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Stochastic Differential Equations Driven by a Jump Markov Process and Their Applications.

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Dedication

I dedicate this work:

To my mother, my sister, and my husband.

To all my family.

Acknowledgment

First of all, I thank Allah who gave me the will and the courage to be able to realize this work.

I want to thank my thesis director, Pr. Nabil KHELFALLAH, for the quality of his supervision. A big thank you for all his encouragement, his endless support, and his advice during the last years.
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ملخص

نهتم في هذه المذكرة بدراسة فئة من المعادلات التفاضلية التراجعية العشوائية المولدة من القياسات العشوائية المتعلقة بعملية مركوف ذو القفزات الصافية. أولاً نثبت وجود حل وحيد لهذا النوع من المعالات باستخدام مولدات تحقق شرط ليبشيتز. بالإضافة إلى ذلك، نقدم ونبرهن نظرية خاصة بمقارنة للحلول التي لها دور مهم في هذا العمل. ثم نقترح إضعاف شرط ليبشيتز ونعالج ثلاث حالات على أنها ثلاث مواضيع مختلفة

في الحالة الأولى ، نهتم بفئة من المعادلات التفاضلية التراجعية العشوائية المولدة بمتغير وينر و عملية ماركوف. نبدأ بإثبات وجود حل (أدنى) لـلمعادلة حيث يكون المعامل مستمرًا ويحقق شرط التزايد الخطي ثم نضعف هذا الشرط ونثبت وجود حل لـلمعادلة عندما يكون المولد ببساطة مستمرًا من اليسار، متزايد ومحدود.

في الحالة الثانية ، ندرس المعادلات التفاضلية التراجعية العشوائية التي يكون مولدها محليًا من ليبشيتز و تكون مولدة بعملية ماركوف . نبين وجود ووحدانية حلول هذه المعادلة بتقريب المشكلة الأولية بمعادلات ذات مولدات تحقق شرط ليبشيتز بحيث هذه الأخيرة لها حل وحيد. بالانتقال إلى النهايات ، نظهر وجود ووحدانية الحلول للمشكلات الأولية ، ثم نثبت وجود حل وحيد لمعادلة كولموغوروف.

في الحالة الثالثة ، نعطي نتيجة وجود ووحدانية الحلول لفئة من المعادلات المولدة بعملية ماركوف مع مولد يحقق شرط النمو اللوغاريتمي ، ثم نطبق هذه النتيجة لإثبات وجود ووحدانية الحل لمجموعة من المعادلات التفاضلية التراجعية العشوائية ذات النمو التربيعي.

Résumé

Dans cette thèse, notre intérêt se porte sur une classe d'équations différentielles stochastiques rétrogrades dirigées par un processus Markovien de saut pur (EDSRs en abrégé). Nous prouvons d'abord un résultat d'existence et d'unicité pour ce type d'EDSRs avec des générateurs globalement Lipschitzien aussi bien qu'un théorème de comparaison pour les solutions. Ensuite, nous proposons d'affaiblir la condition de Lipschitz et nous traitons trois cas faisant l'objet de trois sujets différents.

Dans le premier cas, nous étudions une classe d'équations différentielles stochastiques rétrogrades qui sont dirigées par un processus Markovien de saut et un processus de Wiener. Pour commencer, nous démontrons l'existence d'un résultat dans le cas où le générateur de l'EDSR est continu et satisfait la condition de croissance linéaire. Ensuite, lorsque le générateur est simplement continu à gauche, croissant et borné. La technique utilisé consiste à trouver une suite croissante de processus dont la limite est la solution souhaitée. Enfin, nous démontrons que, sous l'hypothèse de continuité et de croissance linéaire du générateur, l'EDSR étudiée peut avoir soit une seule soit un nombre non dénombrable de solutions.

Dans le deuxième cas, nous étudions une EDS rétrograde qui est dirigée par un processus Markovien de saut dont le générateur peut être localement Lipschitzien. Nous établissons des théorèmes d'existence, d'unicité et de stabilité pour ces EDSRs. Nous approximons essentiellement le problème initial en construisant une suite d'EDSRs avec des générateurs globalement Lipschitzien pour lesquels l'existence et l'unicité des solutions sont vérifiées. En passant aux limites, nous montrons l'existence et l'unicité des solutions au problème initial. Finalement, nous prouvons l'existence d'une solution unique à l'équation de Kolmogorov associée.

Dans le troisième cas nous focalisons au même type d'EDSRs avec un générateur continu et de croissance logarithmique, c'est une croissance entre linéaire et quadratique. En utilisant une méthode de localisation, pour démontrer l'existence et l'unicité de la solution. La méthode consiste à approximer le générateur de l'EDSR en utilisant une suite de générateurs Lipschitziens, ce qui nous permet d'obtenir l'existence de la solution en faisant un passage à la limite. Finalement, nous présentons une application aux EDSRs quadratiques.

Mots clés: Equation differentielle stochastique progressive rétrograde, processus de Markov à saut, mesure aléatoire, principe de comparaison, equation de Kolmogorov.

Abstract

In the present thesis we are interested in the well-posedness problem to a wide class of backward stochastic differential equations driven by Brownian motion and independent random measures related to pure jump Markov processes (BSDEJs for short). We first prove an existence and uniqueness result for this type of BSDEJs with globally Lipschitz generators along with a comparison theorem for the solutions. Then, we propose to relax the Lipschitz framework in three directions as three different topics.

The first topic is devoted to the study such BSDEJs with continuous generators (not necessarily Lipschitz) allowing a linear growth condition. We start by proving the existence of at least one (minimal) solution. Then, we extend this later result to the case when the generator is merely left continuous, increasing, and bounded. Finally, we prove that if the generator is assumed to be continuous and of linear growth in $(y, z, k(\cdot))$ The BSDEJ has one or uncountable solutions.

In the second topic we are concerned with locally Lipschitz setting. We establish an existence, uniqueness and stability theorems to such BSDEJs. We approximate the initial problem by a sequence of BSDEJs with globally Lipschitz generators, such that for each integer n the previous BSDEJ has a unique solution $(Y^n, K^n(\cdot))$. Then by passing to the limits, we show that the initial problem has a unique solution $(Y, K(\cdot))$ as a limit of a Cauchy sequence $(Y^n, K^n(\cdot))$ in a Banach space to be determined later. Finally, we prove the existence of a unique solution to a Kolmogorov equation.

In the third topic we give a result of existence and uniqueness to a class of BSDEJs driven by a jump Markov process with a generator allowing a logarithmic growth. Then, we apply this result to prove the existence of a unique solution to one type of quadratic BSDEJs.

Keywords: Backward stochastic differential equations (BSDE), jump Markov process, comparison principle, Random measure, Kolmogorov equation.

Contents

List of Symbols and Abbreviations

The different symbols and abbreviations used in this thesis.

$a.e$	:	almost everywhere.
$a.s$	:	almost surely.
$\mathbb{R}$	:	real numbers.
τ	:	is a stopping time.
$\bar{A}$	:	the closure of the set A.
$\mathbb{1}_A$	:	the indicator function of the set A.
$\sigma(A)$	:	σ-algebra generated by A.
$(\Gamma, \mathcal{E})$	:	measurable space.
$(\Omega, \mathcal{F}, \mathbb{P})$	:	probability space.
$\{\mathcal{F}_t\}_{t\in[0,T]}$	:	filtration.
$(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$	:	filtered probability space.
$\mathcal{N}$	:	the totality of the $\mathbb{P}$-negligible sets.
$\mathbb{E}(.)$	:	The mathematic expectation.
$\mathbb{E}(\cdot \mid \mathcal{F})$	:	conditional expectation.
$B = (B_t)_{t\in[0,T]}$	:	is a Brownian motion.
$X = (X_t)_{t\in[0,T]}$	:	is a given Markov process.

- $BSDEs$: Backward stochastic differential equations.

- $\mathbb{P}\otimes dt$: the product measure of $\mathbb{P}$ with the Lebesgue measure dt.

- $\mathcal{L}^m(p)$: denote the space of real function $W_s(\omega, \theta)$ defined on $\Omega \times [t, \infty[\times \Gamma$,and $\mathcal{P} \otimes \mathcal{E}$-measurable such that

$$\mathbb{E}\int_t^T \int_\Gamma |W_s(\theta)|^m \, \mathrm{p}(\mathrm{d}s, \mathrm{d}\theta) = \mathbb{E}\int_t^T \int_\Gamma |W_s(y)|^m \, v(s, X_s, \mathrm{d}\theta)\mathrm{d}s < \infty.$$

- $\mathcal{L}^1_{loc}(p^t)$: the space of the real functions W such that $W1.\mathrm{I}_{]0,\tau_n]} \in \mathcal{L}^1(p^t)$ for some increasing sequence of $\mathbb{F}^t$-stopping times τ_n diverging to $+\infty$.

- $\mathcal{M}^2$: the space of real valued square integrable progressively measurable and predictable processes ϕ=$\{\phi_u : u \in [0, T]\}$ such that

$$\|\phi\|^2 = \mathbb{E}\int_t^T |\phi_u|^2 \, \mathrm{d}u < +\infty.$$

- $\mathcal{S}_p$: $p \geq 1$ the space of real-valued and $Prog^t-$measurable processes Y on $[t, T]$ such that

$$\mathbb{E}\left[\int_t^T |Y_r|^p \, \mathrm{d}r\right] < \infty.$$

- $\mathcal{S}^2$: is the space of $\mathcal{F}_t$-adapted and right-continuous with the left limit processes Y, such that

$$\mathbb{E}\left[\sup_{t\in[0,T]} |Y_t|^2\right] < \infty.$$

- $L^2(\Gamma, \mathcal{E}, \nu(., x, \mathrm{d}\theta))$: the space of processes k: $\Gamma \to \mathbb{R}$ such that

$$\|k(\cdot)\|_\nu = \left(\int_\Gamma |k(\theta)|^2 \, \nu(\cdot, x, \mathrm{d}\theta)\right)^{\frac{1}{2}} < \infty.$$

- $\mathcal{B}$: is a Banach space.

General Introduction

The theory of Backward Stochastic Differential Equations (BSDEs for short) is an important and vital field of modern Mathematics. This sort of equation has found many applications in finance, economics, homogenization, partial differential equations, stochastic control, etc. Seminal survey papers in this context are [10, 16, 20, 23, 24, 31].

It is well known that the linear BSDE driven by continuous Brownian motion goes back to the work of J.M. Bismut [15], in 1973, as an adjoint equation of the stochastic version of Pontrayagin stochastic maximum principle. Nonetheless, the theory of nonlinear BSDEs was developed in 1990 by Pardoux and Peng, in their paper [44]. From this work, many authors attempt to relax the assumptions on the generator. Good references for this are [6, 27, 32, 38].

A generally acknowledged fact that the Brownian motion can be seen as the most basic model for describing random phenomena whose value varies continuously. However, when describing, for example, physical phenomena or in the field of finance and insurance, the observed processes may present discontinuities whose location and amplitude are random. The counting of the events that cause these discontinuities is classically described by Poisson processes. Overall, the use of jump processes in modeling stochastic systems with jumps provides valuable insights into the behavior of these systems and helps in making predictions and decisions.

Many papers have also studied BSDEs driven by random jumps processes. Among them, Becherer [13], M. Royer [47]. I. Kharroubi *et al.* [36], E. Bandini and F. Confortola

[12]. We refer the reader to the following list of primordial papers for more literature on this subject [4, 5, 14, 26, 28, 29, 30, 37, 39, 40, 43].

Motivated by all the aforementioned references, we aim in this Ph.D. dissertation to deal with a class of BSDEJs driven by Markov jump processes. The first paper where this type of equation is studied is due to Confortola and Fuhrman, in [19]. The authors provided existence and uniqueness results for globally Lipschitz BSDEJ driven by a pure Markov jump process of the following type

$$Y_s = h(X_T) + \int_s^T f(r, X_r, Y_r, K_r(\cdot))\mathrm{d}r - \int_s^T \int_\Gamma K_r(\theta)\mathrm{q}(\mathrm{d}r, \mathrm{d}\theta), \tag{0.1}$$

for all $s \in [t, T]$ where $t \in [0, T]$, X is a jump Markov process defined on a complete filtered probability space $(\Omega, F, (\mathcal{F}_t)_{t\in[0,T]}, \mathbb{P})$, $\mathrm{q}(\mathrm{d}r, \mathrm{d}\theta)$ stands for a random measure associated to the jump Markov process X, f is the generator and $h(X_T)$ the terminal condition. They further applied their own results to study nonlinear variants of the Kolmogorov equation of the Markov process and also to solve some optimal control problems. Then, in [18] they studied a class of backward stochastic differential equations driven by a marked point process. Under appropriate assumptions they proved the well-posedness and continuous dependence of the solution on the data. Subsequently, Confortola [17] proved the existence and uniqueness of L^p–solutions $(p > 1)$ to a BSDEJ driven by a marked point process, on a bounded time interval.

To the best of our knowledge the Lipschitz condition is the strangest condition that ensures the existence and uniqueness of solution to BSDEJs. The natural question that arises then is: can we get the existence or the uniqueness of solutions to such BSDEJs under a set of conditions weaker than the Lipschitz condition? Fortunately, the answer to this question is yes. Therefore, throughout this dissertation, we want to explore some possible extensions. First to more general versions of this equation driven by jump Markov process and independent Brownian motion, then to BSDEJs with continuous or locally Lipschitz or logarithmic growth generators.

To set the stage for contributions of this dissertation, we first recall some existing results in the literature that cover the same regions mentioned above as possible generalizations for BSDEs driven by continuous Brownian motion (without the jump part). Lepeltier and San Martin [38] studied one-dimensional BSDE with a bounded terminal condition and only a continuous generator which satisfies the linear growth conditions. The first result concerned with multidimensional BSDE with continuous generator is due to Hamadène [33]. In this reference, an existence result has been proved under assumptions that the generator f is uniformly continuous with respect to y, z and the i^{th} component f_i of f depends only on the i^{th} row of z. As a second result in this framework, using the so-called $L^{\mathbf{2}}$-domination technique, Hamadène & Mu [34] proved an existence result for a multidimensional Markovian BSDE with continuous generator and stochastic linear growth. Subsequently, the later result was extended to a coupled BSDEs system in Mu & Wu [41].

The theory of locally Lipschitz BSDEJs driven by continuous Brownian motion started with Hamadene in his seminal paper [32]; in which one-dimensional BSDE with a bounded terminal condition is studied. Then, Bahlali [6] generalized the previous result to the multidimensional case with square integrable terminal data. Subsequently, the last work has been extended by Auguste and N'zi [3] to non-linear Volterra integral equations.

In order to highlight the logarithmic growth case, we present some papers. Bahlali in [6] proved the existence, uniqueness, and stability of the solution for multidimensional BSDEs with locally monotone coefficients. This is done with an almost quadratic growth coefficient and a square-integrable terminal datum, also Bahlali *et al.* in [9] studied the existence and uniqueness of BSDEs with Logarithmic growth in z and L^p-integrable terminal value. Bahlali *et al.* in [11] proved the existence and uniqueness of solutions of BSDEs with generator allowing a logarithmic growth $(|y||ln|y||+|z||ln|z||)$ in the variables y and z with an $L^{\mathbf{p}}$- integrable terminal value.

Another avenue of generalization to the forenamed results concerned the driver process itself, which could contain a jump part. In this setting, El Otmani [25] has studied BSDEJs driven by a simple Lévy process and proved the existence of a (minimal) solution to BSDEJs with continuous or left continuous increasing and bounded generators. Later Yin and Mao [48], dealt with a class of BSDEJ with Poisson jumps and random terminal times. They proved the existence of a unique solution along with two comparison theorems for such BSDE under non-Lipschitz assumptions on the coefficient. These results have been applied to investigate the existence and uniqueness of a minimal solution to one–dimensional BSDE with jumps in the case where its generator is merely continuous and of linear growth. Subsequently, Qin and Xia [46] studied one-dimensional BSDEJ driven by Poisson point processes with continuous and discontinuous coefficients. By means of the comparison theorem, the authors proved the existence of a (minimal) solution for such BSDEJ where the coefficient is continuous and satisfies an improved linear growth assumption. Then, they extended the result to BSDEJ with left or right continuous coefficients. More recently Eddahbi *et al.* [22] investigated existence results to multidimensional Markovian BSDE driven by a Poisson random measure and independent Brownian motion. They got their results in two different cases by assuming that the BSDEJ's generator is totally or partially continuous with respect to state variables and satisfies the usual linear growth condition. As opposed to the case of BSDEJs with a continuous generator, there are only a few papers in the locally Lipschitz or the logarithmic growth settings. To the best of our knowledge, the first extension to the case of the jump is due to Bahlali *et al.* [7], where they treat BSDEJs driven by a family of Teugels martingales and independent Brownian motion. Then, Bahlali *et al.* [21], established an existence and uniqueness of the solution to a reflected multidimensional BSDE in a d-dimensional convex region with locally Lipschitz generator and squared integrable terminal condition. Finally, K Oufdil [42] studied one-dimensional backward stochastic differential equations under logarithmic growth in the z-variable.

This thesis presents advancements in four areas related to the driver process or to the set of conditions satisfied by the BSDEJ's generator or the terminal datum. This will make the content of the four chapters of this dissertation and give rise to the theory of BSDEJs.

In the first chapter, inspired by Confortola and Fuhrman [19], we prove an existence and uniqueness result to a class of BSDEJs driven by both a jump Markov process and an independent Wiener process of the following form

$$\begin{aligned} Y_s &= h(X_T) + \textstyle\int_s^T f(r, X_r, Y_r, Z_r, K_r(\cdot))\mathrm{d}r \\ &\quad - \textstyle\int_s^T Z_r \mathrm{d}B_r - \int_s^T \int_\Gamma K_r(\theta)\mathrm{q}(\mathrm{d}r, \mathrm{d}\theta), \end{aligned} \tag{0.2}$$

where f is globally Lipschitz function. We further give a new demonstration of comparison theorem which is one of the principal tools in the theory of BSDEs. This theorem allows us to compare the solutions of two BSDEJs whenever we can compare their inputs: if

$$h^1(X_T) \leq h^2(X_T) \text{ and } f^1(s,x,y,z,k(\cdot)) \leq f^2(s,x,y,z,k(\cdot))$$

$ds \otimes d\mathbb{P}$–a.s. on $[0,T]\times\Omega$, then

$$Y_s^1 \leq Y_s^2, \quad \forall\ s\in[0,T], \quad \mathbb{P}\text{–a.s.} \tag{0.3}$$

In the second Chapter we deal with BSDEJs with only continuous generators (not necessarily Lipschitz). Firstly, we prove the existence of a (minimal) solution for BSDEJ (0.2) where the generator f is continuous in (y,z), Lipschitz in $k(\cdot)$ and satisfies the following linear growth condition: for all $(s,\omega,x,y,z)\in[0,T]\times\Omega\times\Gamma\times\mathbb{R}\times\mathbb{R}$ and $k(\cdot)\in L^2(\Gamma,\mathcal{E},\nu(s,x,d\theta))$ we have

$$|f(s,x,y,z,k(\cdot))| \leq \lambda\left(1+|y|+|z|+\|(k\varphi_s)(\cdot)\|_\nu\right),$$

where $\varphi_s(\theta):\Omega\times[0,T]\times\Gamma\longrightarrow\mathbb{R}$ is $\mathcal{P}\otimes\mathcal{E}$–measurable and satisfies $a<\varphi_s(\theta)<b$. The main tools are the comparison theorem and the approximation technique. As the second result, we weaker the continuous conditions and we prove the existence of a (minimal) solution for BSDEJ (0.2) when f is only left continuous in y and bounded by using again an approximation of the generator by increasing sequences of Lipschitz functions. We also prove that if the generator is continuous and of linear growth in (y,z) and Lipschitz in $k(\cdot)$ the BSDEJ (0.2) has one or uncountable solutions. Finally, we use the first result to show the existence of an unnecessarily unique solution to the quadratic BSDEJ of the form

$$\begin{aligned} Y_s &= h(X_T) - \int_s^T Z_r \mathrm{d}B_r - \int_s^T\int_\Gamma K_r(\theta)\mathrm{q}(\mathrm{d}r,\mathrm{d}\theta) \\ &\quad + \int_s^T H(r,X_r,Y_r,Z_r,K_r(\cdot))\mathrm{d}r, \end{aligned}$$

where

$$H(r,X_r,y,z,k(\cdot)) = f(r,X_r,y,z,k(\cdot)) + \psi(y)|z|^2 + \left[K_{r,X_{r-},y}\right]_\psi$$

and

$$[k_{s,x,y}]_\psi := \int_\Gamma \frac{F(y+k(\theta))-F(y)-F'(y)k(\theta)}{F'(y)}\nu(s,x,\mathrm{d}\theta),$$

such that ψ is a measurable continuous function that belongs to $\mathbb{L}^1(\mathbb{R})$ and F is a one to one function from $\mathbb{R}$ onto $\mathbb{R}$ belongs to $\mathcal{C}^2(\mathbb{R})$ defined as follow

$$F(x) = \int_0^x \exp\left(2\int_0^y \psi(t)\mathrm{d}t\right)\mathrm{d}y.$$

In the third Chapter, we deal with a class of BSDEJs when the generator is merely locally Lipschitz: for every integer $M > 1$, there exist two constants $L_M > 0$ and $\acute{L}_M > 0$ such that, for a.e. $s \in [0, T]$,

$$\left|f(s, x, y, k(\cdot)) - f(s, x, \acute{y}, \acute{k}(\cdot))\right| \leq \acute{L}_M\,|y - \acute{y}| + L_M\left\|k(\cdot) - \acute{k}(\cdot)\right\|_\nu,$$

and for all y, $\acute{y}$, $k(\cdot)$, $\acute{k}(\cdot)$ such that $|y| \leq M$, $|\acute{y}| \leq M$, $\|k(.)\|_\nu \leq M$, $\left\|\acute{k}(\cdot)\right\|_\nu \leq M$.

We give an existence and uniqueness theorems to such BSDEJs, we essentially approximate the initial problem by constructing a suitable sequence of BSDEJs with globally Lipschitz generators for which the existence and uniqueness of solutions hold. By passing to the limits, we show the existence and uniqueness of solutions to the original problems. The second main result of this chapter is the stability theorem which claims that: if $f_n \to f$ and $h_n(X_T) \to h(X_T)$ as $n \to \infty$ than $(Y^n, K^n(\cdot)) \to (Y, K(\cdot))$ as $n \to \infty$, such that $(f_n)_{n\in\mathbb{N}}$ is a sequence of *Prog*–measurable functions, $(h_n)_{n\in\mathbb{N}}$ is a sequence of $\mathcal{F}_{[t,T]}$–measurable and square-integrable random variables.

This Chapter can be regarded as an extension of the papers [6, 19], where Bahlali in [6] assumed that the locally Lipschitz constant w.r.t y and z are the same and they behave as $\sqrt{\log M}$ in the ball $B(0, M)$, while in our setting the locally Lipschitz constant on y behaves as $\log M$, whereas of z behaves as $\sqrt{\log M}$ On the other hand, under the square integrability assumption on the terminal data we mention that technically the sub-linear growth condition of f

$$|f(t, x, y, k(\cdot))| \leq \lambda\left[1 + |y|^\alpha + \|k(\cdot)\|_\nu^\alpha\right], a.e.\ (t, x) \in [0, T] \times \Gamma,$$

such that $\lambda > 0$, and $\alpha \in [0, 1[$ is only needed in the case where $|Y|$ and $\|Z(\cdot)\|_\nu$ are sufficiently large. Besides, by virtue of the boundedness of the terminal data, we can trade off the sub-linear growth in y by the linear growth see Remark 1 and Corollary 1 of Section 3 in Chapter 3. We can also allow f to be of super-linear growth in y under some appropriate conditions: $yf(t, x, y, k(\cdot)) \leq C\left(1 + |y|^2 + |y|\,\|k(\cdot)\|_\nu\right), a.e.\ (t, x) \in [0, T]\times\Gamma$, see remark 2 and Example 2 of Section 3 in Chapter 3. Obviously, those improvements increase the choices in selecting the generators that satisfy those hypotheses. We also present a parabolic backward equation associated to a Markov process X of the following form

$$\begin{aligned} u(t,x) &= h(x) + \int_t^T \mathcal{L}_r u(r,x)\,\mathrm{d}r \\ &\quad + \int_t^T f(r,x,u(r,x),u(r,.)-u(r,x))\mathrm{d}r, \end{aligned} \tag{0.4}$$

where $t \in [0,T]$, $x \in \Gamma$, $u : [0,T] \times \Gamma \to \mathbb{R}$ is an unknown function such that the function $t \to u(t,x)$ is absolutely continuous on $[0,T]$ such that $(u(s,X_{s-}), u(s,\theta) - u(s,X_{s-})) \in \mathcal{B}^t_{2,2}$, f and h are two given functions, $\mathcal{L}_r$ denote the generator of X of the form

$$\mathcal{L}_r(\varphi(x)) = \int_\Gamma (\varphi(\theta) - \varphi(x))\,\nu(r,x,\mathrm{d}\theta),$$

such that $\varphi : \Gamma \to \mathbb{R}$ is a measurable function, we apply Theorem 3.5 to prove the existence of a unique solution u, Moreover for every $t \in [0,T]$, $x \in \Gamma$ we have $Y^{t,x}_s = u(s,X_s)$, $K^{t,x}_s(\theta) = u(s,\theta) - u(s,X_{s-})$, so that in particular $u(t,x) = Y^{t,x}_t$.

In the fourth Chapter, we study a class of BSDEJs (0.1) with Logarithmic growth in y and k of the type

$$\mid f(t,x,y,k) \mid \leq \eta_t + \acute{C}|y||\ln|y|| + c_0 \left\|k(\cdot)\right\|_\nu \sqrt{|\ln(\|k(\cdot)\|_\nu)|},$$

where c_0 and $\acute{C}$ are two positive constants. We extend the work of Bahlali [11] to the jump case, we prove an existence and uniqueness result under an exponential integrability condition on the terminal data. It is worth mentioning that neither the uniform continuity nor the locally Lipschitz condition will be needed, then we prove that the quadratic BSDEJ with exponential moments

$$\begin{aligned} Y_s &= h(X_T) + \int_s^T \Big(Y_r + Z_r\sqrt{|\ln|Z_r| + Y_r|} \\ &\quad + \left(e^{K_r(\cdot)} - 1\right)\sqrt{|\ln|(e^{K_r(\cdot)}-1)| + Y_r|} + \frac{1}{2}|Z_r|^2 + [K_{r,X_r}] \Big)\,\mathrm{d}s \\ &\quad - \int_s^T Z_r \mathrm{d}B_r - \int_s^T \int_\Gamma K_r(\theta)\mathrm{q}(\mathrm{d}r,\mathrm{d}\theta), \end{aligned}$$

has a unique solution $(Y,Z,K(\cdot))$ if and only if

$$(y_r, z_r, k_r(\theta)) = \left(e^{Y_r}, e^{Y_r}Z_r, e^{Y_r}(e^{K_r(\theta)}-1)\right),$$

for any $r \in [0,T]$ and $\theta \in \Gamma$ is a unique solution of the following equation

$$\begin{aligned} y_s &= e^{\gamma h(X_T)} + \int_s^T \left(y_r \ln y_r + z_r \sqrt{|\ln |z_r||} + k_r\left(\cdot\right) \sqrt{|\ln(\|k_r(\cdot)\|_\nu)|} \right) \mathrm{d}r \\ &\quad - \int_s^T z_r dB_r - \int_s^T \int_\Gamma k_r(\theta) \mathrm{q}(\mathrm{d}r, \mathrm{d}\theta). \end{aligned}$$

To finish this introduction, let us recall that the content of this thesis is the subjects of the papers [1, 2]:

1) **Abdelhadi, K., Eddahbi, M., Khelfallah, N., & Almualim, A. (2022).** Backward Stochastic Differential Equations Driven by a Jump Markov Process with Continuous and Non-Necessary Continuous Generators. Fractal and Fractional, 6(6), 331.

2) **Abdelhadi, K., & Khelfallah, N. (2022).** Locally Lipschitz BSDE with jumps and related Kolmogorov equation. Stochastics and Dynamics, 2250021.

Communications

1. International Workshop on Perspectives On High dimensional Data Analysis (HDDA-2018), Marrakesh, poster presentation entitled: On the Solution of Locally Lipschitz BSDE Associated to Jump Markov Process.

2. Applied Mathematics Days, Biskra, poster presentation entitled: On the Solution of Globally Lipschitz BSDE Associated to Lévy Process.

3. National Training Days for Ph.D. students in mathematics, ElOued, oral presentation entitled: Backward stochastic differential equation associated to Lévy processes with Lipschitz coefficients.

4. Congress of Algerian Mathematicians, Boumerdès, oral presentation entitled: On the Solution of Globally Lipschitz BSDE Associated to Jump Markov Process.

5. International conference on mathematics, Istanbul, Turkey, poster presentation entitled: BSDE Driven by Jump Markov Processes with Continuous Coefficient.

6. International Conference on Mathematics, Istanbul, Turkey, poster presentation entitled: BSDE Driven by Jump Markov Processes with Locally Lipschitz Coefficient.

7. Colloque TAMTAM (2019), Telemcen, oral presentation entitled: The existence and uniqueness of solutions for BSDEs associated with a jump Markov process with locally Lipschitz coefficients.

8. International conference on computational methods in applied sciences, oral presentation entitled: Backward stochastic differential equations associated with to jump Markov process.

Chapter 1

BSDEJ with Lipschitz Coefficients

(Joint work with N. Khelfallah, A. Almualim and M. Eddahbi)

1.1 Introduction

In this chapter, we are interested in the class of backward stochastic differential equations driven by a jump Markov process and an independent Brownian motion with Lipschitz coefficients. In section 1, we give a brief introduction to the jump Markov process theory. In section 2, we give an existence and uniqueness result of BSDEJs (0.2) with globally Lipschitz coefficients. In Section 3, we prove a comparison theorem which will play an important role in the next chapter. Section 4 is devoted to the study of the existence of a unique solution to the Kolmogorov equation related to the underlying BSDEJ.

1.2 Overview of a jump Markov Process

A Markov process is a stochastic process having the Markov property, that is, the law of the future conditionally to the present is independent of that of the past. We recall that the jump processes, which play a crucial role in the theory of probability, are in fact members of a wide class called the Lévy processes. The most well-known examples of Lévy processes are the Wiener process, often called the Brownian motion process, Poisson process and Gamma process.

A Markov jump process (or Markov chain in continuous time) generalizes the notion of the Poisson process in the case where the jump occurring at a random instant is itself random. Moreover, this type of process combines a Poisson process and a Markov chain. They were initiated in 1902 by the Russian mathematician Andrey Andreyevich Markov (1856–1922), and they are named after him.

Throughout this thesis, the real positive number T stands for the horizon, and $(\Omega, \mathcal{F}, \mathbb{P})$ stands for a complete probability space. Let $(\Gamma, \mathcal{E})$ be a measurable space such that $\mathcal{E}$ contains all one-point sets and let X be a normal jump Markov process and B a standard Weiner process. We denote by $\mathbb{F}^t := \left(\mathcal{F}_{[t,s]}\right)_{s\in[t,+\infty[}$ the filtration such that $\left(\mathcal{F}_{[t,s]}\right)_{s\in[t,+\infty[}$ is the right-continuous increasing family of $\mathcal{F}$ defined by $\mathcal{F}_{[t,s]} := \sigma\left(X_r, r \leq s\right) \vee \sigma\left(B_r, t \leq r \leq s\right) \vee \mathcal{N}$, where $\mathcal{N}$ is the totality of $\mathbb{P}$–null sets.

