta}(t) \mid^p\right) < \infty \qquad \forall p \in \mathbb{N}_+. \tag{2.5}$$

we have $y_0^\theta = 0$ et $y^\theta(t)$ fulfills the following SDE

$$\begin{aligned} dy^\theta(t) &= \frac{1}{\theta}\left(f\left(\widehat{x}(t)+\theta\left(y^\theta(t)+z(t)\right),\mathbb{E}(\Psi\left(\widehat{x}(t)+\theta y^\theta(t)+z(t)\right),\widehat{u}(t)+\theta\upsilon(t)\right)-\widehat{f}\left(t\right)\right)dt \\ &\quad -\left(\widehat{f}_x\left(t\right)z(t)+\widehat{f}_y\left(t\right)\mathbb{E}\left(\widehat{\Psi}_x\left(t\right)z(t)\right)+\widehat{f}_\upsilon\left(t\right)\upsilon(t)\right)dt \\ &\quad +\frac{1}{\theta}\left(\sigma\left(\widehat{x}(t)+\theta\left(y^\theta(t)+z(t)\right),\mathbb{E}(\Phi\left(\widehat{x}(t)+\theta\left(y^\theta(t)+z(t)\right)\right),\widehat{u}(t)+\theta\upsilon_t\right)-\widehat{\sigma}\left(t\right)\right)dB_t \\ &\quad -\left(\widehat{\sigma}_x\left(t\right)z(t)+\widehat{\sigma}_y\left(t\right)\mathbb{E}\left(\widehat{\Phi}_x\left(t\right)z(t)\right)+\widehat{\sigma}_\upsilon\left(t\right)\upsilon_t\right)dB_t. \end{aligned} \tag{2.6}$$

Noting that

$$\begin{aligned} &\frac{d}{d\lambda}f\left(\cdot,\mathbb{E}\left(\Psi\left(\widehat{x}(t)+\lambda\theta\left(y^\theta(t)+z(t)\right)\right)\right),\cdot\right) \\ &=f_y\left(\cdot,\mathbb{E}\left(\Psi\left(\widehat{x}(t)+\lambda\theta\left(y^\theta(t)+z(t)\right)\right)\right),\cdot\right)\mathbb{E}\left(\Psi_x\left(\widehat{x}(t)+\lambda\theta\left(y^\theta(t)+z(t)\right)\right)\left(y^\theta(t)+z(t)\right)\right)\theta. \end{aligned}$$

We proceed as in [10], and write

$$\begin{aligned} &\frac{1}{\theta}\left(f\left(\widehat{x}(t)+\theta\left(y^\theta(t)+z(t)\right),\mathbb{E}\Psi\left(\widehat{x}(t)+\theta\left(y^\theta(t)+z(t)\right)\right),\widehat{u}(t)+\theta\upsilon_t\right)-\widehat{f}\left(t\right)\right)dt \\ &=\int_0^1 f_x\left(\widehat{x}(t)+\lambda\theta\left(y^\theta(t)+z(t)\right),\mathbb{E}\Psi\left(\widehat{x}(t)+\lambda\theta\left(y^\theta(t)+z(t)\right)\right),\widehat{u}(t)+\lambda\theta\upsilon_t\right)\left(y^\theta(t)+z(t)\right)d\lambda \\ &+\int_0^1 f_y\left(\widehat{x}(t)+\lambda\theta\left(y^\theta(t)+z(t)\right),\mathbb{E}\Psi\left(\widehat{x}(t)+\lambda\theta\left(y^\theta(t)+z(t)\right)\right),\widehat{u}(t)+\lambda\theta\upsilon_t\right) \\ &\quad .\mathbb{E}\left(\Psi_x\left(\widehat{x}(t)+\lambda\theta\left(y^\theta(t)+z(t)\right)\right)\left(y^\theta(t)+z(t)\right)\right)d\lambda \\ &+\int_0^1 f_\upsilon\left(\widehat{x}(t)+\lambda\theta\left(y^\theta(t)+z(t)\right),\mathbb{E}\Psi\left(\widehat{x}(t)+\lambda\theta\left(y^\theta(t)+z(t)\right)\right),\widehat{u}(t)+\lambda\theta\upsilon_t\right)\upsilon_t d\lambda. \end{aligned} \tag{2.7}$$

Denoting $x(t)=\widehat{x}(t)+\lambda\theta\left(y^\theta(t)+z(t)\right)$ and $u^{\lambda,\theta}(t)=\widehat{u}(t)+\lambda\theta\upsilon_t$ for notational convenience, we may insert (2.7) into (3.51) to obtain the equality

$$\begin{aligned} &\frac{1}{\theta}\left(f\left(\widehat{x}(t)+\theta\left(y^\theta(t)+z(t)\right),\mathbb{E}\Psi\left(\widehat{x}(t)+\theta\left(y^\theta(t)+z(t)\right)\right),\widehat{u}(t)+\theta\upsilon_t\right)-\widehat{f}\left(t\right)\right) \\ &-\left(\widehat{f}_x\left(t\right)z(t)+\widehat{f}_y\left(t\right)\mathbb{E}\left(\widehat{\Psi}_x\left(t\right)z(t)\right)+\widehat{f}_\upsilon\left(t\right)\upsilon_t\right) \\ &=\int_0^1 f_x\left(x(t),\mathbb{E}\Psi\left(x(t)^{\lambda,\theta}\right),u^{\lambda,\theta}(t)\right)y^\theta(t)d\lambda \\ &+\int_0^1 f_y\left(x^{\lambda,\theta}(t),\mathbb{E}\Psi\left(x^{\lambda,\theta}(t)\right),u^{\lambda,\theta}(t)\right)\mathbb{E}\left(\Psi_x\left(x^{\lambda,\theta}(t)\right)y^\theta(t)\right)d\lambda \\ &+\int_0^1\left(f_x\left(x^{\lambda,\theta}(t),\mathbb{E}\Psi\left(x^{\lambda,\theta}(t)\right),u^{\lambda,\theta}(t)\right)-\widehat{f}_x\left(t\right)\right)z(t)d\lambda \end{aligned}$$

$$\begin{aligned} &+\int_0^1\left(f_y\left(x^{\lambda,\theta}(t),\mathbb{E}\Psi\left(x^{\lambda,\theta}(t)\right),u^{\lambda,\theta}(t)\right)\mathbb{E}\left(\Psi_x\left(x^{\lambda,\theta}(t)\right)z(t)\right)-\widehat{f}_y\left(t\right)\mathbb{E}\left(\widehat{\Psi}_x\left(t\right)z(t)\right)\right)d\lambda \\ &+\int_0^1\left(f_\upsilon\left(x^{\lambda,\theta}(t),\mathbb{E}\Psi\left(x^{\lambda,\theta}(t)\right),u^{\lambda,\theta}(t)\right)-\widehat{f}_\upsilon\left(t\right)\right)\upsilon_t d\lambda. \end{aligned} \tag{2.8}$$

The three last terms tend to 0 in $L^2(\Omega \times [0,T])$ as $\theta \to 0$. To see this, we rewrite the second to last term above as

$$\begin{aligned}
I_t &:= \int_0^1 \left(f_y\left(x^{\lambda,\theta}(t), \mathbb{E}\Psi\left(x^{\lambda,\theta}(t)\right), u^{\lambda,\theta}(t)\right) - f_y\left(\widehat{x}(t), \mathbb{E}\Psi\left(x^{\lambda,\theta}(t)\right), u^{\lambda,\theta}(t)\right)\right) \mathbb{E}\left(\Psi_x\left(x^{\lambda,\theta}(t)\right) z(t)\right) d\lambda \\
&+ \int_0^1 \left(f_y\left(\widehat{x}(t), \mathbb{E}\Psi\left(x^{\lambda,\theta}(t)\right), u^{\lambda,\theta}(t)\right) - f_y\left(\widehat{x}(t), \mathbb{E}\Psi\left(\widehat{x}(t)\right), u^{\lambda,\theta}(t)\right)\right) \mathbb{E}\left(\Psi_x\left(x^{\lambda,\theta}(t)\right) z(t)\right) d\lambda \\
&+ \int_0^1 \left(f_y\left(\widehat{x}(t), \mathbb{E}\Psi\left(\widehat{x}(t)\right), u^{\lambda,\theta}(t)\right) - \widehat{f}_y(t)\right) \mathbb{E}\left(\Psi_x\left(x^{\lambda,\theta}(t)\right) z(t)\right) d\lambda \\
&+ \widehat{f}_y(t) \int_0^1 \left(\mathbb{E}\left(\Psi_x\left(x^{\lambda,\theta}(t)\right) z(t)\right) - \mathbb{E}\left(\widehat{\Psi}_x(t)\, z(t)\right)\right) d\lambda.
\end{aligned}$$

By using the Lipschitz continuity and boundedness of the functions as well as Cauchy-Schwarz inequality, we obtain the following estimate of the $L^2(\Omega \times [0,T])$ norm of the expression above ($K > 0$ is a constant).

$$\begin{aligned}
\mathbb{E}\int_0^T |\, I_t \,|^2 \, dt \le K \Bigg\{ &\left(\int_0^T \int_0^1 \mathbb{E} \,|\, \lambda\theta\left(y^\theta(t) + z(t)\right) |^4 \, d\lambda dt\right)^{\frac{1}{2}} \left(\int_0^T \mathbb{E} \,|\, z(t) \,|^4 \, dt\right)^{\frac{1}{2}} \\
&+ \left(\int_0^T \int_0^1 \mathbb{E} \,|\, \lambda\theta\upsilon_t \,|^4 \, d\lambda dt\right)^{\frac{1}{2}} \left(\int_0^T \mathbb{E} \,|\, z(t) \,|^4 \, dt\right)^{\frac{1}{2}} \Bigg\},
\end{aligned}$$

which converges to 0 as $\theta \to 0$ since the expected values are finite. Similar estimations for the thirdand fifth terms in (2.8) show that these terms also converge to 0 in $L^2(\Omega \times [0,T]))$. Now, rewriting the diffusion part in (3.51) in the same way and using the Burkholder-Davis-Gundy inequality, we have bythe boundedness of the functions and Jensen's inequality that

$$\begin{aligned}
\mathbb{E}\left|y^\theta(T)\right|^{*,2} &\le K\left(\int_0^T \mathbb{E}\left|y^\theta(t)\right|^{*,2} dt + \int_0^T \sup_{s\in[0,t]} \left|\mathbb{E}\left(y^\theta(s)\right)\right|^2 dt\right) + \rho^\theta \\
&\le K\int_0^T \mathbb{E}\left|y^\theta(t)\right|^{*,2} dt + \rho^\theta,
\end{aligned}$$

where $K > 0$ is a constant and$\rho^\theta \to 0$ as $\theta \to 0$. Applying Gronwall's lemma gives the result. ■

Lemma 2.2

The Gateaux derivative of the cost functional J is given by

$$\begin{aligned}
\left.\frac{d}{d\theta} J(\widehat{u} + \theta\upsilon)\right|_{\theta=0} &= \mathbb{E}\left(\int_0^T \left(\widehat{l}_x(t)\, z(t) + \widehat{l}_y(t)\, \mathbb{E}\left(\widehat{\varphi}_x(t)\, z(t)\right) + \widehat{l}_\upsilon(t)\, \upsilon_t\right) dt\right) \\
&+ \mathbb{E}\left(\widehat{h}_x(T)\, z_T + \widehat{h}_y(T)\, \mathbb{E}\left(\chi(T)\, z_T\right)\right).
\end{aligned}$$

Proof : By the definition of the derivative of Gateaux, and by using the notation $h(x_T) = h(x_T, \mathbb{E}(\chi(x_T)))$, we have

$$\begin{aligned}
\frac{d}{d\theta}\mathbb{E}\left(h\left(x_T^\theta\right)\right)\Big|_{\theta=0} &= \lim_{\theta\to 0}\mathbb{E}\left(\frac{h\left(x_T^\theta\right) - h\left(\widehat{x}_T\right)}{\theta}\right)\\
&= \lim_{\theta\to 0}\mathbb{E}\int_0^1 h_x\left(\widehat{x}_T + \lambda\left(x_T^\theta - \widehat{x}_T\right)\right)\frac{x_T^\theta - \widehat{x}_T}{\theta}d\lambda\\
&+ \mathbb{E}\int_0^1 h_y\left(\widehat{x}_T + \lambda\left(x_T^\theta - \widehat{x}_T\right)\right)\mathbb{E}\left(\chi_x\left(\widehat{x}_T + \lambda\left(x_T^\theta - \widehat{x}_T\right)\right)\frac{x_T^\theta - \widehat{x}_T}{\theta}\right)d\lambda\\
&= \mathbb{E}\left(\widehat{h}_x(T)\, z_T + \widehat{h}_y(T)\,\mathbb{E}\left(\widehat{\chi}_x(T)\, z_T\right)\right).
\end{aligned}$$

Similarly one can show that

$$\frac{d}{d\theta}\mathbb{E}\left(\int_0^T l\left(x^\theta(t), u^\theta(t)\right)dt\right)\Big|_{\theta=0} = \mathbb{E}\left(\int_0^T\left(\widehat{l}_x(t)\, z(t) + \widehat{l}_y(t)\,\mathbb{E}\left(\widehat{\varphi}_x(t)\, z(t)\right) + \widehat{l}_\upsilon(t)\,\upsilon_t\right)dt\right).$$

From the definitions of the cost function and the perturbed control, we see that this proves the lemma ■

2.2.2 Adjoint equations and duality

In this subsection , we introduce the mean-field type adjoint equations:

$$\begin{cases}
d\widehat{p}(t) = -\left(\widehat{f}_x(t)\,\widehat{p}(t) + \widehat{\sigma}_x(t)\,\widehat{q}(t) + \widehat{l}_x(t)\right)dt + \widehat{q}(t)dB_t\\
\qquad\qquad - \left(\mathbb{E}\left(\widehat{f}_y(t)\,\widehat{p}(t)\right)\widehat{\Psi}_x(t) + \mathbb{E}\left(\widehat{\sigma}_y\widehat{q}(t)\right)\widehat{\Phi}_x(t) + \mathbb{E}\left(\widehat{l}_y(t)\right)\widehat{\varphi}_x(t)\right)dt,\\
\widehat{p}_T = l_x^*(T) + \mathbb{E}\left(l_y^*(T)\right)\widehat{\chi}_x(T).
\end{cases} \tag{2.9}$$

This equations reduces to the standard one, when the coefficients do not depend explicitly on the marginal law of the underlying diffusion. Under the assumptions $(\mathbf{H}_1) - (\mathbf{H}_2)$, this is a linear mean-field backward SDE with bounded coefficients and it follows from [5], Theorem 3.1, that it has a unique $\mathcal{F}_t$-adapted solution $(p; q)$ such that

$$\mathbb{E}\left|\widehat{p}\right|_T^{*,2} + \mathbb{E}\int_0^T\left|\widehat{q}(t)\right|^2 dt < +\infty. \tag{2.10}$$

The duality relations between p and z displayed in the next lemma, follow immediately from integration by parts via Ito's formula.

