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Dedication

To my beloved parents, for all their sacrifices, love, care, and support throughout my studies.

To my dear sisters, for their constant encouragement and moral support.

To my dear brothers, for their support and encouragement.

To my entire family, especially my aunts, for their unwavering support and understanding. Without their continuous encouragement, this work would not have been possible. I pray to God for their long life and good health

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Contents

Introduction

Introduction

This master's dissertation investigates stochastic optimal control problems concerning forward-backward differential equations. These equations are driven by Teugels martingales and involve a Lévy process with moments of all orders, as well as an independent Brownian motion. Our control problem is stated as follows:

• Given a filtered probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\geq 0})$ and a control $u(\cdot)$ valued in some subset of $\mathbb{R}^k$, we consider the following system

$$\begin{cases} dx^u(t) = b\left(t, x^u(t), u\left(t\right)\right) dt + \sum_{i=1}^{d} g^i\left(t, x^u(t), u\left(t\right)\right) dW^i(t) \\ \qquad +\sum_{i=1}^{\infty} c^i\left(t, x^u\left(t_-\right), u(t)\right) dH^i(t), \\ dy^u(t) = -f(t, x^u(t), y^u(t), z^u(t), r^u(t), u(t))dt + \sum_{i=1}^{d} z^{u,i}(t) dW^i(t) \\ \qquad +\sum_{i=1}^{\infty} r^{u,i}(t_-) dH^i(t), \\ x^u(0) = x_0, \;\; y^u(T) = h(x^u(T)), \end{cases} \tag{1}$$

where $W(\cdot)$ is a d-dimensional $\mathcal{F}_t$-Brownian motion. The Teugels martingales $H(t) = (H^i(t))_{i\geq 1}$ considered in this study are pairwise strongly orthonormal. These martingales are associated with a Lévy process that possesses moments of all orders. The control $u(\cdot) = (u(t))_{t\geq 0}$ is adapted to a subfiltration $(\mathcal{G}_t)_{t\geq 0}$ of $(\mathcal{F}_t)$. $x^u(t_-) = \lim_{s\to t, s<t} x^u(s)$, $t > 0$ the left limit process.

The maps

$$
\begin{aligned}
&b:[0,T]\times\mathbb{R}^n\times\mathbb{A}\to\mathbb{R}^n,\ \ g:[0,T]\times\mathbb{R}^n\times\mathbb{A}\to\mathbb{R}^{n\times d},\\
&c:[0,T]\times\mathbb{R}^n\times\mathbb{A}\to l^2(\mathbb{R}^n),\ \ h:\mathbb{R}^n\to\mathbb{R}^m,\\
&f:[0,T]\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\times\mathbb{R}^m\times\mathbb{A}\to l^2(\mathbb{R}^m),
\end{aligned}
$$

are given deterministic functions.

• The objective is to minimize the expected cost over a class of admissible controls, characterized by the following form:

$$
J(u(\cdot))=E\left\{\int_0^T \ell(t,x^u(t),y^u(t),z^u(t),r^u(t),u(t))dt+\phi(x^u(T))+\varphi(y^u(0))\right\}, \tag{2}
$$

where

$$
\begin{aligned}
&\phi:\mathbb{R}^n\to\mathbb{R},\varphi:\mathbb{R}^m\to\mathbb{R},\\
&\ell:[0,T]\times\mathbb{R}^n\times\mathbb{R}^m\times\mathbb{R}^{m\times d}\times\mathbb{R}^m\times\mathbb{A}\to\mathbb{R},
\end{aligned}
$$

are deterministic functions. Several researchers have examined stochastic processes that are influenced by Teugels martingales, which are associated with specific Lévy processes and independent Brownian motions, see, for example ([2]), ([5]), ([3]), ([6]).

Partial information means that the information available to the controller is possibly less than the whole information. That is, any admissible control is adapted to a sub-filtration $(G_t)_t$ of $(\mathcal{F}_t)_t\,, t\geq 0$. This kind of problem, which has a lot of applications in mathematical finance and economics, arises naturally, because it may fail to obtain an admissible control with full information in real-world applications. Meng ([3]) has investigated a maximum principle for an optimal control problem involving fully coupled forward-backward stochastic systems under partial information. Wang et al. ([7]) established the partial information maximum principle of optimality for SDEs. Partially observed optimal control problems of general McKean–Vlasov differential equations has been studied by Lakhdari et al. ([1]). Necessary conditions for partially observed optimal control of general McKean–Vlasov stochastic differential equations with jumps has been proved by Miloudi et al. ([4]).

This study is based on part of the work of Meherrem and Hafayed ([2]), where the authors proved the stochastic maximum principle for optimal control of McKean-Vlasov FBSDEs with Lévy process via the differentiability with respect to probability law.

This work has three chapters.

Chapter 1 (Stochastic optimal control): This chapter is an introduction that introduces the necessary tools for the second and third chapters.

Chapter 2 (Necessary conditions of optimality): The aim of this chapter is to study the necessary conditions of optimality for partial information stochastic optimal control problem of forward-backward differential equations driven by Teugels martingales associated with some Lévy process having moments of all orders and an independent Brownian motion.

Chapter 3 (Sufficient conditions of optimality): In this chapter, we derive sufficient conditions of optimality for our control problem (1) - (2). Under some additional assumptions, the necessary conditions studied in chapter 2 are sufficient conditions for optimality.

Chapter §.1
Stochastic optimal control

Chapter 1

Stochastic optimal control

This chapter is essentially an introduction, aiming to highlight the tools used in our study. We provide some basic reminders regarding stochastic calculus.

1.1 Stochastic processes

Consider a probability space $(\Omega, \mathcal{F}, P)$, where Ω is the sample space, $\mathcal{F}$ is the σ-algebra of events, and P is the probability measure. Let T be a non-empty index set. A stochastic process is a collection of n-dimensional random variables $\{X(t) : t \in T\}$ that map from $(\Omega, \mathcal{F}, P)$ to the set of real numbers $\mathbb{R}^n$. For any $w \in \Omega$, the function $t \to X(t, w)$ is known as a sample path of the stochastic process.

Definition 1.1.1 ***(Random variable):*** *We call a random variable (real-valued)*

$$X : (\Omega, \mathcal{F}) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})),$$

any measurable function, that is:

$$\{w \in \Omega : X(w) \in B\} = \{X \in B\} \in \mathcal{F}, \forall B \in \mathcal{B}(\mathbb{R}),$$

where $\mathcal{B}(\mathbb{R})$ is the Borel sigma-algebra.

Definition 1.1.2 ***(Generated sigma-algebra):*** *The sigma-algebra generated by a random variable X defined on $(\Omega, \mathcal{F})$ is the set*

$$\sigma(X) = \left\{X^{-1}(A) : A \in \varepsilon\right\},$$

where $X^{-1}(A) = \{w \in \Omega : X(w) \in A\}$, which is the smallest sigma-algebra of Ω that "makes X measurable".

Definition 1.1.3 ***(Natural filtration):*** *Let $X = (X_t, t \geq 0)$ be a stochastic process on the probability space $(\Omega, \mathcal{F}, P)$. The natural filtration of X is denoted by $\mathcal{F}_t^X$ and is defined by $\mathcal{F}_t^X = \sigma(X_s : 0 \leq s \leq t)$. Furthermore, the filtration generated by X is also referred to as $\mathcal{F}_t^X$.*

Definition 1.1.4 ***(Stochastic processes):*** *A stochastic process defined on $(\Omega, \mathcal{F}, P)$ with T as the index set and (E, ξ) as the state space is called a family of random variables $(X_t; t \in T)$. For every $w \in \Omega$, the mapping $t \in [0, T] \to X_t(\omega)$ is called the trajectory (or realization) of the corresponding process X.*

Definition 1.1.5 ***(Brownian motion):*** *A stochastic process $(W(t), t \geq 0)$ is said to be a standard Brownian motion if it satisfies the following conditions:*

- $P[W(0) = 0] = 1$.
- For each $w \in \Omega$, the function $t \to W(t, w)$ is continuous almost surely with respect to the probability measure P.
- For all $s \leq t$, the random variable $W(t) - W(s)$ is normally distributed with mean 0 and variance $(t - s)$, that is, $W(t) - W(s) \sim \mathcal{N}(0, t - s)$.
- For any positive integer n and any non-decreasing sequence of times $0 \leq t_0 \leq t_1 \leq ... \leq t_n$, $\left(W_{t_n} - W_{t_{n-1}}, ..., W_{t_1} - W_{t_0}, W_{t_0}\right)$ are independents.

Definition 1.1.6 ***(Adapt-measurable-progressively measurable):***

- A process X is measurable if the application $(t,w) \to X_t(w)$ from $\mathbb{R}_+ \times \Omega$ to $\mathbb{R}^d$ is measurable with respect to the tribu $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}$ and $\mathcal{B}\left(\mathbb{R}^d\right)$.