Let $Prog^t$ be the progressive σ -algebra on $[t, \infty[\times \Omega$; the same symbols will also denote the restriction to $[t, T] \times \Omega$; let $\mathcal{P}^t$ be the predictable σ -algebra. We define a transition measure (also called a rate measure) $\nu(s, x, A)$, $s \in [t, T]$, $x \in \Gamma$, $A \in \Gamma$ from $[t, \infty) \times \Gamma$ to Γ, such that $\sup\limits_{s\in[0,T],x\in\Gamma} \nu(s, x, \Gamma) < \infty$ and $\nu(s, x, \{x\}) = 0$.

For every $t \geq 0$, we define a sequence $(T_n^t)_{n\geq 0}$ of random variables with values in $[0, \infty]$ as follows

$$\begin{aligned} T_0^t(\omega) &= t, \\ T_{n+1}^t(\omega) &= \inf\left\{s > T_n^t(\omega) : X_s(\omega) \neq X_{T_n^t(\omega)}(\omega)\right\}, \end{aligned}$$

with the convention that $T_n^t(\omega) = \infty$ if the indicated set is empty. Since X is a jump process, we have $T_n^t(\omega) < T_{n+1}^t(\omega)$ if $T_n^t(\omega) < \infty$. Since X is non-explosive, $T_n^t(\omega)$ tends towards infinity with n.

In other words, T_n^t are the jump times of X; we consider the marked point process $\left(T_n^t, X_{T_n^t}\right)$ and the associated random measure

$$\mathrm{p}^t(\mathrm{d}s, \mathrm{d}\theta) := \sum_n \delta_{(T_n, X_{T_n})}(\mathrm{d}s, \mathrm{d}\theta) \text{ on } \left]t, +\infty\right[\times \Gamma,$$

where δ stands for the Dirac measure. The compensator (also called the dual predictable projection) $\tilde{\mathrm{p}}^t$ of p^t is $\tilde{\mathrm{p}}^t(\mathrm{d}s, \mathrm{d}\theta) = v(s, X_{s-}^{t,x}, \mathrm{d}\theta)\mathrm{d}s$, so that $\mathrm{q}^t(\mathrm{d}r, \mathrm{d}\theta) := \mathrm{p}^t(\mathrm{d}r,\mathrm{d}\theta) - \nu(r, X_{r-}, \mathrm{d}\theta)\mathrm{d}r$ is the Ito differential of an $\mathbb{F}^t-$martingale. Notice that

$$\mathbb{E}\int_t^T \int_\Gamma K_s(\theta)\mathrm{p}^t(\mathrm{d}s, \mathrm{d}\theta) = \sum_{n\geq 1, T_n^t \leq s} K_{T_n^t}\left(X_{T_n^t}\right), s \in [t, T],$$

is always well defined since $T_n^t \to \infty$.

We refer the reader to the paper [19] for further information about this subject.

In the remainder of this study, we will work on the following spaces

- For $m \in [1, \infty[$, we define $\mathcal{L}^m(p^t)$ as the space of $\mathcal{P} \otimes \mathcal{E}$-measurable real functions $K_s(\omega, \theta)$ defined on $\Omega \times [t, T] \times \Gamma$, such that

$$\begin{aligned}\mathbb{E}^{t,x} \int_t^T \int_\Gamma |K_r(\theta)|^m \, \mathrm{p}^t(\mathrm{d}r, \mathrm{d}\theta) &= \mathbb{E}^{t,x} \int_t^T \int_\Gamma |K_r(\theta)|^m \, \nu(r, X_{r^-}, \mathrm{d}\theta)\mathrm{d}s \\ &= \mathbb{E}^{t,x} \int_t^T \int_\Gamma |K_r(\theta)|^m \, \nu(r, X_r, \mathrm{d}\theta)\mathrm{d}s < \infty.\end{aligned}$$

- $\mathcal{L}^1_{loc}(p^t)$ is the space of the real functions K such that $K\mathbb{1}_{]0,\tau_n]} \in \mathcal{L}^1(p^t)$ for some increasing sequence of $\mathbb{F}^t$-stopping times τ_n diverging to $+\infty$.

- $L^2(\Gamma, \mathcal{E}, \nu(., x, \mathrm{d}\theta))$ the space of processes $k : \Gamma \to \mathbb{R}$ such that

$$\|k(\cdot)\|_\nu = (\int_\Gamma |k(\theta)|^2 \, \nu(\cdot, x, \mathrm{d}\theta))^{\frac{1}{2}} < \infty.$$

- $\mathcal{S}^2$ is the space of $\mathcal{F}_t$-adapted and right- continuous with the left limit (tell) processes Y, such that

$$\mathbb{E}\left[\sup_{t \in [0,T]} |Y_t|^2\right] < \infty.$$

- $\mathcal{S}_p$, $p \geq 1$ the space of real-valued and $Prog^t$–measurable processes Y on $[t, T]$ such that

$$\mathbb{E}\left[\int_t^T |Y_t|^p \, \mathrm{d}r\right] < \infty.$$

- $\mathcal{H}^2$ the space of the processes $K(\cdot)$ on $[t, T]$ such that $K : \Omega \times [t, T] \times \Gamma \to \mathbb{R}$ is $\mathcal{P}^t \otimes \mathcal{B}(\mathbb{R})$-measurable and

$$\mathbb{E}\left[\int_t^T \|K_r(\cdot)\|_\nu^2 \, \mathrm{d}r\right] < \infty.$$

- $\mathcal{M}^2$ the space of real valued square integrable, progressively measurable and predictable processes $\phi = \{\phi_u : u \in [0, T]\}$ such that

$$\|\phi\|^2 = \mathbb{E} \int_t^T |\phi_u|^2 \, \mathrm{d}u < +\infty.$$

- $\mathcal{B} := \mathcal{S}^2 \otimes \mathcal{M}^2 \otimes \mathcal{H}^2$ is the space of processes $(Y, Z, K(\cdot))$ on $[0, T]$, such that

$$\|(Y, Z, K(\cdot))\|_{\mathcal{B}}^2 = \mathbb{E}\left[\sup_{s\in[0,T]} |Y_s|^2 + \int_0^T |Z_r|^2 \, \mathrm{d}r + \int_0^T \|K_r(\cdot)\|_\nu^2 \, \mathrm{d}r\right] < +\infty.$$

The space $\mathcal{B}$ endowed with this norm, is a Banach space.

Definition 1.1

A solution to equation (0.2) is a triple of processes $(Y, Z, K(\cdot))$ which satisfying BSDEJ (0.2) such that $(Y, Z, K(\cdot)) \in \mathcal{B}$.

Remark 1.1

the stochastic integral $\int_0^T \int_\Gamma K_r(\theta)\, \mathrm{q}^t(\mathrm{d}r, \mathrm{d}\theta)$ is a finite variation martingale if $K \in \mathcal{L}^1(p^t)$.

Now, we give the representation theorem which is one of the important tools to prove the results concerning the existence of solutions. Its proof can be found in ([17] Theorem 2.9).

Proposition 1.2

Given $(t, x) \in [0, T] \times \Gamma$, let M be a square-integrable martingale and $\mathbb{F}^t-$adapted on $[t, T]$. Then there exists two processes $K(\cdot) \in \mathcal{L}^2(p)$ and $Z \in \mathcal{M}^2$ such that

$$M_s = M_t + \int_t^r Z_u \mathrm{d}B_u + \int_t^r \int_\Gamma K_u(\theta) \mathrm{q}^t(\mathrm{d}u, \mathrm{d}\theta), \quad r \in [t, T].$$

In what follows, we recall Girsanov's theorem which plays a key role in the sequel.

Let us denote by $\mathcal{A}^2$ the set of square integrable martingales and by $\mathcal{A}$ the subset:

$$\mathcal{A} = \left\{(M_s)_{s\in[0,T]} \in \mathcal{A}^2 : |\omega_r(\theta)| \leq C, \omega_r(e) > -1, u \in \mathcal{M}^2\right\},$$

such that $M_s = \int_t^s u_r \mathrm{d}B_r + \int_t^s \int_\Gamma \omega_r(\theta)\, \mathrm{q}^t(\mathrm{d}r, \mathrm{d}\theta)$. For all $M \in \mathcal{A}$, the Doleans-Dade exponential is defined as

$$\varepsilon_T(M) = e^{M_T - \frac{1}{2}\langle M^C \rangle_T} \prod_{s\in[0,T]} (1 + \Delta M_s)\, e^{-\Delta M_s}.$$

Proposition 1.3

*(**Girsanov's theorem**) Let $W \in \mathcal{L}^2(p)$, $V \in \mathcal{M}^2$ and $U_s = \int_t^s V_r \mathrm{d}B_r + \int_t^s \int_\Gamma W_r(\theta)\, \mathrm{q}^t(\mathrm{d}r, \mathrm{d}\theta)$. For a given $M \in \mathcal{A}$, we define $\tilde{U}_s = U_s - \langle M, U\rangle_s$, then the process $\tilde{U}$ is a martingale under the probability measure $\mathrm{d}Q := \varepsilon_T(M)\, \mathrm{d}\mathbb{P}$.*

Remark 1.4

For the sake of simplicity, we drop the superscripts $t; x$ and shall state the results and their proofs for $t = 0$.

1.3 BSDEJ with Globally Lipschitz Coefficients

1.3.1 Problem Statement and Main Results

In this section, we tackle existence and uniqueness results for BSDEJ (0.2) in the globally Lipschitz case. The main hypothesis needed in this Section are the following:

Hypothesis 1

($\mathbf{H}_{1.1}$) The final condition $h : \Gamma \longrightarrow \mathbb{R}$ is $\mathcal{E}$–measurable and $\mathbb{E}\left|h(X_T)\right|^2 < \infty$.

($\mathbf{H}_{1.2}$) For every $s \in [0, T]$, $x \in \Gamma$, $r \in \mathbb{R}$, $z \in \mathbb{R}$, $f(s, x, r, z, \cdot)$ is a mapping $L^2(\Gamma, \mathcal{E}, \nu(s, x, d\theta)) \longrightarrow \mathbb{R}$.

($\mathbf{H}_{1.3}$) For every bounded and $\mathcal{E}$–measurable function $k(\cdot) : \Gamma \longrightarrow \mathbb{R}$, the mapping $(s, x, r, z) \longmapsto f(s, x, r, z, k(.))$ is $\mathcal{B}([0, T]) \otimes \mathcal{E} \otimes \mathcal{B}(\mathbb{R})$–measurable.

($\mathbf{H}_{1.4}$) $\mathbb{E}\int_0^T |f(s, X_s, 0, 0, 0)|^2 ds < \infty$.

($\mathbf{H}_{1.5}$) There exists $L \geq 0$ such that for every $s \in [0, T]$, $x \in \Gamma$, r, $\acute{r}$, z, $\acute{z} \in \mathbb{R}$ and $k(\cdot)$, $\acute{k}(\cdot) \in L^2(\Gamma, \mathcal{E}, \nu(s, x, d\theta))$

$$\begin{aligned} &\left|f(s, x, r, z, k(\cdot)) - f(s, x, \acute{r}, \acute{z}, \acute{k}(\cdot))\right| \\ &\leq L\left[(|r - \acute{r}| + |z - \acute{z}|) + \left\|k(\cdot) - \acute{k}(\cdot)\right\|_\nu\right]. \end{aligned}$$

Noting that, under Hypotheses 1, it was shown in Lemma 3.2 in [19] that the mapping $(\omega, s, y) \longmapsto f(s, X_{s-}(\omega), y, K_s(\omega, \cdot))$ is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R})$–measurable if $K \in \mathcal{L}^2(p)$. Furthermore, if Y is *Prog*–measurable process then, $(\omega, s) \longmapsto f(s, X_{s-}(\omega), Y_s(\omega), Z_s(\omega, \cdot))$ is *Prog*–measurable.

Throughout the following theorem we reveal the first main result of this chapter.

Theorem 1.5

*Let **Hypothesis 1** holds. Then, the BSDEJ (0.2) has a unique solution $(Y, Z, K(\cdot))$ in $\mathcal{B}$.*

To prove the above theorem, we shall start by giving and proving the following lemmas.

Lemma 1.6

Suppose that $\mathbf{H}_{1.1}$ holds and $f_r : \Omega \times [0, T] \longrightarrow \mathbb{R}$ is Prog–measurable, such that f_r is square integrable. Then, the following BSDEJ

$$Y_s = h(X_T) + \int_s^T f_r \mathrm{d}r - \int_s^T Z_r \mathrm{d}B_r - \int_s^T \int_\Gamma K_r(\theta) \mathrm{q}(\mathrm{d}r, \mathrm{d}\theta), \tag{1.1}$$

has a unique solution $(Y, Z, K(\cdot)) \in \mathcal{B}$.

Proof: We break down the proof into two steps.

Step 1: We want to prove that there exists a process $(Y, Z, K(\cdot))$ satisfying the equation (1.1). To do so, we consider the following martingale

$$M_s = \mathbb{E}\left[\left(h(X_T) + \int_0^T f_r \mathrm{d}r\right) \mid \mathcal{F}_{[0,s]}\right].$$

The martingale representation property in Proposition 1.2 confirms that there exist two processes $Z \in \mathcal{M}^2$ and $K(\cdot) \in \mathcal{L}^2(p)$ such that

$$M_s = M_0 + \int_0^s Z_r \mathrm{d}B_r + \int_0^s \int_\Gamma K_r(\theta) \mathrm{q}(\mathrm{d}r, \mathrm{d}\theta) \quad s \in [0, T].$$

Define the process Y as follows

$$Y_s = M_s - \int_0^s f_r \mathrm{d}r \quad s \in [0, T].$$

It is worth noting that $Y_T = h(X_T)$, then a simple computation shows that BSDEJ (1.1) is verified. the uniqueness of Y is guaranteed by the uniqueness of Z and $K(\cdot)$.

Step 2: We shall show that $(Y, Z, K(\cdot)) \in \mathcal{B}$. By taking the conditional expectation in (1.1), we arrive at

$$Y_s = \mathbb{E}\left[\left(h(X_T) + \int_s^T f_r \mathrm{d}r\right) \mid \mathcal{F}_{[0,s]}\right],$$

squaring both sides of the former equality, taking account of Jensen and Schwarz inequalities, we obtain

$$|Y_s|^2 \leq C\mathbb{E}\left[\left(|h(X_T)|^2 + \int_0^T |f_r|^2 \mathrm{d}r\right) \mid \mathcal{F}_{[0,s]}\right] < \infty. \tag{1.2}$$

Using Itô's formula for semimartingales (see Theorem 32 in [45] to $|Y_s|^2$ and integrating on the time interval $[s, T]$,

$$\begin{aligned} |Y_s|^2 \;&=\; |h(X_T)|^2 + 2\int_s^T Y_r f_r \mathrm{d}r - \int_s^T |Z_r|^2 \mathrm{d}r \\ &\quad -2\int_s^T Y_r Z_r \mathrm{d}B_r - 2\int_s^T \int_\Gamma Y_{r-} K_r(\theta)\, \mathrm{q}(\mathrm{d}r, \mathrm{d}\theta) \\ &\quad - \sum_{s\leq r\leq T} |\Delta Y_r|^2 . \end{aligned} \tag{1.3}$$

Due the fact that $Y_{r-}K_r(\cdot) \in \mathcal{L}^1(p)$, one can easily check that the process $\left(\int_s^t \int_\Gamma Y_{r-} K_r(\theta)\, q(\mathrm{d}r, \mathrm{d}\theta)\right)_{t\in[s,T]}$, is an $\mathbb{F}$–martingale. Indeed, from Young's inequality and the fact that $\sup_{t\in[0,T],x\in\Gamma} \nu(t, x, \Gamma) < \infty$, we get

$$\begin{aligned} \mathbb{E}\int_s^T \int_\Gamma |Y_{r-}|\, |K_r(\theta)|\, \nu(r, X_r, \mathrm{d}\theta)\mathrm{d}r \;&\leq\; \frac{1}{2}\sup_{t\in[0,T],x\in\Gamma} \nu(t, x, \Gamma)\mathbb{E}\int_0^T |Y_r|^2 \mathrm{d}r \\ &\quad +\frac{1}{2}\mathbb{E}\int_0^T \int_\Gamma |K_r(\theta)|^2 \nu(r, X_r, \mathrm{d}\theta)\mathrm{d}r < \infty. \end{aligned}$$

Since $Y \in \mathcal{S}^2$ and $Z \in \mathcal{M}^2$, we can prove that $\int_s^T Y_r Z_r \mathrm{d}B_r$ is an $\mathbb{F}$–martingale , using the Burkholder–Davis–Gundy inequality, we get

$$\begin{aligned} \mathbb{E}\left[\sup_{0\leq s\leq T}\left|\int_s^T Y_r Z_r \mathrm{d}B_r\right|\right] \;&\leq\; C\mathbb{E}\left[\left(\int_s^T |Y_r|^2 \|Z_r\|^2 \mathrm{d}r\right)^{\frac{1}{2}}\right], \\ &\leq\; C\mathbb{E}\left[\sup_{0\leq s\leq T} |Y_s| \left(\int_s^T \|Z_r\|^2 \mathrm{d}r\right)^{\frac{1}{2}}\right]. \end{aligned}$$

Using the inequality $ab \leq \frac{a^2}{2} + \frac{b^2}{2}$, we obtain

$$\begin{aligned} &\mathbb{E}\left[\sup_{0\leq s\leq T}\left|\int_s^T Y_r Z_r \mathrm{d}B_r\right|\right] \\ &\leq\; C'\left[\left(\mathbb{E}\sup_{0\leq s\leq T} |Y_s|^2\right) + \left(\mathbb{E}\int_s^T \|Z_r\|^2 \mathrm{d}r\right)\right] < \infty. \end{aligned}$$

In addition, we can rewrite the last term in the equality (1.3) as the following:

$$\begin{aligned}\sum_{s\leq r\leq T}|\Delta Y_r|^2 &= \int_s^T\int_\Gamma |K_r(\theta)|^2\,\mathrm{p}(\mathrm{d}r,\mathrm{d}\theta) \\ &= \int_s^T\int_\Gamma |K_r(\theta)|^2\,\mathrm{q}(\mathrm{d}r,\mathrm{d}\theta) \\ &\quad+\int_s^T\int_\Gamma |K_r(\theta)|^2\,\nu(r,X_r,\mathrm{d}\theta)\mathrm{d}r.\end{aligned} \tag{1.4}$$

Then, from (1.3) and (1.4), we get

$$\begin{aligned}|Y_s|^2 &= |Y_T|^2+2\int_s^T Y_rf_r\mathrm{d}r-\int_s^T|Z_r|^2\,\mathrm{d}r \\ &\quad-2\int_s^T Y_rZ_r\mathrm{d}B_r-2\int_s^T\int_\Gamma Y_{r-}K_r(\theta)\mathrm{q}(\mathrm{d}r,\mathrm{d}\theta) \\ &\quad-\int_s^T\int_\Gamma |K_r(\theta)|^2\,\mathrm{q}(\mathrm{d}r,\mathrm{d}\theta)-\int_s^T\|K_r(\cdot)\|_\nu^2\,\mathrm{d}r.\end{aligned} \tag{1.5}$$

Taking the expectation, we obtain

$$\begin{aligned}&\mathbb{E}\,|Y_s|^2+\mathbb{E}\int_s^T|Z_r|^2\,\mathrm{d}r+\mathbb{E}\int_s^T\|K_r(\cdot)\|_\nu^2\,\mathrm{d}r \\ &=\mathbb{E}\,|Y_T|^2+2\mathbb{E}\int_s^T Y_rf_r\mathrm{d}r,\end{aligned}$$

we deduce using (1.2) and Young's inequality: $2xy\leq 2x^2+\frac{y^2}{2}$,

$$\begin{aligned}&\mathbb{E}\,|Y_s|^2+\mathbb{E}\int_s^T|Z_r|^2\,\mathrm{d}r+\mathbb{E}\int_s^T\|K_r(\cdot)\|_\nu^2\,\mathrm{d}r \\ &\leq\mathbb{E}\,|Y_T|^2+C+2\mathbb{E}\int_0^T|f_r|^2\,\mathrm{d}r.\end{aligned}$$

Consequently,

$$\sup_{s\in[0,T]}\mathbb{E}\,|Y_s|^2+\mathbb{E}\int_0^T|Z_r|^2\,\mathrm{d}r+\mathbb{E}\int_0^T\|K_r(\cdot)\|_\nu^2\,\mathrm{d}r\leq C. \tag{1.6}$$

Now, we turn back to (1.5) and using the Burkholder–Davis–Gundy inequality together with 1.6, we obtain

$$\mathbb{E}\left[\sup_{0\leq s\leq T}|Y_s|^2\right]+\mathbb{E}\int_0^T|Z_r|^2\,\mathrm{d}r+\mathbb{E}\int_0^T\|K_r(\cdot)\|_\nu^2\,\mathrm{d}r\leq C.$$

Then, we conclude that $(Y,Z,K(\cdot))\in\mathcal{B}$. This achieves the proof of the Lemma. □

Let us define the following sequence $(Y^n,Z^n,K^n(\cdot))_{n\in\mathbb{N}}$ as follows:

$$Y^0=Z^0=K^0(\cdot)=0,$$

and $(Y^{n+1}, Z^{n+1}, K^{n+1}(\cdot))$ is the solution of the following BSDEJ

$$\begin{aligned} Y_s^{n+1} &= h(X_T) + \int_s^T f(r, X_r, Y_r^n, Z_r^n, K_r^n(\cdot))\mathrm{d}r \\ &\quad - \int_s^T Z_r^{n+1}\mathrm{d}B_r - \int_s^T \int_\Gamma K_r^{n+1}(\theta)\mathrm{q}(\mathrm{d}r, \mathrm{d}\theta), \end{aligned} \tag{1.7}$$

for all $s \in [0, T]$.

Lemma 1.7

*Let **Hypothesis 1** holds true. Then, $(Y^n, Z^n, K^n(\cdot))$ is a Cauchy sequence in the Banach space $\mathcal{B}$.*

Proof: First, let us denote

$$\delta Y^{n,m} = Y^m - Y^n,\ \delta Z^{n,m} = Z^m - Z^n,\ \delta K^{n,m} = K^m(\cdot) - K^n(\cdot),$$

and

$$\delta f^{n,m} = f(r, X_r, Y_r^m, Z_r^m, K_r^m(\cdot)) - f(r, X_r, Y_r^n, Z_r^n, K_r^n(\cdot)).$$

Obviously $\delta Y_T = 0$, so Itô's formula applied to $e^{\beta s}\left|\delta Y_s^{n+1,m+1}\right|^2$ shows that

$$\begin{aligned} &\mathbb{E}[e^{\beta s}\left|\delta Y_s^{n+1,m+1}\right|^2] \\ &+\beta\mathbb{E}\textstyle\int_s^T e^{\beta r}\left|\delta Y_r^{n+1,m+1}\right|^2 \mathrm{d}r + \mathbb{E}\int_s^T e^{\beta r}\left|\delta Z_r^{n+1,m+1}\right|^2 \mathrm{d}r \\ &+\mathbb{E}\textstyle\int_s^T \int_\Gamma e^{\beta r}\left|\delta K_r^{n+1,m+1}(\theta)\right|^2 \nu(r, X_r, \mathrm{d}\theta)\mathrm{d}r \\ &= 2\mathbb{E}\textstyle\int_s^T e^{\beta r}\delta Y_r^{n+1,m+1}\delta f_r^{n,m}\mathrm{d}r. \end{aligned}$$

From the Lipschitz condition on f and the inequality $2xy \le \alpha^2 x^2 + \frac{y^2}{\alpha^2}$, we get

$$\begin{aligned} &\mathbb{E}[e^{\beta s}\left|\delta Y_s^{n+1,m+1}\right|^2] + (\beta - 3L\alpha^2)\mathbb{E}\int_s^T e^{\beta r}\left|\delta Y_r^{n+1,m+1}\right|^2 \mathrm{d}r \\ &+\mathbb{E}\int_s^T e^{\beta r}\left|\delta Z_r^{n+1,m+1}\right|^2 \mathrm{d}r + \mathbb{E}\int_s^T\int_\Gamma e^{\beta r}\left|\delta K_r^{n+1,m+1}(\theta)\right|^2 \nu(r, X_r, \mathrm{d}\theta)\mathrm{d}r, \\ &\le \frac{L}{\alpha^2}\left[\mathbb{E}\int_s^T e^{\beta r}\left|\delta Y_r^{n,m}\right|^2 \mathrm{d}r + \mathbb{E}\int_s^T e^{\beta r}\left|\delta Z_r^{n,m}\right|^2 \mathrm{d}r\right] \\ &+\frac{L}{\alpha^2}\left[\mathbb{E}\int_s^T\int_\Gamma e^{\beta r}\left|\delta K_r^{n,m}(\theta)\right|^2 \nu(r, X_r, \mathrm{d}\theta)\mathrm{d}r\right], \end{aligned}$$

choosing β and α such that $\beta - 3L\alpha^2 = 1$ and $\frac{L}{\alpha^2} = \frac{1}{2}$, we obtain

$$\begin{aligned}
&\mathbb{E}\int_s^T e^{\beta r}\left|\delta Y_r^{n+1,m+1}\right|^2 \mathrm{d}r + \mathbb{E}\int_s^T e^{\beta r}\left|\delta Z_r^{n+1,m+1}\right|^2 \mathrm{d}r \\
&+\mathbb{E}\int_s^T \int_\Gamma e^{\beta r}\left|\delta K_r^{n+1,m+1}(\theta)\right|^2 \nu(r, X_r, \mathrm{d}\theta)\mathrm{d}r \\
&\leq \frac{1}{2}\Bigg[\mathbb{E}\int_s^T e^{\beta r}\left|\delta Y_r^{n,m}\right|^2 \mathrm{d}r + \mathbb{E}\int_s^T e^{\beta r}\left|\delta Z_r^{n,m}\right|^2 \mathrm{d}r \\
&\qquad +\mathbb{E}\int_s^T \int_\Gamma e^{\beta r}\left|\delta K_r^{n,m}(\theta)\right|^2 \nu(r, X_r, \mathrm{d}\theta)\mathrm{d}r\Bigg].
\end{aligned}$$

Using again Itô's formula, Burkholder-Davis-Gundy and Gronwall's lemma, it follows that for all $m > n$, there exists a universal constant M, such that

$$\begin{aligned}
&\mathbb{E}[\sup_{s\in[0,T]} e^{\beta s}\left|\delta Y_s^{n,m}\right|^2] + \mathbb{E}\int_s^T e^{\beta r}\left|\delta Z_r^{n,m}\right|^2 \mathrm{d}r \\
&+\mathbb{E}\int_s^T \int_\Gamma e^{\beta r}\left|\delta K_r^{n,m}(\theta)\right|^2 \nu(r, X_r, \mathrm{d}\theta)\mathrm{d}r \\
\leq\ & \frac{M}{2^n}.
\end{aligned}$$

Hence, $(Y^n, Z^n, K^n(\cdot))$ is a Cauchy sequence in the Banach space $\mathcal{B}$. $\square$

Now, we turn out to give the proof of the first main result of this section.

Proof of Theorem 1.5

Existence part: Thanks to Lemma 1.6, the sequence $(Y^n, Z^n, K^n(\cdot))$ in (1.7) is well defined and due to Lemma 1.7 $(Y^n, Z^n, K^n(\cdot))$ is a Cauchy sequence in the Banach space $\mathcal{B}$. Set $(Y, Z, K(\cdot)) := \lim_{n\to\infty}(Y^n, Z^n, K^n(\cdot))$ using classical limit arguments one can check that $(Y, Z, K(\cdot))$ is a solution of BSDEJ (0.2).

Uniqueness part: To prove the uniqueness let us consider $(Y, Z, K(\cdot))$ and $(\acute{Y}, \acute{Z}, \acute{K}(\cdot))$ as two solutions of equation (0.2), we note:

$$\bar{Y}_s = Y_s - \acute{Y}_s, \quad \bar{K}_s(\cdot) = K_s(\cdot) - \acute{K}_s(\cdot), \quad \bar{Z}_s = Z_s - \acute{Z}_s,$$

and

$$\bar{f}_s \;=\; f(s, X_s, Y_s, Z_s, K_s(\cdot)) - f(s, X_s, \acute{Y}_s, \acute{Z}_r, \acute{K}_s(\cdot)).$$

Noting that $\bar{Y}_T = 0$, Itô's formula applied to $\left|\bar{Y}_s\right|^2$ gives for all $s \in [0, T]$

$$\mathbb{E}\left|\bar{Y}_s\right|^2 + \mathbb{E}\int_s^T \left|\bar{Z}_r\right|^2 \mathrm{d}r + \mathbb{E}\int_s^T \left\|\bar{K}_r(\cdot)\right\|_\nu^2 \mathrm{d}r = 2\mathbb{E}\int_s^T \bar{Y}_r \bar{f}_r \mathrm{d}r.$$

Since f is Lipschitz, we obtain

$$\begin{aligned}
&\mathbb{E}\left|\bar{Y}_s\right|^2+\mathbb{E}\int_s^T\left|\bar{Z}_r\right|^2\mathrm{d}r+\mathbb{E}\int_s^T\left\|\bar{K}_r(\cdot)\right\|_\nu^2\mathrm{d}r\\
\leq\ & 2LE\int_s^T\left|\bar{Y}_r\right|\left[\left|\bar{Y}_r\right|+\left|\bar{Z}_r\right|+\left\|\bar{K}_r(\cdot)\right\|_\nu\right]\mathrm{d}r.
\end{aligned}$$

From the inequality $2xy\leq\alpha^2x^2+\frac{y^2}{\alpha^2}$, we get

$$\begin{aligned}
&\mathbb{E}\left|\bar{Y}_s\right|^2+\mathbb{E}\int_s^T\left|\bar{Z}_r\right|^2\mathrm{d}r+\mathbb{E}\int_s^T\left\|\bar{K}_r(\cdot)\right\|_\nu^2\mathrm{d}r\\
\leq\ & 2LE\int_s^T\left|\bar{Y}_r\right|^2\mathrm{d}r+\alpha^2LE\int_s^T\left|\bar{Y}_r\right|^2\mathrm{d}r+\frac{L}{\alpha^2}\mathbb{E}\int_s^T\left|\bar{Z}_r\right|^2\mathrm{d}r\\
&+\alpha^2LE\int_s^T\left|\bar{Y}_r\right|^2\mathrm{d}r+\frac{L}{\alpha^2}\mathbb{E}\int_s^T\left\|\bar{K}_r(\cdot)\right\|_\nu^2\mathrm{d}r.
\end{aligned}$$

Hence

$$\begin{aligned}
&\mathbb{E}\left|\bar{Y}_s\right|^2+\left(1-\frac{L}{\alpha^2}\right)\left[\mathbb{E}\int_s^T\left|\bar{Z}_r\right|^2\mathrm{d}r+\mathbb{E}\int_s^T\left\|\bar{K}_r(\cdot)\right\|_\nu^2\mathrm{d}r\right]\\
\leq\ & 2L\left(1+\alpha^2\right)\mathbb{E}\int_s^T\left|\bar{Y}_r\right|^2\mathrm{d}r.
\end{aligned}$$

If we choose α such that $\frac{L}{\alpha^2}=\frac{1}{2}$, we obtain after a simple computation

$$\begin{aligned}
&\mathbb{E}\left|\bar{Y}_s\right|^2+\frac{1}{2}\mathbb{E}\int_s^T\left|\bar{Z}_r\right|^2\mathrm{d}r+\frac{1}{2}\mathbb{E}\int_s^T\int_\Gamma\left|\bar{K}_r(\theta)\right|^2\nu(r,X_r,\mathrm{d}\theta)\mathrm{d}r\\
\leq\ & 2L(1+2L)\mathbb{E}\int_s^T\left|\bar{Y}_r\right|^2\mathrm{d}r.
\end{aligned}$$

The uniqueness of the solution follows immediately using Gronwall's lemma. □

Remark 1.8

We get the same result when the Brownian motion $Z=0$, for more detail one can see theorem (3.4) in [19].