Lemma 2.3

We have

$$\mathbb{E}\left(\widehat{p}_T z_T\right) = \mathbb{E}\left(\int_0^T \left(\widehat{p}(t)\hat{f}_\upsilon\left(t\right)\upsilon_t - z(t)\widehat{l}_x\left(t\right) - z(t)\mathbb{E}\left(\widehat{l}_y\left(t\right)\right)\widehat{\varphi}_x\left(t\right) + \widehat{q}(t)\widehat{\sigma}_\upsilon\left(t\right)\upsilon_t\right)dt\right).$$

Proof: In view of (2.3) and (2.9), applying Ito's formula to $\widehat{p}_t z(t)$,

$$\begin{aligned}\widehat{p}(t)z(t) = \int_0^T &\left(\widehat{p}(t)\widehat{f}_x\left(t\right)z(t) + \widehat{p}(t)\widehat{f}_y\left(t\right)\mathbb{E}\left(\Psi_x\left(t\right)z(t)\right) + \widehat{p}(t)\widehat{f}_\upsilon\left(t\right)\upsilon_t - z(t)\hat{f}_x\left(t\right)\widehat{p}(t)\right. \\ &- z(t)\mathbb{E}\left(\widehat{f}_y\left(t\right)\widehat{p}(t)\right)\widehat{\Psi}_x\left(t\right) - z(t)\widehat{\sigma}_x\left(t\right)\widehat{q}(t) - z(t)\mathbb{E}\left(\widehat{\sigma}_y\left(t\right)\widehat{q}(t)\right)\widehat{\Phi}_x\left(t\right) - z(t)\widehat{l}_x\left(t\right) \\ &\left.-z(t)\mathbb{E}\left(\widehat{l}_y\left(t\right)\right)\widehat{\varphi}_x\left(t\right) + \widehat{q}(t)\widehat{\sigma}_x\left(t\right)z(t) + \widehat{q}(t)\widehat{\sigma}_y\left(T\right)\mathbb{E}\left(\widehat{\Phi}_t\left(t\right)z(t)\right) + \widehat{q}(t)\widehat{\sigma}_\upsilon\left(t\right)\upsilon_t\right)dt + M_t,\end{aligned}$$

where M_t is a zero-mean martingale. By taking expectations we are left with

$$\mathbb{E}\left(\widehat{p}_T z_T\right) = \mathbb{E}\int_0^T \left(\widehat{p}(t)\widehat{f}_\upsilon\left(t\right)\upsilon_t - z(t)\widehat{h}_x\left(t\right) - z(t)\mathbb{E}\left(\widehat{h}_y\left(t\right)\right)\widehat{\varphi}_x\left(t\right) + \widehat{q}(t)\widehat{\sigma}_\upsilon\left(t\right)\upsilon_t\right)dt,$$

■

Let us introduce the Hamiltonian :

$$\begin{aligned}H\left(t,x,\mu,u,p,q\right) = l\left(t,x,\int\varphi d\mu,\mu\right) + f\left(t,x,\int\Psi d\mu,u\right)p \\ +\,\sigma\left(t,x,\int\Phi d\mu,u\right)q.\end{aligned}$$

To ease the notation, whenever x is a random variable whose probability law is μ, we use the following notation for the Hamiltonian.

$$\begin{aligned}H\left(t,x,\mu,u,p,q\right) := l\left(t,x,\mathbb{E}\left(\varphi\left(x\right)\right),u\right) + f\left(t,x,\mathbb{E}\left(\Psi\left(x\right)\right),u\right)p \\ +\,\sigma\left(t,x,\mathbb{E}\left(\Phi\left(x\right)\right),u\right)q.\end{aligned}$$

By combining *Lemma 2.3* with *Lemma 2.2* and by observing that

$$\mathbb{E}\left(\widehat{p}_T z_T\right) = \mathbb{E}\left(\hat{l}_x\left(T\right)z_T + \hat{l}_y\left(T\right)\mathbb{E}\left(\widehat{\chi}_x\left(T\right)z_T\right)\right).$$

We obtain the following result.

Corollary 2.1

The Gateaux derivative of the cost functional can be expressed in terms of the Hamiltonian H in the following way

$$\begin{aligned}\frac{d}{d\theta}J\left(\widehat{u}+\theta\upsilon\right)\Big|_{\theta=0} &= \mathbb{E}\left(\int_0^T\left(\widehat{l}_\upsilon\left(t\right)\upsilon_t+\widehat{p}(t)\widehat{f}_\upsilon\left(t\right)\upsilon_t+\widehat{q}(t)\widehat{\sigma}_\upsilon\left(t\right)\upsilon_t\right)dt\right)\\ &= \mathbb{E}\left(\int_0^T\frac{d}{d\upsilon}H\left(t,\widehat{x}(t),\widehat{u}(t),\widehat{p}(t),\widehat{q}(t)\right)\upsilon_t dt\right).\end{aligned}$$

2.2.3 Maximum principle for stochastic optimal control

Since U is convex, we may choose the following convex perturbation

$$u^\theta(t)=u^*(t)+\theta\left(\upsilon_t-u^*(t)\right)\in U \text{ for } \theta\in[0,1],$$

The control u^θ is called perturbed control. Since the control u^* is optimal, we have the inequality

$$\begin{aligned}&\frac{d}{d\theta}J\left(u^*+\theta\left(\upsilon-u^*\right)\right)\Big|_{\theta=0}\\ &=\mathbb{E}\left(\int_0^T\frac{d}{d\upsilon}H\left(t,\widehat{x}(t),u^*(t),\widehat{p}(t),\widehat{q}(t)\right)\left(\upsilon_t-u^*(t)\right)dt\right)\geq 0.\end{aligned}$$

As in [3], we can reduce this to

$$\frac{d}{d\upsilon}H\left(t,\widehat{x}(t),u^*(t),\widehat{p}(t),\widehat{q}(t)\right)\left(\upsilon_t-u^*(t)\right)\geq 0,$$

a.e., P-a.s., for all $\upsilon\in U$.

The following theorem is the main result of this section.

Theorem 2.1 (*Stochastic maximum principe*)

Under assumptions $(\mathbf{H}_1)-(\mathbf{H}_2)$, *if* $\widehat{u}(t)$ *is an optimal solution of control problem, then there exists a pair* $(\widehat{p}(t),\widehat{q}(t))$ *of adapted processes which satisfies* (2.9) *and* (2.10), *such that*

$$\frac{d}{d\upsilon}H\left(t,\widehat{x}(t),\widehat{u}(t),\widehat{p}(t),\widehat{q}(t)\right)\left(\upsilon_t-\widehat{u}(t)\right)\geq 0,\qquad \mathbb{P}-a.s,\ \text{for all } t\in[0,T]. \tag{2.11}$$

2.3 Sufficient conditions for optimality

In this section, we state the sufficient optimality conditions, with the same notations as those of the previous section.

Theorem 2.2

Let Assumptions $(\mathbf{H}_1)-(\mathbf{H}_6)$ *hold. Let* $(\widehat{x},\widehat{u})$ *be an admissible pair* $\widehat{u}\in U$, *and such that there exist solutions* $\widehat{p}(t),\widehat{q}(t)$ *to the adjoint equation (2.9). Then, if*

$$H\left(t,\widehat{x}(t),\widehat{u}(t),\widehat{p}(t),\widehat{q}(t)\right)=\inf_{v\in U}H\left(t,\widehat{x}(t),v,\widehat{p}(t),\widehat{q}(t)\right) \qquad \mathbb{P}-p.s,\forall t\in[0,T]\,, \tag{2.12}$$

so $\widehat{u}$ *is an optimal control.*

Proof: We supposons $f\left(t\right)=f\left(t,x(t),\mathbb{E}\left(\Psi\left(x(t)\right)\right),u(t)\right)$ and the same for the other functions. Moreoverwe denote $H\left(t\right)=H\left(t,x(t),u(t),\widehat{p}(t),\widehat{q}(t)\right)$ and $\widehat{H}\left(t\right)=H\left(t,\widehat{x}(t),\widehat{u}(t),\widehat{p}(t),\widehat{q}(t)\right)$. Since h and χ are convex and $h_y\geq 0$ it holds that

$$\begin{aligned}\mathbb{E}\left(\widehat{h}-h\right)&\leq\mathbb{E}\left(\widehat{h}_x\left(T\right)\left(\widehat{x}_T-x_T\right)+\widehat{h}_y\left(T\right)\mathbb{E}\left(\widehat{\chi}\left(T\right)-\chi\left(T\right)\right)\right)\\&\leq\mathbb{E}\left(\widehat{h}_x\left(T\right)\left(\widehat{x}_T-x_T\right)+\widehat{h}_y\left(T\right)\mathbb{E}\left(\widehat{\chi}_x\left(T\right).\left(\widehat{x}_T-x_T\right)\right)\right)\\&=\mathbb{E}\left(\widehat{p}_T\left(\widehat{x}_T-x_T\right)\right).\end{aligned}$$

By using the formula of integration by part, we obtain by taking the expectations

$$\begin{aligned}&\mathbb{E}\left(\widehat{p}_T\left(\widehat{x}_T-x_T\right)\right)\\&=\mathbb{E}\left(\int_0^T\left(\widehat{x}(t)-x(t)\right)d\widehat{p}(t)+\int_0^T\widehat{p}(t)d\left(\widehat{x}(t)-x(t)\right)+\int_0^T\widehat{q}(t)\left(\widehat{\sigma}\left(t\right)-\sigma\left(t\right)\right)dt\right)\\&=-\mathbb{E}\int_0^T\left(\widehat{x}(t)-x(t)\right)\left(\widehat{f}_x\left(t\right)\widehat{p}(t)+\mathbb{E}\left(\widehat{f}_y\left(t\right)\widehat{p}(t)\right)\right)\widehat{\Psi}_x\left(t\right)+\widehat{\sigma}_x\left(t\right)\widehat{q}(t)\\&\quad+\mathbb{E}\left(\widehat{\sigma}_y\left(t\right)\widehat{q}(t)\right)\widehat{\Phi}_x\left(t\right)+\widehat{l}_x\left(t\right)+\mathbb{E}\left(\widehat{l}_y\left(t\right)\right)\widehat{\varphi}_x\left(t\right)dt\\&\quad+\mathbb{E}\int_0^T\widehat{p}(t)\left(\widehat{f}\left(t\right)-f\left(t\right)\right)dt+\mathbb{E}\int_0^T\widehat{q}(t)\left(\widehat{\sigma}\left(t\right)-\sigma\left(t\right)\right)dt\\&=-\mathbb{E}\int_0^T\left(\widehat{x}(t)-x(t)\right)\left(\widehat{f}_x\left(t\right)\widehat{p}(t)+\mathbb{E}\left(\widehat{f}_y\left(t\right)\widehat{p}(t)\right)\widehat{\Psi}_x\left(t\right)+\widehat{\sigma}_x\left(t\right)\widehat{q}(t)\right.\\&\quad\left.+\mathbb{E}\left(\widehat{\sigma}_y\left(t\right)\widehat{q}(t)\right)\widehat{\Phi}_x\left(t\right)+\widehat{l}_x\left(t\right)+\mathbb{E}\left(\widehat{l}_y\left(t\right)\right)\widehat{\varphi}_x\left(t\right)\right)dt\\&\quad+\mathbb{E}\int_0^T\left(\widehat{H}\left(t\right)-H\left(t\right)\right)dt-\mathbb{E}\int_0^T\left(\widehat{l}\left(t\right)-l\left(t\right)\right)dt,\end{aligned}$$

where, in the last step, we have used the definition of the Hamiltonian H. Next, we differentiate the Hamiltonian and use the convexity of the functions to get for all $t\in$

$[0.T]$, $\mathbb{P}-p.s$,

$$
\begin{aligned}
&\widehat{H}(t)-H(t) \\
&\leq \widehat{H}_x(t)(\widehat{x}(t)-x(t))+\widehat{l}_y(t)\,\mathbb{E}(\widehat{\varphi}(t)-\varphi(t))+\widehat{f}_y(t)\,\mathbb{E}\left(\widehat{\Psi}(t)-\Psi(t)\right)\widehat{p}(t) \\
&+\widehat{\sigma}_y(t)\,\mathbb{E}\left(\widehat{\Phi}(t)-\Phi(t)\right)\widehat{q}(t)+\widehat{H}_u(t)(\widehat{u}(t)-u(t)) \\
&\leq \widehat{H}_x(t)(\widehat{x}(t)-x(t))+\widehat{l}_y(t)\,\mathbb{E}(\widehat{\varphi}_x(t)(\widehat{x}(t)-x(t)))+\widehat{f}_y(t)\,\mathbb{E}\left(\widehat{\Psi}_x(t)(\widehat{x}(t)-x(t))\right)\widehat{p}(t) \\
&+\widehat{\sigma}_y(t)\,\mathbb{E}\left(\widehat{\Phi}_x(t)(\widehat{x}(t)-x(t))\right)\widehat{q}(t)+\widehat{H}_u(t)(\widehat{u}(t)-u(t)) \\
&\leq \widehat{H}_x(t)(\widehat{x}(t)-x(t))+\widehat{l}_y(t)\,\mathbb{E}(\widehat{\varphi}_x(t)(\widehat{x}(t)-x(t)))+\widehat{f}_y(t)\,\mathbb{E}\left(\widehat{\Psi}_x(t)(\widehat{x}(t)-x(t))\right)\widehat{p}(t) \\
&+\widehat{\sigma}_y(t)\,\mathbb{E}\left(\widehat{\Phi}_x(t)(\widehat{x}(t)-x(t))\right)\widehat{q}(t),
\end{aligned}
$$

where in the last step we have used that $\widehat{H}_u(\widehat{u}(t)-u(t)) \leq 0$ due to the minimum condition (2.12). Combining the inequalities above gives us

$$
\begin{aligned}
&J(\widehat{u})-J(u) \\
&=\mathbb{E}\int_0^T\left(\widehat{l}(t)-l(t)\right)dt+\mathbb{E}\left(\widehat{h}(T)-h(T)\right) \\
&\leq \mathbb{E}\int_0^T\left(\widehat{H}(t)-H(t)\right)dt-\mathbb{E}\int_0^T(\widehat{x}(t)-x(t))\left(\widehat{f}_x(t)\,\widehat{p}(t)+\mathbb{E}\left(\widehat{f}_y(t)\,\widehat{p}(t)\right)\widehat{\Psi}_x(t)\right. \\
&\left.+\ \widehat{\sigma}_x(t)\,\widehat{q}(t)+\mathbb{E}(\widehat{\sigma}_y(t)\,\widehat{q}(t))\,\widehat{\Phi}_x(t)+\widehat{l}_x(t)+\mathbb{E}\left(\widehat{l}_y(t)\right)\widehat{\varphi}_x(t)\right)dt \\
&=\mathbb{E}\int_0^T\left(\widehat{H}(t)-H(t)\right)dt-\mathbb{E}\int_0^T(\widehat{x}(t)-x(t))\left(\widehat{H}_x(t)+\mathbb{E}\left(\widehat{f}_y(t)\,\widehat{p}(t)\right)\widehat{\Psi}_x(t)\right. \\
&\left.+\ \mathbb{E}(\widehat{\sigma}_y(t)\,\widehat{q}(t)(t))\,\widehat{\Phi}_x(t)+\mathbb{E}\left(\widehat{l}_y(t)\right)\widehat{\varphi}_x(t)\right)dt \\
&\leq 0,
\end{aligned}
$$

Finaly, we deduce

$$J(\widehat{u})\leq J(u).$$

And thus, the control u^* is optimal. ■

CHAPTER 3

Optimal singular control problem for general McKean-Vlasov differantial equation

3.1 Introduction and brief history

In this chapter, we establish a general necessary optimality conditions for stochastic continuous-singular control of McKean-Vlasov type equations. The coefficients of the state equation depend on the state of the solution process as well as of its probability law and the control variable. The coefficients of the system are nonlinear and depend explicitly on the absolutely continuous component of the control. The control domain under consideration is not assumed to be convex.