- A process X is adapted with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ if, for all $t \geq 0$, X_t is $\mathcal{F}_t$-measurable.

- A process X is progressively measurable with respect to$\{\mathcal{F}_t\}_{t\geq 0}$ if for all $t \geq 0$, the application $(s,w) \to X_s(w)$ from $[0,t] \times \Omega$ to $\mathbb{R}^d$ is measurable with respect to $\mathcal{B}([0,t]) \otimes \mathcal{F}_t$ and $\mathcal{B}\left(\mathbb{R}^d\right)$.

Remark 1.1.1 *A progressively measurable process is measurable and adapted.*

Proposition 1.1.1 *If X is a stochastic process whose trajectories are right-continuous (or left-continuous), then X is measurable and X is progressively measurable if it is also adapted.*

1.2 Conditional expectation

Definition 1.2.1 ***(Conditional expectation with respect to a σ-algebra):*** *Let X be a real-valued random variable $($integrable , i.e., $X \in L^1)$ defined on $(\Omega, \mathcal{F}, P)$, and let $\mathcal{G}$ be a sub-sigma-algebra of $\mathcal{F}$. The conditional expectation $\mathbb{E}[X|\mathcal{G}]$ of X given $\mathcal{G}$ is the unique random variable:*

- *$\mathcal{G}$-measurable.*

- $\int_A \mathbb{E}[X|\mathcal{G}]\, dP = \int_A X\ dP$, $\forall A \in \mathcal{G}$.

It is also the unique (up to almost sure equality) $\mathcal{G}$-measurable random variable such that:

$$\mathbb{E}[Y\mathbb{E}[X|\mathcal{G}]] = \mathbb{E}[XY],$$

for any bounded Y, $\mathcal{G}$-measurable variable.

Proposition 1.2.1 ***(Properties of conditional expectation):*** *Let $(\Omega, \mathcal{F}, P)$ be a given probability space, and let $\mathcal{G}$ be a sub-sigma-algebra of $\mathcal{F}$. Let X and Y be two random variables defined on $(\Omega, \mathcal{F}, P)$*

- Linearity: For any constants a and b, we have

$$\mathbb{E}\left(aX + bY \left| \mathcal{G}\right.\right) = a\mathbb{E}\left(X \left| \mathcal{G}\right.\right) + b\mathbb{E}\left(Y \left| \mathcal{G}\right.\right).$$

- Monotonicity: If X and Y are random variables such that $X \leq Y$, then

$$\mathbb{E}\left(X \left| \mathcal{G}\right.\right) \leq \mathbb{E}\left(Y \left| \mathcal{G}\right.\right).$$

- If X is $\mathcal{G}$-measurable, then $\mathbb{E}\left(X \left| \mathcal{G}\right.\right) = X$.
- $\mathbb{E}\left[\mathbb{E}\left[X \left| \mathcal{G}\right.\right]\right] = \mathbb{E}\left[X\right]$.

Definition 1.2.2 ***(Continuous-time martingale):*** *A process $(X_t)_{t\geq 0}$ adapted with respect to a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ and such that for all $t \geq 0$, $X_t \in L^1$ is called:*

- *A martingale if for $s \leq t : \mathbb{E}\left[X_t \left| \mathcal{F}_s\right.\right] = X_s$.*
- *A supermartingale if for $s \leq t : \mathbb{E}\left[X_t \left| \mathcal{F}_s\right.\right] \leq X_s$.*
- *A submartingale if for $s \leq t : \mathbb{E}\left[X_t \left| \mathcal{F}_s\right.\right] \geq X_s$.*
- If X est une martingale $\mathbb{E}\left[X_t\right] = \mathbb{E}\left[X_0\right], \forall t$.
- If $(X_t, t \leq T)$ is a martingale, the process is completely determined by its terminal value: $X_t = \mathbb{E}\left[X_T \left| \mathcal{F}_t\right.\right]$. This latter property is very frequently used in finance.

Theorem 1.2.1 ***(Brownian martingale representation theorem):*** *Let $(\mathcal{F}_t)_{0\leq t\leq T}$ be the natural filtration of the Brownian motion $(W_t)_{0\leq t\leq T}$. Let M be a square integrable continuous martingale with respect to $(\mathcal{F}_t)_{0\leq t\leq T}$. Then there exists a unique predictable process H such that:*

$$\mathbb{E}\left(\int_t^T H_s^2 ds\right) < +\infty,$$

for all $t \in [0,T]$ and:

$$M_t = M_0 + \int_0^t H_s dW_s, \quad P\text{-}p.s.$$

1.3 Lévy processes

Let $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P\right)$ be a filtered probability space.

Definition 1.3.1 ***(Lévy process):*** *An $\mathcal{F}_t$ adapted process $\{L(t)\}_{t\geq 0} = \{L_t\}_{t\geq 0} \subset \mathbb{R}$ is called a Lévy process if it satisties :*

- $L_0 = 0$ *a.s.*
- L_t *is continuous in probability.*
- L_t *is stationary, independent increments.*

Theorem 1.3.1 *Let $\{L_t\}$ be a Lévy process. Then L_t has a càdlàg version (right continuous with left limits) which is also a Lévy process.*
The jump of L_t at $t > 0$ is defined by

$$\Delta L_t = L_t - L_{t^-}.$$

Let B_0 be the family of Borel sets $U \subset \mathbb{R}$ whose closure $\overline{U}$ does not contain 0. For $U \in B_0$ we define

$$N(t,U) = N(t,U,w) = \sum_{0\leq s\leq t} \chi_u(\Delta L_s).$$

In other words, $N(t,U)$ is the number of jumps of size $\Delta L_t \in U$ which occur before or at time t. $N(t,U)$ is called the Poisson random measure (or jump measure) of $L(t)$ The differential form of this measure is written $N(dt,dz)$.

Example 1.3.1 ***(Brownian motion):***

Brownian motion $\{W(t)\}_{t\geq 0}$ has stationary and independent increments. Thus $W(t)$ is a Lévy process.

Example 1.3.2 ***(The Poisson process):***

The Poisson process $\pi(t)$ of intensity $\lambda > 0$ is a Lévy process taking values in $\mathbb{N} \cup \{0\}$ and such that

$$P\left[\pi(t) = n\right] = \frac{(\lambda t)^n}{n!} e^{-\lambda t}; \qquad n = 0, 1, 2, ...$$

1.4 Itô's process

We give a probability space $(\Omega, \mathcal{F}, P)$ and a Brownian motion W on this space. We denote by $\mathcal{F}_t = \sigma\left(W_s,\ s \leq t\right)$ the natural fltration of the Brownian motion.

Theorem 1.4.1 ***(Itô's Lemma for 1-dimensional Brownian Motion):*** *Let W_t be a Brownian motion on $[0,T]$ and suppose $f(x)$ is a twice continuously differentiable function on $\mathbb{R}$. Then for any $t \leq T$ we have*

$$f\left(W_t\right) = f\left(0\right) + \frac{1}{2}\int_0^t f''\left(W_s\right) ds + \int_0^t f'\left(W_s\right) dW_s. \tag{1.1}$$

Proof. Let $0 = t_0 < t_1 < t_2 < ... < t_n = T$, be a partition of $[0,t]$. Clearly

$$f\left(W_t\right) = f\left(0\right) + \sum_{i=0}^{n-1}\left(f\left(W_{t_{i+1}}\right) - f\left(W_{t_i}\right)\right). \tag{1.2}$$

Taylor's Theorem implies

$$f\left(W_{t_{i+1}}\right) - f\left(W_{t_i}\right) = f'\left(W_{t_i}\right)\left(W_{t_{i+1}} - W_{t_i}\right) + \frac{1}{2} f''\left(\theta_i\right)\left(W_{t_{i+1}} - W_{t_i}\right)^2, \tag{1.3}$$

for some $\theta_i \in \left(W_{t_{i+1}} - W_{t_i}\right)$ Substituting (1.3) into (1.2) we obtain

$$f\left(W_{t_i}\right) = f\left(0\right) + \sum_{i=0}^{n-1} f'\left(W_{t_i}\right)\left(W_{t_{i+1}} - W_{t_i}\right) + \frac{1}{2}\sum_{i=0}^{n-1} f''\left(\theta_i\right)\left(W_{t_{i+1}} - W_{t_i}\right)^2. \tag{1.4}$$

If we let $\delta := \max\left|t_{i+1} - t_i\right| \to 0$ then it can be shown that the terms on the right-hand-side of (1.4) converge to the corresponding terms onthe right-hand-side of (1.1) as desired. (This should not be surprising as we know the quadratic variation of Brownian motion on $[0, t]$ is equal to t).