1.4 A Comparison Principle

In this subsection, we shall compare the solutions of two BSDEJs whenever we can compare their inputs which are described by their generators and terminal conditions. To this end, we consider the following assumptions on the generator f: **Hypothesis 2**

($\mathbf{H}_{2.2}$) There exist two constants a and b, $-1 < a < 0$, $b > 0$ such that for every $s \in [0,T]$, $x \in \Gamma$, $r,\ z \in \mathbb{R}$ and $k,\ \acute{k} \in L^2(\Gamma, \mathcal{E}, \nu(s,x,d\theta))$, we have

$$f(s,x,r,z,k(\cdot)) - f(s,x,r,z,\acute{k}(\cdot)) \leq \int_\Gamma (k(\theta) - \acute{k}(\theta))\varphi_s(\theta)\nu(s,x,\mathrm{d}\theta),$$

where $\varphi : \Omega \times [0,T] \times \Gamma \longrightarrow \mathbb{R}$ is $\mathcal{P} \otimes \mathcal{E}$–measurable and satisfies $a < \varphi_s(\cdot) < b$.

Theorem 1.9

(Comparison Theorem). *Let h^1 and h^2 be two final conditions $\mathcal{E}$–measurable for two BSDEJs driven by f^1 and f^2 respectively such that f^1satisfies $\mathbf{H}_{1.1} - \mathbf{H}_{1.5}$ and f^2 satisfies $\mathbf{H}_{1.1} - \mathbf{H}_{1.3}$, $\mathbf{H}_{1.5}$ and $\mathbf{H}_{2.2}$. We denote by $(Y^1, Z^1, K^1(\cdot))$ and $(Y^2, Z^2, K^2(\cdot))$ the associated solutions in $\mathcal{B}$.*

i) *If*

$$h^1(X_T) \leq h^2(X_T) \quad \text{and} \quad f^1(s,x,y,z,k(\cdot)) \leq f^2(s,x,y,z,k(\cdot))$$

$ds \otimes \mathrm{d}\mathbb{P}$–a.s. on $[0,T] \times \Omega$, then

$$Y_s^1 \leq Y_s^2, \quad \forall\ s \in [0,T], \quad \mathbb{P}\text{–a.s.} \tag{1.8}$$

ii) *Assume that the function $\varphi_s(\cdot)$ defined in $\mathbf{H}_{2.2}$ is non-negative and for all $(s,x,y,z) \in [0,T] \times \Gamma \times \mathbb{R} \times \mathbb{R}$, $k(\cdot) \in L^2(\Gamma, \mathcal{E}, \nu(s,x,d\theta))$ we have*

$$f^2(s,x,y,z,k(\cdot)) = \lambda\left(1 + |y| + |z| + \|(k\varphi_s)(\cdot)\|_\nu\right).$$

Then we get (1.8).

iii) *if $Y_0^1 = Y_0^2$, $\mathbb{P}$–a.s., then $Y_s^1 = Y_s^2\ \forall\ s \in [0,T]$, $\mathbb{P}$–a.s., $Z_s^1 = Z_s^2$ $ds \otimes \mathrm{d}\mathbb{P}$–a.e. and $K_s^1(\theta) = K_s^2(\theta)\ \nu(s,x,d\theta)ds \otimes \mathrm{d}\mathbb{P}$–a.e.*

Proof: Let us first prove i). We denote $\delta Y_s = Y_s^1 - Y_s^2$, $\delta K_s(\cdot) = K_s^1(\cdot) - K_s^2(\cdot)$, $\delta Z_s = Z_s^1 - Z_s^2$ and $\delta h = h^1(X_T) - h^2(X_T)$, thus the difference between the two solutions can be decomposed as follows

$$\begin{aligned}
\delta Y_s &= \delta h + \int_s^T \Big(f^1(r, X_r, Y_r^1, Z_r^1, K_r^1(\cdot)) - f^2(r, X_r, Y_r^1, Z_r^1, K_r^1(\cdot))\Big)\, \mathrm{d}r \\
&\quad + \int_s^T \Big(f^2(r, X_r, Y_r^1, Z_r^1, K_r^1(\cdot)) - f^2(r, X_r, Y_r^1, Z_r^1, K_r^2(\cdot))\Big)\, \mathrm{d}r \\
&\quad + \int_s^T \Big(f^2(r, X_r, Y_r^1, Z_r^1, K_r^2(\cdot)) - f^2(r, X_r, Y_r^1, Z_r^2, K_r^2(\cdot))\Big)\, \mathrm{d}r \\
&\quad + \int_s^T \Big(f^2(r, X_r, Y_r^1, Z_r^2, K_r^2(\cdot)) - f^2(r, X_r, Y_r^2, Z_r^2, K_r^2(\cdot))\Big)\, \mathrm{d}r \\
&\quad - \int_s^T \delta Z_r \mathrm{d}B_r - \int_s^T \int_\Gamma \delta K_r(\theta) \mathrm{q}(\mathrm{d}r, \mathrm{d}\theta).
\end{aligned}$$

We denote $\Lambda_r = e^{\int_0^r \beta_u \mathrm{d}u}$, where

$$\beta_u = \begin{cases} \dfrac{f^2(u, X_u, Y_u^1, Z_u^2, K_u^2(\cdot)) - f^2(u, X_u, Y_u^2, Z_u^2, K_u^2(\cdot))}{Y_u^1 - Y_u^2} & \text{if } \ Y_u^1 - Y_u^2 \neq 0 \\ 0 & \text{otherwise.} \end{cases}$$

We apply Itô's formula to $\Lambda_s \delta Y_s$ between s and T, to get:

$$\begin{aligned}
\Lambda_s \delta Y_s &= \Lambda_T \delta h + \int_s^T \Lambda_r \Big(f^1(r, X_r, Y_r^1, Z_r^1, K_r^1(\cdot)) - f^2(r, X_r, Y_r^1, Z_r^1, K_r^1(\cdot))\Big)\, \mathrm{d}r \\
&\quad + \int_s^T \Lambda_r \Big(f^2(r, X_r, Y_r^1, Z_r^1, K_r^1(\cdot)) - f^2(r, X_r, Y_r^1, Z_r^1, K_r^2(\cdot))\Big)\, \mathrm{d}r \\
&\quad + \int_s^T \Lambda_r \Big(f^2(r, X_r, Y_r^1, Z_r^1, K_r^2(\cdot)) - f^2(r, X_r, Y_r^1, Z_r^2, K_r^2(\cdot))\Big)\, \mathrm{d}r \\
&\quad - \int_s^T \Lambda_r \delta Z_r \mathrm{d}B_r - \int_s^T \int_\Gamma \Lambda_r \delta K_r(\theta) \mathrm{q}(\mathrm{d}r, \mathrm{d}\theta). \qquad (1.9)
\end{aligned}$$

Using $\mathbf{H}_{2.2}$ and the fact that

$$\Lambda_T \delta h + \int_s^T \Lambda_r \Big(f^1(r, X_r, Y_r^1, Z_r^1, K_r^1(\cdot)) - f^2(r, X_r, Y_r^1, Z_r^1, K_r^1(\cdot))\Big)\, \mathrm{d}r \leq 0,$$

we obtain

$$\begin{aligned}
\Lambda_s \delta Y_s &\leq \int_s^T \int_\Gamma \Lambda_r \delta K_r(\theta) \varphi_r(y) \nu(r, X_r, \mathrm{d}\theta) \mathrm{d}r + \lambda \int_s^T \Lambda_r \left|\delta Z_r\right| \mathrm{d}r \qquad (1.10) \\
&\quad - \int_s^T \int_\Gamma \Lambda_r \delta K_r(\theta) \mathrm{q}(\mathrm{d}r, \mathrm{d}\theta) - \int_s^T \Lambda_r \delta Z_r \mathrm{d}B_r.
\end{aligned}$$

Set

$$M_s = \int_0^s \int_\Gamma \varphi_r(\theta) \mathrm{q}(\mathrm{d}r, \mathrm{d}\theta) + \lambda \int_0^s \operatorname{sgn}(\delta Z_r)\, \mathrm{d}B_r.$$

and

$$U_s = \int_0^s \int_\Gamma \Lambda_r \delta K_r(\theta) \mathrm{q}(\mathrm{d}r, \mathrm{d}\theta) + \int_0^s \Lambda_r \delta Z_r \mathrm{d}B_r.$$

Thus

$$\begin{aligned}\tilde{U}_s &= \int_0^s \int_\Gamma \Lambda_r \delta K_r(\theta) \mathrm{q}(\mathrm{d}r, \mathrm{d}\theta) - \int_0^s \int_\Gamma \Lambda_r \delta K_r(\theta) \varphi_r(\theta) \nu(r, X_r, \mathrm{d}\theta) \mathrm{d}r \\ &\quad + \int_0^s \Lambda_r \delta Z_r \mathrm{d}B_r - \lambda \int_0^s \Lambda_r |\delta Z_r| \,\mathrm{d}r.\end{aligned}$$

Girsanov's theorem (see Proposition 2 in Section 1 Chapter 1) claims that the process $\tilde{U}$ is a martingale under the probability measure $\mathrm{d}Q := \mathcal{E}_T(M)\mathrm{d}\mathbb{P}$, taking the conditional expectation under the probability measure Q on both sides of (1.10), we get $\Lambda_s \delta Y_s \leq 0$ Q–a.s. and thus $\mathbb{P}$–a.s. Then $Y_s^1 \leq Y_s^2, \quad \forall \quad s \in [0, T], \quad \mathbb{P}$–a.s.

Next, we proceed to prove (ii). Arguing as in the proof of the assertion (i), one can easily show that

$$\begin{aligned}\Lambda_s \delta Y_s &\leq \int_s^T \int_\Gamma \Lambda_r \left(\left| K^1(\theta) \right| - \left| K^2(\theta) \right| \right) \varphi_r(\theta) \nu(r, X_r, \mathrm{d}\theta) \mathrm{d}r \\ &\quad + \lambda \int_s^T \Lambda_r |\delta Z_r| \,\mathrm{d}r - \int_s^T \int_\Gamma \Lambda_r \delta K_r(\theta) \mathrm{q}(\mathrm{d}r, \mathrm{d}\theta) - \int_s^T \Lambda_r \delta Z_r \mathrm{d}B_r, \\ &\leq \int_s^T \int_\Gamma \Lambda_r |\delta K_r(\theta)| \varphi_r(\theta) \nu(r, X_r, \mathrm{d}\theta) \mathrm{d}r + \int_s^T \Lambda_r |\delta Z_r| \,\mathrm{d}r \\ &\quad - \int_s^T \int_\Gamma \Lambda_r \delta K_r(\theta) \mathrm{q}(\mathrm{d}r, \mathrm{d}\theta) - \int_s^T \Lambda_r \delta Z_r \mathrm{d}B_r. \end{aligned} \tag{1.11}$$

Define the new martingale

$$N_s = \int_0^s \int_\Gamma \mathrm{sgn}\left(\delta K_r(\theta)\right) \varphi_r(\theta) \mathrm{q}(\mathrm{d}r, \mathrm{d}\theta) + \lambda \int_0^s \mathrm{sgn}\left(\delta Z_r\right) \mathrm{d}B_r.$$

Using again Girsanov's theorem, it is not difficult to see that

$$\begin{aligned}\hat{K}_s &= \int_0^s \int_\Gamma \Lambda_r \delta K_r(\theta) \mathrm{q}(\mathrm{d}r, \mathrm{d}\theta) - \int_0^s \int_\Gamma \Lambda_r |\delta K_r(\theta)| \varphi_r(\theta) \nu(r, X_r, \mathrm{d}\theta) \mathrm{d}r \\ &\quad + \int_0^s \Lambda_r \delta Z_r \mathrm{d}B_r - \lambda \int_0^s \Lambda_r |\delta Z_r| \,\mathrm{d}r,\end{aligned}$$

is an $\mathbb{F}$–martingale under the probability measure $\mathrm{d}\hat{Q} := \mathcal{E}_T(N)\mathrm{d}\mathbb{P}$ and thus the result follows immediately by taking the conditional expectation under the probability measure Q in,the both sides of the inequality (1.11). To prove iii), we turn back to (1.9) and take

$s = 0$, thus

$$\begin{aligned}
&\Lambda_T \left(h^2 - h^1\right) \\
&+ \int_0^T \Lambda_r \left(f^2(r, X_r, Y_r^1, Z_r^1, K_r^1(\cdot)) - f^1(r, X_r, Y_r^1, Z_r^1, K_r^1(\cdot))\right) \mathrm{d}r \\
= \;& \int_0^T \Lambda_r \left(f^2(r, X_r, Y_r^1, Z_r^1, K_r^1(\cdot)) - f^2(r, X_r, Y_r^1, Z_r^1, K_r^2(\cdot))\right) \mathrm{d}r \\
&+ \int_0^T \Lambda_r \left(f^2(r, X_r, Y_r^1, Z_r^1, K_r^2(\cdot)) - f^2(r, X_r, Y_r^1, Z_r^2, K_r^2(\cdot))\right) \mathrm{d}r \\
&- \int_0^T \Lambda_r \delta Z_r \mathrm{d}B_r - \int_0^T \int_\Gamma \Lambda_r \delta K_r(\theta) \mathrm{q}(\mathrm{d}r, \mathrm{d}\theta),
\end{aligned}$$

by the fact that the right-hand side of the above equality is an $\mathbb{F}$–martingale under the probability measure $\mathrm{d}\hat{Q} := \mathcal{E}_T(N)\mathrm{d}\mathbb{P}$, taking the expectation we get

$$\mathbb{E}^{\hat{Q}} \left[\Lambda_T \left(h^2 - h^1\right)\right] = 0,$$

and

$$\mathbb{E}^{\hat{Q}} \left[\int_0^T \Lambda_r \left(f^2(r, X_r, Y_r^1, Z_r^1, K_r^1(\cdot)) - f^1(r, X_r, Y_r^1, Z_r^1, K_r^1(\cdot))\right) \mathrm{d}r\right] = 0.$$

Hence, $h^2 = h^1$ $\mathbb{P}$–a.s. and $f^1 = f^2$ $\mathrm{d}t \otimes \mathrm{d}\mathbb{P}$–a.e., which implies that $Y_s^1 = Y_s^2$ $\mathbb{P}$–a.s. for all $s \in [0, T]$. Therefore $Z_s^1 = Z_s^2$ $ds \otimes \mathrm{d}\mathbb{P}$–a.e. and $K_s^1(\theta) = K_s^2(\theta)$ $\nu(s, x, d\theta)$ $ds \otimes \mathrm{d}\mathbb{P}$–a.e.
□

1.5 Kolmogorov Equation

In this section, we prove the existence of a unique solution to the Kolmogorov equation under the Lipschitz condition. We further assume that the jump Markov process X, defined in section 1 of chapter 1, satisfies the following conditions:

1. $\mathbb{P}^{t,x}(X_t = x) = 1$ for every $t \in [0, \infty[$, $x \in \Gamma$.

2. For every $0 \leq t \leq s$ and $A \in \mathcal{E}$ the function $x \to \mathbb{P}^{t,x}(X_s \in A)$ is $\mathcal{E}$-measurable.

3. For every $0 \leq r \leq t \leq s$, $A \in \mathcal{E}$ we have

$$\mathbb{P}^{r,x}(X_s \in A \mid \mathcal{F}_{[r,t]}) = \mathbb{P}^{t,X_t}(X_s \in A). \quad \mathbb{P}^{r,x}\text{-a.s.}$$

4. All the trajectories of the pure jump process X have the right limits when Γ is endowed with its discrete topology (the one where all subsets are open). In other words, for every $\omega \in \Omega$ and $t \geq 0$ there exists $\delta > 0$ such that $X_s(\omega) = X_t(\omega)$ for $s \in [t, t+\delta]$.

5. For every $\omega \in \Omega$ the number of jumps of the trajectory $t \to X_t(\omega)$ is finite on every bounded interval, which implies that X is a non-explosive process.

Let

$$\begin{aligned} u\,(t,x) &= h(x) + \int_t^T \mathcal{L}_r u\,(r,x)\,\mathrm{d}r \\ &\quad + \int_t^T f(r,x,u\,(r,x)\,,u\,(r,.) - u\,(r,x))\mathrm{d}r, \end{aligned} \tag{1.12}$$

the parabolic differential equation on the state space Γ (called Kolmogorov equation) where $\mathcal{L}_r$ denote the generator of X of the form

$$\mathcal{L}_r\,(\varphi\,(x)) = \int_\Gamma (\varphi\,(\theta) - \varphi\,(x))\,\nu\,(r,x,\mathrm{d}\theta)\,,$$

such that $\varphi : \Gamma \to \mathbb{R}$ is a measurable function, f and h are two given functions, $u : [0,T] \times \Gamma \to \mathbb{R}$ is an unknown function such that the function $t \to u\,(t,x)$ is absolutely continuous on $[0,T]$ such that
$(u\,(s,X_{s-})\,,u\,(s,\theta) - u\,(s,X_{s-})\,s \in [0,T], \theta \in \Gamma) \in \mathcal{S}_2 \otimes \mathcal{H}^2$ and

$$\begin{cases} \partial_t u(t,x) + \mathcal{L}_t u\,(t,x) + f(t,x,u\,(t,x)\,,u\,(t,.) - u\,(t,x)) = 0, \\ u\,(T,x) = h(x). \end{cases}$$

Now, we give the first lemma in this Section which claims an existence and uniqueness result under some appropriate bounded conditions on f and h.

Lemma 1.10

Suppose that f and h verify Hypothesis 1 and in addition,

$$\sup_{t\in[0,T],x\in\Gamma} (|h\,(x)| + |f\,(t,x,0,0)|) < \infty.$$

Then the nonlinear Kolmogorov equation has a unique solution in the class of measurable bounded functions.

Proof: see [19]

Definition 1.2

We say that a measurable function $u : [0,T] \times \Gamma \to \mathbb{R}$ is a solution of the nonlinear Kolmogorov equation (1.13) if for every $t \in [0,T]$, $x \in \Gamma$

i. $\mathbb{E}^{t,x} \int_s^T \int_\Gamma |u(r,\theta) - u(r,X_r)|^2 \nu(r,X_r,\mathrm{d}\theta)\, dr < \infty.$

ii. $\mathbb{E}^{t,x} \int_s^T |u(r,X_r)|^2 \mathrm{d}r < \infty.$

iii. *(1.13) is satisfied.*

We introduce the following BSDEJ

$$\begin{aligned} Y_s^{t,x} &= h(X_T^{t,x}) + \int_s^T f(r, X_r, Y_r^{t,x}, K_r^{t,x}(\cdot))\mathrm{d}r \\ &\quad - \int_s^T \int_\Gamma K_r^{t,x}(\theta)\mathrm{q}(\mathrm{d}r,\mathrm{d}\theta), \end{aligned} \tag{1.13}$$

which will play a basic role in this result, under **Hypotheses 1** ,Theorem(3.4) in [19] shows that BSDEJ (1.14) has a unique solution $(Y_r^{t,x}, K_r^{t,x}(\cdot))_{s\in[t,T]}$. Note that $Y_t^{t,x}$ is deterministic. Now we are able to state and prove the main result of this section.

Theorem 1.11

*Under **Hypotheses 1** the nonlinear Kolmogorov equation (1.13) has a unique solution u. Moreover for every $t \in [0,T]$, $x \in \Gamma$ we have*

$Y_s^{t,x} = u(s, X_s)$.

$K_s^{t,x}(\theta) = u(s,\theta) - u(s, X_{s^-})$.

so that in particular $u(t,x) = Y_t^{t,x}$.

Proof of uniqueness: Let $u \in \mathcal{L}^2(p)$ be a solution to (1.13), applying Itô's formula to $u(s, X_s^{t,x})$, we get

$$\begin{aligned} u(T, X_T^{t,x}) - u(s, X_s^{t,x}) &= \int_s^T \left(\partial_s u(r, X_r^{t,x}) + \mathcal{L}_r u(r, X_r^{t,x})\right) \mathrm{d}r \\ &\quad + \int_s^T \int_\Gamma \left(u(r,\theta) - u(r, X_{r-}^{t,x})\right) \mathrm{q}^t(\mathrm{d}r,\mathrm{d}\theta). \end{aligned}$$

Taking account that u satisfies (1.13), we obtain

$$\partial_s u(s, X_s^{t,x}) + \mathcal{L}_s u(s, X_s^{t,x}) + f(s, X_s^{t,x}, u\left(s, X_s^{t,x}\right), u(s,.) - u\left(s, X_s^{t,x}\right)) = 0.$$

For all $s \in [t,T]$, and the fact that $u(T, X_T^{t,x}) = h(X_T^{t,x})$, we arrive at

$$\begin{aligned} u(s, X_s^{t,x}) = h(X_T^{t,x}) &+ \int_s^T f(r, X_s^{t,x}, u\left(s, X_r^{t,x}\right), u(r,\cdot) - u\left(s, X_r^{t,x}\right))\mathrm{d}r \\ &- \int_s^T \int_\Gamma \left(u(r,\theta) - u(r, X_{r-}^{t,x})\right) \mathrm{q}^t(\mathrm{d}r,\mathrm{d}\theta). \end{aligned}$$

We have $(u(s, X_s), u(s,.) - u(s, X_{s-}))$ is another solution to the BSDEJ (1.14), theorem 1.5 confirms that BSDEJ (1.14) has a unique solution, hence, we get $Y_s^{t,x} = u(s, X_s)$ $\mathbb{P}^{t,x} - a.s.$ and $K_s^{t,x}(\theta) = u(s,\theta) - u(s, X_{s-})$ $\nu(s,x,d\theta)ds \otimes d\mathbb{P}$–a.e for every $\theta \in \Gamma$. In particular, $u(t,x) = Y_t^{t,x}$

Proof of the Existence: Let $f^n = (f \wedge n) \vee (-n)$, $h^n = (h \wedge n) \vee (-n)$ the truncation of f and h at level n, then we obtain the following family of approximating systems

$$\begin{aligned} u^n(t,x) = h^n(x) + \int_t^T \mathcal{L}_s u^n(s,x)\,\mathrm{d}s \\ + \int_t^T f^n(s,x,u^n(s,x), u^n(s,\cdot) - u^n(s,x))\mathrm{d}s. \end{aligned} \tag{1.14}$$

$$\begin{aligned} Y_s^{(t,x)n} = h^n(X_T) + \int_s^T f^n(r, X_r, Y_r^{(t,x)n}, Z_r^{(t,x)n}(\cdot))\mathrm{d}r \\ - \int_s^T \int_\Gamma K_r^{(t,x)n}(\theta)\mathrm{q}^t(\mathrm{d}r,\mathrm{d}\theta). \end{aligned} \tag{1.15}$$

From lemma (1.10) there exist a unique bounded solution u^n to the Kolmogorov equation (1.14) and a unique solution $(Y^{(t,x)n}, K^{(t,x)n}(\cdot))$ of BSDEJ (1.15) such that $Y_s^{(t,x)n} = u^n(s, X_s)$ $\mathbb{P}^{t,x} - a.s.$, $K_s^{(t,x)n}(\theta) = u^n(s,\theta) - u^n(s, X_{s-})$ $\nu(s,x,d\theta)ds \otimes d\mathbb{P}$–a.e. for any $\theta \in \Gamma$ and $Y_t^{(t,x)n} = u^n(t,x)$.

Proposition 3.5 in [19] and the monotone convergence leads to

$$\begin{aligned} &\sup_{s\in[0,T]} \mathbb{E}^{t,x}\left|Y_s^{(t,x)n} - Y_s^{(t,x)}\right|^2 + \mathbb{E}^{t,x}\int_s^T \left|Y_r^{(t,x)n} - Y_r^{(t,x)}\right|^2 \mathrm{d}r \\ &+ \mathbb{E}^{t,x}\int_s^T \|\left(K_r^{(t,x)n} - K_r^{(t,x)}\right)(\cdot)\|_\nu^2 \mathrm{d}r \\ &\leq c\mathbb{E}^{t,x}\left|h(X_T) - h^n(X_T)\right|^2 + c\mathbb{E}^{t,x}\int_s^T f^n(r, X_r, Y_r^{(t,x)n}, K_r^{(t,x)n}(\cdot))\mathrm{d}r \\ &- c\mathbb{E}^{t,x}\int_r^T f(r, X_r, Y_r^{t,x}, K_r^{t,x}(\cdot))\mathrm{d}r \\ &\to 0. \end{aligned}$$

Then we deduce that

$$\begin{aligned} \left|u(t,x) - u^n(t,x)\right|^2 &= \left|Y_t^{t,x} - Y_t^{(t,x)n}\right|^2 \\ &\leq \sup_{s\in[t,T]} \mathbb{E}^{t,x}\left|Y_s^{t,x} - Y_s^{(t,x)n}\right|^2 \to 0. \end{aligned}$$

Using the previous convergence and Fatou's lemma to get

$$
\begin{aligned}
&\mathbb{E}^{t,x}\int_t^T\left(\left|Y_s^{t,x}-u\left(s,X_s\right)\right|^2+\|K_s^{t,x}\left(\cdot\right)-u\left(s,\cdot\right)+u\left(s,X_s\right)\|_\nu^2\right)\mathrm{d}s\\
&\leq \lim_{n\to\infty}\inf\mathbb{E}^{t,x}\int_t^T\left|Y_s^{t,x}-u^n\left(s,X_s\right)\right|^2\mathrm{d}s\\
&+\lim_{n\to\infty}\inf\mathbb{E}^{t,x}\int_t^T\|K_s^{t,x}\left(\theta\right)-u^n\left(s,\theta\right)+u^n\left(s,X_s\right)\|_\nu^2\mathrm{d}s\\
&=\lim_{n\to\infty}\inf\mathbb{E}^{t,x}\int_t^T\left(\left|Y_s^{t,x}-Y_s^{(t,x)n}\right|^2+\|\left(K_s^{t,x}-K_s^{(t,x)n}\right)\left(\cdot\right)\|_\nu^2\right)\mathrm{d}s\\
&=0.
\end{aligned}
$$

Which prove that $Y_s^{t,x}=u\left(s,X_s\right),K_s^{t,x}(\theta)=u\left(s,\theta\right)-u\left(s,X_{s^-}\right).$

To prove that u satisfies (1.13), it remains to prove that

$$
\int_t^T\mathcal{L}_su^n\left(s,x\right)\mathrm{d}s\to\int_t^T\mathcal{L}_su\left(s,x\right)\mathrm{d}s, \tag{1.16}
$$

and

$$
\begin{aligned}
&\int_t^T f^n(s,x,u^n\left(s,x\right),u^n\left(s,\cdot\right)-u^n\left(s,x\right))\mathrm{d}s\\
&\to\int_t^T f(s,x,u\left(s,x\right),u\left(s,\cdot\right)-u\left(s,x\right))\mathrm{d}s.
\end{aligned} \tag{1.17}
$$

Using the definition of $\mathcal{L}_s$, we get

$$
\begin{aligned}
&\mathbb{E}^{t,x}\left|\int_t^T\mathcal{L}_su^n\left(s,X_s\right)\mathrm{d}s-\int_t^T\mathcal{L}_su\left(s,X_s\right)\mathrm{d}s\right|\\
&=\mathbb{E}^{t,x}\left|\int_t^T\int_\Gamma(u\left(s,\theta\right)-u\left(s,X_s\right)-u^n\left(s,\theta\right)+u^n\left(s,X_s\right))\nu(s,X_s,\mathrm{d}\theta)\mathrm{d}s\right|,\\
&=\mathbb{E}^{t,x}\left|\int_t^T\int_\Gamma\left(K_s^{t,x}(\theta)-K_s^{(t,x)n}(\theta)\right)\nu(s,X_s,\mathrm{d}\theta)\mathrm{d}s\right|\\
&\leq(T-t)^{\frac12}\left(\sup_{t..x}\nu(t,x,\Gamma)\right)^{\frac12}\left(\mathbb{E}^{t,x}\int_t^T\|\left(K_s^{(t,x)n}-K_s^{(t,x)}\right)(\cdot)\|_\nu^2\mathrm{d}s\right)^{\frac12}\\
&\to 0.
\end{aligned}
$$

Therefore, there exists a subsection (still denoted u^n) such that

$$
\int_t^T\mathcal{L}_su^n\left(s,X_s\right)\mathrm{d}s\to\int_t^T\mathcal{L}_su\left(s,X_s\right)\mathrm{d}s,\ \mathbb{P}^{t,x}-a.s.
$$

We have the first jump T_1^t has an exponential law see [19] hence the set $A:=\{\omega\in\Omega:T_1^t\left(\omega\right)>T\}$, has positive $\mathbb{P}^{t,x}$, for each $\omega\in A$, we have $X_r\left(\omega\right)=x$, then we get (1.16).