The proof of our general maximum principle is based on the first and second-order derivatives with respect to measure in Wasserstein space of probability measures, and by using variational mehtod with some estimations.

Stochastic differential equations of the McKean-Vlasov type are Itô's stochastic differential equations, where the coefficients of the state equation depend on the state of the solution process as well as of its probability law. Optimal control problems for McKean-Vlasov type SDEs have been studied by many authors; see, for example, [11, 82, 74, 31, 48, 37, 49, 13, 15, 64, 75, 20, 70, 50, 23]. A Peng's type necessary conditions in the form of maximum principle for SDEs of mean-field type have proved by Buckdahn et al. [11]. The necessary optimality conditions for SDEs has been established by Wang et al. [82]. Stochastic optimal control of mean-field jump-diffusion systems with delay has been studied by Meng and Shen [74]. The necessary and sufficient conditions for mean-field SDEs governed by Teugels martingales associated to Lévy process have been studied in [31, 48]. The singular optimal control for mean-field SDEs has been investigated by Hafayed [37]. Maximum principle for McKean-Vlasov FBSDEs of mean-field type has been studied by Hafayed et al. [49]. The mean-field maximum principle for SDEs has been established in Buckdahn et al. [15]. A general mathematical modeling approach for high-dimensional systems corresponding to a large number of particles has been introduced by Lazry and Lions [64]. The maximum principle for mean-field stochastic delay differential equations and its application to finance have been investigated in Shen et al. [75]. In 1990, a general maximum principle for optimal stochastic control has been established in a recent work by Peng [60]. Controlled McKean-Vlasov type forward-backward stochastic differential equations (FBSDEs) have been studied by Carmona and Delarue. [20]. Linear quadratic optimal control problem for conditional McKean-Vlasov equation with random coefficients with applications has been investigated by Pham [70]. Singular optimal control problem for general controlled nonlinear SDEs, in which the coefficients depend on the state of the solution process as well as of its law and control has been investigated by Hafayed et al [50]. Infinite horizon optimal control problem for mean-field delay system with semi-Markov modulated jump-diffusion processes has been investigated in Deepa and Muthukumar [23].

Necessary conditions for optimal stochastic singular control have been investigated by many authors, see for instance [31, 48, 1, 3, 4, 8, 17, 22, 25, 26, 24, 52]. Necessary conditions for optimal singular stochastic control systems with variable delay have been studied in Aghayeva and Morali [1]. The first version of maximum principle for singular stochastic control problems was obtained by Cadenillas and Haussmann [17]. Necessary conditions for singular optimal control have been derived by Bahlali and Mezerdi [8]. In Dufour and Miller [22], the authors derived stochastic maximum principle where the singular part has a linear form by using a time transformation. Sufficient conditions for existence of optimal singular control and the connection between the singular control and optimal stopping problems have been studied in Dufour and Miller [25]. Necessary conditions for general optimal singular stochastic control problems have been derived by Dufour and Miller [26]. When first-order necessary condition is singular in some sense, second-order necessary conditions for optimal stochastic control with recursive utilities has been studied by Dong and Meng [24]. A second-order maximum principle for singular optimal controls with recursive utilities of stochastic delay systems has been considered by Hao and Meng [51]. We refer to Haussmann and Suo [52] and the references cited therein for the recent developments of stochastic singular control problems.

In this thesis, we establish a general Peng's type necessary conditions for McKean-Vlasov optimal continuous-singular control problem (3.14)-(3.15). The derivative with respect to probability measures in *Wasserstein space* and the associate Itô formula are applied to derive our results. Noting that the McKean-Vlasov dynamics (3.14) occur naturally in the probabilistic analysis of financial optimization problems. So, we have based ourselves on the notion of first and second-order derivative with respect to the probability measure which was introduced by Lions [63], see also, [11, 18]. Our optimal singular control problem is strongly motivated by the recent study of the mean-field games and play an important role in different fields of economics, finance and physics.

The main technical issue to prove the maximum principle for stochastic optimal control without the convexity conditions on either the control domain or the Hamiltonian function, especially in the case where the law of the state and the control variable enters the diffusion coefficient is the need to consider a second-order variational equation, or equivalently, a second-order Taylor expansion (see Peng [60]]), which naturally involves the second-order derivatives of all spatial variables in the coefficients.

3.2 Novelty in this work

Our work distinguishes itself from the above ones in the following aspects.

1. First, we study the more general controlled nonlinear McKean-Vlasov type system, where the coefficients of the equation depend on the state of the solution process $X^{u,\eta}$ as well as of its probability measures $P_{X^{u,\eta}(t)}$.

2. Second, we apply the first and second-order derivatives with respect to probability measures to establish our Peng's type necessary optimality conditions.

3. Third, we study the general continuous-singular control problem, where the control domain is not assumed to be convex.

4. Forth, the second-order derivative with respect to probability measures in *Wasserstein space* is applied to establish our result without convexity conditions.

5. Our McKean-Vlasov control problem occur naturally in the probabilistic analysis of financial optimization problems. Moreover, the above mathematical McKean-Vlasov approaches play an important role in different fields of economics, finance, physics, chemistry and game theory.

The main purpose of this chapter is to prove the general McKean-Vlasov necessary conditions of the optimal continuous-singular control without the convexity assumption. Finally, we extend the maximum principle of Buckdahn et al., [11] to singular control problems.

3.3 Differentiability with respect to measure

We now recall briefly an important notion in McKean-Vlasov control problems: the differentiability with respect to probability measures, in *Wasserstein space* which was introduced by Lions [63]. The main idea is to identify a distribution $\mu \in Q_2(\mathbb{R}^n)$ with a random variable $X \in \mathbb{L}^2(\mathcal{F}, \mathbb{R}^n)$ so that $\mu = P_X$. We assume that probability space $(\Omega, \mathcal{F}, P)$ is *rich-enough* in the sense that for every $\mu \in Q_2(\mathbb{R}^n)$, there is a random variable $X \in \mathbb{L}^2(\mathcal{F}, \mathbb{R}^n)$ such that $\mu = P_X$. We suppose that there is a sub-$\sigma-$field $\mathcal{F}_0 \subset \mathcal{F}$ such that $\mathcal{F}_0$ is *rich-enough i.e,*

$$Q_2(\mathbb{R}^n) \triangleq \left\{ P_X : X \in \mathbb{L}^2(\mathcal{F}_0, \mathbb{R}^n) \right\}. \tag{3.1}$$

By $\mathcal{F} = (\mathcal{F}_t)_{t\in[0,T]}$ we denote the filtration generated by $B(\cdot)$, completed and augmented by $\mathcal{F}_0$.

Next, for any function $g : Q_2(\mathbb{R}^n) \to \mathbb{R}$ we define a function $\widetilde{g} : \mathbb{L}^2(\mathcal{F}, \mathbb{R}^n) \to \mathbb{R}$ such that

$$\widetilde{g}(X) = g(P_X), \; X \in \mathbb{L}^2(\mathcal{F}, \mathbb{R}^n). \tag{3.2}$$

Clearly, the function $\widetilde{g}$, called the *lift* of g, depends only on the law of $X \in \mathbb{L}^2(\mathcal{F}, \mathbb{R}^n)$ and is independent of the choice of the representative X. (see [11])

Definition 3.1

A function $g : Q_2(\mathbb{R}^n) \to \mathbb{R}$ is said to be differentiable at a distribution $\mu_0 \in Q_2(\mathbb{R}^n)$ if there exists $X_0 \in \mathbb{L}^2(\mathcal{F}, \mathbb{R}^n)$, with $\mu_0 = P_{X_0}$ such that its lift $\widetilde{g}$ is Fréchet-differentiable *at X_0. More precisely, there exists a continuous linear functional $\mathcal{D}\widetilde{g}(X_0) : \mathbb{L}^2(\mathcal{F}, \mathbb{R}^n) \to \mathbb{R}$ such that*

$$\begin{aligned} \widetilde{g}(X_0 + \zeta) - \widetilde{g}(X_0) &= \langle \mathcal{D}\widetilde{g}(X_0) \cdot \zeta \rangle + o(\|\zeta\|_2) \\ &= \mathcal{D}_\zeta g(\mu_0) + o(\|\zeta\|_2), \end{aligned} \tag{3.3}$$

where $\langle . \cdot . \rangle$ is the dual product on $\mathbb{L}^2(\mathcal{F}, \mathbb{R}^n)$. We called $\mathcal{D}_\zeta g(\mu_0)$ the Fréchet-derivative *of g at μ_0 in the direction ξ. In this case we have*

$$\mathcal{D}_\zeta g(\mu_0) = \langle \mathcal{D}\widetilde{g}(X_0) \cdot \zeta \rangle = \frac{\mathrm{d}}{\mathrm{d}t} \widetilde{g}(X_0 + t\zeta)\Big|_{t=0}, \text{ with } \mu_0 = P_{X_0}. \tag{3.4}$$

By applying Riesz representation theorem, *there is a unique random variable $\Theta_0 \in \mathbb{L}^2(\mathcal{F}, \mathbb{R}^n)$ such that $\langle \mathcal{D}\widetilde{g}(X_0) \cdot \zeta \rangle = (\Theta_0 \cdot \zeta)_2 = \mathbb{E}[(\Theta_0 \cdot \zeta)_2]$ where $\zeta \in \mathbb{L}^2(\mathcal{F}, \mathbb{R}^n)$. It was shown (see [11], [19]) that there exists a Boral function $\Phi[\mu_0](\cdot) : \mathbb{R}^n \to \mathbb{R}^n$, depending only on the law $\mu_0 = P_{X_0}$ but not on the particular choice of the representative X_0 such that*

$$\Theta_0 = \Phi[\mu_0](X_0). \tag{3.5}$$

Thus we can write (3.3) as

$$\begin{aligned} g(P_X) - g(P_{X_0}) = (\Phi[\mu_0](X_0) \cdot X - X_0)_2 + o(\|X - X_0\|_2), \\ \forall X \in \mathbb{L}^2(\mathcal{F}, \mathbb{R}^n). \end{aligned}$$

We denote

$$\partial_\mu g(P_{X_0}, x) = \Phi[\mu_0](x), \; x \in \mathbb{R}^n.$$

Moreover, we have the following identities

$$\mathcal{D}\widetilde{g}(X_0) = \Theta_0 = \Phi[\mu_0](X_0) = \partial_\mu g(P_{X_0}, X_0), \tag{3.6}$$

and

$$\mathcal{D}_\xi g(P_{X_0}) = \langle \partial_\mu g(P_{X_0}, X_0) \cdot \zeta \rangle, \tag{3.7}$$

where $\zeta = X - X_0$.

Remark 3.1

For each $\mu \in Q_2(\mathbb{R}^n)$, $\partial_\mu g(P_X, \cdot) = \Phi[P_X](\cdot)$ is only defined in a $P_X(\mathrm{d}x) - a.e$ sense where $\mu = P_X$.

Among the different notions of differentiability of a function g defined over $Q_2(\mathbb{R}^n)$, we apply for our control problem that introduced by *Lions* [63] and revised in the notes by Cardaliaguet [18], we refer the reader to Buckdahn et al.,[11] and Carmona and Delarue [19].

Definition 3.2

We say that the function $g \in \mathbb{C}_b^{1,1}(Q_2(\mathbb{R}^n))$ if for all $X \in \mathbb{L}^2(\mathcal{F}, \mathbb{R}^n)$ there exists a P_X-modification of $\partial_\mu g(P_X, \cdot)$ (denoted by $\partial_\mu g$) such that $\partial_\mu g : Q_2(\mathbb{R}^n) \times \mathbb{R}^n \to \mathbb{R}^n$ is bounded and Lipschitz continuous. That is for some $C > 0$, it holds that

1. $|\partial_\mu g(\mu, x)| \leq C, \forall \mu \in Q_2(\mathbb{R}^n), \forall x \in \mathbb{R}^n.$

2. *The derivatives $\partial_\mu g$ satisfied the following*

$$|\partial_\mu g(\mu, x) - \partial_\mu g(\mu', x')| \leq C\left[\mathbb{T}(\mu, \mu') + |x - x'|\right],$$
$$\forall \mu, \mu' \in Q_2(\mathbb{R}^n), \forall x, x' \in \mathbb{R}^n.$$

Noting that if $g \in \mathbb{C}_b^{1,1}(Q_2(\mathbb{R}^n))$ the version of $\partial_\mu g(P_X, \cdot)$, $X \in \mathbb{L}^2(\mathcal{F}, \mathbb{R}^n)$ indicate in *Definition 3.2* is unique (see [11, *Remark*2.2], and [18]). We shall denote by $\partial_\mu g(t, x, \mu_0)$ the derivative with respect to μ computed at μ_0 whenever all the other variables (t, x) are held fixed.

Second-order derivatives with respected to probability law:

We present a second order derivatives with respected to measure of probability.

Let $g \in \mathbb{C}_b^{1,1}(Q_2(\mathbb{R}^n))$ and consider the mapping $\left(\partial_\mu g(\cdot,\cdot)_1, \partial_\mu g(\cdot,\cdot)_2, ..., \partial_\mu g(\cdot,\cdot)_n\right)^\top :$ $Q_2(\mathbb{R}^n) \times \mathbb{R}^n \to \mathbb{R}^n$.

Definition 3.3

We say that the function $g \in \mathbb{C}_b^{2,1}(Q_2(\mathbb{R}^n))$ *if* $g \in \mathbb{C}_b^{1,1}(Q_2(\mathbb{R}^n))$ *such that* $\partial_\mu g(\cdot, x) : Q_2(\mathbb{R}^n) \to \mathbb{R}^n$

1. $\partial_\mu g(\cdot, y) \in \mathbb{C}_b^{1,1}(Q_2(\mathbb{R}^n))$, $\forall y \in \mathbb{R}^n$ *and* $i \in \{1, 2, ..., n\}$.
2. $\partial_\mu g(\mu, \cdot) : \mathbb{R}^n \to \mathbb{R}^n$ *is differentiable, for evry* $\mu \in Q_2(\mathbb{R}^n)$.
3. *The mapps* $\partial_x \partial_\mu g(\cdot, \cdot) : Q_2(\mathbb{R}^n) \times \mathbb{R}^n \to \mathbb{R}^n \otimes \mathbb{R}^n$ *and* $\partial_\mu^2 g(P_{X_0}, y, Z) : Q_2(\mathbb{R}^n) \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \otimes \mathbb{R}^n$ *are bounded and Lipshitz continuous, where*

$$\partial_\mu^2 g(P_{X_0}, y, Z) = \partial_\mu \left[\partial_\mu g(\cdot, y)\right](P_{X_0}, Z).$$

3.3.1 Second-order Taylor expansion

Now, we give a second-order Taylor expansion that plays an essential role to establish our maximum principle.