A more general version of Itô's Lemma can be stated for Itô processes. ■

Theorem 1.4.2 ***(Itô's Lemma for 2-dimensional Itô process):*** *Let X_t be 1-dimensional Itô process satisfying the SDE*

$$dX_t = \mu_t dt + \sigma_t dW_t.$$

If $f(t.x) : [0, \infty) \times \mathbb{R} \to \mathbb{R}$ is a $C^{1,2}$ function and $Y_t := f(t, X_t)$ then

$$\begin{aligned} dY_t &= \frac{\partial f}{\partial t}\left(t, X_t\right) dt + \frac{\partial f}{\partial x}\left(t, X_t\right) dX_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\left(t, X_t\right)\left(dX_t\right)^2 \\ &= \left(\frac{\partial f}{\partial t}\left(t, X_t\right) + \frac{\partial f}{\partial x}\left(t, X_t\right)\mu_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}\left(t, X_t\right)\sigma_t^2\right) dt + \frac{\partial f}{\partial x}\left(t, X_t\right)\sigma_t dW_t. \end{aligned}$$

The "Box" calculus

In the statement of Itô's Lemma, we implicitly assumed that $\left(dX_t\right)^2 = \sigma_t^2 dt$. The box calculus is a series of simple rules for calculating such quantities. In particular, we use the rules

$$dt \times dt = dt \times dW_t = 0,$$

$$\text{and } dW_t \times dW_t = dt,$$

when determining quantities such as $(dW_t)^2$ in the statement of Itô's Lemma above. When we have two correlated Brownian motions, $W_t^{(1)}$ and $W_t^{(2)}$, with correlation coeficient ρ_t, then we easily obtain that $dW_t^{(1)} \times dW_t^{(2)} = \rho_t dt$. We use the **box calculus** for computing the quadratic variation of Itô processes.

Proposition 1.4.1 ***(Integration by parts formula):*** *Let $x_i(t)$ be stochastic processes for $i = 1, 2$ and $t \in [0, T]$ satisfying:*

$$\begin{cases} dx_i(t) &= f(t, x_i(t), v(t))\, dt + \sigma(t, x_i(t), v(t))\, dW(t), \\ x_i(0) &= 0. \end{cases}$$

Then we get

$$\begin{aligned} \mathbb{E}(x_1(T)\, x_2(T)) = \mathbb{E}&\left[\int_0^T x_1(t)\, dx_2(t) + \int_0^T x_2(t)\, dx_1(t)\right] \\ &+ \mathbb{E}\left[\int_0^T \sigma^{\intercal}(t, x_1(t), v(t))\, \sigma(t, x_2(t), v(t))\, dt\right]. \end{aligned}$$

1.5 Some classes of stochastic controls

let $\left(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P\right)$ be a complete filtred probability space.

Definition 1.5.1 ***(Admissible control):*** *$\mathcal{F}_t$-adapted processes $v(t)$ with values in a borelian $A \subset \mathbb{R}^n$ is An admissible control adapted processes*

$$\mathcal{U} := \{v(\cdot) : [0, T] \times \Omega \to A : v(t) \text{ is } \mathcal{F}_t - \text{adapted}\}.$$

Definition 1.5.2 ***(Optimal control):*** *The goal of the optimal control problem is to minimize a cost function $J(v)$ over the set of admissible control $\mathcal{U}$. The control $u(\cdot)$ is an optimal control if*

$$J(u(t)) \leq J(v(t)), \text{ for all } v(\cdot) \in \mathcal{U}.$$

Definition 1.5.3 ***(Feedback control):*** *$v(\cdot)$ is a feedback control if the control $v(\cdot)$ depends on the state variable $X(\cdot)$. If $\mathcal{F}_t^X$ the natural filtration generated by the process X, then $v(\cdot)$ is a feedback control if $v(\cdot)$ is $\mathcal{F}_t^X$-adapted.*

Chapter §.2
Necessary conditions of optimality

Chapter 2

Necessary conditions of optimality

In this chapter, we are interested in the following control problem, on a given filtered probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\geq 0})$:

$$\begin{cases} dx^u(t) = b\left(t, x^u(t), u\left(t\right)\right) dt + \sum_{i=1}^{d} g^i\left(t, x^u(t), u\left(t\right)\right) dW^i(t) \\ \qquad +\sum_{i=1}^{\infty} c^i\left(t, x^u\left(t_-\right), u(t)\right) dH^i(t), \\ dy^u(t) = -f(t, x^u(t), y^u(t), z^u(t), r^u(t), u(t))dt + \sum_{i=1}^{d} z^{u,i}(t)dW^i(t) \\ \qquad +\sum_{i=1}^{\infty} r^{u,i}(t_-)dH^i(t), \\ x^u(0) = x_0,\ y^u(T) = h(x^u(T)), \end{cases} \tag{2.1}$$

where $W(\cdot)$ is a d-dimensional $\mathcal{F}_t$-Brownian motion. The Teugels martingales $H(t) = (H^i(t))_{i\geq 1}$ considered in this study are pairwise strongly orthonormal. These martingales are associated with a Lévy process that possesses moments of all orders. The control $u(\cdot) = (u(t))_{t\geq 0}$ is adapted to a subfiltration $(\mathcal{G}_t)_{t\geq 0}$ of $(\mathcal{F}_t)$. $x^u(t_-) = \lim_{s\to t, s<t} x^u(s)$, $t > 0$ the left limit process.

The maps

$$b : [0, T] \times \mathbb{R}^n \times \mathbb{A} \to \mathbb{R}^n,\ g : [0, T] \times \mathbb{R}^n \times \mathbb{A} \to \mathbb{R}^{n\times d},$$

$$c : [0, T] \times \mathbb{R}^n \times \mathbb{A} \to l^2(\mathbb{R}^n),\ h : \mathbb{R}^n \to \mathbb{R}^m,$$

$$f : [0, T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times \mathbb{R}^m \times \mathbb{A} \to l^2(\mathbb{R}^m),$$

are given deterministic functions.

The expected cost to be minimized takes the following structure

$$J(u(\cdot)) = E\left\{\int_0^T \ell(t, x^u(t), y^u(t), z^u(t), r^u(t), u(t))dt + \phi(x^u(T)) + \varphi(y^u(0))\right\}, \quad (2.2)$$

where

$$\phi : \mathbb{R}^n \to \mathbb{R},\ \varphi : \mathbb{R}^m \to \mathbb{R},$$
$$\ell : [0,T] \times \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times \mathbb{R}^m \times \mathbb{A} \to \mathbb{R},$$

are deterministic functions.

2.1 Notation and formulation of the problem

Let T is a fixed terminal time and $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\geq 0})$ be a fixed filtered probability space equiped with a P-completed right continuous filtration on which a d-dimensional Brownian motion $W(\cdot) = (W(t))_t$ is defined. Let $\mathbb{A}$ be a nonempty compact convex subset of $\mathbb{R}^n$ and let $M(\cdot) \triangleq \{(M(t)) : t \in [0,T]\}$ be a $\mathbb{R}$-valued Lévy process, independent of the Brownian motion $W(\cdot)$, of the form $M(t) = \lambda(t) + bt$, where $\lambda(t)$ is a pure jump process. Suppose that the Lévy measure $\mu(dx)$ associated with the Lévy process $\lambda(t)$ achieve the following conditions.

(i) There exist $\gamma > 0$ such that for every $\delta > 0 : \int_{(-\delta,\delta)} \exp(\gamma |x|)\, \mu(dx) < \infty$.

(ii) $\int_{\mathbb{R}} (1 \wedge x^2)\, \mu(dx) < \infty$.

We denote $\mathcal{A}_{\mathcal{G}}([0,T])$ the set of all admissible controls and we consider the P-augmentation of the natural filtration $\mathcal{F}_t$ denoted by $\left(\mathcal{F}_t^{(W,M)}\right)_{t\in[0,T]}$, which is defined in the following

$$\mathcal{F}_t^{(W,M)} \triangleq \mathcal{F}_t^W \vee \sigma\{M(s) : 0 \leq s \leq t\} \vee \mathcal{F}_0,$$

where $\mathcal{F}_t^{(W)} \triangleq \sigma\{W(s) : 0 \leq s \leq t\}$, $\mathcal{F}_0$ denotes the totality of P-null sets, and $\mathcal{F}_1 \vee \mathcal{F}_2$ denotes the σ-field generated by $\mathcal{F}_1 \cup \mathcal{F}_2$. Let $\mathcal{G}_t$ be a subfiltration of $\mathcal{F}_t : t \in [0,T]$.