We pass now to prove (1.17), we have

$$\begin{aligned}
&\mathbb{E}^{t,x}\left|\int_t^T (f(s, X_s, u(s, X_s), u(s, \cdot) - u(s, X_s))\right.\\
&-f^n(s, X_s, u^n(s, X_s), u^n(s, \cdot) - u^n(s, X_s))\mathrm{d}s|\\
&= \mathbb{E}^{t,x}\left|\int_t^T f(s, X_s, Y_s^{(t,x)}, K_s^{(t,x)}(\cdot)) - f^n(s, X_s, Y_s^{(t,x)n}, K_s^{(t,x)n}(\cdot))\mathrm{d}s\right|\\
&\leq \mathbb{E}^{t,x}\int_t^T \left|f(s, X_s, Y_s^{(t,x)}, K_s^{(t,x)}(\cdot)) - f^n(s, X_s, Y_s^{(t,x)}, K_s^{(t,x)}(\cdot))\right|\mathrm{d}s\\
&+ \mathbb{E}^{t,x}\int_t^T \left|f^n(s, X_s, Y_s^{(t,x)}, K_s^{(t,x)}(\cdot)) - f^n(s, X_s, Y_s^{(t,x)n}, K_s^{(t,x)n}(\cdot))\right|\mathrm{d}s.
\end{aligned}$$

By the monotone convergence theorem, we have

$$\mathbb{E}^{t,x}\int_t^T \left|f(s, X_s, Y_s^{(t,x)}, K_s^{(t,x)}(\cdot)) - f^n(s, X_s, Y_s^{(t,x)}, K_s^{(t,x)}(\cdot))\right|\mathrm{d}s \to 0.$$

Since f^n is a truncation of f we can get from the Lipschitz condition

$$\begin{aligned}
&\mathbb{E}^{t,x}\int_t^T \left|f^n(s, X_s, Y_s^{(t,x)}, K_s^{(t,x)}(\cdot)) - f^n(s, X_s, Y_s^{(t,x)n}, K_s^{(t,x)n}(\cdot))\right|\mathrm{d}s\\
&\leq L\mathbb{E}^{t,x}\int_t^T \left(\left|Y_s^{t,x} - Y_s^{(t,x)n}\right| + \int_\Gamma \left|K_s^{t,x}(\theta) - K_s^{(t,x)n}(\theta)\right|\nu(s, X_s, \mathrm{d}\theta)\right)\mathrm{d}s\\
&\leq L\left((T-t)\,\mathbb{E}^{t,x}\int_t^T \left|Y_s^{t,x} - Y_s^{(t,x)n}\right|^2\mathrm{d}s\right)^{\frac{1}{2}}\\
&+ L(T-t)^{\frac{1}{2}}\left(\sup_{t..x}\nu(t, x, \Gamma)\right)^{\frac{1}{2}}\left(\mathbb{E}^{t,x}\int_t^T \|\left(K_s^{(t,x)n} - K_s^{(t,x)}\right)(\cdot)\|_\nu^2\mathrm{d}s\right)^{\frac{1}{2}} \to 0.
\end{aligned}$$

Hence, there exists a subsection (still denoted u^n) such that

$$\begin{aligned}
&\int_t^T f^n(s, X_s, u^n(s, X_s), u^n(s, \cdot) - u^n(s, X_s))\mathrm{d}s\\
&\to \int_t^T f(s, X_s, u(s, X_s), u(s, \cdot) - u(s, X_s))\mathrm{d}s.
\end{aligned}$$

For each $\omega \in A$, we have $X_r(\omega) = x$ then we get (1.17) which achieves the proof. □

CHAPTER 2

BSDEJs with non-Lipschitz Generators

(Joint work with N. Khelfallah, A. Almualim, and M. Eddahbi.)

2.1 Introduction

In this chapter, we tackle a class of BSDEJs driven by both Wiener and jump Markov processes with non-Lipschitz generators. In Section 1, we give an existence result to BSDEJ (0.2) with a continuous coefficient. In Section 2, we show that BSDEJ (0.2) has either one or a set of countable solutions. In Section 3, we prove that BSDEJ (0.2) with a left continuous and increasing coefficient has at least one solution. Finally, in Section 4, we apply the result of Section 1 to solve one type of BSDEJ whose generator is of a quadratic growth in the variable z and terminal condition in $L.^2$

2.2 BSDEJs with Continuous Coefficients

The purpose of this section is to prove an existence result to BSDEJ(0.2), covering the case where the generator f is continuous in (y, z), Lipschitz in $k(\cdot)$ and satisfies the following linear growth type condition:

($\mathbf{H}_{3.1}$) For all $(s, \omega, x, y, z) \in [0, T] \times \Omega \times \Gamma \times \mathbb{R} \times \mathbb{R}$ and $k(\cdot) \in L^2(\Gamma, \mathcal{E}, \nu(s, x, d\theta))$ we have

$$|f(s, x, y, z, k(\cdot))| \leq \lambda \left(1 + |y| + |z| + \|(k\varphi_s)(\cdot)\|_\nu\right).$$

where the function $\varphi_s(\cdot)$ is defined in Theorem 1.9 ii) in Chapter 1 Section 3.

Theorem 2.1

Let Assumptions $\mathbf{H}_{1.1}-\mathbf{H}_{1.3}$, $\mathbf{H}_{3.1}$ *and* $\mathbf{H}_{2.2}$ *hold true. Then, BSDEJ (0.2) has at least a minimal solution* $(Y, Z, K(\cdot))$.

To prove this theorem, we approximate the generator by an increasing sequence of Lipschitz functions $(f_n)_{n\geq 0}$ which will be defined by the following lemma.

Lemma 2.2

For all $n \geq \lambda$, *we consider* $(f_n)_{n\geq\lambda}$ *defined by*

$$f_n(s,x,y,z,k(\cdot)) = \inf_{(\acute{a},\acute{b})\in\mathbb{Q}\times\mathbb{Q}} \left\{ f(s,x,a,b,k(\cdot)) + n\left(|\acute{a}-y| + \left|\acute{b}-z\right|\right)\right\}.$$

The sequence $(f_n)_{n\geq\lambda}$ *has the following properties:*

For all $(s,x) \in [0,T]\times\Gamma$ *and* $(y,z,k(\cdot)), (\acute{y},\acute{z},\acute{k}(\cdot)) \in \mathbb{R}\times\mathbb{R}\times L^2(\Gamma,\mathcal{E},\nu(s,x,d\theta))$.

$(\mathbf{A}_1)$ *There exists* $n \geq 0$ *such that*

$$\begin{aligned}&\left|f_n(s,x,y,z,k(\cdot)) - f_n(s,x,\acute{y},\acute{z},\acute{k}(\cdot))\right|\\ &\leq n\left[|y-\acute{y}| + |z-\acute{z}| + \|k(\cdot)-\acute{k}(\cdot)\|_\nu\right].\end{aligned}$$

$(\mathbf{A}_2)$ $f_{n+1}(s,x,y,z,k(\cdot)) \geq f_n(s,x,y,z,k(\cdot))$.

$(\mathbf{A}_3)$ *There exist two constants* a *and* b, $-1 < a < 0$, $b > 0$ *such that for every* $s \in [0,T]$, $x \in \Gamma$, r, $z \in \mathbb{R}$ *and* $k(\cdot)$, $\acute{k}(\cdot) \in L^2(\Gamma,\mathcal{E},\nu(s,x,d\theta))$, *we have*

$$f_n(s,x,r,z,k(\cdot)) - f_n(s,x,r,z,\acute{k}(\cdot)) \leq \int_\Gamma (k(\theta)-\acute{k}(\theta))\varphi_s(\theta)\nu(s,x,\mathrm{d}\theta),$$

where $\varphi_s(\theta) : \Omega\times[0,T]\times\Gamma \longrightarrow \mathbb{R}$ *is* $\mathcal{P}\otimes\mathcal{E}$*–measurable and satisfies* $a < \varphi_s(\theta) < b$.

$(\mathbf{A}_4)$ $|f_n(s,x,y,z,k(\cdot))| \leq \lambda\,(1+|y|+|z|+\|k(\cdot)\varphi_s(\cdot)\|_\nu)$.

$(\mathbf{A}_5)$ *If* $(y_n,z_n) \underset{n\to\infty}{\longrightarrow} (y,z)$ *then* $f_n(s,x,y_n,z_n,k(\cdot)) \underset{n\to\infty}{\longrightarrow} f(s,x,y,z,k(\cdot))$.

Proof : The proof can be performed as that of Lemma 1 in [35]. □

We note that for all $n \geq \lambda$, the function f_n verifies the Hypothesis 1.1 which implies that there exists a unique solution $(Y^n, Z^n, K^n(\cdot))$ of BSDEJ with data $(f_n, h(X_T))$. We establish priori estimates on the sequence $(Y^n, Z^n, K^n(\cdot))$.

Lemma 2.3

There exists a constant $C > 0$ depending only on h, T, λ^2 such that for all $n \geq 1$

$$\sup_{n\geq\lambda}\left(\mathbb{E}\sup_{s\in[0,T]}|Y_s^n|^2 + \mathbb{E}\int_0^T |Z_r^n|^2\,\mathrm{d}r + \mathbb{E}\int_0^T \|K_r^n(\cdot)\|_\nu^2\,\mathrm{d}r\right) \leq C.$$

Proof: From Itô's formula applied to $|Y_s^n|^2$, it follows that

$$\begin{aligned}|Y_s^n|^2 &= |h(X_T)|^2 + 2\int_s^T Y_r^n f_n(r, X_r, Y_r^n, Z_r^n, K_r^n(\cdot))\mathrm{d}r \\ &\quad -2\int_s^T\int_\Gamma Y_{r-}^n K_r^n(\theta) q(\mathrm{d}r,\mathrm{d}\theta) - 2\int_s^T Y_r^n Z_r^n \mathrm{d}B_r \\ &\quad -\int_s^T |Z_r^n|^2\,\mathrm{d}r - \int_s^T\int_\Gamma |K_r^n(\theta)|^2\,\mathrm{q}(\mathrm{d}r,\mathrm{d}\theta) - \int_s^T \|K_r^n(\cdot)\|_\nu^2\,\mathrm{d}r.\end{aligned} \tag{2.1}$$

Taking the expectation in both sides of the previous inequality, we obtain

$$\begin{aligned}&\mathbb{E}|Y_s^n|^2 + \mathbb{E}\int_s^T |Z_r^n|^2\,\mathrm{d}r + \mathbb{E}\int_s^T \|K_r^n(\cdot)\|_\nu^2\,\mathrm{d}r \\ =\ &\mathbb{E}|h(X_T)|^2 + 2\mathbb{E}\int_s^T Y_r^n f_n(r, X_r, Y_r^n, Z_r^n, K_r^n(\cdot))\mathrm{d}r.\end{aligned}$$

Therefore, we obtain from $(\mathbf{A}_4)$, Young's inequality $2xy \leq 2x^2 + \frac{y^2}{2}$,

$$\begin{aligned}&\mathbb{E}|Y_s^n|^2 + \mathbb{E}\textstyle\int_s^T |Z_r^n|^2\,\mathrm{d}r + \mathbb{E}\int_s^T \|K_r^n(\cdot)\|_\nu^2\,\mathrm{d}r \\ &\leq 2\left(C + 2\lambda^2 T\right) + 2\left(\tfrac{1}{2} + 4\lambda^2 + 2\lambda\right)\mathbb{E}\textstyle\int_s^T |Y_r^n|^2\,\mathrm{d}r.\end{aligned}$$

Hence, Gronwall's lemma yields

$$\sup_{s\in[0,T]}\mathbb{E}|Y_s^n|^2 + \mathbb{E}\int_0^T |Z_r^n|^2\,\mathrm{d}r + \mathbb{E}\int_0^T \|K_r^n(\cdot)\|_\nu^2\,\mathrm{d}r \leq C. \tag{2.2}$$

Now, returning to (2.1), using $(\mathbf{A}_4)$, Young's inequality $2xy \leq 2x^2 + \frac{y^2}{2}$ and Burkholder-Davis-Gundy inequality to obtain

$$\begin{aligned}&\mathbb{E}\sup_{s\in[0,T]}|Y_s^n|^2] + \mathbb{E}\int_0^T |Z_r^n|^2\,\mathrm{d}r + \mathbb{E}\int_0^T \|K_r^n(\cdot)\|_\nu^2\,\mathrm{d}r \\ &\leq \mathbb{E}|h(X_T)|^2 + 2\lambda^2 T + \left(\frac{1}{2} + 2\lambda^2 + 2\lambda\right)\mathbb{E}\int_0^T |Y_s^n|^2\,\mathrm{d}r \\ &+\frac{1}{2}\left(\mathbb{E}\int_0^T |Z_r^n|^2\,\mathrm{d}r + \mathbb{E}\int_0^T \|K_r^n(\cdot)\|_\nu^2\,\mathrm{d}r\right) \\ &+4\frac{C}{\kappa}\mathbb{E}\left(\sup_{s\in[0,T]}|Y_s|^2\right) + 4\kappa C E\left(\int_0^T |Z_r|^2\,\mathrm{d}r + \int_0^T \|K_r^n(\cdot)\|_\nu^2\,\mathrm{d}r\right).\end{aligned}$$

Then

$$\left(1-4\tfrac{C}{\kappa}\right)\mathbb{E}[\sup_{t\in[0,T]}|Y_t^n|^2]+\mathbb{E}\left(\int_0^T|Z_r^n|^2\,\mathrm{d}r+\int_0^T\|K_r^n(\cdot)\|_\nu^2\,\mathrm{d}r\right)$$
$$\leq \acute{C}+2\lambda^2T+\left(\tfrac{1}{2}+4\kappa C\right)+\left(\tfrac{1}{2}+2\lambda^2+2\lambda\right)\int_0^T\mathbb{E}\sup_{t\in[0,r]}|Y_t^n|^2\,\mathrm{d}r.$$

Finally, Gronwall's lemma gives the desired result.

Proof of Theorem 2.1

We split the proof into the following three steps:

Step 1: In this step we prove that there exists a process $Y\in\mathcal{S}^2$ as the infimum limit of Y^n. Set

$$g(s,x,y,z,k(\cdot))=\lambda\left(1+|y|+|z|+\|(k\varphi_s)(\cdot)\|_\nu\right).$$

Let $(\acute{Y},\acute{Z},\acute{K}(\cdot))$ be the unique solution of the BSDEJ with data $(g,h(X_T))$, there is insured by Theorem 1.5. Remember that for each $n,(Y^n,Z^n,K^n(\cdot))$ is the unique solution of BSDEJ with data $(f_n,h(X_T))$. Now, thanks to $(\mathbf{A}_2)$ and Theorem 1.9 that the sequence $(Y_s^n)_{n\geq1}$ is non decreasing and bounded by Y_s. Therefore there exists a stochastic process Y as the limit of the sequence Y_s^n: $Y_s=\lim_{n\to\infty}Y_s^n$. From Lemma 2.3, we have $\mathbb{E}[\sup_{s\in[0,T]}|Y_s^n|^2]\leq C$, then, Fatou's Lemma gives

$$\begin{aligned}\mathbb{E}[\sup_{s\in[0,T]}|Y_s|^2] &= \mathbb{E}\left[\sup_{s\in[0,T]}\left|\lim_{n\to\infty}Y_s^n\right|^2\right]\\ &\leq \liminf_{n\to\infty}\mathbb{E}[\sup_{s\in[0,T]}|Y_s^n|^2]=C,\end{aligned}$$

which implies that $Y\in\mathcal{S}^2$. Then, from Lebesgue's dominated convergence theorem, we obtain

$$\mathbb{E}\int_0^T|Y_r^n-Y_r|^2\,\mathrm{d}r\underset{n\to\infty}{\longrightarrow}0.$$

Step 2: In this step, we shall prove that $(Z^n,K^n(.))_{n\geq1}$ is a Cauchy sequence on $\mathcal{M}^2\otimes L^2(\Gamma,\mathcal{E},\nu(.,x,d\theta))$. Using Itô's formula and Hölder's inequality, we get for $n,\ m\geq1$:

$$\mathbb{E}|Y_s^n-Y_s^m|^2+\mathbb{E}\int_s^T|Z_r^n-Z_r^m|^2\,\mathrm{d}r+\mathbb{E}\int_s^T\|K_r^n(.)-K_r^m(\cdot)\|_\nu^2\,\mathrm{d}r$$
$$\leq 2\left(\mathbb{E}\int_s^T|Y_r^n-Y_r^m|^2\,\mathrm{d}r\right)^{\frac{1}{2}}\left(\mathbb{E}\int_s^T|f_n(r,X_r,Y_r^n,Z_r^n,K_r^n(\cdot))|^2\,\mathrm{d}r\right)^{\frac{1}{2}}$$
$$+2\left(\mathbb{E}\int_s^T|Y_r^n-Y_r^m|^2\,\mathrm{d}r\right)^{\frac{1}{2}}\left(\mathbb{E}\int_s^T|f_m(r,X_r,Y_r^m,Z_r^m,K_r^m(\cdot))|^2\,\mathrm{d}r\right)^{\frac{1}{2}}.$$

Thanks to $(\mathbf{A}_4)$, Lemma 2.3 and $(a+b+c+\mathrm{d})^2 \leq 4\left(a^2+b^2+c^2+\mathrm{d}^2\right)$, we obtain

$$\begin{aligned}
&\mathbb{E}\int_s^T |f_m(r,X_r,Y_r^m,Z_r^m,K_r^m(\cdot))|^2\,\mathrm{d}r \\
&\leq 4\lambda\mathbb{E}\int_s^T \mathrm{d}r + \mathbb{E}\int_s^T |Y_r^m|^2\,\mathrm{d}r + \mathbb{E}\int_s^T |Z_r^m|^2\,\mathrm{d}r + \mathbb{E}\int_s^T \|K_r^m(\cdot)\|_\nu^2\,\mathrm{d}r \\
&\leq 4\lambda\left(T+\tilde{C}\right) \;=\; C.
\end{aligned}$$

Thus

$$\begin{aligned}
&\mathbb{E}\,|Y_s^n - Y_s^m|^2 + \mathbb{E}\int_s^T |Z_r^n - Z_r^m|^2\,\mathrm{d}r + \mathbb{E}\int_s^T \|K_r^n(\cdot) - K_r^m(\cdot)\|_\nu^2\,\mathrm{d}r \\
&\leq C\left(\mathbb{E}\int_s^T |Y_r^n - Y_r^m|^2\,\mathrm{d}r\right)^{\frac{1}{2}} \underset{n\to\infty}{\longrightarrow} 0.
\end{aligned}$$

Therefore, $(Z^n, K^n(\cdot))_{n\geq 1}$ is a Cauchy sequence on $\mathcal{M}^2 \otimes L^2(\Gamma,\mathcal{E},\nu(s,x,d\theta))$, and thus, there exists a process $(Y,Z,K(\cdot)) \in \mathcal{B}$ as limit of the sequence $(Y^n,Z^n,K^n(\cdot))$.

Step 3: In this step, we show that $(Y,Z,K(\cdot))$ satisfies BSDEJ (0.2), we have $(Y^n,Z^n,K^n(\cdot)) \underset{n\to\infty}{\longrightarrow} (Y,Z,K(\cdot))$ in the space $\mathcal{B}$. Passing to a sub-sequence we get the convergence $\mathrm{d}t\otimes\mathrm{d}\mathbb{P}$ a.s. to $(Y,Z,K(\cdot))$. Then from $(\mathbf{A}_5)$, we have

$$f_n(s,X_s,Y_s^n,Z_s^n,K_s(\cdot)) \underset{n\to\infty}{\longrightarrow} f(s,X_s,Y_s,Z_s,K_s(\cdot)) \quad \mathbb{P}\text{–a.s.}$$

Set $(G_s,H_s) = \sup_{n\geq\lambda}(|Y_s^n|, |Z_s^n|)$, then, from $(\mathbf{A}_4)$ we obtain

$$\sup_{n\geq\lambda}|f_n(s,X_s,Y_s^n,Z_s^n,K_s(\cdot))| \leq \lambda\left(1+G_s+H_s+\|(K_s\varphi_s)(\cdot)\|_\nu\right) \in \mathbb{L}^1(\Omega).$$

By subtracting and adding $f_n(r,X_r,Y_r^n,Z_r^n,K_r(\cdot))$, we get

$$\begin{aligned}
&\mathbb{E}\textstyle\int_0^T |f_n(r,X_r,Y_r^n,Z_r^n,K_r^n(\cdot)) - f(r,X_r,Y_r,Z_r,K_r(\cdot))|^2\,\mathrm{d}r \\
&\leq \mathbb{E}\textstyle\int_0^T |f_n(r,X_r,Y_r^n,Z_r^n,K_r^n(\cdot)) - f_n(r,X_r,Y_r^n,Z_r^n,K_r(\cdot))|^2\,\mathrm{d}r \\
&+\mathbb{E}\textstyle\int_0^T |f_n(r,X_r,Y_r^n,Z_r^n,K_r(\cdot)) - f(r,X_r,Y_r,Z_r,K_r(\cdot))|^2\,\mathrm{d}r.
\end{aligned}$$

Since f_n is Lipschitz in $k(\cdot)$ the first term in the right-hand side of the above equality tends to 0 as n goes to infinity, then the dominated convergence theorem yields the convergence of the second term to 0. Therefore, there exists a subsection (still indexed by n) such that

$$\int_0^T f_n(r,X_r,Y_r^n,Z_r^n,K_r^n(\cdot))\mathrm{d}r \underset{n\to\infty}{\longrightarrow} \int_0^T f(r,X_r,Y_r,Z_r,K_r(\cdot))\mathrm{d}r \quad \mathbb{P}\text{–a.s.}$$

Then, Burkholder-Davis-Gundy inequality, we show that

$$\begin{aligned}
&\int_0^T Z_r^n \mathrm{d}B_r \underset{n\to\infty}{\longrightarrow} \int_0^T Z_r \mathrm{d}B_r,\\
&\int_0^T \int_\Gamma K_r^n(\theta)\mathrm{q}(\mathrm{d}r,\mathrm{d}\theta) \underset{n\to\infty}{\longrightarrow} \int_0^T \int_\Gamma K_r(\theta)\mathrm{q}(\mathrm{d}r,\mathrm{d}\theta).
\end{aligned}$$

It remains to verify that $Y_s^n \underset{n\to\infty}{\longrightarrow} Y_s$ $\mathbb{P}$–a.s.

$$\begin{aligned}
&\mathbb{E}\left|Y_s^n - Y_s\right|^2\\
= \;& \mathbb{E}\left|\int_s^T f_n(r,X_r,Y_r^n,Z_r^n,K_r^n(\cdot))\mathrm{d}r - \int_s^T Z_r^n \mathrm{d}B_r - \int_s^T \int_\Gamma K_r^n(\theta)\mathrm{q}(\mathrm{d}r,\mathrm{d}\theta)\right.\\
&\left. - \int_s^T f(r,X_r,Y_r,Z_r,K_r(\cdot))\mathrm{d}r + \int_s^T Z_r \mathrm{d}B_r + \int_s^T \int_\Gamma K_r(\theta)\mathrm{q}(\mathrm{d}r,\mathrm{d}\theta)\right|^2.
\end{aligned}$$

Since $(a+b+c)^2 \leq 3\left(a^2+b^2+c^2\right)$, the above equality becomes

$$\begin{aligned}
&\mathbb{E}\left|Y_s^n - Y_s\right|^2\\
\leq \;& 3\mathbb{E}\left|\int_s^T \left|f_n(r,X_r,Y_r^n,Z_r^n,K_r^n(\cdot)) - f(r,X_r,Y_r,Z_r,K_r(\cdot))\right| \mathrm{d}r\right|^2\\
&+3\mathbb{E}\left|\int_s^T \left(Z_r^n - Z_r\right) \mathrm{d}B_r\right|^2\\
&+3\mathbb{E}\left|\int_s^T \int_\Gamma \left(K_r^n(\theta) - K_r(\theta)\right) \mathrm{q}(\mathrm{d}r,\mathrm{d}\theta)\right|^2 \underset{n\to\infty}{\longrightarrow} 0.
\end{aligned}$$

Hence, the Theorem is proved.

Remark 2.4

Using similar techniques we can prove that BSDEJ (0.2) has at least a maximal solution $\left(Y^{\max}, Z^{\max}, K^{\max}(\cdot)\right)$.

Remark 2.5

Any solution $\left(Y,Z,K(\cdot)\right)$ *of BSDEJ (0.2) must satisfy* $Y_t^{\min} \leq Y_t \leq Y_t^{\max}$, *a.s., for all* $t \in [0,T]$.

2.3 On the set of Solutions of BSDEJ

In this subsection we draw our attention to the set of solutions of a one-dimensional BSDEJ with jumps when the drift term is assumed to be continuous and of linear growth in $(y, z, k(\cdot))$. We prove then that there exists either one or uncountably many solutions of the equation (0.2). We note $(Y^{\max}, Z^{\max}, K^{\max}(\cdot))$ the maximal solution and $(Y^{\min}, Z^{\min}, K^{\min}(\cdot))$ the minimal solution of BSDEJ (0.2).

Hypothesis 3.2 We assume that

$(H_{3.2})$ For every $s \in [0,T]$, $x \in \Gamma$ the mapping $r, z, k(.) \longmapsto f(s,x,r,z,k(\cdot))$ is continuous and there exists $L \geq 0$ such that for every r, $z \in \mathbb{R}$, $k(\cdot)$, $\acute{k}(\cdot) \in L^2(\Gamma, \mathcal{E}, \nu(s,x,d\theta))$

$$\left| f(s,x,r,z,k(\cdot)) - f(s,x,r,z,\acute{k}(\cdot)) \right| \leq L \| k(\cdot) - \acute{k}(\cdot.) \|_{\nu}.$$

Theorem 2.6

We assume that $\mathbf{H}_{1.1} - \mathbf{H}_{1.3}$ *and* $\mathbf{H}_{3.1} - \mathbf{H}_{3.2}$ *hold true. Then, for each* $t_0 \in [0,T]$ *and* $\xi \in L^2(\Omega, \mathcal{F}_{t_0}, \mathbb{P})$ *such that* $Y_{t_0}^{\min} \leq \xi \leq Y_{t_0}^{\max}$ *a.s., there exists at least one solution* $(Y, Z, K(\cdot)) \in \mathcal{B}$ *to BSDEJ (0.2),satisfying* $Y_{t_0} = \xi$.

Proof: We consider the following BSDEJ for any $t \in [0, t_0]$

$$\begin{aligned} Y_t^1 &= \xi + \int_t^{t_0} f(r, X_r, Y_r^1, Z_r^1, K_r^1(\cdot)) \mathrm{d}r \\ &\quad - \int_t^{t_0} Z_r^1 \mathrm{d}B_r - \int_t^{t_0} \int_\Gamma K_r^1(\theta) \mathrm{q}(\mathrm{d}r, \mathrm{d}\theta). \end{aligned}$$

From Theorem 2.1 the previous equation has at least one solution $(Y^1, Z^1, K^1(\cdot))$, we also consider the following SDE for any $t \in [t_0, T]$

$$\begin{aligned} Y_t^2 &= \xi - \int_{t_0}^{t} f(r, X_r, Y_r^2, Z_r^2, K_r^2(\cdot)) \mathrm{d}r \\ &\quad + \int_{t_0}^{t} Z_r^2 \mathrm{d}B_r + \int_{t_0}^{t} \int_\Gamma K_r^2(\theta) \mathrm{q}(\mathrm{d}r, \mathrm{d}\theta), \end{aligned} \tag{2.3}$$

for a fixed Z^2, $K^2 \in \mathcal{M}^2 \times L^2(\Gamma, \mathcal{E}, \nu(s,x,d\theta))$, let $(Y^2)_{t \in [t_0,T]}$ be a solution to the previous SDE.

Now, we define a stopping time $\tau = \inf\{t \geq t_0 : Y_t^2 \notin (Y_t^{\min}, Y_t^{\max})\}$, such that $Y_T^{\min} = Y_T^{\max}$, and

$$\begin{aligned}(Y_t, Z_t, K_t) &= \mathbf{1}_{[0,t_0[}\left(Y_t^1, Z_t^1, K_t^1(\cdot)\right) + \mathbf{1}_{[t_0,\tau[}\left(Y_t^2, Z_t^2, K_t^2(\cdot)\right) \\ &\quad + \mathbf{1}_{[\tau,T[}\left(Y_t^{\max}, Z_t^{\max}, K_t^{\max}(\cdot)\right)\mathbf{1}_{\{Y_\tau = Y_\tau^{\max}\}} \\ &\quad + \mathbf{1}_{[\tau,T[}\left(Y_t^{\min}, Z_t^{\min}, K_t^{\min}(\cdot)\right)\mathbf{1}_{\{Y_\tau < Y_\tau^{\max}\}},\end{aligned}$$

is a solution to BSDEJ (0.2) with $Y_T = h(X_T)$ and $Y_{t_0} = \xi$. $\square$

This result is an extension of the one obtained by Jia and Peng [35] corresponding to the Brownian setting to BSDEs with jumps.

2.4 BSDEJ with left Continuous and Increasing Coefficients

The aim of this subsection is to prove that BSDEJ 0.2 has at least one solution, which belongs to the Banach space $\mathcal{B}$, assuming that f is only left continuous in y and bounded. We fix a deterministic terminal time $T > 0$ and we assume further that:

($\mathbf{H}_{3.2}$) There exist $L \geq 0$ such that for every $s \in [0,T]$, $x \in \Gamma$, $r \in \mathbb{R}$, z, $\acute{z} \in \mathbb{R}$, $k(\cdot)$, $\acute{k}(\cdot) \in L^2(\Gamma, \mathcal{E}, \nu(s,x,d\theta))$, we have

$$\left| f(s,x,r,z,k(\cdot)) - f(s,x,r,\acute{z},\acute{k}(.)) \right| \leq L\left(|z-\acute{z}| + \|k(\cdot) - \acute{k}(\cdot)\|_\nu\right).$$

($\mathbf{H}_{3.3}$) The function $y \longmapsto f(s,x,y,z,k(\cdot))$ is left continuous and increasing.

($\mathbf{H}_{3.4}$) There exists $M > 0$ such that for all $(s,x,y,z,k(\cdot))$, $|f(s,x,y,z,k(\cdot))| \leq M$.

Theorem 2.7

Let Assumptions $(\mathbf{H}_{1.1}) - (\mathbf{H}_{1.3})$, $(\mathbf{H}_{2.2})$ *and* $(\mathbf{H}_{3.2}) - (\mathbf{H}_{3.4})$ *hold true. Then, BSDEJ (0.2) has at least one solution* $(Y, Z, K(\cdot)) \in \mathcal{B}$.

To prove this Theorem we use the classical approximation by convolution on the generator f. We define $(f_n)_{n\geq 0}$ by

$$f_n(s,x,y,z,k(\cdot)) = n\int_{y-\frac{1}{n}}^{y} f(s,x,r,z,k(\cdot))\,\mathrm{d}r.$$

The sequence $(f_n)_{n\geq 0}$ enjoins the following properties:

(P_1) There exist $C_n \geq 0$ for each n such that

$$\begin{aligned}&\Big|f_n(s,x,y,z,k(.)) - f_n(s,x,\acute{y},\acute{z},\acute{k}(\cdot))\Big|\\&\leq C_n(|y-\acute{y}|+|z-\acute{z}|+\|k(\cdot)-\acute{k}(\cdot)\|_\nu.\end{aligned}$$

(P_2) The sequence $(f_n)_{n\geq 0}$ is increasing.

(P_3) There exist two constants a and b, $-1 < a < 0$, $b > 0$ such that for every $s \in [0,T]$, $x \in \Gamma$, r, $z \in \mathbb{R}$ and $k(\cdot)$, $\acute{k}(\cdot) \in L^2(\Gamma, \mathcal{E}, \nu(s,x,d\theta))$, we have

$$\begin{aligned}&f_n(s,x,r,z,k(\cdot)) - f_n(s,x,r,z,\acute{k}(\cdot))\\&\leq C\int_\Gamma (k(\theta)-\acute{k}(\theta))\varphi_s(\theta)\nu(s,x,\mathrm{d}\theta).\end{aligned}$$

(P_4) $\forall\, \zeta \in [0,T]\times\mathbb{R}\times\mathbb{R}\times L^2(\Gamma,\mathcal{E},\nu(s,x,d\theta))$, $\sup_{n\geq 1}|f_n(\zeta)| \leq M$.

(P_5) if $(y_n, z_n) \underset{n\to\infty}{\longrightarrow} (y,z)$ then $f_n(s,x,y_n,z_n,k(\cdot)) \underset{n\to\infty}{\longrightarrow} f(s,x,y,z,k(\cdot))$.