Let $g \in \mathbb{C}_b^{2,1}(Q_2(\mathbb{R}^n))$, for $j \in \{1, 2, ..., n\}$, we obtain

$$\begin{aligned}
\mathcal{D}\widetilde{g_j}(X_0) - \mathcal{D}\widetilde{g_j}(X_0 - \xi) &= \left[\partial_\mu g\right]_j (P_{X_0}, X_0)) - \left[\partial_\mu g\right]_j (P_{X_0-\xi}, X_0 - \xi)) \\
&= \left[\partial_\mu g\right]_j (P_{X_0}, X_0)) - \left[\partial_\mu g\right]_j (P_{X_0-\xi}, Z))\Big|_{Z=X_0-\xi} \\
&+ \left[\partial_\mu g\right]_j (P_{X_0}, Z)\Big|_{Z=X_0} - \left[\partial_\mu g\right]_j (P_{X_0}, Z)\Big|_{Z=X_0-\xi} \\
&= \int_0^1 \left\langle \mathcal{D}\widetilde{[\partial_\mu g]_j} (X_0 + \theta\xi, Z) \cdot \xi \right\rangle d\theta\Big|_{Z=X_0} \\
&+ \left(\partial_x \left[\partial_\mu g\right]_j (P_{X_0}, X_0), \xi\right) + o\left(\|\xi\|_2\right).
\end{aligned} \tag{3.8}$$

then, we obtain

$$\begin{aligned}
\mathcal{D}\widetilde{[\partial_\mu g]_j} (X_0, y) &= \partial_\mu \left[\left[\partial_\mu g\right]_j (\cdot, y)\right] (P_{X_0}, X_0) \\
&= \left[\partial_\mu^2 g\right]_j (P_{X_0}, y, Z)\Big|_{Z=X_0}.
\end{aligned}$$

Second-order derivatives of f *at a measure* μ_0. Let $(\widehat{\Omega}, \widehat{\mathcal{F}}, \widehat{P})$ be a copy of the probability space $(\Omega, \mathcal{F}, P)$. For any pair of random variable $(Z, \xi) \in \mathbb{L}^2(\mathcal{F}, \mathbb{R}^d) \times \mathbb{L}^2(\mathcal{F}, \mathbb{R}^d)$, we let $(\widehat{Z},\widehat{\xi})$ be an independent copy of (Z,ξ) defined on $(\widehat{\Omega},\widehat{\mathcal{F}},\widehat{P})$. We consider the product probability space $(\Omega\times\widehat{\Omega},\mathcal{F}\otimes\widehat{\mathcal{F}},P\otimes\widehat{P})$ and setting $(\widehat{Z},\widehat{\xi})(w,\widehat{w})=(Z(\widehat{w}),\xi(\widehat{w}))$ for any $(w,\widehat{w})\in\Omega\times\widehat{\Omega}$.

Let $(\widehat{u^*}(t),\widehat{X^*}(t))$ is an independent copy of $(u^*(t),X^*(t))$, so that $P_{X^*(t)}=\widehat{P}_{\widehat{X^*}(t)}$. We denote by $\widehat{\mathbb{E}}$ the expectation under probability measure $\widehat{P}$.

Remark 3.2

The expectation $\widehat{\mathbb{E}}(\cdot)$ acts only on random variables marked with a "$\widehat{...}$", where

$$\widehat{\mathbb{E}}(X)=\int_{\widehat{\Omega}}X(\widehat{w})\mathrm{d}\widehat{P}(\widehat{w}).$$

Now, for any $\mu_0\in Q_2(\mathbb{R}^n)$, in the direction ξ, we define the second-order derivatives of a function g at μ_0 with $\mu_0=P_{X_0}$

$$\begin{aligned}\mathcal{D}^2_{\xi}g(\mu_0)&=\left\langle\left\langle\widetilde{\mathcal{D}[\partial_\mu g]_j}(\cdot,y)(P_{X_0},Z)\mid_{Z=\widehat{X_0}}\cdot\widehat{\xi}\right\rangle\mid_{y=\widehat{X_0}},\xi\right\rangle\\&+\langle(\partial_y\partial_\mu g)(P_{X_0},X_0)\xi\cdot\xi\rangle,\\&=\mathbb{E}\left[\widehat{\mathbb{E}}\left[tr\left(\partial^2_\mu g(P_{X_0},X_0,\widehat{X_0})\widehat{\xi}\otimes\xi\right)\right]\right]\\&+\mathbb{E}\left[tr\left(\partial_y\partial_\mu g(P_{X_0},X_0)\xi\otimes\xi\right)\right],\end{aligned}\tag{3.9}$$

where

$$\begin{aligned}&\widehat{\mathbb{E}}\left[tr\left(\partial^2_\mu g(P_{X_0},X_0,\widehat{X_0})\widehat{\xi}\otimes\xi\right)\right]\\&=\int_{\widehat{\Omega}}tr\left[\partial^2_\mu g(P_{X_0},X_0(w),\widehat{X_0}(\widehat{w}))\widehat{\xi}\otimes\xi(w,\widehat{w})\right]\mathrm{d}\widehat{P}(\widehat{w}).\end{aligned}\tag{3.10}$$

and

$$\begin{aligned}&\mathbb{E}\left[\widehat{\mathbb{E}}\left[tr\left[\partial^2_\mu g(P_{X_0},X_0,\widehat{X_0})\widehat{\xi}\otimes\xi\right]\right]\right]\\&=\int_{\Omega}\int_{\widehat{\Omega}}tr\left[\partial^2_\mu g(P_{X_0},X_0(w),\widehat{X_0}(\widehat{w}))\widehat{\xi}\otimes\xi(w,\widehat{w})\right]\mathrm{d}(P\otimes\widehat{P})(w,\widehat{w}).\end{aligned}\tag{3.11}$$

For convenience, we will use the following notations throughout the chapter, for $\varphi=f,\sigma,\ell,h$:

$$\begin{aligned}\delta\varphi(t)&=\varphi(t,X^*(t),P_{X^*(t)},u^*(t))-\varphi(t,X^*(t),P_{X^*(t)},u(t));\\\varphi_x(t)&=\frac{\partial\varphi}{\partial x}(t,X^*(t),P_{X^*(t)},u^*(t));\\\widehat{\varphi}_\mu(t)&=\partial_\mu\varphi(t,X^*(t),P_{X^*(t)},u^*(t);\widehat{X^*}(t)),\\\widehat{\varphi}^*_\mu(t)&=\partial_\mu\varphi(t,\widehat{X^*}(t),P_{X^*(t)},\widehat{u^*}(t);X^*(t)),\end{aligned}\tag{3.12}$$

and similarly, we denote the second derivative processes:

$$\begin{aligned}\varphi_{xx}(t) &= \frac{\partial^2\varphi}{\partial x^2}(t, X^*(t), P_{X^*(t)}, u^*(t)),\\ \widehat{\varphi}_{\mu\mu}(t) &= \partial^2_\mu\varphi(t, X^*(t), P_{X^*(t)}, u^*(t); X^*(t), \widehat{X^*}(t)),\\ \varphi_{x\mu}(t) &= \partial_x\partial_\mu\varphi(t, X^*(t), P_{X^*(t)}, u^*(t); X^*(t)),\\ \widehat{\varphi}^*_{x\mu}(t) &= \partial_x\partial_\mu\varphi(t, \widehat{X^*}(t), P_{X^*(t)}, \widehat{u^*}(t); \widehat{X^*}(t)).\end{aligned} \tag{3.13}$$

3.4 Formulation of the continuous-singular control problem

Let us formulate the optimal continuous-singular control problem. Let T be a fixed strictly positive real number and $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in[s,T]}, P)$ be a fixed filtered probability space satisfying the usual conditions in which *one*−dimensional Brownian motion $B(t) = \{B(t) : 0 \leq t \leq T\}$ and $B(0) = 0$ is defined. We study general stochastic continuous-singular control problem driven by stochastic differential equation of McKean-Vlasov type of the form:

$$\begin{cases} dX^{u,\eta}(t) = f\left(t, X^{u,\eta}(t), P_{X^{u,\eta}(t)}, u(t)\right)dt + \sigma\left(t, X^{u,\eta}(t), P_{X^{u,\eta}(t)}, u(t)\right)\mathrm{d}B(t)\\ \qquad\qquad + G(t)\mathrm{d}\eta(t),\\ X^{u,\eta}(0) = x_0,\end{cases} \tag{3.14}$$

The criteria to be minimized over the class of admissible controls has the form

$$J(u(\cdot), \eta(\cdot)) = \mathbb{E}\left[\int_0^T l(t, X^{u,\eta}(t), P_{X^{u,\eta}(t)}, u(t))\mathrm{d}t + h(X^{u,\eta}(T), P_{X^{u,\eta}(t)}) + \int_{[0,T]} M(t)\mathrm{d}\eta(t)\right]. \tag{3.15}$$

We consider the following sets:

- $\mathbb{U}_1$: is a a non empty subset of $\mathbb{R}^n$,
- $\mathbb{U}_2 = ([0, +\infty))^m$.

- $\mathcal{U}_1$ the class of measurable, adapted processes $u(\cdot) : [0,T] \times \Omega \to \mathbb{U}_1$.
- $\mathcal{U}_2$ the class of measurable, adapted processes $\eta(\cdot) : [0,T] \times \Omega \to \mathbb{U}_2$ such that $\eta(\cdot)$ is of bounded variation, nondecreasing continuous on the left with right limits and $\eta(0) = 0$.

Since we are interested in continuous-singular stochastic control, we give here the precise definition of an admissible continuous-singular control. (see [52])

Definition 3.4

An admissible continuous-singular control is a pair $(u(\cdot), \eta(\cdot))$ of measurable $\mathbb{U}_1 \times \mathbb{U}_2$-valued, $\mathcal{F}_t-$adapted processes, such that

(1) $\eta(\cdot)$ *is of bounded variation process, nondecreasing, continuous on the left with right limits and* $\eta(0) = 0$.

(2) $\mathbb{E}\left[\sup_{t\in[0,T]} |u(t)|^2 + |\eta(T)|^2\right] < \infty.$

In this chapter, we denote by $\mathcal{U}_1 \times \mathcal{U}_2([0,T])$, the set of all admissible continuous-singular controls. We note that since $\mathrm{d}\eta(t)$ may be singular with respect to Lebesgue measure $\mathrm{d}t$, we call $\eta(\cdot)$ the singular part of the control and the process $u(\cdot)$ its absolutely continuous part. This construction allows us to define integrals of the form

$$\int_{[0,T]} G(t)\mathrm{d}\eta(t) \text{ and } \int_{[0,T]} M(t)\mathrm{d}\eta(t),$$

where $\int_{[0,T]} G(t)\mathrm{d}\eta(t) = \int_0^T G(t)\mathrm{d}\eta(t)$ and $\int_{[0,T]} M(t)\mathrm{d}\eta(t) = \int_0^T M(t)\mathrm{d}\eta(t)$.
We remark that the criteria to be minimized (3.15) over the class of admissible controls involves the law of the solution in a nonlinear way. We note that the integral $\int_{[0,T]} M(t)\mathrm{d}\eta(t)$ called the intervention cost.
Any admissible control $(u^*(\cdot), \eta^*(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2([0,T])$ satisfying

$$J(u^*(\cdot), \eta^*(\cdot)) = \inf_{(u(\cdot),\eta(\cdot))\in\mathcal{U}_1\times\mathcal{U}_2([0,T])} J(u(\cdot), \eta(\cdot)), \tag{3.16}$$

is called an optimal continuous-singular control.

The maps

$$f : [0,T] \times \mathbb{R}^n \times Q_2(\mathbb{R}^n) \times \mathbb{U}_1 \rightarrow \mathbb{R}^n$$
$$\sigma : [0,T] \times \mathbb{R}^n \times Q_2(\mathbb{R}^n) \times \mathbb{U}_1 \rightarrow \mathbb{M}^{n\times d}(\mathbb{R})$$
$$\ell : [0,T] \times \mathbb{R}^n \times Q_2(\mathbb{R}^n) \times \mathbb{U}_1 \rightarrow \mathbb{R}$$
$$h : \mathbb{R}^n \times Q_2(\mathbb{R}^n) \longrightarrow \mathbb{R}$$
$$G : [0,T] \rightarrow \mathbb{M}^{n\times m}(\mathbb{R})$$
$$M : [0,T] \rightarrow ([0,+\infty))^m,$$

are given deterministic functions, where $Q_2(\mathbb{R}^n)$ is Wasserstein space of probability measures on $(\mathbb{R}^n, \mathcal{B}(\mathbb{R}^n))$ with finite second-moment, i.e; $\int_{\mathbb{R}^n} |x|^2 \mu(\mathrm{d}x) < \infty$, endowed with the following 2− *Wasserstein metric*: for $\mu_1, \mu_2 \in Q_2(\mathbb{R}^n)$,

$$\mathbb{T}(\mu_1,\mu_2) = \inf\left\{\left[\int_{\mathbb{R}^{2n}} |x-y|^2 \rho(\mathrm{d}x,\mathrm{d}y)\right]^{\frac{1}{2}},\ \rho \in Q_2(\mathbb{R}^{2n}), \rho(\cdot,\mathbb{R}^n) = \mu_1, \rho(\mathbb{R}^n,\cdot) = \mu_2\right\}. \tag{3.17}$$

Remark 3.3

In order not to over complicate the already notational heavy presentation of this chapter, in what follows we shall assume all processes are one-dimensional (i.e., $n = d = m = 1$). We define a metric $d_1(\cdot,\cdot)$ on the space of admissible controls $\mathcal{U}_1([0,T])$ such that $(\mathcal{U}_1([0,T]), d_1)$ becomes a complete metric space. For any $u(\cdot)$ and $v(\cdot) \in \mathcal{U}_1([0,T])$ we set

$$d_1(u(\cdot), v(\cdot)) = P\otimes dt\{(w,t) \in \Omega \times [0,T] : u(w,t) \neq v(w,t)\}, \tag{3.18}$$

where $P\otimes dt$ is the product measure of P with the Lebesgue measure dt on $[0,T]$. Moreover, it has been shown in the book by Yong and Zhou ([89], 146-147) that $(\mathcal{U}_1([0,T]), d_1)$ is a complete metric space.

3.5 Main results

3.5.1 Assumptions

We will always take the following assumptions in this chapter.