An admissible control is defined as a function $u(\cdot):[0,T]\times\Omega\rightarrow\mathbb{A}$, which is G_t-pedictable, $E\int_0^T|u(s)|^2\,ds<\infty$, such that the equation (2.1) has a unique solution. An admissible control $u^*(\cdot)\in A_{\mathcal{G}}([0,T])$ is called optimal if

$$J(u^*(\cdot))\triangleq\inf_{u(.)\in\mathcal{A}_{\mathcal{G}}([0,T])}J(u(\cdot)). \tag{2.3}$$

Noting that, the jumps of $x^u(t)$ due to the influence of the Lévy process can be described as power jump processes, which are defined by

$$\begin{cases} M_{(k)}(t)\triangleq\sum_{0<t_j\leq t}(\Delta M(t_j))_k : k>1 \\ M_{(1)}(t)\triangleq M(t), \end{cases}$$

where $\Delta M(t)\triangleq M(t)-M(t_-)$, and $M(t_-)\triangleq\lim_{s\rightarrow t,s<t}M(s), t>0$. Furthermore, we can define the continuous component of $M_{(k)}(t)$ by eliminating the discontinuities or jumps present in $M(t)$:

$$M_{(k)}^{(c)}(t)\triangleq M_{(k)}(t)-\sum_{0<t_j\leq t}(\Delta M(t_j))_k : k>1.$$

Remark 2.1.1 *Let us consider the following* $N_{(k)}(t)=M_{(k)}(t)-E\left[M_{(k)}(t)\right] : k\geq 1$*. The jumps of* $x^u(t)$*, and* $y^u(t)$ *caused by the Lévy martingales are defined by*

$$\Delta_M x^u(t)\triangleq c(t,x^u(t_-),u(t))\Delta M(t)\cdot\Delta_M y^u(t)\triangleq\sum_{i=1}^{\infty}r^{u,i}(t_-)\Delta M^j(t)\ (\textit{respectively}).$$

and $H^i(t)\triangleq\sum_{1<k\leq i}\alpha_{ik}N_k(t)$*, where the coefficients* α_{ik} *associated with the orthonormalization of the polynomials* $\{1,x,x^2,...\}$ *with respect to the measure* $m(dx)=x^2\mu(dx)+g^2\delta_0(dx)$*. The Teugels martingales* $(H^i(\cdot))_{i\geq 1}$ *are pathwise strongly orthogonal and their predictable quadratic variation processes are given by* $\langle H^i(t),H^j(t)\rangle=\delta_{ij}t$*.*

Now, we introduce the fundamental notations.

- $\mathcal{M}^{n\times m}(\mathbb{R})$ denotes the space of $n\times m$ real matrices.

- $\mathbb{L}^2(\Omega, \mathcal{F}, P, \mathbb{R}^n)$ is the Banach space of $\mathbb{R}^n$-valued, square integrable random variables on $(\Omega, \mathcal{F}, P)$.

- l^2 is the Hilbert space of real-valued sequences $x = (x_n)_{n\geq 0}$ such that

$$\|x\| \triangleq \left[\sum_{n=1}^{\infty} x_n\right]^2 < \infty.$$

- $l^2(\mathbb{R}^n)$ is the space of $\mathbb{R}^n$-valued $(f_n)_{n\geq 1}$ such that

$$\|f\|_{l^2(\mathbb{R}^n)} \triangleq \left[\sum_{n=1}^{\infty} \|f_n\|_{\mathbb{R}^n}^2\right]^{\frac{1}{2}} < \infty.$$

- $\mathbb{L}^2_{\mathcal{F}}([0,T];\mathbb{R}^n)$ is the Banach space of $\mathcal{F}_t$-predictable processes f such that

$$\|f\|_{\mathbb{L}^2_{\mathcal{F}}([0,T];\mathbb{R}^n)} \triangleq E\left(\int_0^T \sum_{n=1}^{\infty} \|f_n\|_{\mathbb{R}^n}^2 \, dt\right)^{\frac{1}{2}} < \infty,$$

  where $f = \{f_n(t,w) : (t,w) \in [0,T] \times \Omega, n = 1, ..., \infty\}$.

- $\mathbb{M}^2_{\mathcal{F}}([0,T];\mathbb{R}^n)$ is the space of all $\mathbb{R}^n$-valued and $\mathcal{F}_t$-adapted processes f such that

$$\|f\|_{\mathbb{M}^2_{\mathcal{F}}([0,T]\times\Omega)} \triangleq E\left(\int_0^T \|f(t)\|_{\mathbb{R}^n}^2 \, dt\right)^{\frac{1}{2}} < \infty,$$

  where $f = \{f(t,w) : (t,w) \in [0,T] \times \Omega\}$.

- $\mathbb{S}^2_{\mathcal{F}}([0,T];\mathbb{R}^n)$ is the Banach space of $\mathcal{F}_t$-adapted and cadlag processes f such that

$$\|f\|_{\mathbb{S}^2_{\mathcal{F}}([0,T];\mathbb{R}^n)} \triangleq E(\sup_{0\leq t\leq T} \|f\|_{\mathbb{R}^n})^{\frac{1}{2}} < \infty,$$

  where $f = \{f(t,w) : (t,w) \in [0,T] \times \Omega\}$.

- $\mathbb{E}^{\mathcal{G}_t}[X]$ is the conditional expectation of X with respect to $\mathcal{G}_t$, $\mathbb{E}^{\mathcal{G}_t}(X) = \mathbb{E}(X \,|\mathcal{G}_t)$.

Assumptions

We will utilize the following foundational assumptions throughout our discussion.

Assumption (**A1**)

(i) The functions b, g, c are continuously differentiable in (x,u) and f, ℓ are continuously differentiable in (x,y,z,r,u), bounded by $C(1+|x|+|y|+|z|+|r|+|u|)$.

The function ϕ, h is continuously differentiable in x, and the function φ is continuously differentiable in y.

(ii) The derivatives b_x, b_u, g_x, g_u, c_x, c_u are bounded. The derivatives of ℓ with respect to (x,y,z,r,u) are bounded by $C(1+|x|^2+|y|^2+|z|^2+|r|^2+|u|^2)$. The derivatives ϕ_x bounded by $C(1+|x|^2)$, and φ_y is dominated by $C(1+|y|^2)$. The terminal value $y(T) \in l^2_{\mathcal{F}}([0,T];\mathbb{R}^n)$.

(iii) For all $t \in [0,T]$, $b(\cdot,0,0) \in \mathbb{L}^2_{\mathcal{F}}([0,T];\mathbb{R}^n)$, $f(\cdot,0,0,0,0,0) \in \mathbb{L}^2_{\mathcal{F}}([0,T];\mathbb{R}^n)$, $c(\cdot,0,0) \in \mathbb{L}^2_{\mathcal{F}}([0,T];\mathbb{R}^n)$, $g(\cdot,0,0) \in \mathbb{M}^2_{\mathcal{F}}([0,T];\mathbb{R}^{n\times d})$.

Under the above Assumption (**A1**), for each $u(\cdot) \in A_{\mathcal{G}}([0,T])$, Eq. (2.1) admits a unique strong solution $(x^u(\cdot), y^u(\cdot), z^u(\cdot), r^u(\cdot)) \in \mathbb{R}^n \times \mathbb{R}^m \times \mathcal{M}^{m\times d}(\mathbb{R}) \times l^2(\mathbb{R}^m)$ such that

$$
\begin{aligned}
x^u(t) &= x_0 + \int_0^t b\left(s, x^u(s), u(s)\right) ds + \sum_{i=1}^{d} \int_0^t g^i(s, x^u(s), u(s)) dW^i(s) \\
&\quad + \sum_{i=1}^{\infty} \int_0^t c^i(t, x^u(s_-), u(s)) dH^i(s), \\
y^u(t) &= y^u(T) - \int_t^T f(s, x^u(s), y^u(s), z^u(s), r^u(t), u(s)) ds \\
&\quad + \sum_{i=1}^{d} \int_t^T z^{u,i}(s) dW^i(s) + \sum_{i=1}^{\infty} \int_t^T r^{u,i}(s) dH^i(s).
\end{aligned}
$$

Let us present the adjoint equations associated with the stochastic maximum principle for the control problem (2.1)-(2.2).