For all $n \geq 1$, the function f_n verifies: Hypothesis 2, then from Theorem 1.5 there is a unique solution $(Y^n, Z^n, K^n(\cdot))$ of BSDEJ with data $(f_n, h(X))$. We establish priori estimates on the sequence $(Y^n, Z^n, K^n(\cdot))$ which will be needed in the sequel.

Lemma 2.8

There exists a constant $C > 0$ depending only on h, T and M such that for all $n \geq 1$

$$\mathbb{E}[\sup_{s\in[0,T]}|Y_s^n|^2] + \mathbb{E}\int_0^T |Z_r^n|^2\,\mathrm{d}r + \mathbb{E}\int_0^T \|K_r^n(\cdot)\|_\nu^2\,\mathrm{d}r \leq C.$$

Proof: It goes as the proof of Lemma 2.3. □

Proof of Theorem 2.7:

In a way similar to that in the proof of Theorem 2.1 we deduce that there exists a process $Y \in \mathcal{B}$ as the infimum limit of the sequence Y^n:

$$Y_r = \lim_{n\to\infty} Y_r^n \quad \text{and} \quad \lim_{n\to\infty}\mathbb{E}\int_0^T |Y_r^n - Y_r|^2\,\mathrm{d}r = 0.$$

Now, we show that $(Z^n, K^n(\cdot))$ is a Cauchy sequence in $\mathcal{B}$, for $n,\ m \geq 1$ we use Itô's formula to get

$$\mathbb{E}\left|Y_s^n - Y_s^m\right|^2 + \mathbb{E}\int_s^T \left|Z_r^n - Z_r^m\right|^2 \mathrm{d}r + \mathbb{E}\int_s^T \left\|K_r^n(\cdot) - K_r^m(\cdot)\right\|_\nu^2 \mathrm{d}r$$
$$\leq 2\mathbb{E}\int_s^T \left|Y_r^n - Y_r^m\right| \left|f_n(r, X_r^n, Y_r^n, Z_r^n, K_r^n(\cdot)\right| \mathrm{d}r$$
$$+2\mathbb{E}\int_s^T \left|Y_r^n - Y_r^m\right| \left|f_m(r, X_r^m, Y_r^m, Z_r^m, K_r^m(\cdot))\right| \mathrm{d}r.$$

By invoking $\mathbf{P}_4$ and using Hölder's inequality, we obtain

$$\mathbb{E}\left|Y_s^n - Y_s^m\right|^2 + \mathbb{E}\int_s^T \left|Z_r^n - Z_r^m\right|^2 \mathrm{d}r + \mathbb{E}\int_s^T \left\|K_r^n(\cdot) - K_r^m(\cdot)\right\|_\nu^2 \mathrm{d}r$$
$$\leq 4M\sqrt{T}\left(\mathbb{E}\int_0^T \left|Y_r^n - Y_r^m\right|^2 \mathrm{d}r\right)^{\frac{1}{2}} \underset{n\to\infty}{\longrightarrow} 0.$$

So $(Y^n, Z^n, K^n(\cdot))_{n\geq 1}$ is a Cauchy sequence on $\mathcal{B}$, then there exists a process $(Y, Z, K(\cdot)) \in \mathcal{B}$ as a limit of the sequence $(Y^n, Z^n, K^n(\cdot))$. To prove that $(Y, Z, K(\cdot))$ verifies (0.2), we use the same way to that in the proof of Theorem 2.1. $\square$

2.5 Application to Quadratic BSDEJs

In this section, we aim to go beyond the linear growth condition of the BSDEJ's generator. More precisely, we use Theorem 2.1 to show the existence of an unnecessarily unique solution to one kind of quadratic BSDEJ. We define the following function which plays a crucial role in the proof of the Theorem 2.9 below. It allows us to eliminate both the additive quadratic and the exponential terms of the BSDEJ (2.4).

Let ψ be a measurable continuous function that belongs to $L^1(\mathbb{R})$. Define the following function

$$F(x) = \int_0^x \exp\left(2\int_0^y \psi(t)\mathrm{d}t\right) \mathrm{d}y.$$

It was shown in [8] that the function which belongs to $\mathcal{C}^2(\mathbb{R})$ enjoys the following properties:

i) for a.e. x, $F''(x) - 2\psi(x)F'(x) = 0$.

ii) F is a quasi-isometry, that is : there exist two positive constants m and M such that, for any $x, y \in \mathbb{R}$

$$m\left|x - y\right| \leq \left|F(x) - F(y)\right| \leq M\left|x - y\right|.$$

iii) F is a one to one function from $\mathbb{R}$ onto $\mathbb{R}$. Both F and its inverse function F^{-1} belong to $\mathcal{C}^2(\mathbb{R})$.

Next, we will use Theorem 2.1 to solve the following quadratic BSDEJ

$$\begin{aligned} Y_s &= h(X_T) - \int_s^T Z_r \mathrm{d}B_r - \int_s^T \int_\Gamma K_r(\theta)\mathrm{q}(\mathrm{d}r,\mathrm{d}\theta) \\ &+ \int_s^T H(r, X_r, Y_r, Z_r, K_r(\cdot))\mathrm{d}r, \end{aligned} \tag{2.4}$$

where $H(r, X_r, y, z, k(\cdot)) = f(r, X_r, y, z, k(\cdot)) + \psi(y)|z|^2 + \left[K_{r,X_{r-},y}\right]_\psi$ and

$$[k_{s,x,y}]_\psi := \int_\Gamma \frac{F(y + k(\theta)) - F(y) - F'(y)k(\theta)}{F'(y)}\nu(s, x, \mathrm{d}\theta). \tag{2.5}$$

Theorem 2.9

Assume that h satisfies $\mathbf{H}_{1.1}$ and f satisfies assumptions $\mathbf{H}_{1.2}$ and $\mathbf{H}_{2.2}$. Then the equation (2.4) has at least one solution.

Proof : Let $(Y, Z, K(\cdot))$be a solution of BSDEJ (2.4). Then, Itô's formula applied to $F(Y_s)$ shows that

$$\begin{aligned} &F(Y_s) = F(h(X_T)) + \int_s^T F'(Y_{r-})f(r, X_r, Y_r, Z_r, K_r(\cdot))\mathrm{d}r \\ &+ \int_s^T \left(F'(Y_{r-})\left(\psi(Y_r)|Z_r|^2 + \left[K_{r,X_{r-},Y_{r-}}\right]_\psi\right) - \frac{1}{2}F''(Y_r)|Z_r|^2\right)\mathrm{d}r \\ &- \int_s^T F'(Y_{r-})Z_r\mathrm{d}B_r - \int_s^T \int_\Gamma F'(Y_{r-})K_r(\theta)\mathrm{q}(\mathrm{d}r, \mathrm{d}\theta) \\ &- \sum_{s<r\le T} \left(F(Y_r) - F(Y_{r-}) - F'(Y_{r-})\Delta Y_r\right), \end{aligned}$$

since $F'(x)\psi(x) - \frac{1}{2}F''(x) = 0$, and

$$\begin{aligned} &\sum_{s<r\le T} \left(F(Y_r) - F(Y_{r-}) - F'(Y_{r-})\Delta Y_r\right) \\ &= \int_s^T \int_\Gamma \left(F(Y_{r-} + K_r(\theta)) - F(Y_{r-}) - F'(Y_{r-})K_r(\theta)\mathrm{p}(\mathrm{d}r, \mathrm{d}\theta)\right), \end{aligned}$$

moreover, by adding and subtracting the same term

$$\int_s^T \int_\Gamma \left(F(Y_r) - F(Y_{r-}) - F'(Y_{r-})K_r(\theta)\right)\nu(r, X_r, \mathrm{d}\theta)\mathrm{d}r,$$

we obtain

$$\begin{aligned}
&F(Y_s) = F(h(X_T))\\
&+\int_s^T F'(Y_{r-})f(r,X_r,Y_r,Z_r,K_r(\cdot))\mathrm{d}r - \int_s^T F'(Y_r)Z_r\mathrm{d}B_r\\
&+\int_s^T F'(Y_r)\left[K_{r,X_{r-},Y_{r-}}\right]_\psi \mathrm{d}r - \int_s^T\int_\Gamma F'(Y_{r-})K_r(\theta)\mathrm{q}(\mathrm{d}r,\mathrm{d}\theta)\\
&-\int_s^T\int_\Gamma \left(F(Y_{r-}+K_r(\theta)) - F(Y_{r-}) - F'(Y_{r-})K_r(\theta)\mathrm{p}(\mathrm{d}r,\mathrm{d}\theta)\right)\\
&+\int_s^T\int_\Gamma \left(F(Y_{r-}+K_r(\theta)) - F(Y_{r-}) - F'(Y_{r-})K_r(\theta)\right)\nu(r,X_r,\mathrm{d}\theta)\mathrm{d}r\\
&-\int_s^T\int_\Gamma \left(F(Y_{r-}+K_r(y)) - F(Y_{r-}) - F'(Y_{r-})K_r(\theta)\right)\nu(r,X_r,\mathrm{d}\theta)\mathrm{d}r,
\end{aligned}$$

this implies

$$\begin{aligned}
&F(Y_s) = F(h(X_T))\\
&+\int_s^T F'(Y_r)f(r,X_r,Y_r,Z_r,K_r(\cdot))\mathrm{d}r - \int_s^T F'(Y_r)Z_r\mathrm{d}B_r\\
&+\int_s^T F'(Y_r)\left[K_{r,X_{r-},Y_{r-}}\right]_\psi \mathrm{d}r - \int_s^T\int_\Gamma F'(Y_{r-})K_r(\theta)\mathrm{q}(\mathrm{d}r,\mathrm{d}\theta)\\
&-\int_s^T\int_\Gamma \left(F(Y_{r-}+K_r(\theta)) - F(Y_{r-}) - F'(Y_{r-})K_r(\theta)\right)\mathrm{q}(\mathrm{d}r,\mathrm{d}\theta)\\
&-\int_s^T\int_\Gamma \left(F(Y_{r-}+K_r(y)) - F(Y_{r-}) - F'(Y_{r-})K_r(\theta)\right)\nu(r,X_r,\mathrm{d}\theta)\mathrm{d}r.
\end{aligned}$$

According to Lemma 2.3 and the definition of $\left[K_{r,X_{r-},Y_{r-}}\right]_\psi$ we get,

$$\begin{aligned}
F(Y_s) &= F(h(X_T)) + \int_s^T F'(Y_{r-})f(r,X_r,Y_r,Z_r,K_r(\cdot))\mathrm{d}r\\
&\quad - \int_s^T F'(Y_{r-})Z_r\mathrm{d}B_r \qquad (2.6)\\
&\quad - \int_t^T\int_\Gamma \left(F(Y_{r-}+K_r(\theta)) - F(Y_{r-})\right)q(\mathrm{d}r,\mathrm{d}\theta).
\end{aligned}$$

If we take

$$y_r = F(Y_r),\ z_r = F'(Y_{r-})Z_r \text{ and } k_r(\theta) = F(Y_{r-}+K_r(\theta)) - F(Y_{r-}),$$

and

$$\begin{aligned}
&g(r,x,y,z,k)\\
&:= F'(F^{-1}(y))f(r,x,F^{-1}(y+k),\frac{z_r}{F'(F^{-1}(y))},F(F^{-1}(y+k))-F(y))
\end{aligned}$$

we can write the previous equation in the following form

$$\begin{aligned}
y_s &= F(h(X_T)) + \int_s^T g(r,X_r,y_r,z_r,k_r(\cdot))\,\mathrm{d}r \qquad (2.7)\\
&\quad - \int_s^T z_r\mathrm{d}B_r - \int_s^T\int_\Gamma k_r(\theta)q(\mathrm{d}r,\mathrm{d}\theta)).
\end{aligned}$$

Conversely, let $(y, z, k(\cdot))$ be a solution to BSDEJ(2.7), then Itô's formula applied to $Y_s = F^{-1}(y_s)$ shows that

$$\begin{aligned} F^{-1}(y_s) &= h(X_T) + \int_s^T (F^{-1})'(y_{r-}) \left(g\left(r, X_r, y_r, z_r, k_r\left(\cdot\right)\right)\right) \mathrm{d}r \\ &\quad - \int_s^T (F^{-1})'(y_{r-}) z_r \mathrm{d}B_r - \int_s^T \int_\Gamma (F^{-1})'(y_{r-}) k_r(\theta) \mathrm{q}(\mathrm{d}r, \mathrm{d}\theta) \\ &\quad - \frac{1}{2} \int_s^T (F^{-1})''(y_r) \left|z_r\right|^2 \mathrm{d}r \\ &\quad - \sum_{s<r\leq T} \left(F^{-1}(y_r) - F^{-1}(y_{r-}) - (F^{-1})'(y_{r-}) \Delta y_r \right), \end{aligned}$$

then

$$\begin{aligned} Y_s &= h(X_T) - \int_s^T (F^{-1})'(y_{r-}) z_r \mathrm{d}B_r - \int_s^T \int_\Gamma (F^{-1})'(y_{r-}) k_r(\theta) \mathrm{q}(\mathrm{d}r, \mathrm{d}\theta) \\ &\quad + \int_s^T (F^{-1})'(y_{r-}) \left(g\left(r, X_r, y_r, z_r, k_r\left(\cdot\right)\right)\right) \mathrm{d}r - \frac{1}{2} \int_s^T (F^{-1})''(y_r) \left|z_r\right|^2 \mathrm{d}r \\ &\quad - \int_s^T \int_\Gamma \left(F^{-1}(y_{r-} + k_r(\theta)) - F^{-1}(y_{r-}) - (F^{-1})'(y_{r-}) k_r(\theta) \right) \mathrm{p}(\mathrm{d}r, \mathrm{d}\theta) \\ &= h(X_T) - \int_s^T (F^{-1})'(y_{r-}) z_r \mathrm{d}B_r - \int_s^T \int_\Gamma (F^{-1})'(y_{r-}) k_r(\theta) \mathrm{q}(\mathrm{d}r, \mathrm{d}\theta) \\ &\quad + \int_s^T (F^{-1})'(y_{r-}) \left(g\left(r, X_r, y_r, z_r, k_r\left(\cdot\right)\right)\right) \mathrm{d}r - \frac{1}{2} \int_s^T (F^{-1})''(y_r) \left|z_r\right|^2 \mathrm{d}r \\ &\quad + \int_s^T \int_\Gamma \left(F^{-1}(y_{r-} + k_r(\theta)) - F^{-1}(y_{r-}) - (F^{-1})'(y_{r-}) k_r(\theta) \right) \nu(r, X_r, \mathrm{d}\theta) \mathrm{d}r \\ &\quad - \int_s^T \int_\Gamma \left(F^{-1}(y_{r-} + k_r(\theta)) - F^{-1}(y_{r-}) - (F^{-1})'(y_{r-}) k_r(\theta) \right) \mathrm{q}(\mathrm{d}r, \mathrm{d}\theta). \end{aligned} \tag{2.8}$$

Notice that

$$\begin{aligned} (F^{-1})'(x) = \frac{1}{F'(F^{-1}(x))} \text{ and } (F^{-1})''(x) \\ = -\frac{F''(F^{-1}(x))}{(F'(F^{-1}(x)))^2} (F^{-1})'(x) = -\frac{F''(F^{-1}(x))}{(F'(F^{-1}(x)))^3}. \end{aligned}$$

Set $Z_s = (F^{-1})'(y_{s-}) z_s$ and $K_s(\theta) = F^{-1}(y_{s-} + k_s(\theta)) - F^{-1}(y_{s-})$ this implies

$$\begin{aligned} \frac{1}{2} (F^{-1})''(y_r) \left|z_r\right|^2 &= -\tfrac{1}{2} \tfrac{F''(Y_r)}{(F'(Y_r))^3} \tfrac{\left|Z_r\right|^2}{((F^{-1})'(y_{r-}))^2} \\ &\overset{\mathrm{d}s \text{ a.e.}}{=} -\tfrac{1}{2} \tfrac{F''(Y_r)}{F'(Y_r)} \left|Z_r\right|^2 = -\psi(Y_r) \left|Z_r\right|^2 \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} &\textstyle\int_\Gamma \left(F^{-1}(y_{r-} + k_r(\theta)) - F^{-1}(y_{r-}) - (F^{-1})'(y_{r-}) k_r(\theta) \right) \nu(r, X_r, \mathrm{d}\theta) \\ &= \textstyle\int_\Gamma \left(K_r(\theta) - \frac{1}{F'(Y_r)} \left(F(Y_r) - F(Y_{r-}) \right) \right) \nu(r, X_r, \mathrm{d}\theta) \\ &= -\textstyle\int_\Gamma \left(\frac{F(Y_r) - F(Y_{r-}) - F'(Y_r) K_r(\theta)}{F'(Y_r)} \right) \nu(r, X_r, \mathrm{d}\theta) = -\left[K_{r, X_{r-}, Y_{r-}} \right]_\psi, \end{aligned} \tag{2.10}$$

and

$$
\begin{aligned}
& (F^{-1})'(y_{s-})\left(g\left(r, X_r, y_r, z_r, k_r\left(\cdot\right)\right)\right) && (2.11)\\
& = f(r, X_r, F^{-1}\left(y_r\right), \frac{z_r}{F'(F^{-1}\left(y_r\right))}, F^{-1}\left(y_{r-} + k_r(\theta)\right) - F^{-1}\left(y_{r-}\right)) \\
= \quad & f(r, X_r, Y_r, Z_r, K_r\left(\cdot\right)).
\end{aligned}
$$

Substituting (2.10), (2.10) and (2.11) in (2.8) we end up with

$$
\begin{aligned}
Y_t = h(X_T) + \int_s^T & H(r, X_r, Y_r, Z_r, K_r\left(\cdot\right))\mathrm{d}r \\
- \int_s^T Z_r \mathrm{d}B_r - \int_s^T & \int_\Gamma K_r(\theta)\mathrm{q}(\mathrm{d}r, \mathrm{d}\theta).
\end{aligned}
$$

So far we have shown that BSDEJ(2.4) has a solution if and only if BSDEJ(2.7) has a solution. Therefore, since h satisfies $\mathbf{H}_{1.1}$ and F is a Lipschitz function, it is easy to see that $F \circ h$ also satisfies $\mathbf{H}_{1.1}$. On the other hand, using the fact F' is bounded and f satisfies $\mathbf{H}_{1.1}$, $\mathbf{H}_{1.2}$ and $\mathbf{H}_{2.2}$, one can show that g also satisfies assumptions $\mathbf{H}_{1.1}$, $\mathbf{H}_{1.2}$ and $\mathbf{H}_{2.2}$. Therefore, Theorem 2.1 confirmed that BSDEJ(2.7) has at least one solution $(y, z, k\left(\cdot\right)) \in \mathcal{B}$. This implies that BSDEJ(2.4) also has at least one solution $(Y, Z, K\left(\cdot\right))$ which also belongs to $\mathcal{B}$. Indeed, thanks to the Lipschitz continuity of F^{-1} we can easily show that

$$
\left|Y_s\right| \leq M\left|y_s\right|, \ \left|Z_s\right| \leq M\left|z_s\right| \ \text{ and } \ \left\|K(\cdot)\right\|_\nu \leq M\left\|k(\cdot)\right\|_\nu.
$$

□

Chapter 3

On the Solution of Locally Lipschitz BSDE Associated to a Jump Markov Process

(Joint work with N. Khelfallah)

3.1 Introduction

In this chapter we give an existence and uniqueness result (theorem 3.5) on top of the stability (theorem 3.8) of the solution to the BSDEJ (0.1) driven by a Markov jump process with locally Lipschitz generator. These results extend the papers [6, 18] by involving random measures associated with the jump Markov process in the state BSDEJ. To exploit the Markov propriety, we apply Theorem 3.5 to prove the existence of a unique solution to some non-linear variants of Kolmogorov equation with a locally Lipschitz driver. In fact, we construct this solution via a family of BSDEJs of the type (0.1) having the existence and uniqueness propriety.

This chapter is organized as follows. In section 1 we give a priori Estimates and results. In section 2 we study BSDEJs with locally Lipschitz coefficients. In Section 3 we give a stability result. In Section 4 we give a result of the existence and uniqueness of Kolmogorov equations.

Throughout this chapter, we will work on the following Banach space $\mathcal{B}_{2,2} := \mathcal{S}_2 \otimes \mathcal{H}^2$, and we need the following assumptions on the coefficients.

Hypotheses 1

($\mathbf{H_{1.1}}$) For every $s \in [0,T]$, $x \in \Gamma$, $y \in \mathbb{R}$, $f(s,x,y,\cdot)$ is a function from $L^2(\Gamma,\mathcal{E},\nu(s,x,\mathrm{d}\theta))$ to $\mathbb{R}$.

($\mathbf{H_{1.2}}$) For every bounded and $\mathcal{E}$-measurable function $k:\Gamma \to \mathbb{R}$, the function $(s,x,y) \to f(s,x,y,k(\cdot))$ is $\mathcal{B}([0,T]) \otimes \mathcal{E} \otimes \mathcal{B}(\mathbb{R})$-measurable.

Hypotheses 2

($\mathbf{H_{2.1}}$) The function f is continuous in (y,z) for almost all t.

($\mathbf{H_{2.2}}$) There exist two constant $\lambda > 0$ and $\alpha \in [0,1[$ such that

$$|f(s,x,y,k(.))| \leq \lambda \left[1 + |y|^{\alpha} + \|k(\cdot)\|_{\nu}^{\alpha}\right], a.e.t \in [0,T].$$

($\mathbf{H_{2.3}}$) For every integer $M > 1$, there exist two constants $L_M > 0$ and $\acute{L}_M > 0$ such that,

$$|f(s,x,y,k(\cdot)) - f(s,x,\acute{y},\acute{k}(\cdot))| \leq \acute{L}_M|y - \acute{y}| + L_M\|k(\cdot) - \acute{k}(\cdot)\|_{\nu},$$

a.e. t $\in [0,T]$ and for all y, $\acute{y}$, $k(\cdot)$, $\acute{k}(\cdot)$ such that $|y| \leq M$, $|\acute{y}| \leq M$, $\|k(\cdot)\|_{\nu} \leq M$, $\|\acute{k}(\cdot)\|_{\nu} \leq M$.

($\mathbf{H_{2.4}}$) The function $h:\Gamma \to \mathbb{R}$ is $\mathcal{E}$- measurable function and satisfies $\mathbb{E}\left|h(X_T)\right|^2 < \infty$.

When f satisfies ($\mathbf{H}_{2.1}$) and ($\mathbf{H}_{2.2}$), we can define the family of semi–norms $(\rho_n(f))_{n\in\mathbb{N}}$

$$\rho_n(f) = \left(\mathbb{E}\int_0^T \sup_{|y|,\|k(.)\|_{\nu}\leq n} |f(s,X_s,y,k(\cdot))|^2 \,\mathrm{d}s\right)^{\frac{1}{2}}. \tag{3.1}$$

Noticing that, under **Hypotheses 1**, Lemma 3.2 in [19] shows that the function $(\omega,s,y) \to f\left(s,X_{s^-}(\omega),y,K_s(\omega,\cdot)\right)$ is $\mathcal{P}\otimes B(\mathbb{R})$-measurable, if $K \in \mathcal{L}^2(p)$. Furthermore, if Y is $Prog^t$-measurable process then, $(\omega,s) \to f\left(s,X_{s-}(\omega),Y_s(\omega),K_s(\omega,\cdot)\right)$ is $Prog^t$-measurable.

3.2 Priori Estimates and Results

In this section, we give the following useful three Lemmas. They involve some priori estimates of solutions for BSDEJ (0.1) on top of some estimates between two solutions. For later use, we denote BSDEJ (0.1) by BSDEJ (f, ξ) where $\xi := h(X_T)$.

Lemma 3.1

Let $(Y, K(\cdot))$ be a solution of BSDEJ (f, ξ). Assume that f satisfies ***Hypotheses 1*** *and* $(\mathbf{H}_{2.2})$*, then we have the following estimates*

(i) *There exists a positive constant $C := C(\lambda, \xi, T)$, such that for every $s \in [t, T]$, we have*

$$\mathbb{E}\left|Y_s\right|^2 + \mathbb{E}\int_s^T \left\|K_r(\cdot)\right\|_\nu^2 \mathrm{d}r \leq C.$$

. ***(ii)*** *Moreover, if ξ is bounded, then there exists a positive constant $\acute{C}$, such that*

$$\sup_{s\in[0,T]} \left|Y_s\right|^2 \leq \acute{C}.$$

Proof : We first prove (**i**). For the sake of simplicity we drop the superscript t and we write the results and their proofs in the case $t = 0$. Using Itô's formula for semimartingales (cf.Theorem 32 in [45] to $|Y_s|^2$ and integrating on the time interval $[s, T]$ we get

$$\begin{aligned} \left|Y_s\right|^2 &= |\xi|^2 + 2\int_s^T Y_r f\left(r, X_r, Y_r, K_r(\cdot)\right) \mathrm{d}r && (3.2)\\ &\quad -2\int_s^T \int_\Gamma Y_{r^-} K_r(\theta)\, \mathrm{q}(\mathrm{d}r, \mathrm{d}\theta) - \sum_{s\leq r\leq T} \left|\Delta Y_r\right|^2. && (3.3)\end{aligned}$$

We can rewrite the last term in the equality (3.3) as the following,

$$\begin{aligned} \sum_{s\leq r\leq T} \left|\Delta Y_r\right|^2 &= \int_s^T \int_\Gamma \left|K_r(\theta)\right|^2 \mathrm{p}(\mathrm{d}r, \mathrm{d}\theta) && (3.4)\\ &= \int_s^T \int_\Gamma \left|K_r(\theta)\right|^2 \mathrm{q}(\mathrm{d}r, \mathrm{d}\theta) + \int_s^T \int_\Gamma \left|K_r(\theta)\right|^2 \nu(r, X_r, \mathrm{d}\theta)\mathrm{d}r. \end{aligned}$$

Keeping in mind that the stochastic processes q is an $\mathbb{F}-$ martingale, plugging the equality (3.4) into (1.3) and taking the expectation, one can get for $s \in [0, T]$

$$\mathbb{E}(|Y_s|^2) + \mathbb{E}\int_s^T \left\|K_r(\cdot)\right\|_\nu^2 \mathrm{d}r = \mathbb{E}\left|\xi\right|^2 + 2\mathbb{E}\int_s^T Y_r f_r(r, X_r, Y_r, K_r(\cdot))\mathrm{d}r.$$

By invoking $(\mathbf{H}_{2.2})$, using the inequality $|y|^{\alpha} \leq 1+|y|$ for each $\alpha \in [0,1[$ along with Young's inequality, to get

$$\mathbb{E}(|Y_s|^2) \leq \mathbb{E}\,|\xi|^2 + 9T + (1+3\lambda^2)\mathbb{E}\int_s^T |Y_r|^2\,\mathrm{d}r. \tag{3.5}$$

Thanks to Gronwall's Lemma, we get

$$\mathbb{E}(|Y_s|^2) \leq \left(\mathbb{E}\,|\xi|^2 + 18T\right)\exp\left((1+3\lambda^2)T\right) := C_1. \tag{3.6}$$

Once again, the inequalities (3.5), (3.6) and Young's inequality yield

$$\mathbb{E}\int_s^T \|K_r(\cdot)\|_{\nu}^2\,\mathrm{d}r \leq 2\left(\mathbb{E}\,|\xi|^2 + 9T\right) + T(1+9\lambda^2)C_1.$$

We proceed now to prove **(ii).** Again, by replacing the equality (3.4) into (3.3), taking the conditional expectation with respect to $\mathcal{F}_{[0,s]}$, using Assumption $(\mathbf{H}_{2.2})$, the inequality $|y|^{\alpha} \leq 1+|y|$ for $\alpha \in [0,1[$ together with Young's inequality, we deduce

$$|Y_s|^2 \leq C + 9T + \left(2\lambda^2 + 2\lambda\right)\int_s^T \mathbb{E}\left(|Y_r|^2 \mid \mathcal{F}_{[0,s]}\right)\mathrm{d}r.$$

For any time $t \leq s$, using once again $\mathbb{E}\left(\cdot \mid \mathcal{F}_{[0,T]}\right)$ in both sides of the previous inequality and Gronwall's Lemma, to get

$$\mathbb{E}\left(|Y_s|^2 \mid \mathcal{F}_{[0,t]}\right) \leq [C+9T]\exp\left[\left(2\lambda^2+2\lambda\right)(T-s)\right] := \acute{C}.$$

In particular, if $t=s$, we immediately find (**ii**). □

Lemma 3.2

*Let f_1 and f_2 be two functions, $(Y^1, K^1(\cdot))$ [resp. $(Y^2, K^2(\cdot))$] be a solution of the BSDEJ (f_1, ξ_1) [resp. BSDEJ (f_2, ξ_2)], where $\xi_1 := h_1(X_T)$ and $\xi_2 := h_2(X_T)$ are two final conditions such that h_1 and h_2 satisfy $(\mathbf{H}_{2.4})$. If f_1 and f_2 satisfy **Hypotheses 1,** $(\mathbf{H}_{2.1})$ and $(\mathbf{H}_{2.2})$. Then, for every locally Lipschitz function f and every $M>1$, the following estimates hold:*

$$\begin{aligned}\mathbb{E}(\left|Y_r^1 - Y_r^2\right|^2) \;\leq\; & \Big[\mathbb{E}\,|\xi_1-\xi_2|^2 + \rho_M^2(f-f_2) + \rho_M^2(f_1-f) \\ & + \frac{C(\xi_1,\xi_2,\lambda)}{\left(L_M^2 + 2\acute{L}_M + 2\right)M^{2(1-\alpha)}}\Big]\exp\left[(4+4L^{'}+2L_M^2)(T-s)\right],\end{aligned} \tag{3.7}$$

and

$$\mathbb{E}\int_s^T \|K_r^1(\cdot)-K_r^2(\cdot)\|_\nu^2 \mathrm{d}r \le C(\xi_1,\xi_2,\lambda)\left[\mathbb{E}\left|\bar{\xi}\right|^2+\left(\mathbb{E}\int_s^T |Y_r^1-Y_r^2|^2\mathrm{d}r\right)^{\frac{1}{2}}\right]. \quad (3.8)$$

Proof : We set $\bar{Y}=Y^1-Y^2$, $\bar{K}(\cdot)=K^1(\cdot)-K^2(\cdot)$,

$\bar{f}_s=f_1(s,X_s,Y_s^1,K_r^1(\cdot))-f_2(s,X_s,Y_s^2,K_r^2(\cdot))$, $\bar{\xi}=\xi^1-\xi^2$. By Itô's formula we have

$$\mathbb{E}(|\bar{Y}_s|^2)+\mathbb{E}\int_s^T \|\bar{K}_r(\cdot)\|^2\mathrm{d}r=\mathbb{E}\left|\bar{\xi}\right|^2+2\mathbb{E}\int_s^T \bar{Y}_r\bar{f}_r\mathrm{d}r \quad s\in[t,T]. \quad (3.9)$$