Assumption (H1) The coefficients f, σ, ℓ, h are measurable in all variables. Moreover, for all $(u(t), \eta(t)) \in \mathbb{U}_1 \times \mathbb{U}_2$, $f(\cdot,\cdot,u)$, $\sigma(\cdot,\cdot,u)$, $\ell(\cdot,\cdot,u) \in \mathbb{C}_b^{1,1}(\mathbb{R} \times Q_2(\mathbb{R}^d);\mathbb{R})$, $h(\cdot,\cdot) \in \mathbb{C}_b^{1,1}(\mathbb{R} \times Q_2(\mathbb{R}^n);\mathbb{R})$. More precisely, for each $u(t) \in \mathbb{U}_1$, denoting $\varphi(x,\mu) = f(t,x,\mu,u)$, $\sigma(t,x,\mu,u)$, $l(t,x,\mu,u)$, $h(x,\mu)$, the function $\varphi(\cdot,\cdot)$ enjoys the following properties:

(1) For fixed $\mu \in Q_2(\mathbb{R})$, $\varphi(\cdot,\mu)$ continuously differentiable with respect to x;

(2) For fixed $x \in \mathbb{R}$, $\varphi(x,\cdot) \in \mathbb{C}_b^{1,1}(Q_2(\mathbb{R}))$;

(3) All the derivatives $\partial_x\varphi$ and $\partial_\mu\varphi : \varphi = l, \sigma, \ell, h$, are bounded and Lipschitz continuous, with Lipschitz constants independent of $(u(t), \eta(t))$.

Assumption (H2) The coefficients f, σ, ℓ, h satisfy assumption (H1). Furthermore, for all $u(t) \in \mathbb{U}_1$, $f(t,\cdot,\cdot,u)$, $\sigma(t,\cdot,\cdot,u)$, $\ell(t,\cdot,\cdot,u) \in \mathbb{C}_b^{2,1}(\mathbb{R} \times Q_2(\mathbb{R});\mathbb{R})$, $h(\cdot,\cdot) \in \mathbb{C}_b^{2,1}(\mathbb{R} \times Q_2(\mathbb{R});\mathbb{R})$. More precisely, for each $u(t) \in \mathbb{U}_1$, the derivatives of f, σ, ℓ, h, denoted by a generic function $\varphi(t,x,\mu)$, enjoy the following properties:

(1) $\partial_x\varphi(t,\cdot,\cdot) \in \mathbb{C}_b^{1,1}(\mathbb{R} \times Q_2(\mathbb{R}))$;

(2) $\partial_\mu\varphi(t,\cdot,\cdot) \in \mathbb{C}_b^{1,1}(\mathbb{R} \times Q_2(\mathbb{R}) \times \mathbb{R})$;

(3) All the second-order derivatives of f, σ, ℓ, h, are bounded and Lipschitz continuous with Lipschitz constants independent of $(u(t), \eta(t))$.

Assumption (H3) The functions $G(\cdot) : [0,T] \to \mathbb{R}$, and $M(\cdot) : [0,T] \to \mathbb{R}^+$ are continuous and bounded.

Under the assumptions (**H1**)−(**H3**), for each $(u(\cdot), \eta(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2([0,T])$, Eq-(3.14) has unique strong solution $X^{u,\eta}(\cdot)$ given by

$$
\begin{aligned}
X^{u,\eta}(t) = x_0 &+ \int_0^t f(r, X^{u,\eta}(r), P_{X^{u,\eta}(r)}, u(r))\mathrm{d}r + \int_0^t \sigma(r, X^{u,\eta}(r), P_{X^{u,\eta}(r)}, u(r))\mathrm{d}B(r) \\
&+ \int_{[0,t]} G(r)\mathrm{d}\eta(r),
\end{aligned}
$$

such that $\mathbb{E}\left[\sup_{t\in[0,T]} |X^{u,\eta}(t)|^n\right] < C_n$, where C_n is a constant depending only on n and the functional $J(\cdot,\cdot))$ is well defined.

Remark 3.4

As in Peng's maximum principle [60] and Buckdahn et al.'s maximum principle [11], we do not require any differentiability assumptions of the coefficients f, σ, ℓ, h on the control variable $u(\cdot)$. Also, we assume that the control set is a general open set that is not necessary convex.

Let $(u^*(\cdot), \eta^*(\cdot)) \in \mathcal{U}_1 \times \mathcal{U}_2([0,T])$ is an optimal continuous-singular control, the corresponding state process $X^{u^*,\eta^*}(\cdot)$, solution of McKean-Vlasov dynamic (3.14) is denoted by $X^*(\cdot) = X^{u^*,\eta^*}(\cdot)$.

Finally, we define for $t \in [0,T]$:

$$\begin{aligned}\mathcal{L}_{xx}(t,\varphi,z) &= \frac{1}{2}\partial_{xx}\varphi(t, X^*(t), P_{X^*(t)}, u^*(t))z^2, \\ \mathcal{L}_{y\mu}(t,\widehat{\varphi},z) &= \frac{1}{2}\partial_y\partial_\mu\varphi(t, X^*(t), P_{X^*(t)}, u^*(t); \widehat{X^*})z^2.\end{aligned} \tag{3.19}$$

3.5.2 Hamiltonian

Let us define the Hamiltonian associated to our continuous-singular control problem. For any $(t,x,\mu,u,p,q) \in [0,T] \times \mathbb{R} \times Q_2(\mathbb{R}) \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}$

$$H(t,x,\mu,u,p,q) = f(t,x,\mu,u)p + \sigma(t,x,\mu,u)q - \ell(t,x,\mu,u). \tag{3.20}$$

where $(p(\cdot), q(\cdot))$ be a pair of adapted processes, solution of the first-order adjoint equation (3.23). Since the coefficients $G(\cdot)$ and $M(\cdot)$ are independent to $X(\cdot)$, the Hamiltonian functional H is independent to singular control $\eta(\cdot)$.

We denote

$$H(t) = H(t, X^*(t), P_{X^*(t)}, u^*(t), p(t), q(t)). \tag{3.21}$$

We define

$$\begin{aligned}\delta H(t) &= \delta f(t)p_t + \delta\sigma(t)\cdot q_t - \delta\ell(t); \\ H_x(t) &= f_x(t)p(t) + \sigma_x(t)q(t) - \partial_x\ell(t); \\ H_{xx}(t) &= f_{xx}(t)p(t) + \sigma_{xx}(t) \otimes q(t) - \ell_{xx}(t).\end{aligned} \tag{3.22}$$

We introduce the adjoint equations involved in the stochastic maximum principle for our continuous-singular control problem.

3.5.3 Adjoint equation

First-order adjoint equation. We consider the first-order adjoint equation, which is the following McKean-Vlasov linear BSDE:

$$\begin{cases} -\mathrm{d}p(t) = \Big[f_x(t)p(t) + \widehat{\mathbb{E}}\left[\widehat{f}_\mu^*(t)(t)\widehat{p}(t)\right] + \sigma_x(t)q(t) + \widehat{\mathbb{E}}\left[\widehat{\sigma}_\mu^*(t)\widehat{q}(t)\right] - \ell_x(t) \\ \qquad\qquad -\widehat{\mathbb{E}}\left[\widehat{\ell}_\mu^*(t)(t)\right]\Big]\,\mathrm{d}t - q(t)\mathrm{d}B(t), \\ p(T) = h_x(T) + \widehat{\mathbb{E}}[\widehat{h}_\mu^*(T)]. \end{cases} \tag{3.23}$$

Here, from (3.13), $t \in [0,T]$, for $\varphi = f, \sigma, \ell$, we obtain

$$\begin{aligned} \widehat{\mathbb{E}}\left[\partial_\mu \widehat{\varphi^*}(t)\right] &= \widehat{\mathbb{E}}\left[\partial_\mu \varphi(t, \widehat{X}(t), P_{X^*(t)}, \widehat{u}^*(t); z)\right]\Big|_{z=X^*(t)} \\ &= \int_{\widehat{\Omega}} \partial_\mu \varphi(t, \widehat{X}(t,\widehat{w}), P_{X^*(t,w)}, \widehat{u}^*(t,\widehat{w}); X^*(t,w))\mathrm{d}\widehat{P}(\widehat{w}), \end{aligned} \tag{3.24}$$

and the same argument allows to show that

$$\begin{aligned} \widehat{\mathbb{E}}\left[\partial_\mu \widehat{h}^*(T)\right] &= \widehat{\mathbb{E}}\left[\partial_\mu h(\widehat{X}(T), P_{X^*(T)}; z)\right]\Big|_{z=X^*(t)} \\ &= \int_{\widehat{\Omega}} \partial_\mu h(\widehat{X}(T,\widehat{w}), P_{X(T,w)}; X^*(T,w))\mathrm{d}\widehat{P}(\widehat{w}). \end{aligned} \tag{3.25}$$

Second-order adjoint equation. Consider the following standard linear BSDE

$$\begin{cases} \mathrm{d}P(t) = -\Big\{2(f_x(t) + \widehat{\mathbb{E}}[\widehat{f}_\mu^*(t)])P(t) + [\sigma_x(t) + \widehat{\mathbb{E}}(\widehat{\sigma}_\mu^*(t))]^2 P(t) \\ \qquad\qquad + 2(\sigma_x(t) + \widehat{\mathbb{E}}[\widehat{\sigma}_\mu^*(t)])Q(t) + (H_{xx}(t) + \widehat{\mathbb{E}}[\widehat{H}_{\mu y}^*(t)])\Big\}\,\mathrm{d}t + Q(t)\mathrm{d}B(t), \\ P(T) = -(h_{xx}(T) + \widehat{\mathbb{E}}[\widehat{h}_{\mu y}^*(T)]). \end{cases} \tag{3.26}$$

Similar to (3.24) and (3.25), we have

$$\begin{aligned} \widehat{\mathbb{E}}[\widehat{H}_{\mu y}^*(t)]) &= \widehat{\mathbb{E}}\left[\partial_\mu \partial_y H(t, \widehat{X}(t), P_{X^*(t)}, \widehat{u}^*(t), \widehat{p}(t), \widehat{q}(t); y)\right]\Big|_{y=X^*(t)} \\ &= \int_{\widehat{\Omega}} \partial_\mu \partial_y H(t, \widehat{X}(t,\widehat{w}), P_{X^*(t)}, \widehat{u}^*(t,\widehat{w}), \widehat{p}(t), \widehat{q}(t); X^*(t))\mathrm{d}\widehat{P}(\widehat{w}). \end{aligned}$$

We have the following remarks:

Remark 3.5

1. *Since the coefficients* $G(\cdot)$ *and* $M(\cdot)$ *are independent to* $X(\cdot)$, *the adjoint process* $(p(\cdot), q(\cdot))$ *and* $(P(\cdot), Q(\cdot))$ *are independent to singular control* $\eta(\cdot)$, *and it is readily seen that the adjoint equations* (3.23), (3.26) *coincides with the results in* [11].

2. *If the coefficients* f, σ, ℓ, h *do not explicitly depend on law of the solution, the McKean-Vlasov BSDE-*(3.23) *and* (3.26) *reduce to a standard BSDE (see Peng* [60, $Equation 19, page 974$]*), or Buckdahn et al.,* [11]*).*

3. *Since the derivatives* $f_x, f_\mu, \sigma_x, \sigma_\mu, \ell_x, \ell_\mu, h_x, h_\mu$ *are bounded, by assumption H1-*(3)*, the McKean-Vlasov BSDE* (3.23) *admits a unique* $\mathcal{F}_t$*-adapted solution* $(p(\cdot), q(\cdot))$ *which satisfies the following estimate*

$$\mathbb{E}\left[\sup_{t\in[0,T]} |p(t)|^2 + \int_0^T |q(t)|^2 \, dt\right] < +\infty. \tag{3.27}$$

4. *From the boundedness of the first and second derivatives of the coefficients* f, σ, ℓ, h *with respect to*(x, μ), *(see Assumption H2), the linear BSDE-*(3.26) *has a unique* $\mathcal{F}_t-$*adapted solution* $(P(\cdot), Q(\cdot))$ *such that*

$$\mathbb{E}\left[\sup_{t\in[0,T]} |P(t)|^2 + \int_0^T |Q(t)|^2 \, dt\right] < +\infty. \tag{3.28}$$

3.5.4 Necessary conditions of optimal singular control

The purpose of the stochastic maximum principle is to establish necessary conditions for optimality satisfied by an optimal control. In this section, we establish a set of general Peng's type necessary conditions for the optimal continuous-singular control, where the system evolves according to controlled McKean-Vlasov SDEs.

Our result is proved by applying *spike variation method* for continuous parts of the control and *convex perturbation technique* for singular parts.

Let $(u^*(\cdot), \eta^*(\cdot), X^*(\cdot))$ is an optimal solution of the McKean-Vlasov control problem

(3.14)-(3.15). We introduce the following variational equations for our continuous-singular control problem. Let $Y^{u^\varepsilon,\eta^\varepsilon}(\cdot)$ and $Z^\varepsilon(\cdot)$ be the solutions of (3.34), (3.30) associated to $(u^*(\cdot),\eta^*(\cdot))$ respectively.

First-order variational equation: let $E_\varepsilon=[0,\varepsilon]$, $t\in[0,T]$

$$\begin{cases} \mathrm{d}Y^{u^\varepsilon,\eta^\varepsilon}(t)=\left[f_x(t)Y^{u^\varepsilon,\eta^\varepsilon}(t)+\widehat{\mathbb{E}}[\widehat{f}_\mu(t)\widehat{Y}^{u^\varepsilon,\eta^\varepsilon}(t)]+\delta f(t)1_{E_\varepsilon}(t)\right]\mathrm{d}t \\ \qquad +\left[\sigma_x(t)Y^\varepsilon(t)+\widehat{\mathbb{E}}[\widehat{\sigma}_\mu(t)\widehat{Y}^{u^\varepsilon,\eta^\varepsilon}(t)]+\delta\sigma(t)1_{E_\varepsilon}(t)\right]\mathrm{d}B(t) \\ \qquad +G(t)\mathrm{d}(\eta^\varepsilon-\eta^*)(t), \\ Y^{u^\varepsilon,\eta^\varepsilon}(0)=0. \end{cases} \tag{3.29}$$

Here the process $Y^{u^\varepsilon,\eta^\varepsilon}(\cdot)$ is called the *first-order variational process*, associated to $(u^\varepsilon(\cdot),\eta^\varepsilon(\cdot))$ which is depend explicitly to singular control. The process $\eta^\varepsilon(\cdot)$ is the convex perturbed control given by $\eta^\varepsilon(t)=\eta^*(t)+\varepsilon(\eta(t)-\eta^*(t))$.

Second-order variational equation:

$$\begin{cases} \mathrm{d}Z^\varepsilon(t)=\left[f_x(t)Z^\varepsilon(t)+\widehat{\mathbb{E}}[\widehat{f}_\mu(t)\widehat{Z}^\varepsilon(t)]+\mathcal{L}_{xx}(t,f,Y^\varepsilon)+\mathcal{L}_{\mu x}(t,\widehat{f},\widehat{Y}^\varepsilon)\right]\mathrm{d}t \\ \qquad +\left[\sigma_x(t)Z^\varepsilon(t)+\widehat{\mathbb{E}}[\widehat{\sigma}_\mu(t)\widehat{Z}^\varepsilon(t)]+\mathcal{L}_{xx}(t,\sigma,Y^\varepsilon)+\mathcal{L}_{\mu x}(t,\widehat{\sigma},\widehat{Y}^\varepsilon)\right]\mathrm{d}B(t), \\ \qquad +\left[\delta f_x(t)Y^\varepsilon(t)+\widehat{\mathbb{E}}[\delta\widehat{f}_\mu(t)\widehat{Y}^\varepsilon(t)]\right]1_{E_\varepsilon}(t)\,\mathrm{d}t \\ \qquad +\left[\delta\sigma_x(t)Y^\varepsilon(t)+\widehat{\mathbb{E}}[\delta\widehat{\sigma}_\mu(t)\widehat{Y}^\varepsilon(t)]\right]1_{E_\varepsilon}(t)\,\mathrm{d}B(t), \\ Z^\varepsilon(0)=0. \end{cases} \tag{3.30}$$

Here the process $Z^\varepsilon(\cdot)$ is called the *second-order variational process.*

To prove our main result, we need the following technical Lemmas.