We consider the following adjoint equations:

$$\begin{cases} d\Phi(t) = -\left\{ b_x(t)\Phi(t) + \sum_{i=1}^{d} g_x^i Q^i(t) + \sum_{i=1}^{\infty} c_x^i(t) G^i(t) - f_x(t) K(t) + \ell_x(t) \right\} dt \\ \qquad\qquad + \sum_{i=1}^{d} Q^i(t) dW^i(t) + \sum_{i=1}^{\infty} G^i(t) dH^i(t), \\ \Phi(T) = -h_x x(T)\mathcal{K}(T) + \Phi_x x(T). \\ d\mathcal{K}(t) = \left[f_y(t)\mathcal{K}(t) - \ell_y(t) \right] dt + \sum_{i=1}^{d} \left[f_z^i(t)\mathcal{K}^i(t) - \ell_z^i(t) \right] dW^i(t) \\ \qquad\qquad + \sum_{i=1}^{\infty} \left[f_r^i(t)\mathcal{K}^i(t) - \ell_r^i(t) \right] dH^i(t), \\ \mathcal{K}(0) = -\varphi_y y(0). \end{cases} \tag{2.4}$$

Under Assumption (**A1**), the FBSDE (2.4) admits a unique $\mathcal{F}$-adapted solution $(\Phi(\cdot), Q(\cdot), G(\cdot), \mathcal{K}(\cdot))$. The Hamiltonian function $\mathcal{H}$ associated with the stochastic control problem (2.1)−(2.2) defined as follows:

$$\begin{aligned} &\mathcal{H}(t, x, y, z, r, u, \Phi(\cdot), Q(\cdot), G(\cdot), \mathcal{K}(\cdot)) \\ &= \ell(t, x, y, z, r, u) + \Phi(t) b(t, x, u) + \sum_{i=1}^{d} Q^i(t) g^i(t, x, u) \\ &+ \sum_{i=1}^{\infty} G^i(t) c^i(t, x, u) - \mathcal{K}(t) f(t, x, y, z, r, u), \end{aligned} \tag{2.5}$$

where $(\Phi(\cdot), Q(\cdot), G(\cdot), \mathcal{K}(\cdot))$ solution of (2.4).

Then we can rewrite the adjoint Equation (2.4) as

$$\begin{cases} d\Phi(t) = -\mathcal{H}_x(t)dt + \sum_{i=1}^{d} Q^i(t) dW^i(t) + \sum_{i=1}^{\infty} G^i(t) dH^i(t), \\ \Phi(T) = -h_x x(T)\mathcal{K}(T) + \phi_x x(T)). \\ d\mathcal{K}(t) = -\mathcal{H}_y(t)dt - \sum_{i=1}^{d} \mathcal{H}_z(t) dW^i(t) - \sum_{i=1}^{\infty} \mathcal{H}_r(t) dH^i(t), \\ \mathcal{K}(0) = -\varphi_y y(0), \end{cases} \tag{2.6}$$

where $\mathcal{H}(t)=\mathcal{H}(t,x,y,z,r,u,\Phi\left(\cdot\right),Q\left(\cdot\right),G\left(\cdot\right),K(\cdot))$.

Under Assumption (**A1**), the adjoint Equation (2.4) or (2.6) admits a unique solution.

2.2 Stochastic Maximum principle

In this section, we derive necessary conditions for the optimal control using the maximum principle. The control system under consideration is governed by FBSDEs (Forward-Backward Stochastic Differential Equations) driven by orthogonal Teugels martingales, which are associated with certain Lévy processes possessing moments of all orders. We need to make the following Assumptions.

Assumption (**A2**)

For all $0\leq t\leq t+\tau\leq T$, $i=1,...,k$ and all bounded $\mathcal{G}_t$-measurable $\beta=\beta(w)$, the control $\gamma(t)=(0,...,0,\gamma_i(t),0,...0)\in\mathbb{A}$ with

$$\gamma_i(s)=\beta_i 1_{[t,t+\tau]}(s),\ s\in[0,T],\ \text{for } i=1,...,k,$$

belong to $\mathcal{A}_{\mathcal{G}}([0,T])$. Here, $1_{[t,t+\tau]}(s)$ is the indicator function on the set $[t,t+\tau]$.

Assumption (**A3**)

For all $u(\cdot),\gamma(\cdot)\in\mathcal{A}_{\mathcal{G}}([0,T])$ with $\gamma(\cdot)$ bounded, there exist $\delta>0$ such that $u(\cdot)+\theta\gamma(\cdot)\in\mathcal{A}_G([0,T])$ for all $\theta\in(-\delta,\delta)$.

Now, for a given $u(\cdot),\gamma(\cdot)\in\mathcal{A}_{\mathcal{G}}([0,T])$ with $\gamma(\cdot)$ bounded, we define

$$\begin{aligned}
X_1(t) &= X_1^{u^*,\gamma}(t):=\frac{d}{d\theta}\left[x^{u^*+\theta\gamma}(t)\right]\Big|_{\theta=0}, \qquad (2.7)\\
Y_1(t) &= Y_1^{u^*,\gamma}(t):=\frac{d}{d\theta}\left[y^{u^*+\theta\gamma}(t)\right]\Big|_{\theta=0},\\
Z_1(t) &= Z_1^{u^*,\gamma}(t):=\frac{d}{d\theta}\left[z^{u^*+\theta\gamma}(t)\right]\Big|_{\theta=0},\\
Q_1(t) &= Q_1^{u^*,\gamma}(t):=\frac{d}{d\theta}\left[r^{u^*+\theta\gamma}(t)\right]\Big|_{\theta=0},
\end{aligned}$$

where $(x^{u^*+\theta\gamma}(\cdot), y^{u^*+\theta\gamma}(\cdot), z^{u^*+\theta\gamma}(\cdot), r^{u^*+\theta\gamma}(\cdot))$ the solution of Equation (2.1) corresponding to $(u^* + \theta\gamma)$.

Now, we consider the following linear FBSDEs:

$$\begin{cases} dX_1(t) = \left[b_x\left(t\right)X_1\left(t\right) + b_u(t)\gamma(t)\right]dt \\ \qquad\qquad +\sum_{i=1}^{d}\left[g_x^i X_1(t) + g_u^j(t)\gamma(t)\right]dW^i(t) \\ \qquad\qquad +\sum_{i=1}^{\infty}\left[c_x^i X_1(t) + c_u^i(t)\gamma(t)\right]dH^i(t), \\ dY_1(t) = \left[f_x\left(t\right)Y_1\left(t\right) + f_y\left(t\right)Y_1\left(t\right) + f_z\left(t\right)Y_1\left(t\right) + f_u(t)\gamma(t)\right]dt \\ \qquad\qquad +\sum_{i=1}^{d} Z_1^i(t)dW^i(t) + \sum_{i=1}^{\infty} Q_1^i(t)dH^i(t), \\ X_1\left(0\right) = 0,\ Y_1(T) = h_x x(T)X_1(T). \end{cases} \tag{2.8}$$

Let $u^*(\cdot)$ represent a locale minimum for te cost functional J over $\in \mathcal{A}_{\mathcal{G}}([0,T])$, indicating that, for any bounded $\gamma(\cdot) \in \mathcal{A}_{\mathcal{G}}([0,T])$, there exist $\delta > 0$ such that $((u^*(\cdot) + \theta\gamma(\cdot)) \in \mathcal{A}_{\mathcal{G}}([0,T])$ for all $\theta \in (-\delta, \delta)$ and $\mathcal{J}(\theta) = J((u^*(\cdot) + \theta\gamma(\cdot))$ achieves its minimum at $\theta = 0$

$$\frac{d}{d\theta}[\mathcal{J}(\theta)]\Big|_{\theta=0} = \frac{d}{d\theta}J(u^*(t) + \theta\gamma(t))\Big|_{\theta=0} = 0,\ \text{for all } \theta \in (-\delta, \delta). \tag{2.9}$$

Theorem 2.2.1 *Let $(x^*(\cdot), y^*(\cdot), z^*(\cdot), r^*(\cdot))$ be the solution of the FBSDE (2.1) corresponding to $u^*(\cdot)$. Let Assumptions* (**A1**)**,**(**A2**) *and* (**A3**) *hold. Then, there exists a unique adapted process $(\Phi^*(\cdot), Q^*(\cdot), G^*(\cdot), \mathcal{K}^*(t))$ solution of (2.4) such that $u^*(\cdot)$ is a stationary point for $E^{\mathcal{G}_t}(\mathcal{H})$ such that*

$$E^{\mathcal{G}_t}[\mathcal{H}_u(t, x^*, y^*, z^*, r^*, u^*, \Phi^*(\cdot), Q^*(\cdot), G^*(\cdot), \mathcal{K}^*(t))] = 0, \quad a.e. \tag{2.10}$$

In order to establish the validity of the above theorem, the following technical lemma plays a significant role in the subsequent analysis.