For a given $M>1$, we use the notations

$$D_M:=\left\{(s,\omega):\sum_{i=1}^2\left(|Y_s^i|^2+\|K_s^i(\cdot)\|_\nu^2\right)\ge M^2\right\},\bar{D}_M:=\Omega\backslash D_M,$$

to rewrite (3.9) as the following

$$\begin{aligned}&\mathbb{E}(|\bar{Y}_s|^2)+\mathbb{E}\textstyle\int_s^T \|\bar{K}_r(\cdot)\|_\nu^2\mathrm{d}r=\mathbb{E}\left|\xi\right|^2+2\mathbb{E}\int_s^T \bar{Y}_r\bar{f}_r1\!\!1_{D_M}\mathrm{d}r\\&+2\mathbb{E}\textstyle\int_s^T \bar{Y}_r\bar{f}_r1\!\!1_{\bar{D}_M}\mathrm{d}r \quad s\in[t,T],\end{aligned} \quad (3.10)$$

where $1\!\!1_{D^M}$ stands for the indicator function of the set D. We first estimate the last term in the previous equality

$$\begin{aligned}&2\mathbb{E}\textstyle\int_s^T \bar{Y}_r\bar{f}_r1\!\!1_{\bar{D}_M}\mathrm{d}r\le 2\mathbb{E}\int_s^T \bar{Y}_r[\,(f_1-f)\,(r,X_r,Y_r^1,K_r^1(\cdot))]1\!\!1_{\bar{D}_M}\mathrm{d}r\\&+2\mathbb{E}\textstyle\int_s^T \bar{Y}_r[\,(f-f_2)\,(r,X_r,Y_r^2,K_r^2(\cdot))]1\!\!1_{\bar{D}_M}\mathrm{d}r\\&+2\mathbb{E}\textstyle\int_s^T \bar{Y}_r[f(r,X_r,Y_r^1,K_r^1(\cdot))-f(r,X_r,Y_r^2,K_r^2(\cdot))]1\!\!1_{\bar{D}_M}\mathrm{d}r\\&=I_1+I_2+I_3.\end{aligned} \quad (3.11)$$

Then, from the definition of the semi-norm (3.1) and by using the inequality $2xy\le x^2+y^2$, one can get

$$I_1+I_2\le 2\mathbb{E}\int_s^T |\bar{Y}_r|^2\mathrm{d}r+\rho_M^2(f_1-f)+\rho_M^2(f-f_2). \quad (3.12)$$

Since f is Lipschitz in the ball $B(0,M)$, we obtain

$$I_3\le 2\acute{L}_M\mathbb{E}\int_s^T |\bar{Y}_r|^2\mathrm{d}r+2L_M\mathbb{E}\int_s^T |\bar{Y}_r|\|\bar{K}_r(\cdot)\|_\nu\mathrm{d}r.$$

Then, the inequality $2xy \leq \frac{\gamma^2}{2}x^2 + \frac{2}{\gamma^2}y^2$ for $\gamma > 0$, leads to

$$\begin{aligned} I_3 &\leq 2\acute{L}_M \mathbb{E}\int_s^T \left|\bar{Y}_r\right|^2 \mathrm{d}r + \tfrac{\gamma^2}{2}\mathbb{E}\int_s^T \left|\bar{Y}_r\right|^2 \mathrm{d}r + \tfrac{2L_M^2}{\gamma^2}\mathbb{E}\int_s^T \left\|\bar{K}_r(\cdot)\right\|_\nu^2 \mathrm{d}r, \\ &\leq \left(2\acute{L}_M + \tfrac{\gamma^2}{2}\right)\mathbb{E}\int_s^T \left|\bar{Y}_r\right|^2 \mathrm{d}r + \tfrac{2L_M^2}{\gamma^2}\mathbb{E}\int_s^T \left\|\bar{K}_r(\cdot)\right\|_\nu^2 \mathrm{d}r. \end{aligned} \tag{3.13}$$

From the inequalities (3.12) and (3.13), one can get

$$\begin{aligned} &2\mathbb{E}\int_s^T \bar{Y}_r \bar{f}_r \mathbb{1}_{\bar{D}_M}\mathrm{d}r \leq \rho_M^2(f - f_2) + \rho_M^2(f_1 - f) \\ &+ \left(2 + 2\acute{L}_M + \tfrac{\gamma^2}{2}\right)\mathbb{E}\int_s^T \left|\bar{Y}_r\right|^2 \mathrm{d}r + \tfrac{2L_M^2}{\gamma^2}\mathbb{E}\int_s^T \left\|\bar{K}_r(\cdot)\right\|_\nu^2 \mathrm{d}r. \end{aligned} \tag{3.14}$$

Now, we turn out to estimate the second term in the inequality (3.10). Using the inequality
$2xy \leq \beta^2 x^2 + \frac{y^2}{\beta^2}$ for $\beta > 0$, we get

$$\begin{aligned} 2\mathbb{E}\int_s^T \bar{Y}_r \bar{f}_r \mathbb{1}_{D_M}\mathrm{d}r \;\leq\; & \beta^2\mathbb{E}\int_s^T \left|\bar{Y}_r\right|^2 \mathbb{1}_{D_M}\mathrm{d}r \\ & + \frac{2}{\beta^2}\sum_{i=1}^2 \mathbb{E}\int_s^T \left|f_i(s, X_s, Y_s^i, K_r^i(\cdot))\right|^2 \mathbb{1}_{D_M}\mathrm{d}r. \end{aligned}$$

A simple computation shows that, using $(\mathbf{H}_{2.2})$ and the inequality $(a+b+c)^2 \leq 3\left(a^2+b^2+c^2\right)$

$$\begin{aligned} 2\mathbb{E}\int_s^T \bar{Y}_r \bar{f}_r \mathbb{1}_{D_M}\mathrm{d}r \;\leq\; & \beta^2\mathbb{E}\int_s^T \left|\bar{Y}_r\right|^2 \mathbb{1}_{D_M}\mathrm{d}r \\ & + \frac{6C\lambda^2}{\beta^2}\sum_{i=1}^2 \mathbb{E}\int_s^T \left[2 + \left|Y_r^i\right|^{2\alpha} + \left\|K_r^i(\cdot)\right\|_\nu^{2\alpha}\right] \mathbb{1}_{D_M}\mathrm{d}r. \end{aligned}$$

From Lemma 3.1, Hölder's inequality and the fact that

$$\mathbb{1}_{D_M} \leq M^{-2}\sum_{i=1}^2 \left[\left|Y_s^i\right|^2 + \left\|K_s^i(\cdot)\right\|_\nu^2\right],$$

we arrive at

$$\begin{aligned} \mathbb{E}\int_s^T \left|Y_r^1\right|^{2\alpha}\mathbb{1}_{D_M}\mathrm{d}r \;&\leq\; \left(\mathbb{E}\int_s^T \left|Y_r^1\right|^2 \mathrm{d}r\right)^\alpha \left(\mathbb{E}\int_s^T \mathbb{1}_{D_M}\mathrm{d}r\right)^{1-\alpha}, \\ &\leq \left(\mathbb{E}\int_s^T \left|Y_r^1\right|^2 \mathrm{d}r\right)^\alpha \frac{1}{M^{2(1-\alpha)}}\left[\mathbb{E}\sum_{i=1}^2\int_s^T \left(\left|Y_r^i\right|^2 + \left\|K_r^i(\cdot)\right\|_\nu^2\right)\mathrm{d}r\right]^{1-\alpha}, \\ &\leq \frac{C}{M^{2(1-\alpha)}}. \end{aligned}$$

Applying the same method to each one of the terms $\mathbb{E}\int_s^T \left|Y_r^2\right|^{2\alpha}\mathbb{1}_{D_M}\mathrm{d}r$, $\mathbb{E}\int_s^T \left\|K_r^i(\cdot)\right\|_\nu^{2\alpha}\mathbb{1}_{D_M}\mathrm{d}r$ for $i = 1, 2$, we get

$$2\mathbb{E}\int_s^T \bar{Y}_r \bar{f}_r \mathbb{1}_{D_M}\mathrm{d}r \leq \beta^2\mathbb{E}\int_s^T \left|\bar{Y}_r\right|^2 \mathrm{d}r + \frac{C(\xi_1, \xi_2, \lambda)}{\beta^2 M^{2(1-\alpha)}}. \tag{3.15}$$

Choosing $\gamma^2 = 2L_M^2$ and $\beta^2 = \left(L_M^2 + 2\acute{L}_M + 2\right)$, and plugging the inequalities (3.14) and (3.15) into (3.10), we find

$$\begin{aligned}
\mathbb{E}(\left|\bar{Y}_s\right|^2) &\leq \mathbb{E}\left|\bar{\xi}\right|^2 + \rho_M^2(f - f_2) + \rho_M^2(f_1 - f) \\
&+ \frac{C(\xi_1, \xi_2, \lambda)}{\left(L_M^2 + 2\acute{L}_M + 2\right) M^{2(1-\alpha)}} \\
&+ (4 + 4\acute{L}_M + 2L_M^2)\mathbb{E}\int_s^T \left|\bar{Y}_r\right|^2 \mathrm{d}r.
\end{aligned}$$

Then (3.7) follows immediately from Gronwall's Lemma. To prove the second inequality, we go back to the equality (3.9), and we use Schwartz inequality. This achieves the proof of Lemma 2. □

Remark 3.3

We can allow f in Lemma 2 to be of linear growth in y and of sub-linear growth in k as long as we consider the boundedness of the terminal data.

The proof of the following can be shown via truncation argument, and we refer the reader to [7, Lemma 4.4] for its detailed proof.

Lemma 3.4

*Let f be a function satisfies **Hypotheses 1** and **Hypotheses 2**. Then, there exists a sequence of functions f_n such that,*

(i) *For each n, f_n is globally Lipschitz function satisfying **Hypotheses 1** and $(\mathbf{H}_{2.2})$.*

(ii) *There exist tow constants $\lambda > 0$ and $\alpha \in [0, 1[$ such that*

$$\sup_n |f_n(s, x, y, z(\cdot))| \leq |f(s, x, y, z(.))| \leq \lambda \left[1 + |y|^\alpha + \|k(\cdot)\|_\nu^\alpha\right], a.e.\ s \in [t, T].$$

(iii) *For every $M > 1$, $\rho_M^2(f_n - f) \to 0$ as $n \to \infty$.*

3.3 The main Theorems and Results

3.3.1 Existence and Uniqueness

Theorem 3.5

*Suppose that **Hypotheses 1** and **Hypotheses 2** hold true. Assume further that there exist two positive constant L and $\acute{L}$ such that $L_M \leq L + \sqrt{\log M}$ and $\acute{L}_M \leq \acute{L} + \log M$. Then, the BSDEJ (0.1) has a unique solution $(Y, K(\cdot))$ which belongs to $\mathcal{B}_{2,2}$.*

Proof:

This proof is enlightened by Bahlali [6] for locally Lipschitz BSDEs driven by continuous Brownian motion. He first proved the result assuming that the Lipschitz constant L_N is bounded by $\sqrt{(1-\alpha)\log M}$, then he extended it to the case where $L_N \leq \sqrt{\log M}$. Herein, we give the proof directly and differently without passing by those steps.

Suppose that there exist two solutions of the BSDEJ (f, ξ): $(Y^1, K^1(\cdot))$ and $(Y^2, K^2(\cdot))$. The proof of the uniqueness is straight forward of Lemma 3.2 applied with $f_1 = f_2 = f$, $\xi_1 = \xi_2 = h(X_T)$.

To prove the existence, we define a family of approximating BSDEJs obtained by replacing the generator f in BSDEJ (0.1 by f_n defined in Lemma 3.4

$$Y_s^n = h(X_T) + \int_s^T f^n(r, X_r, Y_r^n, K_r^n(\cdot))\mathrm{d}r - \int_s^T \int_\Gamma K_r^n(\theta)\mathrm{q}(\mathrm{d}r, \mathrm{d}\theta), \quad s \in [t, T].$$

In view of Theorem 3.4 in [19], the above BSDEJ has a unique solution $(Y^n, K^n(\cdot))$, for each integer n. Using similar arguments as in the proof of Lemma 3.1, one can easily find

$$\sup_n \mathbb{E}\left(|Y_r^n|^2 + \int_s^T \|K_r^n(\cdot)\|_\nu^2 \,\mathrm{d}r\right) \leq C. \tag{3.16}$$

We split the remainder of the proof into three steps:

Step 1: In this first step, we assume that T is small enough such that $T < \frac{(1-\alpha)}{6}$. Then, we prove that $(Y^n, K^n(\cdot))$ is a Cauchy sequence in the Banach space $(\mathcal{B}_{2,2}. \|\cdot\|_{\mathcal{B}_{2,2}})$. For the sake of simplicity we assume (without loss the generality) that $L = \acute{L} = 0$, so that $L_M \leq \sqrt{\log M}$ and $\acute{L}_M \leq \log M$. We apply (3.7) in Lemma 3.2 to

$(Y^1, K^1(\cdot), f^1, \xi^1) = (Y^n, K^n(\cdot), f_n, \xi)$ and $(Y^2, K^2(.), f^2, \xi^2) = (Y^m, K^m(\cdot), f_m, \xi)$, to obtain

$$\begin{aligned}\mathbb{E}\,|Y_r^n - Y_r^m|^2 \;\; \leq \;\; & \Big[\rho_M^2(f_n - f) + \rho_M^2(f - f_m) \\ & + \frac{C(\xi, \lambda)}{(L_M^2 + 2\acute{L}_M + 2)M^{2(1-\alpha)}}\Big] \\ & \exp\Big[(4 + 4\acute{L}_M + 2L_M^2)T\Big].\end{aligned}$$

Since $L_M \leq \sqrt{\log M}$ and $\acute{L}_M \leq \log M$, we get

$$\begin{aligned}\mathbb{E}\,|Y_r^n - Y_r^m|^2 \;\; & \leq N(M, \alpha)\,[\rho_M^2(f_n - f) + \rho_M^2(f - f_m) \\ & + \tfrac{C(\xi,\lambda)}{(3\log M + 2)M^{2(1-\alpha)}}\Big],\end{aligned}$$

where $N(M, \alpha) = M^{(1-\alpha)} \exp\left(\frac{2}{3}(1 - \alpha)\right)$. Passing to the limits successively on n, m, M, we obtain $\mathbb{E}\,|Y_r^n - Y_r^m|^2 \underset{n,m,M\to\infty}{\to} 0$. We use (3.16) and the Lebesgue's dominated convergence Theorem to get $\mathbb{E}\int_s^T |Y_r^n - Y_r^m|^2\, dr \underset{n,m\to\infty}{\to} 0$. Therefore, in virtue of the previous limit and using the (3.7) in Lemma 3.2, we obtain $\mathbb{E}\int_s^T \|K_r^n(\cdot) - K_r^m(\cdot)\|_\nu^2\, \mathrm{d}r \underset{n,m\to\infty}{\to} 0$.

Hence, $(Y^n, K^n(\cdot))$ is a Cauchy sequence in the Banach space $(\mathcal{B}_{2,2}\, \|\cdot\|_{\mathcal{B}_{2,2}})$. That is,

$$\exists (Y, K(\cdot)) \in \mathcal{B}_{2,2} \text{ such that } \lim_{n\to\infty} \|(Y^n, K^n(\cdot)) - (Y, K(\cdot))\|_{\mathcal{B}_{2,2}} = 0. \tag{3.17}$$

Step 2: In this step, we assume that T is an arbitrary large time duration. Then, we will prove $(Y^n, K^n(.))$ is a Cauchy sequence in the Banach space $(\mathcal{B}_{2,2}\, \|\cdot\|_{\mathcal{B}_{2,2}})$. Firstly, let$([T_i, T_{i+1}])_{i=0}^{i=k}$ be a subdivision of $[0, T]$, such that for any $0 \leq i \leq k$, $|T_{i+1} - T_i| \leq \delta$, where δ is a strictly positive number satisfy $\delta < \frac{(1-\alpha)}{6}$. Now, for $s \in [T_{k-1}, T_k]$, we consider the following BSDEJ

$$Y_s^n = h(X_T) + \int_s^{T_k} f^n(r, X_r, Y_r^n, K_r^n(\cdot))\mathrm{d}r - \int_s^{T_k}\int_\Gamma K_r^n(\theta)\mathrm{q}(\mathrm{d}r, \mathrm{d}\theta). \tag{3.18}$$

It is obvious from step 1 that (3.17) remains valid on the small interval time $[T_{k-1}, T_k]$. Next, for $s \in [T_{k-2}, T_{k-1}]$, we consider the following BSDEJ

$$Y_s^n = Y_{T_{k-1}}^n + \int_s^{T_{k-1}} f^n(r, X_r, Y_r^n, K_r^n(\cdot))\mathrm{d}r - \int_s^{T_{k-1}}\int_\Gamma K_r^n(\theta)\mathrm{q}(\mathrm{d}r, \mathrm{d}\theta). \tag{3.19}$$

Since $T_{k-1} \in [T_{k-1}, T_k]$, $Y_{T_{k-1}}^n$converges to $Y_{T_{k-1}}$, and thus, $(Y^n, K^n(\cdot))$ is also a Cauchy sequence in $\mathcal{B}_{2,2}$, on the small time interval $[T_{k-2}, T_{k-1}]$. Repeating this procedure backwardly for$i = k, ..., 1$, we obtain the desired result on the whole time interval $[0, T]$.

Step 3: In this step, we shall prove the convergence of $f_n(.,X,Y^n,K^n(\cdot))$ to $f(.,X,Y,K(\cdot))$ in $L^1(\Omega,[0,T])$. Denoting $\bar{Y^n}=Y^n-Y$, $\bar{K^n}(\cdot)=K^n(\cdot)-K(\cdot)$ and we set for $M>1$

$$A_M^n:=\{(s,\omega):|Y_s|+\|K_s(\cdot)\|_\nu+|Y_s^n|+\|K_s^n(\cdot)\|_\nu\geq M\},\bar{A^n}_M:=\Omega\backslash A_M^n.$$

First, it is not difficult to see that

$$\begin{array}{l}\mathbb{E}\int_s^T|f_n(r,X_r,Y_r^n,K_r^n(\cdot))-f(r,X_r,Y_r,K_r(\cdot))|\,\mathrm{d}r\\ \leq\mathbb{E}\int_s^T|(f_n-f)(r,X_r,Y_r^n,K_r^n(\cdot))|\left(\mathbb{1}_{\bar{A}_M^n}+\mathbb{1}_{A_M^n}\right)\mathrm{d}r\\ +\mathbb{E}\int_s^T|f(r,X_r,Y_r^n,K_r^n(\cdot))-f(r,X_r,Y_r,K_r(\cdot))|\left(\mathbb{1}_{\bar{A}_M^n}+\mathbb{1}_{A_M^n}\right)\mathrm{d}r.\end{array}$$

Using the inequality $|y|^\alpha\leq 1+|y|$ for each $\alpha\in[0,1]$ and the Assumptions $(\mathbf{H}_{2.2})$ and $(\mathbf{H}_{2.3})$, we obtain

$$\begin{array}{rl} & \mathbb{E}\displaystyle\int_s^T|f_n(r,X_r,Y_r^n,K_r^n(\cdot))-f(r,X_r,Y_r,K_r(\cdot))|\,\mathrm{d}r\\ \leq & \mathbb{E}\displaystyle\int_0^T\sup_{|y|,\|z(.)\|\leq M}|(f_n-f)(r,X_r,y,z(\cdot))|\,\mathrm{d}r\\ & +2\lambda\mathbb{E}\displaystyle\int_s^T[4+|Y_r^n|+\|K_r^n(\cdot)\|_\nu]\,\mathbb{1}_{A_M^n}\mathrm{d}r\\ & +\acute{L}_M\mathbb{E}\displaystyle\int_s^T\left|\bar{Y_r^n}\right|\mathrm{d}r+L_M\mathbb{E}\int_s^T\left\|\bar{K^n}_r(\cdot)\right\|_\nu\mathrm{d}r\\ & +\lambda C\mathbb{E}\displaystyle\int_s^T[6+|Y_r|+\|K_r(\cdot)\|_\nu+|Y_r^n|+\|K_r^n(\cdot)\|_\nu]\,\mathbb{1}_{A_M^n}\mathrm{d}r.\end{array}\tag{3.20}$$

Using the inequality $\mathbb{1}_{A_M^n}\leq M^{-1}[|Y_s|+\|K_s(\cdot)\|_\nu+|Y_s^n|+\|K_s^n(\cdot)\|_\nu]$, Schwarz inequality and Lemma 3.1 we show the existence of a constant C which is independent of M such that the second and the last term in 3.21 are bounded by $CM^{-\frac{1}{2}}$ and thus

$$\begin{array}{l}\mathbb{E}\int_s^T|f_n(r,X_r,Y_r^n,K_r^n(\cdot))-f(r,X_r,Y_r,K_r(\cdot))|\,\mathrm{d}r\\ \leq\mathbb{E}\int_0^T\sup\limits_{|y|,\|z(.)\|_\nu\leq M}|(f_n-f)(r,X_r,y,k(\cdot))|\,\mathrm{d}r\\ +\acute{L}_M\mathbb{E}\int_s^T\left|\bar{Y_r^n}\right|\mathrm{d}r+L_M\mathbb{E}\int_s^T\left\|\bar{K^n}_r(\cdot)\right\|_\nu\mathrm{d}r+CM^{-\frac{1}{2}}.\end{array}$$

Thanks to Lemma 3.4 (**iii**) and Schwarz inequality, the first term in the previous inequality tends to 0 as n goes to infinity. Then, by using Schwarz inequality and step 1, the second and the third terms tend to be 0. Finally, since the constant C is independent of M, the last term goes to 0 by sending M to infinity. The theorem is proved. □

Example 3.1

We consider the following BSDEJ

$$\begin{aligned} Y_t &= h(X_T) + \int_t^T \left(2\sqrt{1+\log Y_r} + \log \left| \frac{\sqrt{1+\log Y_r} - 1}{\sqrt{1+\log Y_r} + 1} \right| \right) \mathrm{d}r \\ &\quad - \int_t^T \int_\Gamma k_r(\theta) \mathrm{q}(\mathrm{d}r, \mathrm{d}\theta), \end{aligned}$$

where the function h satisfies $(\mathbf{H}_{2.4})$. It is easy to see that

$$f(y) := 2\sqrt{1+\log y} + \log \left| \frac{\sqrt{1+\log y} - 1}{\sqrt{1+\log y} + 1} \right|$$

*satisfies **Hypotheses 2**. And thus, in view of Theorem 3.5, it has a unique solution.*

Before claiming the following corollary, we need to state the following hypothesis:

$(\mathbf{H}_{2.5})$ There exist two constants $\lambda > 0$, and $\alpha \in [0, 1[$ such that

$$|f(s, x, y, k(\cdot))| \leq \lambda \left[1 + |y| + \|k(\cdot)\|_\nu^\alpha\right], \; a.e.t \in [0, T].$$

Corollary 3.1

Let $(\mathbf{H}_{1.1})$, $(\mathbf{H}_{1.2})$, $(\mathbf{H}_{2.1})$, $(\mathbf{H}_{2.3})$ and $(\mathbf{H}_{2.5})$ be satisfied and $h : \Gamma \to \mathbb{R}$ be a $\mathcal{E}$- measurable and bounded function. Assume further that there exist two positive constants L, $\acute{L}$ such that $L_M \leq L + \sqrt{\log M}$ and $\acute{L}_M \leq \acute{L} + \log M$. Then, the BSDEJ (0.1) has a unique solution.

Proof: Arguing as in the proof Theorem 3.5 and using Remark 3.3, we get the desired result. □

Remark 3.6

Corollary 3.1 remain true if we replace $(\mathbf{H}_{2.5})$ by

$$yf(t, y, k(.)) \leq C \left(1 + |y|^2 + |y| \|k(\cdot)\|_\nu\right), a.e.t \in [0, T]. \tag{3.21}$$

where C is a positive constant. Indeed, arguing as in the proof of Lemma 1, with the help of the condition (3.21) and the boundedness of h, we get the boundedness of Y, and the rest of the proof is similar to that of Theorem3.5.

Example 3.2

Let $h : \Gamma \to \mathbb{R}$ be $\mathcal{E}$- measurable bounded function. The following BSDEJ has a unique solution

$$Y_t = h(X_T) - \int_t^T (1 + |Y_r|) \log |1 + |Y_r|| \, \mathrm{d}r - \int_t^T \int_\Gamma K_r(\theta) \mathrm{q}(\mathrm{d}r, \mathrm{d}\theta).$$

Indeed, it is not difficult to that the generator $f(y) := -(1+y)\log(1+y)$ is locally Lipshitz on $\mathbb{R}_+$ and does not satisfy $(\mathbf{H}_{2.5})$ because we have $|f(y)| \leq 1 + \frac{1}{\epsilon}|y|^{1+\epsilon}$ for all $\epsilon > 0$. On the other hand, we have that $yf(y) \leq C\left(1 + \frac{1}{\epsilon}|y|^{1+\epsilon}\right)$, and thus f satisfies (3.21).

Now we will extend the same result to BSDEJ (0.2) for that we need the following assumptions on the coefficients.

Hypotheses b

$(\mathbf{H}_{b.1})$ The function f is continuous in (y, z, k) for almost all t.

$(\mathbf{H}_{b.2})$ There exist two constants $\lambda > 0$ and $\alpha \in [0, 1[$ such that

$$|f(s, x, y, z, k(\cdot))| \leq \lambda \left[1 + |y|^\alpha + |z|^\alpha + \|k(\cdot)\|_\nu^\alpha\right], a.e. t \in [0, T].$$

$(\mathbf{H}_{b.3})$ For every integer $M > 1$, there exist two constants $L_M > 0$ and $\acute{L}_M > 0$ such that, a.e. $t \in [0, T]$ we have

$$\begin{aligned} &\left|f(s, x, y, z, k(\cdot)) - f(s, x, \acute{y}, \acute{z}, \acute{k}(\cdot))\right| \\ \leq \; &\acute{L}_M |y - \acute{y}| + L_M |z - \acute{z}| + L_M \|k(\cdot) - \acute{k}(\cdot)\|_\nu, \end{aligned}$$

and for all y, $\acute{y}$, z ,$k(\cdot)$, $\acute{k}(\cdot)$ such that $|y| \leq M$, $|\acute{y}| \leq M$, $|z| \leq M$, $|\acute{z}| \leq M$, $\|k(\cdot)\|_\nu \leq M$, $\|\acute{k}(\cdot)\|_\nu \leq M$.

$(\mathbf{H}_{b.4})$ The function $h : \Gamma \to \mathbb{R}$ is $\mathcal{E}$- measurable function and satisfies $\mathbb{E}|h(X_T)|^2 < \infty$.

When f satisfies $(\mathbf{H}_{a.1})$ and $(\mathbf{H}_{b.2})$, we can define the family of semi–norms $(\rho_n(f))_{n\in\mathbb{N}}$

$$\rho_n(f) = \left(\mathbb{E} \int_0^T \sup_{|y|, \|z(.)\|_\nu \leq n} |f(s, X_s, y, z, k(\cdot))|^2 \, \mathrm{d}s\right)^{\frac{1}{2}}. \tag{3.22}$$

Theorem 3.7

Assume that $\mathbf{H}_{1.1}$_$\mathbf{H}_{1.3}$ *in Chapter 1 and* ***Hypotheses b*** *are satisfied. Assume also that there exist two positive constant* L *and* $\acute{L}$ *such that* $L_M \leq L + \sqrt{\log M}$ *and* $\acute{L}_M \leq \acute{L} + \log M$. *Then, the BSDEJ (0.1) has a unique solution* $(Y, K(\cdot))$ *which belongs to* $\mathcal{B}$.

Proof: We can prove this theorem using the same method of the proof of Theorem 3.5.

□

3.4 Stability of the Solutions for Locally Lipschitz BSDEJs

Next, we will give a stability theorem for the solution to BDSE (f, ξ). Our starting point is to define a sequence $(f_n)_{n\in\mathbb{N}}$ of *Prog*–measurable functions, a sequence of $\mathcal{F}_{[t,T]}$–measurable and square-integrable random variables $(\xi_n)_{n\in\mathbb{N}}$ such that for each integer n, $\xi_n := h_n(X_T)$. Moreover, we suppose that each BSDEJ (f_n, ξ_n) has a (not necessarily unique) solution which will be denoted by $(Y^n, K^n(\cdot))$. Assume further that (f_n, ξ_n) satisfies the following assumptions:

Hypotheses 3

($\mathbf{H}_{3.1}$) For every M, $\rho_M(f_n - f) \to 0$ as $n \to \infty$.

($\mathbf{H}_{3.2}$) $\mathbb{E}\left|\xi_n - \xi\right|^2 \to 0$ as $n \to \infty$.

($\mathbf{H}_{3.3}$) There exist two constant, $\lambda > 0$ and $\alpha \in [0, 1[$ such that

$$\sup_n \left|f_n(s, x, y, k(\cdot))\right| \leq \lambda\left[1 + |y|^\alpha + \|k(\cdot)\|_\nu^\alpha\right], a.e.\ s \in [0, T],$$

and for all y, $k(\cdot)$, such that $|y| \leq M$, $\|k(\cdot)\|_\nu \leq M$.

Theorem 3.8

*(**Stability Theorem**) Suppose that* (f, ξ) *satisfies* ***Hypotheses 1*** *and* ***Hypotheses 2***, *and* (f_n, ξ_n) *satisfies* ***Hypotheses 3***. *Then we have*

$$\lim_{n\to\infty} \mathbb{E}\int_s^T \left(|Y_r^n - Y_r|^2\right) + \|K_r^n(\cdot) - K_r(\cdot)\|_\nu^2\right) \mathrm{d}r = 0.$$

Proof: We apply Lemma 3.2 to $(Y^1, K^1(\cdot), f^1, \xi^1) = (Y, K(.), f, \xi)$ and $(Y^2, K^2(\cdot), f^2, \xi^2) = (Y^n, K^n(\cdot), f^n, \xi^n)$, the result follows immediately by passing to the limits, first on n and next on M. $\square$

3.5 BSDEJs and Kolmogorov equations

In this section, we shall apply Theorem 3.5 to prove the existence of a unique solution to the Kolmogorov equation. Let us assume that the pure jump process X satisfies the assumption 1-4 given in section 4 in Chapter 1, and then we define the following parabolic differential equation on the state space Γ (called Kolmogorov equation)

$$\begin{aligned} u\left(t,x\right) &= h(x) + \int_s^T \mathcal{L}_r u\left(r,x\right) \mathrm{d}r \\ &\quad + \int_s^T f(r, x, u\left(r,x\right), u\left(r,\cdot\right) - u\left(r,x\right)) \mathrm{d}r, \end{aligned} \tag{3.23}$$

where $s \in [0,T]$, $x \in \Gamma$, $u : [0,T] \times \Gamma \to \mathbb{R}$ is an unknown function such that the function $t \to u\left(t,x\right)$ is absolutely continuous on $[0,T]$ such that $\left(u\left(s, X_{s-}\right), u\left(s,\theta\right) - u\left(s, X_{s-}\right)\right) \in \mathcal{B}^t_{2,2}$, f and h are two given functions.

Definition 3.1

We say that a measurable function $u : [0,T] \times \Gamma \to \mathbb{R}$ is a solution to the Kolmogorov equation (3.23), if for every $(t,x) \in [0,T] \times \Gamma$,

$$\mathbb{E}^{t,x} \int_t^T \int_\Gamma \left|\left(u\left(s,\theta\right) - u\left(s,X_s\right)\right)\right|^2 \nu\left(r,x,\mathrm{d}\theta\right) \mathrm{d}s < +\infty,$$

$$\mathbb{E}^{t,x} \int_t^T \left|u\left(s,X_s\right)\right|^2 \mathrm{d}s < +\infty,$$

and (3.23) is satisfied.