Lemma 3.1

Let $X^\varepsilon(\cdot)=X^{u^\varepsilon,\eta^\varepsilon}(\cdot)$ be the solutions of (3.14) corresponding to continuous-singular control $(u^\varepsilon(\cdot),\eta^\varepsilon(\cdot))$. Let assumptions $(H1)$ and $(H2)$ hold. Then we have

$$\lim_{\varepsilon\to 0}\mathbb{E}(\sup_{t\in[0,T]}|X^\varepsilon(t)-X^*(t)|^2)=0.$$

Proof : From standard estimates and the *Burkholder-Davis-Gundy inequality,* we obtain

$$\begin{aligned}
&\mathbb{E}(\sup_{t\in[0,T]} |X^{\varepsilon}(s) - X^{*}(s)|^{2}) \\
&\leq \mathbb{E}\int_{0}^{t} \left| f\left(s, X^{\varepsilon}(s), P_{X^{\varepsilon}(s)}, u^{\varepsilon}(s)\right) - f\left(s, X^{*}(s), P_{X^{*}(s)}, u^{*}(s)\right) \right|^{2} \mathrm{d}s \\
&+ \mathbb{E}\int_{0}^{t} \left| \sigma\left(s, X^{\varepsilon}(s), P_{X^{\varepsilon}(s)}, u^{\varepsilon}(s)\right) - \sigma\left(s, X^{*}(s), P_{X^{*}(s)}, u^{*}(s)\right) \right|^{2} \mathrm{d}s \\
&+ \mathbb{E}\left| \int_{[0,t]} G(s)\mathrm{d}\left(\eta^{\varepsilon} - \eta^{*}\right)(s) \right|^{2},
\end{aligned}$$

by applying assumption $(H1)$ and the Lipschitz conditions on the coefficients f, σ with respect to x, μ, with the help of (3.18), $u^{\varepsilon}(t,B) \neq u^{*}(t,B)$ we get

$$\mathbb{E}(\sup_{0\leq t\leq T} |X^{\varepsilon}(t) - X^{*}(t)|^{2}) \leq C_{T}\mathbb{E}\int_{0}^{t} \sup_{\tau\in[0,s]} |X^{\varepsilon}(\tau) - X^{*}(\tau)|^{2} \mathrm{d}s + C_{T}\varepsilon^{2},$$

by applying *Gronwall's Lemma*, the desired result follows immediately by letting ε tends to zero. ■

Lemma 3.2

Let $X^{u^{\varepsilon},\eta^{*}}(\cdot)$ be the solution of (3.14), corresponding to $(u^{\varepsilon}(\cdot), \eta^{*}(\cdot))$. Let $Y^{\varepsilon}(\cdot)$ be the solution of (3.29), corresponding to $(u^{\varepsilon}(\cdot), \eta^{*}(\cdot))$, then the following estimation holds

$$\lim_{\varepsilon\to 0} \mathbb{E}\left[\sup_{0\leq t\leq T} \left|X^{u^{\varepsilon},\eta^{*}}(t) - X^{*}(t)\right|^{2}\right] = 0. \tag{3.31}$$

$$\lim_{\varepsilon\to 0} \mathbb{E}\left[\sup_{0\leq t\leq T} \left|X^{\varepsilon}(t) - X^{u^{\varepsilon},\eta^{*}}(t)\right|^{2}\right] = 0. \tag{3.32}$$

$$\lim_{\varepsilon\to 0} \mathbb{E}\left[\sup_{0\leq t\leq T} \left|X^{u^{\varepsilon},\eta^{*}}(t) - X^{*}(t) - Y^{\varepsilon}(t)\right|^{2}\right] = 0. \tag{3.33}$$

Proof : Let $Y^{\varepsilon}(\cdot) = Y^{u^{\varepsilon},\eta^{*}}(\cdot)$ the first-order adjoint process corresponding to $(u^{\varepsilon}(\cdot), \eta^{*}(\cdot))$ defined by the following SDE:

$$\begin{cases}
\mathrm{d}Y^{\varepsilon}(t) = \left[f_{x}(t)Y^{\varepsilon}(t) + \widehat{\mathbb{E}}[\widehat{f}_{\mu}(t)\widehat{Y}^{\varepsilon}(t)] + \delta f(t)1_{E_{\varepsilon}}(t)\right]\mathrm{d}t \\
\qquad\qquad + \left[\sigma_{x}(t)Y^{\varepsilon}(t) + \widehat{\mathbb{E}}[\widehat{\sigma}_{\mu}(t)\widehat{Y}^{\varepsilon}(t)] + \delta\sigma(t)1_{E_{\varepsilon}}(t)\right]\mathrm{d}B(t) \\
Y^{\varepsilon}(0) = 0.
\end{cases} \tag{3.34}$$

A simple computation shows that

$$\begin{aligned}
&\left[X^{u^{\varepsilon},\eta^{*}}(t)-X^{*}(t)\right]\\
&=\int_{0}^{t}[f(s,X^{u^{\varepsilon},\eta^{*}}(s),P_{X^{u^{\varepsilon},\eta^{*}}(s)},u^{\varepsilon}(s))-f(s,X^{*}(s),P_{X^{*}(s)},u^{*}(s))]\mathrm{d}s\\
&+\int_{0}^{t}[\sigma(s,X^{u^{\varepsilon},\eta^{*}}(s),P_{X^{u^{\varepsilon},\eta^{*}}(s)},u^{\varepsilon}(s))-\sigma\left(s,X^{*}(s),P_{X^{*}(s)},u^{*}(s)\right)]\mathrm{d}B(s),
\end{aligned}$$

and

$$\begin{aligned}
&\left[X^{\varepsilon}(t)-X^{u^{\varepsilon},\eta^{*}}(t)\right]\\
&=\int_{0}^{t}[f(s,X^{\varepsilon}(s),P_{X^{\varepsilon}(s)},u^{\varepsilon}(s))-f(s,X^{u^{\varepsilon},\eta^{*}}(s),P_{X^{u^{\varepsilon},\eta^{*}}(s)},u^{\varepsilon}(s))]\mathrm{d}s\\
&+\int_{0}^{t}[\sigma(s,X^{\varepsilon}(s),P_{X^{\varepsilon}(s)},u^{\varepsilon}(s))-\sigma(s,X^{u^{\varepsilon},\eta^{*}}(s),P_{X^{u^{\varepsilon},\eta^{*}}(s)},u^{\varepsilon}(s))]\mathrm{d}B(s)\\
&+\int_{[0,t]}G(s)\mathrm{d}\left(\eta^{\varepsilon}-\eta^{*}\right)(s).
\end{aligned}$$

Since $\left[X^{u^{\varepsilon},\eta^{*}}(t)-X^{*}(t)\right]$ is independent to the singular control $\eta^{*}(\cdot)$, the proof of (3.31) follows immediately, (see $[11, Proposition 4.2, estimate (4.8)]$. The proof of (3.32) follows directly from Assumption $(H1)$, $(H3)$ and by applying *Gronwall's Lemma* and *Burkholder-Davis-Gundy inequality.*

Now, we proceed to estimate (3.33). We set $t\in[0,T]$,

$$\beta^{\varepsilon}(t)=\left[X^{u^{\varepsilon},\eta^{*}}(t)-X^{*}(t)\right]-Y^{\varepsilon}(t). \tag{3.35}$$

By standard arguments, it can be proved that

$$\begin{aligned}
\beta^{\varepsilon}(t)&=\int_{0}^{t}\{(f(r,X^{\varepsilon}(r),P_{X^{\varepsilon}(r)},u^{\varepsilon}(r))-f(r,X^{*}(r),P_{X^{*}(r)},u^{*}(r)))\\
&-\left(f_{x}(r)Y^{\varepsilon}(r)+\widehat{\mathbb{E}}[\widehat{f}_{\mu}(r)\widehat{Y}^{\varepsilon}(r)]+\delta f(r)1_{E_{\varepsilon}}(r)\right)\}\mathrm{d}r\\
&+\int_{0}^{t}\{(\sigma(r,X^{\varepsilon}(r),P_{X^{\varepsilon}(r)},u^{\varepsilon}(r))-\sigma(r,X^{*}(r),P_{X^{*}(r)},u^{*}(r))\\
&-\left(\sigma_{x}(r)Y^{\varepsilon}(r)+\widehat{\mathbb{E}}[\widehat{\sigma}_{\mu}(r)\widehat{Y}^{\varepsilon}(r)]+\delta\sigma(r)1_{E_{\varepsilon}}(r)\right)\})\mathrm{d}B(r)\\
&=\int_{0}^{t}\left[a_{1}^{\varepsilon}(r)+f_{x}(r)\beta^{\varepsilon}(r)+\widehat{\mathbb{E}}[\widehat{f}_{\mu}(r)\widehat{\beta}^{\varepsilon}(r)]\right]\mathrm{d}r\\
&+\int_{0}^{t}\left[a_{2}^{\varepsilon}(r)+\sigma_{x}(r)\beta^{\varepsilon}(r)+\widehat{\mathbb{E}}[\widehat{\sigma}_{\mu}(r)\widehat{\beta}^{\varepsilon}(r)]\right]\mathrm{d}B(r),
\end{aligned} \tag{3.36}$$

where

$$\begin{aligned}
a_{1}^{\varepsilon}(r)&=(f(r,X^{\varepsilon}(r),P_{X^{\varepsilon}(r)},u^{\varepsilon}(r))-f(r,X^{*}(r),P_{X^{*}(r)},u^{*}(r))\\
&-f_{x}(r)\left[X^{u^{\varepsilon},\eta^{*}}(r)-X^{*}(r)\right]-\widehat{\mathbb{E}}[\widehat{f}_{\mu}(r)(\widehat{X}^{u^{\varepsilon},\eta^{*}}(r)-\widehat{X}^{*}(r))]\\
&-\delta f(r)1_{E_{\varepsilon}}(r),
\end{aligned}$$

and

$$
\begin{aligned}
a_2^\varepsilon(r) &= (\sigma(r, X^\varepsilon(r), P_{X^\varepsilon(r)}, u^\varepsilon(r)) - \sigma(r, X^*(r), P_{X^*(r)}, u^*(r)) \\
&- \sigma_x(r)\left[X^{u^\varepsilon,\eta^*}(r) - X^*(r)\right] - \widehat{\mathbb{E}}[\widehat{\sigma}_\mu(r)(\widehat{X}^{u^\varepsilon,\eta^*}(r) - \widehat{X}^*(r))] \\
&- \delta\sigma(r)1_{E_\varepsilon}(r).
\end{aligned}
$$

Since

$$
\delta f(r)1_{E_\varepsilon}(r) = (f(r, X^*(r), P_{X^*(r)}, u^\varepsilon(r)) - f(r, X^*(r), P_{X^*(r)}, u^*(r)),
$$

and

$$
\delta\sigma(r)1_{E_\varepsilon}(r) = (\sigma(r, X^*(r), P_{X^*(r)}, u^\varepsilon(r)) - \sigma(r, X^*(r), P_{X^*(r)}, u^*(r)),
$$

we deduce

$$
\begin{aligned}
\int_0^t a_1^\varepsilon(r)\mathrm{d}r &= \int_0^t \{(f(r, X^\varepsilon(r), P_{X^\varepsilon(r)}, u^\varepsilon(r)) - f(r, X^*(r), P_{X^*(r)}, u^\varepsilon(r)) \\
&- f_x(r)\left[X^{u^\varepsilon,\eta^*}(r) - X^*(r)\right] - \widehat{\mathbb{E}}[\widehat{f}_\mu(r)(\widehat{X}^{u^\varepsilon,\eta^*}(r) - \widehat{X}^*(r))]\}\mathrm{d}r \\
&= \int_0^t \int_0^1 \{[f_x(r, X^*(r) + \lambda\left[X^{u^\varepsilon,\eta^*}(r) - X^*(r)\right], P_{(X^*(r)+\lambda(X^{u^\varepsilon,\eta^*}(r)-X^*(r)))}, u^\varepsilon(r)) - f_x(r)] \\
&\times \left[X^{u^\varepsilon,\eta^*}(r) - X^*(r)\right]\}\mathrm{d}\lambda\mathrm{d}r \qquad (3.37) \\
&+ \int_0^1 \{\widehat{\mathbb{E}}[\widehat{f}_x(r, \widehat{X}^*(r) + \lambda[\widehat{X}^{u^\varepsilon,\eta^*}(r) - \widehat{X}^*(r)], P_{(\widehat{X}^*(r)+\lambda(\widehat{X}^{u^\varepsilon,\eta^*}(r)-\widehat{X}^*(r))}, u^\varepsilon(r)) - \widehat{f}_x(r)] \\
&\times (\widehat{X}^{u^\varepsilon,\eta^*}(r) - \widehat{X}^*(r))]\mathrm{d}\lambda\mathrm{d}r.
\end{aligned}
$$

The same argument allows to show that

$$
\begin{aligned}
\int_0^t a_2^\varepsilon(r)\mathrm{d}B(r) &= \int_0^t \{(\sigma(r, X^\varepsilon(r), P_{X^\varepsilon(r)}, u^\varepsilon(r)) - \sigma(r, X^*(r), P_{X^*(r)}, u^\varepsilon(r)) \\
&- \sigma_x(r)\left[X^{u^\varepsilon,\eta^*}(r) - X^*(r)\right] - \widehat{\mathbb{E}}[\widehat{\sigma}_\mu(r)(\widehat{X}^{u^\varepsilon,\eta^*}(r) - \widehat{X}^*(r))]\}\mathrm{d}B(r) \\
&= \int_0^t \int_0^1 \{[\sigma_x(r, X^*(r) + \lambda\left[X^{u^\varepsilon,\eta^*}(r) - X^*(r)\right], P_{(X^*(r)+\lambda\left[X^{u^\varepsilon,\eta^*}(r)-X^*(r)\right])}, u^\varepsilon(r)) - \sigma_x(r)] \\
&\times \left[X^{u^\varepsilon,\eta^*}(r) - X^*(r)\right]\}\mathrm{d}\lambda\mathrm{d}B(r) \qquad (3.38) \\
&+ \int_0^1 \{\widehat{\mathbb{E}}[\widehat{\sigma}_x(r, \widehat{X}^*(r) + \lambda[\widehat{X}^{u^\varepsilon,\eta^*}(r) - \widehat{X}^*(r)], P_{(\widehat{X}^*(r)+\lambda(\widehat{X}^{u^\varepsilon,\eta^*}(r)-\widehat{X}^*(r))}, u^\varepsilon(r)) - \widehat{\sigma}_x(r)] \\
&\times (\widehat{X}^{u^\varepsilon,\eta^*}(r) - \widehat{X}^*(r))]\mathrm{d}\lambda\mathrm{d}B(r).
\end{aligned}
$$