Lemma 2.2.1 *Using Itô's formula to* $\Phi^*(t)X_1(t), \mathcal{K}^*(t)Y_1(t)$ *and take expectation, we get*

$$
\begin{aligned}
&E(\Phi^*(T)X1(T)) + E(\mathcal{K}^*(T)Y_1(T)) \qquad (2.11)\\
&= -E\left[\varphi_y y\left(0\right)Y_1\left(0\right)\right] - E\int_0^T \{X_1(t)\ell_x\left(t\right) + Y_1\left(t\right)\ell_y\left(t\right)\\
&+ Z_1\left(t\right)\ell_z\left(t\right) + Q_1(t)r\left(t\right) + \ell_u\left(t\right)\gamma\left(t\right)\}dt + E\int_0^T \mathcal{H}_u\left(t\right)\gamma\left(t\right)dt.
\end{aligned}
$$

Proof. By applying Itô's formula to $\Phi^*(t)X_1(t)$ and take expectation, we get

$$
\begin{aligned}
&E(\Phi^*(T)X_1(T)) \qquad (2.12)\\
&= E\int_0^T \Phi^*(t)dX_1(t) + E\int_0^T X_1(t)d\Phi^*(t)\\
&+ E\int_0^T \sum\nolimits_{i=1}^{d} Q^{j*}\left(t\right)\left[g_x^j\left(t\right)X_1\left(t\right) + g_x^j\left(t\right)\gamma\left(t\right)\right]dt\\
&+ E\int_0^T \sum\nolimits_{i=1}^{\infty} G^{j*}\left(t\right)\left[c_x^j\left(t\right)X_1\left(t\right) + c_u^j\left(t\right)\gamma\left(t\right)\right]dt\\
&= I_{1,1} + I_{1,2} + I_{1,3} + I_{1,4},
\end{aligned}
$$

where

$$
\begin{aligned}
I_{1,1} &= E\int_0^T \Phi^*(t)dX_1(t) \qquad (2.13)\\
&= E\int_0^T \Phi^*(t)\left[b_x\left(t\right)X_1(t) + b_u\left(t\right)\gamma\left(t\right)\right]dt\\
&= E\int_0^T \Phi^*(t)b_x\left(t\right)X_1(t)dt + E\int_0^T \Phi^*(t)b_u\left(t\right)\gamma\left(t\right)dt.
\end{aligned}
$$

$$
\begin{aligned}
I_{1,2} = E\int_0^T X_1(t)d\Phi^*(t) = -E\int_0^T X_1(t)\{b_x\left(t\right)\Phi^*(t) \qquad (2.14)\\
+ \sum\nolimits_{i=1}^{d} g_x^j\left(t\right)Q^{j*}\left(t\right) + \sum\nolimits_{i=1}^{\infty} c_x^j\left(t\right)G^{j*}\left(t\right)\\
- f_x\left(t\right)\mathcal{K}\left(t\right) + \ell_x\left(t\right)\}dt.
\end{aligned}
$$

Using a similar reasoning as mentioned earlier, one can demonstrate that

$$I_{1,3} = E\int_0^T \sum\nolimits_{i=1}^{d} Q^{j*}(t)\left[g_x^j(t)\,X_1(t) + g_u^j(t)\,\gamma(t)\right]dt \tag{2.15}$$
$$= E\int_0^T \sum\nolimits_{i=1}^{d} Q^{j*}(t)\,g_x^j(t)\,X_1(t)\,dt$$
$$+ E\int_0^T \sum\nolimits_{i=1}^{d} Q^{j*}(t)\,g_u^j(t)\,\gamma(t)\,dt.$$

To obtain an estimate for $I_{1,4}$, one can readily deduce using the same line of reasoning that

$$I_{1,4} = E\int_0^T \sum\nolimits_{i=1}^{d} G^{j*}(t)\left[c_x^j(t)\,X_1(t) + c_u^j(t)\,\gamma(t)\right]dt \tag{2.16}$$
$$= E\int_0^T \sum\nolimits_{i=1}^{d} G^{j*}(t)\,c_x^j(t)\,X_1(t)\,dt$$
$$+ E\int_0^T \sum\nolimits_{i=1}^{d} G^{j*}(t)\,c_u^j(t)\,\gamma(t)\,dt.$$

Combining (2.12) with (2.16), we get

$$E\left(\Phi^*(T)X_1(T)\right) = E\int_0^T \Phi^*(t)b_u(t)\,\gamma(t)\,dt + E\int_0^T \sum_{j=1}^{d} Q^{j*}(t)\,g_u^j(t)\,\gamma(t)\,dt + \tag{2.17}$$
$$E\int_0^T \sum_{j=1}^{\infty} G^{j*}(t)\,c_u^j(t)\,\gamma(t)\,dt - E\int_0^T X_1(t)\,\ell_x(t)\,dt - E\int_0^T X_1(t)\,f_x(t)\,\mathcal{K}(t)\,dt.$$

Similarly, by applying Itô's formula to $\mathcal{K}^*(t)Y_1(t)$ and take expectation, we obtain

$$E\left(\mathcal{K}^*(T)Y_1(T)\right) = -E\left\{\varphi_y y(0)\,Y_1(0)\right\} \tag{2.18}$$
$$+ E\int_0^T \{\mathcal{K}^*(t)\,f_x(t)\,X_1(t)$$
$$+ \mathcal{K}^*(t)\,f_u(t)\,\gamma(t) - Y_1(t)\,\ell_y(t)$$
$$- Z_1(t)\,\ell_z(t) - Q_1(t)\,\ell_r(t)\}dt.$$

Finally, by combining equations (2.17) and (2.18), we can successfully achieve the desired outcome as stated in equation (2.11). ∎

Proof of Theorem 2.2.1. From (2.9) and by differentiating $\mathcal{J}\left(\theta\right)$with respect to θ at $\theta=0$, we get

$$\begin{aligned}&\frac{d}{d\theta}\left[\mathcal{J}\left(\theta\right)\right]\Big|_{\theta=0} \\ &=E\int_0^T\left[\ell_x\left(t\right)X_1\left(t\right)+\ell_y\left(t\right)Y_1\left(t\right)+\ell_z\left(t\right)Z_1\left(t\right)+\ell_r\left(t\right)Q_1\left(t\right)+\ell_u\left(t\right)\gamma\left(t\right)\right]dt \\ &+E\left[\Phi_x x^*\left(T\right)X_1\left(T\right)\right]+E\left[\varphi_y y^*\left(0\right)Y_1\left(0\right)\right]=0.\end{aligned} \tag{2.19}$$

From (2.19) and (2.11), we obtain

$$E\int_0^T\left[\Phi^*\left(t\right)b_u\left(t\right)+\sum_{i=1}^{d}Q^{*i}\left(t\right)g_u^i\left(t\right)+\sum_{i=1}^{\infty}G^{*i}\left(t\right)c_u^i\left(t\right)+\mathcal{K}\left(t\right)f_u\left(t\right)+\ell_u\left(t\right)\right]\gamma\left(t\right)dt=0.$$

From (2.5), we have

$$E\int_0^T\mathcal{H}_u\left(t,x^*,y^*,z^*,r^*,u^*,\Phi^*\left(\cdot\right),Q^*\left(\cdot\right),G^*\left(\cdot\right),\mathcal{K}^*\left(t\right)\right)\gamma\left(t\right)dt=0, \tag{2.20}$$

fix $t\in[0,T]$, and apply the above to $\gamma\left(t\right)=\left(0,...,\gamma_i\left(t\right),...,0\right)$, where $\gamma_i\left(s\right)=\beta_i\left(s\right)1_{[t,t+\tau]}\left(s\right),$ $s\in\left[0,T\right],t+\tau\leq T$ and $\beta_i\left(s\right)$ is bounded, $\mathcal{G}_t$-measurable.

Then, from (2.20), we get

$$E\int_t^{t+\tau}\mathcal{H}_{u_i}\left(s,x^*,y^*,z^*,r^*,u^*\left(s\right),\Phi^*\left(s\right),Q^*\left(s\right),G^*\left(s\right),\mathcal{K}^*\left(s\right)\right)\beta_i\left(s\right)ds=0.$$

Now, by differentiating (2.21) with respect to τ at $\tau=0$, we obtain $t\in[0,T]$

$$\mathcal{H}_{u_i}\left(t,x^*,y^*,z^*,r^*,u^*\left(t\right),\Phi^*\left(t\right),Q^*\left(t\right),G^*\left(t\right),\mathcal{K}^*\left(t\right)\right)\beta_i\left(t\right)dt=0. \tag{2.21}$$

Since (2.21) remains true for all bounded $\mathcal{G}_t$-measurable $\beta_i(\cdot)$, we have $t \in [0, T]$

$$E^{\mathcal{G}_t}\left[\mathcal{H}_u\left(t, x^*, y^*, z^*, r^*, u^*(t), \Phi^*(t), Q^*(t), G^*(t), \mathcal{K}^*(t)\right)\right] = 0, \quad a.e.,$$

then (2.10) is fulfilled. This completes the proof of Theorem 2.2.1. ■

Chapter §.3
Sufficient conditions of optimality

Chapter 3

Sufficient conditions of optimality

Our objective is to establish sufficient conditions for optimal control in relation to the control problem defined by equations (2.1)-(2.2). We demonstrate that, under certain additional assumptions, the necessary conditions outlined in Theorem 2.2.1 can serve as sufficient conditions for optimality. A function $f : \mathbb{R} \longrightarrow \mathbb{R}^n$ is convex if, for every $(x, x\prime) \in \mathbb{R}$, $f(x\prime) - f(x) \geqslant f_x(x)(x\prime - x)$.