Let us also introduce the following BSDEJ

$$\begin{aligned} Y_s^{t,x} &= h(X_T^{t,x}) + \int_s^T f(r, X_r, Y_r^{t,x}, K_r^{t,x}(\cdot)) \mathrm{d}r \\ &\quad - \int_s^T \int_\Gamma K_r^{t,x}(\theta) \mathrm{q}(\mathrm{d}r, \mathrm{d}\theta). \end{aligned} \tag{3.24}$$

Under Hypotheses 1 and Hypotheses 2, theorem 3.5 shows that BSDEJ (3.24) has a unique solution $(Y_r^{t,x}, K_r^{t,x}(\cdot)) \in \mathcal{B}_{2,2}$. Now we are able to state the main result of the section

Theorem 3.9

Let Hypotheses 1 and Hypotheses 2 hold. Then, the Kolmogorov equation (3.23) has a unique solution u. Furthermore, for every $(t,x) \in [0,T] \times \Gamma$ *we have* $u(t,x) = Y_t^{t,x}$.

Proof: It goes as the proof of Theorem 1.11 in Section 4 of Chapter 1. □

Chapter 4

BSDEJs with Logarithmic Growth

(Joint work with N. Khelfallah)

4.1 Introduction

Throughout this chapter, we use a localization procedure to establish an existence and uniqueness result to a non-necessary locally Lipschitz one-dimensional BSDEJ driven by jump Markov process whose generator f is defined by

$$f(t,x,y,k) := f(t,x,y,\int_{\Gamma} k(\theta)v(.,x,\mathrm{d}\theta)),$$

and shows a logarithmic growth of the type $(|y||\ln|y|| + \|k(\cdot)\|_{\nu}\sqrt{|\ln\|k(\cdot)\|_{\nu}|})$ and the terminal data is exponentially integrable. This last condition is stronger than the square integrability one and is enough to ensure the main results of this chapter.

The rest of this chapter is organized as follows. In section 1, we give some auxiliary results. In section 2, we tackle a result of the existence and uniqueness of solutions to Logarithmic Growth BSDEJ. In Section 3, we give an application to Quadratic BSDEs.

To begin with, we give the main needed hypothesis in this chapter

Hypotheses 4

(H$_{4.1}$) There exists a positive constant λ which is large enough such that $\mathbb{E}[|\xi|^{e^{\lambda T}+1}] < +\infty$,

(H$_{4.2}$)

i) f is continuous in (y,k) for almost all (t,w),

ii) There exists a positive process η_t satisfying $\mathbb{E}[\int_0^T \eta_s^{e^{\lambda T}+1} ds] < +\infty$, and two positive constants c_0 and $\acute{C}$ such that for every t,x,y,k:

$$| f(t,x,y,k) | \leq \eta_t + \acute{C}|y||\ln|y|| + c_0\|k(\cdot)\|_{\nu}\sqrt{|\ln(\|k(\cdot)\|_{\nu})|}.$$

($\mathbf{H}_{4.3}$) There exists a real-valued sequence $(A_M)_{M>1}$ and constants $M_2 \in \mathbb{R}_+$, $r > 0$ such that:

i) $\forall M > 1, \quad 1 < \Lambda_M \leq M^r$,

ii) $\lim_{M\to\infty} \Lambda_M = \infty$,

iii) For every $M \in \mathbb{N}$, $x \in \Gamma$ and every y, y', k, k' such that $|y|$, $|y'|$, $\|k(\cdot)\|_\nu$, $\|k'(\cdot)\|_\nu \leq M$, we have

$$\begin{aligned}
&(y-y')\left(f(t,x,y,k) - f(t,x,y',k')\right)\\
&\leq \; M_2|y-y'|^2 \ln \Lambda_M + M_2|y-y'| \left\|(k-k')(\cdot)\right\|_\nu \sqrt{\ln \Lambda_M}\\
&+M_2\frac{\ln \Lambda_M}{\Lambda_M}.
\end{aligned}$$

4.2 Auxiliary Results

Lemma 4.1

Let $(Y, K(\cdot))$ be a solution of the BSDEJ (0.1). Let $\lambda \geq 2\acute{C}+1$. Assume moreover that (ξ, f) satisfies conditions ($\mathbf{H}_{4.1}$) and ($\mathbf{H}_{4.2}$). Then there exists a constant $C(T, c_0, \acute{C})$, such that:

$$\begin{aligned}
i)\ \mathbb{E}\left(\int_0^T |Y_t|^{e^{\lambda t}+1} \mathrm{d}s\right) &\leq C\left(T, c_0, \acute{C}\right)\mathbb{E}\left(|\xi|^{e^{\lambda T}+1} + \int_0^T \eta_s^{e^{\lambda s}+1}\mathrm{d}s\right),\\
ii)\ \mathbb{E}\left[\int_0^T \|K_s(\cdot)\|_\nu^2 \,\mathrm{d}s\right] &\leq C\left(T, c_0, \acute{C}\right)\mathbb{E}\left[|\xi|^2 + \int_0^T |Y_t|^{e^{\lambda t}+1}\mathrm{d}s + \int_0^T |\eta_s|^2\,\mathrm{d}s\right].
\end{aligned}$$

Proof: To begin with, we give the proof of i).
Set $u(t,x) := |x|^{e^{\lambda t}+1}$ and $\operatorname{sgn}(x) := -\mathbb{1}_{\{x\leq 0\}} + \mathbb{1}_{\{x>0\}}$, we have

$$\frac{\partial u}{\partial t} = \lambda e^{\lambda t}\ln(|x|)\,|x|^{e^{\lambda t}+1}, \text{ and } \frac{\partial u}{\partial x} = \left(e^{\lambda t}+1\right)|x|^{e^{\lambda t}}\operatorname{sgn}(x).$$

For $n \geq 0$, let τ_n be the stopping time defined as follows:

$$\tau_n := \inf\left\{t \geq 0, \left[\int_0^t (e^{\lambda s}+1)^2\,|Y_s|^{2e^{\lambda s}}\,\|K(\cdot)\|_\nu^2\,ds\right] \vee |Y_t| \geq n\right\} \wedge T.$$

Itô's formula leads to

$$
\begin{aligned}
&|Y_{t\wedge\tau_n}|^{e^{\lambda(t\wedge\tau_n)}+1}\\
&= |Y_{T\wedge\tau n}|^{e^{\lambda(T\wedge\tau_n)}+1} - \int_{t\wedge\tau_n}^{T\wedge\tau_n} \lambda e^{\lambda s}\ln(|Y_s|)\,|Y_s|^{e^{\lambda s}+1}\,\mathrm{d}s\\
&+ \int_{t\wedge\tau_n}^{T\wedge\tau_n}(e^{\lambda s}+1)\,|Y_s|^{e^{\lambda s}}\,\mathrm{sgn}(Y_s)f(s,X_s,Y_s,K_s)\mathrm{d}s\\
&- \int_{t\wedge\tau_n}^{T\wedge\tau_n}\int_\Gamma(e^{\lambda s}+1)\,|Y_s|^{e^{\lambda s}}\,\mathrm{sgn}(Y_s)K_s(\theta)\,\mathrm{q}(\mathrm{d}s,\mathrm{d}\theta),\\
&- \int_{t\wedge\tau_n}^{T\wedge\tau_n}\int_\Gamma\Big(|Y_s|^{e^{\lambda s}+1} - |Y_{s-}|^{e^{\lambda s}+1} - (e^{\lambda s}+1)\,|Y_{s-}|^{e^{\lambda s}}\,\mathrm{sgn}(Y_{s-})K_s(\theta)\Big)\,\mathrm{p}(\mathrm{d}s,\mathrm{d}\theta),\\
&\le |Y_{T\wedge\tau_n}|^{e^{\lambda(T\wedge\tau_n)}+1} - \int_{t\wedge\tau_n}^{T\wedge\tau_n} \lambda e^{\lambda s}\ln(|Y_s|)\,|Y_s|^{e^{\lambda s}+1}\,\mathrm{d}s\\
&+ \int_{t\wedge\tau_n}^{T\wedge\tau_n}(e^{\lambda s}+1)\,|Y_s|^{e^{\lambda s}}\Big(\eta_s + \acute{C}\,|Y_s|\ln(|Y_s|) + c_0\,\|K_s(\cdot)\|_\nu\sqrt{|\ln(\|K_s(\cdot)\|_\nu)|}\Big)\,\mathrm{d}s\\
&- \int_{t\wedge\tau_n}^{T\wedge\tau_n}\int_\Gamma(e^{\lambda s}+1)\,|Y_s|^{e^{\lambda s}}\,\mathrm{sgn}(Y_s)K_s(\theta)\,\mathrm{q}(\mathrm{d}s,\mathrm{d}\theta),\\
&- \int_{t\wedge\tau_n}^{T\wedge\tau_n}\int_\Gamma\Big(|Y_s|^{e^{\lambda s}+1} - |Y_{s-}|^{e^{\lambda s}+1} - (e^{\lambda s}+1)\,|Y_{s-}|^{e^{\lambda s}}\,\mathrm{sgn}(Y_{s-})K_s(\theta)\Big)\,\mathrm{p}(\mathrm{d}s,\mathrm{d}\theta).
\end{aligned}
$$

By Young's inequality, it holds:

$$(e^{\lambda s}+1)\,|Y_s|^{e^{\lambda s}}\,\eta_s \le |Y_s|^{e^{\lambda s}+1} + (e^{\lambda s}+1)^{e^{\lambda s}+1}\eta_s^{e^{\lambda s}+1}.$$

For $|Y_s|$ large enough and thanks to the last inequality we have:

$$
\begin{aligned}
&|Y_{t\wedge\tau_n}|^{e^{\lambda(t\wedge\tau_n)+1}}\\
&\le |Y_{T\wedge\tau_n}|^{e^{\lambda(T\wedge\tau_n)}+1} - \int_{t\wedge\tau_k}^{T\wedge\tau_n}\lambda e^{\lambda s}(\ln|Y_s|)\,|Y_s|^{(e^{\lambda s}+1)}\,\mathrm{d}s + \int_{t\wedge\tau_k}^{T\wedge\tau_n}|Y_s|^{e^{\lambda s}+1}\mathrm{d}s\\
&+ \int_{t\wedge\tau_n}^{T\wedge\tau_n}(e^{\lambda s}+1)^{e^{\lambda s}+1}\eta_s^{e^{\lambda s}+1}\mathrm{d}s + \int_{t\wedge\tau_n}^{T\wedge\tau_n}\acute{C}(e^{\lambda s}+1)|Y_s|^{e^{\lambda s}+1}\ln(|Y_s|)\mathrm{d}s\\
&+ \int_{t\wedge\tau_n}^{T\wedge\tau_n}c_0(e^{\lambda s}+1)\,|Y_s|^{e^{\lambda s}}\,\|k(\cdot)\|_\nu\sqrt{|\ln(\|k(\cdot)\|_\nu)|}\mathrm{d}s\\
&- \int_{t\wedge\tau_n}^{T\wedge\tau_n}\int_\Gamma(e^{\lambda s}+1)\,|Y_s|^{e^{\lambda s}}\,\mathrm{sgn}(Y_s)K_s(\theta)\,\mathrm{q}(\mathrm{d}s,\mathrm{d}\theta)\\
&- \int_{t\wedge\tau_n}^{T\wedge\tau_n}\int_\Gamma\Big(|Y_s|^{e^{\lambda s}+1} - |Y_{s-}|^{e^{\lambda s}+1} - (e^{\lambda s}+1)\,|Y_{s-}|^{e^{\lambda s}}\,\mathrm{sgn}(Y_{s-})K_s(\theta)\Big)\,\mathrm{p}(\mathrm{d}s,\mathrm{d}\theta).
\end{aligned}
$$

Note that for $\lambda > 2\acute{C}+1$, we have $(\lambda e^{\lambda s} - \acute{C}(e^{\lambda s}+1) - 1) > 0$ and hence; using the inequality (3.2) in [11, Lemma 3.1], which claims that for every $C_1 > 0$ we have ,

$$
\begin{aligned}
&C_1(e^{\lambda s}+1)\,|Y_s|\,\|K_s(\cdot)\|_\nu\sqrt{|\ln(\|K_s(\cdot)\|_\nu)|}\\
\le\ &(e^{\lambda s}+1)e^{\lambda s}\frac{\|K_s(\cdot)\|_\nu}{2} + 3(\lambda e^{\lambda s} - \acute{C}(e^{\lambda s}+1) - 1)\ln(|Y_s|)\,|Y_s|^2.
\end{aligned}
\tag{4.1}
$$

Furthermore, for $C_1 = c_0 3^{e^{\lambda s}}$, one can easily check that

$$
\begin{aligned}
c_0(e^{\lambda s}+1)\,|Y_s|\,\|K_s(\cdot)\|_\nu\sqrt{|\ln(\|K_s(\cdot)\|_\nu)|} \ \le\ &3^{-e^{\lambda s}}(e^{\lambda s}+1)e^{\lambda s}\,\|K_s(\cdot)\|_\nu^2\\
&+(\lambda e^{\lambda s} - \acute{C}(e^{\lambda s}+1) - 1)\ln(|Y_s|)\,|Y_s|^2.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&|Y_{t\wedge\tau_n}|^{e^{\lambda(t\wedge\tau_n)}+1} \\
&\leq |Y_{T\wedge\tau_n}|^{e^{\lambda(T\wedge\tau_n)}+1} + \int_{t\wedge\tau_n}^{T\wedge\tau_n} (e^{\lambda s}+1)^{e^{\lambda s}+1}\eta_s^{e^{\lambda s}+1}\mathrm{d}s \\
&\quad + \int_{t\wedge\tau_n}^{T\wedge\tau_n} -e^{\lambda s}\left(e^{\lambda s}+1\right)3^{-e^{\lambda s}}|Y_{s-}|^{e^{\lambda s}-1}\|K_s(\cdot)\|_\nu^2\,\mathrm{d}s \\
&\quad - \int_{t\wedge\tau_n}^{T\wedge\tau_n}\int_\Gamma (e^{\lambda s}+1)\,|Y_s|^{e^{\lambda s}}\operatorname{sgn}(Y_s)K_s(\theta)\,\mathrm{q}(\mathrm{d}s,\mathrm{d}\theta) \\
&\quad - \int_{t\wedge\tau_n}^{T\wedge\tau_n}\int_\Gamma \left(|Y_s|^{e^{\lambda s}+1} - |Y_{s-}|^{e^{\lambda s}+1} - (e^{\lambda s}+1)\,|Y_{s-}|^{e^{\lambda s}}\operatorname{sgn}(Y_{s-})K_s(\theta)\right)\mathrm{p}(\mathrm{d}s,\mathrm{d}\theta).
\end{aligned}
$$

Next, similar steps as in the proof of Proposition 2 in [37] show that

$$
\begin{aligned}
&- \int_{t\wedge\tau_n}^{T\wedge\tau_n}\int_\Gamma (e^{\lambda s}+1)\,|Y_s|^{e^{\lambda s}}\operatorname{sgn}(Y_s)K_s(\theta)\mathrm{q}(\mathrm{d}s,\mathrm{d}\theta) \\
&- \int_{t\wedge\tau_n}^{T\wedge\tau_n}\int_\Gamma \left(|Y_s|^{e^{\lambda s}+1} - |Y_{s-}|^{e^{\lambda s}+1} - (e^{\lambda s}+1)\,|Y_{s-}|^{e^{\lambda s}}\operatorname{sgn}(Y_{s-})K_s(\theta)\right)\mathrm{p}(\mathrm{d}s,\mathrm{d}\theta) \\
&\leq \int_{t\wedge\tau_n}^{T\wedge\tau_n} -e^{\lambda s}\left(e^{\lambda s}+1\right)3^{-e^{\lambda s}}|Y_{s-}|^{e^{\lambda s}-1}\|K_s(\cdot)\|_\nu^2\,\mathrm{d}s \\
&- \int_{t\wedge\tau_n}^{T\wedge\tau_n}\int_\Gamma \left(|Y_s|^{e^{\lambda s}+1} - |Y_{s-}|^{e^{\lambda s}+1}\right)\mathrm{q}(\mathrm{d}s,\mathrm{d}\theta).
\end{aligned}
$$

Plugging the above inequality in the previous one and taking the expectation we get

$$
\mathbb{E}(|Y_{t\wedge\tau_n}|^{e^{\lambda(t\wedge\tau_n)}+1}) \leq \mathbb{E}(|Y_{T\wedge\tau_n}|^{e^{\lambda(T\wedge\tau_n)}+1}) + (e^{\lambda T}+1)^{e^{\lambda T}+1}\mathbb{E}\int_0^T \eta_s^{e^{\lambda s}+1}\mathrm{d}s.
$$

Fatou's Lemma leads to, by passing to the limits in n

$$
\mathbb{E}(|Y_t|^{e^{\lambda t}+1}) \leq \mathbb{E}(|\xi|^{e^{\lambda T}+1}) + (e^{\lambda T}+1)^{e^{\lambda T}+1}\mathbb{E}\int_0^T \eta_s^{e^{\lambda s}+1}\mathrm{d}s.
$$

The proof of **i**) is completed by integrating both sides of the last inequality.

We proceed now to prove ii). Itô's formula shows that

$$
\begin{aligned}
&|Y_t|^2 + \int_t^T \|K_s(\cdot)\|_\nu^2\,\mathrm{d}s \\
&= |\xi|^2 + 2\int_t^T Y_s f(s,X_s,Y_s,K_s)\mathrm{d}s - 2\int_t^T\int_\Gamma Y_{s-}K_s(\theta)\mathrm{q}(\mathrm{d}s,\mathrm{d}\theta) \\
&\leq |\xi|^2 + 2\int_t^T |Y_s|\left(\eta_s + \acute{C}\,|Y_s|\,|\ln(|Y_s|)| + c_0\,\|K_s(\cdot)\|_\nu\sqrt{|\ln(\|K_s(\cdot)\|_\nu)|}\right)\mathrm{d}s \\
&\quad -2\int_t^T\int_\Gamma Y_{s-}K_s(\theta)\mathrm{q}(\mathrm{d}s,\mathrm{d}\theta).
\end{aligned}
$$

Since for $|Y_s|$ large enough, we have for any $\varepsilon > 0$, $|Y_s|^2 |\ln(|Y_s|)| \leq |Y_s|^{2+\varepsilon}$, we use again the inequality (3.1) in ([11, Lemma 3.1], Lemma 3.1) to show the existence of a positive constant $\acute{C}_1$ depending up on c_0 and $\acute{C}$ such that:

$$\begin{aligned}\frac{1}{2}\int_t^T \|K_s(\cdot)\|_\nu^2 \mathrm{d}s &\leq |\xi|^2 + \int_t^T |Y_s|^2 \mathrm{d}s + \int_t^T |\eta_s|^2 \mathrm{d}s \\ &\quad + \left(2\acute{C} + \bar{C}\right)\int_t^T |Y_s|^{2+\varepsilon} \mathrm{d}s 2\int_t^T \int_\Gamma Y_{s-}K_s(\theta)\mathrm{q}(\mathrm{d}s, \mathrm{d}\theta).\end{aligned}$$

Since $|Y_s|^{2+\varepsilon} \geq |Y_s|^2$ for $|Y_s|$ and λ large enough, then there exists a positive constant $C_2 = C_2(T, c_2, \acute{C})$ such that:

$$\begin{aligned}\int_t^T \|K_s(\cdot)\|_\nu^2 \mathrm{d}s &\leq C_2\Bigg(|\xi|^2 + \int_t^T |\eta_s|^2 \mathrm{d}s + 2\int_t^T |Y_s|^{2+\varepsilon} \mathrm{d}s \\ &\quad -2\int_t^T \int_\Gamma Y_s K_s(\theta)\mathrm{q}(\mathrm{d}s, \mathrm{d}\theta)\Bigg).\end{aligned}$$

If we put $\varepsilon = e^{\lambda t} - 1$ and taking the expectation, we get the desired result. □

Lemma 4.2

There exists a sequence of functions (f_n) such that,

(a) *For each n, f_n is bounded and globally Lipschitz in $(y, k(\cdot))$ a.e. $t \in [0, T]$.*

(b) $\sup_n |f_n(t, \omega, y, k)| \leq \eta_t + \acute{C}|y||\ln(|y|)| + c_0 \|k(\cdot)\|_\nu \sqrt{|\ln(\|k(\cdot)\|_\nu|)|}$, *a.e. $t \in [0, T]$.*

(c) *For every M, $\rho_M(f_n - f) \longrightarrow 0$ as $n \longrightarrow \infty$.*

Proof:

We define a sequence of smooth functions with compact support $\tilde{\Phi}_n : \mathbb{R}^2 \to \mathbb{R}_+$, which approximate the Dirac measure at 0 and $\int \tilde{\Phi}_n(r)\mathrm{d}r = 1$. We also define a sequence of smooth functions $A_n : \mathbb{R}^2 \to \mathbb{R}_+$, such that $0 \leq |A_n| \leq 1$, and

$$\begin{cases} A_n(r) = 1 \text{ for } |r| \leq n \\ A_n(r) = 0 \text{ for } |r| \geq n+1. \end{cases}$$

Let for $n \in \mathbb{N}^*$, $\beta_{p,n}(t, x, y, k) = \int f(t, x, (y, k) - r)\tilde{\Phi}_p(r)\mathrm{d}r A_n(y, k(\cdot))$ such that $p(n)$ be an integer where $p(n) \geq n + n^\alpha$, we see that $\beta_{p,n}(t, x, y, k)$ satisfies the hypothesis $(H_{4.1})$ _ $(H_{4.3})$, putting $f_n := \beta_{p(n),n}$ we get the result. □

Lemma 4.3

Let $(\mathbf{H}_{4.1})$, $(\mathbf{H}_{4.2})$–(ii) *be satisfied. Then, for every* $1 < \alpha < 2$

$$\mathbb{E}\int_0^T |f(s, X_s, Y_s, K_s)|^{\bar{\alpha}} \mathrm{d}s \ \leq \acute{C}\mathbb{E}\left(\int_0^T \left(\eta_s^2 + |Y_s|^2\right) \mathrm{d}s + \mathbb{E}\int_0^T \|K_s(\cdot)\|_\nu^2 \, \mathrm{d}s\right).$$

where $\bar{\alpha} = \frac{2}{\alpha}$ *and* $\acute{C}$ *is a positive constant.*

Proof: Assumption $(H_{4.2})$ implies that there exist two positives constants $c, \tilde{c}$ and α with $1 < \alpha < 2$, such that

$$|f(t, x, y, k)| \leq \eta_t + c\,|y|^\alpha + \tilde{c}\,\|k\,(\cdot)\|_\nu^\alpha \,. \tag{4.2}$$

Using the previous inequality to get

$$\begin{aligned}
&\mathbb{E}\int_0^T |f(s, X_s, Y_s, K_s)|^{\bar{\alpha}} \mathrm{d}s \\
&\leq \mathbb{E}\int_0^T \left(\eta_s + c|Y_s|^\alpha + \tilde{c}\,\|K_s(\cdot)\|_\nu^\alpha\right)^{\bar{\alpha}} \mathrm{d}s \\
&\leq 3\left(1 + c^{\bar{\alpha}} + \tilde{c}^{\bar{\alpha}}\right)\mathbb{E}\int_0^T \left(\eta_s^{\bar{\alpha}} + |Y_s|^{\alpha\bar{\alpha}} + \|K_s(\cdot)\|_\nu^{\alpha\bar{\alpha}}\right) \mathrm{d}s.
\end{aligned}$$

Since $\bar{\alpha} = \frac{2}{\alpha}$, we obtain

$$\begin{aligned}
&\mathbb{E}\int_0^T |f(s, X_s, Y_s, K_s)|^{\bar{\alpha}} \mathrm{d}s \\
&\leq 3\left(1 + c^{\bar{\alpha}} + \tilde{c}^{\bar{\alpha}}\right)\mathbb{E}\int_0^T \left(\eta_s^2 + |Y_s|^2 + \|K_s(\cdot)\|_\nu^2\right) \mathrm{d}s < \infty.
\end{aligned}$$

Putting $\acute{C} = 3\,(1 + c^{\bar{\alpha}} + \tilde{c}^{\bar{\alpha}})$, we get the result. □

Lemma 4.4

For every $\beta \in]1, 2[$, $A > 0$, $(y)_{i=1..d} \subset \mathbb{R}$, $(z)_{i=1..d, j=1..r} \subset \mathbb{R}$ *we have,*

$$A|y||k(\cdot)| - \frac{1}{2}|z|^2 + \frac{2-\beta}{2}|y|^{-2}|yk(\cdot)|^2 \leq \frac{1}{\beta - 1}A^2|y|^2 - \frac{\beta - 1}{4}|k(\cdot)|^2.$$

Proof: It goes by using Hölder's and Schwarz's inequalities □

Arguing as in the proofs of Lemma 4.1, Lemma 4.2 and standard arguments of BSDEJs, one can prove the following estimates.

Lemma 4.5

Let f and ξ be as in Theorem 4.8. Let (f_n) be the sequence of functions associated to f by Lemma 4.2. Denote by $(Y^n, K^n(\cdot))$ the solution of equation (E^{f_n}). Then, there exist constants $\bar{C}_1$, $\bar{C}_2$, $\bar{C}_3$ such that

a) $\sup_n \mathbb{E} \int_0^T \|K^n(.)\|_\nu^2 \, \mathrm{d}s \leq \bar{C}_1$.

b) $\sup_n \mathbb{E} \int_0^T |Y_t^n|^{e^{\lambda T}+1} \mathrm{d}s \leq \bar{C}_2$.

c) $\sup_n \mathbb{E} \int_0^T |f_n(s, X_s, Y_s^n, K_s^n)|^{\frac{2}{\alpha}} \mathrm{d}s \leq \bar{C}_3$.

4.3 Existence and Uniqueness of Logarithmic Growth BSDEJ

Proposition 4.6

For every $R \in \mathbb{N}$, $\beta \in]1, 2[$, $\delta < (\beta - 1) \min\left(\frac{1}{4M_2^2}, \frac{3-\alpha-\beta}{2rM_2^2\beta}\right)$ and $\varepsilon > 0$, there exists $M_0 > R$ such that for all $M > M_0$ and $T' \leq T$:

$$\begin{aligned}\limsup_{n,m\to+\infty} &\left(\sup_{(T'-\delta)^+ \leq t \leq T'} \mathbb{E}|Y_t^n - Y_t^m|^\beta + \mathbb{E}\int_{(T'-\delta)^+}^{T'} \frac{\|(K_s^n - K_s^m)(\cdot)\|_\nu^2}{(|Y_s^n - Y_s^m|^2 + \nu_R)^{\frac{2-\beta}{2}}}\right) \mathrm{d}s \\ &\leq \varepsilon + \frac{4}{\beta(\beta-1)} e^{C_M \delta} \limsup_{n,m\to+\infty} \mathbb{E}|Y_{T'}^n - Y_{T'}^m|^\beta,\end{aligned}$$

where $\nu_R = \sup\{(\Lambda_M)^{-1}, M \geq R\}$, $C_M = \frac{2M_2^2\beta}{(\beta-1)} \ln \Lambda_M$.

To prove the previous Proposition, we need the following lemma. Let $C > 0$ and for $M \in \mathbb{N}^\star$, we set $\psi_t := |Y_t^n - Y_t^m|^2 + (\Lambda_M)^{-1}$.