Finally, the desired result (3.33) follows immediately by combining (3.33), (3.37), (3.38), Gronwall Lemma and estimates (3.31). This completes the proof of Lemma 4.2. ■

Proposition 3.1

Let $Y^{\varepsilon}(t)$ solution (3.30) associated to $(u^{\varepsilon}(\cdot), \eta^{*}(\cdot))$. Under assumption H1, the following estimate holds

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[\sup_{0\leq t\leq T}\left|X^{u^{\varepsilon},\eta^{*}}(t)-X^{*}(t)-Y^{\varepsilon}(t)-Z^{\varepsilon}(t)\right|^{2}\right]=0. \tag{3.39}$$

Proof: Since $\delta f(t)1_{E_{\varepsilon}}(t)=f^{\varepsilon}(t)-f^{*}(t)$ and $\delta\sigma(t)1_{E_{\varepsilon}}(t)=\sigma^{\varepsilon}(t)-\sigma(t)$, then a simple calculation shows that

$$\begin{aligned}Y^{\varepsilon}(t)&=\int_0^t\left[f_x(s)Y^{\varepsilon}(s)+\widehat{\mathbb{E}}[\widehat{f}_{\mu}(s)\widehat{Y}^{\varepsilon}(s)]+f^{\varepsilon}(s)-f^{*}(s)\right]\mathrm{d}s\\&+\int_0^t\left[\sigma_x(s)Y^{\varepsilon}(s)+\widehat{\mathbb{E}}[\widehat{\sigma}_{\mu}(s)\widehat{Y}^{\varepsilon}(s)]+\sigma^{\varepsilon}(s)-\sigma(s)\right]\mathrm{d}B(s),\end{aligned}$$

and

$$\begin{aligned}Z^{\varepsilon}(t)=&\int_0^t[f_x^{\varepsilon}(s)-f_x(s)]X_2^{\varepsilon}(s)+f_x(s)X_2^{\varepsilon}(s)\\&+\frac{1}{2}f_{xx}(s)X_1^{\varepsilon}(s)^2)]\mathrm{d}s\\&+\int_0^t[\sigma_x^{\varepsilon}(s)-\sigma_x^{*}(s)]X_2^{\varepsilon}(s)+\sigma_x^{*}(s)X_2^{\varepsilon}(s)\\&+\frac{1}{2}\sigma_{xx}(s)X_1^{\varepsilon}(s)^2)]\mathrm{d}B(s).\end{aligned}$$

Now, it is clear that $X^{u^{\varepsilon},\eta^{*}}(t)-X^{*}(t)-Y^{\varepsilon}(t)-Z^{\varepsilon}(t)$ depend only on the continuous component of the control and independent to singular control. The result follows by applying the same proof as in [11, *Proposition*5.1, *page*526]. This completes the proof of Proposition 3.1. ∎

The following theorem constitutes the main contribution of this chapter.

Theorem 3.1 (*Stochastic maximum principle*)

Let $(u^{*}(\cdot),\eta^{*}(\cdot),X^{*}(\cdot))$ is an optimal solution of the McKean-Vlasov control problem (3.14)-(3.15). Let assumptions $(H1)$, $(H2)$ and $(H3)$ hold. Then there are two pairs of F_t-adapted processes $(p(\cdot),q(\cdot))$ and $(P(\cdot),Q(\cdot))$ that satisfy (3.23) and (3.26) respectively, such that for all $(u(t),\eta(t))\in U_1\times U_2$, we have:

$$\begin{aligned}
&0 \leq H(t, X^*(t), P_{X^*(t)}, u^*(t), p^*(t), q^*(t)) - H(t, x^*(t), P_{X^*(t)}, u(t), p^*(t), q^*(t)) \\
&- \frac{1}{2} P(t) \left(\sigma(t, X^*(t), P_{X^*(t)}, u(t))) - \sigma \left(t, X^*(t), P_{X^*(t)}, u^*(t) \right) \right)^2 \\
&+ \mathbb{E} \int_{[0,T]} (M(t) + G(t)p(t)) \mathrm{d} \left(\eta - \eta^* \right)(t).
\end{aligned} \tag{3.40}$$

$$P-a.s.,\ a.e.\ t \in [0, T].$$

Proof: To derive our main result, the approach that we use is based on a double perturbation of the optimal continuous-singular control. The first perturbation is a *spike variation*, on the absolutely continuous part of the control and the second one is *convex variation method*, on the singular component. This perturbation is described as follows:

Let $(u^*(\cdot), \eta^*(\cdot))$ is an optimal control and $(u(\cdot), \eta(\cdot))$ is an arbitrary element of $\mathcal{F}_t-$measurable random variable with values in $\mathbb{U}_1 \times \mathbb{U}_2$ which we consider as fixed from now on. We define a perturbed control $(u^\varepsilon(t), \eta^\varepsilon(t))$ as follows. Let $E_\varepsilon = [0, \varepsilon]$

$$u^\varepsilon(t) = \begin{cases} u(t), & t \in E_\varepsilon, \\ u^*(t), & t \in E_\varepsilon^c, \end{cases} \tag{3.41}$$

and

$$\eta^\varepsilon(t) = \eta^*(t) + \varepsilon \left(\eta(t) - \eta^*(t) \right). \tag{3.42}$$

By combining (3.41), (3.42), we define

$$\left(u^\varepsilon(t), \eta^\varepsilon(t)\right) = \begin{cases} \left(u(t), \eta^*(t) + \varepsilon \left(\eta(t) - \eta^*(t) \right)\right) : t \in E_\varepsilon \\ \left(u^*(t), \eta^*(t) + \varepsilon \left(\eta(t) - \eta^*(t) \right)\right) : t \in E_\varepsilon^c, \end{cases} \tag{3.43}$$

where ε a sufficiently small $\varepsilon > 0$. Then, we derive the variational inequality (3.40) in several steps. From the optimality of $(u^*(\cdot), \eta^*(\cdot))$, we have

$$J\left(u^\varepsilon(\cdot), \eta^\varepsilon(\cdot)\right) - J\left(u^*(\cdot), \eta^*(\cdot)\right) \geq 0. \tag{3.44}$$

Now, we separate the above inequality into two parts

$$J_1^\varepsilon = J\left(u^\varepsilon(\cdot), \eta^\varepsilon(\cdot)\right) - J\left(u^\varepsilon(\cdot), \eta^*(\cdot)\right), \tag{3.45}$$

$$J_2^\varepsilon = J\left(u^\varepsilon(\cdot), \eta^*(\cdot)\right) - J\left(u^*(\cdot), \eta^*(\cdot)\right), \tag{3.46}$$

where $J\left(u^\varepsilon(\cdot), \eta^\varepsilon(\cdot)\right) - J\left(u^*(\cdot), \eta^*(\cdot)\right) = J_1^\varepsilon + J_2^\varepsilon$. The variational inequality will be derived from the fact that

$$\lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \left(J_1^\varepsilon + J_2^\varepsilon \right) \geq 0. \tag{3.47}$$

We proceed to estimate the left hand side of inequality(3.47). ■

Lemma 3.3

The following estimate holds, for any $\eta(\cdot) \in \mathcal{U}_2([0,T])$

$$\begin{aligned}\lim_{\varepsilon\to 0}\frac{J_1}{\varepsilon} &= \mathbb{E}\left[h_x(X^*(T))\mathcal{Z}(T) + \widehat{\mathbb{E}}(\widehat{h}_\mu(X^*(T))\widehat{\mathcal{Z}}(T))\right] \\ &+ \mathbb{E}\int_0^T \ell_x(X^*(t))\mathcal{Z}(t)\mathrm{d}t + \mathbb{E}\int_0^T \widehat{\mathbb{E}}(\widehat{h}_\mu(X^*(t))\widehat{\mathcal{Z}}(t)) \\ &+ \mathbb{E}\int_{[0,T]} M(t)\mathrm{d}\left(\eta-\eta^*\right)(t),\end{aligned} \tag{3.48}$$

where $\mathcal{Z}(\cdot)$ *solution of the following SDEs:*

$$\begin{cases}\mathrm{d}\mathcal{Z}(t) = \left[f_x(t)\mathcal{Z}(t) + \widehat{\mathbb{E}}[\widehat{f}_\mu(t)\widehat{\mathcal{Z}}(t)]\right]\mathrm{d}t + \left[\sigma_x(t)\mathcal{Z}(t) + \widehat{\mathbb{E}}[\widehat{\sigma}_\mu(t)\widehat{\mathcal{Z}}(t)]\right]\mathrm{d}B(t) \\ \qquad\qquad + G(t)\mathrm{d}\left(\eta-\eta^*\right)(t) \\ \mathcal{Z}(0) = 0.\end{cases} \tag{3.49}$$

Proof: Under assumptions H1 and H3, Eq-(3.49) admits a unique strong solution $\mathcal{Z}(t)$ given by

$$\begin{aligned}\mathcal{Z}(t) &= \int_0^t \left[f_x(s)\mathcal{Z}(s) + \widehat{\mathbb{E}}[\widehat{f}_\mu(s)\widehat{\mathcal{Z}}(s)]\right]\mathrm{d}s \\ &+ \int_0^t \left[\sigma_x(s)\mathcal{Z}(s) + \widehat{\mathbb{E}}[\widehat{\sigma}_\mu(s)\widehat{\mathcal{Z}}(s)]\right]\mathrm{d}B(s) \\ &+ \int_{[0,T]} G(t)\mathrm{d}\left(\eta-\eta^*\right)(s).\end{aligned}$$

Now, it follows easily by the same arguments developed in Lemma 3.2, the condition (3.33) implies that

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[\sup_{0\le t\le T}\left|\frac{X^{u^\varepsilon,\eta^*}(t) - X^*(t)}{\varepsilon} - \mathcal{Z}(t)\right|^2\right] = 0. \tag{3.50}$$

By a simple computations, we get

$$\begin{aligned}
\frac{J_1^\varepsilon}{\varepsilon} &= \frac{1}{\varepsilon}\left[J\left(u^\varepsilon(\cdot),\eta^\varepsilon(\cdot)\right) - J\left(u^\varepsilon(\cdot),\eta^*(\cdot)\right)\right] \\
&= \frac{1}{\varepsilon}\mathbb{E}[[h(X^{u^\varepsilon,\eta^\varepsilon}(T),P_{X^{u^\varepsilon,\eta^\varepsilon}(T)}) - h(X^{u^\varepsilon,\eta^*}(T),P_{X^{u^\varepsilon,\eta^*}(T)})] \\
&+ \frac{1}{\varepsilon}\mathbb{E}\int_0^T [\ell(t,X^{u^\varepsilon,\eta^\varepsilon}(t),P_{X^{u^\varepsilon,\eta^\varepsilon}(t)},u^\varepsilon(t)) \\
&- \ell(t,X^{u^\varepsilon,\eta^*}(t),P_{X^{u^\varepsilon,\eta^*}(t)},u^\varepsilon(t))]\mathrm{d}t \\
&+ \frac{1}{\varepsilon}\mathbb{E}\int_{[0,T]} G(t)\mathrm{d}\left(\eta^\varepsilon-\eta^*\right)(t).
\end{aligned}$$

By Tylor's formula, the fact that

$$\frac{\eta^\varepsilon(t)-\eta^*(t)}{\varepsilon} = \left(\eta(t)-\eta^*(t)\right),$$

for any $\eta(\cdot)\in\mathcal{U}_2\left([0,T]\right)$ and for real $\lambda\in[0,1]$, we have

$$\begin{aligned}
\frac{J_1^\varepsilon}{\varepsilon} &= \mathbb{E}\int_0^1 h_x(X^{u^\varepsilon,\eta^\varepsilon}(T)+\lambda\left(X^\varepsilon(T)-X^{u^\varepsilon,\eta^*}(T)\right),P_{X^{u^\varepsilon,\eta^\varepsilon}(T)+\lambda(X^\varepsilon(T)-X^{u^\varepsilon,\eta^*}(T))}) \\
&\times\left(X^\varepsilon(T)-X^{u^\varepsilon,\eta^*}(T)\right)\varepsilon^{-1}\mathrm{d}\lambda \\
&+\mathbb{E}\int_0^1 \widehat{\mathbb{E}}[h_\mu(\widehat{X}^{u^\varepsilon,\eta^\varepsilon}(T)+\lambda(\widehat{X}^\varepsilon(T)-\widehat{X}^{u^\varepsilon,\eta^*}(T)),P_{\widehat{X}^{u^\varepsilon,\eta^\varepsilon}(T)+\lambda(\widehat{X}^\varepsilon(T)-\widehat{X}^{u^\varepsilon,\eta^*}(T))})] \\
&\times(\widehat{X}^\varepsilon(T)-\widehat{X}^{u^\varepsilon,\eta^*}(T))\varepsilon^{-1}\mathrm{d}\lambda \\
&+\mathbb{E}\int_0^T\int_0^1 \ell_x(t,X^{u^\varepsilon,\eta^\varepsilon}(t)+\lambda\left(X^\varepsilon(t)-X^{u^\varepsilon,\eta^*}(t)\right),P_{X^{u^\varepsilon,\eta^\varepsilon}(T)+\lambda(X^\varepsilon(T)-X^{u^\varepsilon,\eta^*}(T))},u^\varepsilon(t)]) \\
&\times\left(X^\varepsilon(t)-X^{u^\varepsilon,\eta^*}(t)\right)\varepsilon^{-1}\mathrm{d}\lambda\mathrm{d}t \\
&+\mathbb{E}\int_0^T\int_0^1 \widehat{\mathbb{E}}[\ell_\mu(t,\widehat{X}^{u^\varepsilon,\eta^\varepsilon}(t)+\lambda(\widehat{X}^\varepsilon(t)-\widehat{X}^{u^\varepsilon,\eta^*}(t)),P_{\widehat{X}^{u^\varepsilon,\eta^\varepsilon}(t)+\lambda(\widehat{X}^\varepsilon(t)-\widehat{X}^{u^\varepsilon,\eta^*}(t))},u^\varepsilon(t))] \\
&\times(\widehat{X}^\varepsilon(t)-\widehat{X}^{u^\varepsilon,\eta^*}(t))\varepsilon^{-1}\mathrm{d}\lambda\mathrm{d}t \\
&+\mathbb{E}\int_{[0,T]} M(t)\mathrm{d}\left(\eta-\eta^*\right)(t).
\end{aligned}$$

Finally, since the derivatives h_x, h_μ, ℓ_x and ℓ_μ are continuous and bounded, the result follows from (3.50) and by letting ε going to zero. This completes the proof of Lemma ■

Lemma 3.4

Let $\mathcal{Z}(\cdot)$ be the solution of (3.49) and $p(\cdot)$ the solution of equation (3.23).

$$\begin{aligned}&\mathbb{E}(h_x(T)\mathcal{Z}(T))+\mathbb{E}(\widehat{\mathbb{E}}(\widehat{h}_\mu^*(T)\mathcal{Z}(T)))\\&=-\mathbb{E}\int_0^T\mathcal{Z}(t)\ell_x(t)\mathrm{d}t-\mathbb{E}\int_0^T\mathcal{Z}(t)\widehat{\mathbb{E}}[\widehat{\ell}_\mu^*(t)(t)]\mathrm{d}t.\\&+\mathbb{E}\int_{[0,T]}G(t)p(t)\mathrm{d}\left(\eta-\eta^*\right)(t).\end{aligned}\tag{3.51}$$