3.1 Additional Assumption (A4)

(i) The functional $\mathcal{H}(t, \cdot, \cdot, \cdot, \cdot, \cdot, \Phi^*(t), Q^*(t), G^*(t), \mathcal{K}^*(.))$ is convex with respect to (x, y, z, r, u) for $a.\ e.\ t \in [0, T]$, $P - a.\ s.$

$$
\begin{aligned}
&\mathcal{H}(t, x\prime, y\prime, z\prime, r\prime, u\prime, \Phi^*(t), Q^*(t), G^*(t), \mathcal{K}^*(\cdot)) \\
&- \mathcal{H}(t, x, y, z, r, u, \Phi^*(t), Q^*(t), G^*(t), \mathcal{K}^*(\cdot)) \\
&\geq \mathcal{H}_x(t, x, y, z, r, u, \Phi^*(t), Q^*(t), G^*(t), \mathcal{K}^*(\cdot))(x' - x) \\
&+ \mathcal{H}_y(t, x, y, z, r, u, \Phi^*(t), Q^*(t), G^*(t), \mathcal{K}^*(\cdot))(y' - y) \\
&+ \mathcal{H}_z(t, x, y, z, r, u, \Phi^*(t), Q^*(t), G^*(t), \mathcal{K}^*(\cdot))(z' - z)
\end{aligned}
$$

$$+\mathcal{H}_r\left(t,x,y,z,r,u,\Phi^*\left(t\right),Q^*\left(t\right),G^*\left(t\right),\mathcal{K}^*(\cdot)\left(r'-r\right)\right.$$

$$+\mathcal{H}_u\left(t,x,y,z,r,u,\Phi^*\left(t\right),Q^*\left(t\right),G^*\left(t\right),\mathcal{K}^*\left(\cdot\right)\right)\left(u'-u\right).$$

(ii) The function $\phi\left(\cdot,\cdot\right)$ is convex with respect to (x) and the function $\varphi\left(\cdot,\cdot\right)$ is convex with respect to (y).

3.2 Sufficient conditions under partial information

Theorem 3.2.1 *For any admissible control* $u\left(\cdot\right)\in\mathcal{A}_{\mathcal{G}}\left([0,T]\right)$, *and by Assumptions* $(\mathbf{A1})$–$(\mathbf{A4})$, *the following relation holds*

$$E^{\mathcal{G}_t}\left[\mathcal{H}_u\left(t,x^u,y^u,z^u,r^u,u\left(t\right),\Phi^u\left(t\right),Q^u\left(t\right),G^u\left(t\right),\mathcal{K}^u\left(t\right)\right)\right]=0\ a.e., \tag{3.1}$$

then we have

$$inf\left\{J\left(v\left(\cdot\right)\right):v\left(\cdot\right)\in\mathcal{A}_{\mathcal{G}}\left([0,T]\right)\right\}=J\left(u\left(\cdot\right)\right), \tag{3.2}$$

In other words, the admissible control $u(\cdot)$ can be considered as a partial-information optimal control for the problem described by equations (2.1)-(2.2).

In order to establish Theorem 2.2.1, we require the following auxiliary result, which examines the duality relationships between $\Phi^u\left(t\right),\left[x\left(t\right)-x^u\left(t\right)\right]$, and between $\mathcal{K}^u\left(t\right),\left[y(t)-y^u(t)\right]$.

This lemma holds significance in establishing our sufficient conditions for optimality.

Lemma 3.2.1 *Let $(x(\cdot), y(\cdot), z(\cdot), r(\cdot))$ be the solution of (2.1) corresponding to any admissible control $v(\cdot)$.*

$$
\begin{aligned}
&E\left[\Phi^{u}(T)\left(x(T)-x^{u}(T)\right)\right] \\
&=E \int_{0}^{T} \Phi^{u}(t)\left[b\left(t, x(t), v(t)\right)-b\left(t, x^{u}(t), u(t)\right)\right] dt \\
&+E \int_{0}^{T} \mathcal{H}_{x}(t)\left(x(t)-x^{u}(t)\right) dt \\
&+E \int_{0}^{T} \sum_{i=1}^{d} Q^{i,u}(t)\left[g^{i}\left(t, x(t), v(t)\right)-g^{i}\left(t, x^{u}(t), u(t)\right)\right] dt \\
&+E \int_{0}^{T} \sum_{i=1}^{\infty} G^{i,u}(t)\left[c^{i}\left(t, x(t), v(t)\right)-c^{i}\left(t, x^{u}(t), u(t)\right)\right] dt,
\end{aligned}
$$

where $\mathcal{H}(t)=\mathcal{H}\left(t, x^{u}, y^{u}, z^{u}, r^{u}, u(t), \Phi^{u}(t), Q^{u}(t), G^{u}(t), \mathcal{K}^{u}(t)\right)$.

Similarly,

$$
\begin{aligned}
&E\left[\mathcal{K}^{u}(T)\left(y(T)-y^{u}(T)\right)\right] \\
&=-E\left(\varphi_{y}(y(0))\left(y^{u}(0)-y(0)\right)\right) \\
&+E \int_{0}^{T} \mathcal{K}^{u}(t)\left[f\left(t, x(t), y(t), z(t), r(t), v(t)\right)\right. \\
&\left.-f\left(t, x^{u}(t), y^{u}(t), z^{u}(t), r^{u}(t), u(t)\right)\right] dt+E \int_{0}^{T} \mathcal{H}_{y}(t)\left(y(t)-y^{u}(t)\right) dt \\
&+E \int_{0}^{T} \sum_{i=1}^{d} \mathcal{H}_{z^{i}}(t)\left(z^{i}(t)-z^{i,u}(t)\right) dt+E \int_{0}^{T} \sum_{i=1}^{\infty} \mathcal{H}_{r^{i}}(t)\left(r^{i}(t)-r^{i,u}(t)\right) dt,
\end{aligned}
\tag{3.3}
$$

and

$$
\begin{aligned}
&E\left[\Phi^{u}(T)\left(x(T)-x^{u}(T)\right)\right]+E\left[\mathcal{K}^{u}(T)\left(y(T)-y^{u}(T)\right)\right] \\
&+E\left[\varphi_{y}(y(0))\left(y^{u}(0)-y(0)\right)\right] \\
&=E \int_{0}^{T} \Phi^{u}(t)\left(b\left(t, x(t), v(t)\right)-b\left(t, x^{u}(t), u(t)\right)\right) dt
\end{aligned}
\tag{3.4}
$$

$$+\int_0^T \sum_{i=1}^{d} Q^{u,i}(t)\left[g^i(t,x(t),v(t)) - g^i(t,x^u(t),u(t))\right]dt$$
$$+E\int_0^T \sum_{i=1}^{\infty} G^{u,i}(t)\left[c^i(t,x(t),v(t)) - c^i(t,x^u(t),u(t))\right]dt$$
$$+E\int_0^T K^u(t)\left[f(t,x(t),y(t),z(t),r(t),v(t)) - f(t,x^u(t),y^u(t),z^u(t),r^u(t),u(t))\right]dt$$
$$+E\int_0^T \mathcal{H}_x(t)(x(t)-x^u(t))\,dt + E\int_0^T \mathcal{H}_y(t)(y(t)-y^u(t))\,dt$$
$$+E\int_0^T \sum_{i=1}^{d} \mathcal{H}_{z^i}(t)\left(z^i(t)-z^{i,u}(t)\right)dt + E\int_0^T \sum_{i=1}^{\infty} \mathcal{H}_{r^i}(t)\left(r^i(t)-r^{i,u}(t)\right)dt.$$

Proof.