Lemma 4.7

Let assumptions of Proposition 4.6 be satisfied and let $\kappa := 3 - \alpha - \beta$. Then, for any $C > 0$ we have,

$$\begin{aligned} & e^{Ct}\psi_t^{\frac{\beta}{2}} + C\int_t^{T'} e^{Cs}\psi_s^{\frac{\beta}{2}} \mathrm{d}s \\ \leq\ & e^{CT'}\psi_{T'}^{\frac{\beta}{2}} - \frac{\beta(\beta-1)}{2}\int_t^{T'} e_s^{Cs}\psi_s^{\frac{\beta}{2}-1} \|(K_s^n - K_s^m)(\cdot)\|_\nu^2 \, \mathrm{d}s \\ & -\int_t^{T'}\int_\Gamma e^{Cs}\left(\psi_s^{\frac{\beta}{2}} - \psi_{s-}^{\frac{\beta}{2}}\right) \mathrm{q}(\mathrm{d}s, \mathrm{d}\theta) + J_1 + J_2 + J_3, \end{aligned}$$

where

$$J_1 : = \frac{\beta e^{CT'}}{M^\kappa}\int_t^{T'} \psi_s^{\frac{\beta-1}{2}} \Phi^\kappa(s)|f_n(s, X_s, Y_s^n, K_s^n) - f_m(s, X_s, Y_s^m, K_s^m)|\mathrm{d}s,$$

$$
\begin{aligned}
J_2 : &= \beta e^{CT'}[2M^2+\nu_1]^{\frac{\beta-1}{2}} \left[\int_t^{T'} \sup_{|y|,\|k(.)\|_\nu|\leq M} |\left(f_n-f\right)(s,X_s,y,k)|\mathrm{d}s \right.\\
&\left. + \int_t^{T'} \sup_{|y|,\|k(\cdot)\|_\nu\leq M} |\left(f_m-f\right)(s,X_s,y,k)|\mathrm{d}s\right],
\end{aligned}
$$

and

$$
J_3 : = \beta M_2 \int_t^{T'} e^{Cs}\psi_s^{\frac{\beta}{2}-1}\left(\psi_s \ln\Lambda_M + |Y_s^n - Y_s^m|\|K_s^n(\cdot) - K_s^m(\cdot)\|_\nu\sqrt{\ln\Lambda_M}\right)\mathrm{d}s.
$$

Proof: Itô's formula applied to $e^{Ct}\psi_t^{\frac{\beta}{2}}$ shows that,

$$
\begin{aligned}
& e^{Ct}\psi_t^{\frac{\beta}{2}} + C\int_t^{T'} e^{Cs}\psi_s^{\frac{\beta}{2}}\mathrm{d}s = e^{CT'}\psi_{T'}^{\frac{\beta}{2}} \\
& +\beta\int_t^{T'} e^{Cs}\psi_s^{\frac{\beta}{2}-1}\left(Y_s^n-Y_s^m\right)\left(f_n(s,X_s,Y_s^n,K_s^n)-f_m(s,X_s,Y_s^m,K_s^m)\right)\mathrm{d}s \\
& -\beta\int_t^{T'}\int_\Gamma e^{Cs}\psi_s^{\frac{\beta}{2}-1}\left(Y_s^n-Y_s^m\right)\left(K_s^n(\cdot)-K_s^m(\cdot)\right)\mathrm{q}(\mathrm{d}s,\mathrm{d}\theta) \\
& -\int_t^{T'}\int_\Gamma e^{Cs}\left(\psi_s^{\frac{\beta}{2}}-\psi_{s-}^{\frac{\beta}{2}}-\beta\psi_{s-}^{\frac{\beta}{2}-1}\left(Y_s^n-Y_s^m\right)\left(K_s^n(\cdot)-K_s^m(\cdot)\right)\right)\mathrm{p}(\mathrm{d}s,\mathrm{d}\theta).
\end{aligned}
$$

By similar arguments, as in the proof of Lemma 9 in [37], we can rewrite the jump parts as the following

$$
\begin{aligned}
& -\beta\int_t^{T'}\int_\Gamma e^{Cs}\psi_s^{\frac{\beta}{2}-1}\left(Y_s^n-Y_s^m\right)\left(\left(K_s^n-K_s^m\right)(\theta)\right)\mathrm{q}(\mathrm{d}s,\mathrm{d}\theta) \\
& -\int_t^{T'}\int_\Gamma e^{Cs}\left(\psi_s^{\frac{\beta}{2}}-\psi_{s-}^{\frac{\beta}{2}}-\beta\psi_{s-}^{\frac{\beta}{2}-1}\left(Y_s^n-Y_s^m\right)\left(\left(K_s^n-K_s^m\right)(\theta)\right)\right)\mathrm{p}(\mathrm{d}s,\mathrm{d}\theta) \\
& = -\int_t^{T'}\int_\Gamma e^{Cs}\left(\psi_s^{\frac{\beta}{2}}-\psi_{s-}^{\frac{\beta}{2}}-\beta\psi_{s-}^{\frac{\beta}{2}-1}\left(Y_s^n-Y_s^m\right)\left(\left(K_s^n-K_s^m\right)(\theta)\right)\right)\nu(s,X_s,\mathrm{d}\theta)\mathrm{d}s \\
& -\int_t^{T'}\int_\Gamma e^{Cs}\left(\psi_s^{\frac{\beta}{2}}-\psi_{s-}^{\frac{\beta}{2}}\right)\mathrm{q}(\mathrm{d}s,\mathrm{d}\theta). \\
\leq\ & -\frac{\beta\left(\beta-1\right)}{2}\int_t^{T'}\left(e^{Cs}\left\|\left(K_s^n-K_s^m\right)(\cdot)\right\|_\nu^2\right. \\
& \qquad\left.\left(|Y_t^n-Y_t^m|^2\vee\left|Y_{t-}^n-Y_{t-}^m\right|^2+\left(\Lambda_M\right)^{-1}\right)^{\frac{\beta}{2}-1}\right)\mathrm{d}s \\
& -\int_t^{T'}\int_\Gamma e^{Cs}\left(\psi_s^{\frac{\beta}{2}}-\psi_{s-}^{\frac{\beta}{2}}\right)\mathrm{q}(\mathrm{d}s,\mathrm{d}\theta). \\
=\ & -\frac{\beta\left(\beta-1\right)}{2}\int_t^{T'} e_s^{Cs}\psi_s^{\frac{\beta}{2}-1}\left\|\left(K_s^n-K_s^m\right)(\cdot)\right\|_\nu^2\mathrm{d}s \\
& -\int_t^{T'}\int_\Gamma e^{Cs}\left(\psi_s^{\frac{\beta}{2}}-\psi_{s-}^{\frac{\beta}{2}}\right)\mathrm{q}(\mathrm{d}s,\mathrm{d}\theta).
\end{aligned}
$$

By taking account of the above inequality, we get

$$
\begin{aligned}
& e^{Ct}\psi_t^{\frac{\beta}{2}} + C\int_t^{T'} e^{Cs}\psi_s^{\frac{\beta}{2}}\mathrm{d}s \\
&= e^{CT'}\psi_{T'}^{\frac{\beta}{2}} - \frac{\beta(\beta-1)}{2}\int_t^{T'} e_s^{Cs}\psi_s^{\frac{\beta}{2}-1}\left\|(K_s^n - K_s^m)(\cdot)\right\|_\nu^2 \mathrm{d}s \\
& - \int_t^{T'}\int_\Gamma e^{Cs}\left(\psi_s^{\frac{\beta}{2}} - \psi_{s-}^{\frac{\beta}{2}}\right)\mathrm{q}(\mathrm{d}s,\mathrm{d}\theta) + \acute{J}_1 + \acute{J}_2 + \acute{J}_3 + \acute{J}_4,
\end{aligned}
$$

where

$$
\begin{aligned}
\acute{J}_1 : &= \beta\int_t^{T'} e^{Cs}\psi_s^{\frac{\beta}{2}-1}(Y_s^n - Y_s^m) \\
& \quad \times \left(f_n(s,X_s,Y_s^n,K_s^m) - f_m(s,X_s,Y_s^m,K_s^m)\right)\mathbb{1}_{\{\Phi(s)>M\}}\mathrm{d}s, \\
\acute{J}_2 : &= \beta\int_t^{T'} e^{Cs}\psi_s^{\frac{\beta}{2}-1}(Y_s^n - Y_s^m) \\
& \quad \times \left(f_n(s,X_s,Y_s^n,K_s^n) - f(s,X_s,Y_s^n,K_s^n)\right)\mathbb{1}_{\{\Phi(s)\le M\}}\mathrm{d}s, \\
\acute{J}_3 : &= \beta\int_t^{T'} e^{Cs}\psi_s^{\frac{\beta}{2}-1}(Y_s^n - Y_s^m) \\
& \quad \times \left(f(s,X_s,Y_s^n,K_s^n) - f(s,X_s,Y_s^m,K_s^m)\right)\mathbb{1}_{\{\Phi(s)\le M\}}\mathrm{d}s, \\
\acute{J}_4 : &= \beta\int_t^{T'} e^{Cs}\psi_s^{\frac{\beta}{2}-1}(Y_s^n - Y_s^m) \\
& \quad \times \left(f(s,X_s,Y_s^m,K_s^m\ ) - f_m(s,X_s,Y_s^m,K_s^m)\right)\mathbb{1}_{\{\Phi(s)\le M\}}\mathrm{d}s,
\end{aligned}
$$

with the shorthand $\Phi(s) = |Y_s^n| + |Y_s^m| + \|K_s^n(\cdot)\|_\nu + \|K_s^m(\cdot)\|_\nu$. By using the fact that $|Y_s^n - Y_s^m| \le \psi_s^{\frac{1}{2}}$ a simple computation shows that $\acute{J}_1 \le J_1$ and $\acute{J}_2 + \acute{J}_4 \le J_2$. Finally, by using assumption $(\mathbf{H}_{4.3})$, we get

$$
\begin{aligned}
\acute{J}_3 &\le \beta M_2\int_t^{T'} e^{Cs}\psi_s^{\frac{\beta}{2}-1}\Big[|Y_s^n - Y_s^m|^2\ln\Lambda_M \\
& \quad + |Y_s^n - Y_s^m|\left\|(K_s^n - K_s^m)(.)\right\|_\nu^2\sqrt{\ln\Lambda_M} + \frac{\ln\Lambda_M}{\Lambda_M}\Big]\mathbb{1}_{\{\Phi(s)<M\}}\mathrm{d}s \\
&\le J_3.
\end{aligned}
$$

Which achieves the proof of the lemma. □

Proof of Proposition 4.6: Now we choose $C := C_M := \dfrac{2M_2^2\beta}{\beta-1}\ln\Lambda_M$, and let $\gamma = \frac{2M_2^2\delta\beta}{\beta-1}$. We use the definition of J_3, making use of Lemmas 4.2, 4.4 and 4.5 one can show that there exists a such that for any $\delta > 0$ and $M > R$,

$$
\begin{aligned}
& e^{C_M t}\psi_t^{\frac{\beta}{2}} + \frac{\beta(\beta-1)}{4}\int_t^{T'} e_s^{C_M s}\psi_s^{\frac{\beta}{2}-1}\left\|\left(K_s^n - K_s^m\right)(\cdot)\right\|_\nu^2 \mathrm{d}s \\
& \leq e^{\delta C_M}\psi_{T'}^{\frac{\beta}{2}} - \int_t^{T'}\int_\Gamma e^{C_M s}\left(\psi_s^{\frac{\beta}{2}} - \psi_{s-}^{\frac{\beta}{2}}\right)\mathrm{q}(\mathrm{d}s,\mathrm{d}\theta) \\
& +\beta e^{\delta C_M}\frac{1}{M^\kappa}\int_t^{T'}\psi_s^{\frac{\beta-1}{2}}\Phi^\kappa(s)|f_n(s,X_s,Y_s^n,K_s^n) - f_m(s,X_s,Y_s^m,K_s^m)|\mathrm{d}s, \\
& \beta e^{\delta C_M}\left[2M^2+\nu_1\right]^{\frac{\beta-1}{2}}\left[\int_t^{T'}\sup_{|y|,\|k(\cdot)\|_\nu|\leq M}|\left(f_n - f\right)(s,X_s,y,k)|\mathrm{d}s\right. \\
& \left. + \int_t^{T'}\sup_{|y|,\|k(\cdot)\|_\nu\leq M}|\left(f_m - f\right)(s,X_s,y,k)|\mathrm{d}s\right].
\end{aligned}
$$

And, thus

$$
\begin{aligned}
& \mathbb{E}|Y_t^n - Y_t^m|^\beta + \mathbb{E}\int_{(T'-\delta)^+}^{T'}\frac{\left\|\left(K_s^n - K_s^m\right)(\cdot)\right\|_\nu^2}{\left(|Y_s^n - Y_s^m|^2 + \nu_R\right)^{\frac{2-\beta}{2}}}\mathrm{d}s \\
& \leq \frac{4}{\beta(\beta-1)}e^{C_M\delta}\mathbb{E}|Y_{T'}^n - Y_{T'}^m|^\beta + \frac{4}{\beta(\beta-1)}\frac{\Lambda_M^\gamma}{(\Lambda_M)^{\frac{\beta}{2}}} \\
& +\frac{4}{(\beta-1)}4\acute{C}_3^{\frac{\alpha}{2}}\left(4\acute{C}_2 + T\nu_R\right)^{\frac{\beta-1}{2}}\left(8\acute{C}_2 + 8\acute{C}_1\right)^{\frac{\kappa}{2}}\frac{\Lambda_M^\gamma}{(\Lambda_M)^{\frac{\kappa}{r}}} \\
& +\frac{4}{(\beta-1)}e^{C_M\delta}\left[2M^2+\nu_1\right]^{\frac{\beta-1}{2}}\left[\rho_M(f_n - f) + \rho_M(f_m - f)\right].
\end{aligned}
$$

Taking $\delta < (\beta-1)\min\left(\frac{1}{4M_2^2}, \frac{\kappa}{2rM_2^2\beta}\right)$, we derive

$$
\frac{\Lambda_M^\gamma}{(\Lambda_M)^{\frac{\beta}{2}}} + \frac{\Lambda_M^\gamma}{(\Lambda_M)^{\frac{\kappa}{r}}} \underset{M\to\infty}{\longrightarrow} 0,
$$

To finish the proof of Proposition 4.6 we pass to the limits first on n, m and next on M using assertion (c) of Lemma 4.2. $\square$

Theorem 4.8

*Assume that **Hypotheses 1** in Chapter 3 and **Hypotheses 4** are satisfied. Then, BSDEJ(0.1) has a unique solution in $\mathcal{S}_{e^{\lambda T}+1}\otimes\mathcal{H}^2$.*

Proof:

Proof of the Existence: Taking successively $T' = T$, $T' = (T-\delta)^+$, $T' = (T-2\delta)^+ =$ ... in Proposition 4.6, with the help of Lebegue's dominated theorem, we show that for any $\beta \in]1, 2[$

$$\limsup_{n,m\to+\infty} \mathbb{E}\int_0^T \left(|Y_s^n - Y_s^m|^\beta + \frac{\|(K_s^n - K_s^m)(\cdot)\|_\nu^2}{(|Y_s^n - Y_s^m|^2 + \nu_R)^{\frac{2-\beta}{2}}} \right) \mathrm{d}s = 0.$$

Using Schwarz's inequality we have,

$$\begin{aligned}
&\mathbb{E}\textstyle\int_0^T \|(K_s^n - K_s^m)(\cdot)\|_\nu \,\mathrm{d}s \\
&\leq \mathbb{E}\textstyle\int_0^T \left(\dfrac{\|(K_s^n - K_s^m)(\cdot)\|_\nu^2}{(|Y_s^n - Y_s^m|^2 + \nu_R)^{\frac{2-\beta}{2}}} \mathrm{d}s \right)^{\frac{1}{2}} \left(\mathbb{E}\textstyle\int_0^T (|Y_s^n - Y_s^m|^2 + \nu_R)^{\frac{2-\beta}{2}} \mathrm{d}s \right)^{\frac{1}{2}},
\end{aligned}$$

Lemma 4.5 shows that

$$\left[\mathbb{E}\int_0^T \left(|Y_s^n - Y_s^m|^2 + \nu_R \right)^{\frac{2-\beta}{2}} \mathrm{d}s \right]^{\frac{1}{2}} < \infty.$$

It follows that

$$\lim_{n,m\to+\infty} \mathbb{E}\int_0^T \left(|Y_s^n - Y_s^m|^\beta + \|(K_s^n - K_s^m)(\cdot)\|_\nu \right) \mathrm{d}s = 0.$$

Hence, there exists $(Y, K(\cdot))$ satisfying

$$\mathbb{E}\int_0^T \left(|Y_s|^\beta + \|K_s(\cdot)\|_\nu \right) \mathrm{d}s < \infty,$$

and

$$\lim_{n\to+\infty} \left(|Y_t^n - Y_t|^\beta + \|(K_t^n - K_t)(\cdot)\|_\nu \right) = 0.$$

In particular, there exists a subsequence, which is still denoted by $(Y^n,\ K^n(\cdot))$, such that

$$\lim_{n\to+\infty} (|Y_t^n - Y_t| + \|(K_s^n - K_s)(\cdot)\|_\nu) = 0 \quad a.e.\ (t; w)\,. \tag{4.3}$$

It remains to prove that $\int_0^T [f_n(s, X_s, Y_s^n, K_s^n - f(s, X_s, Y_s, K_s)]\mathrm{d}s$ tends in probability to 0 as n tends to ∞. First, the triangular inequality gives

$$\begin{aligned}
&\mathbb{E}\int_0^T |f_n(s, X_s, Y_s^n, K_s^n) - f(s, X_s, Y_s, K_s)|\mathrm{d}s \\
&\leq \mathbb{E}\int_0^T |(f_n - f)(s, X_s, Y_s^n, K_s^n()|\mathrm{d}s \\
&\quad + \mathbb{E}\int_0^T |f(s, X_s, Y_s^n, K_s^n) - f(s, X_s, Y_s, K_s)|\mathrm{d}s
\end{aligned}$$

Furthermore, by using the fact that

$$\mathbb{1}_{\{|Y_s^n|+\|K_s^n(\cdot)\|_\nu \geq M\}} \leq \frac{(|Y_s^n| + \|K_s^n(\cdot)\|_\nu)^{(2-\alpha)}}{M^{(2-\alpha)}},$$

we get

$$\begin{aligned}
&\mathbb{E}\int_0^T |\left(f_n - f\right)(s, X_s, Y_s^n, K_s^n)|\mathrm{d}s \\
&\leq \mathbb{E}\int_0^T |\left(f_n - f\right)(s, X_s, Y_s^n, K_s^n)|\mathbb{1}_{\{|Y_s^n|+\|K_s^n(\cdot)\|_\nu \leq M\}}\mathrm{d}s \\
&+\mathbb{E}\int_0^T |\left(f_n - f\right)(s, X_s, Y_s^n, K_s^n)|\frac{(|Y_s^n| + \|K_s^n(\cdot)\|_\nu)^{(2-\alpha)}}{M^{(2-\alpha)}}\mathrm{d}s \\
&\leq \rho_M(f_n - f) + \frac{2\bar{C}_3^{\frac{\alpha}{2}}\left[\bar{C}_2 + \bar{C}_1\right]^{1-\frac{\alpha}{2}}}{M^{(2-\alpha)}}.
\end{aligned}$$

Passing to the limit first on n and next on M in the previous inequality, we get,

$$\lim_n \mathbb{E}\int_0^T |\left(f_n - f\right)(s, X_s, Y_s^n, K_s^n)|\mathrm{d}s = 0.$$

Taking account of the limit We use the limit 4.3 and the fact that the function f is continuous in $(y, k(\cdot))$ for all $(t, x) \in [0, T] \times \Gamma$, we get

$$\lim_n |f(s, X_s, Y_s^n, K_s^n) - f(s, X_s, Y_s, K_s)| = 0. \quad a.e.\,(t; w)\,.$$

Moreover, Lemma 4.3 and the assertions (a) and (b) in Lemma 4.5 show that the sequence $|f(s, X_s, Y_s^n, K_s^n) - f(s, X_s, Y_s, K_s)|$ is uniformly integrable. Therefore

$$\lim_n \mathbb{E}\int_0^T |f(s, X_s, Y_s^n, K_s^n) - f(s, X_s, Y_s, K_s)|\mathrm{d}s = 0.$$

The existence is proved.

Proof of the Uniqueness: Let $(Y, K(\cdot))$ and $(Y', K'(\cdot))$ be two solutions of BSDEJ (0.1). Arguing as the proof of Proposition 4.6, one can show that: for every $R > 2$, $\beta \in]1, 2[$, $\delta < (\beta - 1)\min\left(\frac{1}{4M_2^2}, \frac{\kappa}{2rM_2^2\beta}\right)$ and $\varepsilon > 0$, there exists $M_0 > R$ such that for every $M > M_0$ and every $T' \leq T$

$$\begin{aligned}
&\mathbb{E}|Y_t - Y_t^{'}|^\beta + \mathbb{E}\textstyle\int_{(T'-\delta)^+}^{T'}\left[\left\|\left(K_s - K_s^{'}\right)(\cdot)\right\|_\nu^2 (|Y_s^n - Y_s^m|^2 + \nu_R)^{\frac{\beta-2}{2}}\right]\mathrm{d}s \\
&\leq \varepsilon + \tfrac{2}{\beta(\beta-1)}e^{C_N\delta}\mathbb{E}|Y_{T'} - Y_{T'}^{'}|^\beta.
\end{aligned}$$

We successively take $T' = T$, $T' = (T' - \delta)^+$, ... one can easily demonstrate that $Y_t = Y_t^{'}$ $\mathrm{d}\mathbb{P} - a.s$ and $K_s(\cdot) = K_s^{'}(\cdot)\ \nu(t, x, \mathrm{d}\theta)\,\mathrm{d}t \otimes \mathrm{d}\mathbb{P} - a.e.$ the proof is finished. $\square$

Remark 4.9

To wrap up this section we reveal a possible extension to Theorem 4.8, to a wide class of BSDE driven by both Brownian motion and a jump Markov process of the type of BSDEJ (0.2).

For that, we need the following assumptions:

$(\mathbf{H}_{4.a.1})$ f is continuous in (y, z, k) for almost all (t, w),

and two positive constants c_0 and K such that for every t, x, y, k:

$$| f(t, x, y, k) | \leq \eta_t + \acute{C}|y|| \ln |y|| + c_0|z|\sqrt{|\ln |z||} + c_0 \left\| k(\cdot) \right\|_\nu \sqrt{|\ln(\|k(\cdot)\|_\nu)|}.$$

$(\mathbf{H}_{4.a.3})$ For every $M \in \mathbb{N}$, and every $y,\ y',\ z,\ \acute{z},\ k\left(\cdot\right),\ k'\left(\cdot\right)$ such that

$|y|,\ |y'|,\ |z|,\ |z'| \leq M,\ \|k(\cdot)\|_\nu,\ \|k'(\cdot)\|_\nu \leq M$, we have

$$\begin{aligned} & (y - y')\left(f(t, x, y, k(\cdot)) - f(t, x, y', k'(\cdot))\right) \\ \leq\ & M_2|y - y'|^2 \ln \Lambda_M + \ M_2|y - y'||z - z'| \ln \Lambda_M \\ & + M_2|y - y'| \left\| (k - k')\left(\cdot\right) \right\|_\nu^2 \sqrt{\ln \Lambda_M} + M_2 \frac{\ln \Lambda_M}{\Lambda_M}. \end{aligned}$$

Theorem 4.10

Assume that $\mathbf{H}_{1.1}$_$\mathbf{H}_{1.3}$ *in Chapter 1,* **(**$\mathbf{H}_{4.1}$**)***,* **(**$\mathbf{H}_{4.2}$*) (ii),* **(**$\mathbf{H}_{4.3}$**) *(i) (ii)*, (**$\mathbf{H}_{4.a.1}$**), (**$\mathbf{H}_{4.a.3}$*) hold true. Then, BSDEJ (0.1) has a unique solution in* $\mathcal{S}^{e^{\lambda T}+1} \otimes \mathcal{M}^2 \otimes \mathcal{H}^2$.

Proof: The proof of the above theorem can be performed as the proof of Theorem 4.8.

□

4.4 Application to Quadratic BSDEs

In this section, we study the existence and uniqueness of solutions to one kind of quadratic BSDEJ that shows exponential growth with respect to jump variable, for that we introduce the following positive predictable process

$$[K_{s,x}] := \int_{\Gamma} \left(e^{k(\theta)} - 1 - k(\theta)\right) \nu(s,x,\mathrm{d}\theta), \tag{4.4}$$

for a given real number θ and k in $\mathcal{L}^2(p)$, noting that this process is a particular case of (2.5) where $F(x) = e^x$.

Our goal is to solve the following quadratic BSDEJ:

$$\begin{aligned} Y_s &= h(X_T) + \int_s^T \Big(Y_r + Z_r\sqrt{|\ln|Z_r| + Y_r|} \\ &\quad + \left(e^{K_r(\cdot)} - 1\right)\sqrt{|\ln|(e^{K_r(\cdot)} - 1)| + Y_r|} \\ &\quad + \frac{1}{2}|Z_r|^2 + [K_{r,X_r}]\Big)\,\mathrm{d}s - \int_s^T Z_r \mathrm{d}B_r - \int_s^T \int_{\Gamma} K_r(\theta)\mathrm{q}(\mathrm{d}r,\mathrm{d}\theta). \end{aligned} \tag{4.5}$$

Let us first consider the following BSDEJ with logarithmic growth which plays an important role in the sequel

$$\begin{aligned} y_s &= e^{h(X_T)} + \int_s^T \left(y_r \ln y_r + z_r\sqrt{|\ln|z_r||} + k_r(\cdot)\sqrt{|\ln|k_r(\cdot)||}\right)\mathrm{d}r \\ &\quad - \int_s^T z_r \mathrm{d}B_r - \int_s^T \int_{\Gamma} k_r(\theta)\mathrm{q}(\mathrm{d}r,\mathrm{d}\theta). \end{aligned} \tag{4.6}$$

We pass now to the main result of this section

Theorem 4.11

Let $h(X_T)$ be a $\mathcal{E}$–measurable and square integrable random variable such that $\exp(h(X_T))$ belongs to $L^2(\Omega)$, then $(Y,Z,K(\cdot))$ is a unique solution to BSDEJ(4.5) if and only if $(y_r, z_r, k_r(\theta)) = \left(e^{Y_r}, e^{Y_r}Z_r, e^{Y_r}(e^{K_r(\theta)} - 1)\right)$, for any $r \in [0,T]$ and $\theta \in \Gamma$, is a solution to (4.6).

Proof: Putting $z_s = e^{Y_s}Z_s$, $k_s(\cdot) = e^{Y_s}\left(e^{K_s(\cdot)} - 1\right)$ and applying Itô's formula to the function $y_s = e^{Y_s}$, the BSDEJ(4.5) transformed to

$$\begin{aligned} e^{Y_s} &= e^{h(X_T)} + \int_s^T e^{Y_{r-}} \Big(Y_r + Z_r \sqrt{|\ln |Z_r| + Y_r|} + \\ &+ \left(e^{K_r(\cdot)} - 1\right) \sqrt{|\ln |(e^{K_r(\cdot)} - 1)| + Y_r|} + \frac{1}{2} |Z_r|^2 + [K_{r,X_r}] \Big) \mathrm{d}r \\ &- \int_s^T e^{Y_{r-}} Z_r \mathrm{d}B_r - \int_s^T \int_\Gamma e^{Y_{r-}} K_r(\theta) \mathrm{q}(\mathrm{d}r, \mathrm{d}\theta) - \frac{1}{2} \int_s^T e^{Y_{r-}} |Z_r|^2 \mathrm{d}r \\ &- \sum_{0<r\leq T} (e^{Y_r} - e^{Y_{r-}} - e^{Y_{r-}} \Delta Y_r), \end{aligned}$$

then,

$$\begin{aligned} e^{Y_s} &= e^{h(X_T)} + \int_s^T \Big[e^{Y_r} Y_r + e^{Y_r} Z_r \sqrt{|\ln |Z_r| + Y_r|} \\ &+ e^{Y_{r-}} \left(e^{K_r(\cdot)} - 1\right) \sqrt{|\ln |(e^{K_r(\cdot)} - 1)| + Y_r|} + e^{Y_r} \left(\frac{1}{2} |Z_r|^2 + [K_{r,X_r}] \right) \Big] \mathrm{d}r \\ &- \int_s^T e^{Y_r} Z_r \mathrm{d}B_r - \int_s^T \int_\Gamma e^{Y_{r-}} K_r(\theta) \mathrm{q}(\mathrm{d}r, \mathrm{d}\theta) \\ &- \frac{1}{2} \int_s^T e^{Y_r} |Z_r|^2 \mathrm{d}r - \sum_{0<r\leq T} (e^{Y_r} - e^{Y_{r-}} - e^{Y_{r-}} \Delta Y_r), \end{aligned}$$

which implies

$$\begin{aligned} e^{Y_s} &= e^{h(X_T)} + \int_s^T \Big[y_r \ln y_r + z_r \sqrt{|\ln |z_r||} + k_r(\cdot) \sqrt{|\ln |k_r(\cdot)||} \, \mathrm{d}r \\ &+ e^{Y_{r-}} \left(\frac{1}{2} |Z_r|^2 + [K_{r,X_r}] \right) \Big] \mathrm{d}r \\ &- \int_s^T e^{Y_{r-}} Z_r \mathrm{d}B_r - \int_s^T \int_\Gamma e^{Y_{r-}} K_r(\theta) \mathrm{q}(\mathrm{d}r, \mathrm{d}\theta) \\ &- \frac{1}{2} \int_s^T e^{Y_{r-}} |Z_r|^2 \mathrm{d}r - \sum_{0<r\leq T} (e^{Y_r} - e^{Y_{r-}} - e^{Y_{r-}} \Delta Y_r), \end{aligned}$$

Using (4.4), we get

$$\begin{aligned} y_s &= e^{h(X_T)} + \int_s^T \left(y_r \ln y_r + z_r \sqrt{|\ln |z_r||} + k_r(\cdot) \sqrt{|\ln |k_r(\cdot)||} \right) \mathrm{d}r \\ &+ \int_s^T y_{r-} \left(\frac{1}{2} |Z_r|^2 + [K_{r,X_r}] - \frac{1}{2} |Z_r|^2 - [K_{r,X_r}] \right) \mathrm{d}s - \int_s^T y_r Z_r \mathrm{d}B_r \\ &- \int_s^T \int_\Gamma y_{r-} \left(e^{K_r(\theta)} - 1 - K_r(\theta) \right) \mathrm{q}(\mathrm{d}r, \mathrm{d}\theta) - \int_s^T \int_\Gamma y_{r-} K_r(\theta) \mathrm{q}(\mathrm{d}r, \mathrm{d}\theta), \end{aligned}$$

and, thus

$$\begin{aligned} y_s &= e^{h(X_T)} + \int_s^T \left(y_r \ln y_r + z_r \sqrt{\ln |z_r|} + k_r(\cdot) \sqrt{\ln |k_r(\cdot)|} \right) \mathrm{d}r \\ &- \int_s^T y_{r-} Z_r \mathrm{d}B_r - \int_s^T \int_\Gamma y_{r-} \left(e^{K_r(\theta)} - 1 \right) \mathrm{q}(\mathrm{d}r, \mathrm{d}\theta). \end{aligned} \tag{4.7}$$

Since $e^{h(X_T)}$ belongs to $L^2(\Omega)$, and if we take

$$\begin{aligned} z_r &= y_{r-}Z_r, k_r = y_{r-}\left(e^{K_r(\theta)}-1\right) \\ g\left(r, X_r, y_r, z_r, k_r\right) &= y_r \ln y_r + z_r\sqrt{\left|\ln |z_r|\right|} + k_r\left(\cdot\right)\sqrt{\left|\ln |k_r\left(\cdot\right)|\right|}, \end{aligned}$$

we can write

$$\begin{aligned} y_s &= e^{h(X_T)} + \int_s^T g\left(r, X_r, y_r, z_r, k_r\right) \mathrm{d}r \\ &\quad - \int_s^T z_r \mathrm{d}B_r - \int_s^T \int_\Gamma k_r(\theta) \mathrm{q}(\mathrm{d}r, \mathrm{d}\theta), \end{aligned} \tag{4.8}$$

it is clear that from theorem (4.8) the above equation has a unique solution $(y, z, k\left(\cdot\right))$. From (4.7) and (4.8), we get, taking into account that
$y_s = e^{Y_s} > 0$ for any $s \in [0, T]$

$$Z_s = \frac{z_s}{y_{s-}},$$

and

$$K_s\left(\cdot\right) = \ln\left(1 + \frac{k_s(\cdot)}{y_{s-}}\right).$$

We deduce that (4.5) admits a unique solution if and only if (4.8) admits a unique solution.
□

Conclusion

Throughout this Ph.D. dissertation, we aimed to highlight some novel existence and uniqueness results to backward stochastic differential equations driven by a jump Markov process. We have either extended some knowledge results on BSDE systems or weakened the Lipschitz condition on the generator to study a class of BSDEJs driven by a jump Markov process. The main results of this thesis are summarized as follows

1) In the Globally Lipschitz framework, we proved the existence and uniqueness of a solution to one type of BSDEJs driven by both a Brownian motion and a jump Markov process. Then, in the same context, we investigated a comparison theorem which claims that we can compare the solutions of two BSDEJs whenever we can compare their inputs. The ideas of the proofs are classical but have been exploited to extend some results to the case where the generator has less regularity. In particular, the result of the comparison principle, which is proved using the well-known Girsanov theorem, is interesting and allows us to construct a suitable sequence of BSDEJ from which we can extract a convergent sub-sequence.

2) In the non-Lipschitz framework, based on the approximating technique and the limits argument, we have tackled the following three existence and/or uniqueness results

a) We have used the first result to study the existence of a (minimal) solution for BSDE when the coefficient is continuous and satisfies the linear growth condition. We have also proved the existence of a solution for BSDE with a left continuous, increasing, and bounded generator. Finally, we have applied general results to solve a class of quadratic BSDEJ.

b) We have established an existence, uniqueness, and stability result in the case where the underlying BSDEJ's generator satisfies a local Lipschitz condition. We have applied a series of technical results to improve existing ones from the continuous case to the jump setting.

c) We gave an existence and uniqueness result in the case when the generator is of a logarithmic growth in y and k and the terminal data is exponentially integrable. Note that, the existence of this type of equation under the square integrability condition on the terminal data is still an open problem. This result can be considered as a non-trivial extension of the work of Bahlali *et al.* [11] to jump case. Roughly speaking, the set of serious technical difficulties that we have faced came from the Markov Jump part and required new techniques to get around them. As an application, the main result was used to prove the existence and uniqueness of the solution to one kind of quadratic BSDEs.

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