Proof: By applying Itô's formula to $p(t)\mathcal{Z}(t)$ and taking expectation, we get

$$\begin{aligned}&\mathbb{E}(p(T)\mathcal{Z}(T))=\mathbb{E}\int_0^T p(t)d\mathcal{Z}(t)+\mathbb{E}\int_0^T\mathcal{Z}(t)\mathrm{d}p(t)\\&+\mathbb{E}\int_0^T q(t)[\sigma_x(t)\mathcal{Z}(t)+\widehat{\mathbb{E}}[\widehat{\sigma}_\mu(t)\widehat{\mathcal{Z}}(t)]]\mathrm{d}t\\&=\mathbf{I}_1\left(T\right)+\mathbf{I}_2\left(T\right)+\mathbf{I}_3\left(T\right).\end{aligned}\tag{3.52}$$

From (3.50), we obtain

$$\begin{aligned}\mathbf{I}_1\left(T\right)&=\mathbb{E}\int_0^T p(t)\mathrm{d}\mathcal{Z}(t)\\&=\mathbb{E}\int_0^T p(t)[f_x(t)\mathcal{Z}(t)+\widehat{\mathbb{E}}[\widehat{f}_\mu(t)\widehat{\mathcal{Z}}(t)]]\mathrm{d}t\\&+\mathbb{E}\int_{[0,T]}G(t)p(t)\mathrm{d}\left(\eta-\eta^*\right)(t)\\&=\mathbb{E}\int_0^T p(t)f_x(t)\mathcal{Z}(t)dt+\mathbb{E}\int_0^T p(t)\widehat{\mathbb{E}}[\widehat{f}_\mu(t)\widehat{\mathcal{Z}}(t)]\mathrm{d}t\\&+\mathbb{E}\int_{[0,T]}G(t)p(t)\mathrm{d}\left(\eta-\eta^*\right)(t).\end{aligned}\tag{3.53}$$

By applying (3.23), we have

$$\begin{aligned}\mathbf{I}_2\left(T\right)&=\mathbb{E}\int_0^T\mathcal{Z}(t)\mathrm{d}p(t)\\&=-\mathbb{E}\int_0^T\mathcal{Z}(t)[f_x(t)p(t)+\widehat{\mathbb{E}}[\widehat{f}_\mu^*(t)(t)\widehat{p}(t)]+\sigma_x(t)q(t)+\widehat{\mathbb{E}}[\widehat{\sigma}_\mu^*(t)\widehat{q}(t)]\\&-\ell_x(t)-\widehat{\mathbb{E}}[\widehat{\ell}_\mu^*(t)(t)]\Big]\,\mathrm{d}t\\&=-\mathbb{E}\int_0^T\mathcal{Z}(t)f_x(t)p(t)\mathrm{d}t-\mathbb{E}\int_0^T\mathcal{Z}(t)\widehat{\mathbb{E}}\left[\widehat{f}_\mu^*(t)(t)\widehat{p}(t)\right]\mathrm{d}t\\&-\mathbb{E}\int_0^T\mathcal{Z}(t)\sigma_x(t)q(t)\mathrm{d}t-\mathbb{E}\int_0^T\mathcal{Z}(t)\widehat{\mathbb{E}}\left[\widehat{\sigma}_\mu^*(t)\widehat{q}(t)\right]\mathrm{d}t\\&-\mathbb{E}\int_0^T\mathcal{Z}(t)\ell_x(t)\mathrm{d}t-\mathbb{E}\int_0^T\mathcal{Z}(t)\widehat{\mathbb{E}}\left[\widehat{\ell}_\mu^*(t)(t)\right]\mathrm{d}t.\end{aligned}\tag{3.54}$$

By a simple computation, we obtain

$$\begin{aligned}\mathbf{I}_3(T) &= \mathbb{E}\int_0^T q(t)[\sigma_x(t)\mathcal{Z}(t)+\widehat{\mathbb{E}}[\widehat{\sigma}_\mu(t)\widehat{\mathcal{Z}}(t)]]\mathrm{d}t \\ &= \mathbb{E}\int_0^T q(t)[\sigma_x(t)\mathcal{Z}(t)]\mathrm{d}t+\mathbb{E}\int_0^T q(t)\widehat{\mathbb{E}}[\widehat{\sigma}_\mu(t)\widehat{\mathcal{Z}}(t)]]\mathrm{d}t.\end{aligned} \tag{3.55}$$

By combining (3.52)-(3.55), with some direct computation using Fubini's theorem, we get

$$\begin{aligned}\mathbb{E}(p(T)\mathcal{Z}(T)) = -\mathbb{E}\int_0^T \mathcal{Z}(t)\ell_x(t)\mathrm{d}t-\mathbb{E}\int_0^T \mathcal{Z}(t)\widehat{\mathbb{E}}[\widehat{\ell}_\mu^*(t)(t)]\mathrm{d}t. \\ +\mathbb{E}\int_{[0,T]} G(t)p(t)\mathrm{d}(\eta-\eta^*)(t).\end{aligned} \tag{3.56}$$

Finally, the result follows from (3.56) and (3.23). This completes the proof of Lemma 3.4 ■

Proposition 3.2

Let $\eta^*(\cdot)$ the optimal singular control. For any $\eta(\cdot)\in\mathcal{U}_1([0,T])$, we have

$$\lim_{\varepsilon\to 0}\frac{J_1^\varepsilon}{\varepsilon}=\mathbb{E}\int_{[0,T]}(M(t)+G(t)p(t))\mathrm{d}(\eta-\eta^*)(t). \tag{3.57}$$

Proof : The result follows from Lemma 3.3 and Lemma 3.4, by substituting (3.51) in (3.48). This completes the proof of Proposition 3.2.

We proceed to estimate the second term J_2^ε. From (3.46), we have

$$\begin{aligned}J_2^\varepsilon &= [J(u^\varepsilon(\cdot),\eta^*(\cdot))-J(u^*(\cdot),\eta^*(\cdot))] \\ &= \mathbb{E}[[h(X^{u^\varepsilon,\eta^*}(T),P_{X^{u^\varepsilon,\eta^*}(T)})-h(X^*(T),P_{X^*(T)})] \\ &+\mathbb{E}\int_0^T[\ell(t,X^{u^\varepsilon,\eta^*}(t),P_{X^{u^\varepsilon,\eta^*}(t)},u^\varepsilon(t))-\ell(t,X^*(t),P_{X^*(t)},u^*(t))]\mathrm{d}t.\end{aligned} \tag{3.58}$$

Noting that J_2^ε in is independent to singular component of the control. By applying the similar proof as in [11, $Eq-(6.5)$] with the help of Proposition 3.1, and since $J(u^*,\eta^*)\leq$

$J\left(u^{\varepsilon},\eta^{*}\right)$, we get

$$\begin{aligned}J_{2}^{\varepsilon}&=J\left(u^{\varepsilon}\left(\cdot\right),\eta^{*}(\cdot)\right)-J\left(u^{*}(\cdot),\eta^{*}(\cdot)\right)\\&=-\mathbb{E}\left[\int_{0}^{T}(H(t,X^{*}(t),P_{X^{*}(t)},u,p^{*}(t),q^{*}(t))\right.\\&-H(t,x^{*}(t),P_{X^{*}(t)},u^{*}(t),p^{*}(t),q^{*}(t))\\&\left.+\frac{1}{2}P_{t}\left(\delta\sigma\left(t\right)\right)^{2}\right]1_{E_{\varepsilon}}\left(t\right))\mathrm{d}t+o(\varepsilon)\\&=-\mathbb{E}\int_{0}^{T}(\delta H\left(t\right)+\frac{1}{2}P_{t}\left(\delta\sigma\left(t\right)\right)^{2})1_{E_{\varepsilon}}\left(t\right)\mathrm{d}t+o(\varepsilon)\geq 0,\end{aligned}\tag{3.59}$$

where $o(\varepsilon)=C\varepsilon\rho(\varepsilon)$, and $\rho(\varepsilon)\to 0$ as $\varepsilon\to 0$. By letting ε going to zero in (3.59), we get

$$\begin{aligned}&0\leq\lim_{\varepsilon\to 0}\frac{J_{2}^{\varepsilon}}{\varepsilon}=-\mathbb{E}\int_{0}^{T}(\delta H\left(t\right)+\frac{1}{2}P(t)\left(\delta\sigma\left(t\right)\right)^{2})\mathrm{d}t\\&=-\mathbb{E}\int_{0}^{T}[H(t,X^{*}(t),P_{X^{*}(t)},u(t),p^{*}(t),q^{*}(t))\\&-H(t,x^{*}(t),P_{X^{*}(t)},u^{*}(t),p^{*}(t),q^{*}(t))\\&+\frac{1}{2}P(t)\left(\sigma(t,X^{*}(t),P_{X^{*}(t)},u(t)))-\sigma\left(t,X^{*}(t),P_{X^{*}(t)},u^{*}(t)\right)\right)^{2}]\mathrm{d}t.\end{aligned}\tag{3.60}$$

From (3.47), (3.57), (3.60), we obtain

$$\begin{aligned}0\leq\;&\mathbb{E}\int_{0}^{T}[H(t,X^{*}(t),P_{X^{*}(t)},u^{*}(t),p^{*}(t),q^{*}(t))-H(t,x^{*}(t),P_{X^{*}(t)},u(t),p^{*}(t),q^{*}(t))\\&-\frac{1}{2}P(t)\left(\sigma(t,X^{*}(t),P_{X^{*}(t)},u(t)))-\sigma\left(t,X^{*}(t),P_{X^{*}(t)},u^{*}(t)\right)\right)^{2}]\mathrm{d}t\\&+\mathbb{E}\int_{[0,T]}(M(t)+G(t)p(t))\mathrm{d}\left(\eta-\eta^{*}\right)(t).\end{aligned}\tag{3.61}$$

Finally by applying the Lebesgue differentiation theorem to (3.60), we deduce from (3.61) that, for all $(u(t)),\eta(t))\in\mathbb{U}_{1}\times\mathbb{U}_{2}$, a.e., $t\in[0,T]$, it holds P-almost surely,

$$\begin{aligned}&H(t,X^{*}(t),P_{X^{*}(t)},u^{*}(t),p^{*}(t),q^{*}(t))-H(t,X^{*}(t),P_{X^{*}(t)},u(t),p^{*}(t),q^{*}(t))\\&-\frac{1}{2}P(t)\left(\sigma(t,X^{*}(t),P_{X^{*}(t)},u(t)))-\sigma\left(t,X^{*}(t),P_{X^{*}(t)},u^{*}(t)\right)\right)^{2}\\&+\mathbb{E}\int_{[0,T]}(M(t)+G(t)p(t))\mathrm{d}\left(\eta-\eta^{*}\right)(t)\geq 0.\end{aligned}\tag{3.62}$$

This completes the proof of Theoem 3.1 ■

Conclusion

In this thesis, we have discussed a general Peng's type necessary conditions in the form of Pontryagin stochastic maximum principle of optimal continuous-singular control for nonlinear controlled McKean-Vlasov stochastic differential equation. If the coefficients of the singular parts $G(t) = M(t) = 0$, our stochastic maximum principle (Theorem 3.1) coincides with maximum principle developed in Buckdahn et al. [11, $Theorem 3.5$].
Apparently, there are many problems left unsolved such as:

A. One possible problem is to study the general Peng's type maximum principle for optimal control for SDE, the coefficients of the singular parts $G(\cdot)$ and $M(\cdot)$ depend explicitly to the state of the solution process $X^{u,\eta}$ of the form

$$\begin{cases} dX^{u,\eta}(t) = f\left(t, X^{u,\eta}(t), u(t)\right) dt + \sigma\left(t, X^{u,\eta}(t), u(t)\right) dW(t) \\ \qquad\qquad + G(t, X^{u,\eta}) d\eta(t), \\ X^{u,\eta}(0) = x_0, \end{cases}$$

and the cost functional of the form

$$J\left(u(\cdot), \eta(\cdot)\right) = E\left[\int_0^T f(t, X^{u,\eta}(t), u(t))dt + h(X^{u,\eta}(T)) + \int_{[0,T]} M(t, X^{u,\eta}) d\eta(t)\right],$$

B. It would be interesting to investigate the McKean-Vlasov maximum principle (local version via Bensoussan's convex method and general Peng's maximum principle) for optimal continuous-singular control for McKean-Vlasov SDE, the coefficients of the singular parts $G(\cdot)$ and $M(\cdot)$ of the state equation depend on the state of the solution process as well as of its probability law and the control variable.of the form

$$\begin{cases} dX^{u,\eta}(t) = f\left(t, X^{u,\eta}(t), P_{X^{u,\eta}(t)}, u(t)\right) dt + \sigma\left(t, X^{u,\eta}(t), P_{X^{u,\eta}(t)}, u(t)\right) \mathrm{d}W(t) \\ \qquad\qquad + G(t, X^{u}, P_{X^{u,\eta}(t)})\mathrm{d}\eta(t), \\ X^{u,\eta}(0) = x_0, \end{cases}$$

and the expected cost has the form

$$J(u(\cdot), \eta(\cdot)) = E\left[\int_0^T f(t, X^{u,\eta}(t), P_{X^{u,\eta}(t)}, u(t))\mathrm{d}t + h(X^{u,\eta}(T), P_{X^{u,\eta}(t)}) \right.$$
$$\left. + \int_{[0,T]} M(t, X^{u,\eta}, P_{X^{u,\eta}(t)})\mathrm{d}\eta(t)\right].$$

C. Another challenging problem left unsolved is to derive a various maximum principles in the case where the coefficients f, σ, ℓ, G and M depend on the state of the solution process $X^{u,\eta}(\cdot)$, the continuous control variable $u(\cdot)$ as well as of probability law of the pair $P_{(X^{u,\eta}(t), u(t))}$. So we investigate the problem:

$$\begin{cases} dX^{u,\eta}(t) = f\left(t, X^{u,\eta}, u(t), P_{(X^{u,\eta}(t), u(t))}\right) dt + \sigma\left(t, X^{u,\eta}(t), u(t), P_{(X^{u,\eta}(t), u(t))}\right) \mathrm{d}W(t) \\ \qquad\qquad + G(t, X^{u,\eta}, u(t), P_{(X^{u,\eta}(t), u(t))})\mathrm{d}\eta(t), \\ X^{u,\eta}(0) = x_0, \end{cases}$$

and the cost functional has the general form

$$J(u(\cdot), \eta(\cdot)) = E\left[\int_0^T f(t, X^{u,\eta}, u(t), P_{(X^{u,\eta}(t), u(t))})\mathrm{d}t + h(X^{u,\eta}(T), P_{X^{u,\eta}(t)}) \right.$$
$$\left. + \int_{[0,T]} M(t, X^{u,\eta}, u(t), P_{(X^{u,\eta}(t), u(t))})\mathrm{d}\eta(t)\right].$$

We hope to study these interesting new problems in forthcoming works.

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