First, by simple computations, we get

$$\begin{aligned} d(x(t)-x^u(t)) &= (b(t,x(t),v(t)) - b(t,x^u(t),u(t)))\,dt \\ &\quad + \sum_{i=1}^{d} g^i(t,x(t),v(t)) - g^i(t,x^u(t),u(t))\,dW^i(t) \\ &\quad + \sum_{i=1}^{\infty} c^i(t,x(t),v(t)) - c^i(t,x^u(t),u(t))\,dH^i(t), \end{aligned} \tag{3.5}$$

and

$$\begin{aligned} d(y(t)-y^u(t)) &= (f(t,x(t),y(t),z(t),r(t),v(t)) \\ &\quad - f(t,x^u(t),y^u(t),z^u(t),r^u(t),u(t))\,dt \\ &\quad + \sum_{i=1}^{d} \left(z^i(t)-z^{i,u}(t)\right)dW^i(t) \\ &\quad + \sum_{i=1}^{\infty} \left(r^i(t)-r^{i,u}(t)\right)dH^i(t). \end{aligned} \tag{3.6}$$

$$\begin{aligned} E\left[\Phi^{u}(T)(x(T)-x^{u}(T))\right] &= E\int_{0}^{T}\Phi^{u}(t)\,d(x(t)-x^{u}(t)) \\ &+E\int_{0}^{T}(x(t)-x^{u}(t))\,d\Phi^{u}(t) \\ &+E\int_{0}^{T}\sum_{i=1}^{d}Q^{i,u}(t)\left[g^{i}(t,x(t),v(t))-g^{i}(t,x^{u}(t),u(t))\right]dt \\ &+E\int_{0}^{T}\sum_{i=1}^{\infty}G^{i,u}(t)\left[c^{i}(t,x(t),v(t))-c^{i}(t,x^{u}(t),u(t))\right]dt \\ &= I_{2,1}+I_{2,2}+I_{2,3}+I_{2,4}. \end{aligned} \tag{3.7}$$

From (3.5), we obtain

$$\begin{aligned} I_{2,1} &= E\int_{0}^{T}\Phi^{u}(t)\,d(x(t)-x^{u}(t)) \\ &= E\int_{0}^{T}\Phi^{u}(t)\left[b(t,x(t),v(t))-b(t,x^{u}(t),u(t))\right]dt. \end{aligned} \tag{3.8}$$

Similarly, by applying (2.6), we get

$$\begin{aligned} I_{2,2} &= E\int_{0}^{T}(x(t)-x^{u}(t))\,d\Phi^{u}(t) \\ &= E\int_{0}^{T}\mathcal{H}_{x}(t)(x(t)-x^{u}(t))\,dt \\ &= E\int_{0}^{T}\mathcal{H}_{x}(t)(x(t)-x^{u}(t))\,dt. \end{aligned} \tag{3.9}$$

By standard arguments, we obtain

$$I_{2,3}=E\int_{0}^{T}\sum_{i=1}^{d}Q^{i,u}(t)\left[g^{i}(t,x(t),v(t))-g^{i}(t,x^{u}(t),u(t))\right]dt, \tag{3.10}$$

and

$$I_{2,4} = E\int_0^T \sum_{i=1}^{\infty} G^{i,u}(t)\left[c^i(t,x(t),v(t)) - c^i(t,x^u(t),u(t))\right]dt, \tag{3.11}$$

We now turn our attention to second Equation (3.3). By applying integration by parts formula to $(y^u(t) - y(t))\mathcal{K}^u(t)$, we get

$$\begin{aligned} E(\mathcal{K}^u(T)(y^u(T) - y(T))) &= E[\mathcal{K}^u(0)(y^u(0) - y(0))] \\ &+ E\int_0^T \mathcal{K}^u(t)\,d(y(t) - y^u(t)) + E\int_0^T (y(t) - y^u(t))\,dK^u(t) \\ &+ E\int_0^T \sum_{i=1}^{d}\left(z^i(t) - z^{i,u}(t)\right)[\mathcal{H}_{z^i}(t)]\,dt \\ &+ E\int_0^T \sum_{i=1}^{\infty}\left(r^i(t) - r^{i,u}(t)\right)[\mathcal{H}_{r^i}(t)]\,dt \\ &= I_{3,1} + I_{3.2} + I_{3,3} + I_{3,4} + I_{3.5}. \end{aligned} \tag{3.12}$$

Let us turn to the second term $I_{3.2}$. From (3.6), we have

$$\begin{aligned} I_{3.2} &= E\int_0^T \mathcal{K}^u(t)\,d(y(t) - y^u(t)) \\ &= E\int_0^T \mathcal{K}^u(t)\,[f(t,x(t),y(t),z(t),r(t),v(t)) \\ &- f(t,x^u(t),y^u(t),z^u(t),r^u(t),u(t))]dt, \end{aligned} \tag{3.13}$$

from (2.6), we obtain

$$I_{3,3} = E\int_0^T (y(t) - y^u(t))\,d\mathcal{K}^u(t) = E\int_0^T \mathcal{H}_y(t)(y(t) - y^u(t))\,dt, \tag{3.14}$$

$$I_{3,4} = E\int_0^T \sum_{i=1}^{d} \mathcal{H}_{z^i}(t)\left(z^i(t) - z^{i,u}(t)\right)dt, \tag{3.15}$$

and similarly, we obtain

$$I_{3.5} = E\int_0^T \sum_{i=1}^{\infty} \mathcal{H}_{r^i}(t)\left(r^i(t) - r^{i,u}(t)\right)dt. \tag{3.16}$$

From (2.4) and the fact that

$$\begin{aligned} I_{3,1} &= E\left(\mathcal{K}^u(0)\left(y^u(0) - y(0)\right)\right) \\ &= -E\left\{\varphi_y(0) + y^u(0) - y(0)\right\}, \end{aligned} \tag{3.17}$$

the duality relation (3.3) follows immediately by combining (3.13) − (3.17) together with (3.12). Finally, inequality (3.4) is fulfilled from (3.3).

Finally, The Proof of Theorem 3.2.1 is based on the convexity condition on $\phi(\cdot,\cdot)$ and $\varphi(\cdot,\cdot)$, Lemma 3.2.1, and by the convexity of the functional $\mathcal{H}$.

Conclusion

In this master's dissertation, we have discussed the necessary and sufficient conditions for optimal control of FBSDEs driven by Teugels martingale associated with Lévy processes. This kind of problem, which has a lot of applications in mathematical finance and economics.

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Appendix A: Abbreviations and Notations

The different abbreviations and notations used throughout this thesis are explained below:

$(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\geq 0})$	Complete probability space.
$\mathcal{F}_t^W$	Filtration generated by the Brownian motion W.
$\mathcal{F}_t^M$	Filtration generated by the Lévy process M.
W	Brownian motion.
$a.e.$	Almost everywhere.
$a.s.$	Almost surely.
$\mathcal{H}(t)$	Hamiltonian function.
SDE	Stochastic differential equations.
BSDE	Backward stochastic differential equation.
ODE	Ordinary differential equation.
$\mathbb{L}^2(\otimes, \mathcal{F}, P, \mathbb{R}^n)$	Banach space of $\mathbb{R}^n$-valued, square integrable random variables.
$\mathcal{A}_{\mathcal{G}}([0,T])$	Set of all admissible controls.
$\mathbb{L}^2_{\mathcal{F}}([0,T];\mathbb{R}^n)$	Banach space of $\mathcal{F}_t$-predictable processes.
$\mathbb{M}^2_{\mathcal{F}}([0,T];\mathbb{R}^n)$	Space of all $\mathbb{R}^n$-valued and $\mathcal{F}_t$-adapted processes.
$\mathbb{S}^2_{\mathcal{F}}([0,T];\mathbb{R}^n)$	Banach space of $\mathcal{F}_t$-adapted and cadlag processes.
$E^{\mathcal{G}_t}[X]$	Conditional expectation of X.
l^2	Hilbert space of real-valued sequences.

Abstract

This master's dissertation investigates stochastic optimal control problems concerning forward-backward differential equations FBSDEs. These equations are driven by Teugels martingales and involve a Lévy process with moments of all orders, as well as an independent Brownian motion. In this work, we studied the necessary and the sufficient conditions of optimality for partial information stochastic optimal control problem.

Keywords: Forward-backward stochastic systems with levy process, optimal stochastic control, Teugels martingales, necessary and sufficient stochastic maximum principle.

Résume

Ce mémoire étudie les problèmes de contrôle optimal stochastique concernant les équations différentielles FBSDEs. Ces équations sont pilotées par des martingales de Teugels et impliquent un processus de Lévy avec des moments de tous ordres, ainsi qu'un mouvement brownien indépendant. Dans ce travail, nous avons étudié les conditions nécessaires et suffisantes d'optimalité pour le problème de contrôle optimal stochastique à information partielle.

Mots clés : Systèmes stochastiques avec un processus de Levy, contrôle stochastique optimal, martingales de Teugels, principe de maximum stochastique nécessaire et suffisant.

الملخص

في هذه الأطروحة نهتم بمشاكل التحكم الأمثل العشوائية المتعلقة بالمعادلات التفاضلية الأمامية والخلفية. هذه المعادلات مدفوعة بواسطة توقيليس مارتينقال وتتضمن عملية ليفي مع لحظات من جميع الطلبات ، بالإضافة إلى حركة براونية مستقلة. في هذا العمل ، درسنا الشروط اللازمة والكافية للأمثل لمشكلة التحكم الأمثل العشوائية للمعلومات الجزئية.

الكلمات المفتاحية: أنظمة الحوكمة العشوائية الأمامية و الخلفية مع عملية ليفي, التحكم العشوائي الأمثل, مارتنجلات توغيلز, مبدا الحد الأقصى الضروري و الكافي.